RELATIVE REGULAR RIEMANN-HILBERT CORRESPONDENCE

LUISA FIOROT, TERESA MONTEIRO FERNANDES, AND CLAUDE SABBAH

ABSTRACT. On the product of a complex manifold X by a complex curve S considered as a parameter space, we show a Riemann-Hilbert correspondence between regular holonomic relative $\mathcal{D}$-modules (resp. complexes) on the one hand and relative perverse complexes (resp. S-$\mathbb{C}$-constructible complexes) on the other hand.

CONTENTS

INTRODUCTION

Let X and T be complex manifolds. Relative $\mathcal{D}$-modules are modules over the sheaf $\mathcal{D}_{X \times T/T}$ of differential operators relative to the projection

$p_X : X \times T \to T$. *Holonomic* $\mathcal{D}_{X \times T/T}$-modules encode holomorphic families of holonomic $\mathcal{D}_X$-modules parametrized by T whose characteristic variety is contained in a fixed Lagrangian subset $\Lambda \subset T^*X$. This is a strong condition avoiding confluence phenomena. It describes nevertheless the generic behaviour of a deformation of a holonomic $\mathcal{D}_X$-module. From this point of view, it is natural to emphasize those $\mathcal{D}_{X \times T/T}$-modules which are $p_X^{-1}\mathcal{O}_T$-flat, that we call *strict*.

There exists a solution functor from the category of *holonomic* $\mathcal{D}_{X \times T/T}$-modules to that of perverse $\mathbb{C}$-constructible complexes of $p_X^{-1}\mathcal{O}_T$-modules, objects defined in [15]. Strictness (i.e., flatness) on the $\mathcal{D}_{X \times T/T}$-module side corresponds to the property that the Verdier dual of a perverse complex (as a complex of $p_X^{-1}\mathcal{O}_T$-modules) is also perverse.

In their previous work [17], Monteiro Fernandes and Sabbah have introduced the notion of relative regular holonomic $\mathcal{D}_{X \times T/T}$-modules. For example, if $T = \mathbb{C}^*$, a $\mathcal{D}_{X \times T/T}$-module underlying a regular mixed twistor $\mathcal{D}$-module (cf. [13]) is regular holonomic in this sense.

Convention and notation. In this article, we mainly consider the case where the parameter space T has dimension one, and we emphasize this assumption by denoting it by S. Therefore, throughout this article, S will denote a complex curve. On the other hand, we will set $d_X = \dim X$.

Our main result is a Riemann-Hilbert correspondence for regular holonomic $\mathcal{D}_{X \times S/S}$-modules, in the following form. We let $\mathsf{D}_{\mathrm{rhol}}^{\mathrm{b}}(\mathcal{D}_{X \times S/S})$ denote the full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_{X \times S/S})$ consisting of complexes having regular holonomic cohomology modules (cf. Section 1.d for details) and $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$ the category of $\mathbb{C}$-constructible complexes of $p_X^{-1}\mathcal{O}_S$-modules (cf. Section 1.b and [15]).

Theorem 1. *The functors*

$$^{\mathrm{p}}\mathrm{Sol}_X : \mathsf{D}_{\mathrm{rhol}}^{\mathrm{b}}(\mathcal{D}_{X \times S/S}) \longrightarrow \mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$$

$$\mathrm{RH}_X^S : \mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S) \longrightarrow \mathsf{D}_{\mathrm{rhol}}^{\mathrm{b}}(\mathcal{D}_{X \times S/S})$$

are quasi-inverse equivalences of categories.

The functor $^{\mathrm{p}}\mathrm{Sol}_X$ is the solution functor shifted by the dimension of X (cf. [15, §3.3] and Section 1.c), and RH_X^S is the relative Riemann-Hilbert functor (cf. [17, §3.4] and Section 2). One direction of the correspondence, namely $\mathrm{Id} \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{Sol}_X \circ \mathrm{RH}_X^S$, was proved in [17, Th. 3], and a particular case of this correspondence was obtained as Theorem 5 in loc. cit., namely, if $\mathcal{M}$ underlies a regular mixed twistor $\mathcal{D}$-module, then $\mathcal{M}$ can be recovered from $^{\mathrm{p}}\mathrm{Sol}_X(\mathcal{M})$ up to isomorphism by the formula $\mathcal{M} \simeq \mathrm{RH}_X^S({}^{\mathrm{p}}\mathrm{Sol}_X(\mathcal{M}))$.

The methods used in the present paper rely on the previous works [15], [17] as well as [19], [14], [3]. The main tools in the proof given in [17] are the good functorial properties satisfied by holonomic $\mathcal{D}$-modules underlying mixed twistor $\mathcal{D}$-modules, which include stability under pullback, localization along an hypersurface and direct image by projective morphisms.

The main problem to extend these results to more general situations is the bad behaviour of $\mathcal{D}_{X \times S/S}$-holonomicity by pullback in general ([16, Ex. 2.4]). We realized however that we can avoid these general arguments for *regular* holonomic $\mathcal{D}_{X \times S/S}$-modules. We replace them by proving that regular holonomiciy behaves well with respect to pullback:

Theorem 2. *Let $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$ and let $f : Y \to X$ be a morphism of complex manifolds, then $_{\mathrm{D}}f^* \mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{Y \times S/S})$.*

Let us indicate the main points in the proof of Theorem 1 in Section 3. The first tool is [17, Th. 3] which asserts that there exists a natural transformation

$$\mathrm{Id}_{\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)} \xrightarrow{\ \alpha\ } {}^{\mathrm{p}}\mathrm{Sol}_X \circ \mathrm{RH}^S_X$$

providing a functorial isomorphism

$$F \xrightarrow[\sim]{\ \alpha_F\ } {}^{\mathrm{p}}\mathrm{Sol}_X(\mathrm{RH}^S_X(F))$$

for any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$. We are then reduced to proving that there exists a natural transformation

$$\mathrm{Id}_{\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})} \xrightarrow{\ \beta\ } \mathrm{RH}^S_X \circ {}^{\mathrm{p}}\mathrm{Sol}_X$$

such that, for any $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$, denoting by

$$\beta_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathrm{RH}^S_X({}^{\mathrm{p}}\mathrm{Sol}_X(\mathcal{M})))$$

the unique morphism such that ${}^{\mathrm{p}}\mathrm{Sol}_X(\beta_{\mathcal{M}}) \circ \alpha_{{}^{\mathrm{p}}\mathrm{Sol}_X(\mathcal{M})} = \mathrm{Id}_{{}^{\mathrm{p}}\mathrm{Sol}_X(\mathcal{M})}$, we have

$$\mathcal{M} \xrightarrow[\sim]{\ \beta_{\mathcal{M}}\ } \mathrm{RH}^S_X({}^{\mathrm{p}}\mathrm{Sol}_X(\mathcal{M})).$$

The proof of the existence of such a β is reduced to that of the following result, which is equivalent to Theorem 1 by an argument already used in [17, §4.3]:

Theorem 3. *For any $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$ and for any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ the complex $R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathrm{RH}^S_X(F))$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$.*

The proof of Theorem 3 follows the ideas of [11, Lem. 4.1.4] (see also Kashiwara's proof [5, §8.3] in the absolute case).

Although we directly prove Theorem 1 in the torsion-free (strict) case and in the torsion case separately, the general case needs Theorem 3 in order to apply induction on the dimension of the support. The proof of Theorem 3 uses the torsion-free case of Theorem 1 for the induction step and proceeds by considering the case of $\mathcal{D}_{X \times S/S}$-modules of D-type (normal crossing case).

We know (cf. [15, §3]) that the functor ${}^{\mathrm{p}}\mathrm{Sol}_X$ transforms duality on $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X \times S/S})$ to Poincaré-Verdier duality on $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$. A consequence of the Riemann-Hilbert correspondence of Theorem 1 and the full faithfulness of RH^S_X is the good behaviour of the functor RH^S_X with respect to Poincaré-Verdier duality on the one hand, and duality for $\mathcal{D}_{X \times S/S}$-modules on the other hand.

Corollary 4. *For any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$, there exists an isomorphism in* $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$

$$\boldsymbol{D}(\mathrm{RH}^S_X(F)) \simeq \mathrm{RH}^S_X(\boldsymbol{D}F)$$

which is functorial in F.

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1. Review on the relative holonomic $\mathcal{D}_{X \times T/T}$-modules and constructible complexes

In this section, we review the main definitions and properties of the objects entering the relative Riemann-Hilbert correspondence. We refer to [15, 17, 16] for details. We also give supplementary properties that will happen to be useful in the proof of the main results of this article.

1.a. **Holonomic $\mathcal{D}_{X \times T/T}$-modules.** We denote by $\mathcal{D}_{X \times T/T}$ the subsheaf of $\mathcal{D}_{X \times T}$ of relative differential operators with respect to the projection

$$p_X : X \times T \longrightarrow T,$$

that we simply denote by p when there is no ambiguity. This is a Noetherian sheaf of rings. A coherent $\mathcal{D}_{X \times T/T}$-module $\mathcal{M}$ is said to be *holonomic* if $\mathrm{Char}(\mathcal{M}) \subseteq \Lambda \times T$ for some closed conic Lagrangian complex analytic subset Λ of T^*X (see [3, Lem. 2.10] for a more precise description of the characteristic variety).

Example 1.1. Let $f_1, \ldots, f_p : X \to \mathbb{C}$ be holomorphic functions. We let here $T = \mathbb{C}^p$ with coordinates $s_1, \ldots, s_p$ and we consider the partially algebraic version $\mathcal{D}_X[s_1, \ldots, s_p]$ of $\mathcal{D}_{X \times T/T}$. Let M be a holonomic $\mathcal{D}_X$-module and let m be a local section of M. Extending [4] (which is the case $p = 1$), one considers the $\mathcal{D}_X[s_1, \ldots, s_p]$-submodule $\mathcal{M} = \mathcal{D}_X[s_1, \ldots, s_p] \cdot m \cdot f_1^{s_1} \cdots f_p^{s_p}$ of $M[(\prod_i f_i)^{-1}][s_1, \ldots, s_p]$. It is proved in [12, Prop. 13] that $\mathcal{M}$ is relatively holonomic.

Example 1.2. Set $S = \mathbb{C}^*$ with coordinate z. Any mixed twistor $\mathcal{D}$-module, in the sense of [13], which consists of data parametrized by $\mathbb{P}^1$, gives rise, when restricting the parameter to $S = \mathbb{C}^*$, to a holonomic $\mathcal{D}_{X \times S/S}$-module (this is by definition, cf. [18, Chap. 1]).

We denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X \times T/T})$ the full subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{X \times T/T})$ whose complexes have holonomic cohomologies.

Given $t_o \in T$, let i_{t_o} denote the inclusion $X \times \{t_o\} \hookrightarrow X \times T$. Following [15], we denote by

$$Li^*_{t_o} : \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X \times T/T}) \longrightarrow \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$$

the derived functor

$$(1) \qquad\qquad p^{-1}(\mathcal{O}_T/\mathfrak{m}_{t_o}) \overset{L}{\otimes}_{p_X^{-1}\mathcal{O}_T} (\bullet),$$

where $\mathfrak{m}_{t_o}$ is the maximal ideal of functions vanishing at t_o. Thanks to the variant of Nakayama's lemma [17, Prop. 1.9 & Cor. 1.10], the family of functors $Li_{t_o}^*$ on $\mathsf{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_{X \times T/T})$, for $t_o \in T$, is a conservative family, i.e., if $\phi : \mathcal{M} \to \mathcal{N}$ is a morphism in $\mathsf{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_{X \times T/T})$ such that $Li_{t_o}^* \phi$ is an isomorphism in $\mathsf{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_X)$ for each $t_o \in T$ then ϕ is an isomorphism (or, equivalently, using the mapping cone: if $\mathcal{M} \in \mathsf{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_{X \times T/T})$ is such that $Li_{t_o}^* \mathcal{M} = 0$ for each $t_o \in T$ then $\mathcal{M} = 0$).

Recall (cf. [15]) that a coherent $\mathcal{D}_{X \times T/T}$-module is said to be *strict* if it is $p_X^{-1}\mathcal{O}_T$-flat. If $\mathcal{M}$ is strict, $Li_{t_o}^* \mathcal{M}$ consists of a single coherent $\mathcal{D}_X$-module $i_{t_o}^* \mathcal{M}$ (in degree zero). For example (recall that $\dim S = 1$) a coherent $\mathcal{D}_{X \times S/S}$-module $\mathcal{M}$ is strict if and only if it has no $p_X^{-1}\mathcal{O}_S$-torsion. If $\mathcal{M}$ is possibly not strict, we shall denote by $t(\mathcal{M})$ its (coherent) submodule consisting of germs of sections which are torsion elements for the $p_X^{-1}\mathcal{O}_S$-action, and $f(\mathcal{M}) := \mathcal{M}/t(\mathcal{M})$ is called its strict (or torsion-free) quotient. Therefore, the $\mathcal{D}_{X \times S/S}$-module $\mathcal{M}$ is strict if and only if $\mathcal{M} \simeq f(\mathcal{M})$.

Given $\mathcal{M} \in \mathsf{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_{X \times T/T})$, the functor

$$\boldsymbol{D}(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}_{X \times T/T}}(\mathcal{M}, \mathcal{D}_{X \times T/T} \otimes_{\mathcal{O}_{X \times T}} \Omega_{X \times T/T}^{\otimes^{-1}})[d_X]$$

provides a duality in $\mathsf{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_{X \times T/T})$ but, contrary to the absolute case (i.e., $\dim T = 0$), this functor is not t-exact. For example, when the parameter space has dimension one, the lack of exactness of the dual functor on $\mathsf{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_{X \times S/S})$ is due to the fact that the dual of a torsion holonomic $\mathcal{D}_{X \times S/S}$-module $\mathcal{M}$ is not concentrated in degree zero: if $\mathcal{M} \simeq t(\mathcal{M})$ we have $\boldsymbol{D}(\mathcal{M}) \simeq \mathcal{H}^1(\boldsymbol{D}(\mathcal{M}))[-1]$. On the other hand, if $\mathcal{N}$ is a strict holonomic $\mathcal{D}_{X \times T/T}$-module, then $\boldsymbol{D}(\mathcal{N}) \simeq \mathcal{H}^0(\boldsymbol{D}(\mathcal{N}))$ and $\mathcal{H}^0(\boldsymbol{D}(\mathcal{N}))$ is strict (see [17, Prop. 2]), that is, $\mathcal{N}$ is also dual holonomic: recall that a complex $\mathcal{N}$ in $\mathsf{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_{X \times T/T})$ is called *dual holonomic* if it is in the heart of the t-structure Π (see [3, §2]) which, by definition, is the t-structure dual to the canonical t-structure.

The following Lemma will be used in the next Theorem 1.4 which plays an important role in Section 3.b.

