Optimal Transport in Systems and Control

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Xxxx. Xxx. Xxx. Xxx. YYYY. AA:1–26

https://doi.org/10.1146/((please add article doi))

Keywords

Optimal transport, Schrödinger bridge, covariance control, stochastic control, Markov decision processes

Abstract

Optimal transport began as the problem to efficiently redistribute goods between production and consumers, and evolved into a far reaching geometric variational framework for studying flows of distributions on metric spaces. This theory interests in enabling ways a class of stochastic control problems, to regulate dynamical systems so as to limit uncertainty to within specifications. Representative control examples include the landing of a spacecraft aimed probabilistically towards a target, and the suppression of undesirable effects of thermal noise on resonators; in either, the goal is to regulate the flow of the distribution of the random state. Thence, a most unlikely link turned up between transport of probability distributions and a "maximum entropy" inference problem of E. Schrödinger, where the latter is seen as an entropy-regularized version of the former. These intertwined topics, of optimal transport, stochastic control, and inference, are the subject of the current review; it aims to highlight connections, insights, and computational tools, while at the same time making contact with quadratic regulator theory and probabilistic flows on discrete spaces/networks.

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1. INTRODUCTION

In 350 BCE, Aristotle, in his chief cosmological treatise, $\Pi\epsilon\rho\acute{\iota}\ o\upsilon\rho\alpha\nu o\acute{\upsilon}$, states: "Of all curves enclosing a given area the circle has the shortest perimeter". This *isoareal* problem has a celebrated dual version: The Phoenician princess Dido, when arriving in Northern Africa around 820 BCE, was offered by the Numidian king Jarbas as much land as she could enclose with an ox-hide to found Carthage (Vergil's Aeneid, Book 4). Dido had the hide cut into very fine strips and with these encircled a hill which, in time, became the city's citadel and known as Byrsa Hill after the Greek word for ox-hide. This is the oldest *isoperimetric problem*. Was the circle (or semi-circle along the coast) truly enclosing the maximum area? Although this was believed since ancient times, it took the development of the calculus of variations in the late seventeen and eighteen centuries, mostly thanks to Newton, the Bernoulli brothers, de L'Hôpital, Euler and Lagrange, to *prove* this result. Let us recall the so-called *simplest problem in calculus of variations* (1), formulated as follows. Let $L : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be of class C^1, let

$$\mathcal{X} := \{x \in C^1[t_0, t_1] | x(t_0) = x_0,\ x(t_1) = x_1\},$$

and let the functional $I : \mathcal{X} \to \mathbb{R}$ be given by

$$I(x) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt, \quad \dot{x}(t) = \frac{dx}{dt}(t).$$

Problem 1

$$\text{Minimize } I \text{ over } \mathcal{X}.$$

For instance, if $n = 1$ and $L(t, x, \dot{x}) = \sqrt{1 + \dot{x}^2}$ corresponding to arclength, the problem consists in finding the shortest path joining two points in the (t, x)-plane. The latter problem generalizes to the search for geodesics on a Riemannian manifold. Suppose now we make the following essentially cosmetic transformation: We turn Problem 1 into the optimal control problem:

$$\min_{(x,u)\in(\mathcal{X}\times\mathcal{U})} J(x, u) = \int_{t_0}^{t_1} L(t, x(t), u(t))dt \qquad \text{1a.}$$

$$\text{subject to } \dot{x}(t) = u(t), \qquad \text{1b.}$$

where $\mathcal{U} := \{u : [t_0, t_1] \to \mathbb{R}^n, u \text{ continuous}\}$. This has the form of a *steering problem* between the two given points x_0 and x_1.

Suppose further that we only know these two points approximatively, in that we have a probability density for each of them, say ρ_0 and ρ_1, respectively. Our problem has now become a *stochastic control* problem where the only source of uncertainty comes from the state boundary conditions. If noise is present in (1b), then we have an additional source of uncertainty to deal with. These two formulations of stochastic control are the subject matter of this paper, with roots in optimal mass transport on one side and the theory of Schrödinger Bridges on the other. The second problem is also connected to work in stochastic control by Wendell Fleming and others from the mid eighties on (1, 2), controllability of the Fokker-Plank equation (3), and ensemble control (4, 5). In the case when the system is linear (not necessarily an integrator) and the boundary distributions are Gaussian, these problems also relate to contributions by Skelton and his co-workers on covariance control (6, 7, 8). The latter concerned the infinite horizon stationary control, whereas advances in the finite horizon case are more recent (9, 10, 11, 12, 13, 14). Either case can be thought of as the *steering of probability distributions* –the problem to control uncertainty. This paradigm has emerged in recent times (15, 16) as an important variant of stochastic control with several modern applications to guidance, sensing, control of robot swarms, and so on (9, 10, 17, 18, 19, 20, 21, 22, 23, 24).

The paper is outlined as follows. In Section 2 we give a crash course on optimal mass transport and Schrödinger bridge theory; the latter can be viewed as a regularization of the former. In Section 3, we reformulate the optimal mass transport problems as density control problems for some simple dynamics. The extensions to more general dynamics and scenarios are developed in Section 4 and Section 5. In particular, Section 4 focuses on linear-quadratic Gaussian cases, which extends the covariance control theory. The cases with general marginal distributions and nonlinear control-affine dynamics are studied in Section 5. This is followed in Section 6 by the discussion of a discrete counterpart of the density control problem over Markov decision processes.

2. Preliminaries on optimal transport

Optimal mass transport (OMT) theory is concerned with transporting mass from a source distribution to a target distribution with minimum effort. Given two nonnegative measures μ_0, μ_1 on $\mathbb{R}^n$ with equal total mass[1], Monge's (25) formulation of OMT seeks a transport map $T : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto T(x)$ that moves mass from distribution μ_0 to μ_1 in the sense that $T_\sharp \mu_0 = \mu_1$, i.e., $\mu_1(E) = \mu_0(T^{-1}(E))$ for every Borel set E in $\mathbb{R}^n$, and meanwhile minimizes the total cost of transportation

$$\int_{\mathbb{R}^n} c(x, T(x))\mu_0(dx). \qquad 2.$$

OMT is also known as Earth mover's problem

Here, $c(x, y)$ denotes the transportation cost per unit mass from point x to y; popular choices are $c(x, y) = \|x-y\|^2$ and $c(x, y) = c(x-y)$ for some strongly convex function[2] $c(\cdot)$.

[1]Without loss of generality, we take μ_0 and μ_1 to be probability distributions; when these are absolutely continuous with respect to the Lebesgue measure, later on, we use ρ to denote the corresponding density, e.g., $\mu_i(dx) = \rho_i(x)dx$, $i \in \{0, 1\}$.

[2]Monge's original choice was for $c(x, y) = \|x - y\|$, which leads to a challenging problem in that both existence and uniqueness of solution are not guaranteed (26).

The highly nonlinear dependence of the transportation cost on the transport map T has resisted early attempts to conquer Monge's optimal transport problem (26). In 1942, Kantorovich (27) presented a relaxed formulation that instead searches for a distribution $\pi \in \Pi(\mu_0, \mu_1)$ on $\mathbb{R}^n \times \mathbb{R}^n$, referred to as "coupling" of μ_0 and μ_1, that solves

$$\inf_{\pi \in \Pi(\mu_0,\mu_1)} \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x,y)\pi(dxdy). \qquad 3.$$

Here, $\Pi(\mu_0, \mu_1)$ denotes the set of joint distributions with marginals μ_0 and μ_1. Clearly, when the coupling π is induced by a feasible transport map, that is, $\pi = (\mathrm{Id} \times T)_\sharp \mu_0$, the objective function of the Kantorovich formulation 3. coincides with that in the Monge's OMT problem 2.. Here Id stands for the identity map. Kantorovich's most important contribution was the following duality theorem which he established in[3] 1942.

Theorem 1 *Assume that the cost function c is lower semicontinuous. Then there exists a solution to Problem* 3. *Moreover*

$$\min_{\pi \in \Pi(\mu_0,\mu_1)} \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x,y)d\pi(x,y) = \sup_{(\varphi,\psi)\in\Phi_c} \left[\int \varphi d\mu_0 + \int \psi d\mu_1 \right]$$

$$\Phi_c = \{(\varphi,\psi) | \varphi \in L^1(\mu_0), \psi \in L^1(\mu_1), \varphi(x) + \psi(y) \le c(x,y)\}.$$

When μ_0, μ_1 are absolutely continuous with respect to the Lebesgue measure, and $c(x,y) = c(x-y)$ for some strongly convex c, the Monge's OMT problem 2. has a unique (28, 26, 29) solution $T^\star$ and it is equivalent to 3. in the sense that

$$\pi^\star = (\mathrm{Id} \times T^\star)_\sharp \mu_0$$

solves 3.. For the most part of this paper, we assume that $c(x,y) = \|x-y\|^2$, in which case the unique optimal transport $T^\star$ is the gradient of a convex function ϕ, cf. (28, 26), i.e.,

$$T^\star(x) = \nabla\phi(x). \qquad 4.$$

With this quadratic cost, the square root of the optimal cost in 3. defines the celebrated Wasserstein metric[4] (30, 31, 26, 32, 29) over the space of probability distributions.

Clearly, Kantorovich's formulation 3. may be seen as a special, yet infinite-dimensional, linear programming (LP) problem[5]. In spite of abundance of linear programming algorithms, 3. remains a challenging problem when the state dimension n is large since the size of the discretization grid grows exponentially with n. A partial remedy is to solve regularized OMT problems for an approximate solution, with entropy regularization being the most popular and effective. Including an entropy regularizer, the OMT problem 3. becomes

$$\inf_{\pi \in \Pi(\mu_0,\mu_1)} \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x,y)\pi(x,y)dxdy + \epsilon \int_{\mathbb{R}^n \times \mathbb{R}^n} \pi(x,y)\log\pi(x,y)dxdy. \qquad 5.$$

It turns out that, in fact, this regularized OMT problem coincides with the classical Schrödinger bridge problem (SBP) (33, 34) that we discuss next.

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[3]This was several years before von Neumann's duality theorem in the finite dimensional setting.

[4]More precisely, it defines the Wasserstein-2 metric. The general Wasserstein-p metric is defined similarly with the unit transport cost being $c(x,y) = \|x-y\|^p$.

[5]Interestingly, Kantorovich's early contributions to LP also included a form of the simplex method to solve the finite-dimensional problem.

