MULTIPLICATIVE PROPERTIES OF INTEGER VALUED POLYNOMIALS OVER SPLIT-QUATERNIONS

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Abstract. We study localization properties and the prime spectrum of the integer-valued polynomial ring $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$, where $\mathbb{P}_{\mathbb{Z}}$ (respectively $\mathbb{P}_{\mathbb{Q}}$) is the algebra of split-quaternion over $\mathbb{Z}$ (respectively over $\mathbb{Q}$).

Introduction

In [14] N. Werner studied the ring of integer-valued polynomials in a noncommutative setting, by considering quaternion algebras. Precisely, he considered the algebras $\mathbb{H}_{\mathbb{Z}}$ and $\mathbb{H}_{\mathbb{Q}}$ (respectively over $\mathbb{Z}$ and over $\mathbb{Q}$) generated by the unit elements 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, linked by the relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{i}\,\mathbf{j} = \mathbf{k} = -\mathbf{j}\,\mathbf{i}$, $\mathbf{j}\,\mathbf{k} = \mathbf{i} = -\mathbf{k}\,\mathbf{j}$ and $\mathbf{k}\,\mathbf{i} = \mathbf{j} = -\mathbf{i}\,\mathbf{k}$, and considered the set $\mathrm{Int}_{\mathbb{H}_{\mathbb{Q}}}(\mathbb{H}_{\mathbb{Z}})$ of all polynomials $f \in \mathbb{H}_{\mathbb{Q}}[x]$ such that $f(\mathbb{H}_{\mathbb{Z}}) \subseteq \mathbb{H}_{\mathbb{Z}}$. After proving that $\mathrm{Int}_{\mathbb{H}_{\mathbb{Q}}}(\mathbb{H}_{\mathbb{Z}})$ is indeed a noncommutative ring (which strictly contains $\mathbb{H}_{\mathbb{Q}}[x]$), he investigated the ideal structure of this ring, describing some prime ideals above the zero and the maximal ideals of $\mathbb{H}_{\mathbb{Z}}$.

Moving from these ideas, in [3] A. Cigliola, K.A. Loper and N. Werner focused on similar problems in a different setting: instead of $\mathbb{H}_{\mathbb{Z}}$ they considered the set of integer split-quaternions $\mathbb{P}_{\mathbb{Z}}$, i.e. the $\mathbb{Z}$-algebra generated by the unit elements 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ with the relations $-\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1$ and $\mathbf{i}\,\mathbf{j}\,\mathbf{k} = 1$ (see Definition 1.1).

In this paper, we continue the study of the ring $\mathbb{P}_{\mathbb{Z}}$ (Section 1) and of the ring $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$ of integer-valued polynomials over $\mathbb{P}_{\mathbb{Z}}$ (Section 2). We study some denominator sets of $\mathbb{P}_{\mathbb{Z}}$ and $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$ that are not subsets of $\mathbb{Z}$ (in particular, they are not central) and their ring of fractions. Thus, we partially answer to one of the open questions posed in [3, §5] which asks whether it is possible to find and to localize $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$ with respect to noncentral sets. We then study the ring

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$\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ of the polynomials in $\mathbb{Q}[x]$ that are integer valued over $\mathbb{P}_{\mathbb{Z}}$. There is a strict connection between the prime spectrum of this ring and the prime spectrum of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$. This allows to calculate the Krull dimension of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$, for an odd prime integer p, starting from the dimension of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ and thus to get a partial but interesting information about the Krull dimension of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$. Finally, in Section 4, we study in more detail the ideal $p\,\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ and show that it is not prime. In this last Section we will be able to construct explicitely some polynomials of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$.

Throughout the paper, all the rings we consider are unitary but not necessarily commutative.

1. Localizations of $\mathbb{P}_{\mathbb{Z}}$

We recall some definitions and basic properties.

Definition 1.1. Let R be a commutative ring. We denote by $\mathbb{P}_R$ the R-algebra generated by the four unit elements 1, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ with the relations

$$-\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\,\mathbf{j}\,\mathbf{k} = 1.$$

Formally $\mathbb{P}_R := \{\mathrm{q} = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \mid a, b, c, d \in R\}$.

We call $\mathbb{P}_R$ the *ring of split-quaternions* over R.

Let $\mathrm{q} = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \in \mathbb{P}_R$, then:

(a) a, b, c, and d are the *coefficients* of q, and a is the *real part* of q;
(b) the *bar conjugate* of q is $\overline{\mathrm{q}} := a - b\,\mathbf{i} - c\,\mathbf{j} - d\,\mathbf{k}$;
(c) the *norm* of q is $N(\mathrm{q}) := \mathrm{q}\overline{\mathrm{q}} = a^2 + b^2 - c^2 - d^2$;
(d) the *trace* of q is $T(\mathrm{q}) = \mathrm{q} + \overline{\mathrm{q}} = 2a$;
(e) the *minimal polynomial* of q is ([3, Definition 2.4])

$$m_{\mathrm{q}}(x) := \begin{cases} x - q & \text{if } \mathrm{q} \in R \\ x^2 - T(\mathrm{q})x + N(\mathrm{q}) & \text{if } \mathrm{q} \in \mathbb{P}_R \setminus R. \end{cases}$$

$m_{\mathrm{q}}(x)$ is minimal in the way that $m_{\mathrm{q}}(\mathrm{q}) = 0$ and that $m_{\mathrm{q}}(x)$ is the monic polynomial of least degree having q as a root.

In this section, we study some localizations of $\mathbb{P}_{\mathbb{Z}}$. We start with the description of its prime and maximal ideals. Recall that an ideal P of a (not necessarily commutative) ring R is *prime* if, given $a, b \in R$ such that $aPb \subseteq P$, then $a \in P$ or $b \in P$.

Theorem 1.2. [3, Theorem 2.11]. *The prime ideals of $\mathbb{P}_{\mathbb{Z}}$ are:*

(i) (0);
(ii) $p\mathbb{P}_{\mathbb{Z}}$ *where p is an odd prime of $\mathbb{Z}$;*

(iii) $\mathscr{M} = (1+\boldsymbol{i}; 1+\boldsymbol{j})$.

Moreover, the primes $p\mathbb{P}_{\mathbb{Z}}$ *and* $\mathscr{M}$ *are maximal, and* $\mathscr{M}$ *is the only prime ideal containing* 2.

Lemma 1.3. *Let* $\mathrm{q} \in \mathbb{P}_{\mathbb{Z}}$ *such that* $2 \mid \mathrm{N}(\mathrm{q})$. *Then* $\mathrm{q} \in \mathscr{M}$. *In particular* $\mathscr{M}$ *contains all the zero-divisors of* $\mathbb{P}_{\mathbb{Z}}$.

Proof. Let $\mathrm{q} = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ be such that $\mathrm{N}(\mathrm{q}) = a^2 + b^2 - c^2 - d^2 = 2m$, for some $m \in \mathbb{Z}$. By hypothesis, q must have zero, two or four even coefficients. In the case that all coefficients are even, then trivially $\mathrm{q} \in (2) \subseteq \mathscr{M}$. If q has exactly two even coefficients, then q is congruent modulo $2\mathbb{P}_{\mathbb{Z}}$ to the sum of two of 1, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, and all of them are elements of $\mathscr{M}$. Finally, if all coefficients of q are odd, then $\mathrm{q} \equiv 1 + \mathbf{i} + \mathbf{j} + \mathbf{k} (\mathrm{mod}\ 2\mathbb{P}_{\mathbb{Z}})$, and so $\mathrm{q} \in \mathscr{M}$ since $1 + \mathbf{i} + \mathbf{j} + \mathbf{k} = (1+\mathbf{i})(1+\mathbf{j}) \in \mathscr{M}$. □

Definition 1.4. Let R be a ring and S a multiplicative subset in R. We say that S is a *right denominator set* if:

(i) for any $a \in R$ and $s \in S$, $aS \cap sR \neq \emptyset$ (this condition is known as *right Ore condition* and S is called a *right Ore set*);
(ii) for $a \in R$, if $s'a = 0$ for some $s' \in S$, then $as = 0$ for some $s \in S$ (we say that S is *right reversible*).

Remark 1.5. (a) We can define left denominator sets in a completely symmetrical way.
(b) Condition (ii) (reversibility) is automatically satisfied when S does not contain zero-divisors.
(c) It is easily seen that the multiplicative subsets contained in the center of R are always denominator subsets.

By [9, Theorem 10.6], if R is a ring and S a multiplicative subset of R, then R has a right ring of fractions with respect to S (namely, the ring $RS^{-1} := \{\, as^{-1} \mid a \in R,\ s \in S \,\}$) if and only if S is a right denominator set. Similarly we can construct the ring $S^{-1}R := \{\, s^{-1}a \mid a \in R,\ s \in S \,\}$ if and only if S is a left denominator set. If S is both a right and left denominator set, then $RS^{-1} \simeq S^{-1}R$ by [9, Corollary 10.14].

