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# On "marcov" inequalities

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**Abstract**

As colleagues and friends we wish to dedicate these pages to Marco Vianello (who has used since time immorial the username `marcov@math.unipd.it`) on the occasion of his 60th birthday, which is on **October 26, 2021**. Marco has made many important contributions to approximation theory and beyond. Here we briefly summarize some of them in the spirit of the occasion.

## 1 From the first inequalities to the Padua points

**by Stefano De Marchi**

We were both students of Mathematics at the University of Padova in the eighties and we received the degree, in Italian "laurea", the same day on July, 3rd, 1987 (see Fig. 1 below). During the university time we did not hang out, mainly because Marco was in love with Giulia, who became later his wife. Giulia was also student of mathematics appassioned of logic. I remember only this annedoct. There was a football match "3rd year vs 4th year" of mathematics. I played with the third year and I also scored. Marco was with the fourth year and in one contrast fall down and broke his arm.

He started to work on his thesis with prof. Renato Zanovello who was our teacher of "Calcoli numerici e grafici" (the present numerical calculus course). Prof. Zanovello had esteem of me for religious reasons and was our "bridge" in stimulating the passion for numerical analysis. The same year, he became the first director of the doctorate school and pushed both of us to enroll for the PhD in computational mathematics. This is moreless the moment in which we started to hang out more often, discuss of mathematics, share our interests. Marco attracted me for his great and deep knowledge of mathematics, wolrdwide geography, history and also politics.

There is a common destiny that has marked our lives but at the beginning was not clear, at least to me. In fact, after graduating and after a scholarship for some months I left academia and went to work in a computer science firm. Marco instead had clear ideas and started his Ph.D. in "Matematica Computazionale e Informatica Matematica' at the University of Padova. He

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**Figure 1:** Degree in Mathematics cerimony, July 3rd, 1987. Marco is the third from the right, Stefano on his right.

received the doctorate in 1992 defending his PhD dissertation "Extensions and Numerical Applications of the Liouville-Green Approximation" under the supervision of Prof. Renato Spigler. Marco was the first PhD student of Prof. Spigler and also the one with the most academic descendants (3). In 1991 I began my PhD in Computational Mathematis in Padova and we then started our scientific collaboration as well.

Our first paper [22] announced our newfound collaboration. Maybe my influence slowly attracted Marco's interests to approximation theory and in 1996 we wrote the paper on the Peano kernel for vector-valued functions [25]. The main result of that paper was a generalization of the well-known Peano Kernel Theorem to linear operators $L : \mathcal{C}^{n+1}([a,b],X) \to X$, $X$ being a Banach space, which vanish on all abstract polynomials of degree $\leq n$. The resulting theorem, called "An Abstract Peano Kernel Theorem", provided the representation

$$L(f) = \int_a^b f^{n+1}(t) K_n(f) dt$$

where $K_n(t) = (n!)^{-1} \hat{L}_x \{(x-t)_+^n\}$ is the usual scalar Peano kernel, $\hat{L} : \mathcal{W}^{n,1}([a,b];\mathbb{R}) \to \mathbb{R}$ on the Banach space $W^{n,1}([a,b];\mathbb{R}) = \{f \,:\, D^j f \in L^1([a,b];\mathbb{R}), j = 0,...,m\}$ with $D^j$ the $j-th$ weak-derivative and $\hat{L}_x$ denoted the functional applied to the truncated power $(x-t)_+^n$ seen as a function of $x$. The main applications of the result were the estimation of errors in abstract interpolation, quadrature and the trapezoidal rule for approximating classical solutions of linear parabolic evolution equations in Hilbert spaces. We then continued to work on this topic and we extended the same idea, but in a weak form, on normed spaces [23].

In 1995, we both had a daughter. Also in this situation we were "almost synchronized": my daughter Silvia born on July 14, and Marianna, Marco's first daughter, on August 14. I then became assistant professor at the University of Udine and our collaboration started to be more difficult, mainly due to the fact that Marco enjoyed to work in presence, in front to the blackboard. Only in 2001, we published together a new paper in which was more evident the interest of Marco to numerical aspects of approximation theory. In the paper "Approximating the approximant: a numerical code for the polynomial compression of discrete integral operators", appeared in Numer. Algorithms 28 (2001), where we described a Matlab toolbox (`chebcoint`) which implemented a new compression of integral operators by sampling on Chebyshev-Lobatto points. This two level approximation of integral operators allowed us to formulate the inequality

