

## The genus of the subgroup graph of a finite group

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For a finite group  $G$  denote by  $\gamma(L(G))$  the genus of the subgroup graph of  $G$ . We prove that  $\gamma(L(G))$  tends to infinity as either the rank of  $G$  or the number of prime divisors of  $|G|$  tends to infinity.

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### 1. Introduction

The subgroup graph  $L(G)$  of a finite group  $G$  is the graph whose vertices are the subgroups of the group and two vertices,  $H_1$  and  $H_2$ , are connected by an edge if and only if  $H_1 \leq H_2$  and there is no subgroup  $K$  such that  $H_1 \leq K \leq H_2$ . A graph is said to be embedded in a surface  $S$  when it is drawn on  $S$  so that no two edges intersect.

The genus  $\gamma(\Gamma)$  of a graph  $\Gamma$  is the minimum  $g$  such that there exists an embedding of  $\Gamma$  into the orientable surface  $S_g$  of genus  $g$  (or in other words the minimum number  $g$  of handles which must be added to a sphere so that  $G$  can be embedded on the resulting surface).

In [2], the authors investigate the case  $\gamma(L(G)) = 0$ , characterizing the finite groups having a planar subgroup graph. It turns out that there are seven infinite families of such groups, and three additional isolated groups. All these groups have

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order divisible by at most three different primes and their Sylow subgroups have rank at most 2 (recall that the rank of a finite group  $G$  is the minimal number  $r$  such that every subgroup of  $G$  can be generated by  $r$  elements). In this paper, we generalize this result proving that for every non-negative integer  $k$ , there exist two integers  $a_k$  and  $b_k$  such that if  $\gamma(L(G)) \leq k$ , then the order of  $G$  is divisible by at most  $a_k$  different primes and the rank of any Sylow  $p$ -subgroup of  $G$  is at most  $b_k$ . Let  $r_p(G)$  be the rank of a Sylow  $p$ -subgroup of  $G$ ,  $r(G)$  the rank of  $G$ ,  $\pi(G)$  the set of the prime divisors of  $|G|$  and  $\rho(G) = \max_{p \in \pi(G)} r_p(G)$ . By the main result in [3, 7], if  $r_p(G) \leq d$  for every  $p \in \pi(G)$ , then  $r(G) \leq d + 1$  and therefore  $\rho(G) \leq r(G) \leq \rho(G) + 1$ . Hence, we may state our results as follows.

**Theorem 1.** *Let  $G$  be a finite group. Then  $\gamma(L(G))$  tends to infinity as either the rank of  $G$  or the number of prime divisors of  $|G|$  tends to infinity.*

Our proof uses the classification of the finite non-Abelian simple groups. In particular we prove the following result, of independent interest.

**Theorem 2.** *For every  $k \in \mathbb{N}$ , there exist only finitely many non-Abelian finite simple groups  $S$ , with  $\gamma(L(S)) \leq k$ .*

Notice that the previous theorem cannot be deduced from Theorem 1. For example  $\rho(\text{PSL}(2, p)) = 2$  for every prime  $p$  and it follows from [1, Corollary 4.2] that there are infinitely many primes  $p$  such that  $|\text{PSL}(2, p)|$  is divisible by at most 20 primes.

## 2. Proofs of Our Results

If  $\Omega$  is a family of subgroups of  $G$ , we may consider the graph  $L_\Omega(G)$  whose vertices are the subgroups in  $\Omega$  and two vertices,  $H_1$  and  $H_2$ , are connected by an edge if and only if  $H_1 \leq H_2$  and there is no subgroup  $K \in \Omega$  such that  $H_1 \leq K \leq H_2$ . Clearly,  $\gamma(L_\Omega(G)) \leq \gamma(L(G))$  for every choice of  $\Omega$ . Moreover, for any choice of  $\Omega$ , the graph  $\gamma(L_\Omega(G))$  is triangle-free. An easy consequence of Euler's formula (see for example [4, Corollary 11.17(b)]) is that if  $\Gamma$  is triangle-free then

$$\gamma(\Gamma) \geq \frac{|E(\Gamma)|}{4} - \frac{|V(\Gamma)|}{2} + 1.$$

So we have.

