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STABILIZABILITY IN OPTIMIZATION PROBLEMS WITH UNBOUNDED DATA

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Abstract. In this paper we extend the notions of sample and Euler stabilizability to a set of a control system to a wide class of systems with unbounded controls, which includes nonlinear control-polynomial systems. In particular, we allow discontinuous stabilizing feedbacks, which are unbounded approaching the target. As a consequence, sampling trajectories may present a chattering behaviour and Euler solutions have in general an *impulsive character*. We also associate to the control system a cost and provide sufficient conditions, based on the existence of a special Lyapunov function, which allow for the existence of a stabilizing feedback that keeps the cost of all sampling and Euler solutions starting from the same point below the same value, in a uniform way.

1. Introduction.

In the last decades, the problem of the feedback stabilization of a nonlinear control system $\dot{x} = f(x, u)$ to a point or, more in general, to a set $\mathcal{C}$, has been the subject of intense research and the theory is now well established. In particular, it is well know that a continuous stabilizing feedback fails to exist in general, and a smooth Lyapunov function, which guarantees the asymptotic controllability of the system, may not exist either. For these reasons, nonsmooth Lyapunov functions, discontinuous feedback laws K, and a "sample and hold" solution concept for $\dot{x} = f(x, K(x))$, similar to that used in differential games [13], have been introduced (see, e.g.[3, 1, 26, 25, 9, 27, 7, 8, 28, 14]). In particular, semiconcave Lyapunov functions have proven to be a powerful tool for the explicit construction of stabilizing feedback strategies [23, 24]. (For a broader overview of the topic, see review paper [10]). A key hypothesis in these results is that the vector field $f(x, K(x))$ associated

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to the stabilizing feedback K is bounded in a neighborhood of the target. In fact, it is usually assumed that the feedback K itself is bounded close to $\mathcal{C}$.

One of the main goals of this article is, given a (nondifferentiable) Lyapunov function, to construct directly a (discontinuous) stabilizing feedback K and to introduce a notion of solution to $\dot{x} = f(x, K(x))$ in the case of *unbounded* dynamics, which include, for instance, nonlinear control-polynomial systems (the precise assumptions are stated in Subsection 1.2). This necessarily leads to feedbacks which may be unbounded approaching the target. For example, in [2] the authors exhibit some applications to Lagrangian mechanics, where the system is quadratically dependent on (the derivatives of) the control and stabilization can only be achieved by "vibrating controls", i. e. allowing unbounded inputs.

More generally, since control systems are often associated with costs, we aim to build up feedback strategies which, besides stabilizing to a target $\mathcal{C}$ the system

$$\dot{x} = f(x, u), \qquad u(t) \in U, \tag{1}$$

also provide an upper bound for an integral cost of the form

$$\int_0^{T_x} l(x(\tau), u(\tau))\, d\tau. \tag{2}$$

Here the control set U is a closed, possibly unbounded subset of $\mathbb{R}^m$, the target set $\mathcal{C} \subseteq \mathbb{R}^n$ is closed with compact boundary, the Lagrangian l is ≥ 0. $T_x \leq +\infty$ denotes the first exit-time of x from $\mathbb{R}^n \setminus \mathcal{C}$. In this case, when the dynamics are unbounded and the cost is "cheap" (i. e., there are no coercivity hypotheses, which would make the use of unbounded controls "disadvantageous") it could be necessary to implement an unbounded feedback to stabilize the system and keep the cost finite, even if there exists a bounded stabilizing feedback. In this regard, see Example 1, Section 2.

The generalization of the classical stabilizability theory is therefore twofold: besides an extension of the concepts of sampling and Euler solutions to unbounded dynamics, we introduce a suitable notion of associated cost and of *stabilizability with regulated cost*. Furthermore, we obtain an explicit construction of stabilizing feedbacks with regulated cost based upon the existence of a special Lyapunov function, known as a Minimum Restraint function. The original notions of Sampling and Euler stabilizability associated to a discontinuous feedback and their relationship with the existence of a Lyapunov function can be found in [7, 8], where the target is zero and the dynamics are assumed to be bounded near the origin. Later, these results have been extended to more general targets, but always for f (and K) bounded close to the target (see e.g. [14] and the references therein). Minimum Restraint functions were first introduced in [19], where the existence of a function of this type was shown to guarantee global asymptotic controllability to a set, with regulated cost (see also [18]). The problem of defining a stabilizing feedback law with regulated cost through the use of a Minimum Restraint function has been addressed only recently in [16], just in the case of bounded data. The extension to unbounded dynamics is not achieved by refining the techniques already used. Rather, our strategy is to associate an equivalent, *rescaled*, problem with the starting problem, under assumptions that include and generalize those most used in the study of problems with unbounded data.

More in detail, we first assume f and l merely continuous on $(\mathbb{R}^n \setminus \mathcal{C}) \times U$. Hence, sampling trajectories can have a finite blow-up time and chattering phenomena may occur (see Subsection 2.1). As a consequence, classical Euler solutions –defined as uniform limits of sampling solutions– may not exist. This leads us to propose in Section 3 a notion of *weak* Euler solutions and costs, given by the pointwise limit of a sequence of suitably truncated sampling trajectories and costs. In support of the well-posedness of these definitions, we show that: (i) when they exist, classical Euler cost-solution pairs are weak Euler cost-solutions pairs; (ii) the sample stabilizability with regulated cost implies the weak Euler stabilizability with regulated cost (see Theorem 3.4); (iii) when the system is sample stabilizable with regulated cost and the data meet some conditions of weak coercivity –quite usual in optimization problems with unbounded controls, see e.g. [20, 22, 21] and the references therein–, (stabilizing) weak Euler cost-solution pairs do exist (see Proposition 4, Section 3).

Furthermore, we suppose in addition that there exists some continuous *rescaling function* $\nu = \nu(x, u) \geq 0$ such that the *rescaled dynamics and Lagrangian*, given by

$$\bar{f} := f/(1+\nu), \qquad \bar{l} := l/(1+\nu),$$

respectively, are bounded and uniformly continuous on $(B_R(\mathcal{C}) \setminus \mathcal{C}) \times U$, for some $R > 0$ (see hypotheses **(H.1-2)** below). Under this assumption, in Section 2 we can prove the equivalence between the sample stabilizability to the target with regulated cost of (1)-(2) and that of the *rescaled problem*, where $\bar{f}$ and $\bar{l}$ replace f and l, respectively (see Theorem 2.5). This result is crucial to establish in Section 3 sufficient conditions for sample (and therefore, weak Euler) stabilizability with regulated cost and to build explicit feedback strategies, by means of Minimum Restraint functions for the original or for the rescaled problem (see Theorems 4.2, 4.4). Finally, in Theorem 4.6 we show how to implement the previous feedback construction starting from a Lipschitz continuous (not necessarily semiconcave) Minimum Restraint function, when the data are Lipschitz continuous in the state variable. In particular, by choosing $l \equiv 0$ in (2), Theorems 4.4, 4.6 imply that, given a semiconcave or a Lipschitz continuous Lyapunov function for the unbounded control system (1), respectively, we build a (possibly unbounded) stabilizing feedback.

The introduction of $\bar{f}$ and $\bar{l}$ can be seen as a generalization of well-known compactification techniques usually exploited to deal with unbounded data. For instance, if $\nu := |(f, l)|$, then $(\bar{f}, \bar{l})$ coincides with the so-called Erdmann transform of (f, l), used e.g. in [18], while in case f and l are functions with a maximal u-growth $\tilde{\nu}(|u|)$, by choosing $\nu(x, u) := \tilde{\nu}(|u|)$, we can recover the extended Lagrangian and dynamics considered in impulsive control (see e.g. [22, 20, 12]). The assumptions considered in this paper allow for a vast class of dynamics and Lagrangians, including those with a polynomial dependence on $u_1, \cdots, u_m$, $|u_1|, \cdots, |u_m|$, $|u|$, and compositions of polynomials with exponential and Lipschitz continuous functions.

The paper is organized as follows. In the rest of the Introduction we provide some preliminary definitions and the precise assumptions. In Section 2 we introduce the notion of sample stabilizability with regulated cost and Example 1, and prove the equivalence Theorem 2.5. Section 3 is devoted to define weak Euler solutions and costs, derive their main properties, and discuss the concept of weak Euler stabilizability with regulated cost. Section 4 deals with sufficient conditions for sample, Euler and weak Euler stabilizability with regulated cost.

1.1. **Notations and preliminaries.** For every $r \geq 0$ and $\Omega \subset \mathbb{R}^n$, we set $B_r(\Omega) := \{x \in \mathbb{R}^n \mid d(x,\Omega) \leq r\}$, where d is the usual Euclidean distance. When $\Omega = \{z\}$ for some $z \in \mathbb{R}^n$, we also make use of the notation $B(z,r) := B_r(\{z\})$. We use $\overline{\Omega}$ to denote the closure of Ω. For $a, b \in \mathbb{R}$, $a \vee b := \max\{a,b\}$, $a \wedge b := \min\{a,b\}$. For any $F : \Omega \to \mathbb{R}^M$ we call *modulus (of continuity) of* F any increasing, continuous function $\omega : [0,+\infty) \to [0,+\infty)$ such that $\omega(0) = 0$, $\omega(r) > 0$ for every $r > 0$ and $|F(x_1) - F(x_2)| \leq \omega(|x_1 - x_2|)$ for all x_1, $x_2 \in \Omega$. We say that a map $F : I \to J$, I, J real intervals, is increasing (decreasing) when it is monotone nondecreasing (nonincreasing). We use $\mathcal{KL}$ to denote the set of all continuous functions $\beta : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ such that: (1) $\beta(0,t) = 0$ and $\beta(\cdot,t)$ is strictly increasing and unbounded for each $t \geq 0$; (2) $\beta(r,\cdot)$ is strictly decreasing for each $r \geq 0$; (3) $\beta(r,t) \to 0$ as $t \to +\infty$ for each $r \geq 0$.

Let us summarize some basic notions in nonsmooth analysis – see e.g. [4, 6, 29] for a thorough treatment. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set.
A continuous function $F : \overline{\Omega} \to \mathbb{R}$ is said *positive definite on* Ω if $F(x) > 0$ $\forall x \in \Omega$ and $F(x) = 0$ $\forall x \in \partial\Omega$. The function F is called *proper on* Ω if the pre-image $F^{-1}(K)$ of any compact set $K \subset [0,+\infty)$ is compact.
Let $F : \Omega \to \mathbb{R}$ be a locally Lipschitz function. For every $x \in \Omega$, $\partial_P F(x)$ is defined as the *proximal subdifferential* of F at x: $p \in \partial_P F(x)$ if and only if there exist ρ, $\eta > 0$ such that

$$F(y) - F(x) + \rho|y - x|^2 \geq \langle p, y - x\rangle \qquad \forall y \in B(x,\eta) \ (\subset \Omega).$$

The *limiting subdifferential* $\partial_L F(x)$ at x, is given by

$$\partial_L F(x) := \Big\{ \lim p_i : \ p_i \in \partial_P F(x_i), \ \lim x_i = x \Big\}.$$

The proximal subdifferential $\partial_P F(x)$ may be empty at some point; nevertheless, the set of such points has zero measure. The limiting subdifferential $\partial_L F(x)$ instead, is nonempty at every point. The Clarke generalized gradient can be derived as $\operatorname{co} \partial_L F(x)$ at any x.
We will consider also the *set of reachable gradients* of F at x:

$$D^*F(x) := \Big\{ w \in \mathbb{R}^n : \quad w = \lim_k \nabla F(x_k), \quad x_k \in DIFF(F) \setminus \{x\}, \quad \lim_k x_k = x \Big\}$$

where ∇ denotes the classical gradient operator and $DIFF(F)$ is the set of differentiability points of F. The set-valued map $x \rightsquigarrow D^*F(x)$ is upper semicontinuous on Ω, with nonempty, compact values, and $D^*F(x)$ is in general not convex.
A continuous function $F : \Omega \to \mathbb{R}$ is said to be *semiconcave on* Ω if there exist $\rho > 0$ such that

$$F(x) + F(\hat{x}) - 2F\left(\frac{x + \hat{x}}{2}\right) \leq \rho|x - \hat{x}|^2,$$

for all x, $\hat{x} \in \Omega$ such that $[x,\hat{x}] \subset \Omega$. The constant ρ above is called a *semiconcavity constant* for F in Ω. F is said to be *locally semiconcave on* Ω if it semiconcave on every compact subset of Ω. We remind that locally semiconcave functions are locally Lipschitz. Actually, they are twice differentiable almost everywhere.
When F is a locally semiconcave function, then $D^*F(x) = \partial_L F(x)$ for any $x \in \Omega$.

1.2. **Assumptions.** Through the whole paper, $U \subseteq \mathbb{R}^m$ and $\mathcal{C} \subset \mathbb{R}^n$ are closed, nonempty sets and the boundary $\partial\mathcal{C}$ is compact. Given $\mathbf{f} : (\mathbb{R}^n \setminus \mathcal{C}) \times U \to \mathbb{R}^n$ and $\mathbf{l} : (\mathbb{R}^n \setminus \mathcal{C}) \times U \to [0,+\infty)$, we will consider the following sets of hypotheses:

(H.1) *the functions* $\mathbf{f}$, $\mathbf{l}$ *are continuous on* $(\mathbb{R}^n \setminus \mathcal{C}) \times U$*;*

(H.2) *there exists a continuous function* $\nu : (\mathbb{R}^n \setminus \mathcal{C}) \times U \to [0, +\infty)$*, that we call* a rescaling function*, such that the* rescaled functions $\bar{\mathbf{f}}$*,* $\bar{\mathbf{l}}$*, defined by*

$$(\bar{\mathbf{f}}(x,u), \bar{\mathbf{l}}(x,u)) := \left(\frac{\mathbf{f}(x,u)}{1+\nu(x,u)}, \frac{\mathbf{l}(x,u)}{1+\nu(x,u)} \right) \quad \forall (x,u) \in (\mathbb{R}^n \setminus \mathcal{C}) \times U, \tag{3}$$

are uniformly continuous on $\mathcal{K} \times U$ *for every compact subset* $\mathcal{K} \subset \mathbb{R}^n \setminus \mathcal{C}$*, and for any* $R > 0$ *there is some* $M(R) > 0$ *such that*

$$|\bar{\mathbf{f}}(x,u)| \le M(R), \quad \bar{\mathbf{l}}(x,u) \le M(R) \qquad \forall (x,u) \in (B_R(\mathcal{C}) \setminus \mathcal{C}) \times U. \tag{4}$$

In the following, we set $\mathbf{d}(x) := d(x, \mathcal{C})$.

