Multiplicative properties of integer valued polynomials over split-quaternions
MULTIPLICATIVE PROPERTIES OF INTEGER VALUED POLYNOMIALS OVER SPLIT-QUATERNIONS

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Abstract. We study localization properties and the prime spectrum of the integer-valued polynomial ring \( \text{Int}_{\mathbb{P}_Z}(\mathbb{P}_Z) \), where \( \mathbb{P}_Z \) (respectively \( \mathbb{P}_Q \)) is the algebra of split-quaternion over \( \mathbb{Z} \) (respectively over \( \mathbb{Q} \)).

Introduction

In [14] N. Werner studied the ring of integer-valued polynomials in a noncommutative setting, by considering quaternion algebras. Precisely, he considered the algebras \( \mathbb{H}_Z \) and \( \mathbb{H}_Q \) (respectively over \( \mathbb{Z} \) and over \( \mathbb{Q} \)) generated by the unit elements 1, \( i, j, k \), linked by the relations \( i^2 = j^2 = k^2 = -1 \), \( ij = k = -ji \), \( jk = i = -kj \) and \( ki = j = -ik \), and considered the set \( \text{Int}_{\mathbb{H}_Q}(\mathbb{H}_Z) \) of all polynomials \( f \in \mathbb{H}_Q[x] \) such that \( f(\mathbb{H}_Z) \subseteq \mathbb{H}_Z \). After proving that \( \text{Int}_{\mathbb{H}_Q}(\mathbb{H}_Z) \) is indeed a noncommutative ring (which strictly contains \( \mathbb{H}_Q[x] \)), he investigated the ideal structure of this ring, describing some prime ideals above the zero and the maximal ideals of \( \mathbb{H}_Z \).

Moving from these ideas, in [3] A. Cigliola, K.A. Loper and N. Werner focused on similar problems in a different setting: instead of \( \mathbb{H}_Z \) they considered the set of integer split-quaternions \( \mathbb{P}_Z \), i.e. the \( \mathbb{Z} \)-algebra generated by the unit elements 1, \( i, j, k \) with the relations \( -i^2 = j^2 = k^2 = 1 \) and \( ijk = 1 \) (see Definition 1.1).

In this paper, we continue the study of the ring \( \mathbb{P}_Z \) (Section 1) and of the ring \( \text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Z) \) of integer-valued polynomials over \( \mathbb{P}_Z \) (Section 2). We study some denominator sets of \( \mathbb{P}_Z \) and \( \text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Z) \) that are not subsets of \( \mathbb{Z} \) (in particular, they are not central) and their ring of fractions. Thus, we partially answer to one of the open questions posed in [3, §5] which asks whether it is possible to find and to localize \( \text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Z) \) with respect to noncentral sets. We then study the ring

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Int\(_Q(P_Z)\) of the polynomials in \(Q[x]\) that are integer valued over \(P_Z\).

There is a strict connection between the prime spectrum of this ring and the prime spectrum of \(\text{Int}_{P_Q}(P_Z)\). This allows to calculate the Krull dimension of \(\text{Int}_{P_Q}(P_{Z(p)})\), for an odd prime integer \(p\), starting from the dimension of \(\text{Int}_Q(P_Z)\) and thus to get a partial but interesting information about the Krull dimension of \(\text{Int}_{P_Q}(P_Z)\). Finally, in Section 4, we study in more detail the ideal \(p\text{Int}_Q(P_Z)\) and show that it is not prime. In this last Section we will be able to construct explicitly some polynomials of \(\text{Int}_Q(P_{Z(p)})\).

Throughout the paper, all the rings we consider are unitary but not necessarily commutative.

1. LOCALIZATIONS OF \(P_Z\)

We recall some definitions and basic properties.

**Definition 1.1.** Let \(R\) be a commutative ring. We denote by \(P_R\) the \(R\)-algebra generated by the four unit elements 1, \(i\), \(j\) and \(k\) with the relations

\[ -i^2 = j^2 = k^2 = ijk = 1. \]

Formally \(P_R := \{q = a + bi + cj + dk \mid a, b, c, d \in R\}\).

We call \(P_R\) the **ring of split-quaternions** over \(R\).

Let \(q = a + bi + cj + dk \in P_R\), then:

(a) \(a, b, c,\) and \(d\) are the **coefficients** of \(q\), and \(a\) is the **real part** of \(q\);
(b) the **bar conjugate** of \(q\) is \(\overline{q} := a - bi - cj - dk\);
(c) the **norm** of \(q\) is \(N(q) := q\overline{q} = a^2 + b^2 - c^2 - d^2\);
(d) the **trace** of \(q\) is \(T(q) = q + \overline{q} = 2a\);
(e) the **minimal polynomial** of \(q\) is ([3, Definition 2.4])

\[ m_q(x) := \begin{cases} x - q & \text{if } q \in R \\ x^2 - T(q)x + N(q) & \text{if } q \in P_R \setminus R. \end{cases} \]

\(m_q(x)\) is minimal in the way that \(m_q(q) = 0\) and that \(m_q(x)\) is the monic polynomial of least degree having \(q\) as a root.

In this section, we study some localizations of \(P_Z\). We start with the description of its prime and maximal ideals. Recall that an ideal \(P\) of a (not necessarily commutative) ring \(R\) is **prime** if, given \(a, b \in R\) such that \(aPb \subseteq P\), then \(a \in P\) or \(b \in P\).

**Theorem 1.2.** [3, Theorem 2.11]. The prime ideals of \(P_Z\) are:

(i) \(0\);
(ii) \(pP_Z\) where \(p\) is an odd prime of \(\mathbb{Z}\);
(iii) $\mathcal{M} = (1 + i; 1 + j)$.

Moreover, the primes $p\mathbb{P}_Z$ and $\mathcal{M}$ are maximal, and $\mathcal{M}$ is the only prime ideal containing 2.

Lemma 1.3. Let $q \in \mathbb{P}_Z$ such that $2 \mid N(q)$. Then $q \in \mathcal{M}$. In particular $\mathcal{M}$ contains all the zero-divisors of $\mathbb{P}_Z$.

Proof. Let $q = a + bi + cj + dk$ be such that $N(q) = a^2 + b^2 - c^2 - d^2 = 2m$, for some $m \in \mathbb{Z}$. By hypothesis, $q$ must have zero, two or four even coefficients. In the case that all coefficients are even, then trivially $q \in (2) \subseteq \mathcal{M}$. If $q$ has exactly two even coefficients, then $q$ is congruent modulo $2\mathbb{P}_Z$ to the sum of two of $1, i, j$ and $k$, and all of them are elements of $\mathcal{M}$. Finally, if all coefficients of $q$ are odd, then $q \equiv 1 + i + j + k (\mod 2\mathbb{P}_Z)$, and so $q \in \mathcal{M}$ since $1 + i + j + k = (1 + i)(1 + j) \in \mathcal{M}$. □

Definition 1.4. Let $R$ be a ring and $S$ a multiplicative subset in $R$. We say that $S$ is a right denominator set if:

(i) for any $a \in R$ and $s \in S$, $aS \cap sR \neq \emptyset$ (this condition is known as right Ore condition and $S$ is called a right Ore set);

(ii) for $a \in R$, if $s'a = 0$ for some $s' \in S$, then $as = 0$ for some $s \in S$ (we say that $S$ is right reversible).

Remark 1.5. (a) We can define left denominator sets in a completely symmetrical way.

(b) Condition (ii) (reversibility) is automatically satisfied when $S$ does not contain zero-divisors.