Lemma 1.3. *Let $\mathcal{M}$ be a coherent (resp. holonomic) $\mathcal{D}_{X \times T/T}$-module and let Z be a closed analytic subset of X. Then $\Gamma_{Z \times T}\mathcal{M}$ is $\mathcal{D}_{X \times T/T}$-coherent (resp. holonomic).*

Proof. The question is local. Let $F_\bullet \mathcal{M}$ be a good filtration of $\mathcal{M}$. According e.g. to [20, Prop. 1.9], $\Gamma_{Z \times T}F_k\mathcal{M}$ is $\mathcal{O}_{X \times T}$-coherent for each k, and thus $\mathcal{D}_{X \times T/T} \cdot \Gamma_{Z \times T}F_k\mathcal{M}$ is a $\mathcal{D}_{X \times T/T}$-coherent submodule of $\mathcal{M}$. Since $\Gamma_{Z \times T}\mathcal{M} = \bigcup_k \mathcal{D}_{X \times T/T} \cdot \Gamma_{Z \times T}F_k\mathcal{M}$, we conclude that $\Gamma_{Z \times T}\mathcal{M}$ is $\mathcal{D}_{X \times T/T}$-coherent since $\mathcal{D}_{X \times T/T}$ is Noetherian. The statement for holonomic modules obviously follows, since the characteristic variety of $\Gamma_{Z \times T}\mathcal{M}$ is contained in that of $\mathcal{M}$.

q.e.d.

Theorem 1.4. *Let* $\mathcal{M}, \mathcal{N}$ *be strict holonomic* $\mathcal{D}_{X \times S/S}$-*modules. Then the* $p_X^{-1}\mathcal{O}_S$-*module* $\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{N})_{(x,s)}$ *is finitely generated for any point* $(x, s) \in X \times S$.

Proof. We adapt the proof of [6, Th. 4.45]. Let $(X_\alpha)_{\alpha \in A}$ be a μ-stratification of X compatible with $\mathcal{M}$ and $\mathcal{N}$, let U be the union of strata of maximal dimension on which $\mathcal{M}$ and $\mathcal{N}$ are not identically zero and let Z denotes its closure in X. Then Z is a closed analytic subset of X. We argue by induction on the dimension of Z. Set $Z' = Z \smallsetminus U$. This is a closed analytic subset of Z of dimension $< \dim Z$. Let $i : U \hookrightarrow X' := X \smallsetminus Z'$ denote the inclusion. By Kashiwara's equivalence theorem in the relative setting (cf. [17, Th. 1.5]), we have

$$\mathcal{H}om_{\mathcal{D}_{X' \times S/S}}(\mathcal{M}_{|X' \times S}, \mathcal{N}_{|X' \times S}) \simeq i_* \mathcal{H}om_{\mathcal{D}_{U \times S/S}}({}_\mathrm{D}i^* \mathcal{M}_{|X' \times S}, {}_\mathrm{D}i^* \mathcal{N}_{|X' \times S}).$$

Moreover, for any open subset $\widetilde{U}$ in X such that $\widetilde{U} \cap Z = U$, the restriction $\mathcal{M}_{|\widetilde{U} \times S}$ is holonomic and its characteristic variety is contained in the union of the sets $T_{X_\alpha}^* \widetilde{U} \times S$ for any stratum X_α in U, hence ${}_\mathrm{D}i^* \mathcal{M}_{|X' \times S}, {}_\mathrm{D}i^* \mathcal{N}_{|X' \times S}$ are $\mathcal{O}_{U \times S}$-coherent, and thus of the form $\mathcal{O}_{U \times S} \otimes_{p_U^{-1}\mathcal{O}_S} \mathcal{F}$ and $\mathcal{O}_{U \times S} \otimes_{p_U^{-1}\mathcal{O}_S} \mathcal{G}$ for some $p_U^{-1}\mathcal{O}_S$-coherent S-local systems $\mathcal{F}, \mathcal{G}$ on $U \times S$, according to [1, Th. I.2.23(iii)]. Kashiwara's equivalence implies that

$$\mathcal{H}om_{\mathcal{D}_{U \times S/S}}({}_\mathrm{D}i^* \mathcal{M}_{|X' \times S}, {}_\mathrm{D}i^* \mathcal{N}_{|X' \times S}) \simeq \mathcal{H}om_{p_U^{-1}\mathcal{O}_S}(\mathcal{F}, \mathcal{G}),$$

which is thus also a $p_U^{-1}\mathcal{O}_S$-coherent local system. Therefore,

$$\mathcal{H}om_{\mathcal{D}_{X' \times S/S}}(\mathcal{M}_{|X' \times S}, \mathcal{N}_{|X' \times S}) \simeq i_* \mathcal{H}om_{p_U^{-1}\mathcal{O}_S}(\mathcal{F}, \mathcal{G})$$

is S-$\mathbb{C}$-constructible. Let $j : X' \hookrightarrow X$ denote the open inclusion. According to [15, Cor. 2.8 & Prop. 2.20], the sheaf $j_* \mathcal{H}om_{\mathcal{D}_{X' \times S/S}}(\mathcal{M}_{|X' \times S}, \mathcal{N}_{|X' \times S})$ is S-$\mathbb{C}$-constructible. By considering the exact sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \Gamma_{Z' \times S}\mathcal{N}) \longrightarrow \mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{N})$$
$$\longrightarrow j_* j^{-1} \mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{N}),$$

we are reduced to proving the result in the case where $\mathcal{N}$ is replaced with $\mathcal{N}' := \Gamma_{Z' \times S}\mathcal{N}$, which is holonomic by Lemma 1.3. Thus $\mathcal{N}'$ is strict if $\mathcal{N}$ is so. Since $\mathcal{M}$ is strict holonomic, $\boldsymbol{D}\mathcal{M}$ is also strict holonomic (cf. [17, Cor. 1.12]), hence so is $\Gamma_{Z' \times S}\boldsymbol{D}\mathcal{M}$, as well as $\mathcal{M}' := \boldsymbol{D}\Gamma_{Z' \times S}\boldsymbol{D}\mathcal{M}$. On the other hand, since $\mathcal{N}'$ is strict holonomic, $\boldsymbol{D}\mathcal{N}'$ satisfies the same properties, and the natural morphism

$$\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\boldsymbol{D}\mathcal{N}', \Gamma_{Z' \times S}\boldsymbol{D}\mathcal{M}) \longrightarrow \mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\boldsymbol{D}\mathcal{N}', \boldsymbol{D}\mathcal{M})$$

is an isomorphism. Since $\mathcal{N}'$ and $\Gamma_{Z' \times S}\boldsymbol{D}\mathcal{M}$ are $\mathcal{D}_{X \times S/S}$-coherent, they satisfy the biduality isomorphism and by [15, (3)] we conclude

$$\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}', \mathcal{N}') \xrightarrow{\ \sim\ } \mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{N}').$$

Now the induction hypothesis applies to the pair $(\mathcal{M}', \mathcal{N}')$ with corresponding set Z'.   q.e.d.

We recall the following result in [3, Lem. 2.10]:

Proposition 1.5. *For any holonomic $\mathcal{D}_{X \times T/T}$-module $\mathcal{M}$ we have*

$$\mathrm{Char}(\mathcal{M}) = \bigcup_{i \in I} \Lambda_i \times T_i$$

for some closed $\mathbb{C}^*$-conic irreducible Lagrangian subsets Λ_i of T^*X and some closed analytic subsets T_i of T, and, locally on X, the set I is finite. Moreover $p_X(\mathrm{Supp}(\mathcal{M})) = \bigcup_{i \in I} T_i$, hence it is an analytic subset of T, and

$$\dim \mathrm{Char}(\mathcal{M}) = \dim X + t, \quad where\ t = \dim p_X(\mathrm{Supp}(\mathcal{M})) = \sup_{i \in I} \dim T_i.$$

1.b. **Relative constructible and perverse complexes.**

1.b.1. *Relative local systems.* Following [15] we say that a sheaf $p_X^{-1}\mathcal{O}_T$-module F is T-locally constant coherent if, for each point $(x_0, t_o) \in X \times T$ there exists a neighborhood $U = V_{x_0} \times T_{t_o}$ and a coherent sheaf $G^{(x_0, t_o)}$ of $\mathcal{O}_{T_{t_o}}$-modules such that $F_{|U} \cong p_{V_{x_0}}^{-1}(G^{(x_0, t_o)})$. We refer to [17, App.] for basic properties.

By definition, $\mathsf{D}^{\mathrm{b}}_{\mathrm{lc\,coh}}(p_X^{-1}\mathcal{O}_T)$ is the full subcategory of $\mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_T)$ whose complexes have T-locally constant coherent cohomologies (notice that, for such an F, $F_{|\{x_0\} \times T} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_T)$). We refer to [15, §2] for more properties.

1.b.2. *Relative $\mathbb{R}$-constructibility.* In the following, we will have to consider derived categories $\mathsf{D}^\star$ with $\star = \mathrm{b}$ or $\star = -$. We denote by $\mathsf{D}^\star_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_T)$ the full subcategory of $\mathsf{D}^\star(p_X^{-1}\mathcal{O}_T)$ whose objects F admit a μ-stratification (X_α) of X such that $i_\alpha^{-1}(F) \in \mathsf{D}^\star_{\mathrm{lc\,coh}}(p_{X_\alpha}^{-1}\mathcal{O}_T)$ for any α. We refer to [15, §2] for details.

Objects of $\mathsf{D}^-_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_T)$ can be given a simple representative. Let us denote by $\mathcal{S}$ the full additive subcategory of $\mathsf{Mod}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_T)$ whose objects are sheaves which can be expressed as locally finite direct sums of terms of the form $\mathbb{C}_\Omega \boxtimes \mathcal{O}_V := p_X^{-1}\mathcal{O}_T \otimes \mathbb{C}_{\Omega \times V}$ for some relatively compact open subanalytic subsets Ω in X and V in T. A morphism $\varphi : \mathbb{C}_\Omega \boxtimes \mathcal{O}_V \to \mathbb{C}_{\Omega'} \boxtimes \mathcal{O}_{V'}$ is easily described: setting $\Omega'' = \Omega \cap \Omega'$ and $V'' = V \cap V'$, φ is the extension by zero of its restriction $\varphi_{|\Omega'' \times V''} : p_X^{-1}\mathcal{O}_{T|\Omega'' \times V''} \to p_X^{-1}\mathcal{O}_{T|\Omega'' \times V''}$, which is the multiplication by a section of $p_X^{-1}\mathcal{O}_T$ on each connected component of $\Omega'' \times V''$.

Following the terminology of [8, App. A], we say that an object F of $\mathsf{Mod}(p_X^{-1}\mathcal{O}_T)$ is $\mathcal{S}$-coherent if there exist $L \in \mathcal{S}$ and an epimorphism $L \to F$, and if, for any morphism $L' \to F$ with L' in $\mathcal{S}$, there exist L'' in $\mathcal{S}$ and a morphism $L'' \to L'$ such that $L'' \to L' \to F$ is exact. It follows from [17, Prop. 3.5] that the category of $\mathcal{S}$-coherent objects of $\mathsf{Mod}(p_X^{-1}\mathcal{O}_T)$ is equal to $\mathsf{Mod}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_T)$ and the category $\mathsf{D}^-_{\mathcal{S}\text{-coh}}(\mathsf{Mod}(p_X^{-1}\mathcal{O}_T))$ is nothing but $\mathsf{D}^-_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_T)$. On the other hand, one defines the category $\mathsf{D}^-_{\mathrm{coh}}(\mathcal{S})$ as in [8, p. 63], with the identification $\mathbf{A} = \mathsf{Mod}(p_X^{-1}\mathcal{O}_T)$ and $\mathbf{P} = \mathcal{S}$ (the functor L of loc. cit. is here the inclusion, H is the restriction of $\mathrm{Hom}_{\mathsf{Mod}(p_X^{-1}\mathcal{O}_T)}$ to $\mathcal{S} \times \mathsf{Mod}(p_X^{-1}\mathcal{O}_T)$ and α is the identity). In the present situation, we have $\mathsf{D}^-_{\mathrm{coh}}(\mathcal{S}) = \mathsf{D}^-(\mathcal{S})$.

Proposition 1.6 (cf. [8, Th. A.5]). *The natural functor*

$$L : \mathsf{D}^-(\mathcal{S}) \longrightarrow \mathsf{D}^-_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_T)$$

is an equivalence of categories.

Proof. We only need to check that the conditions for applying [8, Th. A.5] are fulfilled, that is, that the pair $(\mathsf{Mod}(p_X^{-1}\mathcal{O}_T), \mathcal{S})$ satisfies the properties (A.1)–(A.4) for $(\mathbf{A}, \mathbf{P})$ in loc. cit., and only (A.3) is not obvious. It is proved in the lemma below. q.e.d.

Lemma 1.7. *Let F and G be objects of $\mathsf{Mod}(p_X^{-1}\mathcal{O}_T)$, let $\psi : F \to G$ be an epimorphism, let $\mathcal{T} \in \mathcal{S}$ and let $g : \mathcal{T} \to G$ be given. Then there exist an object $\mathcal{T}' \in \mathcal{S}$, an epimorphism $\psi' : \mathcal{T}' \to \mathcal{T}$ and a morphism $g' : \mathcal{T}' \to F$ such that $\psi g' = g \psi'$.*

Proof. By $p_X^{-1}\mathcal{O}_T$-linearity we may reduce to the case $\mathcal{T} = p_X^{-1}\mathcal{O}_T \otimes \mathbb{C}_{\Omega \times V}$, for some relatively compact open subanalytic subsets Ω, V respectively in X and T.

Let e be the section $1 \in \Gamma(\Omega \times V, \mathcal{T})$. By the assumption on ψ, we can cover Ω (resp. V) by a locally finite family of relatively compact open subanalytic sets $\Omega_i \subset X$ (resp. $V_i \subset T$), $i = 1, \ldots, m$, and find sections $f_i \in \Gamma(\Omega_i \times V_i, F)$ such that $\psi|_{\Omega_i \times V_i}(f_i) = g(e)|_{\Omega_i \times V_i}$. The morphism $\mathbb{C}_{|\Omega_i \times V_i} \to F_{|\Omega_i \times V_i}$ extends in a unique way as a morphism $\mathbb{C}_{\Omega_i \times V_i} \to F$ and, by $p_X^{-1}\mathcal{O}_T$-linearity, as a morphism $p_X^{-1}\mathcal{O}_T \otimes \mathbb{C}_{\Omega_i \times V_i} \to F$.

Setting $\mathcal{T}' := \bigoplus_{i=1}^m p_X^{-1}\mathcal{O}_T \otimes \mathbb{C}_{\Omega_i \times V_i}$, we obtain in this way a $p_X^{-1}\mathcal{O}_T$-linear morphism $g' : \mathcal{T}' \to F$. On the other hand, by the covering property, the natural morphism $\mathbb{C}_{\Omega_i \times V_i} \to \mathbb{C}_{\Omega \times V}$ which extends $\mathrm{Id} : \mathbb{C}_{|\Omega_i \times V_i} \to \mathbb{C}_{\Omega \times V|\Omega_i \times V_i}$ induces an epimorphism $\bigoplus_{i=1}^m \mathbb{C}_{\Omega_i \times V_i} \to \mathbb{C}_{\Omega \times V}$ and, by $p_X^{-1}\mathcal{O}_T$-linearity, an epimorphism $\psi' : \mathcal{T}' \to \mathcal{T}$, which clearly satisfies $\psi g' = g \psi'$. q.e.d.