2. Sample stabilizability with regulated cost.

In this section we extend the notion of Sample stabilizability with regulated cost firstly introduced in [16, 17], to more general, unbounded data. Furthermore, in Theorem 2.5 we show that, if f and l satisfy **(H.2)**, the original problem is sample stabilizable with regulated cost if and only if the *rescaled problem* is.

2.1. Sampling processes.

Let $\mathbf{f}$, $\mathbf{l}$ verify **(H.1)**.

Definition 2.1 (Admissible process)**.** We say that a triple (x^0, x, u) is an *admissible process* (for $\mathbf{f}$, $\mathbf{l}$) if there exists $T_x \le +\infty$ such that: the control u belongs to $L^\infty_{loc}([0, T_x), U)$; $x : [0, T_x) \to \mathbb{R}^n \setminus \mathcal{C}$ is a solution of the control system

$$\dot{x}(t) = \mathbf{f}(x(t), u(t)), \quad \text{a.e. } t \in (0, T_x), \tag{5}$$

verifying, if $T_x < +\infty$, $\lim_{t \to T_x^-} \mathbf{d}(x(t)) = 0$; the cost x^0 is given by

$$x^0(t) := \int_0^t \mathbf{l}(x(\tau), u(\tau))\, d\tau, \quad \forall t \in [0, T_x). \tag{6}$$

For every $z \in \mathbb{R}^n \setminus \mathcal{C}$, we call (x^0, x, u) as above an *admissible process from* z, when $x(0) = z$.

A *partition* (of $[0, +\infty)$) is a sequence $\pi = (t_k)$ such that $t_0 = 0$, $t_{k-1} < t_k$ $\forall k \ge 1$, and $\lim_{k \to +\infty} t_k = +\infty$. The value $\operatorname{diam}(\pi) := \sup_{k \ge 1}(t_k - t_{k-1})$ will be called the *diameter* or the *sampling time* of the partition π.

We will call *feedback* any locally bounded function $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \to U$. In particular, when U is unbounded we allow feedbacks $\mathbf{K}$ verifying $\limsup_{x \to \bar{x} \in \partial\mathcal{C}} |\mathbf{K}(x)| = +\infty$.

Definition 2.2 (Sampling process)**.** Given a locally bounded feedback $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \to U$, a partition $\pi = (t_k)$, and a point $z \in \mathbb{R}^n \setminus \mathcal{C}$, we call π*-sampling process* (for $\mathbf{f}$, $\mathbf{l}$) *from* z, a triple (x^0, x, u), where x, called the *sampling trajectory*, is a continuous function defined by recursively solving

$$\dot{x} = \mathbf{f}(x(t), \mathbf{K}(x(t_{k-1}))) \qquad \text{a.e. } t \in [t_{k-1}, t_k], \ (x(t) \in \mathbb{R}^n \setminus \mathcal{C})$$

from the initial time t_{k-1} up to time

$$\tau_k := t_{k-1} \vee \sup\{\tau \in [t_{k-1}, t_k] : \ x \text{ is defined on } [t_{k-1}, \tau)\},$$

such that $x(t_0) = x(0) = z$. In this case, the trajectory x is defined on the right-open interval from time zero up to time $T^- := \inf\{\tau_k : \ \tau_k < t_k\}$. Accordingly, for every $k \ge 1$ and for all $t \in [t_{k-1}, t_k) \cap [0, T^-)$, the *sampling control* is defined as

$$u(t) := \mathbf{K}(x(t_{k-1})) \quad \forall t \in [t_{k-1}, t_k) \cap [0, T^-), \quad k \ge 1. \tag{7}$$

The *sampling cost,* x^0, is given by

$$x^0(t) := \int_0^t \mathbf{l}(x(\tau), u(\tau))\, d\tau \quad t \in [0, T^-). \tag{8}$$

If (x^0, x, u) is an admissible process and $T^- = T_x < +\infty$, we extend x to $[0, +\infty)$ by setting $x(t) := \bar{z} \quad \forall t \geq T_x$, where $\bar{z}$ is a point of the set

$$\omega(x) := \left\{ \lim_{j\to+\infty} x(t_j) : \quad (t_j) \text{ is increasing and } \lim_{j\to+\infty} t_j = T_x \right\}.$$

If $\lim_{t\to T_x^-} x^0(t) < +\infty$, we also extend x^0, by setting $x^0(t) := \lim_{t\to T_x^-} x^0(t) \quad \forall t \geq T_x$.

By the definition of T_x, the set $\omega(x)$ is always not empty, since $\partial\mathcal{C}$ is compact. In general, $\omega(x)$ is not a singleton, unless $\mathbf{f}$ is bounded on a neighborhood of $\mathcal{C}$, uniformly with respect to the control. Notice that for any admissible π-sampling process (x^0, x, u), the trajectory x, possibly extended as above, is always defined on the whole interval $[0, +\infty)$.

Definition 2.3 (Sample stabilizability with regulated cost)**.** A locally bounded feedback $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \to U$ is said to sample stabilize the control system $\dot{x} = \mathbf{f}(x, u)$ to $\mathcal{C}$ if there is a function $\beta \in \mathcal{KL}$, that we call *descent rate*, satisfying the following: for each pair $0 < r < R$ there exists $\delta = \delta(r, R) > 0$, such that, for every partition π with $\mathrm{diam}(\pi) \leq \delta$ and for any initial state $z \in \mathbb{R}^n \setminus \mathcal{C}$ such that $\mathbf{d}(z) \leq R$, any π-sampling process (x^0, x, u) for $\mathbf{f}$, $\mathbf{l}$ with $x(0) = z$ is admissible and verifies:

$$\mathbf{d}(x(t)) \leq \max\{\beta(\mathbf{d}(z), t), r\} \qquad \forall t \in [0, +\infty). \tag{9}$$

We call $\dot{x} = \mathbf{f}(x, u)$ *sample stabilizable (to $\mathcal{C}$)* if it admits a feedback $\mathbf{K}$ as above.

If moreover there exist $p_0 > 0$ and a continuous map $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$, positive definite and proper in $\mathbb{R}^n \setminus \mathcal{C}$, such that (x^0, x, u) also verifies

$$x^0(\bar{T}_x^r) = \int_0^{\bar{T}_x^r} \mathbf{l}(x(\tau), u(\tau))\, d\tau \leq \frac{W(z)}{p_0} \tag{10}$$

where

$$\bar{T}_x^r := \inf\{t > 0 : \mathbf{d}(x(\tau)) \leq r \quad \forall \tau \geq t\}, \tag{11}$$

we say that (5)–(6) is *sample stabilizable (to $\mathcal{C}$) with (p_0, W)-regulated cost.* To unify the notation, when $\dot{x} = \mathbf{f}(x, u)$ is merely sample stabilizable, we say that (5)–(6) is *sample stabilizable with (p_0, W)-regulated cost for $p_0 = 0$.*

Disregarding the cost, if the dynamics $\mathbf{f}$ is bounded on $(B_R(\mathcal{C}) \setminus \mathcal{C}) \times U$ for some $R > 0$, the above notion of sample stabilizability is a slight extension of the original one in [7, 15], consisting of the fact that our target is not necessarily a point and the feedback $\mathbf{K}$ can be unbounded on it. However, when both feedback controls and resulting dynamics are unbounded in a neighborhood of the target we are far beyond the classical theory (see e.g. [8, 14]).

Example 1 (A cheap control problem)**.** This example shows how the presence of a cost can drastically change the choice of a stabilizing feedback. In particular, in the following simple problem there is a continuous and bounded stabilizing feedback, but in order to obtain stabilizability with regulated cost it is necessary to choose an unbounded feedback. We consider the scalar control system

$$\dot{x} = f(x, u) := x^2 u, \qquad u \in \mathbb{R}, \tag{12}$$

with target $\mathcal{C} := \{0\}$ and associated cost

$$\int_0^{T_x} l(x)\,dt, \qquad l(x) := |x|, \tag{13}$$

where T_x is as in Definition 2.1. The bounded feedback $\bar{K}(x) := -\text{sign}(x)$ sample stabilizes the system. Indeed, given r, R, $0 < r < R$, for every partition π of $[0, +\infty)$, any π-sampling solution x associated to $\bar{K}$ with $x(0) = z \neq 0$, $|z| \leq R$, satisfies

$$|x(t)| = \frac{|z|}{|z|\,t + 1} =: \beta(|z|, t) \quad \forall t \in [0, +\infty),$$

and it is immediate to check that β is a $\mathcal{KL}$ function. However, by straightforward calculations one has

$$\int_0^{\bar{T}_x^r} |x(t)|\,dt = \ln\left(1 + \frac{|z| - r}{r}\right) \implies \lim_{r \to 0^+} \int_0^{\bar{T}_x^r} |x(t)|\,dt = +\infty$$

where $\bar{T}_x^r$ is as in (11). So $\bar{K}$ does not sample stabilize the system *with regulated cost*. By similar arguments, it can be shown that there is no bounded stabilizing feedback that gives a regulated cost.

Let us now consider the unbounded feedback $K(x) := -1/x$. Given r, R, $0 < r < R$, for every partition $\pi := (t_i)$ of $[0, +\infty)$ and any π-sampling trajectory of K with $x(0) = z$, $0 < |z| \leq R$, we have

$$x(t) = \frac{x(t_i)}{1 + (t - t_i)} \quad \forall t \in [t_i, t_{i+1}].$$

From this, observing that, fixed $\varepsilon \in (0, 1)$, there exists some $\bar{\delta} > 0$ such that $e^{\varepsilon\delta} \leq 1 + \delta$ for any $\delta \in (0, \bar{\delta}]$, we deduce that,

$$|x(t)| = \frac{|z|}{(1 + t - t_i)\prod_{k=0}^{i-1}(1 + t_{k+1} - t_k)} \leq \frac{|z|}{e^{\varepsilon\, t}} =: \beta(|z|, t) \quad \forall t \in [t_i, t_{i+1}],\ \forall\, i \in \mathbb{N},$$

as soon as $\text{diam}(\pi) \leq \bar{\delta}$. Since the right-hand side of the above inequality is a $\mathcal{KL}$ function of $|z|$ and t, then K is a sample stabilizing feedback to the origin. For the associated cost, we get

$$\int_0^{\bar{T}_x^r} |x(t)|\,dt \leq \int_0^{+\infty} |x(t)|\,dt \leq \int_0^{+\infty} |z|\,e^{-\varepsilon\, t}\,dt = \frac{|z|}{\varepsilon}.$$

Therefore, the feedback $K(x) = -1/x$ stabilizes (12)-(13) to the origin with (ε, W)-regulated cost, where $W(x) := |x|$ for all $x \in \mathbb{R}$.

Incidentally, $W(z) = |z|$ does not coincide with the *value function* of the problem, defined for any $z \neq 0$ as

$$V(z) := \inf_{\{(x,u) \text{ admissible},\ x(0)=z\}} \int_0^{T_x} l(x)\,dt,$$

which is identically zero, as it is not difficult to show. However, for any $n \in \mathbb{N}$, $n \geq 1$, there exists a locally bounded sample stabilizing feedback $K_n : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $(1, W_n)$-regulated cost, such that $\lim_{n \to +\infty} W_n(z) = 0$, for each z. This fact could be easily proved directly, by constructing a suitable sequence of feedbacks with this property. On the other hand, it also follows from Theorem 4.4 (see also Theorem 4.2) below, observing that, for each $n \geq 1$, the function

$$W_n(z) := n^{-1}(|z| \wedge |z|^n) \qquad \forall z \in \mathbb{R},$$

is a 1-MRF (see Definition 4.1 below) for f, l as above.

2.2. **Sample stabilizability with regulated cost: a rescaled problem.** Consider the original data f and l verifying hypotheses **(H.1-2)** for some rescaling function ν and let $\bar{f}$, $\bar{l}$ be the associated rescaled functions.

In the following, we denote by (x^0, x, u) any admissible process for f, l (see Definition 2.1). Precisely, $u \in L^\infty_{loc}([0, T_x), U)$, $x : [0, T_x) \to \mathbb{R}^n \setminus \mathcal{C}$ solves the control system

$$\dot{x}(t) = f(x(t), u(t)), \quad \text{a.e. } t \in (0, T_x), \tag{14}$$

and verifies $\lim_{t \to T_x^-} \mathbf{d}(x(t)) = 0$ as soon as $T_x < +\infty$, while the cost x^0 is given by

$$x^0(t) = \int_0^t l(x(\tau), u(\tau))\, d\tau \qquad \forall t \in [0, T_x). \tag{15}$$

Moreover, we introduce the *rescaled control system*

$$y'(s) = \bar{f}(y(s), v(s)), \quad \text{a.e. } s \in (0, S_y), \tag{16}$$

and the *rescaled cost*

$$y^0(s) = \int_0^s \bar{l}(y(\sigma), v(\sigma))\, d\sigma, \quad \forall s \in [0, S_y), \tag{17}$$

whose admissible processes will be denoted by (y^0, y, v), with domain $[0, S_y)$. In particular, we will call (y^0, y, v) an (admissible) *rescaled process*, y a *rescaled trajectory*, and v a *rescaled control*.

In (16) we use the apex "$\,'\,$" to denote differentiation with respect to the new parameter s, in order to stress that it does not coincide, in general, with the time variable t, of (14). Indeed, any rescaled process is composition of a process of the original problem with a suitable time-scale and vice-versa, as stated in the following lemma. Since every L^1 equivalence class contains Borel measurable representatives, from now on we assume without loss of generality that u and v are Borel measurable.

Lemma 2.4. *Fix $z \in \mathbb{R}^n \setminus \mathcal{C}$.*

(i) *Given an admissible process (x^0, x, u) from z, set*

$$s(t) := \int_0^t (1 + \nu(x(\tau), u(\tau)))\, d\tau \;\; \forall t \in [0, T_x), \quad S_y := \lim_{t \to T_x^-} s(t), \quad t(\cdot) := s^{-1}(\cdot).$$

Then $(y_0, y, v)(s) := (x^0, x, u) \circ t(s)$, $s \in [0, S_y)$, is an admissible rescaled process from z.

(ii) *Vice-versa, let (y^0, y, v) be an admissible rescaled process from z and set*

$$t(s) := \int_0^s (1 + \nu(y(\sigma), v(\sigma))^{-1} d\sigma \;\; \forall s \in [0, S_y), \quad T_x := \lim_{s \to S_y^-} t(s), \quad s(\cdot) := t^{-1}(\cdot).$$

Then $(x^0, x, u)(t) := (y_0, y, v) \circ s(t)$, $t \in [0, T_x)$, is an admissible process from z.

Proof. Claims (i), (ii) can be derived by a standard application of the chain rule, once observed that the inverse of an absolutely continuous real map with derivative > 0 almost everywhere, is absolutely continuous (see e.g. [11, Theorem 2.10.13]). □

In Theorem 2.5 below we establish the equivalence between the sample stabilizability with regulated cost of the original problem and that of the rescaled problem.