Lemma 1.6. *Let* R *be a commutative ring and* S *a multiplicative subset of* $\mathbb{P}_R$, *closed under norm (i.e., if* $s \in S$ *then* $\mathrm{N}(s) \in S$*). Then* S *verifies both the right and the left Ore condition.*

Proof. Fix $a \in \mathbb{P}_R$ and $s \in S$. Since $\mathrm{N}(s) \in R$ is a central element, we have that $a\,\mathrm{N}(s) = \mathrm{N}(s)a$. It follows that $a\,\mathrm{N}(s) = s(\bar{s}a)$, so S is a right Ore set since $aS \cap s\mathbb{P}_R \neq \emptyset$. Analogously, $(a\bar{s})s = \mathrm{N}(s)a$ so S is a left Ore set since $Sa \cap \mathbb{P}_R s \neq \emptyset$. □

By the previous lemma, if $S = \mathcal{R}(R)$ is the set of all (right and left) regular elements of R, then S is a denominator set and RS^{-1} is the *total ring of fractions of* R, which we denote by $\mathcal{Q}(R)$.

For commutative rings, the most important way of constructing localizations of a ring R is through the sets $R \setminus P$, where P is a prime ideal; however, if R is not commutative, the complement of a prime ideal may not be multiplicatively closed. For example, if $p = 2k+1$ is an odd prime number, then $p\mathbb{P}_{\mathbb{Z}}$ is prime, but $\mathbb{P}_{\mathbb{Z}} \setminus p\mathbb{P}_{\mathbb{Z}}$ is not multiplicatively closed since $((k+1)+k\,\mathbf{j})((k+1)-k\,\mathbf{j}) = p \in p\mathbb{P}_{\mathbb{Z}}$.

Following the notation of Goldie in [6], we give the following definition:

Definition 1.7. Let be given a ring R and let Q be a proper prime ideal of R. We set:

$$\mathscr{C}(Q) := \{x \in R \mid xr \notin Q, \forall r \notin Q\},$$

and

$$\mathscr{C}'(Q) := \{x \in R \mid rx \notin Q, \forall r \notin Q\}.$$

Proposition 1.8. *Let R be a ring and let $Q \subsetneq R$ be a prime ideal of R. Then $\mathscr{C}(Q)$ is a multiplicatively closed subset of R containing 1 but not 0, and $\mathscr{C}(Q) \subseteq R \setminus Q$. The same properties hold for $\mathscr{C}'(Q)$.*

Proof. For each $r \notin Q$, we have that $1 \cdot r = r \notin Q$ and that $0 \cdot r \in Q$. Then, by definition, $1 \in \mathscr{C}(Q)$ and $0 \notin \mathscr{C}(Q)$. Take now $a, b \in \mathscr{C}(Q)$ and $r \notin Q$. Since $b \in \mathscr{C}(Q)$, then $br \notin Q$. Again, since $a \in \mathscr{C}(Q)$, we have $a(br) \notin Q$. Thus for all $r \notin Q$ we have $(ab)r = a(br) \notin Q$.

Finally, if $x \in \mathscr{C}(Q)$ then, since $1 \notin Q$, we have $x \cdot 1 = x \notin Q$. Hence, $\mathscr{C}(Q) \subseteq R \setminus Q$. □

Proposition 1.9. *Let R be a ring and let $Q \subsetneq R$ be a prime ideal of R. Then $\mathscr{C}(Q)$ is the set of left regular elements of R modulo Q and $\mathscr{C}'(Q)$ is the set of right regular elements of R modulo Q.*

Proof. Take $x \in R$. Then x is a left zero-divisor modulo Q if and only if there is $r \in R/Q$, $r \neq 0$, such that $xr = 0$. This is equivalent to saying that there is an $r \notin Q$ such that $xr \in Q$. In other words, $x \notin \mathscr{C}(Q)$. Similarly for $\mathscr{C}'(Q)$. □

In particular, we have that $\mathscr{C}(0) = \mathcal{R}_l(R)$ is the set of the left regular elements of R, while $\mathscr{C}'(0) = \mathcal{R}_r(R)$ is the set of the right regular elements of R.

We now focus on some properties of the sets $\mathscr{C}(Q)$ associated to the prime ideals of $\mathbb{P}_{\mathbb{Z}}$.

Proposition 1.10. *Let Q be a prime ideal of $\mathbb{P}_{\mathbb{Z}}$. Then:*
(i) $\mathscr{C}(Q)$ is closed under bar conjugation;
(ii) $\mathscr{C}(Q)$ is closed under norm;
(iii) $\mathscr{C}(Q) = \{x \in \mathbb{P}_{\mathbb{Z}} \mid \mathrm{N}(x) \notin Q\}$;
(iv) $\mathscr{C}(Q)$ does not contain any zero-divisor.

Proof. By [7, Proposition 1.6] $\mathbb{P}_{\mathbb{Z}}$ is a Noetherian ring. Thus, from [6, Section 3], $\mathscr{C}(Q) = \mathscr{C}'(Q)$.

Consider first $Q = (0)$. Then $\mathscr{C}(0)$ equals $\mathcal{R}(\mathbb{P}_{\mathbb{Z}})$, the set of all (two-sided) regular elements, and so

$$\mathscr{C}(Q) = \mathcal{R}(\mathbb{P}_{\mathbb{Z}}) = \{x \in \mathbb{P}_{\mathbb{Z}} \mid \mathrm{N}(x) \neq 0\}.$$

This proves the claim in the case $Q = (0)$.

Let now be $Q = p\mathbb{P}_{\mathbb{Z}}$, for an odd prime integer p. We notice that:

- $\mathscr{C}(Q)(\mathrm{mod}\ Q) = \mathscr{C}(\bar{0})$ in $\mathbb{P}_{\mathbb{Z}}/Q = \mathbb{P}_{\mathbb{Z}_p}$ (apply Proposition 1.9);
- $\mathrm{N}(x)(\mathrm{mod}\ p) = \mathrm{N}(\bar{x})$, for $x \in \mathbb{P}_{\mathbb{Z}}$ and $\bar{x} = x(\mathrm{mod}\ p\mathbb{P}_{\mathbb{Z}})$.

Using these equalities, points (i)-(ii)-(iii) reduce to the case $Q = (0)$, which has been already proved. For $p = 2$, the same reasoning applies reducing modulo $\mathscr{M}$.

For the point (iv), if $p = 2$ the claim follows from Lemma 1.3.

If p is an odd prime, then suppose that $xr' = 0$, for some $x \in \mathscr{C}(Q)$ and $0 \neq r' \in \mathbb{P}_{\mathbb{Z}}$. If we write $r' = p^m r$, for some $r \notin Q$, we get $xr = 0 \in Q$ (since p is not a zero divisor from Lemma 1.3) which is absurd. □

In particular, we observe that $\mathscr{C}(p\mathbb{P}_{\mathbb{Z}}) = \{x \in \mathbb{P}_{\mathbb{Z}} \mid p \nmid \mathrm{N}(x)\}$ and $\mathscr{C}(\mathscr{M}) = \{x \in \mathbb{P}_{\mathbb{Z}} \mid 2 \nmid \mathrm{N}(x)\}$.

We will work with the following multiplicative subsets of $\mathbb{P}_{\mathbb{Z}}$:

- the multiplicative subsets of $\mathbb{Z}$;
- the sets $\mathscr{C}(0)$, $\mathscr{C}(\mathscr{M})$ and $\mathscr{C}(p\mathbb{P}_{\mathbb{Z}})$, for any odd prime integer p.

For a general noncommutative ring, given a prime ideal Q, $\mathscr{C}(Q)$ may not be a denominator set: such an example is given, for instance, in [1, Example 2.3]. However we show that $\mathscr{C}(Q)$ is a denominator sets in $\mathbb{P}_{\mathbb{Z}}$ and also in $\mathrm{Int}(\mathbb{P}_{\mathbb{Z}})$ (Proposition 2.4), for each prime ideal Q of $\mathbb{P}_{\mathbb{Z}}$.

Proposition 1.11. *The sets $\mathbb{Z} \setminus (0)$, $\mathbb{Z} \setminus p\mathbb{Z}$, for p prime, and $\mathscr{C}(Q)$, for Q prime ideal of $\mathbb{P}_{\mathbb{Z}}$, are (right and left) denominator sets of $\mathbb{P}_{\mathbb{Z}}$.*

Proof. Let $S = \mathbb{Z}\setminus(0)$ or $S = \mathbb{Z}\setminus p\mathbb{Z}$, for a prime p. Then the statement easily follows from the fact that S is contained in the center of $\mathbb{P}_{\mathbb{Z}}$.