We note the following, to be used in the course of the proof of Lemma 2.5:

Remark 1.8. Let Φ, Ψ be two triangulated functors from $\mathsf{D}^-_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_T)$ to a triangulated category C. Any morphism of functors $\eta_{\mathcal{S}} : \Phi_{\mathcal{S}} \to \Psi_{\mathcal{S}}$ (with $\Phi_{\mathcal{S}} = \Phi \circ L$, $\Psi_{\mathcal{S}} = \Psi \circ L$) can be extended to a morphism of functors $\eta : \Phi \to \Psi$.

1.b.3. *Relative $\mathbb{C}$-constructibility.* By definition (cf. [15, Def. 2.19]), the full subcategory $\mathsf{D}^b_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_T)$ consists of objects of $\mathsf{D}^b_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_T)$ whose micro-support is $\mathbb{C}^*$-conic. We call these objects *T-$\mathbb{C}$-constructible complexes*.

For any $t_o \in T$, there is a functor

$$Li_{t_o}^* : \mathsf{D}^b(p_X^{-1}\mathcal{O}_T) \longrightarrow \mathsf{D}^b(\mathbb{C}_X)$$

also defined by (1). It sends $\mathsf{D}^b_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_T)$ to $\mathsf{D}^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and $\mathsf{D}^b_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_T)$ to $\mathsf{D}^b_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$.

Recall (cf. [15, Prop. 2.2]) that a variant of Nakayama's lemma holds for complexes F in $\mathsf{D}^b(p_X^{-1}\mathcal{O}_T)$ whose cohomology objects $\mathcal{H}^j F$ have fibers $\mathcal{H}^j F_{(x,s)}$ of finite type over $\mathcal{O}_{T,s}$ for any $(x, s) \in X \times T$. As a consequence,

the family of functors $(Li_{t_o}^*)_{t_o \in T}$ on $\mathsf{D}_{\mathbb{R}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_T)$ (resp. $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_T)$) is a conservative family (in particular, if $F \in \mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_T)$ satisfies $Li_{t_o}^* F = 0$ for each $t_o \in T$, then $F = 0$).

1.b.4. *Perversity.* The category $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_T)$ is endowed with a perverse t-structure defined in [15, §2.7] as the relative analogue to the middle perverse t-structure in the absolute case where $\dim T = 0$:

- ${}^{\mathrm{p}}\mathsf{D}_{\mathbb{C}\text{-c}}^{\leqslant 0}(p_X^{-1}\mathcal{O}_T)$ is the full subcategory of objects F of $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_T)$ such that there exists an adapted μ-stratification (X_α) of X for which $i_x^{-1} F \in \mathsf{D}_{\mathrm{coh}}^{\leqslant -d_{X_\alpha}}(\mathcal{O}_T)$ for any $x \in X_\alpha$ and any α.
- ${}^{\mathrm{p}}\mathsf{D}_{\mathbb{C}\text{-c}}^{\geqslant 0}(p_X^{-1}\mathcal{O}_T)$ is the full subcategory of objects F of $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_T)$ such that there exists an adapted μ-stratification (X_α) of X for which $i_x^! F \in \mathsf{D}_{\mathrm{coh}}^{\geqslant d_{X_\alpha}}(\mathcal{O}_T)$ for any $x \in X_\alpha$ and any α.

The heart of this t-structure is the abelian category of *relative perverse sheaves* denoted by $\mathrm{perv}(p_X^{-1}\mathcal{O}_T)$. We often omit the word "relative".

In analogy with the $\mathcal{D}_{X \times S/S}$-module counterpart ($S$ is a curve), following [3, Prop. 3.12], we say that a perverse sheaf is *torsion* if it belongs to the subcategory $\mathrm{perv}(p_X^{-1}\mathcal{O}_S)_{\mathrm{t}}$ of $\mathrm{perv}(p_X^{-1}\mathcal{O}_S)$ whose objects F satisfy $\mathrm{codim}\, p(\mathrm{Supp}\, F) \geqslant 1$ (cf. [3, Cor. 3.1] for this condition), while a perverse sheaf is called *strictly perverse* if it belongs to the full subcategory $\mathrm{perv}(p_X^{-1}\mathcal{O}_S)_{\mathrm{tf}}$ of $\mathrm{perv}(p_X^{-1}\mathcal{O}_S)$ whose objects F satisfy $Li_s^* F \in \mathrm{perv}(\mathbb{C}_X)$ for all $s \in S$. The category $\mathrm{perv}(p_X^{-1}\mathcal{O}_S)_{\mathrm{t}}$ is a full thick abelian subcategory of the category $\mathrm{perv}(p_X^{-1}\mathcal{O}_S)$.

We denote by $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)_{\mathrm{t}}$ the thick subcategory of $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$ whose objects have support in $X \times T$, where T is a subset of S with $\dim T = 0$ or, equivalently, whose perverse cohomologies belong to $\mathrm{perv}(p_X^{-1}\mathcal{O}_S)_{\mathrm{t}}$.

Given an object F of $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$, the functor $\boldsymbol{D}$ defined by

$$\boldsymbol{D}(F) = R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, p_X^{-1}\mathcal{O}_S)[2d_X]$$

provides a duality in $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$, which is however not t-exact with respect to the perverse t-structure. For example, if F is a torsion perverse sheaf, then $\boldsymbol{D}(F) \simeq {}^{\mathrm{p}}\mathcal{H}^1(\boldsymbol{D}(F))[-1]$ (it is a perverse sheaf shifted in degree 1), while if F is a strictly perverse sheaf, $\boldsymbol{D}(F) \simeq {}^{\mathrm{p}}\mathcal{H}^0(\boldsymbol{D}(F))$ is perverse too.

Let us recall that an object F of $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$ is called *dual perverse* if it is in the heart of the t-structure π, which by definition is the t-structure dual to the perverse t-structure introduced in [15, §2.7]. By [17, Lem. 1.4], a complex $F \in \mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$ is perverse and dual perverse if and only if it is strictly perverse.

1.c. **The relative solution functor.** The solution functor for a coherent $\mathcal{D}_{X \times T/T}$-module or an object of $\mathsf{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{D}_{X \times T/T})$ is defined by

$$
\begin{aligned}
{}^{\mathrm{p}}\mathrm{Sol}_X : \mathsf{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{D}_{X \times T/T}) &\longrightarrow \mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_T) \\
\mathcal{M} &\longmapsto R\mathcal{H}om_{\mathcal{D}_{X \times T/T}}(\mathcal{M}, \mathcal{O}_{X \times T})[d_X]
\end{aligned}
$$

By [15, Th. 3.7], when restricted to $\mathsf{D}^b_{\mathrm{hol}}(\mathcal{D}_{X \times T/T})$ the solution functor $^{\mathrm{P}}\mathrm{Sol}_X$ takes values in $\mathsf{D}^b_{\mathbb{C}\text{-}c}(p_X^{-1}\mathcal{O}_T)$ and by [3, Cor. 4.3], is t-exact with respect to the t-structure Π in $\mathsf{D}^b_{\mathrm{hol}}(\mathcal{D}_{X \times T/T})$ and the perverse one p in $\mathsf{D}^b_{\mathbb{C}\text{-}c}(p_X^{-1}\mathcal{O}_T)$.

1.d. **Regular holonomic complexes of $\mathcal{D}_{X \times S/S}$-modules.** In this section, we review the notion of relative regularity as introduced in [17, §2.1] and recall the fundamental example of relative $\mathcal{D}_{X \times S/S}$-modules of D-type.

Definition 1.9 (Regularity, [17, Def. 2.1]). A holonomic $\mathcal{D}_{X \times S/S}$-module $\mathcal{M}$ is said to be *regular* if, for any $s_o \in S$, the object $Li^*_{s_o}\mathcal{M}$ of $\mathsf{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ has regular holonomic cohomologies.

Example 1.10.
  (a) In Example 1.1 let us assume that M is regular. Then the $\mathcal{D}_{X \times S/S}$-module generated by $\mathcal{M}$ is regular.
  (b) In Example 1.2, assume that the mixed twistor $\mathcal{D}$-module is regular in the sense of [18, Def. 4.1.2]. Then the underlying holonomic $\mathcal{D}_{X \times S/S}$-module $\mathcal{M}$ is regular.

According to [17, §2.1], we say that an object $\mathcal{M} \in \mathsf{D}^b_{\mathrm{hol}}(\mathcal{D}_{X \times S/S})$ is *regular* if each of its cohomology modules is regular.

Remark 1.11. An object $\mathcal{M}$ of $\mathsf{D}^b_{\mathrm{hol}}(\mathcal{D}_{X \times S/S})$ is regular if and only if, for each $s_o \in S$, the object $Li^*_{s_o}\mathcal{M}$ of $\mathsf{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ has regular holonomic cohomology. Indeed, we argue by induction on the amplitude of the complex $\mathcal{M}$. Without loss of generality, we may assume that $\mathcal{M} \in \mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X \times S/S})$ and we consider the following distinguished triangle

$$\mathcal{H}^0\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \tau^{\geqslant 1}\mathcal{M} \xrightarrow{+1}$$

(where $\tau^{\geqslant 1}$ is the truncation functor with respect to the natural t-structure on $\mathsf{D}^b_{\mathrm{hol}}(\mathcal{D}_{X \times S/S})$). We deduce $\mathcal{H}^{-1}Li^*_{s_o}\mathcal{H}^0(\mathcal{M}) \simeq \mathcal{H}^{-1}Li^*_{s_o}(\mathcal{M})$ and an exact sequence

$$0 \longrightarrow \mathcal{H}^0 Li^*_{s_o}\mathcal{H}^0\mathcal{M} \longrightarrow \mathcal{H}^0 Li^*_{s_o}\mathcal{M} \longrightarrow \mathcal{H}^0 Li^*_{s_o}\tau^{\geqslant 1}\mathcal{M} \longrightarrow 0.$$

(Note that $\mathcal{H}^k Li^*_{s_o}\mathcal{H}^0(\mathcal{M}) = 0$ for $k \neq 0, -1$.) The assertion follows from the induction hypothesis and the property that the category of regular holonomic $\mathcal{D}_X$-modules is closed under sub-quotients in the category $\mathsf{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$.

We recall the following result in [17]:

Proposition 1.12 (cf. [17, Cor. 2.4]). *Let $f : Y \to X$ be a morphism of complex manifolds and let us assume that f is proper. Then, for each $\mathcal{M} \in \mathsf{D}^b_{\mathrm{rhol}}(\mathcal{D}_{Y \times S/S})$, the pushforward $_{\mathrm{D}}f_*\mathcal{M}$ is an object of $\mathsf{D}^b_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$.*

1.d.1. *Proof of Theorem 2 for a smooth morphism.* Let $f : Y \to X$ be a smooth morphism and let $\mathcal{M}$ be an object of $\mathsf{D}^b_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$. Due to the locality of the regular holonomic property, we may assume that f is a projection $Y = Z \times X \to X$. In that case, $\mathcal{D}_{(Y \to X) \times S}$ is $f^{-1}\mathcal{D}_{X \times S/S}$-flat, so $\mathcal{H}^j{}_{\mathrm{D}}f^*\mathcal{M} \simeq {}_{\mathrm{D}}f^*\mathcal{H}^j\mathcal{M}$ for every j, and we can assume that $\mathcal{M} = \mathcal{H}^0\mathcal{M}$.

As in the absolute case one checks that $\mathrm{Char}({}_{\mathrm{D}}f^*\mathcal{M}) = T_Z^*Z \times \mathrm{Char}(\mathcal{M})$, so ${}_{\mathrm{D}}f^*\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{Y\times S/S})$. Moreover, the commutativity of

$$
\begin{array}{ccc}
Y & \xrightarrow{\ i_s\ } & Y \times S \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{\ i_s\ } & X \times S
\end{array}
$$

implies that $Li_{s\,\mathrm{D}}^* f^*(\mathcal{M}) \simeq {}_{\mathrm{D}}f^* Li_s^*\mathcal{M}$, and the latter is known to be regular holonomic on Y. Therefore, Theorem 2 is proved for f smooth.   q.e.d.

1.d.2. $\mathcal{D}_{X\times S/S}$-*modules of D-type.* They are the fundamental examples of regular holonomic $\mathcal{D}_{X\times S/S}$-modules, so we recall their definition. Let D be a normal crossing divisor in X and let $j : X^* := X \smallsetminus D \hookrightarrow X$ denote the inclusion (we will also denote by j the morphism $j \times \mathrm{Id}_S$). Let F be a coherent S-locally constant sheaf on $X^* \times S$ and let $(V, \nabla) = (\mathcal{O}_{X^*\times S} \otimes_{p_X^{-1}\mathcal{O}_S} F, \mathrm{d}_{X/S})$ be the associated coherent $\mathcal{O}_{X^*\times S}$-module with flat relative connection. There exists a coherent $\mathcal{O}_S$-module G such that, if U is any contractible open set of X^*, then $F_{|U\times S} \simeq p_U^{-1}G$.

Let $\varpi : \widetilde{X} \to X$ denote the real oriented blowing up of X along the components of D. Denote by $\widetilde{j} : X^* \hookrightarrow \widetilde{X}$ the inclusion, so that $j = \varpi \circ \widetilde{j}$. Let $x^o \in D$, $\widetilde{x}^o \in \varpi^{-1}(x^o)$ and let $s^o \in S$. Choose local coordinates $(x_1, \ldots, x_n)$ at x^o such that $D = \{x_1 \cdots x_\ell = 0\}$ and consider the associated polar coordinates $(\boldsymbol{\rho}, \boldsymbol{\theta}, \boldsymbol{x}') := (\rho_1, \theta_1, \ldots, \rho_\ell, \theta_\ell, x_{\ell+1}, \ldots, x_n)$ so that $\widetilde{x}^o$ has coordinates $\boldsymbol{\rho}^o = 0$, $\boldsymbol{\theta}^o$, $\boldsymbol{x}'^o = 0$.

A local section $\widetilde{v}$ of $(\widetilde{j}_*V)_{(\widetilde{x}^o, s^o)}$ is said to have *moderate growth* if for some system of generator of G_{s^o}, and some neighbourhood

$$U_\varepsilon := \{\|\boldsymbol{\rho}\| < \varepsilon, \|\boldsymbol{x}'\| < \varepsilon, \|\boldsymbol{\theta} - \boldsymbol{\theta}^o\| < \varepsilon\}$$

(ε small enough) on which it is defined, its coefficients on the chosen generators of G_{s^o} (these are sections of $\mathcal{O}(U_\varepsilon^* \times U(s^o))$ for a small enough neighbourhood $U(s^o)$ of s^o in S, and $U_\varepsilon^* := U_\varepsilon \smallsetminus \{\rho_1 \cdots \rho_\ell = 0\}$) are bounded by $C\boldsymbol{\rho}^{-N}$, for some $C, N > 0$. A local section v of $(j_*V)_{(x^o, s^o)}$ is said to have *moderate growth* if for each $\widetilde{x}^o$ in $\varpi^{-1}(x^o)$, the corresponding germ in $(\widetilde{j}_*V)_{(\widetilde{x}^o, s^o)}$ has moderate growth.