Theorem 2.5. *Assume that* f, l *satisfy* **(H.1-2)**. *Then a locally bounded feedback* $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ *sample stabilizes the original problem* (14)–(15) *to* $\mathcal{C}$ *with* (p_0, W)-*regulated cost for some* $p_0 \geq 0$, *if and only if it sample stabilizes the rescaled problem* (16)–(17) *to* $\mathcal{C}$ *with* (p_0, W)-*regulated cost.*

Proof. Let K be a sample stabilizing feedback with (p_0, W)-regulated cost for the rescaled problem. Then there exists a function $\beta \in \mathcal{KL}$ such that for any r, $R > 0$, $r < R$, there is some $\tilde{\delta} = \tilde{\delta}(r, R) > 0$ such that for every $z \in \mathbb{R}^n \setminus \mathcal{C}$ with $\mathbf{d}(z) \leq R$ and every partition $\tilde{\pi} = (s_k)$ of $\mathrm{diam}(\tilde{\pi}) \leq \tilde{\delta}$, any $\tilde{\pi}$-sampling rescaled process (y^0, y, v), with initial datum z, is admissible and verifies

$$\begin{aligned} &\mathbf{d}(y(s)) \leq \max\{\beta(\mathbf{d}(z), s), r\} \qquad \forall s \geq 0, \\ &y^0(s) \leq \frac{W(z)}{p_0} \qquad \forall s \in [0, \bar{S}_y^r] \qquad (\text{if } p_0 > 0), \end{aligned} \tag{18}$$

where $\bar{S}_y^r = \inf\{s \geq 0 : \ \mathbf{d}(y(\sigma)) \leq r \ \forall \sigma \geq s\}$. Let $r' = r'(r) > 0$ verify the relation

$$\beta(2r', 0) = r, \tag{19}$$

and let us define $\tilde{N}(r, R) \geq 0$ and $\tilde{M}(r, R) \geq 1$, as

$$\begin{aligned} \tilde{N}(r, R) &:= \sup\{\nu(x, K(x)) : \ \ r'(r) \leq \mathbf{d}(x) \leq \beta(R, 0) + 2R\}, \\ \tilde{M}(r, R) &:= \sup\{|(f, l)(x, K(x))| : \ \ r'(r) \leq \mathbf{d}(x) \leq \beta(R, 0) + 2R\} \vee 1. \end{aligned} \tag{20}$$

Notice that, by the very definition of β, $\beta(2r', 0) \geq 2r'$, so that $r' \leq \frac{r}{2}$. Hence $\tilde{N}(r, R)$ and $\tilde{M}(r, R)$ are well-defined. Clearly, for every fixed $R > 0$, on $(0, R)$ we can suppose $r \mapsto r'(r)$ strictly increasing and continuous, so that $r \mapsto \tilde{N}(r, R)$, $\tilde{M}(r, R)$, are locally bounded and decreasing (possibly diverging to $+\infty$ as r tends to 0^+). Hence they can be dominated by some r-continuous and strictly decreasing maps $N(r, R) \geq \tilde{N}(r, R)$, $M(r, R) \geq \tilde{M}(r, R)$. We set

$$\delta = \delta(r, R) := \frac{\tilde{\delta}(r, R)}{1 + N(r, R)} \wedge \frac{1}{M(r, R)}. \tag{21}$$

For every $z \in \mathbb{R}^n \setminus \mathcal{C}$ with $\mathbf{d}(z) \leq R$ and every partition $\pi = (t_k)$ of $[0, +\infty)$ with $\mathrm{diam}(\pi) \leq \delta$, let us consider an arbitrary π-sampling process (x^0, x, u) from z for the original problem (14)-(15). Let $[0, T^-)$ be the maximal definition interval of x and let us define

$$\hat{T} = \hat{T}_x(r, R) := \sup\{t \in [0, T^-) : \ r'(r) \leq \mathbf{d}(x(t)) \leq \beta(R, 0) + 2R\}.$$

Since $\mathbf{d}(x(0)) = \mathbf{d}(z) \leq R$ and $\mathbf{d}$ is positive definite and proper on $\mathbb{R}^n \setminus \mathcal{C}$, one has $0 < \hat{T} < T^-$. Set

$$s(t) := \int_0^t (1 + \nu(x(\tau), u(\tau)))\, d\tau \quad \forall t \in [0, T^-), \quad \hat{S} := s(\hat{T}), \ S^- := \lim_{t \to T^-} s(t), \tag{22}$$

and let $t = t(s)$ be the inverse map of $s : [0, T^-) \to [0, S^-)$. By Lemma 2.4, the process $(y^0, y, v) := (x^0, x, u) \circ t$ is (the restriction to $[0, \hat{S}]$ of) a $\tilde{\pi}$-sampling rescaled process with $y(0) = z$ and $\mathrm{diam}(\tilde{\pi}) \leq \tilde{\delta}$. Indeed, setting $\hat{n} := \sup\{i \in \mathbb{N} : \ t_i \leq \hat{T}\}$, $s_k := s(t_k)$ $\forall k = 0, \dots, \hat{n}$, and $s_k = s_{k-1} + \tilde{\delta}$ for all $k > \hat{n}$, then for every $k = 1, \dots, \hat{n}$, one has

$$s_k - s_{k-1} = \int_{t_{k-1}}^{t_k} (1 + \nu(x(\tau), u(\tau)))\, d\tau \leq (1 + N(r, R))\delta(r, R) \leq \tilde{\delta}(r, R). \tag{23}$$

Therefore, any $\tilde{\pi}$-sampling extension of (y, v) (associated to K) to $[0, +\infty)$ satisfies (18), so that, for all $t \in [0, \hat{T}]$, we get

$$\begin{aligned} &\mathbf{d}(x(t)) = \mathbf{d}(y(s(t)) \le \max\{\beta(\mathbf{d}(z), s(t)), r\} \le \max\{\beta(\mathbf{d}(z), t), r\}, \\ &x^0(t) = y^0(s(t)) \le \frac{W(z)}{p_0} \qquad (\text{if } p_0 > 0), \end{aligned} \tag{24}$$

where the last inequality in the first expression holds true since $s(t) \ge t$ and the map $t \mapsto \beta(\mathbf{d}(z), t)$ is decreasing. It remains only to show that x is a sampling trajectory extendable to the whole interval $[0, +\infty)$ as described in Definition 2.2 and such that

$$\mathbf{d}(x(t)) \le r \quad \forall t \ge \hat{T}. \tag{25}$$

In fact, proven (25), we get (9) by the first relation in (24). In addition, (25) also implies that $\bar{T}_x^r = \inf\{t \ge 0 : \ \mathbf{d}(x(\tau)) \le r \ \forall \tau \ge t\} \le \hat{T}$ and this, together with the last relation in (24), yields the cost estimate (10). By the arbitrariness of (x^0, x, u), this concludes the proof.

So, let us check (25). By the definition of $\hat{T}$ and by (24), one has $\mathbf{d}(x(\hat{T})) = r'$ and $T^- > \hat{T}$, where either $T^- = T_x$ or $\lim_{t \to T^-} |x(t)| = +\infty$. Suppose first that $\mathbf{d}(x(t)) \le r$ for all $t \in [0, T^-)$. Then either $T^- = +\infty$ and (25) holds true, or $T^- = T_x$. In this case, x can be extended to $[0, +\infty)$ in such a way that $\mathbf{d}(x(t)) = 0$ for all $t \ge T_x$ as in Definition 2.2 and (25) is proven. Suppose now, by contradiction, that $\mathbf{d}(x(t)) > r$ for some $t \in (0, T^-)$. By the continuity of x and $\mathbf{d}$, there exist some $\hat{t}^0$, $\hat{t}^1$ such that $\hat{T} < \hat{t}^0 < \hat{t}^1 < T^-$, and, for every $t \in [\hat{t}^0, \hat{t}^1]$,

$$r' < \mathbf{d}(x(\hat{t}^0)) = 2r' \le r < \mathbf{d}(x(\hat{t}^1)) < R, \quad \mathbf{d}(x(\hat{t}^0)) \le \mathbf{d}(x(t)) \le \mathbf{d}(x(\hat{t}^1)).$$

Hence by (22), for all $\underline{t}$, $\bar{t} \in [\hat{t}^0, \hat{t}^1]$, $\underline{t} < \bar{t}$, such that $\bar{t} - \underline{t} \le \delta$, one has

$$s(\bar{t}) - s(\underline{t}) \le (1 + N(r, R))(\bar{t} - \underline{t}) \le \tilde{\delta}.$$

So, setting $\hat{s}^0 := s(\hat{t}^0)$, $\hat{s}^1 := s(\hat{t}^1)$, we can see the process

$$(y^0, y, v)(s) := (x^0, x, u)(t(s + \hat{s}^0)) \qquad \text{for } s \in [0, \hat{s}^1 - \hat{s}^0],$$

as the restriction of a $\tilde{\pi}$-sampling rescaled process with $y(0) = x(\hat{t}^0)$ and $\text{diam}(\tilde{\pi}) \le \tilde{\delta}$. But then (18) yields

$$\mathbf{d}(x(t)) = \mathbf{d}(y(s(t) - \hat{s}^0)) \le \beta(\mathbf{d}(y(0)), 0) = \beta(2r', 0) = r \qquad \forall t \in [\hat{t}^0, \hat{t}^1],$$

which contradicts the hypothesis that $\mathbf{d}(x(\hat{t}^1)) > r$.

Suppose now that K is a sample stabilizing feedback to $\mathcal{C}$ with (p_0, W)-regulated cost for the original problem. Let β, $\delta = \delta(r, R)$ for any pair r, R with $0 < r < R$ be as in Definition 2.3, so that every π-sampling process (x^0, x, u) with initial datum $z \in \mathbb{R}^n \setminus \mathcal{C}$, $\mathbf{d}(z) \le R$, and $\text{diam}(\pi) \le \delta$, verifies

$$\mathbf{d}(x(t)) \le \max\{\beta(\mathbf{d}(z), t), r\} \ \ \forall t \ge 0, \quad x^0(t) \le \frac{W(z)}{p_0} \ \ \forall t \in [0, \bar{T}_x^r] \ \ (\text{if } p_0 > 0),$$

where $\bar{T}_x^r = \inf\{t \ge 0 : \ \mathbf{d}(x(\tau)) \le r \ \forall \tau \ge t\}$. Let $r' > 0$, $N(r, R)$, and $M(r, R)$ be defined as in (19), (20).

For any $\tilde{\pi}$-sampling rescaled process (y^0, y, v) with $y(0) = z$ and $\text{diam}(\tilde{\pi}) \le \delta$, consider the time-scaling

$$t(s) := \int_0^s (1 + \nu(y(\sigma), v(\sigma)))^{-1} \, d\sigma \quad \forall s \in [0, S^-), \quad \hat{T} := t(\hat{S}), \quad s(\cdot) := t^{-1}(\cdot),$$

where $[0, S^-)$ is the maximal definition interval of y and

$$\hat{S} = \hat{S}_y(r,R) := \sup\{s \in [0,S^-): \ r'(r) \le \mathbf{d}(y(s)) \le \beta(R,0) + 2R\}.$$

Then $t(s) \ge \frac{s}{1+N(r,R)}$ for all $s \in [0,\hat{S}]$ and, arguing as in the previous step, one can easily conclude that y is a sampling trajectory of the rescaled system extendable to $[0,+\infty)$, and that (y^0, y, v) verifies

$$\begin{aligned} &\mathbf{d}(y(s)) \le \max\{\beta(\mathbf{d}(z), t(s)), r\} \le \max\left\{\beta\left(\mathbf{d}(z), \frac{s}{1+N(r,R)}\right), r\right\}, \\ &y^0(s) \le \frac{W(z)}{p_0} \qquad \forall s \in [0, \bar{S}^r_y] \ \ (\text{if } p_0 > 0), \end{aligned} \tag{26}$$

where $\bar{S}^r_y = \inf\{s \ge 0: \ \mathbf{d}(y(\sigma)) \le r \ \forall \sigma \ge s\}$. Let $S(r,R) > 0$ be the value of s implicitly defined by the equation

$$\beta\left(R, \frac{s}{1+N(r,R)}\right) = r.$$

By the monotonicity and continuity properties of β and N, it follows that $S(\cdot,\cdot)$ is a continuous function on $\{(r,R): \ 0 < r < R\}$, such that $r \mapsto S(r,R)$ is strictly decreasing and $R \mapsto S(r,R)$ is strictly increasing. As a consequence, if we denote by $\rho = \rho(R,s)$ the inverse of the map $\rho \mapsto S(\rho,R)$, it is easy to see that ρ is a $\mathcal{KL}$ function. At this point, observe that, for every $\rho \in [r,R)$, since $\mathrm{diam}(\tilde{\pi}) \le \delta(r,R) \le \delta(\rho,R)$, the first relation in (26) implies

$$\mathbf{d}(y(s)) \le \beta\left(R, \frac{s}{1+N(\rho,R)}\right) \quad \forall s \in [0, S(\rho,R)], \quad \mathbf{d}(y(s)) \le r \quad \forall s > S(r,R),$$

which, substituting $\rho = \rho(R,s)$, yields

$$\begin{aligned} &\mathbf{d}(y(s)) \le \beta\left(R, \frac{s}{1+N(\rho(R,s),R)}\right) = \rho(R,s) \qquad \forall s \le S(r,R), \\ &\mathbf{d}(y(s)) \le r \qquad \forall s > S(r,R). \end{aligned}$$

Since $\mathbf{d}(z) \le R$ implies that $\delta(r,\mathbf{d}(z)) \ge \delta(r,R)$, we finally obtain that, for every $\tilde{\pi}$-process with $\mathrm{diam}(\tilde{\pi}) \le \delta(r,R)$, one has

$$\mathbf{d}(y(s)) \le \rho(\mathbf{d}(z), s) \quad \forall s \le S(r,\mathbf{d}(z)), \quad \mathbf{d}(y(s)) \le r \quad \forall s > S(r,\mathbf{d}(z)),$$

which is trivially equivalent to

$$\mathbf{d}(y(s)) \le \max\{\rho(\mathbf{d}(z), s), r\} \qquad \forall s \ge 0.$$

Together with the second relation in (26), this concludes the proof of the sample stabilizability to $\mathcal{C}$ with (p_0, W)-regulated cost of the rescaled problem. □

Remark 1. In view of the above proof, when there is stabilizability, a descent rate β for the rescaled problem is a descent rate also for the original problem. Instead, given a descent rate β for the original problem, a descent rate ρ for the rescaled problem is in general larger. In this case, we get an explicit construction of ρ.

