

PROJECTIVE COVERS OF FLAT CONTRAMODULES

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ABSTRACT. We show that a direct limit of projective contramodules (over a right linear topological ring) is projective if it has a projective cover. A similar result is obtained for ∞ -strictly flat contramodules of projective dimension not exceeding 1, using an argument based on the notion of the topological Jacobson radical. Covers and precovers of direct limits of more general classes of objects, both in abelian categories with exact and with nonexact direct limits, are also discussed, with an eye towards the Enochs conjecture about covers and direct limits, using locally split (mono)morphisms as the main technique. In particular, we offer a simple elementary proof of the Enochs conjecture for the left class of an n -tilting cotorsion pair in an abelian category with exact direct limits.

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INTRODUCTION

The notion of a projective cover is dual to that of an injective envelope. While injective envelopes exist in all Grothendieck abelian categories, projective covers are more rare. It was shown in the classical paper of Bass [7] that all left modules over an associative ring R have projective covers if and only if all flat left R -modules are projective. Such rings were called *left perfect* in [7]. Subsequently people realized that if a flat module over an associative ring has a projective cover, then such module is projective (see, e. g., [25, Section 36.3]).

The classical *Govorov–Lazard theorem* [12, 16] tells that the flat modules are precisely the direct limits of (finitely generated) projective modules. It is not known whether an analogue of this result holds for contramodules. It is only clear that all the direct limits of projective contramodules are flat. In fact, in Corollary 6.1 we

show that all the direct limits of projective contra-modules belong to a possibly more narrow class of *1-strictly flat* contra-modules.

A *Bass flat module* over an associative ring R is a countable direct limit of copies of the free R -module with one generator $R = R[*]$. All Bass flat modules have projective dimension at most 1. This class of modules and its generalizations played an important role both in Bass' paper [7] and in subsequent works (see, e. g., the recent papers [24, 3]). In this paper, we consider the analogous class of contra-modules over a topological ring. Our results imply that a Bass flat contra-module cannot have a projective cover unless it is projective.

In fact, we prove two different generalizations of the latter claim, provable by very different techniques. On the one hand, Theorem 2.1 tells that an ∞ -strictly flat contra-module of projective dimension not exceeding 1 is projective if it has a projective cover. The proof is based on the concept of the topological Jacobson radical of a topological ring. On the other hand, by Corollary 6.5, the same assertion applies to an arbitrary (not necessarily countable) direct limit of projective contra-modules. The proof is based on considerations of local splitness.

Speaking of the latter, one first of all has to distinguish between *locally split monomorphisms* and *locally split epimorphisms* of modules. These are two different theories (even if related by a vague analogy).

The notion of a locally split monomorphism of modules seems to go back to Chase's exposition of a result of Villamayor [10, Proposition 2.2]. Subsequently the property was studied by Rangaswamy with coauthors [23, 15], who called such monomorphisms (or submodules) "strongly pure". The terminology "locally split submodule" appeared in Azumaya's paper [5]. Zimmermann [28] uses the "strongly pure monomorphism" terminology. It is the locally split *monomorphisms* of modules that are relevant for the categorical generalization developed in the present paper.

Locally split epimorphisms of modules were studied (under various names) by Rangaswamy [22] and Azumaya [4, 6]. An attempt to compare and formulate the connections between the concepts of locally split monomorphisms and locally split epimorphisms is made in the recent paper [26].

The *Enochs conjecture* (or "a question of Enochs") suggests that any covering class of modules is closed under direct limits [13, Section 5.4] (cf. [3, Section 5]). This problem was addressed in the papers [24, 3], where some results in the direction of a positive answer to the question of Enochs were obtained.

The following observation related to precovers and covers of direct limits plays a key role in the present paper. Let \mathbf{C} be a class of modules (over a fixed associative ring A) closed under direct sums and direct summands, and let $(C_x)_{x \in X}$ be a direct system of modules $C_x \in \mathbf{C}$, indexed by a directed poset X . Consider the canonical presentation

$$0 \longrightarrow K \xrightarrow{i} \bigoplus_{x \in X} C_x \xrightarrow{p} \varinjlim_{x \in X} C_x \longrightarrow 0$$

of the direct limit $D = \varinjlim_{x \in X} C_x$. Then the monomorphism i is locally split, that is, for every element $k \in K$ there exists an A -module morphism $g: C' = \bigoplus_{x \in X} C_x \longrightarrow K$

such that $gi(k) = k$. It follows that if the epimorphism p is a \mathbf{C} -precover of D and a morphism $q: Q \rightarrow D$ is a \mathbf{C} -cover of D , then q is an isomorphism and p is a split epimorphism. We extend this observation first to abelian categories with exact direct limits (in Theorem 4.4), and subsequently, in some form, to cocomplete abelian categories with nonexact direct limits (Theorem 5.5). In greater generality, we discuss quasi-split exact sequences and locally split (mono)morphisms in cocomplete abelian categories in connection with covers.

It was shown in [11, Lemma 2.1] that, whenever in the notation above X is a linearly ordered set, the kernel of the morphism p is the union of a chain of direct summands in $C' = \bigoplus_{x \in X} C_x$. Our Proposition 3.1 extends this result to all directed posets X (with the difference that the kernel of p is described as the union of directed poset of direct summands in C'). Moreover, Proposition 3.1 applies to arbitrary cocomplete abelian categories.

We also deduce the following application of topological algebra and contramodule theory to the Enochs conjecture. Suppose that a left A -module M is what we call *weakly countably generated*; e. g., this holds if M is the sum of a countable set of its *dually slender* submodules (in the sense of [27]). As usually, we denote by $\mathbf{Add}(M)$ the class of all direct summands of direct sums of copies of M . Assume further that for any countable direct system $M \rightarrow M \rightarrow M \rightarrow \cdots$ of endomorphisms of M , the canonical epimorphism $\bigoplus_{n=1}^{\infty} M \rightarrow \varinjlim_{n \geq 1} M = D$ is an $\mathbf{Add}(M)$ -precover of D , and that the A -module D has an $\mathbf{Add}(M)$ -cover. Then the class of modules $\mathbf{Add}(M)$ is closed under direct limits. Moreover, the A -module M has a perfect decomposition (in the sense of [2]) in this case. This is the result of our Application 7.6.

As another application to the Enochs conjecture, we demonstrate a simple proof of the following assertion. Let (\mathbf{L}, \mathbf{E}) be a cotorsion pair in an abelian category \mathbf{A} with exact direct limit functors. Suppose that the class \mathbf{E} is closed under direct limits in \mathbf{A} . Let M be an object in the kernel $\mathbf{L} \cap \mathbf{E}$ of the cotorsion pair. Assume that any direct limit of objects from $\mathbf{Add}(M)$ has an $\mathbf{Add}(M)$ -cover in \mathbf{A} . Then the class of objects $\mathbf{Add}(M)$ is closed under direct limits in \mathbf{A} . In the context of an n -tilting cotorsion pair (\mathbf{L}, \mathbf{E}) , it follows that the class of objects $\mathbf{L} \subset \mathbf{A}$ is closed under direct limits whenever it is covering. Thus we recover some of the results of the paper [3] with our elementary methods (see Application 7.3 and Corollary 7.4).

1. PRELIMINARIES

Throughout the paper, by “direct limits” we mean directed colimits, i. e., colimits indexed over directed posets.

We refer to [20, Section 6] or [18, Sections 1–2] (see also [19, Introduction and Sections 5–6] or [17, Section 2]) for the background material. This section only contains a sketch of the basic definitions and a little discussion.

A topological ring is said to be *right linear* if its open right ideals form a base of neighborhoods of zero. A right linear topological ring \mathfrak{R} is said to be *separated* if the natural map $\mathfrak{R} \rightarrow \varprojlim_{\mathfrak{J} \subset \mathfrak{R}} \mathfrak{R}/\mathfrak{J}$, where \mathfrak{J} ranges over the open right ideals in \mathfrak{R} ,

is injective; and \mathfrak{R} is said to be *complete* if this map is surjective. Throughout this paper, \mathfrak{R} denotes a complete, separated right linear topological ring.

For any abelian group A and a set X , we use $A[X] = A^{(X)}$ as a notation for the direct sum of X copies of A . Elements of the group $A[X]$ are interpreted as finite formal linear combinations of elements of X with the coefficients in A . For any set X , we denote by $\mathfrak{R}[[X]] = \varprojlim_{\mathfrak{J} \subset \mathfrak{R}} (\mathfrak{R}/\mathfrak{J})[X] \subset \mathfrak{R}^X$ the set of all infinite formal linear combinations $\sum_{x \in X} r_x x$ of elements of X with the families of coefficients $(r_x \in \mathfrak{R})_{x \in X}$ converging to zero in the topology of \mathfrak{R} . The latter condition means that, for every open right ideal $\mathfrak{J} \subset \mathfrak{R}$, the subset $\{x \in X \mid r_x \notin \mathfrak{J}\} \subset X$ is finite [20, Section 6], [18, Sections 1.5–1.7], [17, Section 2.7], [19, Section 5].

The assignment of the set $\mathfrak{R}[[X]]$ to an arbitrary set X is a covariant endofunctor on the category of sets, $\mathfrak{R}[[\!-\!]]: \mathbf{Sets} \rightarrow \mathbf{Sets}$. For any map of sets $f: X \rightarrow Y$, the induced map of sets $\mathfrak{R}[[f]]: \mathfrak{R}[[X]] \rightarrow \mathfrak{R}[[Y]]$ assigns to a formal linear combination $\sum_{x \in X} r_x x$ the formal linear combination $\sum_{y \in Y} s_y y$ with the coefficients $s_y = \sum_{x \in X}^{f(x)=y} r_x$. Here the infinite sum in the right-hand side is understood as the limit of finite partial sums in the topology of \mathfrak{R} .

For any set X , there is a natural “point measure” map $\epsilon_X: X \rightarrow \mathfrak{R}[[X]]$, assigning to every element $x \in X$ the formal linear combination $\sum_{y \in X} r_y y$ with $r_y = 1$ for $y = x$ and $r_y = 0$ otherwise. Moreover, for any set X there is a natural “opening of parentheses” map $\phi_X: \mathfrak{R}[[\mathfrak{R}[[X]]]] \rightarrow \mathfrak{R}[[X]]$ assigning a formal linear combination to a formal linear combination of formal linear combinations. The map ϕ_X computes the products of pairs of elements in \mathfrak{R} and then the infinite sums of such products, interpreted as the limits of finite partial sums in the topology of \mathfrak{R} .

The functor $\mathfrak{R}[[\!-\!]]$ endowed with the natural transformations ϕ and ϵ is a *monad* on the category of sets. A *left \mathfrak{R} -contramodule* is defined as an algebra (or, in our preferred terminology, a module) over this monad. In other words, a left \mathfrak{R} -contramodule \mathfrak{C} is a set endowed with a *left \mathfrak{R} -contraaction* map $\pi_{\mathfrak{C}}: \mathfrak{R}[[\mathfrak{C}]] \rightarrow \mathfrak{C}$ satisfying the conventional associativity and unitality equations of an algebra/module over a monad $(\mathfrak{R}[[\!-\!]], \phi, \epsilon)$. Informally, one can say that a left \mathfrak{R} -contramodule is a left \mathfrak{R} -module endowed with infinite summation operations with the families of coefficients converging to zero in the topology of \mathfrak{R} .

The category of left \mathfrak{R} -contramodules $\mathfrak{R}\text{-contra}$ is a locally presentable abelian category with enough projective objects. There is an exact, faithful forgetful functor $\mathfrak{R}\text{-contra} \rightarrow \mathfrak{R}\text{-mod}$ from the category of left \mathfrak{R} -contramodules to the category of left \mathfrak{R} -modules; this functor preserves infinite products (but not coproducts). The free \mathfrak{R} -contramodule with one generator $\mathfrak{R}[[\!*\!]] = \mathfrak{R}$ is a natural projective generator of $\mathfrak{R}\text{-contra}$. More generally, the projective \mathfrak{R} -contramodules are precisely the direct summands of the *free \mathfrak{R} -contramodules* $\mathfrak{R}[[X]]$, where X is an arbitrary set and the contraaction map is $\pi_{\mathfrak{R}[[X]]} = \phi_X$. For any left \mathfrak{R} -contramodule \mathfrak{C} and any set X the group of \mathfrak{R} -contramodule morphisms $\mathfrak{R}[[X]] \rightarrow \mathfrak{C}$ is naturally isomorphic to the group of all maps of sets $X \rightarrow \mathfrak{C}$.

A right \mathfrak{R} -module \mathcal{N} is said to be *discrete* if the annihilator of any element in \mathcal{N} is an open right ideal in \mathfrak{R} [20, Section 7.2], [18, Section 1.4], [17, Section 2.3]. Discrete right \mathfrak{R} -modules form a Grothendieck abelian category, which we denote by $\mathbf{discr}\text{-}\mathfrak{R}$.

Given a discrete right \mathfrak{R} -module \mathcal{N} and an abelian group V , the left \mathfrak{R} -module $\mathfrak{C} = \text{Hom}_{\mathbb{Z}}(\mathcal{N}, V)$ has a natural left \mathfrak{R} -contramodule structure with the contraction map given by the rule

$$\left\langle b, \pi_{\mathfrak{C}} \left(\sum_{c \in \mathfrak{C}} r_c c \right) \right\rangle = \sum_{c \in \mathfrak{C}} \langle b r_c, c \rangle \quad \text{for all } b \in \mathcal{N} \text{ and } \sum_{c \in \mathfrak{C}} r_c c \in \mathfrak{R}[[\mathfrak{C}]],$$

where $\langle \cdot, \cdot \rangle: \mathcal{N} \times \text{Hom}_{\mathbb{Z}}(\mathcal{N}, V) \rightarrow V$ denotes the evaluation pairing. The sum in the right-hand side is finite because the annihilator of b is open in \mathfrak{R} and the family of elements $(r_c \in \mathfrak{R})_{c \in \mathfrak{C}}$ converges to zero.

For any discrete right \mathfrak{R} -module \mathcal{N} and any left \mathfrak{R} -contramodule \mathfrak{P} , the *contratensor product* $\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{P}$ is an abelian group constructed as the cokernel of (the difference of) the natural pair of maps

$$\mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{R}[[\mathfrak{P}]] \rightrightarrows \mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{P}.$$

Here one of the two maps is $\text{id}_{\mathcal{N}} \otimes \pi_{\mathfrak{P}}$, while the other one is given by the formula

$$b \otimes \sum_{p \in \mathfrak{P}} r_p p \longmapsto \sum_{p \in \mathfrak{P}} b r_p \otimes p \quad \text{for all } b \in \mathcal{N} \text{ and } \sum_{p \in \mathfrak{P}} r_p p \in \mathfrak{R}[[\mathfrak{P}]],$$

where, once again, the sum in the right-hand side is finite because the annihilator of b is open in \mathfrak{R} and the family of elements $r_p \in \mathfrak{R}$ converges to zero [20, Section 7.2], [18, Section 1.8], [17, Section 2.8], [19, Section 5].

For any discrete right \mathfrak{R} -module \mathcal{N} , any left \mathfrak{R} -contramodule \mathfrak{P} , and an abelian group V there is a natural isomorphism of abelian groups

$$(1) \quad \text{Hom}^{\mathfrak{R}}(\mathfrak{P}, \text{Hom}_{\mathbb{Z}}(\mathcal{N}, V)) \simeq \text{Hom}_{\mathbb{Z}}(\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{P}, V),$$

where $\text{Hom}^{\mathfrak{R}}$ denotes the group of morphisms in the category $\mathfrak{R}\text{-contra}$. For any discrete right \mathfrak{R} -module \mathcal{N} and any set X there is a natural isomorphism of abelian groups

$$(2) \quad \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{R}[[X]] \simeq \mathcal{N}[X].$$

A left \mathfrak{R} -contramodule \mathfrak{F} is said to be *flat* if the functor of contratensor product $-\odot_{\mathfrak{R}} \mathfrak{F}: \mathbf{discr}\text{-}\mathfrak{R} \rightarrow \mathbf{Ab}$, acting from the category of discrete right \mathfrak{R} -modules to the category of abelian groups \mathbf{Ab} , is exact. It is clear from the natural isomorphism (2) that free (hence also projective) left \mathfrak{R} -contramodules are flat. Furthermore, it follows from the adjunction isomorphism (1) that the functor of contratensor product $-\odot_{\mathfrak{R}} -: \mathbf{discr}\text{-}\mathfrak{R} \times \mathfrak{R}\text{-contra} \rightarrow \mathbf{Ab}$ preserves colimits (in both of its arguments). It follows that the class of all flat left \mathfrak{R} -contramodules is closed under direct limits. Hence all the direct limits of projective contramodules are flat.