(c) It is easily seen that the multiplicative subsets contained in the center of $R$ are always denominator subsets.

By [9, Theorem 10.6], if $R$ is a ring and $S$ a multiplicative subset of $R$, then $R$ has a right ring of fractions with respect to $S$ (namely, the ring $RS^{-1} := \{ as^{-1} \mid a \in R, s \in S \}$ if and only if $S$ is a right denominator set. Similarly we can construct the ring $S^{-1}R := \{ s^{-1}a \mid a \in R, s \in S \}$ if and only if $S$ is a left denominator set. If $S$ is both a right and left denominator set, then $RS^{-1} \simeq S^{-1}R$ by [9, Corollary 10.14].

Lemma 1.6. Let $R$ be a commutative ring and $S$ a multiplicative subset of $\mathbb{P}_R$, closed under norm (i.e., if $s \in S$ then $N(s) \in S$). Then $S$ verifies both the right and the left Ore condition.

Proof. Fix $a \in \mathbb{P}_R$ and $s \in S$. Since $N(s) \in R$ is a central element, we have that $aN(s) = N(s)a$. It follows that $aN(s) = s(\bar{s}a)$, so $S$ is a right Ore set since $aS \cap s\mathbb{P}_R \neq \emptyset$. Analogously, $(a\bar{s})s = N(s)a$ so $S$ is a left Ore set since $Sa \cap \mathbb{P}_{RS} \neq \emptyset$. □
By the previous lemma, if \( S = \mathcal{R}(R) \) is the set of all (right and left) regular elements of \( R \), then \( S \) is a denominator set and \( RS^{-1} \) is the total ring of fractions of \( R \), which we denote by \( \mathcal{Q}(R) \).

For commutative rings, the most important way of constructing localizations of a ring \( R \) is through the sets \( R \setminus P \), where \( P \) is a prime ideal; however, if \( R \) is not commutative, the complement of a prime ideal may not be multiplicatively closed. For example, if \( p = 2k + 1 \) is an odd prime number, then \( p \mathbb{P}_{\mathbb{Z}} \) is prime, but \( \mathbb{P}_{\mathbb{Z}} \setminus p \mathbb{P}_{\mathbb{Z}} \) is not multiplicatively closed since \( ((k + 1) + k j)((k + 1) - k j) = p \in p \mathbb{P}_{\mathbb{Z}} \).

Following the notation of Goldie in [6], we give the following definition:

**Definition 1.7.** Let be given a ring \( R \) and let \( Q \) be a proper prime ideal of \( R \). We set:

\[
\mathcal{C}(Q) := \{ x \in R \mid xr \notin Q, \forall r \notin Q \},
\]

and

\[
\mathcal{C}'(Q) := \{ x \in R \mid rx \notin Q, \forall r \notin Q \}.
\]

**Proposition 1.8.** Let \( R \) be a ring and let \( Q \subseteq R \) be a prime ideal of \( R \). Then \( \mathcal{C}(Q) \) is a multiplicatively closed subset of \( R \) containing 1 but not 0, and \( \mathcal{C}(Q) \subseteq R \setminus Q \). The same properties hold for \( \mathcal{C}'(Q) \).

**Proof.** For each \( r \notin Q \), we have that \( 1 \cdot r = r \notin Q \) and that \( 0 \cdot r \in Q \). Then, by definition, \( 1 \in \mathcal{C}(Q) \) and \( 0 \notin \mathcal{C}(Q) \). Take now \( a, b \in \mathcal{C}(Q) \) and \( r \notin Q \). Since \( b \in \mathcal{C}(Q) \), then \( br \notin Q \). Again, since \( a \in \mathcal{C}(Q) \), we have \( a(br) \notin Q \). Thus for all \( r \notin Q \) we have \((ab)r = a(br) \notin Q \).

Finally, if \( x \in \mathcal{C}(Q) \) then, since \( 1 \notin Q \), we have \( x \cdot 1 = x \notin Q \). Hence, \( \mathcal{C}(Q) \subseteq R \setminus Q \). \( \square \)

**Proposition 1.9.** Let \( R \) be a ring and let \( Q \subseteq R \) be a prime ideal of \( R \). Then \( \mathcal{C}(Q) \) is the set of left regular elements of \( R \) modulo \( Q \) and \( \mathcal{C}'(Q) \) is the set of right regular elements of \( R \) modulo \( Q \).

**Proof.** Take \( x \in R \). Then \( x \) is a left zero-divisor modulo \( Q \) if and only if there is \( r \in R/Q \), \( r \neq 0 \), such that \( xr = 0 \). This is equivalent to saying that there is an \( r \notin Q \) such that \( xr \in Q \). In other words, \( x \notin \mathcal{C}(Q) \).

Similarly for \( \mathcal{C}'(Q) \). \( \square \)

In particular, we have that \( \mathcal{C}(0) = \mathcal{R}_l(R) \) is the set of the left regular elements of \( R \), while \( \mathcal{C}'(0) = \mathcal{R}_r(R) \) is the set of the right regular elements of \( R \).
We now focus on some properties of the sets $\mathcal{C}(Q)$ associated to the prime ideals of $\mathbb{P}_Z$.

**Proposition 1.10.** Let $Q$ be a prime ideal of $\mathbb{P}_Z$. Then:

(i) $\mathcal{C}(Q)$ is closed under bar conjugation;
(ii) $\mathcal{C}(Q)$ is closed under norm;
(iii) $\mathcal{C}(Q) = \{ x \in \mathbb{P}_Z \mid N(x) \notin Q \}$;
(iv) $\mathcal{C}(Q)$ does not contain any zero-divisor.

**Proof.** By [7, Proposition 1.6] $\mathbb{P}_Z$ is a Noetherian ring. Thus, from [6, Section 3], $\mathcal{C}(Q) = \mathcal{C}'(Q)$.

Consider first $Q = (0)$. Then $\mathcal{C}(0)$ equals $\mathcal{R}(\mathbb{P}_Z)$, the set of all (two-sided) regular elements, and so

$$\mathcal{C}(Q) = \mathcal{R}(\mathbb{P}_Z) = \{ x \in \mathbb{P}_Z \mid N(x) \neq 0 \}.$$

This proves the claim in the case $Q = (0)$.

Let now be $Q = p\mathbb{P}_Z$, for an odd prime integer $p$. We notice that:

- $\mathcal{C}(Q) \mod Q = \mathcal{C}(0)$ in $\mathbb{P}_Z/Q = \mathbb{P}_{Zp}$ (apply Proposition 1.9);
- $N(x) \mod p = N(\bar{x})$, for $x \in \mathbb{P}_Z$ and $\bar{x} = x \mod p\mathbb{P}_Z$.

Using these equalities, points (i)-(ii)-(iii) reduce to the case $Q = (0)$, which has been already proved. For $p = 2$, the same reasoning applies reducing modulo $\mathcal{M}$.

For the point (iv), if $p = 2$ the claim follows from Lemma 1.3.

If $p$ is an odd prime, then suppose that $x r' = 0$, for some $x \in \mathcal{C}(Q)$ and $0 \neq r' \in \mathbb{P}_Z$. If we write $r' = p^m r$, for some $r \notin Q$, we get $x r = 0 \in Q$ (since $p$ is not a zero divisor from Lemma 1.3) which is absurd. □

In particular, we observe that $\mathcal{C}(p\mathbb{P}_Z) = \{ x \in \mathbb{P}_Z \mid p \nmid N(x) \}$ and $\mathcal{C}(\mathcal{M}) = \{ x \in \mathbb{P}_Z \mid 2 \nmid N(x) \}$.