3. Weak Euler stabilizability with regulated cost.

In our previous results, the controllers are taken to be discontinuous feedbacks, so, in principle, the dynamics is discontinuous in the state variable. However, these results are stated in terms of sampling trajectories, which are classical solutions corresponding to piecewise constant controls. Therefore, the issue of defining a solution concept for discontinuous differential equations has so far been neglected. In this section we address this question and define Euler solutions, weak Euler solutions, and the associated costs. Furthermore, we introduce the notions of Euler and of weak Euler stabilizability with regulated cost, and prove that the sample stabilizability with regulated cost implies both of them.

3.1. Weak Euler solutions and costs.

Let the data $\mathbf{f}$, $\mathbf{l}$ verify assumption **(H.1)**.

Following [16], we define Euler trajectories and costs as locally uniform limits of sampling trajectories and costs.

Definition 3.1 (Euler trajectory and cost)**.** Given a locally bounded feedback $\mathbf{K}:\mathbb{R}^n\setminus\mathcal{C}\to U$, fix $z\in\mathbb{R}^n\setminus\mathcal{C}$ and let (π_i) be a sequence of partitions such that $\delta_i:=\mathrm{diam}(\pi_i)\to 0$ as $i\to\infty$. For every i, let (x_i^0,x_i,u_i) be an admissible π_i-sampling process for the data $\mathbf{f}$, $\mathbf{l}$ with initial condition $x_i(0)=z$, such that x_i is defined on $[0,+\infty)$. If there exists a map $\mathcal{X}:[0,+\infty)\to\mathbb{R}^n$ verifying

$$x_i\to\mathcal{X}\qquad\text{locally uniformly in }[0,+\infty),\tag{27}$$

we call $\mathcal{X}$ an *Euler trajectory from z* of (5). If moreover, every x_i^0 is defined on $[0,+\infty)$ and there is a map $\mathcal{X}^0:[0,+\infty)\to[0,+\infty)$ verifying

$$x_i^0\to\mathcal{X}^0\qquad\text{locally uniformly in }[0,+\infty),\tag{28}$$

we call $\mathcal{X}^0$ an *Euler cost from z* associated to $\mathcal{X}$.

The above notion of solution is well suited for situations where the discontinuous dynamics associated to the feedback are bounded around the target, while it seems too strong in the general case. Indeed, suppose that (5)–(6) is sample stabilizable to $\mathcal{C}$ with (p_0,W)-regulated cost for some $p_0>0$. Then, fixed z, any sequence of π_i-sampling cost-trajectory pairs (x_i^0,x_i) with $\mathrm{diam}(\pi_i)\to 0$ is equibounded. If for any $R>0$ there is some $M(R)>0$ such that

$$|\mathbf{f}(x,K(x))|\le M(R),\quad \mathbf{l}(x,K(x))\le M(R)\qquad\forall x\in B_R(\mathcal{C})\setminus\mathcal{C},$$

the sequence (x_i^0,x_i) is also equi-Lipschitz continuous. Therefore, passing eventually to a subsequence, it converges locally uniformly by Ascoli-Arzelá Theorem and the existence of an Euler solution to (5) and of an associated Euler cost is guaranteed.

When instead the data are truly unbounded, sampling trajectories may approach the target faster and faster and even converge to discontinuous functions. Hence Euler solutions defined as locally uniform limits of sampling solutions as above, may not exist. An analogous remark holds for the associated Euler costs. These considerations lead us to consider the following notions of *weak Euler solution* and *weak Euler cost*, inspired by the impulsive control theory (see also [17]), for which we are able to provide existence under weak coercivity conditions which are satisfied in several applications.

Definition 3.2 (Weak Euler trajectory and cost)**.** Let $\mathbf{K}:\mathbb{R}^n\setminus\mathcal{C}\to U$ be a locally bounded feedback, fix $z\in\mathbb{R}^n\setminus\mathcal{C}$, and let (π_i) be a sequence of partitions such that $\delta_i := \mathrm{diam}(\pi_i) \to 0$ as $i \to \infty$. For every i, let (x_i^0, x_i, u_i) be an admissible π_i-sampling process for $\mathbf{f}, \mathbf{l}$ with $x_i(0) = z$. When there exists a map $\mathcal{X} : [0, +\infty) \to \mathbb{R}^n$, verifying, for some sequence $(r_i) \subset (0, \mathbf{d}(z))$ converging to 0:

$$\tilde{x}_i \to \mathcal{X} \qquad \text{pointwise in } [0, +\infty) \tag{29}$$

where, for each i,

$$\begin{aligned} &T_i := \bar{T}_{x_i}^{r_i} = \inf\{t > 0 : \mathbf{d}(x_i(\tau)) \le r_i \ \ \forall \tau \ge t\} \le +\infty, \\ &\tilde{x}_i(t) := x_i(t \wedge T_i) \ \ \forall t \ge 0, \end{aligned} \tag{30}$$

[1]

we call $\mathcal{X}$ a *weak Euler trajectory from* z of (5). For each i, let us set

$$\tilde{x}_i^0(t) := x_i^0(t \wedge T_i) \quad \forall t \ge 0. \tag{31}$$

When it exists, we call *weak Euler cost* associated to $\mathcal{X}$ a map $\mathcal{X}^0 : [0, +\infty) \to [0, +\infty)$, verifying

$$\tilde{x}_i^0 \to \mathcal{X}^0 \qquad \text{pointwise in } [0, +\infty). \tag{32}$$

In short, we will say that $(\mathcal{X}^0, \mathcal{X})$ a *weak Euler cost-trajectory pair from* z.

Clearly, Euler solutions and costs are also weak Euler solutions and costs, respectively. Some relevant properties of weak Euler solutions and costs are stated in Propositions 1, 2, and 3 below.

Given a weak Euler solution $\mathcal{X}$, let us define the exit-time $T_{\mathcal{X}} \le +\infty$ as

$$T_{\mathcal{X}} := \inf\{t > 0 : \ \mathcal{X}([0, t)) \subset \mathbb{R}^n \setminus \mathcal{C}, \ \ \lim_{\tau \to t^-} \mathbf{d}(\mathcal{X}(\tau)) = 0\} \le +\infty, \tag{33}$$

($T_{\mathcal{X}} := +\infty$ if the set is empty). Notice that the function $\mathcal{X}$ is in general discontinuous and it may happen that $\lim\limits_{t \to +\infty} \mathbf{d}(\mathcal{X}(t)) \ne 0$ or $T_{\mathcal{X}} < +\infty$ and $\mathbf{d}(\mathcal{X}(T + \varepsilon)) = 0$ for some $\varepsilon > 0$, but $\lim_{t \to T_x^-} \mathbf{d}(\mathcal{X}(t)) \ne 0$, despite each sampling process (x_i^0, x_i, u_i) in the definition of $\mathcal{X}$ is admissible and verifies $\lim\limits_{t \to +\infty} \mathbf{d}(x_i(t)) = 0$.

Next result provides a uniform lower bound for the exit time $T_{\mathcal{X}}$. In particular, the local boundedness of the feedbacks prevents the existence of purely impulsive weak Euler trajectories, that jump from the initial state to the target in zero time.

Proposition 1. *Given a locally bounded feedback* $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \to U$*, let* x *be an admissible* π*-sampling trajectory of* $\dot{x} = \mathbf{f}(x, u)$ *from* $z \in \mathbb{R}^n \setminus \mathcal{C}$ *associated to* $\mathbf{K}$*. Then for each* $\varepsilon \in (0, \mathbf{d}(z))$*, one has*

$$0 < T^\varepsilon := \frac{\mathbf{d}(z) - \varepsilon}{M(\varepsilon, \mathbf{d}(z))} \le T_x^\varepsilon \tag{34}$$

where

$$T_x^\varepsilon := \inf\{t > 0 : \ \mathbf{d}(x(t)) \le \varepsilon\} \tag{35}$$

and M *is the function mapping the pairs* (ε, R) *with* $0 < \varepsilon < R$ *to*

$$M(\varepsilon, R) := \sup\{|\mathbf{f}(x, u)| \ | \ \ \varepsilon \le \mathbf{d}(x) \le R, \ u \in \mathbf{K}(x)\}. \tag{36}$$

Moreover, if $\mathcal{X}$ *is a weak Euler solution from* z*, then*

$$0 < T^\varepsilon \le T_{\mathcal{X}}. \tag{37}$$

---

[1] When $T_x \le +\infty$ and $\lim_{t \to T_x^-} \mathbf{d}(x(t)) = 0$, T_i is obviously finite. It may happen that $T_i = +\infty$ only in case $T_x = +\infty$ and $\lim_{t \to +\infty} \mathbf{d}(x(t)) \ne 0$. Obviously, $t \wedge (+\infty) = t$.

Proof. Fix $z \in \mathbb{R}^n \setminus \mathcal{C}$ and let x be an admissible π-sampling trajectory of $\dot{x} = \mathbf{f}(x, u)$ associated to the feedback $\mathbf{K}$ and verifying $x(0) = z$. Given $\varepsilon \in (0, \mathbf{d}(z))$, let $T_x^\varepsilon := \inf\{t \geq 0 : \ \mathbf{d}(x(t)) \leq \varepsilon\}$.
When $T_x^\varepsilon = +\infty$, the lower bound (34) is trivially verified. Let T_x^ε be finite. Then $\mathbf{d}(x(T_x^\varepsilon)) = \varepsilon$ and there is some $z^\varepsilon \in \partial\mathcal{C}$ such that $\varepsilon = \mathbf{d}(x(T_x^\varepsilon)) = |x(T_x^\varepsilon) - z^\varepsilon|$. Moreover, by continuity, there exists some time $t_x^\varepsilon \in [0, T_x^\varepsilon)$, such that

$$\mathbf{d}(z) = \mathbf{d}(x(t_x^\varepsilon)) \geq \mathbf{d}(x(t)) \geq \varepsilon \quad \forall t \in [t_x^\varepsilon, T_x^\varepsilon].$$

Hence T_x^ε verifies (34), since

$$\begin{aligned}\mathbf{d}(z) = \mathbf{d}(x(t_x^\varepsilon)) &\leq |x(t_x^\varepsilon) - z^\varepsilon| \leq |x(t_x^\varepsilon) - x(T_x^\varepsilon)| + |x(T_x^\varepsilon) - z^\varepsilon| \\ &\leq M(\varepsilon, \mathbf{d}(z))\,(T_x^\varepsilon - t_x^\varepsilon) + \varepsilon \leq M(\varepsilon, \mathbf{d}(z))\, T_x^\varepsilon + \varepsilon,\end{aligned}$$

where $M(\varepsilon, \mathbf{d}(z))$ is as in (36). Incidentally, $M(\varepsilon, \mathbf{d}(z)) < +\infty$ by the properties of $\mathbf{f}$, $\mathbf{d}$ and $\mathbf{K}$.

Let $\mathcal{X}$ be a weak Euler solution for $\mathbf{K}$ with initial condition $z \in \mathbb{R}^n \setminus \mathcal{C}$, determined by a sequence (x_i^0, x_i, u_i) of admissible π_i-sampling processes from z with $\operatorname{diam}(\pi_i) = \delta_i$ and by (r_i), as in Definition 3.2. In particular, $\mathcal{X}$ is the pointwise limit of $(\tilde{x}_i)$, where $\tilde{x}_i(t) = x_i(t)$ for any $t \leq \bar{T}_{x_i}^{r_i} := \inf\{t \geq 0 : \ \mathbf{d}(x_i(\tau)) \leq r_i \ \forall \tau \geq t\}$. Given $\varepsilon \in (0, \mathbf{d}(z))$, for any $i \in \mathbb{N}$, set $T_i^\varepsilon := \inf\{t > 0 : \ \mathbf{d}(x_i(t)) \leq \varepsilon\}$. We can assume without loss of generality $r_i < \varepsilon$ for all $i \in \mathbb{N}$, since $r_i \to 0$. Hence, $\bar{T}_{x_i}^{r_i} \geq T_i^\varepsilon$ and by the previous step it follows that

$$\bar{T}_{x_i}^{r_i} \geq T_i^\varepsilon \geq T^\varepsilon = \frac{\mathbf{d}(z) - \varepsilon}{M(\varepsilon, \mathbf{d}(z))}.$$

Hence, for every $t \in [0, T_\varepsilon]$,[2] $\mathbf{d}(\tilde{x}_i(t)) = \mathbf{d}(x_i(t)) \geq \varepsilon$ and passing to the limit as $i \to +\infty$ we get $\mathbf{d}(\mathcal{X}(t)) \geq \varepsilon$. As a consequence, we can conclude that $T_{\mathcal{X}} \geq T^\varepsilon$. □

The sequence (r_i) plays a key role in Definition 3.2. In particular, when, for some i, the time T_i is finite, the truncated functions $\tilde{x}_i^0$, $\tilde{x}_i$, differently from x_i^0 and x_i, cannot have a chattering behaviour. However, the restriction to the interval $[0, T_{\mathcal{X}})$ of a weak Euler cost-trajectory pair $(\mathcal{X}^0, \mathcal{X})$ associated to a sampling sequence (x_i^0, x_i, u_i) does not depend on the choice of the sequence (r_i). Precisely, we have:

Proposition 2. *Given a locally bounded feedback $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \to U$, let $(\mathcal{X}^0, \mathcal{X})$ be a weak Euler cost-trajectory pair with initial condition $z \in \mathbb{R}^n \setminus \mathcal{C}$. Let (x_i^0, x_i, u_i), (δ_i), and (r_i) determine $(\mathcal{X}^0, \mathcal{X})$, as in Definition 3.2. Then the following properties hold true.*

(i) *Setting*

$$\bar{T} := \liminf_{i \to +\infty} T_i \qquad (T_i \text{ as in } (30)), \tag{38}$$

we have that $0 < T_{\mathcal{X}} \leq \bar{T} \leq +\infty$, and

$$(\mathcal{X}^0, \mathcal{X})(t) = \lim_{i \to +\infty} (\tilde{x}_i^0, \tilde{x}_i)(t) = \lim_{i \to +\infty} (x_i^0, x_i)(t) \qquad \forall t \in [0, \bar{T}). \tag{39}$$

(ii) *Let $(\hat{r}_i) \subset (0, \mathbf{d}(z))$ be any sequence converging to zero and such that $\hat{r}_i \geq r_i$ for all $i \in \mathbb{N}$. Define, for each $i \in \mathbb{N}$,*

$$\hat{T}_i := \bar{T}_{x_i}^{\hat{r}_i} = \inf\{t > 0 : \mathbf{d}(x_i(\tau)) \leq \hat{r}_i \ \ \forall \tau \geq t\}, \quad \hat{T} := \liminf_{i \to +\infty} \hat{T}_i. \tag{40}$$

[2] If $T_\varepsilon = +\infty$, we obviously mean $t \in [0, +\infty)$.