On the other hand (cf. [17, Def. 2.10]), a coherent $\mathcal{D}_{X\times S/S}$-module $\mathcal{L}$ is said to be of D-type with singularities along a normal crossing divisor $D \subset X$ if it satisfies the following conditions:

(a) $\mathrm{Char}(\mathcal{L}) \subset (\pi^{-1}(D) \times S) \cup (T_X^*X \times S)$,
(b) $\mathcal{L}$ is regular holonomic and strict,
(c) $\mathcal{L} \simeq \mathcal{L}(*(D \times S))$.

The following result is proved in [17, Th. 2.6, Cor. 2.8 & Prop. 2.11]:

Theorem 1.13.

(a) *The subsheaf $\widetilde{V}$ of j_*V consisting of local sections having moderate growth is stable by ∇ and it is $\mathcal{O}_{X\times S}(*D)$-coherent. Moreover, $\widetilde{V}$ is*

     a regular holonomic $\mathcal{D}_{X \times S/S}$-module with characteristic variety contained in $\Lambda \times S$, where Λ is the union of the conormal spaces of the natural stratification of (X, D).

(b) *A coherent $\mathcal{D}_{X \times S/S}$-module $\mathcal{L}$ is of D-type on (X, D) if and only if it is isomorphic to some $\widetilde{V}$ as in* (a).

2. The relative Riemann-Hilbert functor RH^S

In this section we recall the definition of the relative Riemann-Hilbert functor $\mathrm{RH}^S(\bullet)$ introduced in [17] and state some supplementary results needed in the sequel.

2.a. **Relative subanalytic sites and relative subanalytic sheaves.** For details on this subject we refer to [14] and [2]. We also refer to [9] as a foundational paper and to [10] for a detailed exposition on the general theory of sheaves on sites.

Let X and T be real or complex analytic manifolds. One denotes by $\mathrm{Op}(X \times T)$ the family of open subsets of $X \times T$, by $\mathrm{Op}((X \times T)_{\mathrm{sa}}) \subset \mathrm{Op}(X \times T)$ the family of open subanalytic sets; $\mathcal{T} := \mathrm{Op}^{\mathrm{c}}((X \times T)_{\mathrm{sa}})$ denotes the family of relatively compact open subanalytic subsets of $X \times T$ and $\mathcal{T}' \subset \mathcal{T}$ denotes the family of finite unions of relatively compact open subanalytic sets of the form $U \times V$.

The product $X \times T$ is both a $\mathcal{T}$- as well as a $\mathcal{T}'$-space. The associated sites $(X \times T)_{\mathcal{T}}$ and $(X \times T)_{\mathcal{T}'}$ are, respectively, the subanalytic site $(X \times T)_{\mathrm{sa}}$, for which the coverings of an element $\Omega \in \mathrm{Op}((X \times T)_{\mathrm{sa}})$ are the locally finite coverings with elements in $\mathcal{T}$, and the site denoted by $X_{\mathrm{sa}} \times T_{\mathrm{sa}}$, for which the coverings of $\Omega \in \mathcal{T}'$ are the coverings with elements in $\mathcal{T}'$ which admit a finite subcovering.

We shall denote by ρ_T the natural functor of sites $\rho_T : X \times T \to (X \times T)_{\mathrm{sa}}$ associated to the inclusion $\mathrm{Op}_{\mathrm{sa}}(X \times T) \subset \mathrm{Op}(X \times T)$. Accordingly, we shall consider the associated functors $\rho_{T*}, \rho_T^{-1}, \rho_{T!}$.

We shall also denote by $\rho'_T : X \times T \to X_{\mathrm{sa}} \times T_{\mathrm{sa}}$ the functor of sites associated to the inclusion $\mathcal{T}' \subset \mathrm{Op}(X \times T)$. Following [10] we have functors ρ'_{T*} and $\rho'_{T!}$ from $\mathsf{Mod}(\mathbb{C}_{X \times T})$ to $\mathsf{Mod}(\mathbb{C}_{X_{\mathrm{sa}} \times T_{\mathrm{sa}}})$. We simply denote by ρ, resp. ρ', the previous morphism when there is no ambiguity.

Subanalytic sheaves are defined on the subanalytic site of a real analytic manifold, and relative subanalytic sheaves are defined on the subanalytic site $X_{\mathrm{sa}} \times T_{\mathrm{sa}}$. We refer to [14] for the detailed construction of the relative subanalytic sheaves $\mathfrak{Db}_{X \times T}^{t,T}$ (where X and T are real analytic) and $\mathcal{O}_{X \times T}^{t,T}$ in the complex framework (denoted $\mathfrak{Db}_{X \times T}^{t,T,\natural}$ and $\mathcal{O}_{X \times T}^{t,T,\natural}$ in [14]).

They are both $\rho'_! \mathcal{D}_{X \times T}$-modules (either in the real or the complex case) as well as a $\rho'_* p_X^{-1} \mathcal{O}_T$-modules when T is complex.

If $\mathfrak{Db}_{X \times T}^t$ denotes the subanalytic sheaf of tempered distributions introduced by Kashiwara-Schapira in [9], we have, for $U \in \mathrm{Op}(X_{\mathrm{sa}})$ and

$V \in \mathrm{Op}(T_{\mathrm{sa}})$

$$\Gamma(U \times V; \mathfrak{Db}_{X \times T}^{t,T}) = \varprojlim_{W \Subset V} \Gamma(U \times W; \mathfrak{Db}_{X \times T}^{t})$$
$$\simeq \Gamma(X \times V; \rho'^{-1} \Gamma_{U \times T} \mathfrak{Db}_{X \times T}^{t})$$
$$\simeq \Gamma(X \times V; T\mathcal{H}om(\mathbb{C}_{U \times T}, \mathfrak{Db}_{X \times T})).$$

Moreover, when X is also complex, considering the complex conjugate structure $\overline{X}$ on X (resp. $\overline{T}$ on T) and the underlying real analytic structure $X_{\mathbb{R}}$ (resp. $T_{\mathbb{R}}$), we have

$$\mathcal{O}_{X \times T}^{t,T} = R\mathcal{H}om_{\rho'_! \mathcal{D}_{\overline{X} \times \overline{T}}}(\rho'_! \mathcal{O}_{\overline{X} \times \overline{T}}, \mathcal{D}_{X \times T}^{t,T}),$$

where we omit the reference to the real structures.

2.b. The functors TH^S and RH^S.

2.b.1. *The functor* TH^S. When X is a real analytic manifold and S is a complex curve, we define the triangulated functor

$$\mathrm{TH}_X^S : \mathsf{D}_{\mathbb{R}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)^{\mathrm{op}} \longrightarrow \mathsf{D}^+(\mathcal{D}_{X \times S_{\mathbb{R}}/S})$$

given by

$$F \longmapsto \mathrm{TH}_X^S(F) := \rho'^{-1} R\mathcal{H}om_{\rho'_* p_X^{-1}\mathcal{O}_S}(\rho'_* F, \mathfrak{Db}_{X \times S}^{t,S})$$

where $\mathcal{D}_{X \times S_{\mathbb{R}}/S}$ denotes the sheaf of linear differential operators with real analytic coefficients on $X \times S_{\mathbb{R}}$ which commute with $p_X^{-1}\mathcal{O}_S$.

Recall that, as a consequence of [14, Prop. 4.7] we have

$$\mathrm{TH}_X^S(p_X^{-1}\mathcal{O}_S \otimes \mathbb{C}_{H \times V}) \simeq R\mathcal{H}om(\mathbb{C}_{X \times V}, T\mathcal{H}om(\mathbb{C}_{H \times S}, \mathfrak{Db}_{X \times S}))$$
$$\simeq R\Gamma_{X \times V} T\mathcal{H}om(\mathbb{C}_{H \times S}, \mathfrak{Db}_{X \times S}) \quad ([7, (2.6.9)])$$

for any relatively compact locally closed, resp. open, subanalytic subsets H of X, resp. V of S. If $H = Z$ is closed, we have $T\mathcal{H}om(\mathbb{C}_{Z \times S}, \mathfrak{Db}_{X \times S}) = \Gamma_{Z \times S} \mathfrak{Db}_{X \times S}$ by definition. We conclude:

$$(2) \qquad \mathrm{TH}_X^S(p_X^{-1}\mathcal{O}_S \otimes \mathbb{C}_{Z \times V}) \simeq \Gamma_{Z \times V} \mathfrak{Db}_{X \times S}.$$

On the other hand, if $H = \Omega$ is open, since $T\mathcal{H}om(\mathbb{C}_{\Omega \times S}, \mathfrak{Db}_{X \times S})$ is a c-soft sheaf, we obtain

$$(3) \qquad \mathrm{TH}_X^S(p_X^{-1}\mathcal{O}_S \otimes \mathbb{C}_{\Omega \times V}) \simeq \Gamma_{X \times V} T\mathcal{H}om(\mathbb{C}_{\Omega \times S}, \mathfrak{Db}_{X \times S}).$$

2.b.2. *The functor* RH^S. If X is a complex manifold and S is a complex curve, $\mathrm{RH}_X^S : \mathsf{D}_{\mathbb{R}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathcal{D}_{X \times S/S})$ is given by the assignment

$$(4) \qquad \begin{aligned} F \longmapsto \mathrm{RH}_X^S(F) &:= \rho'^{-1} R\mathcal{H}om_{\rho'_* p_X^{-1}\mathcal{O}_S}(\rho'_* F, \mathcal{O}_{X \times S}^{t,S})[d_X] \\ &\simeq R\mathcal{H}om_{\mathcal{D}_{\overline{X} \times \overline{S}}}(\mathcal{O}_{\overline{X} \times \overline{S}}, \mathrm{TH}_X^S(F))[d_X], \end{aligned}$$

the last isomorphism being called here "realification procedure" for short (cf. [17, (3.16)]).

We collect below some results in [17] which will be useful in the sequel. The first gives the behaviour of Li_s^* with respect to RH^S.

Proposition 2.1 (cf. [17, Prop. 3.29])**.** *For each $s \in S$ there is an isomorphism of functors on $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_S)$*

$$Li_s^* \mathrm{RH}_X^S[-d_X](\bullet) \simeq T\mathcal{H}om(Li_s^*(\bullet), \mathcal{O}_X),$$

where X is identified to $X \times \{s\}$ and X_{sa} is identified to $X_{\mathrm{sa}} \times \{s\}$.

Theorem 2.2 (cf. [17, Th. 3])**.** *Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$. Then $\mathrm{RH}_X^S(F) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$ and we have an isomorphism $F \simeq {}^{\mathrm{p}}\mathrm{Sol}_X(\mathrm{RH}_X^S(F))$ which is functorial in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$.*

Let us fix a normal crossing divisor D.

Theorem 2.3 (cf. [17, Prop. 2.11 & Lem. 4.2])**.** *The category of holonomic $\mathcal{D}_{X \times S/S}$-modules $\mathcal{L}$ of D-type with singularities along D is equivalent to the category of locally free $p_{X^*}^{-1}\mathcal{O}_S$-modules with $X^* := X \smallsetminus D$ under the correspondence*

$$\mathcal{L} \longmapsto \mathcal{H}^0 \mathrm{DR}_X(\mathcal{L})_{|X^* \times S}, \qquad F \longmapsto \mathrm{RH}_X^S(j_!\boldsymbol{D}(F[d_X])) = \mathcal{L}.$$

2.c. **Some functorial properties.** Let $f : Y \to X$ be a morphism of real or complex analytic manifolds. We denote similarly the morphism $f \times \mathrm{Id} : Y \times S \to X \times S$. We consider in this section the corresponding pullback functor.

2.c.1. *Pullback with respect to X for TH^S.*

Proposition 2.4. *For any morphism $f : Y \to X$ of real analytic manifolds there exists a morphism of functors from $\mathsf{D}^-_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_S)^{\mathrm{op}}$ to $\mathsf{D}^+(\mathcal{D}_{X \times S_{\mathbb{R}}/S})$:*

$$(5) \qquad \qquad {}_{\mathrm{D}}f_! \, \mathrm{TH}_Y^S(f^{-1}\bullet) \longrightarrow \mathrm{TH}_X^S(\bullet).$$

Proof. We first define the desired morphism functorially on the category $\mathcal{S}$ introduced in Section 1.b.2. We deduce from (3) that the objects of $\mathcal{S}$ are acyclic for $\mathrm{TH}_X^S(\bullet)$ and for $\mathrm{TH}_Y^S(f^{-1}\bullet)$.

Lemma 2.5. *For any morphism $f : Y \to X$ of real analytic manifolds there exist functors*

$$ {}_{\mathrm{D}}f_! \, \mathrm{TH}_Y^S(f^{-1}\bullet)_{|\mathcal{S}^{\mathrm{op}}} : \mathcal{S}^{\mathrm{op}} \longrightarrow \mathsf{C}^+(\mathcal{D}_{X \times S_{\mathbb{R}}/S}) $$
$$ \mathrm{TH}_X^S(\bullet)_{|\mathcal{S}^{\mathrm{op}}} : \mathcal{S}^{\mathrm{op}} \longrightarrow \mathsf{C}^+(\mathcal{D}_{X \times S_{\mathbb{R}}/S}). $$

and there exists a canonical morphism of functors

$$ {}_{\mathrm{D}}f_! \, \mathrm{TH}_Y^S(f^{-1}\bullet)_{|\mathcal{S}^{\mathrm{op}}} \longrightarrow \mathrm{TH}_X^S(\bullet)_{|\mathcal{S}^{\mathrm{op}}} $$

inducing the morphism (5)*.*

Proof. Let Ω, resp. V, be a relatively compact open subanalytic set in X, resp. S (here, we consider the real analytic structures in X and S, also usually identified by $-_{\mathbb{R}}$). We set $Z = X \smallsetminus \Omega$ and we start by considering the sheaf $G = p_X^{-1}\mathcal{O}_S \otimes \mathbb{C}_{Z \times V}$. According to (2), we have $\mathrm{TH}_X^S(G) = \Gamma_{Z \times V} \mathfrak{Db}_{X \times S}$, regarded as a $\mathcal{D}_{X_{\mathbb{R}} \times S_{\mathbb{R}}/S}$-module.