$$\|S_{m,n} - S_m\|_\infty \leq \Lambda_m \|\Phi_n - \Phi\|_\infty \tag{1}$$

where, $\Phi_n$ denotes the approximation of the integral operator $\Phi$ by a suitable quadrature rule that can be seen as an approximation of $S_m$ (the $m-$th truncated Chebyshev series expansion), $S_{m,n},\ n < m$, the approximating polynomial of $S_m$ and $\Lambda_m$ the Lebesgue constant. On studying two fast methods, based on polynomial approximations for the evaluation of discrete integral transforms of the type (see [24])

$$(T_n u)_i = \sum_{j=1}^n w_j K(x_i, y_i, u_j), \;\; i = 1, ..., m, \;\;\; n \ll m, \tag{2}$$

on Chebyshev and Leja points, we became involved in Leja sequences and potential theory. Meanwhile I moved from the University of Udine to the University of Verona. During those years I started to collaborate with Len Bos who was professor at the University of Calgary. I introduced Marco to him, who immediately was fascinated by his great knowledge of approximation theory, pluripotential theory and mathematical inequalities. It is the time when Marco started to be passionate about multivariate polynomial approximation in particular to the problem of finding "good interpolation points" for polynomial interpolation of total degree on compacts of $\mathbb{R}^d$. To be precise, it was Len who introduced me to that problem, that he was studying since the eighties. I told the main ideas to Marco and we started to think about the problem. We were not confident with the theory, but expert in computations, which alllow to "see" interesting configurations in the square of $\mathbb{R}^2$. This is when we started to find *new nodal sets* for polynomial interpolation, both for tensor product and for total degree, on the square $Q = [-1,1]^2$ (cf. [13]). We computed numerically the Lebsgue constants for different families of points: tensor product Chebsyhev-Lobatto and tensor product Leja sequences, *Dubiner points* and the *geometric greedy points*. We came through the *Morrow-Patterson* points, which are a unisolvent set of points for polynomial interpolation of total degree. These points have closed form

$$x_m = \cos\left(\frac{m\pi}{n+2}\right), \quad y_k = \begin{cases} \cos\left(\frac{2k\pi}{n+3}\right) & m\ odd \\ \cos\left(\frac{(2k-1)\pi}{n+3}\right) & m\ even \end{cases}$$

with $1 \le m \le n$, $1 \le k \le (n+2)/2$. The set of Morrow-Patterson points of degree $n$, $MP_n$, has cardinality equal to that $\mathbb{P}_2^n(Q)$, the space of polynomials of total degree $\le n$. This fantastic set of points, easy to construct and with the growth of the Lebesgue constant (determined in a manuscript by Len Bos) of order $n^6$, so polynomial and not exponential, attracted our attention. In July of 2004, in a hot summer afternoon, me, Marco, Len Bos and Shayne Waldron (who were both visiting us for some days), we played with these points, trying different configurations with the aim of getting a Lebesgue constant with the optimal growth of $(\log^n)^2$. Miracolously, the formula

$$x_m = \cos\left(\frac{(m-1)\pi}{n}\right), \quad y_k = \begin{cases} \cos\left(\frac{(2k-2)\pi}{n+1}\right) & m\ odd \\ \cos\left(\frac{(2k-1)\pi}{n+1}\right) & m\ even \end{cases} \tag{3}$$

shows the desired Lebesgue constant growth. We went out to the bar in front of the Department in Via Belzoni (the Department of Mathematics was in a ugly ex-prison building) and we ordered a *spritz*, the well-known Italian aperitif, to celebrate the finding. The *Padua points* were found in this heuristic way. The rest of the story is recalled below by Alvise.

The new family of unisolvent points, opened new horizons in multivariate polynomial approximation. At the beginning the work was only numerical, but we needed a proof of the optimal growth of the Lebesgue constant. This happened a couple of years later. Again we were in four, but this time in Verona: me, Marco, Len Bos and Yuan Xu. Marco Caliari (who was PhD student of Marco and moved to Verona for his post-doc) observed that the first family of Padua points lie on the parametric curve

$$\gamma(t) = (-\cos((n+1)t), -\cos(nt)), \ t \in [0, \pi],$$

that turns out to be a *Lissajous curve*. Len looked at the curve and after some considerable number of hours he came with the proof and with Yuan we all refined the result (cf. [5]). During one talk in a conference, Len was so enthusiastic of this discovery, that he said to the audience "I was looking for these points for many years, my hair became white and I lost them too, but finally we found them". I can claim now the same. It was really a fantastic period for all of us. Below Alvise is reminding what happened afterward: the CAA research group, the Dolomites meetings (Workshops and Research Weeks), the journal "Dolomites Research Notes on Approximation", the scientific network RITA, the thematic group of the Italian Mathematical Union on Approximation Theory. From 2003 we erupted like a vulcano and we are still like a lava stream.