If $S = \mathscr{C}(Q)$, then S does not contain zero-divisors (Proposition 1.10), so $\mathscr{C}(Q)$ is right and left reversible. Finally, $\mathscr{C}(Q)$ is a right (left) Ore set by Lemma 1.6, since it is closed under bar conjugation (Proposition 1.10). Thus $\mathscr{C}(Q)$ is a right and left denominator set of $\mathbb{P}_{\mathbb{Z}}$. $\square$

Proposition 1.12. *Let $S = \mathscr{C}(0)$ or $S = \mathbb{Z} \setminus (0)$. Then*

$$\mathbb{P}_{\mathbb{Z}}S^{-1} = S^{-1}\mathbb{P}_{\mathbb{Z}} = \mathbb{P}_{\mathbb{Q}} = \mathcal{Q}(\mathbb{P}_{\mathbb{Z}}),$$

which is the total ring of fractions of $\mathbb{P}_{\mathbb{Z}}$.

Proof. By Proposition 1.11, S is a denominator set. So the ring $\mathbb{P}_{\mathbb{Z}}S^{-1}$ exists and its elements are the fractions rs^{-1}, where $r, s \in \mathbb{P}_{\mathbb{Z}}$ and $\mathrm{N}(s) \neq 0$. Then $rs^{-1} = \frac{1}{\mathrm{N}(s)} r\overline{s} \in \mathbb{P}_{\mathbb{Q}}$. Thus $\mathbb{P}_{\mathbb{Z}}S^{-1} \subseteq \mathbb{P}_{\mathbb{Q}}$. Conversely, given $\mathrm{q} \in \mathbb{P}_{\mathbb{Q}}$, write q in the form $\mathrm{p} \cdot a^{-1}$, where $\mathrm{p} \in \mathbb{P}_{\mathbb{Z}}$ and a is a common denominator for the coefficients of q. Obviously, $a \in S$ and so $\mathrm{p}a^{-1} \in \mathbb{P}_{\mathbb{Z}}S^{-1}$, i.e., $\mathbb{P}_{\mathbb{Z}}S^{-1} \supseteq \mathbb{P}_{\mathbb{Q}}$. Thus $\mathbb{P}_{\mathbb{Z}}S^{-1} = \mathbb{P}_{\mathbb{Q}}$. Similarly, $S^{-1}\mathbb{P}_{\mathbb{Z}} = \mathbb{P}_{\mathbb{Q}}$. Finally $\mathbb{P}_{\mathbb{Q}}$ is the total ring of fractions of $\mathbb{P}_{\mathbb{Z}}$ because we localize with respect to the set of regular elements of $\mathbb{P}_{\mathbb{Z}}$. $\square$

Similarly, if we localize $\mathbb{P}_{\mathbb{Z}}$ at $S = \mathbb{Z} \setminus p\mathbb{Z}$ or $S = \mathscr{C}(Q)$, where $Q = p\mathbb{P}_{\mathbb{Z}}$, for a prime number p, we get the algebra of split-quaternions with coefficients in $\mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at the ideal $p\mathbb{Z}$ (as we see in the following Proposition). In the following, $\mathbb{Z}_p$ will denote the field with p elements.

Proposition 1.13. *Let p be a prime number and let $S = \mathbb{Z} \setminus p\mathbb{Z}$ or $S = \mathscr{C}(Q)$, where Q is a prime ideal of $\mathbb{P}_{\mathbb{Z}}$ such that $Q \cap \mathbb{Z} = p\mathbb{Z}$.*

Then

$$\mathbb{P}_{\mathbb{Z}}S^{-1} = S^{-1}\mathbb{P}_{\mathbb{Z}} = \mathbb{P}_{\mathbb{Z}_{(p)}}.$$

Proof. We know that S is a denominator set of $\mathbb{P}_{\mathbb{Z}}$ by Proposition 1.11. So the ring $\mathbb{P}_{\mathbb{Z}}S^{-1}$ exists.

Let $S = \mathbb{Z} \setminus p\mathbb{Z}$. It is easy to see that $\mathbb{P}_{\mathbb{Z}}S^{-1} \subseteq \mathbb{P}_{\mathbb{Z}_{(p)}}$. For the reverse inclusion, notice that the minimum common denominator of any element of $\mathbb{Z}_{(p)}$ is an element of $\mathbb{Z}\setminus p\mathbb{Z}$. So $\mathbb{P}_{\mathbb{Z}}S^{-1} = \mathbb{P}_{\mathbb{Z}_{(p)}}$. Similarly it can be proved that $S^{-1}\mathbb{P}_{\mathbb{Z}} = \mathbb{P}_{\mathbb{Z}_{(p)}}$.

Let $S = \mathscr{C}(Q)$. Since the norm of the elements of S is not divisible by p (Proposition 1.10), a right fraction $\mathrm{p}s^{-1} \in \mathbb{P}_{\mathbb{Z}}S^{-1}$, for some $\mathrm{p} \in \mathbb{P}_{\mathbb{Z}}$ and $s \in S$, can be seen as a rational split-quaternion $\mathrm{q} = \frac{1}{\mathrm{N}(s)}\mathrm{p}\overline{s} = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$, where $a, b, c, d \in \mathbb{Q}$ and their denominators are not divisible by p. Thus $\mathbb{P}_{\mathbb{Z}}S^{-1} \subseteq \mathbb{P}_{\mathbb{Z}_{(p)}}$. For the reverse inclusion let

$\mathrm{q} \in \mathbb{P}_{\mathbb{Z}_{(p)}}$. Taking a common denominator, write $\mathrm{q} = \frac{1}{n}\mathrm{p}$, for some $\mathrm{p} \in \mathbb{P}_{\mathbb{Z}}$ and $n \in \mathbb{Z}$. Since the minimum common denominator of some elements of $\mathbb{Z}_{(p)}$ is an element of $\mathbb{Z} \setminus p\mathbb{Z}$, then n is not divisible by p. Thus neither $n^2 = \mathrm{N}(n)$ is divisible by p. So $n \in S$ and $\mathbb{P}_{\mathbb{Z}} S^{-1} = \mathbb{P}_{\mathbb{Z}_{(p)}}$. In the same manner we can prove that $S^{-1}\mathbb{P}_{\mathbb{Z}} = \mathbb{P}_{\mathbb{Z}_{(p)}}$. □

Imitating Proposition 1.12 we can give this general result.

Proposition 1.14. *Let R be a commutative ring and let $\mathcal{Q}(R)$ be its total ring of fractions. Then*

$$\mathcal{Q}(\mathbb{P}_R) = \mathbb{P}_{\mathcal{Q}(R)}.$$

Proof. Let S be the set of regular elements of R. Then, S is contained in the center of $\mathbb{P}_R$, and thus it is a denominator set of $\mathbb{P}_R$; it is also easy to see that $S^{-1}\mathbb{P}_R = \mathbb{P}_{S^{-1}R} = \mathbb{P}_{\mathcal{Q}(R)}$ (see proof of Propositions 1.12).

We claim that the elements of $\mathbb{P}_{\mathcal{Q}(R)}$ are either invertible or zero-divisors. Take $\mathrm{q} \in \mathbb{P}_{\mathcal{Q}(R)}$. If $N(\mathrm{q})$ is regular, then it is invertible in $\mathcal{Q}(R)$, and thus $\frac{1}{N(\mathrm{q})}\overline{\mathrm{q}} \in \mathbb{P}_{\mathcal{Q}(R)}$ is the inverse of q. Conversely, if $N(\mathrm{q})$ is not regular, then there is $z \in R$, $z \neq 0$, such that $z\,\mathrm{N}(\mathrm{q}) = 0$. If $z\mathrm{q} \neq 0$, then also $z\overline{\mathrm{q}} = \overline{zq} \neq 0$. So we have that:

$$0 = zN(\mathrm{q}) = z\overline{\mathrm{q}}\mathrm{q} = (z\overline{\mathrm{q}})\mathrm{q}$$

hence, q is a zero-divisor.

Thus, $\mathbb{P}_{\mathcal{Q}(R)}$ is a total ring of fractions, and so it is the total ring of fractions of $\mathbb{P}_R$. □

2. Integer-valued polynomials

The ring of *integer-valued polynomials* over $\mathbb{P}_{\mathbb{Z}}$ is

$$\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}}) = \{f(x) \in \mathbb{P}_{\mathbb{Q}}[x] \mid f(\mathbb{P}_{\mathbb{Z}}) \subseteq \mathbb{P}_{\mathbb{Z}}\}.$$

This set is actually a ring ([15, Theorem 1.2]), and in [3] the authors describe explicitly some proper ideals of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$. A similar construction can be done if, instead of $\mathbb{P}_{\mathbb{Z}}$, we use $\mathbb{P}_{\mathbb{Z}_{(p)}}$ or $\mathbb{P}_{\mathbb{Q}}$; in the former case, [15, Theorem 1.2] guarantees that $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ is a ring, while in the latter $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Q}}) = \mathbb{P}_{\mathbb{Q}}[x]$ is the whole ring of polynomials (and, in particular, is a ring).