On the other hand, the integration of distributions induces a morphism

$$\int_f : f_!(\mathfrak{Db}_{Y\times S} \otimes_{\mathcal{A}_{Y\times S}} \omega_{Y\times S/S}) \longrightarrow \mathfrak{Db}_{X\times S} \otimes_{\mathcal{A}_{X\times S}} \omega_{X\times S/S}.$$

One then mimics [8, Prop. 4.3] by replacing the transfer module $\mathcal{D}_{Y\times S\to X\times S}$ for the morphism $Y\times S \to X\times S$ in the absolute sense by the relative one $\mathcal{D}_{(Y\to X)\times S/S} := \mathcal{A}_{Y\times S} \otimes_{f^{-1}\mathcal{A}_{X\times S}} f^{-1}\mathcal{D}_{X\times S_\mathbb{R}/S}$. Let us set $\mathfrak{Db}_{Y\times S}^\vee = \mathfrak{Db}_{Y\times S} \otimes_{\mathcal{A}_{Y\times S}} \omega_{Y\times S/S}$ and let $\mathrm{Sp}_\bullet(\mathcal{D}_{(Y\to X)\times S/S})$ denote the Spencer resolution of $\mathcal{D}_{(Y\to X)\times S/S}$ (we recall that, for $k\in\mathbb{N}$, $\mathrm{Sp}_k(\mathcal{D}_{(Y\to X)\times S/S})$ are locally free over $\mathcal{D}_{Y\times S_\mathbb{R}/S}$). The terms of the complex in $\mathsf{C}^+(\mathcal{D}_{Y\times S_\mathbb{R}/S})$

$$\mathcal{C}_{f^{-1}Z\times V} := (\Gamma_{f^{-1}Z\times V}\,\mathfrak{Db}_{Y\times S}^\vee) \otimes_{\mathcal{D}_{Y\times S_\mathbb{R}/S}} \mathrm{Sp}_\bullet(\mathcal{D}_{(Y\to X)\times S/S})$$

$$= \Gamma_{f^{-1}Z\times V}\big(\mathfrak{Db}_{Y\times S}^\vee \otimes_{\mathcal{D}_{Y\times S_\mathbb{R}/S}} \mathrm{Sp}_\bullet(\mathcal{D}_{(Y\to X)\times S/S})\big)$$

are thus c-soft sheaves. Hence, the object $_\mathrm{D}f_!\,\mathrm{TH}^S(p_Y^{-1}\mathcal{O}_S \otimes \mathcal{C}_{f^{-1}Z\times V})$ is represented by the complex

$$f_!\mathcal{C}_{f^{-1}Z\times V} \simeq \Gamma_{Z\times V}f_!\big(\mathfrak{Db}_{Y\times S}^\vee \otimes_{\mathcal{D}_{Y\times S_\mathbb{R}/S}} \mathrm{Sp}_\bullet(\mathcal{D}_{(Y\to X)\times S/S})\big)$$

in $\mathsf{C}^+(\mathcal{D}_{X\times S_\mathbb{R}/S})$. As in loc. cit., we get a morphism in $\mathsf{C}^+(\mathcal{D}_{X\times S_\mathbb{R}/S})$:

$$\varphi_{Z\times V} : f_!\mathcal{C}_{f^{-1}Z\times V} \longrightarrow \Gamma_{Z\times V}\,\mathfrak{Db}_{X\times S} = \mathrm{TH}_X^S(G).$$

We now consider the object $L = p_X^{-1}\mathcal{O}_S \otimes \mathbb{C}_{\Omega\times V}$ of $\mathcal{S}$. From the short exact sequence

$$0 \longrightarrow \mathbb{C}_{\Omega\times V} \longrightarrow \mathbb{C}_{X\times V} \longrightarrow \mathbb{C}_{Z\times V} \longrightarrow 0,$$

we *define* $f_!\mathcal{C}_{f^{-1}(\Omega\times V)}$ as the cokernel of the natural morphism $f_!\mathcal{C}_{f^{-1}Z\times V} \to f_!\mathcal{C}_{f^{-1}X\times V}$ in $\mathsf{C}^+(\mathcal{D}_{X\times S_\mathbb{R}/S})$.

Therefore, on the one hand, the complex $f_!\mathcal{C}_{f^{-1}(\Omega\times V)}$ is a representative of $_\mathrm{D}f_!\,\mathrm{TH}_Y^S(f^{-1}L)$ and $\mathrm{TH}_X^S(L)$ is the cokernel in $\mathsf{Mod}(\mathcal{D}_{X\times S_\mathbb{R}/S})$ of

$$\mathrm{TH}_X^S(p_X^{-1}\mathcal{O}_S \otimes \mathbb{C}_{Z\times V}) \longhookrightarrow \mathrm{TH}_X^S(p_X^{-1}\mathcal{O}_S \otimes \mathbb{C}_{X\times V}).$$

On the other hand, by completing the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f_!\mathcal{C}_{f^{-1}Z\times V} & \longrightarrow & f_!\mathcal{C}_{f^{-1}X\times V} & \longrightarrow & f_!\mathcal{C}_{f^{-1}(\Omega\times V)} & \longrightarrow & 0 \\
& & \varphi_{Z\times V}\downarrow & & \varphi_{X\times V}\downarrow & & & & \\
0 & \to & \mathrm{TH}_X^S(p_X^{-1}\mathcal{O}_S \otimes \mathbb{C}_{Z\times V}) & \to & \mathrm{TH}_X^S(p_X^{-1}\mathcal{O}_S \otimes \mathbb{C}_{X\times V}) & \longrightarrow & \mathrm{TH}_X^S(L) & \longrightarrow & 0
\end{array}$$

we define a morphism $\varphi_{\Omega\times V} : f_!\mathcal{C}_{f^{-1}(\Omega\times V)} \to \mathrm{TH}_X^S(L)$ in $\mathsf{C}^+(\mathcal{D}_{X\times S_\mathbb{R}/S})$. Functoriality with respect to $\mathcal{S}$ follows from the description of the morphisms in $\mathcal{S}$ (cf. Section 1.b.2). q.e.d.

We can now end the proof of Proposition 2.4. According to Remark 1.8, it is enough to extend the morphism obtained in Lemma 2.5 as a morphism of functors from $\mathsf{D}^-(\mathcal{S})$ to $\mathsf{D}^+(\mathcal{D}_{X\times S_\mathbb{R}/S})$. Functoriality above leads to the definition of a morphism of functors from $\mathsf{C}^-(\mathcal{S})$ to the category of double complexes indexed by $\mathbb{N}^2$ of $\mathsf{Mod}(\mathcal{D}_{X\times S_\mathbb{R}/S})$, and we obtain the desired morphism of functors by passing to the associated simple complexes. q.e.d.

2.c.2. *Behaviour of* RH^S *by localization and pullback with respect to* X.

Proposition 2.6. *Let* Y *be a complex hypersurface of* X. *Then, for any* $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ *there is a natural isomorphism*

$$(6) \qquad \mathrm{RH}^S_X(F)(*(Y \times S)) \xrightarrow{\sim} \mathrm{RH}^S_X(F \otimes \mathbb{C}_{(X \smallsetminus Y) \times S}).$$

In particular, if $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$,

  (a) $\mathrm{RH}^S_X(F)(*(Y \times S))$ *belongs to* $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$.
  (b) *There is a natural isomorphism* $\mathrm{RH}^S_X(F \otimes \mathbb{C}_{Y \times S}) \simeq R\Gamma_{[Y \times S]}(\mathrm{RH}^S_X(F))$ *and so* $R\Gamma_{[Y \times S]}(\mathrm{RH}^S_X(F))$ *also belongs to* $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$.
  (c) *If the natural morphism* $\mathrm{RH}^S_X(F) \to \mathrm{RH}^S_X(F)(*(Y \times S))$ *is an isomorphism, then so is the natural morphism* $F \otimes \mathbb{C}_{(X \smallsetminus Y) \times S} \to F$.

Proof. Let $f = 0$ be a local defining equation of Y. Since this is a local problem we may start by assuming that $F = p_X^{-1}\mathcal{O}_S \otimes \mathbb{C}_{\Omega \times S}$ for a relatively compact open subanalytic subset Ω of X. Noting that f is invertible on $T\mathcal{H}om(\mathbb{C}_{(\Omega \smallsetminus Y) \times S}, \mathfrak{Db}_{X \times S})$, according to [5, Prop. 3.23], the natural restriction morphism

$$T\mathcal{H}om(\mathbb{C}_{\Omega \times S}, \mathfrak{Db}_{X \times S})(*(Y \times S)) \longrightarrow T\mathcal{H}om(\mathbb{C}_{(\Omega \smallsetminus Y) \times S}, \mathfrak{Db}_{X \times S})$$

is an isomorphism. Thus, applying Proposition 1.6, the natural $\mathcal{D}_{\overline{X} \times \overline{S}}$-linear morphism $\mathrm{TH}^S_X(F)(*(Y \times S)) \to \mathrm{TH}^S_X(F \otimes \mathbb{C}_{(X \smallsetminus Y) \times S})$ is an isomorphism for any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_S)$. The existence of the morphism (6) and the fact that it is an isomorphism then follow by (4) and functoriality.

The remaining statements (a) and (b) follow straightforwardly (see also [17, Ex. 3.20]), while (c) is obtained by applying $^{\mathrm{p}}\mathrm{Sol}_X$ to the isomorphism (6), and by using Theorem 2.2.   q.e.d.

Corollary 2.7. *For any* $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ *and for any closed submanifold* Y *of* X, $R\Gamma_{[Y \times S]}(\mathrm{RH}^S_X(F))$ *is a complex with regular holonomic* $\mathcal{D}_{X \times S/S}$-*cohomologies.*

Proof. The statement being local, we may assume that Y is an intersection of smooth hypersurfaces of X and then conclude by Proposition 2.6(b) that $R\Gamma_{[Y \times S]}(\mathrm{RH}^S(F)) \simeq \mathrm{RH}^S(F \otimes \mathbb{C}_{Y \times S})$ which concludes the proof.   q.e.d.

Proposition 2.8. *Let* $f : Y \to X$ *be a smooth morphism of complex manifolds. Then there exists a natural isomorphism in* $\mathsf{D}^{\mathrm{b}}(f^{-1}\mathcal{D}_{X \times S/S})$, *functorial in* $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_S)$:

$$R\mathcal{H}om_{\mathcal{D}_{Y \times S/S}}(\mathcal{D}_{(Y \to X) \times S/S}, \mathrm{RH}^S_Y(f^{-1}F)) \simeq f^{-1}\mathrm{RH}^S_X(F)[d_Y - d_X].$$

The proof is performed by mimicking the proof of [8, Th. 5.8 (5.14)] using Proposition 2.4.

The following result is the relative version of [8, Prop. 5.9 (5.20)] from which we adapt the proof.

Proposition 2.9. *For any $F \in \mathsf{D}^b_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ and for any morphism $f : Y \to X$ of complex manifolds, there exists a natural morphism in $\mathsf{D}^b(\mathcal{D}_{Y \times S/S})$:*

$$(7) \qquad {}_{\mathrm{D}}f^* \, \mathrm{RH}^S_X(F)[d_Y - d_X] \longrightarrow \mathrm{RH}^S_Y(f^{-1}F).$$

Moreover, when $F \in \mathsf{D}^b_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$, this morphism is an isomorphism.

Proof. We start by decomposing f as the graph embedding $Y \to Y \times X$ followed by the projection $Y \times X \to X$, reducing to the case of (i) a closed immersion and (ii) a projection morphism.

Let us treat (i). We shall prove that

$$ {}_{\mathrm{D}}f^* \, \mathrm{RH}^S_X(F)[d_Y - d_X] \simeq \mathrm{RH}^S_Y(f^{-1}F) $$

by a natural isomorphism in $\mathsf{D}^b(\mathcal{D}_{Y \times S/S})$ functorially in F. We start by noticing that

$$(8) \qquad {}_{\mathrm{D}}f^* \, \mathrm{RH}^S_X(F \otimes \mathbb{C}_{(X \setminus Y) \times S}) = 0.$$

To check this local statement we may assume, by induction on $\operatorname{codim} Y$, that Y is smooth of codimension one, and (8) follows from Proposition 2.6. Hence we conclude that

$$
\begin{aligned}
{}_{\mathrm{D}}f^* \, \mathrm{RH}^S_X(F)[d_Y - d_X] &\simeq {}_{\mathrm{D}}f^* \, \mathrm{RH}^S_X(F \otimes \mathbb{C}_{Y \times S})[d_Y - d_X] \\
&\simeq {}_{\mathrm{D}}f^* \, {}_{\mathrm{D}}f_* \, \mathrm{RH}^S_Y(f^{-1}F)[d_Y - d_X] \quad \text{by Proposition 2.6(b)} \\
&\simeq \mathrm{RH}^S_Y(f^{-1}F) \quad \text{since } {}_{\mathrm{D}}f^* {}_{\mathrm{D}}f_*[d_Y - d_X] \simeq \mathrm{Id}_{\mathsf{D}^b(\mathcal{D}_{Y \times S/S})}.
\end{aligned}
$$

Let us now treat (ii). Let $f : Y \to X$ a projection of $Y = X \times Z$ on X. Recall that in that case we have a natural transformation of functors on $\mathsf{D}^b(\mathcal{D}_{Y \times S/S})$

$$ \mathcal{D}_{(Y \to X) \times S/S} \otimes_{f^{-1}\mathcal{D}_{X \times S/S}} R\mathcal{H}om_{\mathcal{D}_{Y \times S/S}}(\mathcal{D}_{(Y \to X) \times S/S}, \cdot) \longrightarrow \mathrm{Id}. $$

Then by Proposition 2.8 we obtain the canonical morphism (7).

Let us now assume that $F \in \mathsf{D}^b_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$. Since the result is true when f is a closed embedding, it remains to consider the case where f is a smooth morphism. Then f is a non-characteristic morphism for any $\mathcal{M} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{D}_X)$ in the sense of [7, Def. 11.2.11], hence, according to Theorem 11.3.5 of loc. cit., we have a functorial isomorphism $f^{-1}\operatorname{Sol}(\mathcal{M}) \simeq \operatorname{Sol}({}_{\mathrm{D}}f^*\mathcal{M})$. If $\mathcal{M} \in \mathsf{D}^b_{\mathrm{rhol}}(\mathcal{D}_X)$ and $F = \operatorname{Sol}(\mathcal{M})$, according to the Riemann-Hilbert correspondence in the absolute case (here denoted by RH), we have $\mathcal{M} \simeq \mathrm{RH}(F)$ and we conclude an isomorphism ${}_{\mathrm{D}}f^*\mathcal{M} \simeq \mathrm{RH}(f^{-1}F)$. As a consequence,

$$ Li_s^* {}_{\mathrm{D}}f^* \, \mathrm{RH}^S_X(F) \simeq {}_{\mathrm{D}}f^* Li_s \, \mathrm{RH}^S_X(F) \simeq {}_{\mathrm{D}}f^* \, T\mathcal{H}om(Li_s^* F, \mathcal{O}_X)[d_X] $$

$$ \overset{(*)}{\simeq} T\mathcal{H}om(f^{-1}Li_s^* F, \mathcal{O}_Y)[d_X] \simeq Li_s^* \, \mathrm{RH}^S_Y(f^{-1}F)[d_X - d_Y], $$

where $(*)$ holds by the absolute case recalled above and the compatibility of our constructions with the similar ones in the absolute case. By applying the variant of Nakayama's Lemma (see 1.a) to the morphism (7) in $\mathsf{D}^b_{\mathrm{rhol}}(\mathcal{D}_{Y \times S/S})$, we obtain that (7) is an isomorphism for any smooth morphism, and thus for any morphism. q.e.d.