For the purposes of the present paper, an (apparently) stronger flatness property of contramodules is relevant. The left derived functor of contratensor product

$$\text{Ctrtor}_*^{\mathfrak{R}}: \mathbf{discr}\text{-}\mathfrak{R} \times \mathfrak{R}\text{-contra} \longrightarrow \mathbf{Ab}, \quad \text{Ctrtor}_0^{\mathfrak{R}}(\mathcal{N}, \mathfrak{C}) = \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{C}$$

is constructed using projective resolutions of its second (contramodule) argument. A left \mathfrak{R} -contramodule \mathfrak{F} is said to be *n-strictly flat* (where $n \geq 1$ is an integer) if $\text{Ctrtor}_i^{\mathfrak{R}}(\mathcal{N}, \mathfrak{F}) = 0$ for all discrete right \mathfrak{R} -modules \mathcal{N} and all $0 < i \leq n$ [18, Section 2]. It suffices to check these conditions for the cyclic discrete right \mathfrak{R} -modules $\mathcal{N} = \mathfrak{R}/\mathfrak{J}$.

An \mathfrak{R} -contramodule is *∞ -strictly flat* if it is *n-strictly flat* for all $n \geq 1$. Obviously, a contramodule of projective dimension $\leq n$ is *n-strictly flat* if and only if it is *∞ -strictly flat*.

The kernel of an epimorphism from an *n-strictly flat* contramodule to an $(n + 1)$ -strictly flat contramodule is *n-strictly flat*. The kernel of an epimorphism from a flat contramodule to a 1-strictly flat contramodule is flat. Any 1-strictly flat contramodule is flat (so one can think of flat contramodules as “0-strictly flat”).

Clearly, for every $n \geq 1$ the class of all *n-strictly flat* left \mathfrak{R} -contramodules is closed under extensions. By [18, Lemma 2.1], the class of all 1-strictly flat contramodules is also closed under coproducts. We will see below in Corollary 6.1 that the class of 1-strictly flat contramodules is closed under direct limits. So all the direct limits of projective contramodules are, in fact, 1-strictly flat.

Over a topological ring \mathfrak{R} with a countable base of neighborhoods of zero, any flat contramodule is *∞ -strictly flat* [19, Remark 6.11 and Corollary 6.15].

We will use the following pieces of notation from [19, Section 5] and [18, Sections 1.5 and 1.10]. Given a closed right ideal $\mathfrak{J} \subset \mathfrak{R}$ and a set X , we denote by $\mathfrak{J}[[X]] \subset \mathfrak{R}[[X]]$ the subgroup of all zero-convergent X -indexed families of elements of \mathfrak{J} in the group of all such families of elements of \mathfrak{R} . For any left \mathfrak{R} -contramodule \mathfrak{C} , we denote by $\mathfrak{J} \ltimes \mathfrak{C} \subset \mathfrak{C}$ the image of the restriction $\mathfrak{J}[[\mathfrak{C}]] \rightarrow \mathfrak{C}$ of the contraction map $\mathfrak{R}[[\mathfrak{C}]] \rightarrow \mathfrak{C}$ to the subgroup $\mathfrak{J}[[\mathfrak{C}]] \subset \mathfrak{R}[[\mathfrak{C}]]$.

When \mathfrak{J} is a closed two-sided ideal in \mathfrak{R} , the subgroup $\mathfrak{J} \ltimes \mathfrak{C}$ is actually an \mathfrak{R} -subcontramodule in \mathfrak{C} . For any closed right ideal $\mathfrak{J} \subset \mathfrak{R}$ and any set X , one has

$$\mathfrak{J} \ltimes (\mathfrak{R}[[X]]) = \mathfrak{J}[[X]] \subset \mathfrak{R}[[X]].$$

For any open right ideal $\mathfrak{J} \subset \mathfrak{R}$ and any left \mathfrak{R} -contramodule \mathfrak{C} , the abelian group $\mathfrak{C}/(\mathfrak{J} \ltimes \mathfrak{C})$ can be interpreted as the contratensor product

$$\mathfrak{C}/(\mathfrak{J} \ltimes \mathfrak{C}) = (\mathfrak{R}/\mathfrak{J}) \odot_{\mathfrak{R}} \mathfrak{C}$$

of the cyclic discrete right \mathfrak{R} -module $\mathfrak{R}/\mathfrak{J}$ with the left \mathfrak{R} -contramodule \mathfrak{C} [18, Section 1.10], [17, Section 2.8], [19, Section 5].

2. JACOBSON RADICAL AND SUPERFLUOUS SUBCONTRAMODULES

Let \mathbf{A} be a category and $\mathbf{L} \subset \mathbf{A}$ be a class of objects. A morphism $l: L \rightarrow A$ in \mathbf{A} is said to be an *L-precover* (of the object A) if $L \in \mathbf{L}$ and for any morphism $l': L' \rightarrow A$ with $L' \in \mathbf{L}$ there exists a morphism $f: L' \rightarrow L$ such that $l' = lf$. An *L-precover* $l: L \rightarrow A$ is said to be an *L-cover* if for any endomorphism $f: L \rightarrow L$ the equation $lf = l$ implies that f is an automorphism.

Let \mathbf{B} be an abelian category with enough projective objects. We denote the full subcategory of projective objects in \mathbf{B} by $\mathbf{B}_{\text{proj}} \subset \mathbf{B}$. Then a morphism $p: P \rightarrow B$ in \mathbf{B} is a projective precover (i. e., a \mathbf{B}_{proj} -precover) if and only if $P \in \mathbf{B}_{\text{proj}}$ and p is an epimorphism. A projective precover $p: P \rightarrow B$ is a projective cover if and only if its kernel K is a superfluous subobject in P . Here a subobject $K \subset P$ of an arbitrary object P in an abelian category \mathbf{B} is said to be *superfluous* if for any subobject $X \subset P$ the equality $K + X = P$ implies $X = P$.

The aim of this section is to prove the following

Theorem 2.1. *Let \mathfrak{F} be an ∞ -strictly flat left \mathfrak{A} -contramodule of projective dimension not exceeding 1. Assume that \mathfrak{F} has a projective cover in $\mathfrak{A}\text{-contra}$. Then \mathfrak{F} is a projective \mathfrak{A} -contramodule.*

Our proof extends to the contramodule realm the argument for a discrete ring R outlined in the now-obsolete preprint [8, Lemma 3.2 and/or Corollary 3.4(a)]. The proof is based on three technical propositions, the first of which is formulated immediately below.

We denote by $\mathfrak{H} = \mathfrak{H}(\mathfrak{A}) \subset \mathfrak{A}$ the topological Jacobson radical of the ring \mathfrak{A} , that is, the intersection of all the open maximal right ideals in \mathfrak{A} [14, Section 3.B], [18, Section 6]. So \mathfrak{H} is a closed two-sided ideal in \mathfrak{A} [18, Lemma 6.1]. The Jacobson radical of the ring \mathfrak{A} viewed as an abstract (nontopological) associative ring is denoted by $H = H(\mathfrak{A}) \subset \mathfrak{A}$. So H is a two-sided ideal in \mathfrak{A} , but we do not know whether it needs to be a closed ideal. Obviously, one has $H(\mathfrak{A}) \subset \mathfrak{H}(\mathfrak{A})$.

Proposition 2.2. *Let \mathfrak{P} be a projective left \mathfrak{A} -contramodule and $\mathfrak{K} \subset \mathfrak{P}$ be a superfluous subcontramodule. Then $\mathfrak{K} \subset \mathfrak{H} \triangleleft \mathfrak{P}$.*

The proof of Proposition 2.2 consists of three lemmas.

Lemma 2.3. *Let \mathbf{B} be an abelian category, $f: P \rightarrow Q$ be a morphism in \mathbf{B} , and $K \subset P$ be a superfluous subobject. Then the image $L = f(K)$ of the subobject K under the morphism f is a superfluous subobject in Q . In particular,*

(a) *if $P, Q \in \mathbf{B}$ are two objects and $K \subset P$ is a superfluous subobject, then $K \oplus 0$ is a superfluous subobject in $F = P \oplus Q$,*

(b) *if $P, Q \in \mathbf{B}$ are two objects, $K \subset F = P \oplus Q$ is a superfluous subobject, and $f: F \rightarrow Q$ is the direct summand projection, then $f(K)$ is a superfluous subobject in Q .*

Proof. Let $Y \subset Q$ be a subobject such that $L + Y = Q$. Then $X = f^{-1}(Y) \subset P$ is a subobject such that $K + X = P$. Hence $X = P$; so $f(P) \subset Y$. It follows that $L = f(K) \subset f(P) \subset Y$ and therefore $Y = Q$. \square

Lemma 2.4. *Let \mathfrak{C} be a left \mathfrak{A} -contramodule and $c \in \mathfrak{C}$ be an element. Then the cyclic \mathfrak{A} -submodule $\mathfrak{A}c \subset \mathfrak{C}$ is an \mathfrak{A} -subcontramodule in \mathfrak{C} .*

Proof. Let $\mathfrak{A} = \mathfrak{A}[[*]]$ be the free left \mathfrak{A} -contramodule with one generator. Then the \mathfrak{A} -contramodule morphisms $\mathfrak{A} \rightarrow \mathfrak{C}$ correspond bijectively to the elements of \mathfrak{C} . In other words, the map $\mathfrak{A} \xrightarrow{c} \mathfrak{C}$ taking every element $r \in \mathfrak{A}$ to the element $rc \in \mathfrak{C}$ is

a left \mathfrak{R} -contramodule morphism (see [20, Section 6.2] or [18, Section 1.7]). Now the cyclic submodule $\mathfrak{R}\mathfrak{c} \subset \mathfrak{C}$ is the image of this contramodule morphism, hence it is a subcontramodule. \square

More generally, any finitely generated \mathfrak{R} -submodule of an \mathfrak{R} -contramodule is a subcontramodule.

Lemma 2.5. *Consider the free left \mathfrak{R} -contramodule with one generator $\mathfrak{R}[[\ast]] = \mathfrak{R}$. Let $\mathfrak{L} \subset \mathfrak{R}$ be a superfluous left \mathfrak{R} -subcontramodule. Then $\mathfrak{L} \subset H(\mathfrak{R})$.*

Proof. Notice first of all that \mathfrak{L} is a left \mathfrak{R} -submodule (i. e., a left ideal) in \mathfrak{R} . Suppose \mathfrak{L} is not contained in $H(\mathfrak{R})$. Then there exists a (not necessarily closed) maximal left ideal $M \subset \mathfrak{R}$ such that \mathfrak{L} is not contained in M , and consequently $\mathfrak{L} + M = \mathfrak{R}$. Let $l \in \mathfrak{L}$ and $m \in M$ be a pair of elements such that $l + m = 1$. Then we have $\mathfrak{L} + \mathfrak{R}m = \mathfrak{R}$ and $\mathfrak{R}m \subsetneq \mathfrak{R}$. By Lemma 2.4, the principal left ideal $\mathfrak{R}m$ is a left \mathfrak{R} -subcontramodule in \mathfrak{R} . (Notice that there is *no* claim about $\mathfrak{R}m$ being a *closed* left ideal in \mathfrak{R} here.) The contradiction with the superfluousness assumption proves that $\mathfrak{L} \subset H(\mathfrak{R})$. \square

Proof of Proposition 2.2. By Lemma 2.3(a), we can assume that \mathfrak{P} is a free left \mathfrak{R} -contramodule, $\mathfrak{P} = \mathfrak{R}[[X]]$. For every element $x \in X$, consider the coordinate projection $f_x: \mathfrak{P} \hookrightarrow \mathfrak{R}^X \rightarrow \mathfrak{R}$ corresponding to x and put $\mathfrak{L}_x = f_x(\mathfrak{R})$. By Lemma 2.3(b), \mathfrak{L}_x is a superfluous left \mathfrak{R} -subcontramodule in \mathfrak{R} . According to Lemma 2.5, we have $\mathfrak{L}_x \subset H \subset \mathfrak{H}$. It follows that $\mathfrak{R} \subset \mathfrak{H}[[X]] = \mathfrak{H} \times \mathfrak{R}[[X]]$. \square

In fact, we have shown that under the assumptions of Proposition 2.2 one has $\mathfrak{R} \subset \overline{H} \times \mathfrak{P}$, where $\overline{H} \subset \mathfrak{R}$ is the topological closure of the abstract Jacobson radical $H \subset \mathfrak{R}$ (but we will not use this fact).

The second main technical ingredient is the next

Proposition 2.6. *Let $0 \rightarrow \mathfrak{R} \rightarrow \mathfrak{P} \rightarrow \mathfrak{F} \rightarrow 0$ be a short exact sequence of left \mathfrak{R} -contramodules, where \mathfrak{R} and \mathfrak{P} are projective \mathfrak{R} -contramodules and \mathfrak{F} is a 1-strictly flat \mathfrak{R} -contramodule. Let $\mathfrak{J} \subset \mathfrak{R}$ be a closed right ideal. Then one has*

$$\mathfrak{R} \cap (\mathfrak{J} \times \mathfrak{P}) = \mathfrak{J} \times \mathfrak{R}.$$

Proof. Let us first consider the case of an open right ideal $\mathfrak{J} \subset \mathfrak{R}$. Then the equation $\mathfrak{R} \cap (\mathfrak{J} \times \mathfrak{P}) = \mathfrak{J} \times \mathfrak{R}$ is equivalent to exactness of the short sequence

$$(3) \quad 0 \longrightarrow \mathfrak{R}/(\mathfrak{J} \times \mathfrak{R}) \longrightarrow \mathfrak{P}/(\mathfrak{J} \times \mathfrak{P}) \longrightarrow \mathfrak{F}/(\mathfrak{J} \times \mathfrak{F}) \longrightarrow 0.$$

For any left \mathfrak{R} -contramodule \mathfrak{C} , we have $\mathfrak{C}/(\mathfrak{J} \times \mathfrak{C}) = (\mathfrak{R}/\mathfrak{J}) \circ_{\mathfrak{R}} \mathfrak{C}$. In view of the homological long exact sequence of the derived functor Ctrtor ,

$$\cdots \longrightarrow \text{Ctrtor}_1^{\mathfrak{R}}(\mathfrak{N}, \mathfrak{F}) \longrightarrow \mathfrak{N} \circ_{\mathfrak{R}} \mathfrak{R} \longrightarrow \mathfrak{N} \circ_{\mathfrak{R}} \mathfrak{P} \longrightarrow \mathfrak{N} \circ_{\mathfrak{R}} \mathfrak{F} \longrightarrow 0,$$

which is defined for any discrete right \mathfrak{R} -module \mathfrak{N} and in particular for $\mathfrak{N} = \mathfrak{R}/\mathfrak{J}$, it follows that the short sequence (3) is exact.