For simplicity of notation, in this Section we will write $\mathrm{Int}(\mathbb{P}_{\mathbb{Z}})$ instead of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$.

A class of ideals of $\mathrm{Int}(\mathbb{P}_{\mathbb{Z}})$ can be constructed in the following way: if $\mathrm{q} = a + bi + cj + dk \in \mathbb{P}_{\mathbb{Z}}$ and I is a principal ideal of $\mathbb{P}_{\mathbb{Z}}$ generated by an element of $\mathbb{Z}$, then

$$\mathfrak{P}_{I,\mathrm{q}} := \{f(x) \in \mathrm{Int}(\mathbb{P}_{\mathbb{Z}}) \mid f(z) \in I\ \forall z \in C(\mathrm{q})\},$$

is an ideal of $\mathrm{Int}(\mathbb{P}_{\mathbb{Z}})$, where $C(\mathrm{q}) = \{a \pm bi \pm cj \pm dk\}$ ([3, Proposition 4.2]).

If P is a prime ideal of $\mathrm{Int}(\mathbb{P}_{\mathbb{Z}})$, then $P \cap \mathbb{P}_{\mathbb{Z}}$ is a prime ideal of $\mathbb{P}_{\mathbb{Z}}$; since we have a classification of the prime ideals of $\mathbb{P}_{\mathbb{Z}}$ (Theorem 1.2), we can study the spectrum of $\mathrm{Int}(\mathbb{P}_{\mathbb{Z}})$ according to the restriction to $\mathbb{P}_{\mathbb{Z}}$.

Proposition 2.1. *The following hold.*

(1) [3, Corollary 4.10] *The prime ideals P of $\mathrm{Int}(\mathbb{P}_{\mathbb{Z}})$ with $P \cap \mathbb{P}_{\mathbb{Z}} = (0)$ are exactly those of the form*

$$P = M(x) \cdot \mathbb{P}_{\mathbb{Q}}[x] \cap \mathrm{Int}(\mathbb{P}_{\mathbb{Z}}) =: P_{M(x)},$$

where $M(x) \in \mathbb{Z}[x]$ is an irreducible polynomial.

In particular, if $m_{\mathrm{q}}(x)$ is the minimal polynomial of an element $\mathrm{q} \in \mathbb{P}_{\mathbb{Z}}$ then $P_{m_{\mathrm{q}}(x)} = \mathfrak{P}_{0,\mathrm{q}}$ is a prime ideal.

(2) [3, Theorem 4.16] *Let $\mathrm{q} := a + b\,\boldsymbol{i} + c\,\boldsymbol{j} + d\,\boldsymbol{k} \in \mathbb{P}_{\mathbb{Z}} \setminus \mathbb{Z}$ and let p be an odd prime. If $\gcd(b, c, d, p) = 1$, then $\mathfrak{P}_{p\mathbb{P}_{\mathbb{Z}},\mathrm{q}}$ is prime if and only if $m_{\mathrm{q}}(x)$ is irreducible mod p, in which case $\mathfrak{P}_{p\mathbb{P}_{\mathbb{Z}},\mathrm{q}}$ is maximal.*

(3) [3, Corollary 4.22] *Let $\mathrm{q} = a + b\,\boldsymbol{i} + c\,\boldsymbol{j} + d\,\boldsymbol{k} \in \mathbb{P}_{\mathbb{Z}}$, and assume that either $b \equiv c \pmod 2$ or $b \equiv d \pmod 2$. Then,*

$$\mathfrak{M}_{\mathrm{q}} := \{f \in \mathrm{Int}(\mathbb{P}_{\mathbb{Z}}) \mid f(q) \in \mathscr{M}\}$$

is a maximal ideal of $\mathrm{Int}(\mathbb{P}_{\mathbb{Z}})$.

Remark 2.2.

(1) While the first case of the proposition completely classifies the prime ideals above (0), the other two merely give some examples of the prime ideals above $p\mathbb{P}_{\mathbb{Z}}$ and $\mathscr{M}$, but not a complete list.

(2) We refer to [3] for some results about the equality among these ideals.

Lemma 2.3. *The following hold:*

(1) *If p is an odd prime number and $\mathrm{q} \in \mathbb{P}_{\mathbb{Z}}$, then $\mathfrak{P}_{m_{\mathrm{q}}(x)} \subset \mathfrak{P}_{p\mathbb{P}_{\mathbb{Z}},\mathrm{q}}$.*

(2) *If $q \in \mathbb{P}_{\mathbb{Z}}$ is as in Propositon 2.1(3), then $\mathfrak{P}_{m_{\mathrm{q}}(x)} \subset \mathfrak{M}_{\mathrm{q}}$.*

Proof. Let $f(x) \in \mathfrak{P}_{m_{\mathrm{q}}(x)}$: then, $f(x) = m_{\mathrm{q}}(x)g(x)$ for some $g(x) \in \mathbb{P}_{\mathbb{Q}}[x]$. Since $m_{\mathrm{q}}(x)$ has coefficients in the center of $\mathbb{P}_{\mathbb{Q}}$, we have $f(\mathrm{q}) = m_{\mathrm{q}}(\mathrm{q})g(\mathrm{q}) = 0$. Hence, $f(x) \in \mathfrak{M}_{\mathrm{q}}$; furthermore, $m_{\mathrm{q}}(\mathrm{q}') = 0$ for all $\mathrm{q}' \in$

$C(\mathrm{q})$ (since the elements of $C(\mathrm{q})$ have the same minimal polynomial of q [3, paragraph after Definition 4.1]) and thus $f(x) \in \mathfrak{P}_{p\mathbb{P}_{\mathbb{Z}},\mathrm{q}}$. Therefore $\mathfrak{P}_{m_{\mathrm{q}}(x)}$ is contained in both $\mathfrak{P}_{p\mathbb{P}_{\mathbb{Z}},\mathrm{q}}$ and $\mathfrak{M}_{\mathrm{q}}$.

By intersecting the ideals with $\mathbb{Z}$, it is easily seen that the inclusions are proper. □

When D is a Noetherian commutative domain, the integer-valued polynomials over D behave well with respect to the localization, that is, if S is a multiplicative subset of D then $S^{-1}\operatorname{Int}(D) = \operatorname{Int}(S^{-1}D)$ ([2, Theorem I.2.3]). In [3, Theorem 3.4] an analogous result has been showed for $\operatorname{Int}(\mathbb{P}_{\mathbb{Z}})$ when S is a multiplicatively closed subset $S \subseteq \mathbb{Z}$ (it is central). In the following we prove that $\operatorname{Int}(\mathbb{P}_{\mathbb{Z}})$ behaves well with respect to localization also for denominator sets whose elements are not necessarily central, as $S = \mathscr{C}(Q)$, where Q is a prime ideal of $\mathbb{P}_{\mathbb{Z}}$.

Theorem 2.4. *Let Q be a prime ideal of $\mathbb{P}_{\mathbb{Z}}$ and let $S = \mathscr{C}(Q)$. Then S is also a denominator set of $\operatorname{Int}(\mathbb{P}_{\mathbb{Z}})$ and $\operatorname{Int}(\mathbb{P}_{\mathbb{Z}})S^{-1} = \operatorname{Int}(\mathbb{P}_{\mathbb{Z}}S^{-1})$.*

Proof. To prove that S is a denominator set of $\operatorname{Int}(\mathbb{P}_{\mathbb{Z}})$ it is sufficient to use the same argument of Lemma 1.6 and Proposition 1.11, observing that $\mathrm{N}(s)$ is in the center of $\operatorname{Int}(\mathbb{P}_{\mathbb{Z}})$ for each $s \in S$.

Let Q be a prime ideal of $\mathbb{P}_{\mathbb{Z}}$, and let $Q \cap \mathbb{Z} = p\mathbb{Z}$ (where p is either a prime number or 0). Set $T := \mathbb{Z} \setminus p\mathbb{Z}$. By Propositions 1.12 and 1.13, we have $\operatorname{Int}(\mathbb{P}_{\mathbb{Z}}T^{-1}) = \operatorname{Int}(\mathbb{P}_{\mathbb{Z}}\mathscr{C}(Q)^{-1}) = \operatorname{Int}(\mathbb{P}_{\mathbb{Z}_{(p)}})$.