We will work with the following multiplicative subsets of $\mathbb{P}_Z$:

- the multiplicative subsets of $\mathbb{Z}$;
- the sets $\mathcal{C}(0)$, $\mathcal{C}(\mathcal{M})$ and $\mathcal{C}(p\mathbb{P}_Z)$, for any odd prime integer $p$.

For a general noncommutative ring, given a prime ideal $Q$, $\mathcal{C}(Q)$ may not be a denominator set: such an example is given, for instance, in [1, Example 2.3]. However we show that $\mathcal{C}(Q)$ is a denominator sets in $\mathbb{P}_Z$ and also in $\text{Int}(\mathbb{P}_Z)$ (Proposition 2.4), for each prime ideal $Q$ of $\mathbb{P}_Z$.

**Proposition 1.11.** The sets $\mathbb{Z} \setminus (0)$, $\mathbb{Z} \setminus p\mathbb{Z}$, for $p$ prime, and $\mathcal{C}(Q)$, for $Q$ prime ideal of $\mathbb{P}_Z$, are (right and left) denominator sets of $\mathbb{P}_Z$. 
Proof. Let $S = \mathbb{Z} \setminus \{0\}$ or $S = \mathbb{Z} \setminus p\mathbb{Z}$, for a prime $p$. Then the statement easily follows from the fact that $S$ is contained in the center of $\mathbb{P}_Z$.

If $S = \mathcal{C}(Q)$, then $S$ does not contain zero-divisors (Proposition 1.10), so $\mathcal{C}(Q)$ is right and left reversible. Finally, $\mathcal{C}(Q)$ is a right (left) Ore set by Lemma 1.6, since it is closed under bar conjugation (Proposition 1.10). Thus $\mathcal{C}(Q)$ is a right and left denominator set of $\mathbb{P}_Z$. □

**Proposition 1.12.** Let $S = \mathcal{C}(0)$ or $S = \mathbb{Z} \setminus \{0\}$. Then

$$\mathbb{P}_Z S^{-1} = S^{-1} \mathbb{P}_Z = \mathbb{P}_Q = \mathcal{Q}(\mathbb{P}_Z),$$

which is the total ring of fractions of $\mathbb{P}_Z$.

Proof. By Proposition 1.11, $S$ is a denominator set. So the ring $\mathbb{P}_Z S^{-1}$ exists and its elements are the fractions $rs^{-1}$, where $r,s \in \mathbb{P}_Z$ and $N(s) \neq 0$. Then $rs^{-1} = \frac{1}{N(s)} r s \in \mathbb{P}_Q$. Thus $\mathbb{P}_Z S^{-1} \subseteq \mathbb{P}_Q$. Conversely, given $q \in \mathbb{P}_Q$, write $q$ in the form $p \cdot a^{-1}$, where $p \in \mathbb{P}_Z$ and $a$ is a common denominator for the coefficients of $q$. Obviously, $a \in S$ and so $pa^{-1} \in \mathbb{P}_Z S^{-1}$, i.e., $\mathbb{P}_Z S^{-1} \supseteq \mathbb{P}_Q$. Thus $\mathbb{P}_Z S^{-1} = \mathbb{P}_Q$. Similarly, $S^{-1} \mathbb{P}_Z = \mathbb{P}_Q$. Finally $\mathbb{P}_Q$ is the total ring of fractions of $\mathbb{P}_Z$ because we localize with respect to the set of regular elements of $\mathbb{P}_Z$. □

Similarly, if we localize $\mathbb{P}_Z$ at $S = \mathbb{Z} \setminus p\mathbb{Z}$ or $S = \mathcal{C}(Q)$, where $Q = p\mathbb{P}_Z$, for a prime number $p$, we get the algebra of split-quaternions with coefficients in $\mathbb{Z}_p$, the localization of $\mathbb{Z}$ at the ideal $p\mathbb{Z}$ (as we see in the following Proposition). In the following, $\mathbb{Z}_p$ will denote the field with $p$ elements.

**Proposition 1.13.** Let $p$ be a prime number and let $S = \mathbb{Z} \setminus p\mathbb{Z}$ or $S = \mathcal{C}(Q)$, where $Q$ is a prime ideal of $\mathbb{P}_Z$ such that $Q \cap \mathbb{Z} = p\mathbb{Z}$.

Then

$$\mathbb{P}_Z S^{-1} = S^{-1} \mathbb{P}_Z = \mathbb{P}_{Z(p)}.$$

Proof. We know that $S$ is a denominator set of $\mathbb{P}_Z$ by Proposition 1.11. So the ring $\mathbb{P}_Z S^{-1}$ exists.

Let $S = \mathbb{Z} \setminus p\mathbb{Z}$. It is easy to see that $\mathbb{P}_Z S^{-1} \subseteq \mathbb{P}_{Z(p)}$. For the reverse inclusion, notice that the minimum common denominator of any element of $\mathbb{Z}_p$ is an element of $\mathbb{Z} \setminus p\mathbb{Z}$. So $\mathbb{P}_Z S^{-1} = \mathbb{P}_{Z(p)}$. Similarly it can be proved that $S^{-1} \mathbb{P}_Z = \mathbb{P}_{Z(p)}$.

Let $S = \mathcal{C}(Q)$. Since the norm of the elements of $S$ is not divisible by $p$ (Proposition 1.10), a right fraction $ps^{-1} \in \mathbb{P}_Z S^{-1}$, for some $p \in \mathbb{P}_Z$ and $s \in S$, can be seen as a rational split-quaternion $q = \frac{1}{N(s)} p s = a + b i + c j + d k$, where $a, b, c, d \in \mathbb{Q}$ and their denominators are not divisible by $p$. Thus $\mathbb{P}_Z S^{-1} \subseteq \mathbb{P}_{Z(p)}$. For the reverse inclusion let
q ∈ \mathbb{P}_{Z(p)}$. Taking a common denominator, write $q = \frac{1}{n}p$, for some $p ∈ \mathbb{P}_Z$ and $n ∈ \mathbb{Z}$. Since the minimum common denominator of some elements of $\mathbb{Z}_{(p)}$ is an element of $\mathbb{Z} \setminus p\mathbb{Z}$, then $n$ is not divisible by $p$. Thus neither $n^2 = N(n)$ is divisible by $p$. So $n ∈ S$ and $\mathbb{P}_ZS^{-1} = \mathbb{P}_{Z(p)}$. In the same manner we can prove that $S^{-1}\mathbb{P}_Z = \mathbb{P}_{Z(p)}$. □

Imitating Proposition 1.12 we can give this general result.

**Proposition 1.14.** Let $R$ be a commutative ring and let $\mathcal{Q}(R)$ be its total ring of fractions. Then

$$\mathcal{Q}(\mathbb{P}_R) = \mathbb{P}_{\mathcal{Q}(R)}.$$  

*Proof. Let $S$ be the set of regular elements of $R$. Then, $S$ is contained in the center of $\mathbb{P}_R$, and thus it is a denominator set of $\mathbb{P}_R$; it is also easy to see that $S^{-1}\mathbb{P}_R = \mathbb{P}_S^{-1}R = \mathbb{P}_{\mathcal{Q}(R)}$ (see proof of Propositions 1.12).