In the general case of a closed right ideal $\mathfrak{J} \subset \mathfrak{R}$, we have

$$\mathfrak{J} = \bigcap_{\mathfrak{J} \subset \mathfrak{J}'} \mathfrak{J}',$$

where the intersection is taken over all the open right ideals \mathfrak{J} in \mathfrak{R} containing \mathfrak{J} . It follows that, for any set X ,

$$\mathfrak{J}[[X]] = \bigcap_{\mathfrak{J} \subset \mathfrak{R}} \mathfrak{J}[[X]] \subset \mathfrak{R}[[X]].$$

In other words, for any free left \mathfrak{R} -contramodule $\mathfrak{G} = \mathfrak{R}[[X]]$ we have

$$\mathfrak{J} \times \mathfrak{G} = \mathfrak{J}[[X]] = \bigcap_{\mathfrak{J} \subset \mathfrak{R}} \mathfrak{J}[[X]] = \bigcap_{\mathfrak{J} \subset \mathfrak{R}} \mathfrak{J} \times \mathfrak{G}.$$

Since any projective left \mathfrak{R} -contramodule \mathfrak{Q} is a direct summand of a free one, we obtain the equality

$$\mathfrak{J} \times \mathfrak{Q} = \bigcap_{\mathfrak{J} \subset \mathfrak{R}} \mathfrak{J} \times \mathfrak{Q} \subset \mathfrak{Q}.$$

In particular, this holds for $\mathfrak{Q} = \mathfrak{P}$ and $\mathfrak{Q} = \mathfrak{K}$. Finally, we can compute

$$\mathfrak{K} \cap (\mathfrak{J} \times \mathfrak{P}) = \mathfrak{K} \cap \bigcap_{\mathfrak{J} \subset \mathfrak{R}} \mathfrak{J} \times \mathfrak{P} = \bigcap_{\mathfrak{J} \subset \mathfrak{R}} \mathfrak{K} \cap (\mathfrak{J} \times \mathfrak{P}) = \bigcap_{\mathfrak{J} \subset \mathfrak{R}} \mathfrak{J} \times \mathfrak{K} = \mathfrak{J} \times \mathfrak{K}.$$

□

Our third main technical result in this section is the following version of Nakayama lemma for projective contramodules. It is the contramodule generalization of the classical [7, Proposition 2.7].

Proposition 2.7. *Let \mathfrak{P} be a nonzero projective left \mathfrak{R} -contramodule. Then*

$$\mathfrak{H} \times \mathfrak{P} \subsetneq \mathfrak{P}.$$

The proof of Proposition 2.7 is based on two lemmas. The first of them expands the list of equivalent conditions characterizing the topological Jacobson radical \mathfrak{H} of a topological ring \mathfrak{R} given in [18, Lemma 6.2].

Lemma 2.8. *Given an element $h \in \mathfrak{R}$, the following three conditions are equivalent:*

- (a) $h \in \mathfrak{H}$;
- (b) *for every open right ideal $\mathfrak{J} \subset \mathfrak{R}$ and every element $r \in \mathfrak{R}$, the right multiplication by $1 - hr$ acts injectively in $\mathfrak{R}/\mathfrak{J}$;*
- (c) *for every open right ideal $\mathfrak{J} \subset \mathfrak{R}$ and every element $r \in \mathfrak{R}$, the right multiplication by $1 - rh$ acts injectively in $\mathfrak{R}/\mathfrak{J}$.*

In particular, for every element $h \in \mathfrak{H}$, the right multiplication with $1 - h$ acts injectively in \mathfrak{R} .

Proof. (a) \implies (b) and (c): since \mathfrak{H} is a two-sided ideal in \mathfrak{R} by [18, Lemma 6.1], it suffices to show that $1 - h$ acts injectively in $\mathfrak{R}/\mathfrak{J}$ for every $h \in \mathfrak{H}$ and any open right ideal \mathfrak{J} . Let $0 \neq s + \mathfrak{J} \in \mathfrak{R}/\mathfrak{J}$. Then there exists an open right ideal $\mathfrak{I} \subset \mathfrak{R}$ such that $s\mathfrak{I} \subset \mathfrak{J}$. By [18, Lemma 6.2(iii) or (iv)], there is an element $t \in \mathfrak{R}$ such that $(1 - h)t + \mathfrak{I} = 1 + \mathfrak{I}$. Multiplying the latter equation by s on the left, we get $s(1 - h)t + s\mathfrak{I} = s + s\mathfrak{I}$, hence $s(1 - h)t + \mathfrak{J} = s + \mathfrak{J} \neq 0$ in $\mathfrak{R}/\mathfrak{J}$. It follows that $s(1 - h) + \mathfrak{J} \neq 0$ in $\mathfrak{R}/\mathfrak{J}$.

(b) or (c) \implies (a): by [18, Lemma 6.2(ii)], for any $h \notin \mathfrak{H}$ there exists a simple discrete right \mathfrak{R} -module \mathfrak{S} and a pair of nonzero elements $x, y \in \mathfrak{S}$ such that $xh = y$. Since \mathfrak{S} is simple, there is also an element $r \in \mathfrak{R}$ such that $yr = x$. Thus $x(1 - hr) =$

$0 = y(1 - rh)$ in \mathcal{S} , and we have shown that neither $1 - hr$ nor $1 - rh$ act injectively in the cyclic discrete right \mathfrak{R} -module \mathcal{S} .

To prove the last assertion of the lemma, suppose that we have $s(1 - h) = 0$ in \mathfrak{R} for some elements $s \in \mathfrak{R}$ and $h \in \mathfrak{H}$. Let $\mathcal{J} \subset \mathfrak{R}$ be an open right ideal. Then we have $s(1 - h) + \mathcal{J} = 0$ in \mathfrak{R}/\mathcal{J} , which implies $s + \mathcal{J} = 0$ in \mathfrak{R}/\mathcal{J} by (b) or (c). Hence $s \in \mathcal{J}$. As this holds for every open right ideal \mathcal{J} and the topological ring \mathfrak{R} is separated by assumption, we can conclude that $s = 0$ in \mathfrak{R} . \square

For the next lemma we need the construction of the topological ring of *row-zero-convergent matrices* $\mathbf{Mat}_Y(\mathfrak{R})$ (see [21, Section 5]). For any set Y , the elements of $\mathbf{Mat}_Y(\mathfrak{R})$ are $Y \times Y$ matrices $(m_{x,y} \in \mathfrak{R})_{x,y \in Y}$ such that for every fixed $x \in Y$ the family of elements $(m_{x,y})_{y \in Y}$ converges to zero in the topology of \mathfrak{R} . The usual matrix multiplication, which is well-defined thanks to the infinite summation in \mathfrak{R} , gives $\mathbf{Mat}_Y(\mathfrak{R})$ a ring structure. There is also a natural topology in $\mathbf{Mat}_Y(\mathfrak{R})$, which makes it a complete, separated topological ring with a base of neighborhoods of zero consisting of open right ideals.

The main motivation for the construction of $\mathbf{Mat}_Y(\mathfrak{R})$ were Morita equivalence type results. In particular, following [21, Proposition 5.2], the categories of discrete right modules over the rings \mathfrak{R} and $\mathbf{Mat}_Y(\mathfrak{R})$ are naturally equivalent. The equivalence is provided by the functor

$$(4) \quad \mathcal{V}_Y: \text{discr-}\mathfrak{R} \longrightarrow \text{discr-}\mathbf{Mat}_Y(\mathfrak{R})$$

assigning to every discrete right \mathfrak{R} -module \mathcal{N} the discrete right $\mathbf{Mat}_Y(\mathfrak{R})$ -module $\mathcal{V}_Y(\mathcal{N}) = \mathcal{N}^{(Y)}$. Elements of the direct sum $\mathcal{N}^{(Y)}$ of Y copies of \mathcal{N} are interpreted as finite rows of elements of \mathcal{N} , and $\mathbf{Mat}_Y(\mathfrak{R})$ acts in $\mathcal{V}_Y(\mathcal{N})$ by the usual right action of matrices on row-vectors (which is well-defined in this case due to the row-zero-convergence condition imposed on the elements of $\mathbf{Mat}_Y(\mathfrak{R})$ and the discreteness condition imposed on \mathcal{N}).

Lemma 2.9. *The topological Jacobson radical of the topological ring $\mathbf{Mat}_Y(\mathfrak{R})$ consists of all the (row-zero-convergent) matrices with entries in the topological Jacobson radical \mathfrak{H} of the ring \mathfrak{R} . So $\mathfrak{H}(\mathbf{Mat}_Y(\mathfrak{R})) = \mathbf{Mat}_Y(\mathfrak{H}(\mathfrak{R}))$.*

Proof. By [18, Lemma 6.2(ii)], the topological Jacobson radical $\mathfrak{H}(\mathfrak{R})$ consists of all the elements $a \in \mathfrak{R}$ annihilating all the simple discrete right \mathfrak{R} -modules, and similarly for $\mathfrak{H}(\mathbf{Mat}_Y(\mathfrak{R}))$.

The equivalence $\mathcal{V}_Y: \text{discr-}\mathfrak{R} \longrightarrow \text{discr-}\mathbf{Mat}_Y(\mathfrak{R})$ in (4), as any equivalence of abelian categories, induces a bijection between the isomorphism classes of simple objects. So the simple discrete right $\mathbf{Mat}_Y(\mathfrak{R})$ -modules are precisely the right $\mathbf{Mat}_Y(\mathfrak{R})$ -modules $\mathcal{V}_Y(\mathcal{S})$, where \mathcal{S} ranges over all the simple discrete right \mathfrak{R} -modules. It remains to observe that a matrix $A = (a_{xy})_{x,y \in X} \in \mathbf{Mat}_Y(\mathfrak{R})$ annihilates all the elements of $\mathcal{V}_Y(\mathcal{S}) = \mathcal{S}^{(Y)}$ if and only if all the entries $a_{x,y}$ of A annihilate all the elements of \mathcal{S} . \square

Proof of Proposition 2.7. We follow the argument in [1, proof of Proposition 17.14] with suitable modifications. Let $\mathfrak{G} = \mathfrak{R}[[X]]$ be a free left \mathfrak{R} -contramodule and \mathfrak{P} be

a direct summand of \mathfrak{G} . We will view \mathfrak{P} as a subcontramodule in \mathfrak{G} and denote by $e: \mathfrak{G} \rightarrow \mathfrak{G}$ an idempotent \mathfrak{R} -contramodule endomorphism of \mathfrak{G} such that $\mathfrak{P} = \mathfrak{G}e$ (for simplicity of notation, we let e act in \mathfrak{G} on the right). Elements of the set X will be viewed as (the basis) elements of \mathfrak{G} .

Let $q = \sum_{x \in X} q_x x$ be an element of \mathfrak{P} . Here $(q_x \in \mathfrak{R})_{x \in X}$ is a family of elements converging to zero in the ring \mathfrak{R} and the sum $\sum_{x \in X} q_x x$ can be understood as the result of applying the contramodule infinite summation operation with the family of coefficients q_x to the X -indexed family of elements $x \in \mathfrak{G}$.

Assuming that $\mathfrak{P} = \mathfrak{H} \ltimes \mathfrak{P}$, we will prove that $q = 0$. Indeed, we have $\mathfrak{H} \ltimes \mathfrak{P} \subset \mathfrak{H} \ltimes \mathfrak{G} = \mathfrak{H}[[X]]$; so $\mathfrak{P} \subset \mathfrak{H}[[X]] \subset \mathfrak{R}[[X]] = \mathfrak{G}$. For every element $x \in X$, we have $xe \in \mathfrak{P}$, hence

$$xe = \sum_{y \in X} a_{x,y} y, \quad a_{x,y} \in \mathfrak{H} \subset \mathfrak{R},$$

where the family of elements $(a_{x,y})_{y \in X}$ converges to zero in \mathfrak{R} for every fixed $x \in X$. Now we can compute that

$$\begin{aligned} 0 = q - qe &= \sum_{x \in X} q_x x - \sum_{x \in X} q_x xe \\ &= \sum_{x \in X} q_x \sum_{y \in Y} \delta_{x,y} y - \sum_{x \in X} q_x \sum_{y \in Y} a_{x,y} y \\ &= \sum_{x \in X} q_x \sum_{y \in Y} (\delta_{x,y} - a_{x,y}) y. \end{aligned}$$

The resulting equation means that

$$(5) \quad \sum_{x \in X} q_x (\delta_{x,y} - a_{x,y}) = 0 \quad \text{for every } y \in X.$$

If the set X is empty, then there is nothing to prove. Otherwise, choose an element $x_0 \in X$, and consider the $X \times X$ matrix $Q = (q_{z,x})_{z,x \in X}$ with the entries $q_{z,x} = q_x$ when $z = x_0$ and $q_{z,x} = 0$ when $z \neq x_0$. In other words, we consider the family of elements $(q_x)_{x \in X}$ as an X -indexed row and build an $X \times X$ matrix in which this row is the only nonzero one. We also consider the $X \times X$ matrices $A = (a_{x,y})_{x,y \in X}$ and $1 = (\delta_{x,y})_{x,y \in X}$. All the three matrices Q , A , and 1 have entries in \mathfrak{R} and zero-convergent rows, so they belong to $\mathfrak{Mat}_X(\mathfrak{R})$; and, of course, 1 is the unit element of the ring $\mathfrak{Mat}_X(\mathfrak{R})$. Then the family of equations (5) can be expressed as a matrix multiplication equation $Q(1 - A) = 0$ in the ring $\mathfrak{Mat}_X(\mathfrak{R})$.

Furthermore, by Lemma 2.9, the matrix A belongs to the topological Jacobson radical of the ring $\mathfrak{Mat}_X(\mathfrak{R})$. By Lemma 2.8, the right multiplication with $1 - A$ acts injectively in $\mathfrak{Mat}_X(\mathfrak{R})$. Thus $Q = 0$, and it follows that $q = 0$, as desired. \square

Proof of Theorem 2.1. Let \mathfrak{F} be a 1-strictly flat left \mathfrak{R} -contramodule of projective dimension not exceeding 1, and $p: \mathfrak{P} \rightarrow \mathfrak{F}$ be its projective cover in $\mathfrak{R}\text{-contra}$. Since there are enough projective objects in $\mathfrak{R}\text{-contra}$, the map p is surjective. Put $\mathfrak{K} = \ker(p)$. Then \mathfrak{K} is a projective \mathfrak{R} -contramodule. By [18, Lemma 3.1], \mathfrak{K} is a superfluous subcontramodule in \mathfrak{P} .

According to Proposition 2.2, we have $\mathfrak{K} \subset \mathfrak{H} \ltimes \mathfrak{P}$. From Proposition 2.6, we know that $\mathfrak{K} \cap (\mathfrak{H} \ltimes \mathfrak{P}) = \mathfrak{H} \ltimes \mathfrak{K}$. Thus $\mathfrak{K} = \mathfrak{H} \ltimes \mathfrak{K}$. By Proposition 2.7, it follows that $\mathfrak{K} = 0$. We can conclude that $\mathfrak{F} \simeq \mathfrak{P}$ is a projective left \mathfrak{R} -contramodule. \square

In particular, by [18, Corollary 2.4], all countable direct limits of projective contramodules are 1-strictly flat (in fact, ∞ -strictly flat) of projective dimension not exceeding 1. Hence it follows from Theorem 2.1 that a countable direct limit of projective left \mathfrak{R} -contramodules is projective if it has a projective cover. The following special case is of interest.

A *Bass flat contramodule* is a countable direct limit of free left \mathfrak{R} -contramodules with one generator, computed in the category of left \mathfrak{R} -contramodules $\mathfrak{R}\text{-contra}$,

$$\mathfrak{B} = \varinjlim_{\omega}^{\mathfrak{R}\text{-contra}} (\mathfrak{R} \xrightarrow{*a_1} \mathfrak{R} \xrightarrow{*a_2} \mathfrak{R} \xrightarrow{*a_3} \dots)$$

where a_1, a_2, \dots is a sequence of elements of \mathfrak{R} and $*a: \mathfrak{R} \rightarrow \mathfrak{R}$ is the left \mathfrak{R} -contramodule morphism of right multiplication with $a \in \mathfrak{R}$.

Corollary 2.10. *If a Bass flat left \mathfrak{R} -contramodule \mathfrak{B} has a projective cover in $\mathfrak{R}\text{-contra}$, then \mathfrak{B} is projective (as a left \mathfrak{R} -contramodule). \square*

The generalization to uncountable direct limits of projective contramodules will be obtained as Corollary 6.5 in Section 6.