To prove the statement it is enough to show that

$$(1) \quad \operatorname{Int}(\mathbb{P}_{\mathbb{Z}})T^{-1} \subseteq \operatorname{Int}(\mathbb{P}_{\mathbb{Z}})\mathscr{C}(Q)^{-1} \subseteq \operatorname{Int}(\mathbb{P}_{\mathbb{Z}_{(p)}}) \subseteq \operatorname{Int}(\mathbb{P}_{\mathbb{Z}})T^{-1}.$$

The first inclusion follows from the fact that $T \subseteq \mathscr{C}(Q)$, while the last one from [3, Theorem 3.4] (it is actually an equality). Thus, we only need to prove that $\operatorname{Int}(\mathbb{P}_{\mathbb{Z}})\mathscr{C}(Q)^{-1} \subseteq \operatorname{Int}(\mathbb{P}_{\mathbb{Z}_{(p)}})$. Again by [3, Theorem 3.4], we have $\operatorname{Int}(\mathbb{P}_{\mathbb{Z}}) \subseteq \operatorname{Int}(\mathbb{P}_{\mathbb{Z}_{(p)}})$; furthermore, each element of $\mathscr{C}(Q)$ becomes invertible in $\mathbb{P}_{\mathbb{Z}_{(p)}}$ and thus in $\operatorname{Int}(\mathbb{P}_{\mathbb{Z}_{(p)}})$. Hence, $\operatorname{Int}(\mathbb{P}_{\mathbb{Z}})\mathscr{C}(Q)^{-1} \subseteq \operatorname{Int}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ and all the containments must be equalities. □

Note that the exact same argument can be used if we localize on the left: if S is $S = \mathscr{C}'(Q)$ then $S^{-1}\operatorname{Int}(\mathbb{P}_{\mathbb{Z}}) = \operatorname{Int}(S^{-1}\mathbb{P}_{\mathbb{Z}})$.

Corollary 2.5. *The following hold.*

(1) If $S = \mathcal{R}(\mathbb{P}_{\mathbb{Z}})$ or $S = \mathbb{Z} \setminus (0)$ then $\operatorname{Int}(\mathbb{P}_{\mathbb{Z}})S^{-1} = \operatorname{Int}(\mathbb{P}_{\mathbb{Q}}) = \mathbb{P}_{\mathbb{Q}}[x]$.

(2) If p is a prime number and $S = \mathbb{Z} \setminus p\mathbb{Z}$ or $S = \mathscr{C}(Q)$, with Q a prime ideal of $\mathbb{P}_{\mathbb{Z}}$ such that $Q \cap \mathbb{Z} = p\mathbb{Z}$, then $\operatorname{Int}(\mathbb{P}_{\mathbb{Z}})S^{-1} = \operatorname{Int}(\mathbb{P}_{\mathbb{Z}_{(p)}})$.

Proof. For the first point, the equality $\mathrm{Int}(\mathbb{P}_\mathbb{Z})S^{-1} = \mathrm{Int}(\mathbb{P}_\mathbb{Q})$ follows from Theorem 2.4 and Proposition 1.12. The equality $\mathrm{Int}(\mathbb{P}_\mathbb{Q}) = \mathbb{P}_\mathbb{Q}[x]$ follows directly from the definitions.

Similarly, the second point follows from Theorem 2.4 and from Proposition 1.13. □

These results allow us to represent $\mathbb{P}_\mathbb{Z}$ and $\mathrm{Int}(\mathbb{P}_\mathbb{Z})$ as intersection of localizations.

Proposition 2.6. *Let $\mathcal{P}$ be the set of prime numbers. Then, the following hold.*

(1) $\mathbb{P}_\mathbb{Z} = \bigcap_{p\in\mathcal{P}} \mathbb{P}_{\mathbb{Z}_{(p)}}$.
(2) $\mathrm{Int}(\mathbb{P}_\mathbb{Z}) = \bigcap_{p\in\mathcal{P}} \mathrm{Int}(\mathbb{P}_{\mathbb{Z}_{(p)}})$.

Proof. (1) The inclusion ($\subseteq$) is obvious since for every prime p, $\mathbb{P}_\mathbb{Z} \subseteq \mathbb{P}_{\mathbb{Z}_{(p)}}$. For the reverse inclusion, take an element $\mathrm{q} = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ of the intersection. Then a, b, c, $d \in \bigcap_p \mathbb{Z}_{(p)} = \mathbb{Z}$ and $\mathrm{q} \in \mathbb{P}_\mathbb{Z}$.

(2) For all primes p, let Q_p be the maximal ideal of $\mathbb{P}_\mathbb{Z}$ above p. We have that $\mathrm{Int}(\mathbb{P}_\mathbb{Z}) \subseteq (\mathrm{Int}(\mathbb{P}_\mathbb{Z}))\mathscr{C}(Q_p)^{-1} = \mathrm{Int}(\mathbb{P}_{\mathbb{Z}_{(p)}})$, and thus $\mathrm{Int}(\mathbb{P}_\mathbb{Z})$ is inside the intersection. Conversely, if $f(x)$ belongs to the intersection and $\mathrm{q} \in \mathbb{P}_\mathbb{Z}$, then $f(\mathrm{q}) \in \mathbb{P}_{\mathbb{Z}_{(p)}}$ for every prime number p, and thus $f(\mathrm{q}) \in \bigcap_p \mathbb{P}_{\mathbb{Z}_{(p)}} = \mathbb{P}_\mathbb{Z}$ (by the previous point) and $f(x) \in \mathrm{Int}(\mathbb{P}_\mathbb{Z})$. □

3. Matrix representations

To study the spectrum of $\mathrm{Int}(\mathbb{P}_\mathbb{Z})$, we introduce the related commutative ring

$$\mathrm{Int}_\mathbb{Q}(\mathbb{P}_\mathbb{Z}) := \{f(x) \in \mathbb{Q}[x] \mid \forall \mathrm{q} \in \mathbb{P}_\mathbb{Z} :\ f(\mathrm{q}) \in \mathbb{P}_\mathbb{Z}\},$$

and we define similarly $\mathrm{Int}_\mathbb{Q}(\mathbb{P}_{\mathbb{Z}_{(p)}})$. These sets are easily seen to be rings by using polynomial evaluation. To avoid confusion in the notation, from now in we will go back to write $\mathrm{Int}_{\mathbb{P}_\mathbb{Q}}(\mathbb{P}_\mathbb{Z})$ and $\mathrm{Int}_{\mathbb{P}_\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ for $\mathrm{Int}(\mathbb{P}_\mathbb{Z})$ and $\mathrm{Int}(\mathbb{P}_{\mathbb{Z}_{(p)}})$, respectively. Note that, if we consider $\mathbb{Q}[x]$ as as subring of $\mathbb{P}_\mathbb{Q}[x]$ in the obvious way, then $\mathrm{Int}_\mathbb{Q}(\mathbb{P}_\mathbb{Z}) = \mathrm{Int}_{\mathbb{P}_\mathbb{Q}}(\mathbb{P}_\mathbb{Z}) \cap \mathbb{Q}[x]$.

The relation between $\mathrm{Int}_\mathbb{Q}(\mathbb{P}_\mathbb{Z})$ and $\mathrm{Int}_{\mathbb{P}_\mathbb{Q}}(\mathbb{P}_\mathbb{Z})$ passes through a matrix representation of the rings $\mathbb{P}_\mathbb{Z}$ and $\mathbb{P}_{\mathbb{Z}_{(p)}}$. We denote by $\mathcal{M}_n(R)$ the ring of matrices of order n over R.

Proposition 3.1. [3, Proposition 2.2] *The following hold.*

(1) Let R be a commutative ring with identity such that 2 is a unit of R. Then, $\mathbb{P}_R \cong \mathcal{M}_2(R)$ as R-algebras.
(2) Let $\mathcal{A} = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a \equiv d,\ b \equiv c \mod 2\} \subseteq \mathcal{M}_2(\mathbb{Z})$. Then, $\mathbb{P}_\mathbb{Z} \cong \mathcal{A}$ as $\mathbb{Z}$-algebras.

Let D be a domain with quotient field K. We define

$$\mathrm{Int}_K(\mathcal{M}_n(D)) := \{f(x) \in K[x] \mid \forall A \in \mathcal{M}_n(D) : f(A) \in \mathcal{M}_n(D)\}$$

and

$$\mathrm{Int}_{\mathcal{M}_n(K)}(\mathcal{M}_n(D)) := \\ = \{f(x) \in \mathcal{M}_n(K)[x] \mid \forall A \in \mathcal{M}_n(D) : f(A) \in \mathcal{M}_n(D)\}.$$

These rings roughly correspond, respectively, to $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ and $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$.