Recall that, for $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$, $\mathrm{RH}_X^S(F)$ is a strict regular holonomic module if and only if F is strictly perverse ([17, Prop. 2]).

Lemma 2.10. *Let $F, G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ and let us assume that G is perverse and F is strictly perverse. Then the $p_X^{-1}\mathcal{O}_S$-module*

$$\mathcal{H}om_{\mathrm{perv}(p_X^{-1}\mathcal{O}_S)}(F, G) := \mathcal{H}^0 R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, G)$$

is strict.

Proof. Let us consider a morphism $\Phi : F \to G$ of perverse sheaves such that, for some $s_o \in S$ and some local coordinate s vanishing at s_o, $s^n \Phi = 0$. Then Φ is the zero morphism away from s_o, so $\mathrm{RH}_X^S(\Phi) : \mathrm{RH}_X^S(G) \to \mathrm{RH}_X^S(F)$ is also zero away from s_o. This means that the image of $\mathrm{RH}_X^S(\Phi)$ is a torsion submodule of the strict module $\mathrm{RH}_X^S(F)$, so it is zero, hence Φ is zero since ${}^{\mathrm{p}}\mathrm{Sol}_X \mathrm{RH}_X^S \simeq \mathrm{Id}$.   q.e.d.

2.d. **Behaviour of RH^S under finite ramification over S.** Let $s_0 \in S$, let N be a natural number and let $\delta : (S', s_0') \to (S, s_0)$ be the ramification of center s_0 of degree N (that is, there exist a local chart on S centered in s_0 and a local chart centered in s_0' such that $\delta(s') = s'^N$). For simplicity we shall keep the notation δ also to denote the morphism $\mathrm{Id}_X \times \delta : X \times S' \to X \times S$.

For a $\mathcal{D}_{X \times S/S}$-module $\mathcal{M}$, resp. an object $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_S)$, the pullback is defined by

$${}_{\mathrm{D}}\delta^*\mathcal{M} := \mathcal{O}_{X \times S'} \otimes_{\delta^{-1}\mathcal{O}_{X \times S}} \delta^{-1}\mathcal{M} \quad \text{resp.} \quad \delta^* F := p^{-1}\mathcal{O}_{S'} \otimes_{p^{-1}\delta^{-1}\mathcal{O}_S} \delta^{-1}F.$$

We remark that if ${}_{\mathrm{D}}\delta_*$ denotes the direct image in the sense of $\mathcal{D}_{X \times S/S}$-modules, then ${}_{\mathrm{D}}\delta_* = \delta_*$, so we simply denote ${}_{\mathrm{D}}\delta^*, {}_{\mathrm{D}}\delta_*$ by δ^*, δ_*. We also remark that $\mathcal{O}_{X \times S'}$ is flat over $\delta^{-1}\mathcal{O}_{X \times S}$ hence we have

$$\delta_* \delta^*\mathcal{M} \simeq \delta_* \mathcal{O}_{X \times S'} \otimes_{\mathcal{O}_{X \times S}} \mathcal{M} \quad \text{resp.} \quad \delta_* \delta^* F := \delta_* \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} F,$$

so that $\mathcal{M}$, resp. F, is a direct summand in $\delta_* \delta^*\mathcal{M}$, resp. in $\delta_* \delta^* F$.

The first pullback induces a well-defined exact functor from $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{X \times S/S})$ to $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{X \times S'/S'})$, as already used in [17] in a particular situation (proof of Corollary 2.8, where δ is denoted by ρ), and the second one a well-defined functor $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_S) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p^{-1}\mathcal{O}_{S'})$. The following results benefited from useful discussions with Luca Prelli.

Lemma 2.11. *There is a well-defined morphism in $\mathsf{D}^{\mathrm{b}}(\rho'_{S*} \delta^{-1}\mathcal{D}_{X \times S/S})$*

$$(9) \qquad\qquad \delta^{-1}\mathcal{O}^{t,S}_{X \times S} \longrightarrow \mathcal{O}^{t,S'}_{X \times S'}.$$

Proof. It is sufficient to prove the existence of such a morphism when replacing $\mathcal{O}^{t,S}_{X \times S}$ with $C^{\infty,t,S}_{X \times S}$. In that case it is obtained by the composition of functions with δ — this does not interfere with the growth conditions — yielding a $\delta^{-1}\mathcal{D}_{X \times S/S}$-linear morphism since the operators in $\mathcal{D}_{X \times S/S}$ do not involve derivations along S.   q.e.d.

Proposition 2.12. *With the notation as above, for any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_S)$, there exists a morphism, functorial in F, in $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_{X\times S'/S'})$*

$$\Psi_F : \delta^* \operatorname{RH}_X^S(F) \longrightarrow \operatorname{RH}_X^{S'}(\delta^* F),$$

which is an isomorphism if F is an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$.

Proof. Following the definition of the relative Riemann-Hilbert functor, we have to construct a natural morphism:

$$(10) \quad \mathcal{O}_{X\times S'} \otimes_{\delta^{-1}\mathcal{O}_{X\times S}} \delta^{-1}\rho_S'^{-1} R\mathcal{H}om_{\rho_{S*}' p_X^{-1}\mathcal{O}_S}(\rho_{S*}'F, \mathcal{O}_{X\times S}^{t,S})$$
$$\longrightarrow \rho_{S'}'^{-1} R\mathcal{H}om_{\rho_{S'*}' p^{-1}\mathcal{O}_{S'}}(\rho_{S'*}'\delta^* F, \mathcal{O}_{X\times S'}^{t,S'}).$$

We have a natural isomorphism of functors on sites $\delta^{-1}\rho_S'^{-1} \simeq \rho_{S'}'^{-1}\delta^{-1}$ which yields a natural isomorphism in $\mathsf{D}^{\mathrm{b}}(\delta^{-1}\mathcal{D}_{X\times S/S})$

$$(11) \quad \delta^{-1}\rho_S'^{-1} R\mathcal{H}om_{\rho_{S*}' p_X^{-1}\mathcal{O}_S}(\rho_{S*}'F, \mathcal{O}_{X\times S}^{t,S})$$
$$\simeq \rho_{S'}'^{-1}\delta^{-1} R\mathcal{H}om_{\rho_{S*}' p_X^{-1}\mathcal{O}_S}(\rho_{S*}'F, \mathcal{O}_{X\times S}^{t,S}).$$

Recall that, for a morphism $g : Z' \to Z$ of manifolds and $\mathcal{A}$ a sheaf of rings on Z, we have a natural morphism of bifunctors on $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ (cf. [7, (2.6.27)]):

$$(12) \qquad f^{-1} R\mathcal{H}om_{\mathcal{A}}(\bullet, \bullet) \longrightarrow R\mathcal{H}om_{f^{-1}\mathcal{A}}(f^{-1}(\bullet), f^{-1}(\bullet)).$$

Since we are working with sheaves on Grothendieck topologies (see [2] and [9]), we have the analogous of (12), that is, in the present situation, we have a natural morphism in $\mathsf{D}^{\mathrm{b}}(\rho_{S'!}'\delta^{-1}\mathcal{D}_{X\times S/S})$

$$\delta^{-1}R\mathcal{H}om_{\rho_{S*}' p_X^{-1}\mathcal{O}_S}(\rho_{S*}'F, \mathcal{O}_{X\times S}^{t,S}) \longrightarrow R\mathcal{H}om_{\delta^{-1}\rho_{S*}' p_X^{-1}\mathcal{O}_S}(\delta^{-1}\rho_{S*}'F, \delta^{-1}\mathcal{O}_{X\times S}^{t,S}),$$

hence a natural morphism in $\mathsf{D}^{\mathrm{b}}(\delta^{-1}\mathcal{D}_{X\times S/S})$

$$(13) \quad \rho_{S'}'^{-1}\delta^{-1} R\mathcal{H}om_{\rho_{S*}' p_X^{-1}\mathcal{O}_S}(\rho_{S*}'F, \mathcal{O}_{X\times S}^{t,S})$$
$$\longrightarrow \rho_{S'}'^{-1} R\mathcal{H}om_{\delta^{-1}\rho_{S*}' p_X^{-1}\mathcal{O}_S}(\delta^{-1}\rho_{S*}'F, \delta^{-1}\mathcal{O}_{X\times S}^{t,S}).$$

According to the morphism (9) in $\mathsf{D}^{\mathrm{b}}(\delta^{-1}\mathcal{D}_{X\times S/S})$, combining with (13) and the commutation $\delta^{-1}\rho_{S*}' \simeq \rho_{S'*}'\delta^{-1}$ we derive a functorial chain of morphisms in $\mathsf{D}^{\mathrm{b}}(\delta^{-1}\mathcal{D}_{X\times S/S})$

$$\rho_{S'}'^{-1}\delta^{-1} R\mathcal{H}om_{\rho_{S*}' p_X^{-1}\mathcal{O}_S}(\rho_{S*}'F, \mathcal{O}_{X\times S}^{t,S})$$
$$\longrightarrow \rho_{S'}'^{-1} R\mathcal{H}om_{\delta^{-1}\rho_{S*}' p_X^{-1}\mathcal{O}_S}(\delta^{-1}\rho_{S*}'F, \mathcal{O}_{X\times S'}^{t,S'})$$
$$\longrightarrow \rho_{S'}'^{-1} R\mathcal{H}om_{\rho_{S*}'\delta^{-1} p_X^{-1}\mathcal{O}_S}(\rho_{S*}'\delta^{-1}F, \mathcal{O}_{X\times S'}^{t,S'})$$
$$\longrightarrow \rho_{S'}'^{-1} R\mathcal{H}om_{\rho_{S'*}' p^{-1}\delta^*\mathcal{O}_S}(\rho_{S'*}'\delta^* F, \mathcal{O}_{X\times S'}^{t,S'}),$$

where the last term results by applying $\mathcal{O}_{S'} \otimes_{\delta^{-1}\mathcal{O}_S} (\cdot)$. We also remark that the last term (which we will name $\mathcal{L}$ for simplicity) is the right term of the desired morphism (10) and is already an object of $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_{X\times S'})$. Hence, by

applying to (11) the functor $\mathcal{O}_{X \times S'} \otimes_{\delta^{-1}\mathcal{O}_{X \times S}} (\cdot)$ we derive a chain of natural morphisms in $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_{X \times S'/S'})$.

$$\mathcal{O}_{X \times S'} \otimes_{\delta^{-1}\mathcal{O}_{X \times S}} \delta^{-1} \rho_{S'}'^{-1} \, R\mathcal{H}om_{\rho_{S*}' p_X^{-1}\mathcal{O}_S}(\rho_{S*}'F, \mathcal{O}_{X \times S}^{t,S})$$
$$\longrightarrow \mathcal{O}_{X \times S'} \otimes_{\delta^{-1}\mathcal{O}_{X \times S}} \mathcal{L} \longrightarrow \mathcal{L}$$

whose composition gives the desired morphism Ψ_F.

Assume now that $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ and let us prove that Ψ_F is an isomorphism.

Since in that case $\mathrm{RH}_X^S(F)$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$, it is clear that $\delta^* \mathrm{RH}_X^S(F)$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}'_{X \times S/S})$. The same holds true with $\mathrm{RH}_X^{S'}(\delta^*F)$ since δ^*F is an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_{S'})$.

It is then sufficient to apply $Li_{s'}^*$ to both sides of the morphism Ψ_F and apply Proposition 2.1, noting that $Li_{s'}^*\delta^* = Li_s^*$, where $s = \delta(s')$. This way, both members become, by reduction to the absolute case, isomorphic to $T\mathcal{H}om(Li_s^*F, \mathcal{O}_X)[d_X]$.   q.e.d.

We shall now come back to the situation described at the beginning of this section, and we keep the same notations.

Proposition 2.13. *Let* $\mathcal{M}, \mathcal{N}$ *be* $\mathcal{D}_{X \times S/S}$*-modules and assume that* $\mathcal{M}$ *is coherent. If the complex* $R\mathcal{H}om_{\mathcal{D}_{X \times S'/S'}}(\delta^*\mathcal{M}, \delta^*\mathcal{N})$ *belongs to* $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_{S'})$, *then the complex* $R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{N})$ *also belongs to* $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$.

Proof. Since $\mathcal{N}$ is a direct summand of $\delta_*\delta^*\mathcal{N}$ it suffices to deduce from the assumption that $R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \delta_*\delta^*\mathcal{N})$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$. Thanks to the adjunction morphism we have in $\mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$

$$R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \delta_*\delta^*\mathcal{N}) \simeq \delta_* \, R\mathcal{H}om_{\delta^{-1}\mathcal{D}_{X \times S/S}}(\delta^{-1}\mathcal{M}, \delta^*\mathcal{N})$$
$$\simeq \delta_* \, R\mathcal{H}om_{\mathcal{D}_{X \times S'/S'}}(\delta^*\mathcal{M}, \delta^*\mathcal{N}).$$

The result is then a consequence of the assumption and the properness of δ.   q.e.d.

The following result will be used in Section 3. Recall (cf. [17, (3.17)]) that, for $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_S)$, there exists a natural morphism

$$\mathrm{RH}_X^S(F) \longrightarrow R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, \mathcal{O}_{X \times S})[d_X].$$

Lemma 2.14. *Let* $\mathcal{L}$ *be a* $\mathcal{D}_{X \times S/S}$*-module of D-type and let* F *be an object of* $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ *be such that* $F \simeq F \otimes \mathbb{C}_{(X \smallsetminus D) \times S}$. *Then the morphism*

$$\beta_{\mathcal{L}, F} : R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{L}, \mathrm{RH}_X^S(F))$$
$$\longrightarrow R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{L}, R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, \mathcal{O}_{X \times S})[d_X])$$

is an isomorphism.