Then $T_{\mathcal{X}} \le \hat{T}$ $(\le \bar{T})$ and the sequence $(\hat{x}_i^0, \hat{x}_i)$, where $(\hat{x}_i^0, \hat{x}_i)(t) := (x_i^0, x_i)(t \wedge \hat{T}_i)$ for all $t \ge 0$ and $i \in \mathbb{N}$, verifies

$$\lim_{i\to+\infty} (\hat{x}_i^0, \hat{x}_i)(t) = (\mathcal{X}^0, \mathcal{X})(t) \qquad \forall t \in [0, \hat{T}).$$

Proof of Proposition 2. (i) By Proposition 1 it follows that, given an arbitrary $\varepsilon \in (0, \mathbf{d}(z))$ one has $\bar{T} \ge T^\varepsilon > 0$. For every $t \in [0, \bar{T})$, it is clear that $t < T_i$ for all i large enough. Hence $(\tilde{x}_i^0, \tilde{x}_i)(t) = (x_i^0, x_i)(t)$ for such i and the definitions (29), (32) imply that

$$\lim_{i\to+\infty} (x_i^0, x_i)(t) = (\mathcal{X}^0, \mathcal{X})(t) \qquad \forall t \in [0, \bar{T}).$$

If $\bar{T} = +\infty$, the proof is concluded. If instead $\bar{T} < +\infty$, it remains to show that $T_{\mathcal{X}} \le \bar{T}$. To this aim, let us consider a subsequence $k \mapsto i_k$ such that $\bar{T} = \lim_{k\to+\infty} T_{i_k}$. For every $\varepsilon_1 > 0$, $T_{i_k} < \bar{T} + \varepsilon_1$ for all k large enough, so that $\mathbf{d}(\tilde{x}_{i_k}(\bar{T} + \varepsilon_1)) = \mathbf{d}(x_{i_k}(T_{i_k})) = r_{i_k}$ for such k. Hence, we get

$$\mathbf{d}(\mathcal{X}(\bar{T} + \varepsilon_1)) \le |\mathcal{X}(\bar{T} + \varepsilon_1) - \tilde{x}_{i_k}(\bar{T} + \varepsilon_1)| + \mathbf{d}(\tilde{x}_{i_k}(\bar{T} + \varepsilon_1)) \to 0 \quad \text{as } k \to \infty.$$

By the arbitrariness of $\varepsilon_1 > 0$, this implies that $T_{\mathcal{X}} \le \bar{T}$, so concluding the proof.

(ii) By the hypothesis that $\hat{r}_i \ge r_i$, it follows that $\hat{T}_i \le T_i$ $\forall i \in \mathbb{N}$. Hence $\hat{T} \le \bar{T}$. The facts that $\hat{T} > 0$ and

$$\lim_{i\to+\infty} (\hat{x}_i^0, \hat{x}_i)(t) = \lim_{i\to+\infty} (x_i^0, x_i)(t) = (\mathcal{X}^0, \mathcal{X})(t) \qquad \forall t \in [0, \hat{T}),$$

can be proved arguing as in the previous step. It remains to show that $T_{\mathcal{X}} \le \hat{T}$. If $\hat{T} = \bar{T}$, the thesis follows by (i). Suppose now $\hat{T} < \bar{T}$. Let $k \mapsto i_k$ be a subsequence such that $\lim_{k\to+\infty} \hat{T}_{i_k} = \hat{T}$. For every $\varepsilon_1 > 0$ such that $\hat{T} < \hat{T} + \varepsilon_1 < \bar{T}$, by the definition of $\bar{T}$ one has $\hat{T} + \varepsilon_1 \le T_{i_k}$ for all k large enough. Therefore, $(\tilde{x}_{i_k}^0, \tilde{x}_{i_k})(\hat{T} + \varepsilon_1) = (x_{i_k}^0, x_{i_k})(\hat{T} + \varepsilon_1)$ for such k, and

$$\begin{aligned}\mathbf{d}(\mathcal{X}(\hat{T} + \varepsilon_1)) &\le |\mathcal{X}(\hat{T} + \varepsilon_1) - \tilde{x}_{i_k}(\hat{T} + \varepsilon_1)| + \mathbf{d}(\tilde{x}_{i_k}(\hat{T} + \varepsilon_1) \\ &= |\mathcal{X}(\hat{T} + \varepsilon_1) - \tilde{x}_{i_k}(\hat{T} + \varepsilon_1)| + \mathbf{d}(x_{i_k}(\hat{T} + \varepsilon_1)).\end{aligned}$$

Since $\hat{T}_{i_k} \le \hat{T} + \varepsilon_1$ as soon as k is large enough, $\mathbf{d}(x_{i_k}(\hat{T} + \varepsilon_1)) \le \hat{r}_{i_k}$ for such k. Taking the limit as $k \to +\infty$ one derives that $\mathbf{d}(\mathcal{X}(\hat{T} + \varepsilon_1)) = 0$ for every $\varepsilon_1 > 0$. Therefore, $T_{\mathcal{X}} \le \hat{T}$ and the proof is concluded. □

If the target is a singleton, the definition of weak Euler solution $\mathcal{X}$ does not depend at all on the sequence (r_i) and can be simplified as follows.

Proposition 3. *Assume that the target $\mathcal{C}$ is reduced to a point. Let $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \to U$ be a locally bounded feedback and let $z \in \mathbb{R}^n \setminus \mathcal{C}$ be given. Then a function $\mathcal{X}$ is a weak Euler solution of (5) from z if and only if there exists a sequence (x_i, u_i) of admissible π_i-sampling processes of (5) from z such that diam$(\pi_i) \to 0$ as $i \to +\infty$, and verifying*

$$x_i \to \mathcal{X} \qquad \textit{pointwise in } [0, +\infty). \tag{41}$$

Proof. Let $\mathcal{C} = \{\bar{z}\}$. The proof consists in showing that, given a sequence (x_i, u_i) of admissible π_i-sampling processes of (5) from z such that $\delta_i := \text{diam}(\pi_i) \to 0$ as $i \to +\infty$, and any sequence $(r_i) \subset (0, \mathbf{d}(z))$ converging to zero, there exists the limit

$$\mathcal{X}(t) := \lim_{i\to+\infty} x_i(t) \qquad \forall t \in [0, +\infty),$$

if and only if there exists the limit

$$\mathcal{X}_1(t) := \lim_{i\to+\infty} \tilde{x}_i(t) \qquad \forall t \in [0,+\infty),$$

where $\tilde{x}_i(t) := x_i(t \wedge T_i)$ and T_i is as in (30). Moreover, $\mathcal{X} \equiv \mathcal{X}_1$.

If $T_i = +\infty$, one has $\tilde{x}_i = x_i$ trivially. For each $i \in \mathbb{N}$ with $T_i < +\infty$, by definition, $\tilde{x}_i(t) = x_i(t)$ for all $t \in [0, T_i]$, $\tilde{x}_i(t) = x_i(T_i)$ for all $t \geq T_i$, and $\mathbf{d}(x_i(T_i)) = |x_i(T_i) - \bar{z}| = r_i$. Moreover, $\mathbf{d}(x_i(t)) = |x_i(t) - \bar{z}| \leq r_i$ for all $t \geq T_i$. Then, for every $t > T_i$, one has

$$|x_i(t) - \tilde{x}_i(t)| = |x_i(t) - x_i(T_i)| \leq |x_i(t) - \bar{z}| + |\bar{z} - x_i(T_i)| \leq 2r_i.$$

Therefore, for every $t \geq 0$, $(x_i(t))$ converges if and only if $(\tilde{x}_i(t))$ converges and $\lim_{i\to+\infty} \tilde{x}_i(t) = \lim_{i\to+\infty} x_i(t)$. $\square$

Definition 3.3 (Euler and weak Euler stabilizability with regulated cost)**.** A locally bounded feedback $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \to U$ is said to Euler [resp., weak Euler] stabilize $\dot{x} = \mathbf{f}(x,u)$ to $\mathcal{C}$ if there is a function $\beta \in \mathcal{KL}$ such that, for each $z \in R^n \setminus \mathcal{C}$, every Euler [resp., weak Euler] trajectory $\mathcal{X}$ of (5) from z verifies

$$\mathbf{d}(\mathcal{X}(t)) \leq \beta(\mathbf{d}(z), t) \qquad \forall t \in [0,+\infty). \tag{42}$$

If moreover there exist $p_0 > 0$ and a continuous map $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0,+\infty)$ which is positive definite and proper in $\mathbb{R}^n \setminus \mathcal{C}$, such that, for any $\mathcal{X}$ as above, every Euler [resp. weak Euler] cost $\mathcal{X}^0$ associated to $\mathcal{X}$ verifies

$$\mathcal{X}^0(t) \leq \frac{W(z)}{p_0} \qquad \forall t \in [0, T_{\mathcal{X}}), \tag{43}$$

where $T_{\mathcal{X}}$ is as in (33), we call (5)–(6) *Euler [resp., weak Euler] stabilizable to $\mathcal{C}$ with (p_0, W)-regulated cost.* When $\dot{x} = \mathbf{f}(x,u)$ is merely Euler [resp., weak Euler] stabilizable to $\mathcal{C}$, we also say that (5)–(6) is *Euler [resp., weak Euler] stabilizable to $\mathcal{C}$ with (p_0, W)-regulated cost for $p_0 = 0$.*

Remark 2. Since any Euler cost-solution pair for (5)–(6) is a weak Euler cost-solution pair, the weak Euler stabilizability with regulated cost implies the Euler stabilizability with regulated cost.

3.2. **Sample and weak Euler stabilizability with regulated cost.**

Sample stabilizability to $\mathcal{C}$ with (p_0, W)-regulated cost implies Euler and weak Euler stabilizability to $\mathcal{C}$ with (p_0, W)-regulated cost.

Theorem 3.4. *Assume that* $\mathbf{f}$, $\mathbf{l}$ *verify assumption* **(H.1)***. If a locally bounded feedback* $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \to U$ *sample stabilizes* (5)–(6) *to* $\mathcal{C}$ *with* (p_0, W)*-regulated cost for some* $p_0 \geq 0$*, then* $\mathbf{K}$ *Euler and weak Euler stabilizes* (5)–(6) *to* $\mathcal{C}$ *with* (p_0, W)*-regulated cost and with the same descent rate.*

Proof. In view of Remark 2, it is sufficient to show that the sample stabilizability with regulated cost implies the weak Euler stabilizability with regulated cost.

Preliminarily, let us reformulate the notion of sample stabilizability with regulated cost. Let β, $\delta = \delta(r, R)$ for $0 < r < R$, be as in Definition 2.3. Since we can assume without loss of generality that

$$r \mapsto \delta(r, R)$$

is a continuous, strictly increasing map, that verifies $\lim_{r\to 0^+}\delta(r,R) = 0$ and $\delta(R) := \lim_{r\to R^-}\delta(r,R) < +\infty$, we can define the inverse map

$$\delta \mapsto r(\delta) \qquad \forall \delta \in [0, \delta(R)]. \tag{44}$$

This function is continuous, strictly increasing and verifies $r(0) = 0$. As a consequence, by the sample stabilizability to $\mathcal{C}$ of (5)-(6) with (p_0, W)-regulated cost, for each $\delta \in (0, \delta(R))$, for every partition π with $\text{diam}(\pi) = \delta$, and for every π-sampling process (x^0, x, u) with $x(0) = z$, $\mathbf{d}(z) = R$, one has x defined in $[0, +\infty)$ and verifying:

$$\mathbf{d}(x(t)) \leq \max\{\beta(\mathbf{d}(z), t), r(\delta)\} \qquad \forall t \geq 0, \tag{45}$$

and, if $p_0 > 0$,

$$x^0(\bar{T}_x^{r(\delta)}) = \int_0^{\bar{T}_x^{r(\delta)}} l(x(\tau), u(\tau))\, d\tau \leq \frac{W(z)}{p_0}, \tag{46}$$

where $\bar{T}_x^{r(\delta)} = \inf\{t > 0 : \mathbf{d}(x(\tau)) \leq r(\delta) \ \ \forall \tau \geq t\}$, as in (11).

Given $z \in \mathbb{R}^n \setminus \mathcal{C}$, let $(\mathcal{X}^0, \mathcal{X})$ be an arbitrary weak Euler cost-solution pair associated to the sampling stabilizing feedback $\mathbf{K}$ and with initial condition $\mathcal{X}(0) = z$. By definition, there are a sequence of partitions (π_i) such that $\delta_i := \text{diam}(\pi_i) \to 0$ as $i \to +\infty$, a sequence of admissible π_i-sampling processes (x_i^0, x_i, u_i) for (5)-(6) with $x_i(0) = z$ for each i, and a vanishing sequence $(r_i) \subset (0, \mathbf{d}(z))$, such that

$$\lim_{i\to+\infty} (\tilde{x}_i^0, \tilde{x}_i)(t) = (\mathcal{X}^0, \mathcal{X})(t) \quad \forall t \in [0, +\infty),$$

where $(\tilde{x}_i^0, \tilde{x}_i)$ is the sequence of truncated functions introduced in Definition 3.2, namely,

$$(\tilde{x}_i^0, \tilde{x}_i)(t) = (x_i^0, x_i)(t \wedge \bar{T}_{x_i}^{r_i}), \quad \bar{T}_{x_i}^{r_i} = \inf\{t > 0 : \mathbf{d}(x_i(\tau)) \leq r_i \ \ \forall \tau \geq t\}.$$

Since $\delta_i \to 0$, we can assume without loss of generality that $\delta_i < \delta(\mathbf{d}(z))$ for all i. Hence, by the previous step it follows that, for every i,

$$\mathbf{d}(x_i(t)) \leq \max\{\beta(\mathbf{d}(z), t), r(\delta_i)\} \qquad \forall t \geq 0 \tag{47}$$

and, if $p_0 > 0$,

$$x_i^0(t) \leq \frac{W(z)}{p_0} \quad \forall t \in [0, \bar{T}_{x_i}^{r(\delta_i)}]. \tag{48}$$

Since $\tilde{x}_i(t) = x_i(t)$ for all $t \in [0, \bar{T}_{x_i}^{r_i}]$ and $\mathbf{d}(\tilde{x}_i(t)) = r_i$ for all $t \geq \bar{T}_{x_i}^{r_i}$, (47) implies

$$\mathbf{d}(\tilde{x}_i(t)) \leq \max\{\beta(\mathbf{d}(z), t), r(\delta_i), r_i\} \quad \forall t \geq 0,$$

and, passing to the limit as $i \to +\infty$, we get

$$\mathbf{d}(\mathcal{X}(t)) \leq \beta(\mathbf{d}(z), t) \qquad \forall t \in [0, +\infty), \tag{49}$$

because r_i and $r(\delta_i)$ tend to 0. By the arbitrariness of the Euler solution $\mathcal{X}$, this proves that the feedback $\mathbf{K}$ weak Euler stabilizes (5) to $\mathcal{C}$, with the same descent rate β of the sample stabilizability.