**Lemma 1.** *Let  $G$  be a finite group and let  $\Omega$  be a family of subgroups of  $G$ . Then*

$$\gamma(L(G)) \geq \gamma(L_\Omega(G)) \geq \frac{|E(L_\Omega(G))|}{4} - \frac{|V(L_\Omega(G))|}{2} + 1.$$

**Lemma 2.** *Let  $G$  be a finite soluble group and let  $t$  be the number of the distinct prime divisors of  $|G|$ . Then*

$$\lim_{t \rightarrow \infty} \gamma(L(G)) = \infty.$$

**Proof.** Suppose that  $p_1, \dots, p_t$  are the distinct prime divisors of the order of  $G$  and let  $P_1, \dots, P_t$  be a Sylow basis of  $G$ . To any subset  $J$  of  $\{p_1, \dots, p_t\}$ , there

corresponds the subgroup  $H_J = \prod_{j \in J} P_j$  of  $G$ . Let  $\Omega = \{H_J \mid J \subseteq \{1, \dots, t\}\}$ . The vertices of  $L_\Omega(G)$  correspond to the subsets of  $\{p_1, \dots, p_t\}$ , so  $|V(L_\Omega(G))| = 2^t$ . If  $J_1, J_2 \subseteq \{p_1, \dots, p_t\}$  and  $|J_1| \leq |J_2|$ , then  $H_{J_1}$  and  $H_{J_2}$  are adjacent if and only if  $J_2 = J_1 \cup \{p\}$  for a prime  $p \notin J_1$ , so

$$|E(L_\Omega(G))| = \sum_{0 \leq i \leq t-1} \binom{t}{i} (t-i).$$

It follows from Lemma 1 that

$$\begin{aligned} \gamma(L(G)) &\geq \sum_{0 \leq i \leq t-1} \binom{t}{i} \frac{(t-i)}{4} - \frac{2^t}{2} + 1 \geq \sum_{0 \leq i \leq \frac{t-1}{2}} \binom{t}{i} \frac{t}{8} - \frac{2^t}{2} + 1 \\ &\geq \frac{2^t \cdot t}{16} - \frac{2^t}{2} + 1 = 2^{t-1} \left( \frac{t}{8} - 1 \right) + 1. \end{aligned} \quad \square$$

Notice that the assumption  $t \geq 3$  in the statement of the previous lemma is necessary. Indeed  $L(A_{p,2}) \cong K_{2,p+1}$  is a planar graph for any choice of  $p$ .

**Lemma 3.** *Let  $p$  be a prime,  $t$  a positive integer and  $A_{p,t}$  the elementary abelian  $p$ -group of rank  $t$ . If  $t \geq 3$ , then  $\gamma(L(A_{p,t}))$  tends to infinity as  $p^t$  tends to infinity.*

**Proof.** Let  $\Omega$  be the family of the subgroups of  $A_{p,t}$  of order  $p$  and  $p^2$  and let  $\Gamma = \Gamma_\Omega(A_{p,t})$ . Notice that  $\Gamma$  is a bipartite graph and

$$|V(\Gamma)| = \begin{bmatrix} t \\ 2 \end{bmatrix}_p + \begin{bmatrix} t \\ 1 \end{bmatrix}_p, \quad |E(\Gamma)| = \begin{bmatrix} t \\ 2 \end{bmatrix}_p \frac{p^2 - 1}{p - 1}.$$

Since

$$\begin{bmatrix} t \\ 1 \end{bmatrix}_p = \begin{bmatrix} t \\ 2 \end{bmatrix}_p \frac{p^2 - 1}{p^{t-1} - 1},$$

we deduce

$$\begin{aligned} \gamma(L(G)) &\geq \gamma(\Gamma) \geq \frac{|E(\Gamma)|}{4} - \frac{|V(\Gamma)|}{2} \\ &= \frac{1}{4} \begin{bmatrix} t \\ 2 \end{bmatrix}_p \left( \frac{p^2 - 1}{p - 1} - 2 \left( 1 + \frac{p^2 - 1}{p^{t-1} - 1} \right) \right) \\ &= \frac{1}{4} \begin{bmatrix} t \\ 2 \end{bmatrix}_p \left( p - 1 - \frac{2(p^2 - 1)}{p^{t-1} - 1} \right). \end{aligned}$$

In particular, if  $t \geq 3$ , then  $\gamma(L(A_{p,t}))$  tends to infinity as  $p^t$  tends to infinity.  $\square$