In 1931/32, Schrödinger (33, 34) considered the following *hot gas Gedankenexperiment*: A large number N of i.i.d. Brownian particles in $\mathbb{R}^n$ are observed to have at time $t = 0$ an empirical distribution approximately equal to ρ_0, and at some later time $t = 1$ an empirical distribution approximately equal to ρ_1. Suppose that ρ_1 considerably differs from what it should be according to the law of large numbers, namely

$$\int q^B(0, x, 1, y)\rho_0(x)dx,$$

where

$$q^B(s, x, t, y) = (2\pi)^{-n/2}(t-s)^{-n/2} \exp\left(-\frac{\|x-y\|^2}{2(t-s)}\right)$$

denotes the Brownian transition probability density kernel. It is apparent that the particles have been transported in an unlikely way. But of the many unlikely ways in which this could have happened, which one is the most likely? In view of Sanov's theorem, see Föllmer (35), Schrödinger's question reduces to determining a probability law $\mathcal{P}(\cdot)$ on $C[0,1]$, the continuous paths on $\mathbb{R}^n$ over the time interval $[0,1]$, that minimizes the relative entropy

$$\mathbb{D}(\mathcal{P}\|\mathcal{Q}) := \int_{C[0,1]} \log\left(\frac{d\mathcal{P}}{d\mathcal{Q}}\right) d\mathcal{P}. \qquad 6.$$

Wiener measure: a class of measures over path space induced by Brownian motion.

Here $\frac{d\mathcal{P}}{d\mathcal{Q}}$ denotes the Radon-Nikodym derivative, $\mathcal{Q}(\cdot)$ is the probability law induced by the prior Markovian evolution (Wiener measure in Schrödinger's original problem). $\mathcal{P}(\cdot)$ is chosen among probability laws that are absolutely continuous with respect to $\mathcal{Q}(\cdot)$ and have the prescribed marginals.

Disintegration of measures

For a given measure $\mathcal{P}$ over path space $C[0, 1]$, let $\mathcal{P}^{xy}$ represents conditioning of $\mathcal{P}$ on paths that take values x and y at $t \in \{0, 1\}$, respectively, and $\mathcal{P}_{01}$ denotes the joint probability for the values of paths at the two ends, $t \in \{0, 1\}$. Then, $\mathcal{P}$ can be disintegrated (36) into

$$\mathcal{P}(\cdot) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{P}^{xy}(\cdot)\mathcal{P}_{01}(dxdy)$$

By disintegration of measures,

$$\mathbb{D}(\mathcal{P}\|\mathcal{Q}) = \mathbb{D}(\mathcal{P}_{01}\|\mathcal{Q}_{01}) + \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbb{D}(\mathcal{P}^{xy}\|\mathcal{Q}^{xy})\mathcal{P}_{01}(dxdy).$$

The second term on the right is nonnegative and the minimum value 0 is achieved when the $\mathcal{P}^{xy}$ is the same as $\mathcal{Q}^{xy}$ for each x, y. Thus, the *Schrödinger Bridge Problem* (SBP), to identify a probability law $\mathcal{P}$ that is in agreement with the specified marginals while minimizing $\mathbb{D}(\mathcal{P}\|\mathcal{Q})$, reduces to

$$\inf_{\mathcal{P}_{01} \in \Pi(\rho_0, \rho_1)} \int_{\mathbb{R}^n \times \mathbb{R}^n} \log\left(\frac{d\mathcal{P}_{01}}{d\mathcal{Q}_{01}}\right) d\mathcal{P}_{01}.$$

The solution to this optimization problem is referred to as the *Schrödinger bridge*. Existence of the minimizer has been proven in various degrees of generality by Fortet (37), Beurling (38), Jamison (39), Föllmer (35) with Jamison's result being stated below, for a general diffusion kernel.

Theorem 2 *Given two probability measures $\mu_0(dx) = \rho_0(x)dx$ and $\mu_1(dy) = \rho_1(y)dy$ on $\mathbb{R}^n$ and the continuous, everywhere positive Markov kernel $q(s,x,t,y)$, there exists a unique pair (up to scaling) of functions $(\hat{\varphi}_0, \varphi_1)$ on $\mathbb{R}^n$ such that the measure $\mathcal{P}_{01}$ on $\mathbb{R}^n \times \mathbb{R}^n$ defined by*

$$\mathcal{P}_{01}(E) = \int_E q(0,x,1,y)\hat{\varphi}_0(x)\varphi_1(y)dxdy \qquad 7.$$

has marginals μ_0 and μ_1. Furthermore, the Schrödinger bridge from μ_0 to μ_1 induces the distribution flow

$$\mathcal{P}_t(dx) = \varphi(t,x)\hat{\varphi}(t,x)dx, \text{ with} \qquad 8a.$$

$$\varphi(t,x) = \int q(t,x,1,y)\varphi_1(y)dy \qquad 8b.$$

$$\hat{\varphi}(t,x) = \int q(0,y,t,x)\hat{\varphi}_0(y)dy. \qquad 8c.$$

When the Markov kernel is associated with a scaled Brownian motion, that is,

$$q = q_\epsilon^B := (2\pi)^{-n/2}((t-s)\epsilon)^{-n/2} \exp\left(-\frac{\|x-y\|^2}{2(t-s)\epsilon}\right), \qquad 9.$$

the Schrödinger bridge problem reduces to

$$\min_{\pi\in\Pi(\mu_0,\mu_1)} \int_{\mathbb{R}^n\times\mathbb{R}^n} \pi(x,y)\log\frac{\pi(x,y)}{q_\epsilon^B(0,x,1,y)}dxdy,$$

which can readily be checked to reduce to 5. with a quadratic cost $c(x,y) = \|x-y\|^2$, after discarding constant terms. Thus, the SBP can be viewed as an entropy regularized OMT problem with quadratic cost.

Since the optimal solution $\mathcal{P}_{01}$ depends only on $\hat{\varphi}_0, \varphi_1$, to solve the SBP, we just need to find a proper pair function $\hat{\varphi}_0, \varphi_1$ such that $\mathcal{P}_{01} \in \Pi(\mu_0, \mu_1)$. Setting $\varphi_0 = \varphi(0,\cdot)$ and $\hat{\varphi}_1 = \hat{\varphi}(1,\cdot)$, then we obtain

$$\rho_0 = \varphi_0(\cdot)\hat{\varphi}_0(\cdot) \qquad 10a.$$

$$\rho_1 = \varphi_1(\cdot)\hat{\varphi}_1(\cdot), \qquad 10b.$$

from 8a., and

$$\varphi_0(x) = \int q(0,x,1,y)\varphi_1(y)dy \qquad 10c.$$

$$\hat{\varphi}_1(y) = \int q(0,x,1,y)\hat{\varphi}_0(x)dx \qquad 10d.$$

from 8b.-8c.. A natural algorithm to solve the SBP was formulated by Robert Fortet in 1940 (37, 40) by tracing the following circular sequence of computations

$$\begin{array}{ccc} \hat{\varphi}_0(\cdot) & \xrightarrow{10d} & \hat{\varphi}_1(\cdot) \\ {\scriptstyle 10a}\uparrow & & \downarrow{\scriptstyle 10b} \\ \varphi_0(\cdot) & \xleftarrow{10c} & \varphi_1(\cdot) \end{array} \qquad 11.$$

or, equivalently, by iterating the composition of maps

$$\hat{\varphi}_0(\cdot) \xrightarrow{10d} \hat{\varphi}_1(\cdot) \xrightarrow{10b} \varphi_1(\cdot) \xrightarrow{10c} \varphi_0(\cdot) \xrightarrow{10a} (\hat{\varphi}_0(\cdot))_{\text{next}}\,. \qquad 12.$$

Fortet established directly convergence of a rather complex scheme involving three different sequences of functions. The iteration may be shown, under appropriate assumptions, to be strictly contractive with respect to a suitable projective metric, namely the Hilbert metric and thus the algorithm converges globally (41). In the discrete setting, these algorithms are named IPF-Sinkhorn; establishing their convergence is much simpler than in the continuous case.

3. Density control

Both 2. and 3. are static formulations of OMT. Their solution specifies what the optimal mass allocation strategy is. It does not provide, however, details on how to achieve this. In 2000, a dynamic (Eulerian) formulation of OMT was discovered in the seminal work (42) which addresses this issue. More specifically, when μ_0 and μ_1 are absolutely continuous, i.e., $\mu_0(dx) = \rho_0(x)dx$, $\mu_1(dy) = \rho_1(y)dy$, with ρ_0, ρ_1 being the corresponding density functions, the dynamic formulation of OMT for quadratic cost $c(x,y) = \|x-y\|^2$ reads

$$\inf_{\rho,v} \int_{\mathbb{R}^n} \int_0^1 \frac{1}{2}\|v(t,x)\|^2 \rho(t,x)dtdx, \qquad 13a.$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(v\rho) = 0, \qquad 13b.$$

$$\rho(0,x) = \rho_0(x), \quad \rho(0,y) = \rho_1(y). \qquad 13c.$$

The minimum is taken over all pairs (ρ, v) satisfying 13b.-13c., and some additional technical assumptions, see (26, Theorem 8.1), (32, Chapter 8). The solution to 13. clarifies that the optimal mass reallocation can be achieved by moving the mass following a time-varying velocity field $v(t,x)$. Moreover, $\rho(t,x)$ clearly describes how the mass evolves over time when the optimal transport plan is utilized.

Equation 13b. is the continuity equation of fluid dynamics. It also describes the evolution of probability distribution of the state for a closed-loop first-order integrator. In particular, the state distribution for the system $\dot{x}^v(t) = v(t, x^v(t))$ with feedback control $v(\cdot,\cdot)$, and initial state $x^v(0) \sim \rho_0$ follows exactly 13b. with initial condition ρ_0.

The objective function 13a. also has the stochastic interpretation

$$\int_{\mathbb{R}^n} \int_0^1 \frac{1}{2}\|v(t,x)\|^2 \rho(t,x)dtdx = \mathbb{E}\left\{\int_0^1 \frac{1}{2}\|v(t,x^v(t))\|^2 dt\right\}.$$

Thus, we arrive at the stochastic control formulation of OMT as

$$\inf_{v\in\mathcal{V}} \mathbb{E}\left\{\int_0^1 \frac{1}{2}\|v(t,x^v(t))\|^2 dt\right\}, \qquad 14a.$$

$$\dot{x}^v(t) = v(t, x^v(t)), \qquad 14b.$$

$$x^v(0) \sim \mu_0, \quad x^v(1) \sim \mu_1, \qquad 14c.$$

where $\mathcal{V}$ represents the family of admissible state feedback control strategies, for which the controlled system 14b. has a unique solution for almost every deterministic initial condition at $t = 0$. Note that we have used μ_0, μ_1 to account for the possibility of singular marginal distributions.