We claim that the elements of $\mathbb{P}_{\mathcal{Q}(R)}$ are either invertible or zero-divisors. Take $q ∈ \mathbb{P}_{\mathcal{Q}(R)}$. If $N(q)$ is regular, then it is invertible in $\mathcal{Q}(R)$, and thus $\frac{1}{N(q)}\overline{q} ∈ \mathbb{P}_{\mathcal{Q}(R)}$ is the inverse of $q$. Conversely, if $N(q)$ is not regular, then there is $z ∈ R$, $z ≠ 0$, such that $zN(q) = 0$. If $zq ≠ 0$, then also $z\overline{q} = z\overline{q} ≠ 0$. So we have that:

$$0 = zN(q) = z\overline{q}q = (z\overline{q})q$$

hence, $q$ is a zero-divisor.

Thus, $\mathbb{P}_{\mathcal{Q}(R)}$ is a total ring of fractions, and so it is the total ring of fractions of $\mathbb{P}_R$. □

2. **Integer-valued polynomials**

The ring of integer-valued polynomials over $\mathbb{P}_Z$ is

$$\text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Z) = \{ f(x) ∈ \mathbb{P}_Q[x] \mid f(\mathbb{P}_Z) ⊆ \mathbb{P}_Z \}.$$  

This set is actually a ring ([15, Theorem 1.2]), and in [3] the authors describe explicitly some proper ideals of $\text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Z)$. A similar construction can be done if, instead of $\mathbb{P}_Z$, we use $\mathbb{P}_{Z(p)}$ or $\mathbb{P}_Q$; in the former case, [15, Theorem 1.2] guarantees that $\text{Int}_{\mathbb{P}_Q}(\mathbb{P}_{Z(p)})$ is a ring, while in the latter $\text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Q) = \mathbb{P}_Q[x]$ is the whole ring of polynomials (and, in particular, is a ring).

For simplicity of notation, in this Section we will write $\text{Int}(\mathbb{P}_Z)$ instead of $\text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Z)$. 

A class of ideals of \( \text{Int}(\mathbb{P}_\mathbb{Z}) \) can be constructed in the following way: if \( q = a + bi + cj + dk \in \mathbb{P}_\mathbb{Z} \) and \( I \) is a principal ideal of \( \mathbb{P}_\mathbb{Z} \) generated by an element of \( \mathbb{Z} \), then
\[
\mathfrak{P}_{I,q} := \{ f(x) \in \text{Int}(\mathbb{P}_\mathbb{Z}) \mid f(z) \in I \forall z \in C(q) \},
\]
is an ideal of \( \text{Int}(\mathbb{P}_\mathbb{Z}) \), where \( C(q) = \{ a \pm bi \pm cj \pm dk \} \) ([3, Proposition 4.2]).

If \( P \) is a prime ideal of \( \text{Int}(\mathbb{P}_\mathbb{Z}) \), then \( P \cap \mathbb{P}_\mathbb{Z} \) is a prime ideal of \( \mathbb{P}_\mathbb{Z} \); since we have a classification of the prime ideals of \( \mathbb{P}_\mathbb{Z} \) (Theorem 1.2), we can study the spectrum of \( \text{Int}(\mathbb{P}_\mathbb{Z}) \) according to the restriction to \( \mathbb{P}_\mathbb{Z} \).

**Proposition 2.1.** The following hold.

1. [3, Corollary 4.10] The prime ideals \( P \) of \( \text{Int}(\mathbb{P}_\mathbb{Z}) \) with \( P \cap \mathbb{P}_\mathbb{Z} = (0) \) are exactly those of the form
\[
P = M(x) \cdot \mathbb{P}_\mathbb{Q}[x] \cap \text{Int}(\mathbb{P}_\mathbb{Z}) =: P_{M(x)},
\]
where \( M(x) \in \mathbb{Z}[x] \) is an irreducible polynomial.

    In particular, if \( m_q(x) \) is the minimal polynomial of an element \( q \in \mathbb{P}_\mathbb{Z} \) then \( P_{m_q(x)} = \mathfrak{P}_{0,q} \) is a prime ideal.

2. [3, Theorem 4.16] Let \( q := a + bi + cj + dk \in \mathbb{P}_\mathbb{Z} \setminus \mathbb{Z} \) and let \( p \) be an odd prime. If \( \gcd(b, c, d, p) = 1 \), then \( \mathfrak{P}_{p\mathbb{P}_\mathbb{Z},q} \) is prime if and only if \( m_q(x) \) is irreducible mod \( p \), in which case \( \mathfrak{P}_{p\mathbb{P}_\mathbb{Z},q} \) is maximal.

3. [3, Corollary 4.22] Let \( q = a + bi + cj + dk \in \mathbb{P}_\mathbb{Z} \), and assume that either \( b \equiv c \pmod{2} \) or \( b \equiv d \pmod{2} \). Then,
\[
\mathfrak{M}_q := \{ f \in \text{Int}(\mathbb{P}_\mathbb{Z}) \mid f(q) \in \mathcal{M} \}
\]
is a maximal ideal of \( \text{Int}(\mathbb{P}_\mathbb{Z}) \).

**Remark 2.2.**

1. While the first case of the proposition completely classifies the prime ideals above \( (0) \), the other two merely give some examples of the prime ideals above \( p\mathbb{P}_\mathbb{Z} \) and \( \mathcal{M} \), but not a complete list.

2. We refer to [3] for some results about the equality among these ideals.

**Lemma 2.3.** The following hold:

1. If \( p \) is an odd prime number and \( q \in \mathbb{P}_\mathbb{Z} \), then \( \mathfrak{P}_{m_q(x)} \subseteq \mathfrak{P}_{p\mathbb{P}_\mathbb{Z},q} \).

2. If \( q \in \mathbb{P}_\mathbb{Z} \) is as in Proposition 2.1(3), then \( \mathfrak{P}_{m_q(x)} \subseteq \mathfrak{M}_q \).

**Proof.** Let \( f(x) \in \mathfrak{P}_{m_q(x)} \): then, \( f(x) = m_q(x)g(x) \) for some \( g(x) \in \mathbb{P}_\mathbb{Q}[x] \). Since \( m_q(x) \) has coefficients in the center of \( \mathbb{P}_\mathbb{Q} \), we have \( f(q) = m_q(q)g(q) = 0 \). Hence, \( f(x) \in \mathfrak{M}_q \); furthermore, \( m_q(q') = 0 \) for all \( q' \in \mathbb{P}_\mathbb{Z} \).
C(q) (since the elements of C(q) have the same minimal polynomial of q [3, paragraph after Definition 4.1]) and thus \( f(x) \in \mathcal{P}_{p \mathbb{Z}, q} \). Therefore \( \mathcal{P}_{m_{q}(x)} \) is contained in both \( \mathcal{P}_{p \mathbb{Z}, q} \) and \( \mathfrak{m}_{q} \).

By intersecting the ideals with \( \mathbb{Z} \), it is easily seen that the inclusions are proper. \( \square \)

When \( D \) is a Noetherian commutative domain, the integer-valued polynomials over \( D \) behave well with respect to the localization, that is, if \( S \) is a multiplicative subset of \( D \) then \( S^{-1} \text{Int}(D) = \text{Int}(S^{-1}D) \) ([2, Theorem I.2.3]). In [3, Theorem 3.4] an analogous result has been showed for \( \text{Int}(\mathbb{P}_Z) \) when \( S \) is a multiplicatively closed subset \( S \subset \mathbb{Z} \) (it is central). In the following we prove that \( \text{Int}(\mathbb{P}_Z) \) behaves well with respect to localization also for denominator sets whose elements are not necessarily central, as \( S = \mathcal{C}(Q) \), where \( Q \) is a prime ideal of \( \mathbb{P}_Z \).