**Lemma 4.**  *$\gamma(L(\text{PSL}(2, q)))$  tends to infinity as  $q$  tends to infinity.*

**Proof.** First assume  $q = 2^t$ . In this case, a Sylow 2-subgroup of  $\text{PSL}(2, q)$  is isomorphic to  $A_{2,t}$ , hence  $\gamma(L(\text{PSL}(2, 2^t))) \geq \gamma(L(A_{2,t}))$  and the conclusion follows

from Lemma 3. Now assume that  $q$  is odd. If  $q \notin \{5, 7, 9, 17\}$ , then there exists  $n \in \{\frac{q-1}{2}, \frac{q+1}{2}\}$  such that  $n$  is divisible by at least two different primes (see, for example [5, Theorem 3]). We factorize  $n = a \cdot b$  where  $a$  and  $b$  are coprime integers properly dividing  $n$ . The group  $\text{PSL}(2, q)$  has a maximal subgroup  $M$  isomorphic to the dihedral group  $D_n$  of order  $2n$ , and inside  $M$  we can find  $n/a$  subgroups  $H_1, \dots, H_{n/a}$  isomorphic to  $D_a$ ,  $n/b$  subgroups  $K_1, \dots, K_{n/b}$  isomorphic to  $D_b$  and  $n$  subgroups  $J_1, \dots, J_n$  that have order 2 and are non-central in  $M$ . Let  $\Omega := \{M, H_i, K_j, J_k, 1 \mid 1 \leq i \leq \frac{n}{a}, 1 \leq j \leq \frac{n}{b}, 1 \leq k \leq n\}$ . We have

$$\begin{aligned} v &= |V(\Gamma_\Omega(\text{PSL}(2, q)))| = 2 + n + \frac{n}{a} + \frac{n}{b}, \\ e &= |E(\Gamma_\Omega(\text{PSL}(2, q)))| = 3n + \frac{n}{a} + \frac{n}{b}, \end{aligned}$$

since  $M$  is adjacent to  $H_i$  and  $K_j$  for  $1 \leq i \leq n/a$  and  $1 \leq j \leq n/b$ ,  $1$  is adjacent to  $J_k$  for  $1 \leq k \leq n$ , any  $H_i$  contains precisely  $a$  non-central subgroups of order 2 and any  $K_j$  contains precisely  $b$  non-central subgroups of order 2. This is shown as follows:

$$\begin{aligned} \gamma(L(\text{PSL}(2, q))) &\geq \gamma(L_\Omega(\text{PSL}(2, q))) \geq \frac{e}{4} - \frac{v}{2} + 1 \\ &= \frac{1}{4} \left( 3n + \frac{n}{a} + \frac{n}{b} \right) - \frac{1}{2} \left( 2 + n + \frac{n}{a} + \frac{n}{b} \right) + 1 \\ &= \frac{n}{4} \left( 1 - \frac{1}{a} - \frac{1}{b} \right) \geq \frac{n}{4} \left( 1 - \frac{1}{2} - \frac{1}{3} \right) \geq \frac{n}{24}. \end{aligned}$$

The conclusion follows from the observation that  $n$  tends to infinity as  $q$  tends to infinity.  $\square$

**Lemma 5.** *If  $G \leq \text{GL}(2, p)$ , where  $p$  is a prime, and  $\gamma(L(G)) \leq k$ , then the number of the prime divisors of  $|G|$  is at most  $\beta_k$ , where  $\beta_k$  is a positive integer depending only on  $k$ .*

**Proof.** By Lemma 2, there exists  $\alpha_k$  such that if  $X$  is a finite soluble group and  $\gamma(L(X)) \leq k$ , then  $|\pi(X)| \leq \alpha_k$ . So may so assume that  $G$  is not soluble. It follows from [6, Hauptsatz 8.27] that there are two cases:

- (a)  $\text{PSL}(2, p)$  is a composition factor of  $G$ . In this case,  $\gamma(L(X)) \leq k$  for every subgroup  $X$  of  $\text{PSL}(2, p)$ . Since  $\text{PSL}(2, p)$  contains two soluble subgroups  $H_1$  and  $H_2$  of order, respectively,  $p+1$  and  $p(p-1)/2$  we deduce from Lemma 2 that  $|\pi(G)| = |\pi(H_1) \cup \pi(H_2)| \leq 2\alpha_k$ .

(b)  $\text{Alt}(5)$  is a composition factor of  $G$ . In this case

$$\frac{G \cap \text{SL}(2, p)}{G \cap Z(\text{SL}(2, p))} \cong \text{Alt}(5),$$

so  $G$  contains a normal subgroup  $N$  such that  $G/N$  is cyclic and  $\pi(N) = \{2, 3, 5\}$ . It follows  $|\pi(G)| \leq \alpha_k + 3$ .  $\square$