Problem 14. is a special case of density/uncertainty control for the simple case of first-order integrator dynamics. In general, the goal of such a density/uncertainty control problem is to drive a dynamical system from a given initial uncertain state to a target one with minimum cost. It differs from standard optimal control in the added constraint on the terminal state distribution and the absence of a terminal penalty in the index. Note that the scenario when μ_0, μ_1 are Dirac measures does fall into scope of standard optimal control. Thus, to some extent, density control can be viewed as a relaxation of optimal control problem, replacing hard state constraints by soft (probabilistic) ones. On the other hand, when viewed as a control problem over the space of probability densities as in 13., it is in fact a standard, albeit infinite dimensional, optimal control problem with hard constraints $\rho(0,\cdot) = \rho_0, \rho(1,\cdot) = \rho_1$ at the two end points.

One strategy (13) to solve this atypical optimal control problem 14. is to transform it into a standard one by adding an artificial terminal cost ψ_1 to 14., without enforcing the terminal constraint $x^v(1) \sim \mu_1$ at the outset. Applying dynamic programming to the resulting problem leads to

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\|\nabla\psi\|^2 = 0 \qquad \text{15a.}$$

with terminal condition $\psi(1,\cdot) = \psi_1$ and associated optimal control being $v(t,x) = \nabla\psi(t,x)$. Substituting back to 14b. yields the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\nabla\psi) = 0. \qquad \text{15b.}$$

To constitute a solution to 14., ρ has to satisfy the boundary conditions

$$\rho(0,\cdot) = \rho_0, \quad \rho(0,\cdot) = \rho_1. \qquad \text{15c.}$$

For a fixed ρ_0, the procedure determines a map from ψ_1 to $\rho(1,\cdot)$. If for some ψ_1, the resulting $\rho(1,\cdot)$ matches the specified boundary distribution ρ_1, then $v(t,x) = \nabla\psi(t,x)$ is in fact a solution to 14..

To find such a ψ_1, one essentially needs to solve the partial differential equation (PDE) system 15.. It turns out that 15. always has a unique solution (up to a constant shift on ψ). This implies that, given a fixed ρ_0, for any target distribution ρ_1, there is a unique terminal cost ψ that can be added to the density control problem 14. such that the solution to the resulting standard optimal control problem also solves 14.. This terminal cost in fact relates to ϕ in 4. as

$$\psi_1(x) = \frac{\|x\|^2}{2} - \phi^*(x) \qquad \text{16.}$$

where ϕ^* denotes the convex conjugate (43) of ϕ. With this ψ_1, the solution to 15a. can be obtained using the Hopf-Lax formula, yielding

$$\psi(t,x) = \inf_y \left\{\psi_1(y) + \frac{\|x-y\|^2}{2(1-t)}\right\}, \quad t\in[0,1).$$

Remark 1 *The PDE system* 15. *can also be obtained by (formally) applying Pontryagin's maximum principle to the fluid dynamic formulation* 13. *(see (15, 13) for more details).*

Such a connection between dynamic programming and the maximum principle for the associated dynamics over state distribution is expected to occur for more general stochastic control problems.

The entropy regularized OMT, or equivalently the SBP, can also be cast as a stochastic control problem. Specifically, the SBP with prior diffusion kernel q_ϵ^B in 9. becomes

$$\inf_{v\in\mathcal{V}} \mathbb{E}\left\{\int_0^1 \frac{1}{2}\|v(t,x^v)\|^2 dt\right\}, \quad \text{17a.}$$

$$dx^v(t) = v(t,x^v(t))dt + \sqrt{\epsilon}dw(t), \quad \text{17b.}$$

$$x^v(0) \sim \rho_0, \quad x^v(1) \sim \rho_1, \quad \text{17c.}$$

where again $\mathcal{V}$ denotes the set of admissible state feedback control laws and dw represents standard white noise. Departing from 14., the underlying dynamics in 17. is a stochastic diffusion process. The derivation of this stochastic control reformulation of SBP is completely different to that of OMT. It builds on the celebrated Girsanov theorem (44), stating that

$$\frac{d\mathcal{P}_{x^v}}{d\mathcal{P}_{x^0}} = \exp\left\{\int_0^1 \frac{1}{\sqrt{\epsilon}} v(t,x^v(t))\cdot dw + \int_0^1 \frac{1}{2\epsilon}\|v(t,x^v(t))\|^2 dt\right\} \quad \text{18.}$$

where $\mathcal{P}_{x^v}, \mathcal{P}_{x^0}$ denote the measures induced by x^v and x^0, respectively (with $x^0 := x^{v(\cdot)=0}$). Substituting into 6. yields (see (45) for more details) a remarkable conclusion that the relative entropy between the controlled process and the uncontrolled one is equal to the control energy (scaled by $1/\epsilon$)! This is summarized in the following theorem.

Theorem 3

$$\mathbb{D}(\mathcal{P}_{x^v}\|\mathcal{P}_{x^0}) = \mathbb{E}\left\{\int_0^1 \frac{1}{2\epsilon}\|v(t,x^v(t))\|^2 dt\right\}. \quad \text{19.}$$

When described in terms of state probability distributions ρ, the stochastic control problem 17. has the following reformulation (12, 46)

$$\inf_{\rho,v} \int_{\mathbb{R}^n}\int_0^1 \frac{1}{2}\|v(t,x)\|^2 \rho(t,x)dtdx, \quad \text{20a.}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(v\rho) - \frac{\epsilon}{2}\Delta\rho = 0, \quad \text{20b.}$$

$$\rho(0,\cdot) = \rho_0, \quad \rho(1,\cdot) = \rho_1, \quad \text{20c.}$$

where 20b. is the Fokker-Planck equation capturing the state distribution evolution. The infimum is over smooth fields v and ρ that solve weakly this Fokker-Planck equation. Formulation 20. is similar to OMT 13. except for the presence of the Laplacian in 20b.. Intuitively, when $\epsilon \searrow 0$, 20. converges to 13.. This connection has been justified in (47, 48, 49, 36), stating that the OMT problem is, in a suitable sense, the limit of the SBP when the diffusion coefficient of the reference Brownian motion q_ϵ^B goes to zero. This echoes with the fact that SBP is a regularized OMT (with regularization intensity ϵ).

Fisher Information Functional regularization

An alternative equivalent reformulation of SBP given in (15) is

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n}\int_0^1 \left[\frac{1}{2}\|v(t,x)\|^2 + \frac{\epsilon^2}{8}\|\nabla \log \rho(t,x)\|^2\right]\rho(t,x)dtdx,$$
$$\frac{\partial \rho}{\partial t} + \nabla\cdot(v\rho) = 0$$
$$\rho(0,x) = \rho_0(x), \quad \rho(1,y) = \rho_1(y),$$

where the Laplacian in the dynamical constraint is traded for a "Fisher information" regularization term in the cost functional. This reformulation answers a question posed by E. Carlen in 2006 investigating the connections between OMT and Nelson's stochastic mechanics (50), cf. (15, Section 5).

The stochastic control formulation of SBP can be solved following a similar strategy as in 14. for OMT, which yields coupled PDE system

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\|\nabla\psi\|^2 + \frac{\epsilon}{2}\Delta\psi = 0 \qquad \text{21a.}$$
$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\nabla\psi) - \frac{\epsilon}{2}\Delta\rho = 0 \qquad \text{21b.}$$
$$\rho(0,\cdot) = \rho_0, \quad \rho(0,\cdot) = \rho_1 \qquad \text{21c.}$$

that resembles 15. with optimal control strategy being $v(t,x) = \nabla\psi(t,x)$. In the above, 21a. is a second-order Hamilton-Jacobi-Bellman (HJB) equation. Applying a logarithmic transformation $\psi = \epsilon\log\varphi$ and $\hat{\varphi} = \rho/\varphi$, the system of equations 21. is cast in the form

$$\frac{\partial \varphi}{\partial t} + \frac{\epsilon}{2}\Delta\varphi = 0 \qquad \text{22a.}$$
$$\frac{\partial \hat{\varphi}}{\partial t} - \frac{\epsilon}{2}\Delta\hat{\varphi} = 0 \qquad \text{22b.}$$
$$\varphi(0,\cdot)\hat{\varphi}(0,\cdot) = \rho_0, \quad \varphi(1,\cdot)\hat{\varphi}(1,\cdot) = \rho_1. \qquad \text{22c.}$$

This is a pair of linear PDEs only coupled through boundary conditions; 22a. is a backward Kolmogrov equation and 22b. is a Fokker-Planck equation. The optimal control to 17. is then given by $v(t,x) = \epsilon\nabla\log\varphi(t,x)$.

Interestingly, 22. is in fact the Schrödinger system for the SBP associated with transition kernel q_ϵ^B; it is easy to see that 22a.-22b. are just PDE's corresponding to 8. for $q = q_\epsilon^B$. The analytic nature of $\hat{\varphi}$, being a harmonic function, and φ a co-harmonic (i.e., harmonic in the reverse time-direction) is noted.

We have seen that the standard OMT and Schrödinger bridge theories provide elegant solutions to the density control problems associated with deterministic/stochastic first-order integrator. From a control theory point of view, a natural direction to pursue is to establish a framework for density control of general stochastic systems

$$dx(t) = f(t,x,u)dt + \sigma(t,x)dw. \qquad \text{23.}$$

Such an approach is on-going and has already led to fruitful results in a number of directions (9, 10, 12, 51, 15, 13, 17, 14, 52, 53). Next, in Section 4, we will focus on the case of linear systems with Gaussian stochastic uncertainty while we briefly mention more general cases in Section 5.

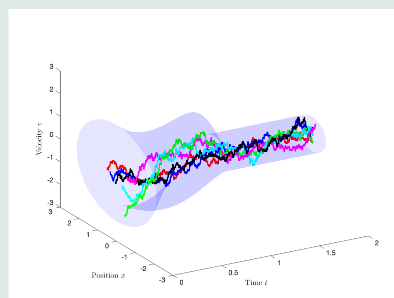
4. Covariance control

In this section, we focus on density control problems for general linear dynamics with Gaussian distributions. This subject is known as covariance control or, alternatively, as covariance steering. The term "covariance control" first arose in the work of Skelton and his coworkers (8, 7, 6) to describe steady state regulation of state statistics, whereas the qualifier "steering" was used later on to describe the class of problems where state statistics are prescribed to constrain a controlled finite time transition (9, 10, 14, 52, 53). This covariance steering/control framework has found use in a range of applications such as active cooling of stochastic oscillators (51).