Proposition 3.2. *Let D be a domain with quotient field K. Then*

$$\mathrm{Int}_{\mathcal{M}_n(K)}(\mathcal{M}_n(D)) \simeq \mathcal{M}_n(\mathrm{Int}_K(\mathcal{M}_n(D))).$$

Moreover:

(i) The ideals of $\mathrm{Int}_{\mathcal{M}_n(K)}(\mathcal{M}_n(D))$ are in $1-1$ correspondence with the sets of the form $\mathcal{M}_n(\mathscr{I})$, where $\mathscr{I}$ is an ideal of $\mathrm{Int}_K(\mathcal{M}_n(D))$.

(ii) The prime ideals of $\mathrm{Int}_{\mathcal{M}_n(K)}(\mathcal{M}_n(D))$ are in $1-1$ correspondence with the sets of the form $\mathcal{M}_n(\mathscr{P})$, where $\mathscr{P}$ is a prime ideal of $\mathrm{Int}_K(\mathcal{M}_n(D))$.

(iii) The maximal ideals of $\mathrm{Int}_{\mathcal{M}_n(K)}(\mathcal{M}_n(D))$ are in $1-1$ correspondence with the sets of the form $\mathcal{M}_n(\mathscr{M})$, where $\mathscr{M}$ is a maximal ideal of $\mathrm{Int}_K(\mathcal{M}_n(D))$.

Proof. See [4, Theorem 7.2] and [4, Theorem 7.3]. The remaining part follows from [8, Theorem 3.1]. □

Putting together these two results, we have the following theorem.

Theorem 3.3. *Let p be an odd prime integer. Then, the prime ideals of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ are in $1-1$ correspondence with the prime ideals of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$.*

Proof. By Proposition 3.1, $\mathbb{P}_{\mathbb{Q}} \simeq \mathcal{M}_2(\mathbb{Q})$, and the isomorphism brings $\mathbb{P}_{\mathbb{Z}_{(p)}}$ into $\mathcal{M}_2(\mathbb{Z}_{(p)})$. By Proposition 3.2,

$$\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}_{(p)}}) \simeq \mathrm{Int}_{\mathcal{M}_2(\mathbb{Q})}(\mathcal{M}_2(\mathbb{Z}_{(p)})) \simeq \\ \simeq \mathcal{M}_2(\mathrm{Int}_{\mathbb{Q}}(\mathcal{M}_2(\mathbb{Z}_{(p)}))) \simeq \mathcal{M}_2(\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}}));$$

thus the prime ideals of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ are in bijective correspondence with the prime ideals of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$, as claimed. □

The main advantage of this theorem is that $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ is a commutative ring properly contained in between the two well-studied rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$.

Proposition 3.4. *The nonzero prime ideals P of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ such that $P \cap \mathbb{Z} = (0)$ are pairwise uncomparable.*

Proof. It is enough to notice that $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})S^{-1} = \mathbb{Q}[x]$, if $S = \mathbb{Z} \setminus (0)$. Then the prime ideals of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ contracting to (0) in $\mathbb{Z}$ are in 1-1 correspondence with prime ideals of $\mathbb{Q}[x]$, which are not comparable (except (0)). This correspondence preserves the order, thus the thesis follows. $\square$

Theorem 3.5. *The rings* $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ *and* $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$*, for any prime* p*, have dimension* 2*.*

Proof. The ring $\mathbb{Z}[x]$ is a Noetherian ring of dimension 2, and thus by [5, Theorem 30.9 and Corollary 30.10] every overring of $\mathbb{Z}[x]$ has dimension at most 2. Since $\mathbb{Z}[x] \subseteq \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}}) \subseteq \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$, it follows that the dimensions of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ and $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ are at most 2.

Fix now a prime p. By Proposition 2.1, we can find an integer split-quaternion $\mathrm{q} = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \in \mathbb{P}_{\mathbb{Z}}$ such that its minimal polynomial $m_{\mathrm{q}}(x)$ is irreducible modulo p and such that $\mathfrak{P}_{p\mathbb{Z},\mathrm{q}}$ (if $p \neq 2$) or $\mathfrak{M}_{\mathrm{q}}$ (if $p = 2$) is a maximal ideal of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$ over p. Then, its restriction $P_2 := \mathfrak{P}_{p\mathbb{Z},\mathrm{q}} \cap \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ or $P_2 := \mathfrak{M}_{\mathrm{q}} \cap \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ is a prime ideal of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$.

By construction, $m_{\mathrm{q}}(x)$ is irreducible in $\mathbb{Q}[x]$, and thus $\mathfrak{P}_{m_{\mathrm{q}}(x)}$ is a prime ideal of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$, which by Lemma 2.3 is contained inside $\mathfrak{P}_{p\mathbb{Z},\mathrm{q}}$ and $\mathfrak{M}_q$; hence, $P_1 := \mathfrak{P}_{m_{\mathrm{q}}(x)} \cap \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ is a prime ideal of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ inside P_2. Therefore, $(0) \subseteq P_1 \subseteq P_2$ is a chain of prime ideals of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$; furthermore, the inclusions are strict since $m_{\mathrm{q}}(x) \in P_1 \setminus (0)$, while $p \in P_2 \setminus P_1$. Hence, $(0) \subsetneq P_1 \subsetneq P_2$ is a chain of length 2, and thus $\dim \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}}) \geq 2$; therefore, the dimension must be exactly 2.

For $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$, it is enough to note that the chain $(0) \subsetneq P_1 \subsetneq P_2$ lifts to a chain in $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ (since this ring is a localization of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$) and thus also $\dim(\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})) \geq 2$ and $\dim(\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})) = 2$. $\square$

Note that this replicates the same pattern of $\mathrm{Spec}(\mathrm{Int}(\mathbb{Z}))$ shown in [2, Proposition V.2.7]. The correspondence with matrix rings allow also to say something about $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$.

Corollary 3.6. *If* p *is an odd prime, then* $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ *has dimension* 2*. Furthermore,* $\dim(\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})) \geq 2$*.*

Proof. By Theorem 3.3, the dimension of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ is the same of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$, which is 2 by Theorem 3.5. The last claim follows since $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ is a localization of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$. $\square$

Theorem 3.3 does not work for $p = 2$, and thus the previous results do not allow to calculate the dimension of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$. However, we conjecture that $\dim(\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})) = \dim(\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}_{(2)}})) = 2$.

An important difference between $\mathrm{Int}(\mathbb{Z})$ and $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ is that the latter is not integrally closed (and thus it is not a Prüfer domain); see Corollary 3.8 below. However, we can describe its integral closure by using algebraic integers.

Given a finite degree extension $\mathbb{Q}(\theta)$ of $\mathbb{Q}$, we indicate by $\mathcal{A}_\theta$ the ring of algebraic integers of $\mathbb{Q}(\theta)$. If $n \in \mathbb{N}$ is positive, the set of all algebraic integers of degree at most n over $\mathbb{Q}$ is

$$\mathcal{A}_n := \bigcup_{[\mathbb{Q}(\theta):\mathbb{Q}]\leqslant n} \mathcal{A}_\theta;$$

similarly, if p is a prime number, we denote by $\mathcal{A}_{n,p}$ is the set of algebraic numbers that are root of a monic irreducible polynomial of degree n over $\mathbb{Z}_{(p)}$.

In [10] the authors define the set of integer-valued polynomials over $\mathcal{A}_n$ with rational coefficients to be the set

$$\mathrm{Int}(\mathcal{A}_n) := \bigcap_{\theta\in\mathcal{A}_n} \mathrm{Int}_{\mathbb{Q}}(\mathcal{A}_\theta).$$

The ring $\mathrm{Int}(\mathcal{A}_n)$ can be seen as the set of all polynomials with rational coefficients that map $\mathcal{A}_n$ into $\mathcal{A}_n$. They also show that $\mathrm{Int}_{\mathbb{Q}}(\mathcal{A}_n)$ is a Prüfer domain for every n ([10, Theorem 3.9]).

Theorem 3.7. *Let p be an odd prime integer. Then* $\mathrm{Int}_{\mathbb{Q}}(\mathcal{A}_2)_{(p)} = \mathrm{Int}_{\mathbb{Q}}(\mathcal{A}_{2,p})$ *is the integral closure of* $\mathrm{Int}_{\mathbb{Q}}\left(\mathbb{P}_{\mathbb{Z}_{(p)}}\right)$ *in* $\mathbb{Q}[x]$.