**Theorem 2.4.** Let \( Q \) be a prime ideal of \( \mathbb{P}_Z \) and let \( S = \mathcal{C}(Q) \). Then \( S \) is also a denominator set of \( \text{Int}(\mathbb{P}_Z) \) and \( \text{Int}(\mathbb{P}_Z)S^{-1} = \text{Int}(\mathbb{P}_Z S^{-1}) \).

**Proof.** To prove that \( S \) is a denominator set of \( \text{Int}(\mathbb{P}_Z) \) it is sufficient to use the same argument of Lemma 1.6 and Proposition 1.11, observing that \( N(s) \) is in the center of \( \text{Int}(\mathbb{P}_Z) \) for each \( s \in S \).

Let \( Q \) be a prime ideal of \( \mathbb{P}_Z \), and let \( Q \cap \mathbb{Z} = p\mathbb{Z} \) (where \( p \) is either a prime number or 0). Set \( T := \mathbb{Z} \setminus p\mathbb{Z} \). By Propositions 1.12 and 1.13, we have \( \text{Int}(\mathbb{P}_Z T^{-1}) = \text{Int}(\mathbb{P}_Z \mathcal{C}(Q)^{-1}) = \text{Int}(\mathbb{P}_{Z(p)}) \).

To prove the statement it is enough to show that

\[
(1) \quad \text{Int}(\mathbb{P}_Z)T^{-1} \subseteq \text{Int}(\mathbb{P}_Z \mathcal{C}(Q)^{-1}) \subseteq \text{Int}(\mathbb{P}_{Z(p)}) \subseteq \text{Int}(\mathbb{P}_Z)T^{-1}.
\]

The first inclusion follows from the fact that \( T \subseteq \mathcal{C}(Q) \), while the last one from [3, Theorem 3.4] (it is actually an equality). Thus, we only need to prove that \( \text{Int}(\mathbb{P}_Z \mathcal{C}(Q)^{-1}) \subseteq \text{Int}(\mathbb{P}_{Z(p)}) \). Again by [3, Theorem 3.4], we have \( \text{Int}(\mathbb{P}_Z) \subseteq \text{Int}(\mathbb{P}_{Z(p)}) \); furthermore, each element of \( \mathcal{C}(Q) \) becomes invertible in \( \mathbb{P}_{Z(p)} \) and thus in \( \text{Int}(\mathbb{P}_{Z(p)}) \). Hence, \( \text{Int}(\mathbb{P}_Z \mathcal{C}(Q)^{-1}) \subseteq \text{Int}(\mathbb{P}_{Z(p)}) \) and all the containments must be equalities. \( \square \)

Note that the exact same argument can be used if we localize on the left: if \( S \) is \( S = \mathcal{C}'(Q) \) then \( S^{-1} \text{Int}(\mathbb{P}_Z) = \text{Int}(S^{-1}\mathbb{P}_Z) \).

**Corollary 2.5.** The following hold.

1. If \( S = \mathcal{R}(\mathbb{P}_Z) \) or \( S = \mathbb{Z} \setminus \{0\} \) then \( \text{Int}(\mathbb{P}_Z)S^{-1} = \text{Int}(\mathbb{P}_Q) = \mathbb{P}_Q[x] \).

2. If \( p \) is a prime number and \( S = \mathbb{Z} \setminus p\mathbb{Z} \) or \( S = \mathcal{C}(Q) \), with \( Q \) a prime ideal of \( \mathbb{P}_Z \) such that \( Q \cap \mathbb{Z} = p\mathbb{Z} \), then \( \text{Int}(\mathbb{P}_Z)S^{-1} = \text{Int}(\mathbb{P}_{Z(p)}) \).
Proof. For the first point, the equality $\text{Int}(\mathbb{P}_Z)S^{-1} = \text{Int}(\mathbb{P}_Q)$ follows from Theorem 2.4 and Proposition 1.12. The equality $\text{Int}(\mathbb{P}_Q) = \mathbb{P}_Q[x]$ follows directly from the definitions.

Similarly, the second point follows from Theorem 2.4 and from Proposition 1.13. □

These results allow us to represent $\mathbb{P}_Z$ and $\text{Int}(\mathbb{P}_Z)$ as intersection of localizations.

**Proposition 2.6.** Let $\mathcal{P}$ be the set of prime numbers. Then, the following hold.

1. $\mathbb{P}_Z = \bigcap_{p \in \mathcal{P}} \mathbb{P}_Z(p)$.
2. $\text{Int}(\mathbb{P}_Z) = \bigcap_{p \in \mathcal{P}} \text{Int}(\mathbb{P}_Z(p))$.

Proof. (1) The inclusion ($\subseteq$) is obvious since for every prime $p$, $\mathbb{P}_Z \subseteq \mathbb{P}_Z(p)$. For the reverse inclusion, take an element $q = a + bi + cj + dk$ of the intersection. Then $a, b, c, d \in \bigcap_p \mathbb{Z}_p = \mathbb{Z}$ and $q \in \mathbb{P}_Z$.

(2) For all primes $p$, let $Q_p$ be the maximal ideal of $\mathbb{P}_Z$ above $p$. We have that $\text{Int}(\mathbb{P}_Z) \subseteq (\text{Int}(\mathbb{P}_Z))^{(Q_p)^{-1}} = \text{Int}(\mathbb{P}_Z(p))$, and thus $\text{Int}(\mathbb{P}_Z)$ is inside the intersection. Conversely, if $f(x)$ belongs to the intersection and $q \in \mathbb{P}_Z$, then $f(q) \in \mathbb{P}_Z(p)$ for every prime number $p$, and thus $f(q) \in \bigcap_p \mathbb{P}_Z(p) = \mathbb{P}_Z$ (by the previous point) and $f(x) \in \text{Int}(\mathbb{P}_Z)$.

3. **Matrix representations**

To study the spectrum of $\text{Int}(\mathbb{P}_Z)$, we introduce the related commutative ring

$$\text{Int}_Q(\mathbb{P}_Z) := \{ f(x) \in Q[x] \mid \forall q \in \mathbb{P}_Z : f(q) \in \mathbb{P}_Z \},$$

and we define similarly $\text{Int}_Q(\mathbb{P}_Z(p))$. These sets are easily seen to be rings by using polynomial evaluation. To avoid confusion in the notation, from now in we will go back to write $\text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Z)$ and $\text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Z(p))$ for $\text{Int}(\mathbb{P}_Z)$ and $\text{Int}(\mathbb{P}_Z(p))$, respectively. Note that, if we consider $Q[x]$ as as subring of $\mathbb{P}_Q[x]$ in the obvious way, then $\text{Int}_Q(\mathbb{P}_Z) = \text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Z) \cap Q[x]$.

The relation between $\text{Int}_Q(\mathbb{P}_Z)$ and $\text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Z)$ passes through a matrix representation of the rings $\mathbb{P}_Z$ and $\mathbb{P}_Z(p)$. We denote by $\mathcal{M}_n(R)$ the ring of matrices of order $n$ over $R$.