For interested readers, here are the links for these initiatives:

- CAA Padova-Verona Research Group: https://www.math.unipd.it/~marcov/CAA.html
- 5th Dolomites Workshop on Constructive Approximation and Applications: https://events.math.unipd.it/dwcaa21/
- Dolomites Research Notes on Approximation: https://drna.padovauniversitypress.it/
- "Rete Italiana di Approssimazione" (RITA): https://sites.google.com/view/ritanetworkapp/
- Unione Matematica Italiana, Thematic Group on Approximation Theory and Applications: https://umi-taa.dmi.unipg.it/

We are still looking for the Padua points in higher dimensions and hope to find them one day. Lissajous curves helped a lot. In 2017 [9], studying these curves trying to find 3-dimensional Padua points, we noticed that these curves are fundamental for generating cubature formulas on a special families of rank-1 Chebsyehv lattices. Moreover they are a good sampling set for computing discrete extremal sets (such as the Approximate Fekete and Leja points) and for *hyperinterpolation*. We showed that on $K = [-1,1]^3$ taking as Chebyshev lattice the points $\mathcal{A}_n = \ell(\theta_s)$, $s = 0, \dots, \mu$, i.e. points on an *admissible Lissajous curve*, for the corresponding hyperinterpolation operator $\mathcal{H}_n$ we get

$$\|p\|_{\infty,K} = \|\mathcal{H}_n p\|_{\infty,K} \le \|\mathcal{H}\| \, \|p\|_{\infty,\mathcal{A}_n} = \mathcal{O}((\log n)^3) \|p\|_{\infty,\mathcal{A}_n}. \tag{4}$$

This last inequality has a particular meaning: it is the most recent one we did together with Marco. We know that it is not the last: we have too many things to do and our "to do" list is full of ideas.

I want to conclude by expressing to Marco a lot of health from the bottom of my heart. Happy birthday, Marco!

## 2 Being a student and colleague of Marco Vianello

**by Alvise Sommariva**

In 1993 I finished all the exams at the university and I had to find an advisor for the degree thesis. I had liked Prof. Renato Zanovello's course on Numerical Analysis and so it was natural to ask him if he had some thesis topics that I could work on. He said it was alright but that he was busy with Ph.D. course administration and that it was necessary to ask for the help of a new young researcher as co-advisor, Marco Vianello.

I went to his office, a small room that he was sharing with other young faculty and he said it was fine with him. Everything went smoothly in those months, but on the day before the delivery to the printing house there was a sudden breakdown. As is normal in these situations, we were busy in his office, trying to finish the thesis in time and at 10 PM, under pressure, I made the huge mistake of saving the manuscript after a thoughtless and destructive `replace all`. The only backup I had was dated two days before. He remained with me in the office until 1 AM, trying to resolve this issue. We still laugh at it, but this says how much he tries to help friends and students when the going gets tough.

Later, he was my Ph.D. advisor. In those early years of his career, many well-known mathematicians were visiting our university, which was important eespecially for young researchers. Very often, Walter Gautschi, a good old friend of Renato Zanovello, was giving seminars and we were delighted by his amazing lectures on quadrature and orthogonal polynomials. In the meantime Stefano De Marchi introduced his good friend Len Bos that acquainted Marco with the wonderful and scary world of multivariate approximation.

While dealing with me on the theoretical and numerical solution of certain Hammerstein integral equations [34], [39], he also had some long term collaborations in very different subjects. He was working with Renato Spigler, who was his Ph.D. advisor, on difficult topics such as for instance, Liouville-Green type approximations [45], [46], with Renato Zanovello on the asymptotic summation of power series [19], [20], and with Stefano De Marchi on Peano kernels [25], showing his deep interest in applicable functional analysis.