Proof. By [10, Theorem 4.6], $\mathrm{Int}_{\mathbb{Q}}(\mathcal{A}_2)$ is the integral closure of $\mathrm{Int}_{\mathbb{Q}}(\mathcal{M}_2(\mathbb{Z}))$. Using Proposition 3.1, and recalling that the localization at prime integers preserves the integral closure, we have that:

$$\begin{aligned}\mathrm{Int}_{\mathbb{Q}}(\mathcal{A}_2)_{(p)} &= \overline{\mathrm{Int}_{\mathbb{Q}}(\mathcal{M}_2(\mathbb{Z}))}_{(p)} = \overline{\mathrm{Int}_{\mathbb{Q}}(\mathcal{M}_2(\mathbb{Z}))_{(p)}} = \\ &= \overline{\mathrm{Int}_{\mathbb{Q}}(\mathcal{M}_2(\mathbb{Z})_{(p)})} = \overline{\mathrm{Int}_{\mathbb{Q}}\left(\mathcal{M}_2(\mathbb{Z}_{(p)})\right)} = \overline{\mathrm{Int}_{\mathbb{Q}}\left(\mathbb{P}_{\mathbb{Z}_{(p)}}\right)}.\end{aligned}$$

Finally, using [13, Theorem 13] with $A = \mathbb{P}_{\mathbb{Z}_{(p)}}$, we have that $\overline{\mathrm{Int}_{\mathbb{Q}}\left(\mathbb{P}_{\mathbb{Z}_{(p)}}\right)}$ is also the integral closure of $\mathrm{Int}_{\mathbb{Q}}(\mathcal{A}_{2,p})$. □

Corollary 3.8. *The ring* $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ *is not integrally closed.*

Proof. If $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ is integrally closed, then its localization at an odd prime p, $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$, is integrally closed too. Thus, from Theorem 3.7, $\overline{\mathrm{Int}_{\mathbb{Q}}\left(\mathbb{P}_{\mathbb{Z}_{(p)}}\right)} = \mathrm{Int}_{\mathbb{Q}}(\mathcal{A}_2)_{(p)}$ and this is Prüfer. Since $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}}) \cong \mathrm{Int}_{\mathbb{Q}}(M_2(\mathbb{Z}_{(p)}))$, it follows that the ring

$$\mathrm{Int}_{\mathbb{Q}}(B, M_2(\mathbb{Z}_{(p)})) := \{f \in \mathbb{Q}[x] \mid f(B) \in M_2(\mathbb{Z}_{(p)})\}$$

is an overring of $\mathrm{Int}_{\mathbb{Q}}(M_2(\mathbb{Z}_{(p)}))$, for every matrix $B \in M_2(\mathbb{Z}_{(p)})$. Taking $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and arguing as in [10, §4], it can be shown that $\mathrm{Int}_{\mathbb{Q}}(B, M_2(\mathbb{Z}_{(p)}))$ is not integrally closed. The claim follows. □

4. The ideal $p\,\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$

In this section we study in more detail the ideal $p\,\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ generated by a prime number p (not necessarily odd). Our first result can be seen as a refinement of the proof of Theorem 3.5.

Proposition 4.1. *Let p be a prime number. Then, every prime ideal of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ containing p is maximal.*

Proof. We follow the proof of [2, Lemma V.1.9].

Let $u_1, \dots, u_k$ be a set of residues of $\mathbb{P}_{\mathbb{Z}}/p\mathbb{P}_{\mathbb{Z}}$ (with $k = p^4$), and let P be a prime ideal of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$ containing p. Take any $a(x) \in \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})$, and let $a_i(x) := a(x) - u_i$. Let $b(x) := a_1(x) \cdots a_k(x)$: by construction, for every $\mathrm{q} \in \mathbb{P}_{\mathbb{Z}}$ there is an i such that $a(\mathrm{q}) \equiv u_i \bmod p\mathbb{P}_{\mathbb{Z}}$.

Since the a_i have coefficients in the commutative ring $\mathbb{Q}$, we have $b(\mathrm{q}) = a_1(\mathrm{q}) \cdots a_k(\mathrm{q})$: hence, $b(\mathrm{q}) \in p\mathbb{P}_{\mathbb{Z}}$ and so $b(x) \in p\,\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}}) \subseteq P$; since P is prime, there must be an i such that $a_i(x) \in P$. However, $a_i(x) \equiv u_i \bmod P$, and thus $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}})/P$ is isomorphic to $\mathbb{P}_{\mathbb{Z}}/p\mathbb{P}_{\mathbb{Z}} \simeq \mathbb{P}_{\mathbb{Z}_p}$. Hence, P is maximal, as claimed. □

Corollary 4.2. *Let p be an odd prime integer. Then, every prime ideal of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ containing p is maximal.*

Proof. It is enough to use Proposition 4.1 and the correspondence of Theorem 3.3. □

Remark 4.3.

(1) The previous two results allow to give an alternative proof of Theorem 3.5. Indeed, if $(0) \subsetneq Q_1 \subsetneq Q_2 \subsetneq Q_3$ is a chain of prime ideals of length 3, then either $Q_1 \cap \mathbb{Z} = Q_2 \cap \mathbb{Z} = (0)$ or $Q_2 \cap \mathbb{Z} = Q_3 \cap \mathbb{Z} = p\mathbb{Z}$, for some prime number p. The latter case is made impossible by Proposition 4.1 (as Q_2 contains p but is not maximal); on the other hand the former case would imply that two nonzero prime ideals of $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$ over (0) are comparable, against Proposition 3.4.

(2) The proof of Proposition 4.1 does *not* work in the ring $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$, since the evaluation of a product of polynomials cannot be done separately for each factor, and thus $b(\mathrm{q}) \neq a_1(\mathrm{q}) \cdots a_k(\mathrm{q})$ in general. Nevertheless, we conjecture (but we don't have a proof) that the same property holds also in $\mathrm{Int}_{\mathbb{P}_{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Z}})$.

A consequence of Proposition 4.1 is that the ideals $p\,\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ are not prime. We now want to find an explicit description of the polynomials in $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ and, as a corollary, to find two polynomials outside $p\,\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ whose product is inside the ideal.

Proposition 4.4. *Let R, S be commutative rings and let $\pi : R \longrightarrow S$ be a homomorphism. Then, the natural map*

$$\begin{aligned} \varphi\colon \mathbb{P}_R &\longrightarrow \mathbb{P}_S \\ a + b\,\boldsymbol{i} + c\,\boldsymbol{j} + d\,\boldsymbol{k} &\longmapsto \pi(a) + \pi(b)\,\boldsymbol{i} + \pi(c)\,\boldsymbol{j} + \pi(d)\,\boldsymbol{k} \end{aligned}$$

is a ring homomorphism. Furthermore, if π is surjective then φ is surjective and $\ker\varphi = (\ker\pi)\mathbb{P}_R = \mathbb{P}_{\ker\pi} = \{a+b\,\boldsymbol{i}+c\,\boldsymbol{j}+d\,\boldsymbol{k} \mid a,b,c,d \in \ker\pi\}$*; in particular, $\mathbb{P}_R/\ker\varphi \simeq \mathbb{P}_S$.*

Proof. Straightforward. $\square$

An important particular case is when $R = \mathbb{Z}$ or $R = \mathbb{Z}_{(p)}$ and $S = \mathbb{Z}_p$: in this case, the kernel of π is generated by p, and thus we obtain the well-known isomorphisms $\dfrac{\mathbb{P}_{\mathbb{Z}}}{p\mathbb{P}_{\mathbb{Z}}} \simeq \dfrac{\mathbb{P}_{\mathbb{Z}_{(p)}}}{p\mathbb{P}_{\mathbb{Z}_{(p)}}} \simeq \mathbb{P}_{\mathbb{Z}_p}$.

In particular, the previous proposition shows that polynomial evaluation behaves well with respect to quotients. Given a surjection $\pi : R \longrightarrow S$ and a polynomial $f(x) = \sum_{t=0}^{n} \mathrm{p}_t\, x^t \in R[x]$, we denote by $\overline{f}(x) = \sum_{t=0}^{n} \pi(\mathrm{p}_t)\, x^t \in S[x]$ the polynomial obtained by reducing the coefficients modulo $\ker\varphi$. Then, for every $\mathrm{q} \in \mathbb{P}_{\mathbb{Z}}$, we have $\pi(f(\mathrm{q})) = \overline{f}(\pi(\mathrm{q}))$.

Proposition 4.5. *Let p be a prime integer. Let $f(x) \in \mathbb{Z}[x]$ and $\overline{f}(x) \in \mathbb{Z}_p[x]$ be as above. Given an integer $n > 1$ such that $n = p^{\alpha}m$ with $p \nmid m$, then $\frac{1}{n}f(x) \in \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ if and only if $f(\mathrm{q}) \in p^{\alpha}\mathbb{P}_{\mathbb{Z}_{(p)}}$, for all $\mathrm{q} \in \mathbb{P}_{\mathbb{Z}_{(p)}}$. In particular if $\alpha = 1$, $\frac{1}{n}f(x) \in \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ if and only if $\overline{f}(\mathrm{q}) = \overline{0}$ in $\mathbb{P}_{\mathbb{Z}_p}$, for all $\mathrm{q} \in \mathbb{P}_{\mathbb{Z}_p}$.*

Proof. We have that

$$\frac{1}{n}f(x) \in \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}}) \Leftrightarrow \frac{1}{n}f(\mathrm{q}) \in \mathbb{P}_{\mathbb{Z}_{(p)}}\ \forall q \in \mathbb{P}_{\mathbb{Z}_{(p)}} \Leftrightarrow f(\mathrm{q}) \in n\mathbb{P}_{\mathbb{Z}_{(p)}}\ \forall q \in \mathbb{P}_{\mathbb{Z}_{(p)}}.$$

Since $p \nmid m$, $n\mathbb{P}_{\mathbb{Z}_{(p)}} = p^{\alpha}\mathbb{P}_{\mathbb{Z}_{(p)}}$.