Active cooling

Newton's laws relate position x and velocity v of particles to friction $-bv(t)$, conservative forces $-\nabla V$, with V representing a potential, stochastic forcing dW, and control action $u(t)$ as in

$$\begin{aligned} dx(t) &= \quad v(t)\,dt, \\ mdv(t) &= -bv(t)\,dt + u(t)dt - \nabla V(x(t))dt + \sigma dW(t), \end{aligned}$$



with $x(t_0) = x_0$ and $v(t_0) = v_0$ a.s. In a variety of applications relating to scientific instrumentation, the task of the control u is to suppress state uncertainty and thus, through control action, ensure a lower effective temperature than what the stochastic excitation dictates.

When the potential V is quadratic, the stationary (Boltzmann) distribution becomes Gaussian and the problem reduces to a covariance steering/control problem. The picture to the right displays typical trajectories in phase space under suitably selected control to steer and maintain the state covariance. The transparent tube represents the 3-σ region of the Gaussian distribution inside which the trajectories should lie with probability at least 99.7%.

4.1. Minimum energy steering

Consider linear dynamics

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + \sqrt{\epsilon}B(t)dw(t) \qquad 24.$$

where the pair A, B is assumed to be controllable in the sense that the reachability Gramian

$$M(t,s) = \int_s^t \Psi(t,\tau)B(\tau)B(\tau)'\Psi(t,\tau)'d\tau$$

with $\Psi(\cdot,\cdot)$ denoting the state transition matrix for A, is nonsingular for all $s < t$. For the sake of simplicity, we don't make the dependence of A, B over t explicit unless necessary. Assume that the initial state $x(0)$ is a random vector with Gaussian distribution $\rho_0 = \mathcal{N}(m_0, \Sigma_0)$. We seek a minimum energy control input over time interval[6] [0, 1] that steers the system to a target state distribution $\rho_1 = \mathcal{N}(m_1, \Sigma_1)$. We assume $\Sigma_1 > 0$; the case where Σ_1 is singular is more challenging and has been addressed in (54). Formally, it reads

$$\inf_{u\in\mathcal{U}} J(u) = \mathbb{E}\left\{\int_0^1 \|u(t)\|^2 dt\right\}, \tag{25a.}$$

$$dx(t) = Ax(t)dt + Bu(t)dt + \sqrt{\epsilon}Bdw(t) \tag{25b.}$$

$$x(0) \sim \mathcal{N}(m_0, \Sigma_0), \quad x(1) \sim \mathcal{N}(m_1, \Sigma_1), \tag{25c.}$$

where the minimization is over the set $\mathcal{U}$ of all admissible control laws. By linearity of this problem, the mean/expectation of the control drives the deterministic part of the dynamics from initial value m_0 to terminal value m_1, and can be obtained independent to the covariances (see (9) for more details). Thus, without loss of generality, for the rest of this paper, we assume $m_0 = m_1 = 0$.

This problem resembles a standard stochastic linear quadratic regulator problem except for the boundary conditions. As in Section 3, we adopt the strategy of adding artificial terminal cost while relaxing the terminal constraint, to bring it into the form of standard optimal control. Then, subsequently, investigate the possibility of selecting the terminal cost to enforce the constraint. To this end we assume that $\{\Pi(t) \mid 0 \le t \le 1\}$ is a differentiable function taking values in the set of $n \times n$ symmetric matrices, and construct an augmented cost

$$\tilde{J}(u) = \mathbb{E}\left\{\int_0^1 \|u(t)\|^2 dt + x(1)'\Pi(1)x(1) - x(0)'\Pi(0)x(0)\right\}, \tag{26.}$$

Then, minimizing $\tilde{J}(u)$ or $J(u)$ over all control strategies while enforcing the boundary conditions 25c. give the same answer, since the added terms are constant and have no effect. But

$$\tilde{J}(u) = \mathbb{E}\left\{\int_0^1 \|u(t)\|^2 dt + \int_0^1 d\left(x(t)'\Pi(t)x(t)\right)\right\}.$$

If we select $\Pi(t)$ on $[0, 1]$ to satisfy the Riccati equation

$$\dot{\Pi}(t) = -A'\Pi(t) - \Pi(t)A + \Pi(t)BB'\Pi(t), \tag{27a.}$$

then

$$\tilde{J}(u) = \mathbb{E}\left\{\int_0^1 \|u(t) + B'\Pi(t)x(t)\|^2 dt + \int_0^1 \frac{\epsilon}{2}\,\mathrm{trace}\left(\Pi(t)BB'\right) dt\right\},$$

by Itô calculus. Clearly, if boundary values for Π can be found so that the choice

$$u^\star(t) = -B'\Pi(t)x(t)$$

ensures that the boundary conditions $\Sigma(0) = \Sigma_0$ and $\Sigma(1) = \Sigma_1$ hold, for the state covariance, in agreement with the Lyapunov equation

$$\dot{\Sigma}(t) = \left(A - BB'\Pi(t)\right)\Sigma(t) + \Sigma(t)\left(A - BB'\Pi(t)\right)' + \epsilon BB', \tag{27b.}$$

[6]Any time window can be converted to [0, 1] by rescaling time. Thus, without loss of generality, we assume a unit time window for notational simplicity.

then this choice of control is indeed optimal. Thus, we seek a solution pair $(\Pi(t), \Sigma(t))$ of the *coupled* system of these equations 27a. and 27b. with split boundary conditions

$$\Sigma(0) = \Sigma_0, \quad \Sigma(1) = \Sigma_1. \tag{27c.}$$

To solve for the pair $(\Pi(t), \Sigma(t))$, when $\epsilon > 0$, we define

$$\mathrm{H}(t) := \epsilon\Sigma(t)^{-1} - \Pi(t),$$

that leads the system of coupled Riccati equations through their boundary values,

$$\dot{\Pi}(t) = - A'\Pi(t) - \Pi(t)A + \Pi(t)BB'\Pi(t) \tag{28a.}$$
$$\dot{\mathrm{H}}(t) = - A'\mathrm{H}(t) - \mathrm{H}(t)A - \mathrm{H}(t)BB'\mathrm{H}(t) \tag{28b.}$$
$$\epsilon\Sigma_0^{-1} = \Pi(0) + \mathrm{H}(0), \quad \epsilon\Sigma_1^{-1} = \Pi(1) + \mathrm{H}(1). \tag{28c.}$$

Expressing 28a.-28b. in terms of Π^{-1}, H^{-1}, we arrive at two Lyapunov equations instead. Based on this transformation, the following closed-form solution to 28. was obtained in (9),

$$\Pi_\epsilon(0) = \frac{\epsilon}{2}\Sigma_0^{-1} + \Psi_{10}'M_{10}^{-1}\Psi_{10} - \Sigma_0^{-1/2}(\frac{\epsilon^2}{4}I + \Sigma_0^{1/2}\Psi_{10}'M_{10}^{-1}\Sigma_1 M_{10}^{-1}\Psi_{10}\Sigma_0^{1/2})^{1/2}\Sigma_0^{-1/2}, \tag{29.}$$

where $M_{10} := M(1,0), \Psi_{10} := \Psi(1,0)$, and the subscript ϵ is used to emphasize its dependences on the value of ϵ. The optimal control is in a state feedback form $u^\star(t,x) = -B'\Pi_\epsilon(t)x$, with $\Pi_\epsilon(t)$ being the solution to the Riccati equation 27a..

Linear-quadratic Gaussian Schrödinger system

Comparing the covariance steering in 25. to the basic SBP 17., the former allows general dynamics but is restricted to Gaussian marginals. For $A = 0$, $B = I$, the correspondence between the solutions of the two problems (by solving systems 28. and 22., respectively) is as follows,

$$\varphi(x) \propto \exp(-\|x\|_\Pi^2) \text{ and } \hat{\varphi}(x) \propto \exp(-\|x\|_\mathrm{H}^2).$$

Note that 28. becomes meaningless when $\epsilon = 0$. To get the solution for $\epsilon = 0$, we can take the limit of 29. by letting $\epsilon \searrow 0$, which leads to

$$\Pi_0(0) = \Sigma_0^{-1/2}[\Sigma_0^{1/2}\Psi_{10}'M_{10}^{-1}\Psi_{10}\Sigma_0^{1/2} - (\Sigma_0^{1/2}\Psi_{10}'M_{10}^{-1}\Sigma_1 M_{10}^{-1}\Psi_{10}\Sigma_0^{1/2})^{1/2}]\Sigma_0^{-1/2}. \tag{30.}$$

The optimal control is once again a state feedback $u^\star(t,x) = -B'\Pi_0(t)x$ with $\Pi_0(t)$ the solution to the Riccati equation 27a.. In fact, $\Pi_0(t)$ has the explicit form

$$\Pi_0(t) = -M(t,0)^{-1}\Psi(t,0)\Big[\Psi_{10}'M_{10}^{-1}\Psi_{10} - \Sigma_0^{-1/2}(\Sigma_0^{1/2}\Psi_{10}'M_{10}^{-1}\Sigma_1 M_{10}^{-1}\Psi_{10}\Sigma_0^{1/2})^{1/2}$$
$$\Sigma_0^{-1/2}\Psi(t,0)'M(t,0)^{-1}\Psi(t,0)\Big]^{-1}\Psi(t,0)'M(t,0)^{-1} - M(t,0)^{-1}.$$

Standard OMT and SBP with Gaussian marginals correspond to $A = 0, B = I$, giving

$$\Pi_\epsilon(0) = \frac{\epsilon}{2}\Sigma_0^{-1} + I - \Sigma_0^{-1/2}(\frac{\epsilon^2}{4}I + \Sigma_0^{1/2}\Sigma_1\Sigma_0^{1/2})^{1/2}\Sigma_0^{-1/2}. \tag{31.}$$

4.2. State penalty

State penalty can also be introduced into density control. In the covariance control setting, we arrive at

$$\inf_{u\in\mathcal{U}} \mathbb{E}\left\{\int_0^1 [\|u(t)\|^2 + x(t)'Q(t)x(t)]dt\right\}, \tag{32a.}$$

$$dx(t) = Ax(t)dt + Bu(t)dt + \sqrt{\epsilon}Bdw(t), \tag{32b.}$$

$$x(0) \sim \mathcal{N}(0,\Sigma_0), \quad x(1) \sim \mathcal{N}(0,\Sigma_1), \tag{32c.}$$

where $Q(\cdot)$ is the weight for the state penalty, that does not need to be nonnegative.