**Proposition 3.1.** [3, Proposition 2.2] The following hold.

1. Let $R$ be a commutative ring with identity such that 2 is a unit of $R$. Then, $\mathbb{P}_R \cong \mathcal{M}_2(R)$ as $R$-algebras.
2. Let $\mathcal{A} = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d, \ b \equiv c \ mod \ 2 \} \subseteq \mathcal{M}_2(\mathbb{Z})$. Then, $\mathbb{P}_Z \cong \mathcal{A}$ as $\mathbb{Z}$-algebras.
Let $D$ be a domain with quotient field $K$. We define
\[
\text{Int}_K(M_n(D)) := \{ f(x) \in K[x] \mid \forall A \in M_n(D) : f(A) \in M_n(D) \}
\]
and
\[
\text{Int}_{M_n(K)}(M_n(D)) := \{ f(x) \in M_n(K)[x] \mid \forall A \in M_n(D) : f(A) \in M_n(D) \}.
\]
These rings roughly correspond, respectively, to $\text{Int}_Q(P_Z)$ and $\text{Int}_{P_Q}(P_Z)$.

**Proposition 3.2.** Let $D$ be a domain with quotient field $K$. Then
\[
\text{Int}_{M_n(K)}(M_n(D)) \simeq M_n(\text{Int}_K(M_n(D))).
\]
Moreover:
(i) The ideals of $\text{Int}_{M_n(K)}(M_n(D))$ are in 1-1 correspondence with the sets of the form $M_n(\mathcal{I})$, where $\mathcal{I}$ is an ideal of $\text{Int}_K(M_n(D))$.
(ii) The prime ideals of $\text{Int}_{M_n(K)}(M_n(D))$ are in 1-1 correspondence with the sets of the form $M_n(\mathcal{P})$, where $\mathcal{P}$ is a prime ideal of $\text{Int}_K(M_n(D))$.
(iii) The maximal ideals of $\text{Int}_{M_n(K)}(M_n(D))$ are in 1-1 correspondence with the sets of the form $M_n(\mathcal{M})$, where $\mathcal{M}$ is a maximal ideal of $\text{Int}_K(M_n(D))$.

**Proof.** See [4, Theorem 7.2] and [4, Theorem 7.3]. The remaining part follows from [8, Theorem 3.1]. □

Putting together these two results, we have the following theorem.

**Theorem 3.3.** Let $p$ be an odd prime integer. Then, the prime ideals of $\text{Int}_{P_Q}(P_{Z(p)})$ are in 1-1 correspondence with the prime ideals of $\text{Int}_Q(P_{Z(p)})$.

**Proof.** By Proposition 3.1, $P_Q \sim M_2(Q)$, and the isomorphism brings $P_{Z(p)}$ into $M_2(Z(p))$. By Proposition 3.2,
\[
\text{Int}_{P_Q}(P_{Z(p)}) \simeq \text{Int}_{M_2(Q)}(M_2(Z(p))) \simeq M_2(\text{Int}_Q(M_2(Z(p)))) \simeq M_2(\text{Int}_Q(P_{Z(p)}));
\]
thus the prime ideals of $\text{Int}_{P_Q}(P_{Z(p)})$ are in bijective correspondence with the prime ideals of $\text{Int}_Q(P_{Z(p)})$, as claimed. □

The main advantage of this theorem is that $\text{Int}_Q(P_{Z(p)})$ is a commutative ring properly contained in between the two well-studied rings $Z[x]$ and $Q[x]$.

**Proposition 3.4.** The nonzero prime ideals $P$ of $\text{Int}_Q(P_Z)$ such that $P \cap Z = (0)$ are pairwise incomparable.
Corollary 3.6. If \( p \) is an odd prime, then \( \text{Int}_{\mathbb{F}_p}(\mathbb{P}_\mathbb{Z}(\mathbb{P}_Z)) \) has dimension 2. Furthermore, \( \dim(\text{Int}_{\mathbb{F}_p}(\mathbb{P}_Z)) \geq 2 \).

**Proof.** By Theorem 3.3, the dimension of \( \text{Int}_{\mathbb{F}_p}(\mathbb{P}_Z) \) is the same of \( \text{Int}_{\mathbb{Q}}(\mathbb{P}_Z) \), which is 2 by Theorem 3.5. The last claim follows since \( \text{Int}_{\mathbb{F}_p}(\mathbb{P}_Z(\mathbb{P}_Z)) \) is a localization of \( \text{Int}_{\mathbb{F}_p}(\mathbb{P}_Z) \). □
An important difference between $\text{Int}(\mathbb{Z})$ and $\text{Int}_Q(\mathbb{P}_Z)$ is that the latter is not integrally closed (and thus it is not a Prüfer domain); see Corollary 3.8 below. However, we can describe its integral closure by using algebraic integers.

Given a finite degree extension $\mathbb{Q}(\theta)$ of $\mathbb{Q}$, we indicate by $\mathcal{A}_\theta$ the ring of algebraic integers of $\mathbb{Q}(\theta)$. If $n \in \mathbb{N}$ is positive, the set of all algebraic integers of degree at most $n$ over $\mathbb{Q}$ is

$$\mathcal{A}_n := \bigcup_{[\mathbb{Q}(\theta):\mathbb{Q}] \leq n} \mathcal{A}_\theta;$$

similarly, if $p$ is a prime number, we denote by $\mathcal{A}_{n,p}$ the set of algebraic numbers that are root of a monic irreducible polynomial of degree $n$ over $\mathbb{Z}(p)$.

In [10] the authors define the set of integer-valued polynomials over $\mathcal{A}_n$ with rational coefficients to be the set

$$\text{Int}(\mathcal{A}_n) := \bigcap_{\theta \in \mathcal{A}_n} \text{Int}_\mathbb{Q}(\mathcal{A}_\theta).$$

The ring $\text{Int}(\mathcal{A}_n)$ can be seen as the set of all polynomials with rational coefficients that map $\mathcal{A}_n$ into $\mathcal{A}_n$. They also show that $\text{Int}_\mathbb{Q}(\mathcal{A}_n)$ is a Prüfer domain for every $n$ ([10, Theorem 3.9]).

**Theorem 3.7.** Let $p$ be an odd prime integer. Then $\text{Int}_\mathbb{Q}(\mathcal{A}_2)_p = \text{Int}_\mathbb{Q}(\mathcal{A}_{2,p})$ is the integral closure of $\text{Int}_\mathbb{Q}(\mathbb{P}_Z(p))$ in $\mathbb{Q}[x]$.

**Proof.** By [10, Theorem 4.6], $\text{Int}_\mathbb{Q}(\mathcal{A}_2)$ is the integral closure of $\text{Int}_\mathbb{Q}(\mathcal{M}_2(\mathbb{Z}))$. Using Proposition 3.1, and recalling that the localization at prime integers preserves the integral closure, we have that:

$$\text{Int}_\mathbb{Q}(\mathcal{A}_2)_p = \text{Int}_\mathbb{Q}(\mathcal{M}_2(\mathbb{Z}))_p = \text{Int}_\mathbb{Q}(\mathcal{M}_2(\mathbb{Z}))_p = \text{Int}_\mathbb{Q}(\mathbb{P}_Z(p)),$$

Finally, using [13, Theorem 13] with $\mathcal{A} = \mathbb{P}_Z(p)$, we have that $\text{Int}_\mathbb{Q}(\mathbb{P}_Z(p))$ is also the integral closure of $\text{Int}_\mathbb{Q}(\mathcal{A}_2,p)$.