His friendship with Luca Bergamaschi, the help of Ángeles Martínez Calomardo, and the arrival of a Ph.D. student, Marco Caliari, opened for him a new fruitful territory, the evaluation of the exponential operator for large, sparse, symmetric matrices and its application for solving advection-diffusion problems [3], [4], [14], [29].

All these collaborations lead him to a more mature career of which I will give more details.

In 2004, during a visit in Padua of Len Bos and Shayne Waldron, together with Marco Caliari and Stefano De Marchi, they started studying good polynomial interpolation sets on the unit square, finally proposing the so called *Padua points* [13]. I know that Shayne is not much cited in this framework, but Marco told me many times that his contribution was fundamental. Denoting by $C_{n+1}$ the set of $n+1$ Chebsyhev-Lobatto points

$$C_{n+1} := \{\cos(j\pi), j = 0 \dots, n\}$$

using an obvious notation, the set $\mathrm{Pad}_n$ of Padua points at degree $n$ is defined as

$$\mathrm{Pad}_n := (\mathrm{C}_{n+1}^{\mathrm{odd}} \times \mathrm{C}_{n+2}^{\mathrm{even}}) \cup (\mathrm{C}_{n+1}^{\mathrm{even}} \times \mathrm{C}_{n+2}^{\mathrm{odd}}) \subset \mathrm{C}_{n+1} \times \mathrm{C}_{n+2}.$$

This unisolvent set $\mathrm{Pad}_n$ shares many properties with Chebyshev-type points, since it has a simple explicit defintion and has a low Lebesgue constant, in this case $O(\log^2(n))$, where $n$ is the interpolation degree. With the help of Yuan Xu, many approaches to Padua points were given [5], [8]. It is important to point out that it was observed later that $\mathrm{Pad}_n$ lies on a particular Lissajous curve, leading to interesting pointsets in higher dimensions [9].

Not least, Lagrange interpolation at the Padua points has been used in several scientific and technological applications, from Computational Chemistry to Image Processing, Materials Science, Mathematical Statistics, Quantum Physics, being implemented even in the `Chebfun2` package.

At the same time, he became more involved with numerical quadrature.

Stefano started being interested in Radial Basis Functions, telling us how much the latest studies on this topic were becoming attractive for our community. In spite of that, we noticed that quadrature on scattered data was being overlooked and made a first paper on this subject, limiting our analysis to a rectangular domain but studying more recently in 2019 the case of polygonal regions. Though one may think that integrating the RBF approximant $\sum_k c_k \phi(\|x - x_k\|)$ on the domain $\Omega$ is easy once its moments $\int_\Omega \phi(\|x - x_k\|) dx$ are available, the evaluation of the latter is not trivial even for basic $\Omega$, especially when the RBF $\phi$ is compactly supported [38], [42] . Noticing a growing interest on this subject, recently, we started a collaboration with some long time friends, Alessandra De Rossi and Roberto Cavoretto, having in mind to use the latest discoveries on Radial Basis Functions to obtain better quadrature results [15]

On a similar subject, I still remember Marco rushing in my office, with a Russian calculus book by N. Piskunov in hand, very proud of himself, saying *I have a promising idea!* I knew already it was working, because when he is so enthusiastic it means that for many days he pondered undercover on the topic, being sure of the result. In this case, he wanted to use the Gauss-Green theorem to numerically approximate definite integrals on polygons, without resorting to a primitive of the integrand but just using a suitable Gauss-Legendre rule, and in this way producing a certain algebraic rule of fixed degree of precision. Unfortunately he did not take into account the fact that a such defined quadrature rule could have many points outside the integration domain $\Omega$, but applying this technique to a suitable rotation of $\Omega$ and mapping back the nodes, we were able to fix the problem, at least for convex polygons [35]. This area became one of his favourite topics, in which the initial idea was modified to handle also domains with curvilinear boundary $\partial\Omega$, as when $\partial\Omega$ can be tracked parametrically by splines, trying to recover, as much as possible, rules with positive weights and internal nodes for all these instances, also providing reliable software.

At the end of 2005, besides all the research being made, new challenges arose. Together with Stefano and many other friends, they decided to make conferences and workshops in Approximation theory in Alba di Canazei, a wonderful village in the Italian Dolomites. Preliminary reconnaissances were fun, since beyond organizational work we could make brief hikes and enjoy the typical good local food. The first conference was given in honor of Walter Gautschi, and later of Annie Cuyt and Robert Schaback. All that experience, little by little, led also to the birth of the electronic journal *Dolomites Research Notes on Approximation* and of RITA, acronym of *Research ITalian network on Approximation*.