Following a similar strategy as for the minimum energy covariance control 25., for $\epsilon > 0$, we obtain two Riccati equations that are coupled through boundary conditions,

$$-\dot{\Pi}(t) = A'\Pi(t) + \Pi(t)A - \Pi(t)BB'\Pi(t) + Q(t), \tag{33a.}$$

$$-\dot{\mathrm{H}}(t) = A'\mathrm{H}(t) + \mathrm{H}(t)A + \mathrm{H}(t)BB'\mathrm{H}(t) - Q(t), \tag{33b.}$$

$$\epsilon\Sigma_0^{-1} = \Pi(0) + \mathrm{H}(0), \quad \epsilon\Sigma_1^{-1} = \Pi(1) + \mathrm{H}(1). \tag{33c.}$$

The corresponding optimal control is once again in a state feedback form

$$u(t,x) = -B(t)'\Pi(t)x. \tag{34.}$$

This new system of coupled Riccati equations 33. is substantially different from 28. in that they can no longer be directly transformed into linear Lyapunov equations. Yet, it is still possible to obtain solutions in closed form by expressing Π in 33a. (and similarly for H) as a matrix fraction $\Pi(t) = Y(t)X(t)^{-1}$, with $[X, Y]$ satisfying the linear dynamics

$$\begin{bmatrix}\dot{X}\\ \dot{Y}\end{bmatrix} = \begin{bmatrix}A(t) & -B(t)B(t)'\\ -Q(t) & -A(t)'\end{bmatrix}\begin{bmatrix}X\\ Y\end{bmatrix}. \tag{35.}$$

Indeed, if we denote the state transition matrix of this linear system by

$$\Phi(t,s) = \begin{bmatrix}\Phi^{11}(t,s) & \Phi^{12}(t,s)\\ \Phi^{21}(t,s) & \Phi^{22}(t,s),\end{bmatrix}, \tag{36.}$$

and for simplicity

$$\begin{bmatrix}\Phi_{10}^{11} & \Phi_{10}^{12}\\ \Phi_{10}^{21} & \Phi_{10}^{22}\end{bmatrix} := \begin{bmatrix}\Phi^{11}(1,0) & \Phi^{12}(1,0)\\ \Phi^{21}(1,0) & \Phi^{22}(1,0)\end{bmatrix},$$

then the system 33c. has a unique solution specified by (14)

$$\Pi_\epsilon(0) = \frac{\epsilon\Sigma_0^{-1}}{2} - (\Phi_{10}^{12})^{-1}\Phi_{10}^{11} - \Sigma_0^{-1/2}\left(\frac{\epsilon^2 I}{4} + \Sigma_0^{1/2}(\Phi_{10}^{12})^{-1}\Sigma_1((\Phi_{10}^{12})^{-1})'\Sigma_0^{1/2}\right)^{1/2}\Sigma_0^{-1/2}. \tag{37.}$$

We leave it as an exercise for the reader to check that 37. reduces to 29. when $Q(\cdot) \equiv 0$.

The optimal control in cases where $\epsilon = 0$ is again a linear state feedback $u(t,x) = -B(t)'\Pi_0(t)x$, with $\Pi_0(\cdot)$ determined from the initial condition

$$\Pi_0(0) = -(\Phi_{10}^{12})^{-1}\Phi_{10}^{11} - \Sigma_0^{-1/2}\left(\Sigma_0^{1/2}(\Phi_{10}^{12})^{-1}\Sigma_1((\Phi_{10}^{12})^{-1})'\Sigma_0^{1/2}\right)^{1/2}\Sigma_0^{-1/2},$$

obtained by letting $\epsilon \searrow 0$ in 37..

4.3. Different input and noise channels

In Section 4.1 and Section 4.2, the noise and control are assumed to enter the system through the same channels, i.e., having identical input matrices. However, in many applications (10), this may not be the case. Thus, we are led to consider covariance control for the system

$$dx(t) = Ax(t)dt + Bu(t)dt + B_1 dw(t), \quad x(0) \sim \mathcal{N}(0, \Sigma_0), \tag{38.}$$

where $B_1 \neq B$. For simplicity, we consider the minimum energy control to drive the above system 38. to a target state distribution $x(1) \sim \mathcal{N}(0, \Sigma_1)$.

In a similar manner as before we arrive at

$$\dot{\Pi} = -A'\Pi - \Pi A + \Pi BB'\Pi \tag{39a.}$$

$$\dot{\mathrm{H}} = -A'\mathrm{H} - \mathrm{H}A - \mathrm{H}BB'\mathrm{H} + (\Pi + \mathrm{H})\left(BB' - B_1 B_1'\right)(\Pi + \mathrm{H}) \tag{39b.}$$

$$\Sigma_0^{-1} = \Pi(0) + \mathrm{H}(0), \quad \Sigma_1^{-1} = \Pi(1) + \mathrm{H}(1). \tag{39c.}$$

When 39. admits a well-defined solution, then, as before, $u(t,x) := -B'\Pi(t)x$ is the optimal control to our covariance control problem (10). However, in contrast to the case where $B = B_1$, which has a closed-form solution, the two Riccati equations in 39. are coupled not only through their boundary values 39c., but also in a nonlinear manner through their dynamics in 39b.. Due to this nonlinear dynamic coupling, the existence and uniqueness of solutions for 39. is not available at present.

While in general it is not known whether the covariance steering problem corresponding to 38. has a minimizing control law, the feasibility of the problem has been established in (10); it is known that as long as (A, B) is a controllable pair, there is at least one (linear) feedback control law that drives the state from initial distribution $\mathcal{N}(0, \Sigma_0)$ to target distribution $\mathcal{N}(0, \Sigma_1)$. We provide below an approach that allows constructing suboptimal controls incurring a cost that is arbitrarily close to $\inf_{u \in \mathcal{U}} J(u)$. This is based on the fact that the covariance steering problem can be recast as an (infinitely dimensional) convex optimization problem (10).

Consider the expected control-energy

$$\mathbb{E}\left\{\int_0^1 u(t)'u(t)dt\right\} = \int_0^1 \mathrm{trace}(K(t)\Sigma(t)K(t)')dt$$

for linear state feedback controls with gain $K(t)$ and the state covariance $\Sigma(\cdot)$ satisfying the Lyapunov equation

$$\dot{\Sigma}(t) = (A + BK(t))\Sigma(t) + \Sigma(t)(A + BK(t))' + B_1 B_1'.$$

The change of variables $U(t) := \Sigma(t)K(t)'$ recasts the expected energy minimization as

$$\min_{U(\cdot),\Sigma(\cdot)} \int_0^1 \mathrm{trace}(U(t)'\Sigma(t)^{-1}U(t))dt \tag{40a.}$$

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A' + BU(t)' + U(t)B' + B_1 B_1' \tag{40b.}$$

$$\Sigma(0) = \Sigma_0, \quad \Sigma(1) = \Sigma_1, \tag{40c.}$$

which is a convex problem in the parameters (U, Σ). The optimization problem can be further converted to a semi-definite program in a standard manner (10).

Although 40. is a convex problem, it is infinite dimensional. The convexity itself is not sufficient to justify the existence of the optimizer. Rigorous analysis is not yet available to show that an optimal control to the covariance steering problem associated with 38. exists. Numerically, this convex optimization is solved by discretization over time. A suboptimal feedback gain is then recovered in the form $K(t) = -U(t)'\Sigma(t)^{-1}$.

4.4. Extensions

It is natural to extend the above discussion on covariance steering/control to infinite-horizon setting. In fact, the covariance control problem was first investigated for infinite-horizon problems in (8, 7, 6), though drawing no connection to OMT. Consider the same dynamical system 38.. Suppose that A, B and B_1 do not depend on time. The goal of covariance control in the infinite-horizon setting is to maintain the state-covariance at a fixed value $\Sigma > 0$. Unlike the finite-horizon cases, it turns out that not all $\Sigma > 0$ are achievable. There exists a constant state feedback law $u(t) = -Kx(t)$ so that $\Sigma > 0$ is a stationary state covariance for the linear stochastic controlled evolution 38. if and only if (10, Theorem 4)

$$\text{rank}\begin{bmatrix} A\Sigma + \Sigma A' + B_1 B_1' & B \\ B & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}.$$

The condition ensures that Σ satisfies the algebraic Lyapunov equation

$$(A - BK)\Sigma + \Sigma(A - BK)' + B_1 B_1' = 0$$

for a suitable K that ensures $A - BK$ being a Hurwitz matrix. Alternative conditions for stationary covariance control were presented in (8, 7, 6). The above rank condition extends (55, Theorem 1).

Another straightforward extension is to setting of output feedback with measurement noise, that is, the case where the feedback control is based on an output process

$$dy = Cxdt + dv$$

with dv being white measurement noise. It was shown in (52) that the achievable covariance $\Sigma(\cdot)$ is bounded below by the minimum estimation error using a Kalman filter.

There are other scenarios that can be considered, including a differential game setting which involves more than one agent (53), a mean-field game setting with many agents (56), and nonlinear covariance control for nonlinear dynamics (57, 20). Lastly, covariance control for discrete dynamics has been extensively studied recently in (19, 18, 22, 8).