Proof. The case where $\mathcal{L} = \mathcal{O}_{X \times S}$ was proved in [17, Lem. 3.19]. We aim at reducing to this case. The statement has a local nature so we choose local coordinates $x_1, \ldots, x_n$ in X such that $D = \{x \in X \mid x_1 \cdots x_\ell = 0\}$,

and we assume that S is a disc of small enough radius with coordinate s. Let $\delta : S' \to S$ be a finite morphism ramified at $s = 0$ only. According to Propositions 2.12 and 2.13, the assertion of the lemma holds for $(\mathcal{L}, F)$ if it holds for $(\delta^* \mathcal{L}, \delta^* F)$. By Theorem 1.13(b) and the same argument as in the proof of [17, Cor. 2.8], there exists δ such that $\delta^* \mathcal{L}$ is isomorphic to

$$\mathcal{D}_{X \times S'/S'} \Big/ \Big(\sum_{i=1}^{\ell} \mathcal{D}_{X \times S'/S'}(x_i \partial_{x_i} - \alpha_i) + \sum_{i=\ell+1}^{n} \mathcal{D}_{X \times S'/S'} \partial_{x_i} \Big)$$

for some holomorphic functions α_i, $i = 1, \dots, \ell$, on S'. We will therefore assume that $\mathcal{L}$ is already of this form. We can replace F with a resolution as given by Proposition 1.6 and, since the result is local, we may reduce to the case $F = p_X^{-1} \mathcal{O}_S \otimes \mathbb{C}_{(\Omega \smallsetminus D) \times S}$ (since $F \simeq F \otimes \mathbb{C}_{(X \smallsetminus D) \times S}$) for some relatively compact open subanalytic subset Ω of X. We have

$$\mathrm{RH}^S(F) \simeq T\mathcal{H}om(\mathbb{C}_{(\Omega \smallsetminus D) \times S}, \mathcal{O}_{X \times S})[d_X]$$

and

$$R\mathcal{H}om_{p_X^{-1} \mathcal{O}_S}(F, \mathcal{O}_{X \times S})[d_X] \simeq R\mathcal{H}om(\mathbb{C}_{(\Omega \smallsetminus D) \times S}, \mathcal{O}_{X \times S})[d_X].$$

Let us consider the automorphism Φ induced on $T\mathcal{H}om(\mathbb{C}_{(\Omega \smallsetminus D) \times S}, \mathfrak{Db}_{X \times S})$ and on $R\mathcal{H}om(\mathbb{C}_{(\Omega \smallsetminus D) \times S}, \mathfrak{Db}_{X \times S}) \simeq \Gamma_{(\Omega \smallsetminus D) \times S}(\mathfrak{Db}_{X \times S})$ by multiplication by the real analytic function $|x_1|^{2\alpha_1(s)} |x_2|^{2\alpha_2(s)} \cdots |x_\ell|^{2\alpha_\ell(s)}$. Then, in $\mathsf{D}^b(p_X^{-1} \mathcal{O}_S)$, Φ induces isomorphisms

$$\Phi_1 : R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{L}, T\mathcal{H}om(\mathbb{C}_{(\Omega \smallsetminus D) \times S}, \mathfrak{Db}_{X \times S}))$$
$$\xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{O}_{X \times S}, T\mathcal{H}om(\mathbb{C}_{(\Omega \smallsetminus D) \times S}, \mathfrak{Db}_{X \times S}))$$

and

$$\Phi_2 : R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{L}, \Gamma_{(\Omega \smallsetminus D) \times S}(\mathfrak{Db}_{X \times S}))$$
$$\xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{O}_{X \times S}, \Gamma_{(\Omega \smallsetminus D) \times S}(\mathfrak{Db}_{X \times S})).$$

We derive a morphism in $\mathsf{D}^b(p_X^{-1} \mathcal{O}_S)$

$$\Phi_2 \mu \Phi_1^{-1} : R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{O}_{X \times S}, T\mathcal{H}om(\mathbb{C}_{(\Omega \smallsetminus D) \times S}, \mathfrak{Db}_{X \times S}))$$
$$\longrightarrow R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{O}_{X \times S}, \Gamma_{(\Omega \smallsetminus D) \times S}(\mathfrak{Db}_{X \times S}))$$

which coincides with the natural one. By the realification procedure (cf. Section 2.b.2), we are thus reduced to the case $\mathcal{L} = \mathcal{O}_{X \times S}$, as wanted.   q.e.d.

3. Relative Riemann-Hilbert correspondence

Proving Theorem 1 is equivalent to proving Theorem 3. Indeed, in one direction, let us recall the method introduced in [17, §4.3] to deduce Theorem 1 from Theorem 3. According to [17, (3.17)], there exists a natural morphism of bifunctors from $\mathsf{D}^b_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})^{\mathrm{op}} \times \mathsf{D}^b_{\mathbb{R}\text{-}c}(p_X^{-1} \mathcal{O}_S)$ to $\mathsf{D}^b(p_X^{-1} \mathcal{O}_S)$:

$$(14) \quad R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathrm{RH}_X^S(F))$$
$$\longrightarrow R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, R\mathcal{H}om_{p_X^{-1} \mathcal{O}_S}(F, \mathcal{O}_{X \times S}[d_X]))$$
$$\simeq R\mathcal{H}om_{p_X^{-1} \mathcal{O}_S}(F, {}^{\mathrm{P}}\mathrm{Sol}_X(\mathcal{M})),$$

where the last isomorphism is an application of [7, (2.6.7)]. Notice that the right-hand side of (14) is an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ provided that $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$. In that case, by Theorem 3, the left-hand side is also an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$. In particular, $R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathrm{RH}^S_X(F))_{(x,s)}$ has $\mathcal{O}_{S,s}$-finitely generated cohomologies for any $(x, s) \in X \times S$. By the variant of Nakayama's lemma recalled in Section 1.b.3, and since $Li^*_{s_o}(14)$ is an isomorphism for any $s_o \in S$ (this is the absolute case, where the result is known (cf. [5, Cor. 8.6]), we conclude that (14) is an isomorphism. Replacing F with $^{\mathrm{p}}\mathrm{Sol}_X \mathcal{M}$, we deduce an isomorphism of functors

$$\mathrm{Id}_{\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})} \xrightarrow{\ \beta\ } \mathrm{RH}^S_X \circ {}^{\mathrm{p}}\mathrm{Sol}_X,$$

concluding Theorem 1 as explained in the introduction.

Conversely, Theorem 1 implies Theorem 3 since the former implies full faithfulness of $^{\mathrm{p}}\mathrm{Sol}$, so we have a natural isomorphism

$$R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathrm{RH}^S_X(F)) \simeq R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, {}^{\mathrm{p}}\mathrm{Sol}_X \mathcal{M})$$

and the right-hand side belongs to $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$.

3.a. **Proof of Theorems 1 and 3 in the torsion case.** Recall that, according to Proposition 1.5 a holonomic $\mathcal{D}_{X\times S/S}$-module $\mathcal{M}$ is torsion if and only if $\mathrm{Supp}(M) \subseteq X \times T$ with $\dim T = 0$. In that case we have the following result.

Proposition 3.1. *Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ be a torsion $\mathcal{D}_{X\times S/S}$-module. Then $\widetilde{\mathcal{M}} := \mathcal{D}_{X\times S} \otimes_{\mathcal{D}_{X\times S/S}} \mathcal{M}$ is a regular holonomic $\mathcal{D}_{X\times S}$-module.*

Proof. The statement being local, we may assume that $\mathrm{Char}(\mathcal{M}) = \Lambda \times \{s_o\}$, where Λ is a Lagrangian $\mathbb{C}^*$-conic closed analytic subset in T^*X, and, taking a local coordinates s on S vanishing at s_o, there exists $n \in \mathbb{N}$ such that $s^n\mathcal{M} = 0$. Since we are dealing with triangulated categories, by an easy argument by induction on n we may assume that $n = 1$. In that case, we have $\mathcal{M} \simeq \mathcal{M}_0 \boxtimes \mathcal{O}_S/\mathcal{O}_S s$, where, by the assumption of relative regularity, $\mathcal{M}_0$ is a regular holonomic $\mathcal{D}_X$-module satisfying $\mathrm{Char}(\mathcal{M}_0) = \Lambda$. By construction $\widetilde{\mathcal{M}} \simeq \mathcal{M}_0 \boxtimes \mathcal{D}_S/\mathcal{D}_S s$ and $\mathrm{Char}(\widetilde{\mathcal{M}}) = \Lambda \times T^*_T S =: \widetilde{\Lambda}$.

Therefore $\widetilde{\mathcal{M}}$ is a regular holonomic $\mathcal{D}_{X\times S}$-module since the category of regular holonomic $\mathcal{D}_{X\times S}$-modules is closed under external tensor product.
                                                                                                q.e.d.

We denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})_{\mathrm{t}}$ the thick subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ whose objects have support in $X \times T$ with $\dim T = 0$.

Proposition 3.2. *The solution functor $^{\mathrm{p}}\mathrm{Sol}$ restricted to $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})_{\mathrm{t}}$ is an equivalence of categories*

$$^{\mathrm{p}}\mathrm{Sol}_X : \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})_{\mathrm{t}} \longrightarrow \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)_{\mathrm{t}}$$

with quasi-inverse the restriction of the functor RH^S_X to $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)_{\mathrm{t}}$.

Proof. It is sufficient to prove that the restriction of RH_X^S to $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)_t$ is fully faithful. Indeed $^p\mathrm{Sol}$ is essentially surjective since, according to Theorem 2.2, for any $F \in \mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$ we have $F \simeq {}^p\mathrm{Sol}_X\,\mathrm{RH}_X^S(F)$, and in the case of a torsion object F in $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)_t$ we have $\mathrm{RH}_X^S(F) \in \mathsf{D}_{\mathrm{rhol}}^{\mathrm{b}}(\mathcal{D}_{X\times S/S})_t$.

For the full faithfulness it is enough to prove that the morphism:

$$R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathrm{RH}_X^S(G))$$
$$\longrightarrow R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(G, \mathcal{O}_{X\times S})[d_X])$$

is an isomorphism for any $\mathcal{M}\in\mathsf{D}_{\mathrm{rhol}}^{\mathrm{b}}(\mathcal{D}_{X\times S/S})_t$ and for any $G\in\mathsf{D}_{\mathbb{R}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$.

The cohomologies of $\mathcal{M}$ are regular holonomic $\mathcal{D}_{X\times S/S}$-modules and, according to Proposition 3.1, $\mathcal{D}_{X\times S} \otimes_{\mathcal{D}_{X\times S/S}} \mathcal{M}$ is a complex whose cohomologies are regular holonomic.

Thanks to Proposition 1.6, we may assume that $G=p_X^{-1}\mathcal{O}_S \otimes \mathbb{C}_{\Omega\times S}$ for some open subanalytic subset Ω of X, hence

$$\mathrm{RH}_X^S(G) = T\mathcal{H}om(\mathbb{C}_{\Omega\times S}, \mathcal{O}_{X\times S})[d_X],$$

which is a complex with $\mathcal{D}_{X\times S}$-modules as cohomologies and we get a chain of isomorphisms

$$R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathrm{RH}_X^S(G)) \simeq R\mathcal{H}om_{\mathcal{D}_{X\times S}}(\mathcal{D}_{X\times S} \otimes_{\mathcal{D}_{X\times S/S}} \mathcal{M}, \mathrm{RH}_X^S(G))$$
$$\overset{(*)}{\simeq} R\mathcal{H}om_{\mathcal{D}_{X\times S}}(\mathcal{D}_{X\times S} \otimes_{\mathcal{D}_{X\times S/S}} \mathcal{M}, R\mathcal{H}om(\mathbb{C}_{\Omega\times S}, \mathcal{O}_{X\times S})[d_X])$$
$$\simeq R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, R\mathcal{H}om(\mathbb{C}_{\Omega\times S}, \mathcal{O}_{X\times S})[d_X])$$
$$\simeq R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(G, \mathcal{O}_{X\times S})[d_X]),$$

where isomorphism $(*)$ follows by [5, Cor. 8.6]. q.e.d.

3.b. **Proof of Theorem 1 in the strict case.**

Proposition 3.3. *Let* $\mathcal{M}, \mathcal{N}$ *be regular holonomic* $\mathcal{D}_{X\times S/S}$*-modules,* $\mathcal{N}$ *being strict. Then the natural morphism*

$$\beta_{\mathcal{M},\mathcal{N}} : \mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{N}) \longrightarrow \mathcal{H}om_{\mathrm{perv}(p_X^{-1}\mathcal{O}_S)}({}^p\mathrm{Sol}_X\,\mathcal{N}, {}^p\mathrm{Sol}_X\,\mathcal{M})$$

is an isomorphism. In particular, if $\mathcal{M}$ *is strict, taking* $\mathcal{N} := \mathrm{RH}_X^S({}^p\mathrm{Sol}_X\,\mathcal{M})$, *there exists an isomorphism* $\beta_{\mathcal{M}} : \mathcal{M} \to \mathrm{RH}_X^S({}^p\mathrm{Sol}_X\,\mathcal{M})$ *which is compatible with Kashiwara's morphism in the absolute case.*

Proof. Since $\mathcal{N}$ is holonomic and strict, $^p\mathrm{Sol}_X\,\mathcal{N}$ is strictly perverse, after [17, Prop. 2], and by Lemma 2.10 the target of $\beta_{\mathcal{M},\mathcal{N}}$, which is S-$\mathbb{C}$-constructible, is strict. By an argument similar to that of Lemma 2.10, the source of $\beta_{\mathcal{M},\mathcal{N}}$ is also strict, and its fibers at any $(x, s) \in X \times S$ are $\mathcal{O}_{S,s}$ modules of finite type according to Theorem 1.4. Hence, in particular, for each $s \in S$ we have

$$Li_s^*\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{N}) \simeq i_s^*\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{N})$$

and

$$Li_s^*(\mathcal{H}om_{\mathrm{perv}(p_X^{-1}\mathcal{O}_S)}(^{\mathrm{P}}\mathrm{Sol}_X \mathcal{N}, {}^{\mathrm{P}}\mathrm{Sol}_X \mathcal{M}))$$

$$\simeq i_s^*\mathcal{H}om_{\mathrm{perv}(p_X^{-1}\mathcal{O}_S)}(^{\mathrm{P}}\mathrm{Sol}_X \mathcal{N}, {}^{\mathrm{P}}\mathrm{Sol}_X \mathcal{M}).$$

Since $i_s^*(\beta_{\mathcal{M},\mathcal{N}})$ is an isomorphism according to the regularity assumption and the absolute case, the first statement follows by the variants of Nakayama's lemma (see Sections 1.a and 1.b.3).

For the second statement, we use the isomorphism α_F recalled in the introduction for $F = {}^{\mathrm{P}}\mathrm{Sol}_X(\mathcal{M})$, so that, for $\mathcal{N} = \mathrm{RH}_X^S(^{\mathrm{P}}\mathrm{Sol}_X \mathcal{M})$, we find $^{\mathrm{P}}\mathrm{Sol}_X \mathcal{N} \simeq {}^{\mathrm{P}}\mathrm{Sol}_X \mathcal{M}$. Note that F is strictly perverse, according to [17, Prop. 2], and therefore $\mathcal{N}$ is also strict (cf. [17, Cor. 4]). We have thus obtained an isomorphism

$$\mathrm{Hom}_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathrm{RH}_X^S(^{\mathrm{P}}\mathrm{Sol}_X \mathcal{M})) \longrightarrow \mathrm{Hom}_{\mathrm{perv}(p_X^{-1}\mathcal{O}_S)}(^{\mathrm{P}}\mathrm{Sol}_X \mathcal{M}, {}^{\mathrm{P}}\mathrm{Sol}_X \mathcal{M}),$$

and we define $\beta_{\mathcal{M}}$ as being the unique morphism corresponding to $\mathrm{Id} \in \mathrm{Hom}_{\mathrm{perv}(p_X^{-1}\mathcal{O}_S)}(^{\mathrm{P}}\mathrm{Sol}_X \mathcal{M}, {}^{\mathrm{P}}\mathrm{Sol}_X \mathcal{M})$. It is an isomorphism by checking the reduction to each $s_o \in S$. \hfill q.e.d.