$\square$

Lemma 4.6. *Let R be a commutative domain. Take $\mathrm{q} \in \mathbb{P}_R \setminus R$ and let $m_{\mathrm{q}}(x) \in R[x]$ be its minimal polynomial over R. If a polynomial $f(x) \in R[x]$ is such that $f(\mathrm{q}) = 0$, then $m_{\mathrm{q}}(x) \mid f(x)$ in $R[x]$.*

Proof. Since $m_{\mathrm{q}}(x)$ is monic we can divide $f(x)$ by $m_{\mathrm{q}}(x)$ obtaining

$$f(x) = g(x)m_{\mathrm{q}}(x) + r(x),$$

for some $g(x), r(x) \in R[x]$. In particular $r(x) = ax + b$ is linear as $m_{\mathrm{q}}(x)$ is of degree two. Since $R[x]$ is contained in the center of $\mathbb{P}_R[x]$, we can evaluate the polynomial relation above in q, obtaining $0 = f(\mathrm{q}) = g(\mathrm{q}) \cdot 0 + a\mathrm{q} + b$. Since R is a domain and $\mathrm{q} \notin R$, necessarily $a = b = 0$. □

We observe that Lemma 4.6 does not hold if $f(x) \in \mathbb{P}_R[x] \setminus R[x]$. For example, consider $\mathbf{i} \in \mathbb{P}_\mathbb{Z}$ and $f(x) = x^3 + \mathbf{i}x + (\mathbf{i} + 1)x + \mathbf{i} + 1$. Then $f(\mathbf{i}) = 0$ but $f(x) = (x^2 + 1)(x + \mathbf{i}) + \mathbf{i}x + 1$ and the remainder is nonzero.

Corollary 4.7. *With the hypothesis and notation of Proposition 4.5, let p be a prime integer and $n = pm$ with $p \nmid m$. Then $\frac{1}{n}f(x) \in$ $\mathrm{Int}_\mathbb{Q}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ if and only if $\overline{f}(x)$ is divided by all the minimal polynomials of the elements of $\mathbb{P}_{\mathbb{Z}_p}$.*

Proof. It is an immediate consequence of Proposition 4.5 and Lemma 4.6. □

Using the previous Corollary we can construct a nontrivial element of $\mathrm{Int}_\mathbb{Q}(\mathbb{P}_{\mathbb{Z}_{(p)}})$.

Example 4.8. The polynomial

$$\Phi_p(x) = \frac{1}{p}(x^p - x)(x^{p^2} - x)$$

belongs to $\mathrm{Int}_\mathbb{Q}(\mathbb{P}_{\mathbb{Z}_{(p)}})$.

By Proposition 4.5, it is sufficient to show that $f(x) = (x^p - x)(x^{p^2} - x) \in \mathbb{Z}[x]$ vanishes over all elements of $\mathbb{P}_{\mathbb{Z}_p}$. Observe that every monic and irreducible polynomial of $\mathbb{Z}_p[x]$ of degree one or two is a factor of $f(x)$. In particular, if $g(x)$ is a linear polynomial then $g(x)^2$ divides $f(x)$, since $g(x)$ divides both $x^p - x$ and $x^{p^2} - x$. By Corollary 4.7, this also means that the minimal polynomial of every split-quaternion of $\mathbb{P}_{\mathbb{Z}_p}$ is a factor of $f(x)$.

In particular we can show that every monic and quadratic polynomial of $\mathbb{Z}_p[x]$ is the minimal polynomial for some element of $\mathbb{P}_{\mathbb{Z}_p}$. The proof is *mutatis mutandis* the same as the proof of [14, Lemma 3.5]. This means that the polynomial $\Phi_p(x)$ does not contain any redundant factor.

Proposition 4.9. *With the above notation we have the following proper inclusions:*

$$\mathbb{Z}_{(p)}[x] \subsetneq \mathrm{Int}_\mathbb{Q}(\mathbb{P}_{\mathbb{Z}_{(p)}}) \subsetneq \mathrm{Int}(\mathbb{Z}_{(p)}).$$

Proof. The first inclusion follows from the fact that $\mathbb{Z} \subseteq \mathbb{P}_{\mathbb{Z}}$ and thus $\mathbb{Z}_{(p)}[x] \subseteq \mathbb{P}_{\mathbb{Z}_{(p)}}[x] \cap \mathbb{Q}[x] \subseteq \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$. It is proper since the polynomial $\Phi_p(x)$ given in Example 4.8 belongs to $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ but not to $\mathbb{Z}_{(p)}[X]$.

The second inclusion is straightforward since $\mathbb{Z}_{(p)} = \mathbb{P}_{\mathbb{Z}_{(p)}} \cap \mathbb{Q}$.

To see that it is proper, consider the "binomial polynomial" $f(x) = \frac{x(x-1)(x-2)\ldots(x-p+1)}{p} \in \mathrm{Int}(\mathbb{Z}_{(p)})$. If $p = 2$ then $f(\mathbf{i}) = \frac{-1-\mathbf{i}}{2} \notin \mathbb{P}_{\mathbb{Z}_{(2)}}$; if p is odd then $pf(x)$ is not divided by x^2+1 (the minimal polynomial of $\mathbf{i}$), and thus $f(\mathbf{i}) \notin \mathbb{P}_{\mathbb{Z}_{(p)}}$ by Corollary 4.7. It follows that $f(x) \notin \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$. □

The fact that the two containments of the previous proposition are strict also follows from [11, Theorem 2.12] (the first one) and [12, Theorem 2.11] (the second one).

Proposition 4.10. *The ideal* $p\,\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ *is not a prime ideal of* $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$.

Proof. Let us consider the polynomials:

$$f(x) = (x^p - x)^2 \in \mathbb{Z}[x],$$
$$g(x) = \frac{1}{p}(x^{p^2} - x)^2 \in \mathbb{Q}[x],$$
$$F(x) = f(x)g(x) = \frac{1}{p}(x^p - x)^2(x^{p^2} - x)^2 \in \mathbb{Q}[x].$$

These three polynomials are elements of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$. Indeed, for $f(x)$ it follows from the inclusion $\mathbb{Z}[x] \subseteq \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$. For $F(x)$ and $g(x)$ observe that they are equal to $\Phi_p(x)$ (Example 4.8) multiplied by a polynomial with integer coefficients.

We claim that $F(x) \in p\,\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ while $f(x)$ and $g(x)$ do not belong to this ideal.

Indeed, $\frac{1}{p}F(x) = (\Phi_p(x))^2 \in \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ and thus $F(x) \in p\,\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$.

As regards $f(x)$, we have that $\overline{f}(x)$ is not divisible by any quadratic irreducible polynomial over $\mathbb{Z}_p$, and thus by Corollary 4.7 $\frac{1}{p}f(x) \notin \mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$.

For $g(x)$, consider $\frac{1}{p}g(x) = \frac{1}{p^2}(x^{p^2} - x)^2$. If $p = 2$ then $\frac{1}{2}g(\mathbf{i}) = -\frac{\mathbf{i}}{2} \notin \mathbb{P}_{\mathbb{Z}_{(2)}}$. If p is odd, then we set $\mathrm{q} := \mathbf{i} + (p-1)\,\mathbf{k}$. We have that $\mathrm{q}^2 = p^2 - 2p$, and if we raise q to an even power greater than 2, we obtain an integer divisible by p^2. Since $\frac{1}{p}g(x)$ is a central polynomial, we can evaluate it in q using its factorization. Thus, we have

$$\frac{1}{p}g(\mathrm{q}) = \frac{(\mathrm{q}^{p^2} - q)^2}{p^2} = \frac{\mathrm{q}^{2p^2} + \mathrm{q}^2 - 2\mathrm{q}^{p^2+1}}{p^2} = m + \frac{p-2}{p} \notin \mathbb{P}_{\mathbb{Z}_{(p)}}$$

for some $m \in \mathbb{Z}$.

Since $F(x) = f(x)g(x)$, we can conclude that $p\,\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$ is not a prime ideal of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}_{(p)}})$. □

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