5. Density steering

Having seen the special cases of density control in the linear Gaussian setting, let us return to general marginal distributions. We again focus on finite-horizon. A treatment in an infinite-horizon setting can be found in (58) and the references therein. Consider the nonlinear control-affine system

$$dx = f(x)dt + \sigma(x)u(t)dt + \sqrt{\epsilon}\sigma(x)dw. \qquad 41.$$

Here we have suppressed the dependences of f, σ over time t for notational simplicity. Denote $a(x) = \sigma(x)\sigma(x)'$. We assume that the system is controllable in the sense that the

Hörmander's condition (59) holds. It is equivalent to the hypoellipticity of the operator

$$\sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} f_i(x)\partial_{x_i} - \partial_t.$$

We are interested in the following density control problem

$$\inf_u \mathbb{E}\left\{\int_0^1 \left[\frac{1}{2}\|u(t)\|^2 + V(x(t))\right] dt\right\} \quad \text{42a.}$$

$$dx = f(x)dt + \sigma(x)u(t)dt + \sqrt{\epsilon}\sigma(x)dw \quad \text{42b.}$$

$$x(0) \sim \rho_0(x), \quad x(1) \sim \rho_1. \quad \text{42c.}$$

It turns out that this problem can also be formally addressed by adding an artificial terminal cost so that the resulting standard optimal control policy generates the specified target distribution. As in Section 3, this pipeline points to the coupled HJB and Fokker-Planck equation system

$$\frac{\partial \psi}{\partial t} + f \cdot \nabla\psi + \frac{1}{2}\nabla\psi \cdot a\nabla\psi + \frac{\epsilon}{2}\sum_{i,j=1}^{n} \frac{a_{ij}\partial^2\psi}{\partial x_i \partial x_j} = V \quad \text{43a.}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot ((f + \sigma\nabla\psi)\rho) - \frac{\epsilon}{2}\sum_{i,j=1}^{n} \frac{\partial^2(a_{ij}\rho)}{\partial x_i \partial x_j} = 0 \quad \text{43b.}$$

$$\rho(0,\cdot) = \rho_0, \quad \rho(0,\cdot) = \rho_1. \quad \text{43c.}$$

And once again, when $\epsilon > 0$, using the logarithmic transformation $\psi = \epsilon \log\varphi$ and $\hat{\varphi} = \rho/\varphi$, we arrive at two PDEs that are only coupled through boundary conditions as in

$$\frac{\partial \varphi}{\partial t} + f \cdot \nabla\varphi + \frac{\epsilon}{2}\sum_{i,j=1}^{n} \frac{a_{ij}\partial^2\varphi}{\partial x_i \partial x_j} = V\varphi \quad \text{44a.}$$

$$\frac{\partial \hat{\varphi}}{\partial t} + \nabla \cdot (f\hat{\varphi}) - \frac{\epsilon}{2}\sum_{i,j=1}^{n} \frac{\partial^2(a_{ij}\hat{\varphi})}{\partial x_i \partial x_j} = -V\hat{\varphi} \quad \text{44b.}$$

$$\varphi(0,\cdot)\hat{\varphi}(0,\cdot) = \rho_0, \quad \varphi(1,\cdot)\hat{\varphi}(1,\cdot) = \rho_1. \quad \text{44c.}$$

It can be shown that the density control problem is equivalent to an SBP associated with prior diffusion being the uncontrolled process 41. (with $u = 0$) with the possibility of "creation" or "killing" of rate V. More specifically, the cost function 42a. is equal to the relative entropy between the controlled process and the uncontrolled one[7]. Consequently, the existence as well as uniqueness of a function pair $(\varphi, \hat{\varphi})$ satisfying 44. and thus the optimal control is guaranteed. Moreover, the PDE system 44. leads to an algorithm that solves the density control problem. It is essentially the same as the iterative algorithm 12.

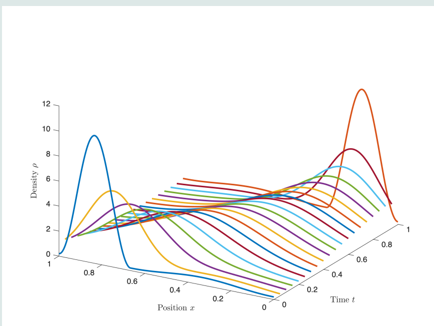
[7]This connection together with Feynman-Kac formula is explored in a different topic in stochastic control, constituting the foundation of path-integral control (60, 61, 62, 63).

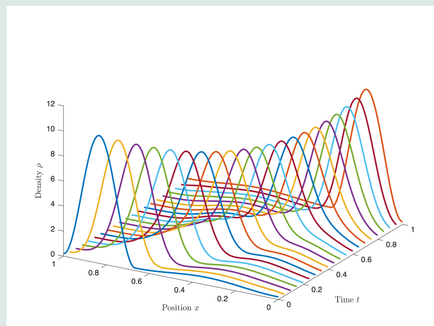
but the step 10c. is achieved by solving 44a. backward, and the step 10d. by solving 44b. forward. A potentially more scalable algorithm was recently proposed in (64).

The solution in case when $\epsilon = 0$ is more delicate. One possibility is to solve the density control problem for $\epsilon > 0$ and then take the limit $\epsilon \searrow 0$. This strategy works well under some strong assumptions such as that σ is square and nonsingular, or 41. is linear and controllable. But for general nonlinear dynamics, it is unclear whether such an approach would work or not. On the other hand, the existence of solution in case of $\epsilon = 0$ for general nonlinear control-affine system was recently established in (65) under some technical assumptions using other techniques.

Illustrative example for density control

Consider a one-dimensional diffusion process $dx(t) = -1.5x(t)dt + u(t)dt + \sqrt{\epsilon}dw(t)$. The goal is to find a feedback control to steer the state from an initial (non-Gaussian) distribution to a target one. The evolution of state distribution is depicted below for different value of ϵ (left: $\sqrt{\epsilon} = 0.5$, right: $\sqrt{\epsilon} = 0.15$)





6. Distribution steering over Markov Decision Processes (MDPs)

The density control problems in previous sections have a counterpart, in the discrete time and space setting. In (23, 66, 67), this counterpart has been explored to robustly transport a single commodity from one distribution to another over a network. For instance, such a network may represent highway connections between cities, and the task is to transport products between cities from a supply distribution to a demand distribution.

In this section, we present an alternative interpretation of this discrete counterpart of density control, as a distribution-steering problem for MDPs –a well-known discrete version of dynamical systems.

An MDP is a 4-tuple $\{\mathcal{X}, \mathcal{U}, P, C\}$ where $\mathcal{X}$ denotes the state space, $\mathcal{U}$ denotes the action space, $P(u)$ specifies the transition probability of the state for a given action $u \in \mathcal{U}$, and $C(x, u)$ denotes a running cost. Usually the goal in MDP is to search for an optimal control strategy, deterministic $u_t = \pi(x_t)$ or stochastic, specified by a distribution of $\mathcal{U}$ for each x, that minimizes a total cumulative cost $\mathbb{E}\{\sum_{t=0}^{\infty} \gamma^t C(x_t, u_t)\}$. Here $0 < \gamma \leq 1$ is a discount factor. This optimal control problem allows for the possibility of discrete states and actions.

We consider a special class of an MDP known as linearly solvable MDP (68), in which the $\mathcal{X} = \{1, 2, \ldots, n\}$ is a finite set with $|\mathcal{X}| = n$, and $\mathcal{U} = \mathbb{R}^n$. The transition kernel is $P(u) = [P_{ij}(u)]$ with

$$P_{ij}(u) \propto \bar{P}_{ij} \exp(u_j), \tag{45.}$$

and the cost is[8]

$$C(i, u) = \mathrm{KL}(P_i(u) \parallel P_i(0)) := \sum_j P_{ij}(u) \log \frac{P_{ij}(u)}{P_{ij}(0)}. \tag{46.}$$

Note $P_i = [P_{ij}]_{j=1}^n$ needs to be a probability vector that describes the state transition. In linear solvable MDPs, the state transition $P_{ij}(u)$ can be anything with proper choice of u as long as it is compatible to the zero-control one $P_{ij}(0) = \bar{P}_{ij}$, in the sense that $P_{ij}(u) = 0$ if $\bar{P}_{ij} = 0$. This structure provides considerable flexibility for the control to affect the behavior of the MDP. The running cost is 0 for zero control action, $u = 0$. For nonzero control, the cost captures the difference of the new transition kernel with respect to the prior one. In (68), the infinite horizon optimal control problem minimizing the total cost $\mathbb{E}\{\sum_{t=0}^{\infty} \gamma^t C(x_t, u_t)\}$ is studied. It turns out that the corresponding Bellman equation can be converted to a linear equation after a logarithmic transformation; this is where the name "linearly solvable MDP" comes from. The resulting linear equation can be solved efficiently and thus improve the scalability of the linearly solvable MDP. Since a KL-divergence cost is being used, this line of research became known as KL-control (68, 69, 70, 63).

Herein, we consider a finite-horizon optimal control problem over this class of MDPs with the objective of steering the state from one distribution μ_0 to a target distribution μ_T at time $t = T$. The cost to minimize is

$$\mathbb{E}\left\{\sum_{t=0}^{T-1} C(x_t, u_t)\right\} = \mathbb{E}\left\{\sum_{t=0}^{T-1} \mathrm{KL}(P_{x_t}(u_t) \parallel P_{x_t}(0))\right\}. \tag{47.}$$

This cost is exactly the KL-divergence (or equivalently, the relative entropy) of the distribution $\mathcal{P}_u$ induced by the controlled MPD on the path space relative to that of the prior MDP without control, $\mathcal{P}_0$, viz., $\mathrm{KL}(\mathcal{P}_u \| \mathcal{P}_0)$. Hence, the distribution steering problem over this class of MDPs is equivalent to a SBP with marginal constraints μ_0, μ_T and prior process being the uncontrolled MDP.

Leveraging the theory of the SBP, we obtain the following characterization of the optimal controlled transition kernel as

$$P_{ij}^\star(u)[t] = \bar{P}_{ij} \frac{\varphi_j(t+1)}{\varphi_i(t)}, \tag{48.}$$

where $\varphi, \hat{\varphi}$ solve

$$\varphi_i(t) = \sum_{j=1}^{n} \bar{P}_{ij} \varphi_j(t+1), \quad t = 0, 1, \ldots, T-1 \tag{49a.}$$

$$\hat{\varphi}_j(t+1) = \sum_{i=1}^{n} \bar{P}_{ij} \hat{\varphi}_i(t), \quad t = 0, 1, \ldots, T-1 \tag{49b.}$$

$$\varphi_i(0)\hat{\varphi}_i(0) = \mu_0(i), \quad \varphi_i(T)\hat{\varphi}_i(T) = \mu_T(i), \text{ for } i \in \{1, \ldots, n\} \tag{49c.}$$

[8]The Kullback-Leibler divergence between two probability vectors is another terminology for the relative entropy between the two. It is commonly used in the discrete setting and so, herein, we follow the convention and use "KL" instead.

The optimal control is thus time-varying as

$$u_t(x_t) = \log \frac{\varphi(t+1)}{\varphi_{x_t}(t)}. \tag{50.}$$

The coupled equation system 49. is a discrete counterpart of the Schrödinger system 8.. It can be shown (23) that it has a unique solution under the assumption that $\bar{P}^T$ has all positive entries; this condition holds when the Markov chain associated with $\bar{P}$ is irreducible and T is sufficiently large, e.g., $T \geq n$. We emphasize that the linear equation 49a. corresponds to the linear equation derived from the Bellman equation in linearly solvable MDPs (68).