Corollary 3.4. *For any strict regular holonomic $\mathcal{D}_{X \times S/S}$-module $\mathcal{M}$, the complexes $R\Gamma_{[Y \times S]}(\mathcal{M})$ and $\mathcal{M}(*(Y \times S))$ belong to $\mathsf{D}_{\mathrm{rhol}}^{\mathrm{b}}(\mathcal{D}_{X \times S/S})$.*

Proof. Since $\mathcal{M}$ is strict we have a natural isomorphism $\mathcal{M} \simeq \mathrm{RH}_X^S(^{\mathrm{P}}\mathrm{Sol}_X \mathcal{M})$ by Proposition 3.3. Thus $\mathcal{M}(*(Y \times S)) \simeq \mathrm{RH}_X^S(^{\mathrm{P}}\mathrm{Sol}_X(\mathcal{M}) \otimes \mathbb{C}_{(X \smallsetminus Y) \times S})$ and $R\Gamma_{[Y \times S]}(\mathcal{M}) \simeq \mathrm{RH}_X^S(^{\mathrm{P}}\mathrm{Sol}_X(\mathcal{M}) \otimes \mathbb{C}_{Y \times S})$ by Proposition 2.6 which concludes the proof. \hfill q.e.d.

A special case of Corollary 4 can now be obtained (the general case is proved in Section 3.e).

Corollary 3.5. *For each $F \in \mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$ which is strictly perverse, there exists a functorial (when restricting to this quasi-abelian category) isomorphism which is compatible with the canonical morphism in the absolute case:*

$$\boldsymbol{D}(\mathrm{RH}_X^S(F)) \simeq \mathrm{RH}_X^S(\boldsymbol{D}F),$$

where in the left-hand side $\boldsymbol{D}$ stands for the duality functor in $\mathsf{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_{X \times S/S})$ while in the right-hand side stands for the duality functor on $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$.

Proof. We consider the isomorphism $\beta_{\mathcal{M},\mathcal{N}}$ in the case of $\mathcal{M} := \boldsymbol{D}\,\mathrm{RH}_X^S(F)$ and $\mathcal{N} := \mathrm{RH}_X^S(\boldsymbol{D}F)$. Since $^{\mathrm{P}}\mathrm{Sol}_X(\mathcal{M}) \simeq {}^{\mathrm{P}}\mathrm{Sol}_X(\mathcal{N}) \simeq \boldsymbol{D}F$, the desired morphism will be the unique one corresponding to the identity of $\boldsymbol{D}F$. The fact that it is an isomorphism is again a consequence of the reduction to the absolute case for each s_o in S. \hfill q.e.d.

3.c. **End of the proof of Theorem 2.** We can argue by induction on the length of $\mathcal{M}$ and then reduce to the cases of a projection and of a closed embedding. The first case was proved in Section 1.d.1. Let us now consider a closed embedding $i : Y \hookrightarrow X$. The strict case is a consequence of Corollary 3.4. It remains to treat the torsion case. If $\mathcal{M} = t(\mathcal{M})$ then

(Proposition 3.2) there exists $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ such that $\mathcal{M} = \mathrm{RH}^S_X(F)$, hence (Proposition 2.9) $_{\mathrm{D}}i^*\mathcal{M} \simeq \mathrm{RH}^S_Y(i^{-1}F)$ and the latter object belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{Y \times S/S})$.   q.e.d.

3.d. **End of the proof of Theorem 3.** We refer to [11, Lem. 4.1.4] which contains the guidelines for the proof of Theorem 3. In what follows, for a complex manifold X and $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$ we consider the statement

$$P_X(\mathcal{M}): \quad R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathrm{RH}^S_X(F)) \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \quad \forall\, F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S),$$

in other words, $\mathcal{M}$ satisfies Theorem 3.

Lemma 3.6. *The statement P satisfies the following properties.*

(a) *For any manifold X and any open covering $(U_i)_{i \in I}$ of X,*

$$P_X(\mathcal{M}) \iff P_{U_i}(\mathcal{M}_{|U_i}) \;\forall\, i \in I.$$

(b) $P_X(\mathcal{M}) \implies P_X(\mathcal{M}[n]) \;\forall\, n \in \mathbb{Z}.$

(c) *For any distinguished triangle $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \xrightarrow{+1}$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X \times S/S})$,*

$$P_X(\mathcal{M}') \wedge P_X(\mathcal{M}'') \implies P_X(\mathcal{M}).$$

(d) *For any regular relative holonomic $\mathcal{D}_{X \times S/S}$-modules $\mathcal{M}$ and $\mathcal{M}'$,*

$$P_X(\mathcal{M} \oplus \mathcal{M}') \implies P_X(\mathcal{M}).$$

(e) *For any projective morphism $f : X \to Y$ and any good regular holonomic $\mathcal{D}_{X \times S/S}$-module $\mathcal{M}$,*

$$P_X(\mathcal{M}) \implies P_Y(_{\mathrm{D}}f_*\mathcal{M}).$$

(f) *If $\mathcal{M} = \mathcal{H}^0(\mathcal{M})$ is torsion, then $P_X(\mathcal{M})$ is true.*

Proof. It is clear that $P_X(\bullet)$ satisfies Properties 3.6(a), (b), (c), (d). Then Property (e) follows by adjunction, Proposition 2.9 and by the stability of S-$\mathbb{C}$-constructibility under proper direct image. Last, Property (f) has been seen in Section 3.a.   q.e.d.

End of the proof of Theorem 3 (and hence that of Theorem 1). We wish to prove that $P_X(\mathcal{M})$ is true for any X and $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$. As recalled in the introduction of Section 3, it is enough to show that, for any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$, the cohomology modules of $R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathrm{RH}^S_X(F))_{(x,s)}$ are finitely generated $\mathcal{O}_{S,s}$-modules for any $(x, s) \in X \times S$.

Let us set

$$\mathrm{Char}(\mathcal{M}) := \bigcup_j \mathrm{Char}(\mathcal{H}^j \mathcal{M}) = \bigcup_{i \in I} \Lambda_i \times T_i$$

(see Proposition 1.5), $\mathrm{Supp}(\mathcal{M}) = \bigcup_{i \in I} Y_i \times T_i$ and $Z_{\mathcal{M}} := \bigcup_{i \in I} Y_i$. We proceed by induction on the dimension of $Z_{\mathcal{M}}$. If $\dim Z_{\mathcal{M}} < 0$ we have $\mathrm{Supp}(\mathcal{M}) = \varnothing$, which implies $\mathcal{M} = 0$ and hence, by 3.6(f), $P_X(0)$ holds true.

Let us suppose $P_X(\mathcal{N})$ true for any $\mathcal{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$ such that $\dim Z_{\mathcal{N}} < k$ (with $k \geqslant 0$) and let us prove the truth of $P_X(\mathcal{M})$ for $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$ with $\dim Z_{\mathcal{M}} = k$.

By 3.6(b) and (c), we are reduced to proving $P_X(\mathcal{M})$ in the case where $\mathcal{M}$ is a regular holonomic $\mathcal{D}_{X \times S/S}$-module with $\dim Z_{\mathcal{M}} = k$.

Following the notation of Section 1.a, let $t(\mathcal{M})$ (respectively $f(\mathcal{M})$) be the torsion part (respectively the strict quotient) of $\mathcal{M}$. According to 3.6(c) (applied to the distinguished triangle $t(\mathcal{M}) \to \mathcal{M} \to f(\mathcal{M}) \xrightarrow{+1}$) and to 3.6(f), we are reduced to proving $P_X(f(\mathcal{M}))$. Notice that $\dim Z_{f(\mathcal{M})} \leqslant k$ since $Z_{f(\mathcal{M})} \subseteq Z_{\mathcal{M}}$. If $\dim Z_{f(\mathcal{M})} < k$, $P_X(f(\mathcal{M}))$ holds true by induction. Hence we are reduced to proving $P_X(\mathcal{M})$ in the case where $\mathcal{M}$ is a strict regular holonomic $\mathcal{D}_{X \times S/S}$-module such that $\dim Z_{\mathcal{M}} = k$.

In this case, locally (recall that $P_X(\mathcal{M})$ is a local statement by 3.6(a)), there exists a hypersurface Y in X such that $Y \cap Z_{\mathcal{M}}$ is of dimension $< k$ and $Y \smallsetminus Y \cap Z_{\mathcal{M}}$ is smooth. We can now apply 3.6(c) to the triangle $R\Gamma_{[Y \times S]}(\mathcal{M}) \to \mathcal{M} \to \mathcal{M}(*(Y \times S)) \xrightarrow{+1}$ (which is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$ by Corollary 3.4). By the induction hypothesis, $P_X(R\Gamma_{[Y \times S]}(\mathcal{M}))$ holds true. We are thus reduced to proving $P_X(\mathcal{M})$ for $\mathcal{M} \simeq \mathcal{M}(*(Y \times S))$.

On the other hand, it is enough to check the property $P_X(\mathcal{M})$ for those $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(p_X^{-1}\mathcal{O}_S)$ such that $F = F \otimes \mathbb{C}_{(X \smallsetminus Y) \times S}$. Indeed, let us check that it holds for those F such that $F = F \otimes \mathbb{C}_{Y \times S}$. For any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(p_X^{-1}\mathcal{O}_S)$, we have

$$R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, R\Gamma_{[Y \times S]}(\mathrm{RH}_X^S(F)))$$
$$\simeq R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, R\Gamma_{[(Y \cap Z_{\mathcal{M}}) \times S]}(\mathrm{RH}_X^S(F)))$$
$$\simeq R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\boldsymbol{D}R\Gamma_{[(Y \cap Z_{\mathcal{M}}) \times S]}(\mathrm{RH}_X^S(F)), \boldsymbol{D}\mathcal{M}),$$

where the first isomorphism holds true because $\mathrm{Supp}(\mathcal{M}) \subset Z_{\mathcal{M}} \times S$ and the second one follows by [15, (3)]. According to Proposition 2.6(b)

$$\mathcal{N} := \boldsymbol{D}R\Gamma_{[(Y \cap Z_{\mathcal{M}}) \times S]}(\mathrm{RH}_X^S(F))$$

belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X \times S/S})$. Thus, by the induction hypothesis, since $\mathcal{N}$ is supported on $(Y \cap Z_{\mathcal{M}}) \times S$ while $\boldsymbol{D}(\mathcal{M}) \simeq \mathrm{RH}_X^S(\boldsymbol{D}\,{}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M})$ (by Proposition 3.3 and Corollary 3.5), $\mathcal{H}^j R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{N}, \boldsymbol{D}(\mathcal{M}))_{(x,s)}$, and thus $\mathcal{H}^j R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, R\Gamma_{[Y \times S]}(\mathrm{RH}_X^S(F)))_{(x,s)}$, is of finite type over $\mathcal{O}_{S,s}$ for all $j \in \mathbb{Z}$ and $(x,s) \in X \times S$. We now notice that, if $F = F \otimes \mathbb{C}_{Y \times S}$, then $\mathrm{RH}_X^S(F) = R\Gamma_{[Y \times S]}(\mathrm{RH}_X^S(F))$, so the same property holds for $R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathrm{RH}_X^S(F))$, which is enough, as noticed at the beginning of the proof.

We thus assume that $\mathcal{M} = \mathcal{M}(*(Y \times S))$ is strict, and $F = F \otimes \mathbb{C}_{(X \smallsetminus Y) \times S}$. There exists a commutative diagram

$$
\begin{array}{ccc}
& X' \smallsetminus Y' \overset{j'}{\hookrightarrow} X' \\
{\scriptstyle \pi_{Z*}} \nearrow & & \downarrow {\scriptstyle \pi} \\
Z_{\mathcal{M}}^* \overset{i}{\hookrightarrow} X \smallsetminus Y \overset{j}{\hookrightarrow} X
\end{array}
$$

where π is a projective morphism, X' is smooth, Y' is a normal crossing hypersurface, $Z_{\mathcal{M}}^* := Z_{\mathcal{M}} \smallsetminus Y$ and π_Z^* is biholomorphic. Note that the assumption on F entails that

$$\pi^{-1}F = \pi^{-1}F \otimes \mathbb{C}_{(X' \smallsetminus Y') \times S} \quad \text{and} \quad \pi_*\pi^{-1}F = F,$$

while $_{\mathrm{D}}\pi^*\mathcal{M}[d_{X'} - d_X]$ is concentrated in degree zero and is of D-type along Y'. Moreover, $\mathcal{M} \simeq {}_{\mathrm{D}}\pi_{*\mathrm{D}}\pi^*\mathcal{M}[d_{X'} - d_X]$. According to Proposition 2.9 and Lemma 2.14,

$$R\pi_* \, R\mathcal{H}om_{\mathcal{D}_{X' \times S/S}}({}_{\mathrm{D}}\pi^*\mathcal{M}, {}_{\mathrm{D}}\pi^* \, \mathrm{RH}_X^S(F)) \simeq R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathrm{RH}_X^S(F))$$

is an object of $\mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$, which ends the proof of Theorem 3.   q.e.d.

3.e. **Proof of Corollary 4.** For any $F \in \mathsf{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(p_X^{-1}\mathcal{O}_S)$, we have functorial isomorphisms

$$^{\mathrm{p}}\mathrm{Sol}_X \, \boldsymbol{D}(\mathrm{RH}_X^S(F)) \simeq \boldsymbol{D} \, ^{\mathrm{p}}\mathrm{Sol}_X(\mathrm{RH}_X^S(F)) \simeq \boldsymbol{D}F \simeq {}^{\mathrm{p}}\mathrm{Sol}_X \, \mathrm{RH}_X^S(\boldsymbol{D}F).$$

Corollary 4 then follows by the full faithfulness of the functor $^{\mathrm{p}}\mathrm{Sol}_X$.   q.e.d.

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(L. Fiorot) Dipartimento di Matematica "Tullio Levi-Civita" Università degli Studi di Padova, Via Trieste, 63, 35121 Padova Italy
E-mail address: luisa.fiorot@unipd.it

(T. Monteiro Fernandes) Centro de Matemática e Aplicações Fundamentais-CIO and Departamento de Matemática da Faculdade de Ciências da Universidade de Lisboa, Bloco C6, Piso 2, Campo Grande, 1749-016, Lisboa Portugal
E-mail address: mtfernandes@fc.ul.pt

(C. Sabbah) CMLS, CNRS, École polytechnique, Institut Polytechnique de Paris, 91128 Palaiseau cedex, France
E-mail address: Claude.Sabbah@polytechnique.edu