**Proof of Theorem 2.** Fix  $k \in \mathbb{N}$  and let  $S$  be a finite non-Abelian simple group with  $\gamma(L(S)) \leq k$ . Since  $r_2(\text{Alt}(n)) \geq \lfloor \frac{n}{2} \rfloor - 1$ , it follows from Lemma 3, that  $\gamma(L(\text{Alt}(n))) > k$  if  $n$  is large enough. So we may assume that  $S$  is of Lie type over the field  $\mathbb{F}_q$ ,  $q = p^f$ . Since  $r_p(S)$  tends to infinity as the Lie rank of  $S$  tends to infinity, it follows from Lemma 3 that the Lie rank of  $S$  is bounded in term of  $k$ . On the other hand  $S$  contains a section isomorphic to  $\text{PSL}(2, q)$ , so, by Lemma 4,  $\gamma(L(S)) \geq \gamma(L(\text{PSL}(2, q)))$  tends to infinity as  $q$  tends to infinity. So we bounded in term of  $k$  either  $q$  as the Lie rank, and consequently the number of possibilities for  $S$  itself.  $\square$

**Proof of Theorem 1.** By Lemma 3, there exists  $\rho_k$  such if  $\gamma(L(G)) \leq k$ , then  $r_p(G) \leq \rho_k$  for every prime divisor  $p$  of  $|G|$ , and consequently  $r(G) \leq \rho_k + 1$ . So it suffices to prove that there exists  $\pi_k$  such if  $\gamma(L(G)) \leq k$ , then  $|\pi(G)| \leq \pi_k$ . Since  $\pi(G) = \pi(G/\text{Frat}(G))$  (see [6, Satz 3.8]) and  $\gamma(L(G)) \geq \gamma(L(G/\text{Frat}(G)))$  we may assume  $\text{Frat}(G) = 1$ . Write

$$X = \text{soc}(G) = A_1 \times \cdots \times A_r \times B_1 \times \cdots \times B_s \times C_1 \times \cdots \times C_t$$

as a direct product of minimal normal subgroups of  $G$  with the property that the factors  $A_i$ 's are Abelian with rank at most 2, the factors  $B_j$ 's are Abelian with rank at least 3 and the factors  $C_l$ 's are non-Abelian. By Theorem 2, the family  $\mathcal{S}_k$  of the finite non-Abelian simple groups  $S$  with  $\gamma(L(S)) \leq k$  is finite. Let  $N$  be a minimal non-Abelian normal subgroup of  $G$ . There exists a non-Abelian simple group  $S$  and a positive integer  $m$  such that  $N \cong S^m$ . Since  $\gamma(L(S)) \leq \gamma(L(N)) \leq \gamma(L(G)) \leq k$  and  $r_2(N) \geq 2 \cdot m$ , it follows that  $S \in \mathcal{S}_k$  and, by Lemma 3,  $m \leq \tau$  for a positive integer  $\tau$  depending only on  $k$ . Moreover by Lemma 3, if  $N \cong C_{p^u}$  is an Abelian minimal normal subgroup of  $G$ , then either  $u \leq 2$  or  $p^u \leq \sigma$  for a positive integer  $\sigma$  depending only on  $k$ . It follows that there exists a finite family  $\mathcal{F}_k$  of finite characteristically simple groups such that  $B_j, C_l \in \mathcal{F}_k$  for every  $1 \leq j \leq s$  and  $1 \leq l \leq t$ . Let  $\Lambda_k$  be the set of the primes dividing  $|Y||\text{Aut } Y|$  for some  $Y \in \mathcal{F}_k$  and set  $\lambda_k = |\Lambda_k|$ . Since  $\text{Frat}(G) = 1$ , we have that  $X$  coincides with the generalized Fitting subgroup of  $G$  and consequently  $C_G(X) = Z(X)$  and every prime dividing  $|G/Z(X)|$  divides  $|G/C_G(N)|$  for some minimal normal subgroup  $N$  of  $G$ . It follows that a prime  $p$  dividing  $|G|$  either divides  $|A_i||G/C_G(A_i)|$  for some  $1 \leq i \leq r$ , or belong to  $\Lambda_k$ . Let  $\Sigma = \cup_{1 \leq i \leq r} \pi(A_i)$ . It follows from Lemma 2, that  $|\Sigma| \leq \alpha_k$  for an integer  $\alpha_k$  depending only on  $k$ . If we denote by  $\Sigma_i$  the set of the prime divisors of

$|G/C_G(A_i)|$ , we have  $\pi(G) \subseteq \Sigma \cup \Lambda_k \cup_{1 \leq i \leq r} \Sigma_i$ . If  $A_i$  is cyclic, then  $G/C_G(A_i)$  is cyclic and again it follows from Lemma 2 that  $|\Sigma_i| \leq \alpha_k$ . Otherwise  $|\Sigma_i| \leq \beta_k$  by Lemma 5. We deduce that  $|\pi(G)| \leq \alpha_k + \lambda_k + \alpha_k \max(\alpha_k, \beta_k)$ .  $\square$

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