3. QUASI-SPLIT EXACT SEQUENCES AND LOCALLY SPLIT MORPHISMS

In the next several sections, we consider two kinds of abelian categories. An abelian category is said to be *Ab3* if it is cocomplete, or in other words, if it has set-indexed coproducts. A cocomplete abelian category is said to satisfy *Ab5* if it has exact direct limit functors.

In particular, the category of left contramodules $\mathfrak{R}\text{-contra}$ over a topological ring \mathfrak{R} is *Ab3*, but usually not *Ab5*. A *Grothendieck abelian category* is an *Ab5* category with a generator. The category of left modules $A\text{-mod}$ over any associative ring A is Grothendieck.

Let \mathbf{A} be an *Ab5* category. We will say that a short exact sequence $0 \rightarrow K \rightarrow C \rightarrow D \rightarrow 0$ in \mathbf{A} is *quasi-split* if it is the direct limit of a direct system of split short exact sequences $0 \rightarrow K_x \rightarrow C \rightarrow D_x \rightarrow 0$, indexed by a directed poset X , where $(K_x)_{x \in X}$ and $(D_x)_{x \in X}$ are direct systems of objects in \mathbf{A} and $(C_x = C)_{x \in X}$ is a constant direct system. If this is the case, the morphism $K \rightarrow C$ is said to be a *quasi-split monomorphism* and the morphism $C \rightarrow D$ is a *quasi-split epimorphism*.

The definition of a quasi-split short exact sequence in an *Ab3* category \mathbf{B} is slightly more complicated. Suppose that we are given a direct system

$$(6) \quad 0 \longrightarrow \mathfrak{K}_x \longrightarrow \mathfrak{C} \longrightarrow \mathfrak{D}_x \longrightarrow 0$$

of split short exact sequences in \mathbf{B} , indexed by a directed poset X , where $(\mathfrak{K}_x)_{x \in X}$ and $(\mathfrak{D}_x)_{x \in X}$ are direct systems of objects and $(\mathfrak{C}_x = \mathfrak{C})_{x \in X}$ is a constant direct system.

The direct limit of a direct system of short exact sequences in \mathbf{B} does not need to be exact, but only right exact; so the direct limit of (6) is a right exact sequence

$$(7) \quad \mathfrak{M} \longrightarrow \mathfrak{C} \longrightarrow \mathfrak{D} \longrightarrow 0.$$

Let \mathfrak{K} be the image of the morphism $\mathfrak{M} \longrightarrow \mathfrak{C}$. Then we will say that the short exact sequence $0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{C} \longrightarrow \mathfrak{D} \longrightarrow 0$ in \mathbf{B} is *quasi-split*, the morphism $\mathfrak{K} \longrightarrow \mathfrak{C}$ is a *quasi-split monomorphism*, and the morphism $\mathfrak{C} \longrightarrow \mathfrak{D}$ is a *quasi-split epimorphism*.

The following proposition (generalizing [11, Lemma 2.1]) introduces the class of examples we are mainly interested in.

Proposition 3.1. *Let $(\mathfrak{c}_{y,x}: \mathfrak{C}_x \rightarrow \mathfrak{C}_y)_{x < y \in X}$ be a diagram in a cocomplete abelian category \mathbf{B} , indexed by a directed poset X , and let $\mathfrak{D} = \varinjlim_{x \in X}^{\mathbf{B}} \mathfrak{C}_x$ be its direct limit in \mathbf{B} . Then the natural epimorphism $\mathfrak{p}: \mathfrak{C}' = \coprod_{x \in X}^{\mathbf{B}} \mathfrak{C}_x \longrightarrow \mathfrak{D}$ is quasi-split.*

Proof. We have a natural right exact sequence

$$(8) \quad \coprod_{x < y \in X} \mathfrak{C}_{x,y} \xrightarrow{\mathfrak{t}} \coprod_{x \in X} \mathfrak{C}_x \xrightarrow{\mathfrak{p}} \mathfrak{D} \longrightarrow 0,$$

where $\mathfrak{C}_{x,y} = \mathfrak{C}_x$ is a copy of the object \mathfrak{C}_x for every pair of elements $x < y$ in X .

Fix an element $z \in X$, and consider the subdiagram $(\mathfrak{C}_x)_{x \in X}^{x \leq z}$ of our diagram $(\mathfrak{C}_x)_{x \in X}$ formed by all the objects \mathfrak{C}_x with $x \leq z$ and the morphisms $\mathfrak{c}_{y,x}: \mathfrak{C}_x \longrightarrow \mathfrak{C}_y$, $x < y \leq z$. Obviously, the object \mathfrak{C}_z is the direct limit of the diagram $(\mathfrak{C}_x)_{x \in X}^{x \leq z}$. So the right exact sequence (8) for the diagram $(\mathfrak{C}_x)_{x \in X}^{x \leq z}$ takes the form

$$(9) \quad \coprod_{x < y \in X}^{y \leq z} \mathfrak{C}_{x,y} \xrightarrow{\mathfrak{t}_z} \coprod_{x \in X}^{x \leq z} \mathfrak{C}_x \xrightarrow{\mathfrak{p}_z} \mathfrak{C}_z \longrightarrow 0.$$

As the element $z \in X$ varies, the right exact sequences (9) form a diagram indexed by the same poset X . Given a pair of elements $z < w \in X$, the related morphisms on the middle and the leftmost terms of (9) are the subcoproduct inclusions corresponding to the inclusions of subsets $\{x \in X \mid x \leq z\} \hookrightarrow \{x \in X \mid x \leq w\}$ and $\{(x, y) \mid x < y \leq z\} \hookrightarrow \{(x, y) \mid x < y \leq w\}$, while the morphism on the rightmost terms is $\mathfrak{c}_{w,z}: \mathfrak{C}_z \longrightarrow \mathfrak{C}_w$. The right exact sequence (8) is the direct limit of the right exact sequences (9) over $z \in X$.

We denote by \mathfrak{K} the image of the morphism \mathfrak{t} (which coincides with the kernel of the morphism \mathfrak{p}) in the sequence (8). Similarly, let us denote by \mathfrak{K}_z the image of the morphism \mathfrak{t}_z (which coincides with the kernel of the morphism \mathfrak{p}_z) in the sequence (9). So we have short exact sequences

$$(10) \quad 0 \longrightarrow \mathfrak{K} \xrightarrow{\mathfrak{i}} \coprod_{x \in X} \mathfrak{C}_x \xrightarrow{\mathfrak{p}} \mathfrak{D} \longrightarrow 0$$

and

$$(11) \quad 0 \longrightarrow \mathfrak{K}_z \xrightarrow{\mathfrak{i}_z} \coprod_{x \in X}^{x \leq z} \mathfrak{C}_x \xrightarrow{\mathfrak{p}_z} \mathfrak{C}_z \longrightarrow 0.$$

As the element $z \in X$ varies, the short exact sequences (11) form a diagram, indexed by the poset X . The morphism \mathbf{p} is the direct limit of the morphisms \mathbf{p}_z , while the object \mathfrak{K} does *not* need to be the direct limit of the objects \mathfrak{K}_z (as the direct limits in \mathbf{B} are only right exact). In fact, the direct limit of the short exact sequences (11) is, generally speaking, a right exact sequence of the form

$$(12) \quad \mathfrak{M} = \varinjlim_{z \in X} \mathfrak{K}_z \xrightarrow{\mathbf{m}} \prod_{x \in X} \mathfrak{C}_x \xrightarrow{\mathbf{p}} \mathfrak{D} \longrightarrow 0,$$

and the object \mathfrak{K} is the image of the morphism \mathbf{m} .

Notice that the object \mathfrak{K}_z is, of course, naturally isomorphic to $\prod_{x \in X}^{x < z} \mathfrak{C}_x$, and the short exact sequence (11) is naturally split. However, the morphism $\mathbf{i}_z: \prod_{x \in X}^{x < z} \mathfrak{C}_x \longrightarrow \prod_{x \in X}^{x \leq z} \mathfrak{C}_x$ is *not* the subcoproduct inclusion related to the inclusion of subsets $\{x \mid x < z\} \hookrightarrow \{x \mid x \leq z\}$. Rather, it is a certain “diagonal” map which can be constructed in terms of the morphisms $\mathfrak{c}_{z,x}: \mathfrak{C}_x \longrightarrow \mathfrak{C}_z$. Given a pair of elements $z < w$, the related morphism between the middle terms of the sequences (11) is the subcoproduct inclusion described above, but the related morphism $\mathfrak{k}_{w,z}: \mathfrak{K}_z \longrightarrow \mathfrak{K}_w$ between the leftmost terms of the sequences (11) is *not* the subcoproduct inclusion.

Now we recall the notation $\mathfrak{C}' = \prod_{x \in X} \mathfrak{C}_x$, and set \mathfrak{D}_z to be the cokernel of the composition of split monomorphisms $\mathfrak{K}_z \longrightarrow \prod_{x \in X}^{x \leq z} \mathfrak{C}_x \longrightarrow \mathfrak{C}'$. Then we have a direct system of split short exact sequences

$$(13) \quad 0 \longrightarrow \mathfrak{K}_z \xrightarrow{i'_z} \mathfrak{C}' \xrightarrow{p'_z} \mathfrak{D}_z \longrightarrow 0,$$

where the objects $(\mathfrak{C}' = \mathfrak{C}')_{z \in X}$ form a constant direct system and the morphism \mathbf{m} is the direct limit of the morphisms i'_z over $z \in X$. Hence the morphism \mathbf{p} is the direct limit of the morphisms p'_z , and we are done. \square

The next concept is more general than quasi-splitness. Before introducing it for arbitrary Ab3 categories, let us recall its definition for the categories of modules. Let A be an associative ring, and $K \subset C$ be (say, left) A -modules. Then the submodule K is said to be *locally split* in C if for every element $k \in K$ there exists an A -module morphism $h: C \longrightarrow K$ such that $h(k) = k$. If this is the case, then for any finite set of elements $k_1, \dots, k_n \in K$ there exists an A -module morphism $g: C \longrightarrow K$ such that $g(k_i) = k_i$ for all $i = 1, \dots, n$ [28, Proposition 1.2 (2) \Rightarrow (1)].

Let $\mathbf{m}: \mathfrak{M} \longrightarrow \mathfrak{C}$ be a morphism in an Ab3 category \mathbf{B} . We will say that \mathbf{m} is *locally split* if there exists a direct system $(\mathfrak{K}_x)_{x \in X}$ in the category \mathbf{B} , indexed by some directed poset X , an epimorphism $\mathfrak{s}: \varinjlim_{x \in X} \mathfrak{K}_x \longrightarrow \mathfrak{M}$, and morphisms $\mathfrak{g}_x: \mathfrak{C} \longrightarrow \mathfrak{M}$, $x \in X$, such that the equation $\mathfrak{g}_x \mathbf{m} \mathfrak{s} \mathfrak{k}_x = \mathfrak{s} \mathfrak{k}_x$ holds for every $x \in X$, where $\mathfrak{k}_x: \mathfrak{K}_x \longrightarrow \varinjlim_{y \in X} \mathfrak{K}_y$ is the canonical morphism. A subobject $\mathfrak{K} \subset \mathfrak{C}$ is said to be *locally split* if its inclusion morphism $\mathbf{i}: \mathfrak{K} \longrightarrow \mathfrak{C}$ is locally split.

Let us emphasize that our definition of a locally split morphism is designed to handle locally split *monomorphisms*. It is *not* relevant to the notion of a locally split epimorphism (which has been also considered in the module theory literature, as per the references in the introduction; for example, a dual version of the above result

about the equivalence of the local splitness for one element and for a finite number of elements can be found in [6, Corollary 2]).

Though a locally split morphism in an Ab3 category, in the sense of our definition, does not need to be a monomorphism, any locally split morphism in an Ab5 category is a monomorphism, as we will see in the next section.

Lemma 3.2. *In any Ab3 category, the image of a locally split morphism is a locally split subobject.*

Proof. Denote by \mathfrak{N} the image of a morphism $\mathfrak{m}: \mathfrak{M} \rightarrow \mathfrak{C}$; so \mathfrak{m} decomposes as $\mathfrak{M} \xrightarrow{\mathfrak{n}} \mathfrak{N} \xrightarrow{\mathfrak{i}} \mathfrak{C}$, where \mathfrak{n} is an epimorphism and \mathfrak{i} is a monomorphism. Assuming that the morphism \mathfrak{m} is locally split, we have to show that the morphism \mathfrak{i} is. Indeed, let $(\mathfrak{K}_x)_{x \in X}$ be a direct system, $\mathfrak{s}: \varinjlim_{x \in X} \mathfrak{K}_x \rightarrow \mathfrak{M}$ be an epimorphism, and $\mathfrak{g}_x: \mathfrak{C} \rightarrow \mathfrak{M}$ be a family of morphisms witnessing the local splitness of the morphism \mathfrak{m} . Then the same direct system $(\mathfrak{K}_x)_{x \in X}$, the epimorphism $\mathfrak{ns}: \varinjlim_{x \in X} \mathfrak{K}_x \rightarrow \mathfrak{N}$, and the morphisms $\mathfrak{ng}_x: \mathfrak{C} \rightarrow \mathfrak{N}$ witness the local splitness of the monomorphism \mathfrak{i} (because the equations $\mathfrak{g}_x \mathfrak{m} \mathfrak{s} \mathfrak{k}_x = \mathfrak{s} \mathfrak{k}_x$ imply the equations $\mathfrak{ng}_x \mathfrak{i} \mathfrak{n} \mathfrak{s} \mathfrak{k}_x = \mathfrak{n} \mathfrak{s} \mathfrak{k}_x$ for all $x \in X$). \square

The following lemma shows that our terminology is consistent with the classical definition for module categories.

Lemma 3.3. *Let A be an associative ring and $i: K \rightarrow C$ be an injective morphism of left A -modules. Then the morphism i is locally split in $A\text{-mod}$, in the sense of the above categorical definition, if and only if it is locally split in the classical sense.*

Proof. “If”: assume that for every finitely generated submodule $M \subset K$ there exists a morphism $g: C \rightarrow K$ such that $gi(m) = m$ for all $m \in M$. Let X denote the poset of all finitely generated submodules in K , ordered by inclusion, and let $K_x \subset K$ be the finitely generated submodule corresponding to an element $x \in X$. Let $s: \varinjlim_{x \in X} K_x \rightarrow K$ be the natural isomorphism. For every $x \in X$, let $g_x: C \rightarrow K$ be a morphism such that $g_x i(k) = k$ for all $k \in K_x$. Then the direct system $(K_x)_{x \in X}$, the isomorphism $s: \varinjlim_{x \in X} K_x \rightarrow K$, and the morphisms $g_x: C \rightarrow K$ witness the local splitness of the monomorphism $i: K \rightarrow C$.

“Only if”: assume that a direct system $(K_x)_{x \in X}$, an epimorphism $s: \varinjlim_{x \in X} K_x \rightarrow K$, and some morphisms $g_x: C \rightarrow K$ witness the local splitness of the monomorphism $i: K \rightarrow C$. Then, for any finitely generated submodule $M \subset K$, there exists an index $x \in X$ such that the submodule $M \subset K$ is contained in the image of the composition $K_x \xrightarrow{k_x} \varinjlim_{y \in X} K_y \xrightarrow{s} K$. It follows that the morphism $g_x: C \rightarrow K$ satisfies the equation $g_x i(m) = m$ for all $m \in M$. \square

The proof of Lemma 3.3 shows that, for direct systems of *finitely generated* modules $(K_x)_{x \in X}$ with an epimorphism $s: \varinjlim_{x \in X} K_x \rightarrow K$, the possibility to satisfy the definition of local splitness by finding a suitable family of morphisms $g_x: C \rightarrow K$ does *not* depend on the choice of a particular direct system. If the local splitness of an injective A -module morphism $i: K \rightarrow C$ is witnessed by a direct system

$(K_x)_{x \in X}$, an epimorphism $s: \varinjlim_{x \in X} K_x \longrightarrow K$, and some morphisms $g_x: C \longrightarrow K$, $x \in X$, then for any direct system of finitely generated A -modules $(L_y)_{y \in Y}$ and an epimorphism $t: \varinjlim_{y \in Y} L_y \longrightarrow K$ one can find morphisms $h_y: C \longrightarrow K$, $y \in Y$ which also witness the local splitness of i .