Remark 2 *The KL-divergence corresponds to the control energy in the continuous setting 17.. When the running cost is* $C(i,u) = \mathrm{KL}(P_i(u) \parallel P_i(0)) + q(i)$*, it becomes a discrete counterpart of* $\mathbb{E}\left\{\int_0^1 \left[\frac{1}{2}\|u(t)\|^2 + V(x(t))\right] dt\right\}$*. Thus, the equivalence with the SBP still holds. However, the prior process becomes a generalized Markov chain with transition kernel* $\bar{P}$ *and the possibility of "creation" or "killing" with rate* $\exp(q)$*.*

Finally, an MDP induces a graph with states corresponding to nodes and allowable transitions between states corresponding to edges. As a result, the control problem over MDPs amounts to a transport problem over networks. The consequent transition probability at each state/node prescribes the transport schedule at that node. The special linearly solvable MDP structure implies that the transport schedule at each node can be assigned arbitrarily. Hence, this framework can be applied to transport problems over networks (23, 66, 67). It should be noted that the framework of transport over networks is versatile, in that, a prior transport plan (uncontrolled transition kernel) can also be taken as an additional design parameter. In fact, selecting as prior the (generalized) Ruelle-Bowen random walk (71, 72), results in transport plans that balance efficiency with robustness. We refer the reader to (23, 66, 67) for more details.

7. Closing comments

We have surveyed a number of topics that highlight the rapidly growing impact of Optimal Transport in Systems Theory and Control Engineering. The overarching theme of our exposition is on ways to control uncertainty in state trajectories of dynamical systems and to specify objectives in terms of soft probabilistic terminal constraints –thereby, the pertinent emerging trend in control theory can be referred to as *Control of Uncertainty*. There are several other applications of OMT in Systems and Control that are not covered in this short survey, such as in inverse problems (73, 74), filtering and estimation (75, 76, 77, 78, 79, 80), path planing (81), and swarm control (82).

In spite of its ancient roots, going back to Monge in 1781, Optimal Transport did not make inroads into the theory of dynamical systems until the 1990's, when Benamou, Brenier, Gangbo, McCann, Otto, and others (26, 29, 32) recast transportation with a quadratic cost in a variational form. Many important works followed. Additional impetus was provided by a far reaching and unlikely link between Optimal Transport and the so called Schrödinger Bridge problem. Schrödinger's problem was conceived as a *Gendankenexpriment* on stochastically driven particles (33, 34), aimed to shine light into the time reversibility of physical laws (49, 36, 15). In the process, E. Schrödinger put forth, along with the foundations of the *maximum entropy* inference method, a variational problem on random trajectories that ultimately turned out to be a model for the optimal steering of stochastically driven dynamical systems.

The deep connection between quadratic control cost and entropy functionals on path trajectories, via large deviation theory, was drawn in the 1990's by Dai Pra and Wakolbinger (45, 83, 47, 48). This link between SBP and OMT, besides expanding the significance of OMT in stochastic control, has provided a popular and efficient algorithm for solving OMT problems (84, 41). While the mathematics of OMT and SBP is now providing a powerful paradigm to attack many diverse problems in engineering, physics, computer science, and so on, the focus of our survey has been on the impact in Systems and Control. Specifically, our starting point was the progression from variational problems of mechanics to stochastic control in the space of state-distributions. This led to expanding classical Quadratic Regulator theory with the new chapter of Uncertainty Control. While it required a new set of technics, solutions turned out to be familiar looking in terms of differential (coupled in this case) Riccati equations. An important offshoot to optimal transport on discrete spaces/networks and the control of Markov Decision Processes was discussed. In both, continuous and discrete spaces, theoretical and computational challenges remain, such as expanding on possible state and control constraints (18, 22, 85), dealing with limits to actuation authority vis-a-vis stochastic noise (cf. Section 4.3), and dealing with high dimensions when only samples of the marginals are known (86). It would be an omission not to note that the impact of OMT and SBP in other disciplines, such as oceanic and atmospheric sciences (87, 88, 89), computer imaging (90), data sciences (91, 92) and machine learning (93), is also rapidly expanding.

In closing, we venture recalling that in 1620, Francis Bacon lists among his *Idola Tribus* (logical fallacies of human nature) in *Novum Organum Scientiarum* the following: "The human understanding is of its own nature prone to suppose the existence of more order and regularity in the world than it finds." Could this just be a consequence of evolution? Indeed, we cannot make any rational analysis nor decision based on chaos. After all, Plato's $\delta\eta\mu\iota o\upsilon\rho\gamma o\varsigma$ (literally: people's worker) does not create, but rather produces *order* from the chaotic preexisting matter. In Schrödinger's original problem for a cloud of Brownian particles, the prior Wiener measure represents, in a pregnant form, *Chaos*. The Schrödinger Bridge approach, of transport under stochastic uncertainty, is the less prejudicial strategy to derive some form of order from chaos, namely a model on which we can base our analysis and decisions. It is most fortunate that this procedure can be formulated as a control problem, in fact as the problem to control uncertainty. This enlarges significantly the scope of control theory, connecting it to other vast areas of science to which OMT and maximum entropy inference methods have been applied. As we have seen in this paper, we can then use and adapt control ideas and techniques to develop new effective ways to attack problems.

SUMMARY POINTS

1. Optimal Mass Transport (OMT) can be cast as a stochastic control problem.
2. Schrödinger Bridge problem (SBP) was conceived as the inference problem of finding the most likely random evolution linking boundary marginal distributions.
3. SBP can also be cast as a stochastic control problem, just as OMT, but with an added source of stochastic uncertainty.
4. In either, the transportation cost to be minimized is the expected value of a quadratic cost over possible trajectories.

5. Applications lead to considering various generalization with regard to the underlying dynamics and terminal state distributions.
6. A discrete space counterpart of either, OMT or SBP, relates to control problems for Markov Decision Processes (MDPs) and transport over networks.

FUTURE ISSUES

1. OMT and SBP represent rapidly developing subjects, with a rich mathematical basis, that impacts a range of scientific disciplines beyond Systems and Control.
2. OMT and SBP have helped launch a new sub-discipline of Stochastic Control, namely, Control of Uncertainty, where many technical and computations issues remain open.

DISCLOSURE STATEMENT

The authors are not aware of any affiliations, memberships, funding, or financial holdings that might be perceived as affecting the objectivity of this review.

ACKNOWLEDGMENTS

This research was partially supported by NSF under grants 1807664, 1839441, 1901599, 1942523 and AFOSR under grant FA9550-20-1-0029,

LITERATURE CITED

1. Fleming W, Rishel R. 1975. *Deterministic and Stochastic Optimal Control*. Springer
2. Fleming WH, Soner HM. 2006. *Controlled Markov processes and viscosity solutions*, vol. 25. Springer Science & Business Media
3. Blaquière A. 1992. Controllability of a Fokker-Planck equation, the Schrödinger system, and a related stochastic optimal control (revised version). *Dynamics and Control* 2:235–253
4. Brockett RW. 2007. Optimal control of the Liouville equation. *AMS IP Studies in Advanced Mathematics* 39:23
5. Brockett R. 2012. Notes on the control of the Liouville equation. In *Control of Partial Differential Equations*. Springer
6. Grigoriadis KM, Skelton RE. 1997. Minimum-energy covariance controllers. *Automatica* 33:569–578
7. Zhu G, Grigoriadis KM, Skelton RE. 1995. Covariance control design for hubble space telescope. *Journal of Guidance, Control, and Dynamics* 18:230–236
8. Collins E, Skelton R. 1985. *Covariance control discrete systems*. In *1985 24th IEEE Conference on Decision and Control*, pp. 542–547. IEEE
9. Chen Y, Georgiou TT, Pavon M. 2016. Optimal steering of a linear stochastic system to a final probability distribution, Part I. *IEEE Trans. on Automatic Control* 61:1158–1169
10. Chen Y, Georgiou TT, Pavon M. 2016. Optimal steering of a linear stochastic system to a final probability distribution, Part II. *IEEE Trans. on Automatic Control* 61:1170–1180
11. Halder A, Wendel EDB. 2016. *Finite Horizon Linear Quadratic Gaussian Density Regulator with Wasserstein Terminal Cost*. In *Proc. American Control Conf.*

12. Chen Y, Georgiou TT, Pavon M. 2015. *Optimal steering of inertial particles diffusing anisotropically with losses*. In *Proc. American Control Conf.*, pp. 1252–1257
13. Chen Y, Georgiou TT, Pavon M. 2017. Optimal transport over a linear dynamical system. *IEEE Transactions on Automatic Control* 62:2137–2152
14. Chen Y, Georgiou TT, Pavon M. 2018. Optimal steering of a linear stochastic system to a final probability distribution, Part III. *IEEE Transactions on Automatic Control* 63:3112–3118
15. Chen Y, Georgiou TT, Pavon M. 2016. On the relation between optimal transport and Schrödinger bridges: A stochastic control viewpoint. *Journal of Optimization Theory and Applications* 169:671–691
16. Chen Y, Georgiou TT, Pavon M. 2020. Stochastic control liasons: Richard Sinkhorn meets Gaspard Monge on a Schroedinger bridge. *arXiv preprint arXiv:2005.10963*
17. Chen Y. 2016. Modeling and control of collective dynamics: From Schrödinger bridges to optimal mass transport. Ph.D. thesis, University of Minnesota
18. Bakolas E. 2016. *Optimal covariance control for discrete-time stochastic linear systems subject to constraints*. In *2016 IEEE 55th Conference on Decision and Control (CDC)*, pp. 1153–1158. IEEE
19. Bakolas E. 2018. Constrained minimum variance control for discrete-time stochastic linear systems. *Systems & Control Letters* 113:109–116
20. Ridderhof J, Okamoto K, Tsiotras P. 2019. *Nonlinear uncertainty control with iterative covariance steering*. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pp. 3484–3490. IEEE
21. Ridderhof J, Tsiotras P. 2019. *Minimum-fuel powered descent in the presence of random disturbances*. In *AIAA Scitech 2019 Forum*, pp. 0646
22. Okamoto K, Goldshtein M, Tsiotras P. 2018. Optimal covariance control for stochastic systems under chance constraints. *IEEE Control Systems Letters* 2:266–271
23. Chen Y, Georgiou T, Pavon M, Tannenbaum A. 2016. Robust transport over networks. *IEEE Transactions on Automatic Control*
24. Carrillo JA, Choi YP, Hauray M. 2014. The derivation of swarming models: mean-field limit and wasserstein distances. In *Collective dynamics from bacteria to crowds*. Springer
25. Monge G. 1781. *Mémoire sur la théorie des déblais et des remblais*. De l'Imprimerie Royale
26. Villani C. 2003. *Topics in Optimal Transportation*. No. 58 in Graduate Studies in Mathematics. American Mathematical Soc.
27. Kantorovich LV. 1942. *On the transfer of masses*. In *Dokl. Akad. Nauk. SSSR*, vol. 37 (No. 2), pp. 227–229
28. Brenier Y. 1991. Polar factorization and monotone rearrangement of vector-valued functions. *Communications on pure and applied mathematics* 44:375–417
29. Villani C. 2008. *Optimal Transport: Old and New*, vol. 338. Springer
30. Jordan R, Kinderlehrer D, Otto F. 1998. The variational formulation of the Fokker–Planck equation. *SIAM journal on mathematical analysis* 29:1–17
31. Otto F. 2001. The geometry of dissipative evolution equations: the porous medium equation. *Communications in Partial Differential Equations*
32. Ambrosio L, Gigli N, Savaré G. 2006. *Gradient flows: in metric spaces and in the space of probability measures*. Springer
33. Schrödinger E. 1931. Über die Umkehrung der Naturgesetze. *Sitzungsberichte der Preuss Akad. Wissen. Phys. Math. Klasse, Sonderausgabe* IX:144–153
34. Schrödinger E. 1932. *Sur la théorie relativiste de l'électron et l'interprétation de la mécanique quantique*. In *Annales de l'institut Henri Poincaré*, vol. 2 (No. 4), pp. 269–310. Presses universitaires de France
35. Föllmer H. 1988. Random fields and diffusion processes. In *École d'Été de Probabilités de Saint-Flour XV–XVII, 1985–87*. Springer
36. Léonard C. 2014. A survey of the Schrödinger problem and some of its connections with optimal