**Corollary 3.8.** The ring $\text{Int}_\mathbb{Q}(\mathbb{P}_Z)$ is not integrally closed.

**Proof.** If $\text{Int}_\mathbb{Q}(\mathbb{P}_Z)$ is integrally closed, then its localization at an odd prime $p$, $\text{Int}_\mathbb{Q}(\mathbb{P}_Z(p))$, is integrally closed too. Thus, from Theorem 3.7, $\text{Int}_\mathbb{Q}(\mathbb{P}_Z(p)) = \text{Int}_\mathbb{Q}(\mathcal{A}_2)_p$ and this is Prüfer. Since $\text{Int}_\mathbb{Q}(\mathbb{P}_Z(p)) \cong \text{Int}_\mathbb{Q}(\mathcal{M}_2(\mathbb{Z}(p)))$, it follows that the ring

$$\text{Int}_\mathbb{Q}(B, \mathcal{M}_2(\mathbb{Z}(p))) := \{ f \in \mathbb{Q}[x] \mid f(B) \in \mathcal{M}_2(\mathbb{Z}(p)) \}$$
is an overring of $\text{Int}_Q(M_2(\mathbb{Z}_p))$, for every matrix $B \in M_2(\mathbb{Z}_p)$. Taking $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and arguing as in [10, §4], it can be shown that $\text{Int}_Q(B, M_2(\mathbb{Z}_p))$ is not integrally closed. The claim follows.

4. The ideal $p\, \text{Int}_Q(\mathbb{P}_Z)$

In this section we study in more detail the ideal $p\, \text{Int}_Q(\mathbb{P}_Z)$ generated by a prime number $p$ (not necessarily odd). Our first result can be seen as a refinement of the proof of Theorem 3.5.

Proposition 4.1. Let $p$ be a prime number. Then, every prime ideal of $\text{Int}_Q(\mathbb{P}_Z)$ containing $p$ is maximal.

Proof. We follow the proof of [2, Lemma V.1.9].

Let $u_1, \ldots, u_k$ be a set of residues of $\mathbb{P}_Z/p\mathbb{P}_Z$ (with $k = p^4$), and let $P$ be a prime ideal of $\text{Int}_Q(\mathbb{P}_Z)$ containing $p$. Take any $a(x) \in \text{Int}_Q(\mathbb{P}_Z)$, and let $a_i(x) := a(x) - u_i$. Let $b(x) := a_1(x) \cdots a_k(x)$: by construction, for every $q \in \mathbb{P}_Z$ there is an $i$ such that $a(q) \equiv u_i \mod p\mathbb{P}_Z$.

Since the $a_i$ have coefficients in the commutative ring $Q$, we have $b(q) = a_1(q) \cdots a_k(q)$: hence, $b(q) \in p\mathbb{P}_Z$ and so $b(x) \in p\, \text{Int}_Q(\mathbb{P}_Z) \subseteq P$; since $P$ is prime, there must be an $i$ such that $a_i(x) \in P$. However, $a_i(x) \equiv u_i \mod P$, and thus $\text{Int}_Q(\mathbb{P}_Z)/P$ is isomorphic to $\mathbb{P}_Z/p\mathbb{P}_Z \cong \mathbb{P}_{Z_p}$. Hence, $P$ is maximal, as claimed.

Corollary 4.2. Let $p$ be an odd prime integer. Then, every prime ideal of $\text{Int}_Q(\mathbb{P}_{Z(p)})$ containing $p$ is maximal.

Proof. It is enough to use Proposition 4.1 and the correspondence of Theorem 3.3.

Remark 4.3.

(1) The previous two results allow to give an alternative proof of Theorem 3.5. Indeed, if $(0) \subsetneq Q_1 \subsetneq Q_2 \subsetneq Q_3$ is a chain of prime ideals of length 3, then either $Q_1 \cap \mathbb{P} = Q_2 \cap \mathbb{P} = (0)$ or $Q_2 \cap \mathbb{P} = Q_3 \cap \mathbb{P} = p\mathbb{P}$, for some prime number $p$. The latter case is made impossible by Proposition 4.1 (as $Q_2$ contains $p$ but is not maximal); on the other hand the former case would imply that two nonzero prime ideals of $\text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Z)$ over $(0)$ are comparable, against Proposition 3.4.

(2) The proof of Proposition 4.1 does not work in the ring $\text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Z)$, since the evaluation of a product of polynomials cannot be done separately for each factor, and thus $b(q) \neq a_1(q) \cdots a_k(q)$ in general. Nevertheless, we conjecture (but we don’t have a proof) that the same property holds also in $\text{Int}_{\mathbb{P}_Q}(\mathbb{P}_Z)$.
A consequence of Proposition 4.1 is that the ideals $p \text{Int}_Q(P_{Z(p)})$ are not prime. We now want to find an explicit description of the polynomials in $\text{Int}_Q(P_{Z(p)})$ and, as a corollary, to find two polynomials outside $p \text{Int}_Q(P_{Z(p)})$ whose product is inside the ideal.

**Proposition 4.4.** Let $R, S$ be commutative rings and let $\pi : R \to S$ be a homomorphism. Then, the natural map

$$\varphi : P_R \to P_S$$

$$a + b \alpha + c \beta + d \gamma \mapsto \pi(a) + \pi(b) \alpha + \pi(c) \beta + \pi(d) \gamma$$

is a ring homomorphism. Furthermore, if $\pi$ is surjective then $\varphi$ is surjective and $\ker \varphi = (\ker \pi)P_R = \ker \pi = \{a + b \alpha + c \beta + d \gamma \mid a, b, c, d \in \ker \pi\}$; in particular, $P_R / \ker \varphi \simeq P_S$.

**Proof.** Straightforward. □

An important particular case is when $R = \mathbb{Z}$ or $R = \mathbb{Z}_p$ and $S = \mathbb{Z}_p$; in this case, the kernel of $\pi$ is generated by $p$, and thus we obtain the well-known isomorphisms $P_{\mathbb{Z}} / pP_{\mathbb{Z}} \simeq P_{\mathbb{Z}_p} / pP_{\mathbb{Z}_p} \simeq \mathbb{Z}_p$.

In particular, the previous proposition shows that polynomial evaluation behaves well with respect to quotients. Given a surjection $\pi : R \to S$ and a polynomial $f(x) = \sum_{t=0}^{n} p_t x^t \in R[x]$, we denote by $\overline{f}(x) = \sum_{t=0}^{n} \pi(p_t) x^t \in S[x]$ the polynomial obtained by reducing the coefficients modulo $\ker \varphi$. Then, for every $q \in P_{\mathbb{Z}}$, we have $\pi(f(q)) = \overline{f}(\pi(q))$.

**Proposition 4.5.** Let $p$ be a prime integer. Let $f(x) \in \mathbb{Z}[x]$ and $\overline{f}(x) \in \mathbb{Z}_p[x]$ be as above. Given an integer $n > 1$ such that $n = p^\alpha m$ with $p \nmid m$, then $\frac{1}{n} f(x) \in \text{Int}_Q(P_{Z(p)})$ if and only if $f(q) \in p^\alpha P_{Z(p)}$, for all $q \in P_{Z(p)}$. In particular if $\alpha = 1$, $\frac{1}{n} f(x) \in \text{Int}_Q(P_{Z(p)})$ if and only if $\overline{f}(q) = 0$ in $P_{\mathbb{Z}_p}$, for all $q \in P_{\mathbb{Z}_p}$.