Lemma 3.4. *Any quasi-split monomorphism in an Ab3 category is locally split.*

Proof. Assume that we are given a direct system of split short exact sequences (6), whose direct limit is a right exact sequence (7), and \mathfrak{K} is the image of the morphism $\mathfrak{m}: \mathfrak{M} \longrightarrow \mathfrak{C}$. Let $\mathfrak{k}_x: \mathfrak{K}_x \longrightarrow \mathfrak{M}$ denote the canonical morphism $\mathfrak{K}_x \longrightarrow \varinjlim_{y \in X} \mathfrak{K}_y$, and let $\mathfrak{h}_x: \mathfrak{C} \longrightarrow \mathfrak{K}_x$ be a morphism splitting the short exact sequence $0 \longrightarrow \mathfrak{K}_x \xrightarrow{\mathfrak{m}\mathfrak{k}_x} \mathfrak{C} \longrightarrow \mathfrak{D}_x \longrightarrow 0$ (6). Then the direct system $(\mathfrak{K}_x)_{x \in X}$, the identity isomorphism $\varinjlim_{x \in X} \mathfrak{K}_x \longrightarrow \mathfrak{M}$, and the morphisms $\mathfrak{g}_x = \mathfrak{k}_x \mathfrak{h}_x: \mathfrak{C} \longrightarrow \mathfrak{M}$ witness the local splitness of the morphism $\mathfrak{m}: \mathfrak{M} \longrightarrow \mathfrak{C}$. Indeed, we have $\mathfrak{h}_x \mathfrak{m}\mathfrak{k}_x = \text{id}_{\mathfrak{K}_x}$ for every $x \in X$, hence $\mathfrak{g}_x \mathfrak{m}\mathfrak{k}_x = \mathfrak{k}_x \mathfrak{h}_x \mathfrak{m}\mathfrak{k}_x = \mathfrak{k}_x$. By Lemma 3.2, the image \mathfrak{K} of a locally split morphism $\mathfrak{m}: \mathfrak{M} \longrightarrow \mathfrak{C}$ is a locally split subobject in \mathfrak{C} , as desired. \square

Lemma 3.5. *Let \mathbf{A} and \mathbf{B} be Ab3 categories, and let $R: \mathbf{B} \longrightarrow \mathbf{A}$ be a colimit-preserving functor. Then*

- (a) *R takes quasi-split epimorphisms in \mathbf{B} to quasi-split epimorphisms in \mathbf{A} ;*
- (b) *R takes locally split morphisms in \mathbf{B} to locally split morphisms in \mathbf{A} .*

Proof. Both the assertions follow immediately from the definitions. \square

The next lemma explains the relevance of the local splitness property to covers.

Lemma 3.6. *Let \mathbf{B} be an Ab3 category and $\mathbf{C} \subset \mathbf{B}$ be a class of objects. Then any \mathbf{C} -cover with a locally split kernel is a monomorphism in \mathbf{B} . In particular, if an epimorphism with a locally split kernel is a cover, then it is an isomorphism.*

Proof. Let $\mathfrak{q}: \mathfrak{Q} \longrightarrow \mathfrak{B}$ be a \mathbf{C} -cover of an object $\mathfrak{B} \in \mathbf{B}$, and let $\mathfrak{j}: \mathfrak{L} \longrightarrow \mathfrak{Q}$ be the kernel of \mathfrak{q} . Assume that a direct system $(\mathfrak{L}_x)_{x \in X}$, an epimorphism $\mathfrak{t}: \varinjlim_{x \in X} \mathfrak{L}_x \longrightarrow \mathfrak{L}$, and some morphisms $\mathfrak{h}_x: \mathfrak{Q} \longrightarrow \mathfrak{L}$ witness the local splitness of \mathfrak{j} . Let $\mathfrak{l}_x: \mathfrak{L}_x \longrightarrow \varinjlim_{y \in X} \mathfrak{L}_y$ be the canonical morphism. Then we have $\mathfrak{h}_x \mathfrak{j}\mathfrak{l}_x = \mathfrak{t}\mathfrak{l}_x$ for all $x \in X$.

Consider the endomorphism $(\text{id}_{\mathfrak{Q}} - \mathfrak{j}\mathfrak{h}_x): \mathfrak{Q} \longrightarrow \mathfrak{Q}$. We have $\mathfrak{q}(\text{id}_{\mathfrak{Q}} - \mathfrak{j}\mathfrak{h}_x) = \mathfrak{q}$, since $\mathfrak{q}\mathfrak{j} = 0$. Since \mathfrak{q} is a cover, it follows that $\text{id}_{\mathfrak{Q}} - \mathfrak{j}\mathfrak{h}_x$ is an automorphism of \mathfrak{Q} . Now the equations

$$(14) \quad (\text{id}_{\mathfrak{Q}} - \mathfrak{j}\mathfrak{h}_x)\mathfrak{j}\mathfrak{l}_x = \mathfrak{j}\mathfrak{l}_x - \mathfrak{j}\mathfrak{h}_x\mathfrak{j}\mathfrak{l}_x = \mathfrak{j}\mathfrak{l}_x - \mathfrak{j}\mathfrak{l}_x = 0$$

imply $\mathfrak{j}\mathfrak{l}_x = 0$. Since \mathfrak{j} is a monomorphism, it follows that $\mathfrak{t}\mathfrak{l}_x = 0$. As this holds for all $x \in X$, we can conclude that $\mathfrak{t} = 0$. Since \mathfrak{t} is an epimorphism by assumption, this means that $\mathfrak{L} = 0$; so \mathfrak{q} is a monomorphism. \square

4. LOCALLY SPLIT MORPHISMS AND COVERS IN AB5 CATEGORIES

In this section, \mathbf{A} denotes a cocomplete abelian category with exact direct limits (i. e., an Ab5 category).

Lemma 4.1. *In an Ab5 category, any locally split morphism is a monomorphism.*

Proof. Let $m: M \rightarrow C$ be a morphism in \mathbf{A} . Assume that a direct system $(K_x)_{x \in X}$, an epimorphism $s: \varinjlim_{x \in X} K_x \rightarrow M$, and some morphisms $g_x: C \rightarrow M$ witness the local splitness of m . Let $k_x: K_x \rightarrow \varinjlim_{y \in X} K_y$ be the canonical morphism. Then the equation $g_x m s k_x = s k_x$ holds for every $x \in X$.

Suppose that $\rho: L \rightarrow M$ is a morphism in \mathbf{A} such that $m\rho = 0$. Denote by $L_x = L \sqcap_M K_x$ the fibered product of the pair of morphisms $\rho: L \rightarrow M$ and $s k_x: K_x \rightarrow M$. In an Ab5 category, fibered products commute with direct limits; so we have $\varinjlim_{x \in X} L_x = L \sqcap_M \varinjlim_{x \in X} K_x$. Since the morphism $s: \varinjlim_{x \in X} K_x \rightarrow M$ is an epimorphism, it follows that the natural morphism $t: \varinjlim_{x \in X} L_x \rightarrow L$ is an epimorphism, too.

Let $l_x: L_x \rightarrow \varinjlim_{y \in X} L_y$ be the canonical morphism. Then the canonical morphism $L_x \rightarrow L$ decomposes as $L_x \xrightarrow{l_x} \varinjlim_{y \in X} L_y \xrightarrow{t} L$. Denote the canonical morphism $L_x \rightarrow K_x$ by ρ_x . Then the diagram

$$\begin{array}{ccccc}
 L_x & \xrightarrow{l_x} & \varinjlim_{y \in X} L_y & \xrightarrow{t} & L \\
 \rho_x \downarrow & & \downarrow & & \downarrow \rho \\
 K_x & \xrightarrow{k_x} & \varinjlim_{y \in X} K_y & \xrightarrow{s} & M
 \end{array}$$

is commutative, so we have $s k_x \rho_x = \rho t l_x: L_x \rightarrow M$ for every $x \in X$. Now we can compute that

$$\rho t l_x = s k_x \rho_x = g_x m s k_x \rho_x = g_x m \rho t l_x = 0,$$

since $m\rho = 0$. As this holds for all $x \in X$, it follows that $\rho t = 0$. Since t is an epimorphism, we can conclude that $\rho = 0$. Thus m is a monomorphism. \square

Lemma 4.2. *In an Ab5 category, any direct summand of a locally split monomorphism is a locally split monomorphism.*

Proof. Let $i: K \rightarrow C$ be a monomorphism in \mathbf{A} . Assume that a direct system $(K_x)_{x \in X}$, an epimorphism $s: \varinjlim_{x \in X} K_x \rightarrow K$, and some morphisms $g_x: C \rightarrow K$ witness the local splitness of i . Let $k_x: K_x \rightarrow \varinjlim_{y \in X} K_y$ be the canonical morphism. Then the equation $g_x i s k_x = s k_x$ holds for every $x \in X$.

Suppose that a (mono)morphism $j: L \rightarrow Q$ in \mathbf{A} is a direct summand of the monomorphism i . Then we have a commutative diagram

$$(15) \quad \begin{array}{ccc} L & \xrightarrow{j} & Q \\ \rho \downarrow & & \downarrow \gamma \\ K & \xrightarrow{i} & C \\ \lambda \downarrow & & \downarrow \beta \\ L & \xrightarrow{j} & Q \end{array}$$

where both the vertical compositions are identity morphisms.

Denote by $L_x = L \sqcap_K K_x$ the fibered product of the pair of morphisms $\rho: L \rightarrow K$ and $sk_x: K_x \rightarrow K$. As in the previous proof, we have $\varinjlim_{x \in X} L_x = L \sqcap_K \varinjlim_{x \in X} K_x$, since \mathbf{A} is an Ab5 category. The natural morphism $t: \varinjlim_{x \in X} L_x \rightarrow L$ is an epimorphism, since the morphism s is.

Following the notation of the previous proof, the canonical morphism $L_x \rightarrow L$ decomposes as $L_x \xrightarrow{l_x} \varinjlim_{y \in X} L_y \xrightarrow{t} L$. Denote the canonical morphism $L_x \rightarrow K_x$ by ρ_x . Similarly to the previous proof, we have $sk_x \rho_x = \rho t l_x: L_x \rightarrow K$.

For every $x \in X$, denote by $h_x: Q \rightarrow L$ the composition

$$(16) \quad Q \xrightarrow{\gamma} C \xrightarrow{g_x} K \xrightarrow{\lambda} L.$$

We claim that the direct system $(L_x)_{x \in X}$, the epimorphism $t: \varinjlim_{x \in X} L_x \rightarrow L$, and the morphisms $h_x: Q \rightarrow L$ witness the local splittness of the morphism j . Indeed,

$$(17) \quad h_x j t l_x = \lambda g_x \gamma j t l_x = \lambda g_x i \rho t l_x = \lambda g_x i s k_x \rho_x = \lambda s k_x \rho_x = \lambda \rho t l_x = t l_x,$$

since $\gamma j = i \rho$ and $\lambda \rho = \text{id}_L$. \square

Corollary 4.3. *Let \mathbf{A} be an Ab5 category and $\mathbf{C} \subset \mathbf{A}$ be a class of objects. Let $p: C \rightarrow D$ be an epimorphism in \mathbf{A} . Assume that*

- (1) *the morphism p is a \mathbf{C} -precover with a locally split kernel;*
- (2) *the object $D \in \mathbf{A}$ has a \mathbf{C} -cover.*

Then $D \in \mathbf{C}$ and the epimorphism p is split.

Proof. Let $q: Q \rightarrow D$ be a \mathbf{C} -cover of D . Since $C \in \mathbf{C}$ and p is an epimorphism, the morphism q is an epimorphism, too. Let $i: K \rightarrow C$ and $j: L \rightarrow Q$ be the kernels of the morphisms p and q , respectively. Since p is a \mathbf{C} -precover and q is a \mathbf{C} -cover, the short exact sequence $0 \rightarrow L \xrightarrow{j} Q \xrightarrow{q} D \rightarrow 0$ is a direct summand of the short exact sequence $0 \rightarrow K \xrightarrow{i} C \xrightarrow{p} D \rightarrow 0$.

In particular, the monomorphism j is a direct summand of the monomorphism i . By assumption, the morphism i is locally split. Applying Lemma 4.2, we conclude that the morphism j is locally split. By Lemma 3.6, it follows that $L = 0$ and q is an isomorphism. Consequently, $D \in \mathbf{C}$ and the epimorphism p is split. \square

Theorem 4.4. *Let \mathbf{A} be an Ab5 category and $\mathbf{C} \subset \mathbf{A}$ be a class of objects closed under coproducts and direct summands. Let $(c_{y,x}: C_x \rightarrow C_y)_{x < y \in X}$ be a diagram of objects $C_x \in \mathbf{C}$, indexed by a directed poset X , and let $D = \varinjlim_{x \in X}^{\mathbf{A}} C_x$ be its direct limit in the category \mathbf{A} . Let $p: \coprod_{x \in X} C_x \rightarrow D$ be the natural epimorphism. Assume that*

- (1) *the morphism p is a \mathbf{C} -precover in \mathbf{A} ;*
- (2) *the object D has a \mathbf{C} -cover in \mathbf{A} .*

Then $D \in \mathbf{C}$ and the epimorphism p is split.

Proof. By Proposition 3.1, the epimorphism p is quasi-split; so its kernel i is a quasi-split monomorphism. According to Lemma 3.4, it follows that the morphism i is locally split. Hence Corollary 4.3 is applicable, and we are done. \square

5. LOCALLY SPLIT MORPHISMS AND COVERS IN AB3 CATEGORIES

In this section we consider an Ab3 category \mathbf{B} , an Ab5 category \mathbf{A} , and a functor $R: \mathbf{B} \rightarrow \mathbf{A}$ preserving all colimits. Equivalently, R is a right exact functor preserving coproducts. Any such functor is additive. We will denote the functor R by $\mathfrak{C} \mapsto \overline{\mathfrak{C}}$ for brevity.

Following [9, Section 8], we say that a short exact sequence $0 \rightarrow \mathfrak{K} \rightarrow \mathfrak{C} \rightarrow \mathfrak{D} \rightarrow 0$ in the category \mathbf{B} is *functor pure* (or *f-pure* for brevity) if, for any Ab5 category \mathbf{A} and any colimit-preserving functor $R: \mathbf{B} \rightarrow \mathbf{A}$, the short sequence $0 \rightarrow R(\mathfrak{K}) \rightarrow R(\mathfrak{C}) \rightarrow R(\mathfrak{D}) \rightarrow 0$ is exact in \mathbf{A} . Equivalently, this means that the morphism $\overline{\mathfrak{K}} \rightarrow \overline{\mathfrak{C}}$ is a monomorphism.

If a short exact sequence $0 \rightarrow \mathfrak{K} \rightarrow \mathfrak{C} \rightarrow \mathfrak{D} \rightarrow 0$ is f-pure, we will say that $\mathfrak{K} \rightarrow \mathfrak{C}$ is an *f-pure monomorphism* and $\mathfrak{C} \rightarrow \mathfrak{D}$ is an *f-pure epimorphism*.

Lemma 5.1. *For any cocomplete abelian category \mathbf{B} , the class of all f-pure epimorphisms in \mathbf{B} is closed under direct limits.*

Proof. The class of all epimorphisms is closed under all colimits in any cocomplete category. In our context, we need to prove a similar property for the direct limits of f-pure epimorphisms in \mathbf{B} .