transport. *Dicrete Contin. Dyn. Syst. A* 34:1533–1574
37. Fortet R. 1940. Résolution d'un système d'équations de M. Schrödinger. *J. Math. Pures Appl.* 83
38. Beurling A. 1960. An automorphism of product measures. *The Annals of Mathematics* 72:189–200
39. Jamison B. 1974. Reciprocal processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 30:65–86
40. Essid M, Pavon M. 2019. Traversing the schrödinger bridge strait: Robert fortet?s marvelous proof redux. *Journal of Optimization Theory and Applications* 181:23–60
41. Chen Y, Georgiou T, Pavon M. 2016. Entropic and displacement interpolation: a computational approach using the Hilbert metric. *SIAM Journal on Applied Mathematics* 76:2375–2396
42. Benamou JD, Brenier Y. 2000. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numerische Mathematik* 84:375–393
43. Rockafellar RT. 1970. *Convex analysis*. No. 28 in Princeton Mathematical Series. Princeton university press
44. Karatzas I, Shreve S. 1988. *Brownian Motion and Stochastic Calculus*. Springer
45. Dai Pra P. 1991. A stochastic control approach to reciprocal diffusion processes. *Applied mathematics and Optimization* 23:313–329
46. Gentil I, Léonard C, Ripani L. 2015. About the analogy between optimal transport and minimal entropy. *arXiv:1510.08230*
47. Mikami T. 2004. Monge's problem with a quadratic cost by the zero-noise limit of h-path processes. *Probability theory and related fields* 129:245–260
48. Mikami T, Thieullen M. 2008. Optimal transportation problem by stochastic optimal control. *SIAM Journal on Control and Optimization* 47:1127–1139
49. Léonard C. 2012. From the Schrödinger problem to the Monge–Kantorovich problem. *Journal of Functional Analysis* 262:1879–1920
50. Carlen E. 2006. Stochastic mechanics: a look back and a look ahead. In *Diffusion, Quantum Theory and Radically Elementary Mathematics*, ed. WG Faris, pp. 117–139, vol. 47. Princeton University Press
51. Chen Y, Georgiou TT, Pavon M. 2015. Fast cooling for a system of stochastic oscillators. *Journal of Mathematical Physics* 56:113302
52. Chen Y, Georgiou T, Pavon M. 2015. *Steering state statistics with output feedback*. In *2015 54th IEEE Conference on Decision and Control (CDC)*, pp. 6502–6507. IEEE
53. Chen Y, Georgiou TT, Pavon M. 2019. Covariance steering in zero-sum linear-quadratic two-player differential games. *arXiv preprint arXiv:1909.05468*
54. Ciccone V, Chen Y, Georgiou TT, Pavon M. 2020. Regularized transport between singular covariance matrices. *arXiv preprint arXiv:2006.10000*
55. Georgiou TT. 2002. The structure of state covariances and its relation to the power spectrum of the input. *Automatic Control, IEEE Trans. on* 47:1056–1066
56. Chen Y, Georgiou TT, Pavon M. 2018. Steering the distribution of agents in mean-field games system. *Journal of Optimization Theory and Applications* 179:332–357
57. Yi Z, Cao Z, Theodorou E, Chen Y. 2019. Nonlinear covariance control via differential dynamic programming. *arXiv preprint arXiv:1911.09283*
58. Bakshi K, Fan DD, Theodorou EA. 2018. Schrödinger approach to optimal control of large-size populations. *arXiv preprint arXiv:1810.06064*
59. Hörmander L. 1967. Hypoelliptic second order differential equations. *Acta Mathematica* 119:147–171
60. Kappen HJ. 2005. Path integrals and symmetry breaking for optimal control theory. *Journal of statistical mechanics: theory and experiment* 2005:P11011
61. Kappen HJ. 2005. Linear theory for control of nonlinear stochastic systems. *Physical review letters* 95:200201
62. Theodorou E, Buchli J, Schaal S. 2010. A generalized path integral control approach to rein-

forcement learning. *The Journal of Machine Learning Research* 11:3137–3181
63. Theodorou EA, Todorov E. 2012. *Relative entropy and free energy dualities: Connections to path integral and KL control*. In *2012 ieee 51st ieee conference on decision and control (cdc)*, pp. 1466–1473. IEEE
64. Caluya KF, Halder A. 2019. Wasserstein proximal algorithms for the schr\"{o} dinger bridge problem: Density control with nonlinear drift. *arXiv preprint arXiv:1912.01244*
65. Elamvazhuthi K, Liu S, Li W, Osher S. 2020. Optimal transport of nonlinear control-affine systems
66. Chen Y, Georgiou TT, Pavon M, Tannenbaum A. 2017. Efficient robust routing for single commodity network flows. *IEEE Transactions on Automatic Control* 63:2287–2294
67. Chen Y, Georgiou TT, Pavon M, Tannenbaum A. 2019. Relaxed schroedinger bridges and robust network routing. *IEEE Transactions on Control of Network Systems*
68. Todorov E. 2007. *Linearly-solvable Markov decision problems*. In *Advances in neural information processing systems*, pp. 1369–1376
69. Todorov E. 2009. Efficient computation of optimal actions. *Proceedings of the national academy of sciences* 106:11478–11483
70. Todorov E. 2009. *Compositionality of optimal control laws*. In *Advances in Neural Information Processing Systems*, pp. 1856–1864
71. Ruelle D. 2004. *Thermodynamic formalism: the mathematical structure of equilibrium statistical mechanics*. Cambridge University Press
72. Delvenne JC, Libert AS. 2011. Centrality measures and thermodynamic formalism for complex networks. *Physical Review E* 83:046117
73. Georgiou TT, Karlsson J, Takyar MS. 2008. Metrics for power spectra: an axiomatic approach. *IEEE Transactions on Signal Processing* 57:859–867
74. Karlsson J, Ringh A. 2017. Generalized sinkhorn iterations for regularizing inverse problems using optimal mass transport. *SIAM Journal on Imaging Sciences* 10:1935–1962
75. Chen Y, Karlsson J. 2018. State tracking of linear ensembles via optimal mass transport. *IEEE Control Systems Letters* 2:260–265
76. Reich S. 2018. Data assimilation: The Schrödinger perspective. *arXiv preprint arXiv:1807.08351*
77. Taghvaei A, Mehta PG. 2019. An optimal transport formulation of the ensemble Kalman filter. *arXiv preprint arXiv:1910.02338*
78. Elvander F, Haasler I, Jakobsson A, Karlsson J. 2020. Multi-marginal optimal transport using partial information with applications in robust localization and sensor fusion. *Signal Processing* 171:107474
79. Haasler I, Ringh A, Chen Y, Karlsson J. 2020. Multi-marginal optimal transport and schr\" odinger bridges on trees. *arXiv preprint arXiv:2004.06909*
80. Singh R, Haasler I, Zhang Q, Karlsson J, Chen Y. 2020. Inference with aggregate data: An optimal transport approach. *arXiv preprint arXiv:2003.13933*
81. Zhai H, Egerstedt M, Zhou H. 2019. Path planning in unknown environments using optimal transport theory. *arXiv preprint arXiv:1909.11235*
82. Krishnan V, Martínez S. 2018. *Distributed optimal transport for the deployment of swarms*. In *2018 IEEE Conference on Decision and Control (CDC)*, pp. 4583–4588. IEEE
83. Wakolbinger A. 1990. *Schrödinger Bridges from 1931 to 1991*. In *Proc. of the 4th Latin American Congress in Probability and Mathematical Statistics, Mexico City*, pp. 61–79
84. Cuturi M. 2013. *Sinkhorn distances: Lightspeed computation of optimal transport*. In *Advances in Neural Information Processing Systems*, pp. 2292–2300
85. Okamoto K, Tsiotras P. 2019. *Input hard constrained optimal covariance steering*. In *2019 IEEE 58th Annual Conference on Decision and Control (CDC)*, pp. 3497–3502. IEEE
86. Pavon M, Tabak EG, Trigila G. 2018. The data-driven Schroedinger bridge. *arXiv preprint arXiv:1806.01364, Comm. Pure Appl. Math.* to appear.

87. Cullen M, Gangbo W. 2001. A variational approach for the 2-dimensional semi-geostrophic shallow water equations. *Archive for rational mechanics and analysis* 156:241–273
88. Cheng J, Cullen M, Feldman M. 2018. Classical solutions to semi-geostrophic system with variable coriolis parameter. *Archive for Rational Mechanics and Analysis* 227:215–272
89. Ning L, Carli FP, Ebtehaj AM, Foufoula-Georgiou E, Georgiou TT. 2014. Coping with model error in variational data assimilation using optimal mass transport. *Water Resources Research* 50:5817–5830
90. Angenent S, Pichon E, Tannenbaum A. 2006. Mathematical methods in medical image processing. *Bulletin of the American mathematical society* 43:365–396
91. Peyré G, Cuturi M, et al. 2019. Computational optimal transport: With applications to data science. *Foundations and Trends® in Machine Learning* 11:355–607
92. Santambrogio F. 2015. Optimal transport for applied mathematicians. *Birkäuser, NY* 55:94
93. Kolouri S, Park SR, Thorpe M, Slepcev D, Rohde GK. 2017. Optimal mass transport: Signal processing and machine-learning applications. *IEEE signal processing magazine* 34:43–59