**Proof.** We have that

$$\frac{1}{n} f(x) \in \text{Int}_Q(P_{Z(p)}) \iff \frac{1}{n} f(q) \in P_{Z(p)} \forall q \in P_{Z(p)} \iff f(q) \in nP_{Z(p)} \forall q \in P_{Z(p)}.$$

Since $p \nmid m$, $nP_{Z(p)} = p^\alpha P_{Z(p)}$. □

**Lemma 4.6.** Let $R$ be a commutative domain. Take $q \in P_R \setminus R$ and let $m_q(x) \in R[x]$ be its minimal polynomial over $R$. If a polynomial $f(x) \in R[x]$ is such that $f(q) = 0$, then $m_q(x) \mid f(x)$ in $R[x]$. 
Proof. Since $m_q(x)$ is monic we can divide $f(x)$ by $m_q(x)$ obtaining

$$f(x) = g(x)m_q(x) + r(x),$$

for some $g(x), r(x) \in R[x]$. In particular $r(x) = ax + b$ is linear as $m_q(x)$ is of degree two. Since $R[x]$ is contained in the center of $\mathbb{P}_R[x]$, we can evaluate the polynomial relation above in $q$, obtaining $0 = f(q) = g(q) \cdot 0 + aq + b$. Since $R$ is a domain and $q \notin R$, necessarily $a = b = 0$. □

We observe that Lemma 4.6 does not hold if $f(x) \in \mathbb{P}_R[x] \setminus R[x]$. For example, consider $i \in \mathbb{P}_Z$ and $f(x) = x^3 + ix + (i + 1)x + i + 1$. Then $f(i) = 0$ but $f(x) = (x^2 + 1)(x + i) + ix + 1$ and the remainder is nonzero.

**Corollary 4.7.** With the hypothesis and notation of Proposition 4.5, let $p$ be a prime integer and $n = pm$ with $p \nmid m$. Then $\frac{1}{n}f(x) \in \text{Int}_Q(\mathbb{P}_{Z_{(p)}})$ if and only if $f(x)$ is divided by all the minimal polynomials of the elements of $\mathbb{P}_{Z_{(p)}}$.

**Proof.** It is an immediate consequence of Proposition 4.5 and Lemma 4.6. □

Using the previous Corollary we can construct a nontrivial element of $\text{Int}_Q(\mathbb{P}_{Z_{(p)}})$.

**Example 4.8.** The polynomial

$$\Phi_p(x) = \frac{1}{p}(x^p - x)(x^{p^2} - x)$$

belongs to $\text{Int}_Q(\mathbb{P}_{Z_{(p)}})$.

By Proposition 4.5, it is sufficient to show that $f(x) = (x^p - x)(x^{p^2} - x) \in Z[x]$ vanishes over all elements of $\mathbb{P}_{Z_p}$. Observe that every monic and irreducible polynomial of $Z_p[x]$ of degree one or two is a factor of $f(x)$. In particular, if $g(x)$ is a linear polynomial then $g(x)^2$ divides $f(x)$, since $g(x)$ divides both $x^p - x$ and $x^{p^2} - x$. By Corollary 4.7, this also means that the minimal polynomial of every split-quaternion of $\mathbb{P}_{Z_p}$ is a factor of $f(x)$.

In particular we can show that every monic and quadratic polynomial of $Z_p[x]$ is the minimal polynomial for some element of $\mathbb{P}_{Z_p}$. The proof is mutatis mutandis the same as the proof of [14, Lemma 3.5]. This means that the polynomial $\Phi_p(x)$ does not contain any redundant factor.

**Proposition 4.9.** With the above notation we have the following proper inclusions:

$$Z_{(p)}[x] \subsetneq \text{Int}_Q(\mathbb{P}_{Z_{(p)}}) \subsetneq \text{Int}(Z_{(p)}).$$
Let us consider the polynomials:

\[
\begin{align*}
\Phi_p(x) &\quad \text{given in Example 4.8 belongs to } \text{Int}_Q(\mathbb{P}_{\mathbb{Z}(p)}) \\
\text{Example 4.8} &\quad \text{The minimal polynomial } \Phi_p(x) \text{ is not divisible by any quadratic polynomial over } \mathbb{Z}_p, \text{ and thus by Corollary 4.7 } f(i) \notin \mathbb{P}_{\mathbb{Z}(p)} \text{ by Corollary 4.7. It follows that } f(x) \notin \text{Int}_Q(\mathbb{P}_{\mathbb{Z}(p)}). \\
\text{The fact that the two containments of the previous proposition are } &\quad \text{strict also follows from } [11, \text{Theorem 2.12}] \text{ (the first one) and } [12, \text{Theorem 2.11}] \text{ (the second one).}
\end{align*}
\]

**Proposition 4.10.** The ideal \( p \text{Int}_Q(\mathbb{P}_{\mathbb{Z}(p)}) \) is not a prime ideal of \( \text{Int}_Q(\mathbb{P}_{\mathbb{Z}(p)}) \).

**Proof.** Let us consider the polynomials:

\[
\begin{align*}
f(x) &= (x^p - x)^2 \in \mathbb{Z}[x], \\
g(x) &= \frac{1}{p} (x^p - x)^2 \in \mathbb{Q}[x], \\
F(x) &= f(x)g(x) = \frac{1}{p} (x^p - x)^2(x^p - x)^2 \in \mathbb{Q}[x].
\end{align*}
\]

These three polynomials are elements of \( \text{Int}_Q(\mathbb{P}_{\mathbb{Z}(p)}) \). Indeed, for \( f(x) \) it follows from the inclusion \( \mathbb{Z}[x] \subseteq \text{Int}_Q(\mathbb{P}_{\mathbb{Z}(p)}) \). For \( F(x) \) and \( g(x) \) observe that they are equal to \( \Phi_p(x) \) (Example 4.8) multiplied by a polynomial with integer coefficients.

We claim that \( F(x) \in p \text{Int}_Q(\mathbb{P}_{\mathbb{Z}(p)}) \) while \( f(x) \) and \( g(x) \) do not belong to this ideal.

Indeed, \( \frac{1}{p} F(x) = (\Phi_p(x))^2 \in \text{Int}_Q(\mathbb{P}_{\mathbb{Z}(p)}) \) and thus \( F(x) \in p \text{Int}_Q(\mathbb{P}_{\mathbb{Z}(p)}) \).

As regards \( f(x) \), we have that \( \mathcal{F}(x) \) is not divisible by any quadratic irreducible polynomial over \( \mathbb{Z}_p \), and thus by Corollary 4.7 \( \frac{1}{p} f(x) \notin \text{Int}_Q(\mathbb{P}_{\mathbb{Z}(p)}) \).

For \( g(x) \), consider \( \frac{1}{p} g(x) = \frac{1}{p^2} (x^p - x)^2 \). If \( p = 2 \) then \( \frac{1}{p} g(i) = -\frac{1}{2} \notin \mathbb{P}_{\mathbb{Z}(2)} \). If \( p \) is odd, then we set \( q := i + (p - 1)k \). We have that \( q^2 = p^2 - 2p \), and if we raise \( q \) to an even power greater than 2, we obtain an integer divisible by \( p^2 \). Since \( \frac{1}{p} g(x) \) is a central polynomial,
we can evaluate it in \( q \) using its factorization. Thus, we have

\[
\frac{1}{p} g(q) = \frac{(q^{p^2} - q)^2}{p^2} = \frac{q^{2p^2} + q^2 - 2q^{p^2 + 1}}{p^2} = m + \frac{p - 2}{p} \notin \mathbb{P}_{\mathbb{Z}(p)}
\]

for some \( m \in \mathbb{Z} \).

Since \( F(x) = f(x)g(x) \), we can conclude that \( p \text{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}(p)}) \) is not a prime ideal of \( \text{Int}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Z}(p)}) \). □

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References

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