Let X be a directed poset, and let

$$(\mathfrak{p}_x)_{x \in X}: (\mathfrak{b}_{y,x}: \mathfrak{B}_x \rightarrow \mathfrak{B}_y)_{x < y \in X} \longrightarrow (\mathfrak{c}_{y,x}: \mathfrak{C}_x \rightarrow \mathfrak{C}_y)_{x < y \in X}$$

be a morphism of X -indexed diagrams in \mathbf{B} such that the morphism $\mathfrak{p}_x: \mathfrak{B}_x \rightarrow \mathfrak{C}_x$ is an f-pure epimorphism for every index $x \in X$. Put $\mathfrak{K}_x = \ker(\mathfrak{p}_x)$; so we have a short exact sequence of diagrams

$$(18) \quad 0 \longrightarrow (\mathfrak{K}_x)_{x \in X} \xrightarrow{(i_x)} (\mathfrak{B}_x)_{x \in X} \xrightarrow{(\mathfrak{p}_x)} (\mathfrak{C}_x)_{x \in X} \longrightarrow 0$$

such that the short exact sequence of objects $0 \rightarrow \mathfrak{K}_x \rightarrow \mathfrak{B}_x \rightarrow \mathfrak{C}_x \rightarrow 0$ is f-pure exact in \mathbf{B} for every $x \in X$.

In any cocomplete abelian category, all colimit functors are right exact. Passing to the direct limit of (18), we obtain a right exact sequence

$$(19) \quad \mathfrak{M} \xrightarrow{\mathfrak{m}} \mathfrak{B} \xrightarrow{\mathfrak{p}} \mathfrak{C} \longrightarrow 0,$$

where $\mathfrak{M} = \varinjlim_{x \in X} \mathfrak{K}_x$, $\mathfrak{B} = \varinjlim_{x \in X} \mathfrak{B}_x$, and $\mathfrak{C} = \varinjlim_{x \in X} \mathfrak{C}_x$. Denote by $\mathfrak{i}: \mathfrak{K} \rightarrow \mathfrak{B}$ the kernel of the epimorphism \mathfrak{p} . Then the object \mathfrak{K} is also the image of the morphism \mathfrak{m} , which factorizes into the composition $\mathfrak{m} = \mathfrak{in}$ of an epimorphism $\mathfrak{n}: \mathfrak{M} \rightarrow \mathfrak{K}$ and the monomorphism $\mathfrak{i}: \mathfrak{K} \rightarrow \mathfrak{B}$.

We need to prove that the morphism $R(\mathfrak{i}) = \bar{\mathfrak{i}}: \bar{\mathfrak{K}} \rightarrow \bar{\mathfrak{B}}$ is a monomorphism in \mathbf{A} . By assumption, the functor R preserves exactness of the short sequences (18); so we get a short exact sequence of diagrams

$$(20) \quad 0 \longrightarrow (\bar{\mathfrak{K}}_x)_{x \in X} \xrightarrow{(\bar{\mathfrak{i}}_x)} (\bar{\mathfrak{B}}_x)_{x \in X} \xrightarrow{(\bar{\mathfrak{p}}_x)} (\bar{\mathfrak{C}}_x)_{x \in X} \longrightarrow 0$$

in the category \mathbf{A} . Direct limits are exact in \mathbf{A} ; so passing to the direct limit of (20) we obtain a short exact sequence

$$(21) \quad 0 \longrightarrow K \xrightarrow{\mathfrak{i}} B \xrightarrow{\mathfrak{p}} C \longrightarrow 0$$

in the category \mathbf{A} . Furthermore, the functor R preserves direct limits; so it takes the right exact sequence (19) to the exact sequence (21).

We have shown that $K = \bar{\mathfrak{M}}$, $B = \bar{\mathfrak{B}}$, and the morphism $\bar{\mathfrak{m}} = \mathfrak{i}: K \rightarrow B$ is a monomorphism. It remains to recall that the morphism \mathfrak{m} decomposes as $\mathfrak{m} = \mathfrak{in}$, where $\mathfrak{n}: \mathfrak{M} \rightarrow \mathfrak{K}$ is an epimorphism and $\mathfrak{i}: \mathfrak{K} \rightarrow \mathfrak{B}$ is a monomorphism. The right exact functor R takes epimorphisms to epimorphisms, so $\bar{\mathfrak{n}}: \bar{\mathfrak{M}} \rightarrow \bar{\mathfrak{K}}$ is an epimorphism. As the composition $\bar{\mathfrak{m}} = \bar{\mathfrak{i}}\bar{\mathfrak{n}}$ is a monomorphism, it follows that $\bar{\mathfrak{n}}$ is an isomorphism and $\bar{\mathfrak{i}}$ is a monomorphism. Thus we have $\bar{\mathfrak{K}} = K$ and the functor R takes the short exact sequence

$$(22) \quad 0 \longrightarrow \mathfrak{K} \xrightarrow{\mathfrak{i}} \mathfrak{B} \xrightarrow{\mathfrak{p}} \mathfrak{C} \longrightarrow 0$$

in the category \mathbf{B} to the short exact sequence (21) in the category \mathbf{A} . So $\mathfrak{p}: \mathfrak{B} \rightarrow \mathfrak{C}$ is an f -pure epimorphism. \square

Lemma 5.2. *In any cocomplete abelian category, the cokernel of any locally split morphism is an f -pure epimorphism. Any locally split monomorphism is f -pure.*

Proof. Let $\mathfrak{m}: \mathfrak{M} \rightarrow \mathfrak{C}$ be a locally split morphism in \mathbf{B} , and let $\mathfrak{p}: \mathfrak{C} \rightarrow \mathfrak{D}$ be the cokernel of \mathfrak{m} . Denote by \mathfrak{K} the image of the morphism \mathfrak{m} . Then the morphism \mathfrak{m} decomposes as $\mathfrak{M} \xrightarrow{\mathfrak{n}} \mathfrak{K} \xrightarrow{\mathfrak{i}} \mathfrak{C}$, where \mathfrak{n} is an epimorphism and \mathfrak{i} is a monomorphism. By Lemma 3.2, the morphism \mathfrak{i} is locally split.

Applying the functor R , we get the morphism $\bar{\mathfrak{m}} = \bar{\mathfrak{i}}\bar{\mathfrak{n}}$, where $\bar{\mathfrak{n}}$ is an epimorphism. By Lemma 3.5(b), both the morphisms $\bar{\mathfrak{m}}$ and $\bar{\mathfrak{i}}$ are locally split in \mathbf{A} . By Lemma 4.1, both the morphisms $\bar{\mathfrak{m}}$ and $\bar{\mathfrak{i}}$ are monomorphisms.

We have shown that the short exact sequence $0 \rightarrow \mathfrak{K} \xrightarrow{\mathfrak{i}} \mathfrak{C} \xrightarrow{\mathfrak{p}} \mathfrak{D} \rightarrow 0$ is f -pure in \mathbf{B} , so \mathfrak{p} is an f -pure epimorphism and \mathfrak{i} is an f -pure monomorphism. We have also shown that the morphism $\bar{\mathfrak{n}}: \bar{\mathfrak{M}} \rightarrow \bar{\mathfrak{K}}$ is an isomorphism in \mathbf{A} . \square

Corollary 5.3. *In any cocomplete abelian category, any quasi-split exact sequence (quasi-split epimorphism, or quasi-split monomorphism) is functor pure.*

Proof. By Lemma 5.1, any direct limit of f-pure epimorphisms is an f-pure epimorphism. In particular, any direct limit of split epimorphisms is an f-pure epimorphism. It follows that any quasi-split epimorphism is f-pure.

Alternatively, by Lemma 5.2, any locally split monomorphism is f-pure. By Lemma 3.4, any quasi-split monomorphism is locally split. Thus any quasi-split monomorphism is f-pure. \square

The next proposition extends the result of Corollary 4.3 to abelian categories with nonexact direct limits.

Proposition 5.4. *Let \mathcal{B} be an $Ab3$ category and $\mathcal{C} \subset \mathcal{B}$ be a class of objects. Let $\mathfrak{q}: \mathfrak{Q} \rightarrow \mathfrak{D}$ be a \mathcal{C} -cover in \mathcal{B} . Put $\mathfrak{L} = \ker(\mathfrak{q})$. Assume that the object $\mathfrak{D} \in \mathcal{B}$ has a \mathcal{C} -precover with a locally split kernel. Then, for any $Ab5$ category \mathcal{A} and any colimit-preserving functor $R: \mathcal{B} \rightarrow \mathcal{A}$, one has $R(\mathfrak{L}) = 0$.*

Proof. Since any (pre)cover is a (pre)cover of its image and the images of all \mathcal{C} -precovers of a given object \mathfrak{D} coincide, without loss of generality we can replace \mathfrak{D} by $\text{im}(\mathfrak{q})$ and assume that \mathfrak{q} is an epimorphism. The argument below is a kind of conjunction of the proofs of Lemmas 3.6 and 4.2.

Let $\mathfrak{p}: \mathfrak{C} \rightarrow \mathfrak{D}$ be a \mathcal{C} -precover of \mathfrak{D} with a locally split kernel $\mathfrak{i}: \mathfrak{K} \rightarrow \mathfrak{C}$. Then the short exact sequence $0 \rightarrow \mathfrak{L} \xrightarrow{j} \mathfrak{Q} \xrightarrow{\mathfrak{q}} \mathfrak{D} \rightarrow 0$ is a direct summand of the short exact sequence $0 \rightarrow \mathfrak{K} \xrightarrow{i} \mathfrak{C} \xrightarrow{\mathfrak{p}} \mathfrak{D} \rightarrow 0$. So we have a diagram of morphisms of short exact sequences

$$(23) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{L} & \xrightarrow{j} & \mathfrak{Q} & \xrightarrow{\mathfrak{q}} & \mathfrak{D} & \longrightarrow & 0 \\ & & \rho \downarrow & & \gamma \downarrow & & \parallel & & \\ 0 & \longrightarrow & \mathfrak{K} & \xrightarrow{i} & \mathfrak{C} & \xrightarrow{\mathfrak{p}} & \mathfrak{D} & \longrightarrow & 0 \\ & & \lambda \downarrow & & \beta \downarrow & & \parallel & & \\ 0 & \longrightarrow & \mathfrak{L} & \xrightarrow{j} & \mathfrak{Q} & \xrightarrow{\mathfrak{q}} & \mathfrak{D} & \longrightarrow & 0 \end{array}$$

where all the vertical compositions are identity maps.

Assume that a direct system $(\mathfrak{K}_x)_{x \in X}$, an epimorphism $\mathfrak{s}: \varinjlim_{x \in X}^{\mathcal{B}} \mathfrak{K}_x \rightarrow \mathfrak{K}$, and some morphisms $\mathfrak{g}_x: \mathfrak{C} \rightarrow \mathfrak{K}$ witness the local splitness of the morphism \mathfrak{i} in the category \mathcal{B} . As in the proof of Lemma 4.2, denote by $\mathfrak{h}_x: \mathfrak{Q} \rightarrow \mathfrak{L}$ the composition

$$\mathfrak{Q} \xrightarrow{\gamma} \mathfrak{C} \xrightarrow{\mathfrak{g}_x} \mathfrak{K} \xrightarrow{\lambda} \mathfrak{L}.$$

As in the proof of Lemma 3.6, consider the endomorphism $(\text{id}_{\mathfrak{Q}} - j\mathfrak{h}_x): \mathfrak{Q} \rightarrow \mathfrak{Q}$. We have $\mathfrak{q}(\text{id}_{\mathfrak{Q}} - j\mathfrak{h}_x) = \mathfrak{q}$, since $\mathfrak{q}j = 0$. Since \mathfrak{q} is a cover, it follows that $\text{id}_{\mathfrak{Q}} - j\mathfrak{h}_x$ is an automorphism of \mathfrak{Q} .

Now we apply the functor R to this whole picture. We will use the diagram (15) as a notation for the image of the leftmost and middle columns of the diagram (23) under the functor R . By Lemma 5.2, the monomorphism \mathfrak{i} is f-pure; so the morphism $\bar{\mathfrak{i}} = i: K \rightarrow C$ is a monomorphism in \mathbf{A} . It follows that the monomorphism \mathfrak{j} is f-pure as well, being a direct summand of \mathfrak{i} ; in other words, the morphism $\bar{\mathfrak{j}} = j: L \rightarrow Q$ is a monomorphism, since it is a direct summand of i .

Applying the functor R to the direct system $(\mathfrak{K}_x)_{x \in X}$, we obtain a direct system $(K_x = \bar{\mathfrak{K}}_x)_{x \in X}$ in the category \mathbf{A} . The functor R preserves direct limits and takes epimorphisms to epimorphisms, so we get an epimorphism $\bar{\mathfrak{s}} = s: \varinjlim_{x \in X}^{\mathbf{A}} K_x \rightarrow K$. Let $\mathfrak{k}_x: \mathfrak{K}_x \rightarrow \varinjlim_{y \in X} \mathfrak{K}_y$ be the canonical morphism; then $k_x = \bar{\mathfrak{k}}_x$ is the canonical morphism $K_x \rightarrow \varinjlim_{y \in X} K_y$. Put $\bar{\mathfrak{g}}_x = g_x: C \rightarrow K$; then the functor R takes the morphism $\mathfrak{h}_x = \lambda \mathfrak{g}_x \gamma$ to the morphism $h_x = \lambda g_x \gamma: Q \rightarrow L$, as in (16).

As in the proofs of Lemmas 4.1 and 4.2, denote by $L_x = L \sqcap_K K_x$ the fibered product of the pair of morphisms $\rho: L \rightarrow K$ and $sk_x: K_x \rightarrow K$. Since \mathbf{A} is an Ab5 category, we have $\varinjlim_{x \in X} L_x = L \sqcap_K \varinjlim_{x \in X} K_x$, and the natural morphism $t: \varinjlim_{x \in X} L_x \rightarrow L$ is an epimorphism. Let $l_x: L_x \rightarrow \varinjlim_{y \in X} L_y$ be the canonical morphism. Now the computation in (17) shows that $h_x j t l_x = t l_x$.

For every index $x \in X$, the morphism $\text{id}_Q - j h_x = R(\text{id}_\Omega - j \mathfrak{h}_x)$ is an automorphism of the object $Q \in \mathbf{A}$, because the morphism $\text{id}_\Omega - j \mathfrak{h}_x$ is an automorphism of the object $\Omega \in \mathbf{B}$ and the functor R (as any functor) takes isomorphisms to isomorphisms. Hence, similarly to (14), the equations

$$(\text{id}_Q - j h_x) j t l_x = j t l_x - j t l_x = 0$$

imply $j t l_x = 0$. Since j is a monomorphism, it follows that $t l_x = 0$ for all $x \in X$, and consequently $t = 0$ and $L = 0$, as desired. \square

Theorem 5.5. *Let \mathbf{B} be an Ab3 category and $\mathbf{C} \subset \mathbf{B}$ be a class of objects closed under coproducts and direct summands. Let $(\mathfrak{c}_{y,x}: \mathfrak{C}_x \rightarrow \mathfrak{C}_y)_{x < y \in X}$ be a diagram of objects $\mathfrak{C}_x \in \mathbf{C}$, indexed by a directed poset X , and let $\mathfrak{D} = \varinjlim_{x \in X}^{\mathbf{B}} \mathfrak{C}_x$ be its direct limit in the category \mathbf{B} . Let $\mathfrak{p}: \coprod_{x \in X}^{\mathbf{B}} \mathfrak{C}_x \rightarrow \mathfrak{D}$ be the natural epimorphism. Assume that*

- (1) *the morphism \mathfrak{p} is a \mathbf{C} -precover in \mathbf{B} ;*
- (2) *the object \mathfrak{D} has a \mathbf{C} -cover $\mathfrak{q}: \Omega \rightarrow \mathfrak{D}$ in \mathbf{B} .*

Put $\mathfrak{L} = \ker(\mathfrak{q})$. Then, for every Ab5 category \mathbf{A} and colimit-preserving functor $R: \mathbf{B} \rightarrow \mathbf{A}$, one has $R(\mathfrak{L}) = 0$.