Suppose $p_0 > 0$. To conclude the proof, it remains to show that

$$\lim_{t\to T_{\mathcal{X}}^-} \mathcal{X}^0(t) \leq \frac{W(z)}{p_0}, \tag{50}$$

where $T_{\mathcal{X}} = \inf\{t \geq 0 : \ \lim_{\tau\to t^-} \mathbf{d}(\mathcal{X}(\tau)) = 0\} > 0$ by Proposition 1. To this aim, for each i, let us define $\hat{r}_i := \max\{r_i, r(\delta_i)\}$, so that $\hat{r}_i \geq r_i$ and $\lim_{i\to+\infty} \hat{r}_i = 0$. In view of Lemma 2, (ii) we have that

$$T_{\mathcal{X}} \leq \hat{T} := \liminf_{i\to+\infty} \hat{T}_i, \quad \hat{T}_i := \inf\{t > 0 : \mathbf{d}(x_i(\tau)) \leq \hat{r}_i \ \forall \tau \geq t\}. \tag{51}$$

Fix $t \in [0, T_{\mathcal{X}})$. By (51), $\hat{T}_i > t$ for all i sufficiently large. Moreover, one has $\hat{T}_i \leq \bar{T}_{x_i}^{\delta(r_i)}$, since $\hat{r}_i \geq r(\delta_i)$. This together with (48) implies that

$$\mathcal{X}^0(t) = \lim_{i\to+\infty} x_i^0(t) \leq \frac{W(z)}{p_0} \quad \forall t \in [0, T_{\mathcal{X}}). \tag{52}$$

Taking the limit as $t \to T_{\mathcal{X}}^-$ we get finally (50). □

The weak coercivity hypothesis **(HC)** below, is sufficient to guarantee that, when (5)–(6) is sample stabilizable to $\mathcal{C}$ with (p_0, W)-regulated cost, the set of the corresponding weak Euler cost-solution pairs is nonempty.

(HC) *For some $\bar{R} > 0$, there exist $C_1 \geq 0$ and $C_2 > 0$ such that $\mathbf{f}$, $\mathbf{l}$ verify*

$$\mathbf{l}(x,u) \geq C_2|\mathbf{f}(x,u)| - C_1 \quad \forall (x,u) \in (B_{\bar{R}}(\mathcal{C}) \setminus \mathcal{C}) \times U.$$

Notice that control-polynomial dynamics and running costs of the form

$$\mathbf{f}(x,u) := f_0(x) + \textstyle\sum_{i=1}^d \left(\sum_{\alpha\in\mathbb{N}^M,\, \alpha_1+\cdots+\alpha_M=i} u_1^{\alpha_1}\cdots u_M^{\alpha_M} f_{\alpha_1,\dots,\alpha_M}(x)\right),$$
$$\mathbf{l}(x,u) \geq l_0(x) + l_1(x)|u| + \cdots + l_{\bar{d}}(x)|u|^{\bar{d}},$$

where the maps f_0, $f_{\alpha_1,\dots,\alpha_M}$, l_i are continuous in $\mathbb{R}^n$ and $l_i \geq 0$, verify hypothesis **(HC)** as soon as $\bar{d} \geq d$ and $l_{\bar{d}}(x) \geq C > 0$ in $B_{\bar{R}}(\mathcal{C}) \setminus \mathcal{C}$, for some $C > 0$.

Proposition 4. *Let $\mathbf{f}$, $\mathbf{l}$ verify hypotheses* **(H.1)**, **(HC)**, *and let $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \to U$ be a locally bounded feedback that sample stabilizes (5)-(6) to $\mathcal{C}$ with (p_0, W)-regulated cost, for some $p_0 > 0$. Then for any $z \in \mathbb{R}^n \setminus \mathcal{C}$ there exists at least one weak Euler cost-solution pair $(\mathcal{X}^0, \mathcal{X})$ of (5)-(6) from z associated to the feedback $\mathbf{K}$.*

Proof. As already observed in the proof of Theorem 3.4, by the sample stabilizability to $\mathcal{C}$ of (5)-(6) with (p_0, W)-regulated cost, it follows that there exist a $\mathcal{KL}$-function β and, for every $R > 0$, a continuous, strictly increasing map $\delta \mapsto r(\delta) \in [0, R]$ for all $\delta \in [0, \delta(R)]$ with $r(0) = 0$ (see (44)), such that, for each $\delta \in (0, \delta(R))$, for every partition π with $\mathrm{diam}(\pi) = \delta$, and for every π-sampling process (x^0, x, u) with $x(0) = z$, $\mathbf{d}(z) = R$, x is defined in $[0, +\infty)$ and (x^0, x, u) verifies (45), (46). In particular, given $z \in \mathbb{R}^n \setminus \mathcal{C}$, for any sequence (δ_i) converging to 0 small enough, for each i, there exists at least one admissible π_i-sampling process (x_i^0, x_i, u_i) with $x_i(0) = z$, $\mathrm{diam}(\pi_i) = \delta_i$, x defined on $[0, +\infty)$, and it verifies:

$$\mathbf{d}(x_i(t)) \leq \max\{\beta(\mathbf{d}(z), t), r(\delta_i)\} \qquad \forall t \geq 0, \tag{53}$$

$$x_i^0(\bar{T}_{x_i}^{r(\delta_i)}) = \int_0^{\bar{T}_{x_i}^{r(\delta_i)}} l(x_i(t), u_i(t))\,dt \leq \frac{W(z)}{p_0}. \tag{54}$$

By Proposition 1, given an arbitrary $\varepsilon \in (0, \mathbf{d}(z))$, for each i, one has

$$\bar{T}_{x_i}^{r(\delta_i)} \geq T^{\varepsilon} > 0, \tag{55}$$

where T^ε is as in (34). Let us set $r_i := r(\delta_i)$ and $T_i := \bar{T}_{x_i}^{r(\delta_i)}$. Taking a subsequence if necessary, we can assume that there exists

$$\bar{T} := \lim_{i\to+\infty} T_i \le +\infty,$$

where $\bar{T} \ge T^\varepsilon > 0$ by (55). We set

$$(\tilde{x}_i^0, \tilde{x}_i)(t) := (x_i^0, x_i)(t \wedge T_i) \qquad \forall t \ge 0. \tag{56}$$

Assume first $\mathbf{d}(z) \le \bar{R}_1$, where $\beta(\bar{R}_1, 0) = \bar{R}$ and $\bar{R}$ is as in hypothesis **(HC)**. Hence, $\bar{R}_1 \le \bar{R}$ by the properties of β, and (53), (54), and **(HC)** imply that, for every $i \in \mathbb{N}$, one has

$$\mathbf{d}(x_i(t)) \le \bar{R} \qquad \forall t \ge 0, \tag{57}$$

and

$$\tilde{x}_i^0(t) = \int_0^t |\dot{\tilde{x}}_i^0(\tau)|\, d\tau = \int_0^{t\wedge T_i} l(x_i(\tau), u_i(\tau))\, d\tau \le \frac{W(z)}{p_0} \qquad \forall t \ge 0,$$

$$\int_0^t |\dot{\tilde{x}}_i(\tau)|\, dt = \int_0^{t\wedge T_i} |f(x_i(\tau), u_i(\tau))|\, d\tau \le \frac{W(z)}{C_2\, p_0} + \frac{C_1}{C_2}\,(t \wedge T_i) \qquad \forall t \ge 0.$$

Moreover, by (57) and the compactness of $\partial\mathcal{C}$, it follows that there is some $M > 0$ such that $|\tilde{x}_i(t)| \le M$ for all $t \ge 0$ and for every i. Hence the sequence $(\tilde{x}_i^0, \tilde{x}_i)$ is equibounded and has equibounded total variation on $[0, t]$ for every $t > 0$, so that Helly's Selection Theorem (see [5, Theorem 15.1]) implies that there exist a subsequence, which we still denote $(\tilde{x}_i^0, \tilde{x}_i)$, and a bounded map $(\mathcal{X}^0, \mathcal{X}) : [0, +\infty) \to [0, +\infty) \times \mathbb{R}^n$ with locally bounded total variation, such that

$$\lim_{i\to+\infty} (\tilde{x}_i^0, \tilde{x}_i)(t) = (\mathcal{X}^0, \mathcal{X})(t) \qquad \forall t \ge 0. \tag{58}$$

In view of Definition 3.2, $(\mathcal{X}^0, \mathcal{X})$ is weak Euler cost-solution pair.

If instead $\mathbf{d}(z) > \bar{R}_1$, for every $i \in \mathbb{N}$, we set

$$\hat{T}_i^{\bar{R}_1} := \inf\{t \ge 0 :\ \mathbf{d}(x_i(t)) \le \bar{R}_1\}, \quad \hat{T}^{\bar{R}_1} := \inf\{t \ge 0 :\ \beta(\mathbf{d}(z), t) \le \bar{R}_1\},$$

where $0 < \hat{T}_i^{\bar{R}_1} \le \hat{T}^{\bar{R}_1}$. Since $r(\delta_i) \to 0$ as $i \to +\infty$, one has $\hat{T}_i^{\bar{R}_1} \le T_i$ for all i large enough. For such i, $\mathbf{d}(\tilde{x}_i(t)) = \mathbf{d}(x_i(t)) \ge \bar{R}_1$ for all $t \le \hat{T}_i^{\bar{R}_1}$, and

$$\mathbf{d}(\tilde{x}_i(t)) = \mathbf{d}(x_i(t)) \le \beta(\bar{R}_1, t - \hat{T}_i^{\bar{R}_1}) \le \beta(\bar{R}_1, 0) = \bar{R} \qquad \forall t \in [\hat{T}_i^{\bar{R}_1}, T_i],$$

by the definition of $\bar{R}_1$ and the properties of β. Hence, **(HC)** yields that

$$\begin{aligned}\int_0^t |\dot{\tilde{x}}_i(t)|\, dt &= \int_0^{t\wedge\hat{T}_i^{\bar{R}_1}} |f(x_i(t), u_i(t))|\, dt + \int_{t\wedge\hat{T}_i^{\bar{R}_1}}^{t\wedge T_i} |f(x_i(t), u_i(t))|\, dt \\ &\le \bar{M}\, \hat{T}^{\bar{R}_1} + \frac{W(z)}{C_2\, p_0} + \frac{C_1}{C_2}\,(t \wedge T_i) \qquad \forall t \ge 0,\end{aligned}$$

where $\bar{M} := \sup\{|\mathbf{f}(x, u)| \mid \bar{R}_1 \le \mathbf{d}(x) \le \beta(\mathbf{d}(z), 0),\ u \in \mathbf{K}(x)\}$. From now on, the proof proceeds as in the case $\mathbf{d}(z) \le \bar{R}_1$ and we omit it. □

Remark 3. From the previous proof we can deduce that the statement of Proposition 4 remains valid even if **(HC)** is replaced by any condition that implies the equiboundedness of the total variation of the sequence $(\tilde{x}_i)$ of the stabilizing sampling trajectories, on any interval $[0, t]$, $\ t > 0$.

4. Sufficient stabilizability conditions in optimal control.

In this section we provide sufficient conditions for the sample, Euler, and weak Euler stabilizability with regulated cost of the original problem (14)-(15). Such conditions rely on the existence of a p_0-*Minimum Restraint function*.

4.1. Main results.

Given arbitrary functions $\mathbf{f}$, $\mathbf{l}$ verifying **(H.1)**, we introduce the *Hamiltonian* $H_{\mathbf{f},\mathbf{l}} : (\mathbb{R}^n \setminus \mathcal{C}) \times \mathbb{R} \times \mathbb{R}^n \to [-\infty, +\infty)$, given by

$$H(x, p_0, p) := \inf_{u \in U} \left\{ \langle p \, , \, \mathbf{f}(x, u) \rangle + p_0 \, \mathbf{l}(x, u) \right\}. \tag{59}$$

Notice that, because of the unboundedness of the data, H may be discontinuous and also equal to $-\infty$ at some points. Following [16, 17], we define a p_0-*Minimum Restraint function* as follows.

Definition 4.1 (p_0-Minimum Restraint Function)**.** Let $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$ be a continuous function, and let us assume that W is locally semiconcave, positive definite, and proper on $\mathbb{R}^n \setminus \mathcal{C}$. We say that W is a p_0-*Minimum Restraint function – in short,* p_0-*MRF – for some* $p_0 \geq 0$ for $\mathbf{f}$, $\mathbf{l}$, if there exists some continuous, strictly increasing function $\gamma : (0, +\infty) \to (0, +\infty)$, that we call a *decrease rate*, verifying the following *decrease condition*:

$$H_{\mathbf{f},\mathbf{l}}(x, p_0, D^*W(x)) \leq -\gamma(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}.^{3} \tag{60}$$

Remark 4. Given $\mathbf{f}$, $\mathbf{l}$, a p_0-MRF W with $p_0 = 0$ is simply a Control Lyapunov function, in short CLF, for the control system $\dot{x} = \mathbf{f}(x, u)$. If $p_0 > 0$, W is still a CLF, since $\mathbf{l} \geq 0$, but condition (60) now includes, for instance, also Petrov-type controllability conditions for the minimum time problem, where $\mathbf{l} = 1$. However, on the one hand, unlike the existence of a CLF, the existence of a p_0-MRF gives cost information. On the other hand, since $\mathbf{l}$ may be zero on an arbitrary set, it is not possible to reformulate the present problem as a minimum time problem for the rescaled dynamics $\mathbf{f}/\mathbf{l}$. For more details on the notion of p_0-MRF and examples we refer to [19, 18, 16].

The existence of a p_0-MRF $\bar{W}$ for the rescaled data $\bar{f}$, $\bar{l}$, guarantees the sample stabilizability to $\mathcal{C}$ with $(p_0, \bar{W})$-regulated cost of the original problem:

Theorem 4.2. *Given f, l verifying* **(H.1-2)***, let $\bar{W}$ be a p_0-MRF with $p_0 \geq 0$ for the rescaled functions $\bar{f}$, $\bar{l}$. Then there exists a locally bounded feedback $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ that sample stabilizes the original problem* (14)–(15) *to $\mathcal{C}$ with $(p_0, \bar{W})$-regulated cost.*

Proof. In view of hypotheses **(H.1-2)**, the rescaled functions $\bar{f}$, $\bar{l}$, satisfy the regularity and boundedness assumptions that make [16, Theorem 1.1] applicable. Hence there exists a locally bounded feedback strategy $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ that sample stabilizes to $\mathcal{C}$ the rescaled problem (16)–(17) with $(p_0, \bar{W})$-regulated cost. The claim on the sample stabilizability to $\mathcal{C}$ with $(p_0, \bar{W})$-regulated cost of (14)–(15) now follows straightforwardly from the equivalence Theorem 2.5. $\square$

It is not difficult to show that any p_0-MRF for the rescaled problem is a p_0-MRF for the original problem. Instead, a p_0-MRF W for f, l may not be a p_0-MRF for $\bar{f}$, $\bar{l}$, but in view of Theorem 4.3 below we can always build an associated p_0-MRF $\bar{W} \geq W$ for the rescaled problem.