Proof. The proof is similar to that of Theorem 4.4. By Proposition 3.1, the epimorphism \mathfrak{p} is quasi-split; so its kernel \mathfrak{i} is a quasi-split monomorphism. By Lemma 3.4, it follows that the morphism \mathfrak{i} is locally split. Hence Proposition 5.4 is applicable. \square

6. COVERS OF DIRECT LIMITS IN CONTRAMODULE CATEGORIES

In this section we specialize the results of Section 5 to the case of the category $\mathbf{B} = \mathfrak{R}\text{-contra}$ of left contramodules over a topological ring \mathfrak{R} .

A short exact sequence of left \mathfrak{R} -contramodules $0 \rightarrow \mathfrak{K} \rightarrow \mathfrak{C} \rightarrow \mathfrak{D} \rightarrow 0$ is said to be *contratensor pure* (or *c-pure* for brevity) if, for every discrete right \mathfrak{R} -module \mathcal{N} , the short exact sequence of abelian groups $0 \rightarrow \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{K} \rightarrow \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{C} \rightarrow \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{D} \rightarrow 0$ is exact, or equivalently, the map $\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{K} \rightarrow \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{C}$ is injective. If a short exact sequence $0 \rightarrow \mathfrak{K} \rightarrow \mathfrak{C} \rightarrow \mathfrak{D} \rightarrow 0$ in $\mathfrak{R}\text{-contra}$ is c-pure, we will say that $\mathfrak{K} \rightarrow \mathfrak{C}$ is a *c-pure monomorphism* and $\mathfrak{C} \rightarrow \mathfrak{D}$ is a *c-pure epimorphism*.

The functor of contratensor product $\mathcal{N} \odot_{\mathfrak{R}} - : \mathfrak{R}\text{-contra} \rightarrow \mathbf{Ab}$ takes values in the category of abelian groups $\mathbf{A} = \mathbf{Ab}$ (which has exact direct limits) and preserves all colimits (being a left adjoint functor). So any f-pure exact sequence (monomorphism, or epimorphism) in $\mathbf{B} = \mathfrak{R}\text{-contra}$ is c-pure.

The next corollary is a generalization of [18, Lemma 2.2].

Corollary 6.1. *The class of all 1-strictly flat left \mathfrak{R} -contramodules is closed under direct limits in $\mathfrak{R}\text{-contra}$.*

Proof. Let $(f_{y,x} : \mathfrak{F}_x \rightarrow \mathfrak{F}_y)_{x < y \in X}$ be a diagram of 1-strictly flat left \mathfrak{R} -contramodules \mathfrak{F}_x , indexed by a directed poset X . By Proposition 3.1 and Corollary 5.3, the short exact sequence of \mathfrak{R} -contramodules

$$0 \longrightarrow \mathfrak{K} \xrightarrow{i} \prod_{x \in X}^{\mathfrak{R}\text{-contra}} \mathfrak{F}_x \xrightarrow{p} \varinjlim_{x \in X}^{\mathfrak{R}\text{-contra}} \mathfrak{F}_x \longrightarrow 0$$

is functor pure, hence (in particular) contratensor pure. Since the \mathfrak{R} -contramodule $\prod_{x \in X}^{\mathfrak{R}\text{-contra}} \mathfrak{F}_x$ is 1-strictly flat by [18, Lemma 2.1], it follows that the \mathfrak{R} -contramodule $\varinjlim_{x \in X}^{\mathfrak{R}\text{-contra}} \mathfrak{F}_x$ is 1-strictly flat (see the discussion in [18, Section 2]). \square

Let $\mathbf{C} \subset \mathfrak{R}\text{-contra}$ be a class of left \mathfrak{R} -contramodules closed under coproducts and direct summands.

Corollary 6.2. *Let $(c_{y,x} : \mathfrak{C}_x \rightarrow \mathfrak{C}_y)_{x < y \in X}$ be a diagram of left \mathfrak{R} -contramodules $\mathfrak{C}_x \in \mathbf{C} \subset \mathfrak{R}\text{-contra}$, indexed by a directed poset X , and let $\mathfrak{D} = \varinjlim_{x \in X}^{\mathfrak{R}\text{-contra}} \mathfrak{C}_x$ be its direct limit in the category $\mathfrak{R}\text{-contra}$. Let $p : \prod_{x \in X}^{\mathfrak{R}\text{-contra}} \mathfrak{C}_x \rightarrow \mathfrak{D}$ be the natural epimorphism. Assume that*

- (1) *the morphism p is a \mathbf{C} -precover in $\mathfrak{R}\text{-contra}$;*
- (2) *the \mathfrak{R} -contramodule \mathfrak{D} has a \mathbf{C} -cover $q : \mathfrak{Q} \rightarrow \mathfrak{D}$ in $\mathfrak{R}\text{-contra}$.*

Put $\mathfrak{L} = \ker(q)$. Then, for every open right ideal $\mathfrak{I} \subset \mathfrak{R}$, one has $\mathfrak{I} \triangleleft \mathfrak{L} = \mathfrak{L}$.

Proof. In the context of Theorem 5.5, set $\mathbf{B} = \mathfrak{R}\text{-contra}$, $\mathbf{A} = \mathbf{Ab}$, and $R = \mathcal{N} \odot_{\mathfrak{R}} -$, where \mathcal{N} is a discrete right \mathfrak{R} -module. Then we can conclude that $\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{L} = 0$. In particular, for $\mathcal{N} = \mathfrak{R}/\mathfrak{I}$ and any left \mathfrak{R} -contramodule \mathfrak{C} , one has $\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{C} = \mathfrak{C}/(\mathfrak{I} \triangleleft \mathfrak{C})$; so we get the desired equation $\mathfrak{L} = \mathfrak{I} \triangleleft \mathfrak{L}$. \square

Corollary 6.3. *Suppose that the topological ring \mathfrak{R} has a countable base of neighborhoods of zero. Let $(\mathfrak{c}_{y,x}: \mathfrak{C}_x \rightarrow \mathfrak{C}_y)_{x < y \in X}$ be a diagram of left \mathfrak{R} -contramodules $\mathfrak{C}_x \in \mathcal{C} \subset \mathfrak{R}\text{-contra}$, indexed by a directed poset X , and let $\mathfrak{D} = \varinjlim_{x \in X}^{\mathfrak{R}\text{-contra}} \mathfrak{C}_x$ be its direct limit in the category $\mathfrak{R}\text{-contra}$. Let $\mathfrak{p}: \coprod_{x \in X}^{\mathfrak{R}\text{-contra}} \mathfrak{C}_x \rightarrow \mathfrak{D}$ be the natural epimorphism. Assume that*

- (1) *the morphism \mathfrak{p} is a \mathcal{C} -precover in $\mathfrak{R}\text{-contra}$;*
- (2) *the \mathfrak{R} -contramodule \mathfrak{D} has a \mathcal{C} -cover in $\mathfrak{R}\text{-contra}$.*

Then one has $\mathfrak{D} \in \mathcal{C}$ and the epimorphism \mathfrak{p} is split.

Proof. By the contramodule Nakayama lemma [19, Lemma 6.14], the equations $\mathfrak{J} \triangleleft \mathfrak{L} = \mathfrak{L}$ for a fixed left \mathfrak{R} -contramodule \mathfrak{L} and all the open right ideals $\mathfrak{J} \subset \mathfrak{R}$ imply $\mathfrak{L} = 0$. So the assertion follows from Corollary 6.2.

Note that this version of contramodule Nakayama lemma does *not* hold without the assumption of a countable base of neighborhoods of zero in \mathfrak{R} , generally speaking. For a counterexample, see [17, Remark 6.3]. \square

A left \mathfrak{R} -contramodule \mathfrak{C} is said to be *separated* if the intersection of its subgroups $\mathfrak{J} \triangleleft \mathfrak{C}$, taken over all the open right ideals $\mathfrak{J} \subset \mathfrak{R}$, vanishes. (See [20, Section 7.3] or [17, Section 5] for the discussion.)

Corollary 6.4. *Suppose that the class $\mathcal{C} \subset \mathfrak{R}\text{-contra}$ consists of separated left \mathfrak{R} -contramodules. Let $(\mathfrak{c}_{y,x}: \mathfrak{C}_x \rightarrow \mathfrak{C}_y)_{x < y \in X}$ be a diagram of left \mathfrak{R} -contramodules $\mathfrak{C}_x \in \mathcal{C} \subset \mathfrak{R}\text{-contra}$, indexed by a directed poset X , and let $\mathfrak{D} = \varinjlim_{x \in X}^{\mathfrak{R}\text{-contra}} \mathfrak{C}_x$ be its direct limit in the category $\mathfrak{R}\text{-contra}$. Let $\mathfrak{p}: \coprod_{x \in X}^{\mathfrak{R}\text{-contra}} \mathfrak{C}_x \rightarrow \mathfrak{D}$ be the natural epimorphism. Assume that*

- (1) *the morphism \mathfrak{p} is a \mathcal{C} -precover in $\mathfrak{R}\text{-contra}$;*
- (2) *the \mathfrak{R} -contramodule \mathfrak{D} has a \mathcal{C} -cover in $\mathfrak{R}\text{-contra}$.*

Then one has $\mathfrak{D} \in \mathcal{C}$ and the epimorphism \mathfrak{p} is split.

Proof. In the context of Corollary 6.2, we have $\mathfrak{L} \subset \mathfrak{D}$ and $\mathfrak{D} \in \mathcal{C}$. Hence

$$\bigcap_{\mathfrak{J} \subset \mathfrak{R}} (\mathfrak{J} \triangleleft \mathfrak{L}) \subset \bigcap_{\mathfrak{J} \subset \mathfrak{R}} (\mathfrak{J} \triangleleft \mathfrak{D}) = 0,$$

where the intersection is taken over all the open right ideals $\mathfrak{J} \subset \mathfrak{R}$. Thus the equations $\mathfrak{J} \triangleleft \mathfrak{L} = \mathfrak{L}$ for all open right ideals \mathfrak{J} imply $\mathfrak{L} = 0$. \square

The next corollary is the main result of this section, and the promised generalization of Corollary 2.10.

Corollary 6.5. *Let $(\mathfrak{f}_{y,x}: \mathfrak{P}_x \rightarrow \mathfrak{P}_y)_{x < y \in X}$ be a diagram of projective left \mathfrak{R} -contramodules $\mathfrak{P}_x \in \mathfrak{R}\text{-contra}_{\text{proj}}$, indexed by a directed poset X , and let $\mathfrak{F} = \varinjlim_{x \in X}^{\mathfrak{R}\text{-contra}} \mathfrak{P}_x$ be its direct limit in the category $\mathfrak{R}\text{-contra}$. Assume that the \mathfrak{R} -contramodule \mathfrak{F} has a projective cover in $\mathfrak{R}\text{-contra}$. Then \mathfrak{F} is a projective left \mathfrak{R} -contramodule.*

Proof. Let $\mathcal{C} = \mathfrak{R}\text{-contra}_{\text{proj}}$ be the class of all projective left \mathfrak{R} -contramodules. All projective \mathfrak{R} -contramodules are separated, so Corollary 6.4 is applicable.

Furthermore, the left \mathfrak{A} -contramodule $\coprod_{x \in X} \mathfrak{P}_x$ is projective and the morphism $\mathfrak{p}: \coprod_{x \in X} \mathfrak{P}_x \rightarrow \mathfrak{F}$ is an epimorphism; so \mathfrak{p} is a projective precover. Hence the condition (1) is satisfied. The condition (2) is satisfied by assumption. We can conclude that $\mathfrak{F} \in \mathbf{C} = \mathfrak{A}\text{-contra}_{\text{proj}}$. \square

7. APPLICATIONS TO THE ENOCHS CONJECTURE

For the benefit of a reader not necessarily familiar with the context, let us recall the statement of the conjecture [13, Section 5.4] (cf. [3, Section 5]).

Conjecture 7.1 (a question of Enochs). *Let A be an associative ring and $\mathbf{L} \subset A\text{-mod}$ be a class of left A -modules. Assume that every left A -module has an \mathbf{L} -cover. Then the class of modules \mathbf{L} is closed under direct limits in $A\text{-mod}$.*

Let \mathbf{A} be a cocomplete abelian category and $M \in \mathbf{A}$ be an object. Then we denote by $\text{Add}(M) \subset A\text{-mod}$ the class of all direct summands of coproducts of copies of M in \mathbf{A} . In this section we mostly discuss certain results in the direction of the Enochs conjecture for the class of objects $\mathbf{L} = \text{Add}(M)$.

7.1. Cotorsion pairs with the right class closed under direct limits. Given a class of objects \mathbf{L} in a cocomplete abelian category \mathbf{A} , we will denote by $\varinjlim^{\mathbf{A}} \mathbf{L} \subset \mathbf{A}$ the class of all direct limits of objects from \mathbf{L} . This means the direct limits in \mathbf{A} of diagrams of objects of \mathbf{L} indexed by directed posets.

For a pair of objects M, N in a cocomplete abelian category \mathbf{A} , let us denote by $\text{PExt}_{\mathbf{A}}^1(M, N) \subset \text{Ext}_{\mathbf{A}}^1(M, N)$ the abelian group of all equivalence classes of f-pure short exact sequences $0 \rightarrow N \rightarrow ? \rightarrow M \rightarrow 0$ in \mathbf{A} . In other words, $\text{PExt}_{\mathbf{A}}^1(-, -)$ is the Ext^1 group in the functor pure exact structure on the category \mathbf{A} (see [9, Section 8]).

Corollary 7.2. *Let \mathbf{A} be an Ab5 category and $M \in \mathbf{A}$ be an object. Suppose that $\text{PExt}_{\mathbf{A}}^1(M, E) = 0$ for all objects $E \in \varinjlim^{\mathbf{A}} \text{Add}(M)$. Let $D \in \varinjlim^{\mathbf{A}} \text{Add}(M)$ be an object having an $\text{Add}(M)$ -cover in \mathbf{A} . Then $D \in \text{Add}(M)$.*

Proof. This is a corollary of Theorem 4.4. Let $(c_{y,x}: C_x \rightarrow C_y)_{x < y \in X}$ be a diagram of objects $C_x \in \text{Add}(M)$ such that $D = \varinjlim_{x \in X}^{\mathbf{A}} C_x$. Then the short exact sequence

$$(24) \quad 0 \longrightarrow K \xrightarrow{i} \prod_{x \in X} C_x \xrightarrow{p} D \longrightarrow 0$$

(cf. (10)) is f-pure exact in \mathbf{A} by Proposition 3.1 and Corollary 5.3 (see also [9, Example 8.4]). Furthermore, following the proof of Proposition 3.1 and keeping in mind that the direct limits in \mathbf{A} are exact by assumption, the object K in the short exact sequence (24) is the direct limit of the objects K_z , which belong to $\text{Add}(M)$. By assumption, it follows that $\text{PExt}_{\mathbf{A}}^1(M, K) = 0$. Since the sequence (24) is f-pure exact, we can conclude that any morphism $M \rightarrow D$ can be lifted to a morphism $M \rightarrow \prod_{x \in X} C_x$. In other words, this means that the morphism $p: \prod_{x \in X} C_x \rightarrow D$ is an $\text{Add}(M)$ -precover. According to Theorem 4.4, we have $D \in \text{Add}(M)$. \square

Let \mathbf{A} be an abelian category. A pair of full subcategories (\mathbf{L}, \mathbf{E}) in \mathbf{A} is said to be a *cotorsion pair* if

- \mathbf{E} is the class of all objects $A \in \mathbf{A}$ such that $\text{Ext}_{\mathbf{A}}^1(L, A) = 0$ for all $L \in \mathbf{L}$; and
- \mathbf{L} is the class of all objects $A \in \mathbf{A}$ such that $\text{Ext}_{\mathbf{A}}^1(A, E) = 0$ for all $E \in \mathbf{E}$.

The intersection $\mathbf{L} \cap \mathbf{E} \subset \mathbf{A}$ is called the *kernel* of the cotorsion pair (\mathbf{L}, \mathbf{E}) .

Application 7.3. *Let \mathbf{A} be an Ab5-category and (\mathbf{L}, \mathbf{E}) be a cotorsion pair in \mathbf{A} . Assume that the class of objects $\mathbf{E} \subset \mathbf{A}$ is closed under direct limits, and let $M \in \mathbf{L} \cap \mathbf{E}$ be an object of the kernel. Let $D \in \varinjlim^{\mathbf{A}} \text{Add}(M)$ be an object having an $\text{Add}(M)$ -cover in \mathbf{A} . Then $D \in \text{Add}(M)$.*

Proof. In any cotorsion pair (\mathbf{L}, \mathbf{E}) in a cocomplete abelian category \mathbf{A} , the left class \mathbf{L} is closed under coproducts, and both the classes \mathbf{L} and \mathbf{E} are closed under direct summands. In the situation at hand, the class $\mathbf{E} \subset \mathbf{A}$ is closed under coproducts by assumption. Hence for an object $M \in \mathbf{L} \cap \mathbf{E}$ we have $\text{Add}(M) \subset \mathbf{L} \cap \mathbf{E}$. Since, moreover, we have assumed that the class $\mathbf{E} \subset \mathbf{A}$ is closed under direct limits, we have $\varinjlim^{\mathbf{A}} \text{Add}(M) \subset \mathbf{E}$. It follows that $\text{Ext}_{\mathbf{A}}^1(M, E) = 0$ for all $E \in \varinjlim^{\mathbf{A}} \text{Add}(M)$, as (\mathbf{L}, \mathbf{E}) is a cotorsion pair and $M \in \mathbf{L}$.

Applying Corollary 7.2, we conclude that $D \in \text{Add}(M)$. □

The Enochs conjecture for the left class \mathbf{L} of an *n-tilting cotorsion pair* in an Ab5-category \mathbf{A} can be deduced from Application 7.3. Let $T \in \mathbf{A}$ be an *n-tilting object* in the sense of the paper [20] (see [9, Section 11] for a brief summary), and let (\mathbf{L}, \mathbf{E}) be the corresponding tilting cotorsion pair in \mathbf{A} .

By [9, Proposition 12.3], the tilting class \mathbf{E} is closed under direct limits in \mathbf{A} . Furthermore, by [20, Theorem 3.4], the tilting cotorsion pair (\mathbf{L}, \mathbf{E}) is *complete* and *hereditary*, its kernel $\mathbf{L} \cap \mathbf{E}$ coincides with the class $\text{Add}(T) \subset \mathbf{A}$ (by [20, Lemma 3.2(b)]), and the *coresolution dimension* of objects of \mathbf{A} with respect to the coresolving subcategory \mathbf{E} does not exceed n (by [20, Lemma 3.1]). We refer to [20, Section 3] for the definitions of the terms involved. These are the properties of the cotorsion pair (\mathbf{L}, \mathbf{E}) that we will use.

Corollary 7.4. *Let \mathbf{A} be an Ab5-category and $T \in \mathbf{A}$ be an n-tilting object (where $n \geq 0$ is an integer). Let (\mathbf{L}, \mathbf{E}) be the n-tilting cotorsion pair in \mathbf{A} induced by T . Assume that all the objects of \mathbf{A} have \mathbf{L} -covers. Then the class of objects $\mathbf{L} \subset \mathbf{A}$ is closed under direct limits.*

Proof. It suffices to assume that every object of the class $\varinjlim^{\mathbf{A}} \text{Add}(T) \subset \mathbf{A}$ has an \mathbf{L} -cover. Then, by [9, Lemma 10.2], every object of $\varinjlim^{\mathbf{A}} \text{Add}(T)$ as an $\text{Add}(T)$ -cover (since $\varinjlim^{\mathbf{A}} \text{Add}(T) \subset \varinjlim^{\mathbf{A}} \mathbf{E} \subset \mathbf{E}$ by [9, Proposition 12.3] and $\mathbf{L} \cap \mathbf{E} = \text{Add}(T)$ by [20, Lemma 3.2(b)]). By Application 7.3, it follows that the class $\text{Add}(T)$ is closed under direct limits in \mathbf{A} . According to [9, Corollary 12.4 (ii) \Rightarrow (i)], this is equivalent to the class $\mathbf{L} \subset \mathbf{A}$ being closed under direct limits. We refer to [9, Theorem 13.2 (1) \Rightarrow (2) \Rightarrow (5) \Rightarrow (6) \Rightarrow (3)] for a more detailed discussion. This argument only uses the basic properties of *n-tilting cotorsion pairs* in abelian categories. □

So we have obtained a rather simple elementary proof of some of the results in (the tilting case of) [3, Corollary 5.5], extended from the module categories to abelian categories with exact direct limits.

7.2. Weakly countably generated modules. Let A be an associative ring and $M \in A\text{-mod}$ be a left A -module.

A result which we call the “generalized tilting theory” establishes an equivalence $\text{Add}(M) \simeq \mathfrak{R}\text{-contra}_{\text{proj}}$ between the full subcategory $\text{Add}(M) \subset A\text{-mod}$ and the full subcategory of projective contramodules $\mathfrak{R}\text{-contra}_{\text{proj}} \subset \mathfrak{R}\text{-contra}$ over a certain topological ring \mathfrak{R} [20, Theorems 7.1 and 9.9], [9, Section 2]. Moreover, this equivalence of categories is obtained as a restriction of a pair of adjoint functors

$$(25) \quad \Psi: A\text{-mod} \rightleftarrows \mathfrak{R}\text{-contra} : \Phi$$

where the right adjoint functor Ψ can be computed as $\Psi(N) = \text{Hom}_A(M, N)$, while the left adjoint functor Φ is the contratensor product $\Phi(\mathfrak{C}) = M \odot_{\mathfrak{R}} \mathfrak{C}$. In particular, the left A -module $M \in \text{Add}(M)$ corresponds to the free left \mathfrak{R} -contramodule with one generator $\mathfrak{R} = \mathfrak{R}[[*]] \in \mathfrak{R}\text{-contra}_{\text{proj}}$ [20, Proposition 7.3], [21, Theorem 3.13].

Here the underlying abstract ring of the topological ring \mathfrak{R} is the ring of endomorphisms $\mathfrak{R} = \text{Hom}_A(M, M)$ of the A -module M . We use the convention in which the ring \mathfrak{R} acts on the module M on the right, making M an A - \mathfrak{R} -bimodule. The composition multiplication in $\text{Hom}_A(M, M)$ is defined accordingly. However, there is a certain flexibility in the choice of a topology on the ring \mathfrak{R} . In particular, one can use the *finite topology* [20, Theorem 7.1], [21, Example 3.7(1)] or the *weakly finite topology* [20, Theorem 9.9], [21, Example 3.10(2)].

Let us briefly recall the definitions of these topologies. A left A -module E is said to be *weakly finitely generated* (or *dually slender* [27]) if the natural map $\bigoplus_{x \in X} \text{Hom}_A(E, N_x) \rightarrow \text{Hom}_A(E, \bigoplus_{x \in X} N_x)$ is an isomorphism for every family of left A -modules $(N_x)_{x \in X}$. In the *finite topology* on the ring $\text{Hom}_A(M, M)$, the annihilators of finitely generated submodules $F \subset M$ form a base of neighborhoods of zero. In the *weakly finite topology* on the ring $\text{Hom}_A(M, M)$, the annihilators of weakly finitely generated submodules $E \subset M$ form a base of neighborhoods of zero. In any one of these two topologies, $\mathfrak{R} = \text{Hom}_A(M, M)$ is a complete, separated right linear topological ring.

Let us say that a left A -module M is *weakly countably generated* if there exists a suitable complete, separated right linear topological ring structure on the ring $\mathfrak{R} = \text{Hom}_A(M, M)$ with a countable base of neighborhoods of zero in \mathfrak{R} . Here a “suitable topological ring structure on the endomorphism ring \mathfrak{R} ” means that there is a pair of adjoint functors Ψ and Φ (25) whose restrictions to the full subcategories $\text{Add}(M) \subset A\text{-mod}$ and $\mathfrak{R}\text{-contra}_{\text{proj}} \subset \mathfrak{R}\text{-contra}$ are mutually inverse equivalences of categories $\text{Add}(M) \simeq \mathfrak{R}\text{-contra}_{\text{proj}}$ assigning the free left \mathfrak{R} -contramodule with one generator $\mathfrak{R} = \mathfrak{R}[[*]]$ to the left A -module M .

Lemma 7.5. *Let M be a left A -module such that $M = \sum_{i=1}^{\infty} E_i$, where $(E_i \subset M)_{i=1}^{\infty}$ is a countable set of submodules in M and all the A -modules E_i are weakly finitely generated. Then the A -module M is weakly countably generated.*

Proof. We consider the weakly finite topology on the ring \mathfrak{R} . In view of [20, Theorem 9.9] and [21, Theorem 3.13], it only remains to show that \mathfrak{R} has a countable base of neighborhoods of zero. We claim that the annihilators of the submodules $E_1 + \cdots + E_n \subset M$, $n \geq 1$, form such a base.

Indeed, a finite sum of weakly finitely generated submodules is clearly weakly finitely generated. So it remains to check that, for any weakly finitely generated submodule $E \subset M$ there exists $n \geq 1$ such that $E \subset \sum_{i=1}^n E_i$. For this purpose, put $N_m = M / \sum_{i=1}^m E_i$ and consider the family of left A -modules $(N_m)_{m=1}^\infty$. Consider the left A -module morphism $f: M \rightarrow \prod_{m=1}^\infty N_m$ whose components are the epimorphisms $M \rightarrow N_m$. Since $M = \sum_{i=1}^\infty E_i$, the image of the map f is contained in the submodule $\bigoplus_{m=1}^\infty N_m \subset \prod_{m=1}^\infty N_m$.

So we have an A -module morphism $g: M \rightarrow \bigoplus_{m=1}^\infty N_m$. Denote by $h = g|_E: E \rightarrow \bigoplus_{m=1}^\infty N_m$ the restriction of the morphism g to the submodule $E \subset M$. Since the A -module E is weakly finitely generated, there exists an integer $n \geq 1$ such that the image of the morphism h is contained in the submodule $\bigoplus_{m=1}^{n-1} N_m \subset \bigoplus_{m=1}^\infty N_m$. This means exactly that $E \subset \sum_{i=1}^n E_i$. \square

Let A be an associative ring and M be a left A -module. Let $M \xrightarrow{f_1} M \xrightarrow{f_2} M \xrightarrow{f_3} \cdots$ be a countable direct system of copies of M . Consider the related telescope short exact sequence of left A -modules

$$(26) \quad 0 \longrightarrow \bigoplus_{n=0}^\infty M \xrightarrow{i} \bigoplus_{n=0}^\infty M \xrightarrow{p} \varinjlim_{n \geq 0} M \longrightarrow 0.$$

Following [9, Section 6], we will say that a left A -module M satisfies the *telescope Hom exactness condition* (THEC) if, for every sequence of endomorphisms $f_1, f_2, f_3, \dots \in \text{Hom}_A(M, M)$ the telescope short exact sequence (26) stays exact after applying the functor $\text{Hom}_A(M, -)$. It is worth noticing that the short exact sequence (26) staying exact after $\text{Hom}_A(M, -)$ is applied means precisely that the epimorphism $\bigoplus_{n=0}^\infty M \xrightarrow{p} D$ is an $\text{Add}(M)$ -precover of the A -module $D = \varinjlim_{n \geq 0} M$.

Furthermore, let us say that an A -module N is Σ -pure-rigid if the pure Ext group $\text{PExt}_A^1(N, N^{(\omega)})$ vanishes. All Σ -pure-rigid modules satisfy THEC (since the short exact sequence (26) is pure and the pullback of a pure exact sequence (26) with respect to any A -module morphism $M \rightarrow D$ is pure). There are also other sufficient conditions for THEC discussed in [9].

A family of left A -modules $(M_\theta)_{\theta \in \Theta}$ is said to be *locally T-nilpotent* if for every sequence of indices $\theta_1, \theta_2, \theta_3, \dots \in \Theta$, every sequence of nonisomorphisms $f_i \in \text{Hom}_A(M_{\theta_i}, M_{\theta_{i+1}})$, $i = 1, 2, 3, \dots$, and every element $b \in M_{\theta_1}$, there exists an integer $n \geq 1$ such that $f_n f_{n-1} \cdots f_1(b) = 0$ in $M_{\theta_{n+1}}$. A left A -module M is said to have a *perfect decomposition* [2] if there exists a locally T-nilpotent family of left A -modules $(M_\theta)_{\theta \in \Theta}$ such that $M \simeq \bigoplus_{\theta \in \Theta} M_\theta$.

Application 7.6. *Let A be an associative ring and M be a weakly countably generated left A -module. Suppose that M satisfies THEC. Assume that all the countable direct limits of copies of M in the category of left A -modules $A\text{-mod}$ have $\text{Add}(M)$ -covers.*

Then the class of objects $\mathbf{Add}(M) \subset A\text{-mod}$ is closed under direct limits, and the A -module M has a perfect decomposition.

Proof. By assumption, we have a natural equivalence $\mathbf{Add}(M) \simeq \mathfrak{R}\text{-contra}_{\text{proj}}$. Under this equivalence, direct systems $M \xrightarrow{f_1} M \xrightarrow{f_2} M \xrightarrow{f_3} \dots$ of copies of the A -module M corresponds to direct systems $\mathfrak{R} \xrightarrow{*a_1} \mathfrak{R} \xrightarrow{*a_2} \mathfrak{R} \xrightarrow{*a_3} \dots$ of copies of the left \mathfrak{R} -contramodule \mathfrak{R} (where $a_n = f_n \in \text{Hom}_A(M, M) = \mathfrak{R}$). The direct limit of the latter direct system is a Bass flat left \mathfrak{R} -contramodule \mathfrak{B} .

Following the proof of [9, Corollary 6.7 (1) \Rightarrow (2)], under our THEC assumption the left A -module $D = \varinjlim_{n \geq 1} M$ has an $\mathbf{Add}(M)$ -cover if and only if the left \mathfrak{R} -contramodule \mathfrak{B} has a projective cover. If this is the case, then by Corollary 2.10 the left \mathfrak{R} -contramodule \mathfrak{B} is projective.

Alternatively, one can use Theorem 4.4 (in the category of left A -modules $\mathbf{A} = A\text{-mod}$) in order to conclude, from the assumptions that p is an $\mathbf{Add}(M)$ -precover and D has an $\mathbf{Add}(M)$ -cover, that the epimorphism of left A -modules $p: \bigoplus_{n=1}^{\infty} M \rightarrow D$ splits. Consequently, we have $D \in \mathbf{Add}(M)$. By [9, Corollary 6.7 (3) \Rightarrow (4)], the left \mathfrak{R} -contramodule \mathfrak{B} is projective.

As this holds for all Bass flat left \mathfrak{R} -contramodules, by [18, Proposition 4.3 and Lemma 6.3] it follows that that all discrete right \mathfrak{R} -modules are coperfect (i. e., all descending chains of cyclic submodules in discrete right \mathfrak{R} -modules terminate). Now [21, Theorem 12.4] is applicable, since the topological ring \mathfrak{R} has a countable base of neighborhoods of zero by assumption. Thus we can conclude that the topological ring \mathfrak{R} is topologically left perfect (that is, the Jacobson radical $\mathfrak{H} \subset \mathfrak{R}$ is topologically left T-nilpotent and strongly closed in \mathfrak{R} , and the quotient ring $\mathfrak{R}/\mathfrak{H}$ in its quotient topology is topologically semisimple).

Hence, according to [21, Theorem 14.1 (iv) \Rightarrow (iii')], the class of all projective left \mathfrak{R} -contramodules is closed under direct limits in $\mathfrak{R}\text{-contra}$. Following [21, Corollary 9.9], this means that the full subcategory $\mathbf{Add}(M) \subset A\text{-mod}$ has split direct limits. In particular, by [21, Lemma 9.2(b)] or [9, Lemma 6.5], $\mathbf{Add}(M)$ is closed under direct limits in $A\text{-mod}$. By [2, Theorem 1.4], split direct limits in $\mathbf{Add}(M)$ imply that the A -module M has perfect decomposition. \square

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