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[3]This means that $H_{\mathbf{f},\mathbf{l}}(x, p_0, p) \leq -\gamma(W(x))$ for every $p \in D^*W(x)$.

Preliminarily, let us show that a p_0-MRF for f, l provides a locally bounded feedback satisfying the decrease condition. This is a direct consequence of the following, more general result.

Proposition 5. *Assume that* $\mathbf{f}$, $\mathbf{l}$ *satisfy* **(H.1)**. *Let* $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$ *be a continuous function, which is locally Lipschitz continuous, proper and positive definite on* $\mathbb{R}^n \setminus \mathcal{C}$ *and verifies the decrease condition*

$$H_{\mathbf{f},\mathbf{l}}(x, p_0, \partial_L W(x)) < -\gamma(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C},$$

for some $p_0 \geq 0$ *and some continuous, strictly increasing function* $\gamma : (0, +\infty) \to (0, +\infty)$. *Then there exist a strictly increasing continuous map* $\tilde{\gamma} : (0, +\infty) \to (0, +\infty)$, $\tilde{\gamma} \leq \gamma$, *and a continuous function* $N : (0, +\infty) \to (0, +\infty)$ *such that*

$$\min_{U \cap B(0, N(W(x)))} \{\langle \partial_L W(x), \mathbf{f}(x, u)\rangle + p_0 \, \mathbf{l}(x, u)\} < -\tilde{\gamma}(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}. \quad (61)$$

Furthermore, for any selection $p(x) \in \partial_L W(x)$, $x \in \mathbb{R}^n \setminus \mathcal{C}$, *there exists a locally bounded feedback* $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \to U$ *verifying for all* $x \in \mathbb{R}^n \setminus \mathcal{C}$,

$$|\mathbf{K}(x)| \leq N(W(x))$$

and

$$\langle p(x), \mathbf{f}(x, \mathbf{K}(x))\rangle + p_0 \, \mathbf{l}(x, \mathbf{K}(x)) < -\tilde{\gamma}(W(x)). \quad (62)$$

The main results of this section rely on:

Theorem 4.3. *Assume that* f, l *satisfy* **(H.1-2)**. *Let* $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$ *be a continuous function, which is locally Lipschitz continuous, proper and positive definite on* $\mathbb{R}^n \setminus \mathcal{C}$ *and verifies for some* $p_0 \geq 0$ *the decrease condition*

$$H_{f,l}(x, p_0, \partial_L W(x)) \leq -\gamma(W(x)) \qquad \forall x \in \mathbb{R}^n \setminus \mathcal{C},$$

where $\gamma : (0, +\infty) \to (0, +\infty)$ *is a continuous, strictly increasing function.*

Then for any $R > 0$ *there exist a continuous function* $\bar{W}_R : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$ *and a continuous, strictly increasing function* $\gamma_R : (0, +\infty) \to (0, +\infty)$ *enjoying the following properties.*

(i) *The function* $\bar{W}_R : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$ *is locally Lipschitz continuous, proper and positive definite on* $\mathbb{R}^n \setminus \mathcal{C}$, $\bar{W}_R \geq W$, *and* $\bar{W}_R(x) = W(x)$ *for all* $x \in B_R(\mathcal{C}) \setminus \mathcal{C}$. *In addition, when* W *is locally semiconcave on* $\mathbb{R}^n \setminus \mathcal{C}$ *or locally Lipschitz continuous on* $\overline{\mathbb{R}^n \setminus \mathcal{C}}$, *so is* $\bar{W}_R$. *One has* $\gamma_R \leq \gamma$.

(ii) $\bar{W}_R$ *and* γ_R *verify the decrease condition*

$$H_{\bar{f},\bar{l}}(x, p_0, \partial_L \bar{W}_R(x)) \leq -\gamma_R(\bar{W}_R(x)) \qquad \forall x \in \mathbb{R}^n \setminus \mathcal{C}. \quad (63)$$

(iii) *Given a selection* $p(x) \in \partial_L W(x)$ *for any* $x \in \mathbb{R}^n \setminus \mathcal{C}$ *and a locally bounded feedback* $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ *as in Proposition 5, the (unique) selection* $\bar{p}(x) \in \partial_L \bar{W}_R(x)$ *associated to* $p(x)$ *verifies*

$$\langle \bar{p}(x), \bar{f}(x, K(x))\rangle + p_0 \, \bar{l}(x, K(x)) \leq -\gamma_R(\bar{W}_R(x)) \qquad \forall x \in \mathbb{R}^n \setminus \mathcal{C}. \quad (64)$$

As a consequence of Proposition 5 and Theorem 4.3, that will be proved in Subsection 4.2, the existence of a p_0-MRF W for the original problem still implies sample stabilizability to $\mathcal{C}$ with (p_0, W)-regulated cost. Precisely, we have:

Theorem 4.4. *Assume that* f, l *verify hypotheses* **(H.1-2)** *and let* W *be a* p_0*-MRF with* $p_0 \geq 0$ *for such* f, l. *Then there exists a locally bounded feedback* $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ *that sample, Euler and weak Euler stabilizes the original problem* (14)–(15) *to* $\mathcal{C}$ *with* (p_0, W)*-regulated cost.*

Proof. We only need to prove that, given W as above, (14)–(15) is sample stabilizable to $\mathcal{C}$ with (p_0, W)-regulated cost, because then the rest of the statement follows from Theorem 3.4.

To this end, fix an arbitrary $R_1 > 0$ and consider $\bar{W} := \bar{W}_{R_1}$, γ_{R_1} and a feedback K as in Theorem 4.3. In particular, $\bar{W}$ is locally semiconcave as W, so that for every $x \in \mathbb{R}^n \setminus \mathcal{C}$, the limiting subdifferential $\partial_L \bar{W}(x)$ coincides with the set of reachable gradients $D^*\bar{W}(x)$ at x. Therefore, $\bar{W}$ is a p_0-MRF for the rescaled problem (16)–(17), with dynamics $\bar{f}$ and lagrangian $\bar{l}$, and by Theorem 4.2 it follows that K is a locally bounded feedback which sample stabilizes (14)–(15) to $\mathcal{C}$, with $(p_0, \bar{W})$-regulated cost. If $p_0 = 0$, this concludes the proof. Otherwise, observe that until now we have shown that there exists some function $\beta \in \mathcal{KL}$ such that, given $0 < r < R$, there is some $\delta = \delta(r, R) > 0$ such that for any $z \in \mathbb{R}^n \setminus \mathcal{C}$ with $\mathbf{d}(z) \leq R$, any π-sampling process (x^0, x, u) with $\mathrm{diam}(\pi) \leq \delta$ and $x(0) = z$ verifies

$$\mathbf{d}(x(t)) \leq \max\{\beta(\mathbf{d}(z), t), r\} \qquad \forall t > 0, \tag{65}$$

and $x^0(t) \leq \bar{W}(z)/p_0$ for all $t \in [0, \bar{T}_x^r]$ ($\bar{T}_x^r$ as in (11)). Since $\bar{W}$ is in general larger than W, it remains to show that we have in fact

$$x^0(t) \leq \frac{W(z)}{p_0} \qquad \forall t \in [0, \bar{T}_x^r]. \tag{66}$$

By Theorem 4.3, there is a map $\bar{W}_{2R}$ which is a p_0-MRF for $\bar{f}$, $\bar{l}$ by the previous arguments, and verifies $\bar{W}_{2R} \equiv W$ on $B_{2R}(\mathcal{C})$. Hence there exist some ($\beta_{2R} \in \mathcal{KL}$ and) $\delta_{2R} = \delta_{2R}(r, R) > 0$ such that all π-sampling process (x^0, x, u) with $\mathrm{diam}(\pi) \leq \tilde{\delta}(r, R) := \delta_{2R}(r, R) \wedge \delta(r, R)$ and $x(0) = z$ verify in particular (65), but also have $x^0(t) \leq \bar{W}_{2R}(z)/p_0$ for all $t \in [0, \bar{T}_x^r]$. The last inequality yields (66), because $\bar{W}_{2R}(z) = W(z)$ for every $z \in \mathbb{R}^n \setminus \mathcal{C}$ with $\mathbf{d}(z) \leq R$. ☐

Whenever the rescaled functions $\bar{f}(\cdot, u)$, $\bar{l}(\cdot, u)$ are locally Lipschitz continuous in $\overline{\mathbb{R}^n \setminus \mathcal{C}}$ uniformly w.r.t. u, sample stabilizability can be achieved under milder regularity assumptions on the p_0-MRFs. In particular, the semiconcavity requirement in the definition of a p_0-MRF can be replaced by local Lipschitz continuity.

Definition 4.5 (Lipschitz continuous p_0-Minimum Restraint Function)**.** We call *Lipschitz continuous p_0-Minimum Restraint Function,* $p_0 \geq 0$*,* for $\mathbf{f}$, $\mathbf{l}$ satisfying hypothesis **(H.1)**, any function $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$ which is locally Lipschitz continuous on $\overline{\mathbb{R}^n \setminus \mathcal{C}}$, positive definite, and proper on $\mathbb{R}^n \setminus \mathcal{C}$, and verifies the decrease condition

$$H(x, p_0, \partial_L W(x)) < -\gamma(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C},$$

for some continuous, strictly increasing function $\gamma : (0, +\infty) \to (0, +\infty)$.

We consider the following strengthened version of hypotheses **(H.1-2)**:

(HL) *The data* $\mathbf{f}$*,* $\mathbf{l}$ *satisfy* **(H.1-2)***. Moreover, the rescaled functions* $\bar{\mathbf{f}}, \bar{\mathbf{l}}$ *can be continuously extended to* $\partial\mathcal{C} \times U$ *and for every compact set* $\mathcal{K} \subset \overline{\mathbb{R}^n \setminus \mathcal{C}}$ *there exists* $L > 0$ *such that*

$$|\bar{\mathbf{f}}(x_1, u) - \bar{\mathbf{f}}(x_2, u)| + |\bar{\mathbf{l}}(x_1, u) - \bar{\mathbf{l}}(x_2, u)| \leq L|x_1 - x_2| \quad \forall (x_1, u), (x_2, u) \in \mathcal{K} \times U. \tag{67}$$

In this setting, the existence of a Lipschitz continuous p_0-MRF W for the rescaled problem or for the original problem, still guarantees sample stabilizability to $\mathcal{C}$ with $(p_0/2, W)$-regulated cost.

Theorem 4.6. *Assume that f, l satisfy* **(HL)** *and let W be a Lipschitz continuous p_0-MRF with $p_0 \geq 0$, either for the rescaled data $\bar{f}$, $\bar{l}$, or for f, l. Then there exists a locally bounded feedback $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ that sample, Euler, and weak Euler stabilizes the original problem (14)–(15) to $\mathcal{C}$ with $(p_0/2, W)$-regulated cost.*

Proof. Suppose first that W is a Lipschitz continuous p_0-MRF for $\bar{f}$, $\bar{l}$. In view of hypothesis **(HL)**, the rescaled problem satisfies the assumptions of [16, Theorem 4.3]. This implies the existence of a (locally semiconcave) $\frac{p_0}{2}$-MRF $W_1 \leq W$ for $\bar{f}$, $\bar{l}$, which by [16, Theorem 1.1] yields the existence of a locally bounded feedback $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ that sample stabilizes to $\mathcal{C}$ the rescaled problem (16)–(17) with $(p_0/2, W)$-regulated cost. Therefore, K sample stabilizes to $\mathcal{C}$ with $(p_0/2, W)$-regulated cost also the original problem (14)–(15), in view of Theorem 2.5.

If instead W is a Lipschitz continuous p_0-MRF for f, l, let us fix $R_1 > 0$. By Theorem 4.3 there exists a Lipschitz continuous p_0-MRF $\bar{W}_{R_1} \geq W$ for $\bar{f}$, $\bar{l}$, and it verifies $\bar{W}_{R_1} \equiv W$ on $B_{R_1}(\mathcal{C})$. Then the existence of a locally bounded feedback $K : \mathbb{R}^n \setminus \mathcal{C} \to U$ that sample stabilizes to $\mathcal{C}$ the original problem (14)–(15) with $(p_0/2, \bar{W}_{R_1})$-regulated cost, can obtained as in the previous case. The fact that the cost is actually $(p_0/2, W)$-regulated, can be proven arguing as in the last part of the proof of Theorem 4.4.
In both cases, the Euler and weak Euler stabilizability with the same regulated cost then follows by Theorem 3.4. □

In the case of control-affine data, the previous results extend the sufficient conditions for sample stabilizability with regulated cost introduced in [17], which require the existence of a MRF for the *rescaled* problem.

4.2. **Proofs of Proposition 5 and of Theorem 4.3.**

Proof of Proposition 5. Let $\{\mu_k\}_{k\in\mathbb{Z}} \subset (0, +\infty)$ be a bi-infinite, strictly increasing sequence such that $\mu_k \to 0$ as $k \to -\infty$ and $\mu_k \to +\infty$ as $k \to +\infty$. Since γ is strictly increasing, by (60) one has for all $k \in \mathbb{Z}$

$$H(x, p_0, \partial_L W(x)) < -\gamma(W(x)) \leq -\gamma(\mu_k) \quad \forall x \in W^{-1}([\mu_k, +\infty)).$$

In particular, for all $\bar{x} \in W^{-1}([\mu_k, +\infty))$ and $\bar{p} \in \partial_L W(\bar{x})$, there exists $\bar{u} \in U$ such that

$$\langle \mathbf{f}(\bar{x}, \bar{u}), \bar{p}\rangle + p_0\, \mathbf{l}(\bar{x}, \bar{u}) < -\gamma(\mu_k). \tag{68}$$

Fix $k \in \mathbb{Z}$ and define

$$\Gamma_k := \{(x, p) \mid x \in W^{-1}([\mu_k, \mu_{k+1}]),\ p \in \partial_L W(x)\}.$$

Notice that the properties of W –in particular, the properness of W and the upper semicontinuity of the set-valued map $x \rightsquigarrow \partial_L W(x)$– imply that Γ_k is a compact set. Then the map $h_k : [0, +\infty) \to \mathbb{R}$ given by

$$h_k(N) := \max_{(x,p)\in\Gamma_k} \min_{U\cap B(0,N)} \{\langle \mathbf{f}(x, u), p\rangle + p_0\, \mathbf{l}(x, u)\}$$

is well defined.

Step 1. Given any $k \in \mathbb{Z}$, we show that there exists a sufficiently large N_k satisfying

$$h_k(N) < -\gamma(\mu_k) \quad \forall N \geq N_k. \tag{69}$$

Indeed, let $\{N^j\}$ be a positive, strictly increasing, diverging sequence of real numbers. Consider a sequence $\{(x^j, p^j)\} \subset \Gamma_k$ such that

$$(x^j, p^j) \in \underset{(x,p)\in\Gamma_k}{\operatorname{argmax}} \left\{ \min_{U\cap B(0,N^j)} \{\langle \mathbf{f}(x,u), p\rangle + p_0\,\mathbf{l}(x,u)\} \right\} \quad \forall j \in \mathbb{N},$$

so that

$$h_k(N^j) = \min_{U\cap B(0,N^j)} \{\langle \mathbf{f}(x^j,u), p^j\rangle + p_0\,\mathbf{l}(x^j,u)\}.$$

Since Γ_k is compact, then, by passing to a subsequence if necessary, (x^j, p^j) converges to some $(\bar{x}, \bar{p}) \in \Gamma_k$ as $j \to \infty$. Choose $\bar{u}$ like in (68). By the continuity of $\mathbf{f}$ and $\mathbf{l}$ there exists a sufficiently large J such that $N^J > |\bar{u}|$ and

$$\langle \mathbf{f}(x^J, \bar{u}), p^J\rangle + p_0\,\mathbf{l}(x^J, \bar{u}) < -\gamma(\mu_k).$$

Since by construction h_k is decreasing, then for all $N \geq N^J$

$$\begin{aligned} h_k(N) \leq h_k(N^J) &= \min_{U\cap B(0,N^J)} \{\langle \mathbf{f}(x^J,u), p^J\rangle + p_0\,\mathbf{l}(x^J,u)\} \\ &\leq \langle \mathbf{f}(x^J,\bar{u}), p^J\rangle + p_0\,\mathbf{l}(x^J,\bar{u}) < -\gamma(\mu_k). \end{aligned}$$

Therefore, setting $N_k := N^J$ we have (69).

Step 2. Let $\tilde{\gamma} : (0, +\infty) \to (0, +\infty)$ be a strictly increasing, continuous map such that, for every $k \in \mathbb{Z}$,

$$\tilde{\gamma}(\mu) \leq \gamma(\mu_k) \qquad \forall \mu \in [\mu_k, \mu_{k+1}]$$

(for instance, $\tilde{\gamma}$ can be obtained by the linear interpolation of the point set $\{(\mu_{k+1}, \gamma(\mu_k))\}_{k\in\mathbb{Z}}$). Let $N : (0, +\infty) \to (0, +\infty)$ be a continuous approximation from above of the piecewise constant function $\bar{N}(\mu) := N_k$ for all $\mu \in [\mu_k, \mu_{k+1})$, $k \in \mathbb{Z}$. With this choice of N and $\tilde{\gamma}$, by (69) it follows that relation (61) is verified.

Step 3. Fixed a selection $p(x) \in \partial_L W(x)$ for every $x \in \mathbb{R}^n \setminus \mathcal{C}$, consider a function $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \to U$ such that

$$\mathbf{K}(x) \in \underset{U\cap B(0,N(W(x)))}{\operatorname{argmin}} \{\langle \mathbf{f}(x,u), p(x)\rangle + p_0\,\mathbf{l}(x,u)\} \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}.$$

Then $|\mathbf{K}(x)| \leq N(W(x))$ and by (61) one has

$$\langle \mathbf{f}(x, \mathbf{K}(x)), p(x)\rangle + p_0\,\mathbf{l}(x, \mathbf{K}(x)) < -\tilde{\gamma}(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}.$$

Furthermore, for any compact set $\mathcal{K} \subset \mathbb{R}^n \setminus \mathcal{C}$, let $\mu_{min} := \min_{x\in\mathcal{K}} W(x)$ and $\mu_{max} := \max_{x\in\mathcal{K}} W(x)$. Set $N_{\mathcal{K}} := \max_{\mu\in[\mu_{min},\mu_{max}]} N(\mu)$. Therefore $\mathbf{K}(x) \in U \cap B(0, N_{\mathcal{K}})$ for all $x \in \mathcal{K}$, so proving that $\mathbf{K}$ is a locally bounded feedback. □

Remark 5. Given any $\sigma > 0$, one can assume without loss of generality the function N in Proposition 5 decreasing in $(0, \sigma]$. It suffices, for instance, to replace N with a continuous approximation from above of the map

$$\tilde{N}_\sigma(\mu) := \begin{cases} \max_{r\in[\mu,2\sigma]} N(r) & \mu \in (0, 2\sigma], \\ N(\mu) & \mu > 2\sigma, \end{cases}$$

which is clearly decreasing on $(0, \sigma]$.

Proof of Theorem 4.3. Let us prove (i). Given W as in the statement of Theorem 4.3, fix $R>0$ and let $\sigma=\sigma(R):=\inf\{\sigma>0:\ \ \{z:\ W(z)\le\sigma\}\supseteq B_R(\mathcal{C})\}$, so that when $\mathbf{d}(x)\le R$ one has $W(x)\le\sigma$. Fix an arbitrary $\bar\sigma>2\sigma$. Let $\tilde\gamma$, N and K be as in Proposition 5 for f and l, and let us assume N decreasing on $(0,\bar\sigma]$, as it is possible thanks to Remark 5. Let $\nu_0:(0,+\infty)\to[0,+\infty)$ be given by

$$\nu_0(\mu):=\begin{cases}\displaystyle\max_{(x,u)\in W^{-1}([\mu,\bar\sigma])\times(U\cap B(0,N(\mu))}\{\nu(x,u)\} & \mu\in(0,\bar\sigma],\\ \displaystyle\max_{(x,u)\in W^{-1}(\{\mu\})\times(U\cap B(0,N(\mu))}\{\nu(x,u)\} & \mu>\bar\sigma.\end{cases}$$

The function ν_0 is well defined and locally bounded, because W is proper and ν, N are continuous. Moreover, ν_0 is decreasing on $(0,\bar\sigma]$. Let ν_1 be a smooth approximation from above of ν_0, decreasing on $(0,2\sigma]$ – like N. Then

$$\nu_1(W(x))\ge\nu(x,u)\quad\forall x\in\mathbb{R}^n\setminus\mathcal{C},\ \ u\in U\cap B(0,N(W(x))). \tag{70}$$

Consider a nonnegative, smooth map $\bar\nu=\bar\nu_R:[0,+\infty)\to[0,+\infty)$ such that $\bar\nu\equiv0$ in $[0,\sigma]$ and $\bar\nu:=\nu_1$ in $[2\sigma,+\infty)$.

We set

$$\begin{aligned}&\xi_R(\mu):=\mu+\int_0^\mu\bar\nu(r)\,dr\ \ \forall\mu\in[0,+\infty),\quad \bar W_R:=\xi_R\circ W,\\ &\tilde\gamma_R(\mu):=\begin{cases}\frac{\tilde\gamma}{1+\nu_1}\circ\xi_R^{-1}(\mu) & \mu\in(0,2\sigma],\\ \tilde\gamma\circ\xi_R^{-1}(\mu) & \mu>2\sigma.\end{cases}\end{aligned}\tag{71}$$

The function ξ_R is the identity in $[0,\sigma]$ and $\xi_R(\mu)\ge\mu$ in $(\sigma,+\infty)$, so that $\bar W_R\ge W$ in the whole set $\mathbb{R}^n\setminus\mathcal{C}$ and $\bar W_R\equiv W$ on $W^{-1}([0,\sigma])$. Hence, in particular, $\bar W_R(x)=W(x)$ when $\mathbf{d}(x)\le R$. Moreover, $\bar W_R$ is locally Lipschitz continuous, proper and positive definite on $\mathbb{R}^n\setminus\mathcal{C}$; it is also locally Lipschitz continuous on $\overline{\mathbb{R}^n\setminus\mathcal{C}}$ or, by [4, Proposition 2.1.12], locally semiconcave on $\mathbb{R}^n\setminus\mathcal{C}$, when W is. By the properties of ξ_R, the decrease rate $\tilde\gamma_R:(0,+\infty)\to(0,+\infty)$ is well defined, strictly increasing, continuous except at point 2σ and $\tilde\gamma_R\le\tilde\gamma$. Since $\tilde\gamma\le\gamma$ by Proposition 5, then there exists a positive, strictly increasing, smooth approximation from below γ_R of $\tilde\gamma_R$ such that $\gamma_R\le\gamma$: this concludes the proof of statement (i).

In order to prove (ii), namely that $\bar W_R$, γ_R verify the decrease condition (63), we make use of the following result:

Lemma 4.7. *Let $\Omega\subset\mathbb{R}^n$ be an open subset and let $W:\Omega\to(0,+\infty)$ be a locally Lipschitz continuous function. If $\xi:(0,+\infty)\to(0,+\infty)$ is a strictly increasing, C^2 function with $\xi'>0$, then for every $x\in\Omega$ one has*

$$\partial_L(\xi\circ W)(x)=\xi'(W(x))\partial_L W(x). \tag{72}$$

Proof. Let us show that, given $x\in\Omega$,

$$\partial_P(\xi\circ W)(x)=\xi'(W(x))\partial_P W(x). \tag{73}$$

Then it immediately follows the thesis (72).

Let us begin by showing that $\partial_P(\xi\circ W)(x)\subseteq\xi'(W(x))\partial_P W(x)$. Let $p\in\partial_P W(x)$. Then there exist a neighborhood of x and some $\bar\rho>0$ and $\bar\epsilon>0$, such that

$$W(y)-W(x)+\bar\rho|y-x|^2\ge\langle p,y-x\rangle\qquad\forall y\in B(x,\bar\epsilon).$$

Since $\xi\in C^2(0,+\infty)$, by the Taylor expansion of ξ at $\mu=W(x)$, for any $\tilde\mu$ in some neighbourhood of μ, one has $\xi(\tilde\mu)-\xi(\mu)=\xi'(\mu)(\tilde\mu-\mu)+\frac{\xi''(\mu)}{2}|\tilde\mu-\mu|^2+o(|\tilde\mu-\mu|^2)$.

For the local Lipschitz continuity of W, possibly reducing $\bar\varepsilon$, for every $y \in B(x,\bar\epsilon)$, one has

$$\begin{aligned}&\xi(W(y)) - \xi(W(x))\\ &\quad = \xi'(W(x))(W(y)-W(x)) + \tfrac{\xi''(W(x))}{2}|W(y)-W(x)|^2 + o(|W(y)-W(x)|^2).\end{aligned}$$

Since $\xi' > 0$, for every $p \in \partial_P W(x)$ we derive that

$$\begin{aligned}\xi(W(y)) - \xi(W(x)) \geq \xi'(W(x))\left[\langle p, y-x\rangle - \bar\rho|y-x|^2\right]&\\ + L_W^2\,\frac{\xi''(W(x))\wedge 0}{2}|y-x|^2 + o(|y-x|^2),&\end{aligned}$$

where $L_W > 0$ denotes the (local) Lipschitz constant of W. Hence

$$\xi(W(y)) - \xi(W(x)) + \rho\,|y-x|^2 \geq \langle \xi'(W(x))\,p, y-x\rangle,$$

as soon as $\rho > 0$ verifies $\rho \geq \left[\xi'(W(x))\bar\rho - L_W^2\,\frac{\xi''(W(x))\wedge 0}{2} + \frac{o(|y-x|^2)}{|y-x|^2}\right]$, so that $\bar p := \xi'(W(x))\,p \in \xi'(W(x))\partial_P W(x)$ and the inclusion $\xi'(W(x))\partial_P W(x) \subseteq \partial_P(\xi\circ W)(x)$ is proved.

The assumption $\xi' > 0$ implies that the inverse function ξ^{-1} is strictly increasing and C^2, as ξ. Hence the opposite inclusion $\partial_P(\xi\circ W)(x) \subseteq \xi'(W(x))\partial_P W(x)$, can be obtained by applying the previous arguments to $\xi\circ W$ and ξ^{-1} in place of W and ξ, respectively. This yields the equality (73) and the proof of the lemma is concluded. □

By Lemma 4.7, for every $x \in \mathbb{R}^n \setminus \mathcal{C}$ we have $\partial_L \bar W_R(x) = (1+\bar\nu(W(x))\partial_L W(x)$, so that given an arbitrary $\bar p \in \partial_L \bar W_R(x)$, there exists some $p \in \partial_L W(x)$ such that

$$\bar p = (1+\bar\nu(W(x))\,p. \tag{74}$$

Let $\bar u \in U \cap B(0, N(W(x))$ satisfy

$$\langle f(x,\bar u), p\rangle + p_0\,l(x,\bar u) = \min_{U\cap B(0,N(W(x)))}\{\langle f(x,u),p\rangle + p_0\,l(x,u)\}.$$

By (61), (70), (71), and (74), when $x \in W^{-1}((0,2\sigma])$ one has $\langle f(x,\bar u), p\rangle < 0$ and

$$\begin{aligned}H_{\bar f,\bar l}(x,p_0,\bar p) &\leq \langle \bar f(x,\bar u), \bar p\rangle + p_0\,\bar l(x,\bar u) = (1+\bar\nu(W(x))\,\langle \bar f(x,\bar u), p\rangle + p_0\,\bar l(x,\bar u)\\ &= \frac{1}{1+\nu(x,\bar u)}\,\left[(1+\bar\nu(W(x))\,\langle f(x,\bar u),p\rangle + p_0\,l(x,\bar u)\right]\\ &< -\frac{\tilde\gamma(W(x))}{1+\nu(x,\bar u)} \leq -\gamma_R(\bar W_R(x)).\end{aligned}$$

If otherwise $x \in W^{-1}((2\sigma,+\infty))$, then $\bar\nu(W(x)) = \nu_1(W(x))$ and, recalling that $\langle f(x,\bar u),p\rangle < 0$ and $p_0\,l(x,\bar u) \geq 0$, we get by (70)

$$\begin{aligned}H_{\bar f,\bar l}(x,p_0,\bar p) &\leq \langle \bar f(x,\bar u), \bar p\rangle + p_0\,\bar l(x,\bar u)\\ &= (1+\ \nu_1(W(x))\,\langle \bar f(x,\bar u), p\rangle + p_0\,\bar l(x,\bar u)\\ &\leq \frac{1+\nu_1(W(x))}{1+\nu(x,\bar u)}\,\left[\langle f(x,\bar u),p\rangle + p_0\,l(x,\bar u)\right]\\ &< -\tilde\gamma(W(x)) \leq -\gamma_R(\bar W_R(x)),\end{aligned}$$

and this implies the validity of (ii).

The proof of statement (iii) follows by the arguments above, by simply replacing $\bar u$ with $K(x)$, where K is a feedback as in Proposition 5. □

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