After the discovery of the Padua points, he started thinking about how to find numerically good interpolation sets, in particular with low Lebesgue constant. There were some deep theoretical studies on special domains $\Omega$ such as the square, the triangle or the sphere, but not so many on nonstandard ones, e.g. on general polygons. Furthermore, *good* point sets were often not available. One day, in 2008, he knocked my office door saying that he made some strange experiments in Matlab. He took a *very dense* subset $X_n = \{x_k\}_{k=1,\dots,M}$ of the unit interval $\Omega = [-1,1]$, computed the moments $\{m_k\}_{k=0,1,\dots,n}$ of the Chebyshev basis of the polynomial space $\mathbb{P}_n(\Omega)$ of degree $n$, the interpolation Vandermonde matrix $V$ relatively to such basis and $X_n$ and solved by Matlab `backslash` the linear system $V^T w = m$. Surprisingly, the vector of the weights $w = \{w\}_k$ had only $n+1$ non zero coordinates, exactly as the dimension of $\mathbb{P}_n(\Omega)$. Next the Lebesgue constant of the points corresponding to these non zero weights was very good. It was looking like a miracle, too good to be true. In most cases, the happiness of the discovery persists for a couple of days, finding later that something is wrong or of limited interest. This time everything went differently. The previous tests were tried on the unit-square $\Omega$, again using dense discrete sets $X_n$ and everything was still fine. Numerical experiments were showing that these arguments could be generalized to more general compact domains. Of course, the core was how Matlab was implementing the `backslash` command and how to choose a suitable set $X_n$ [36]. After some research, it turned out that this set had to be a member of a *(weakly) admissible mesh* (sometimes called *norming set*) that, as introduced and analysed by his good friends Jean-Paul Calvi and Norm Levenberg, is such that

$$\|p\|_\Omega \le C_n \|p\|_{X_n}, \quad \forall p \in \mathbb{P}_n(\Omega),$$

with $C_n$ that grows at most algebraically with $n$, see [6].

The intuition was saying that the extracted set was sharing some properties with the so called *Fekete points* that have a *slow* growth of the Lebesgue constant. It took a lot of work by him and some close collaborators, to connect this problem to some results in pluripotential theory, to prove that these pioneering ideas were right, later extending them to sets with different properties, i.e. the *Discrete Leja Points* [7]. Still there was the problem of finding good starting *(weakly) admissible meshes* $X_n$, a set that is also of main importance for polynomial least squares approximation. This problem led him and the new Ph.D. student Federico Piazzon to study the properties of such $X_n$, also determining many of them for rather general compact domains $\Omega$, see e.g. [32], [33], [50], a subject that is still one of his main interests, with applications even in polynomial optimization [31], [49].

More recently, his studies were devoted to numerical approximation of continuous functions on subintervals of $[-\omega,\omega] \subseteq [0,2\pi]$, by means of trigonometric polynomials of degree $n$, say $\mathbb{T}_n([-\omega,\omega])$. The fact the $[-\omega,\omega]$ is a symmetric interval is not actually a problem, since it may be easily generalized to more general instances. A first result was achieved with Len Bos in 2012, introducing the set $\{\theta_j\}_{j=1,\dots,2n+1}$

$$\theta_j = \theta_j(n,\omega) = 2\arcsin(\sin(\omega/2)\tau_j) \in (-\omega,\omega), \quad \tau_j = \cos\left(\frac{(2j-1)\pi}{2(2n+1)}\right)$$

that is unisolvent for interpolation in $\mathbb{T}_n([-\omega,\omega])$ with Lebesgue constant $O(\log(n))$ (see [18]), as well as an interpolatory quadrature rule with $2n+1$ positive weights and nodes [12]. Next, he and Gaspare Da Fies wrote a paper on trigonometric Gaussian quadrature on subintervals of the period, whose nodes and weights are available as soon as the nodes and the weights of the Gaussian rule w.r.t.

$$w(x) = \frac{2\sin(\omega/2)}{\sqrt{1-\sin^2(\omega/2)x^2}}, \quad x \in (-1,1)$$

are at hand [17]. These results, somehow crossing with those of other scientists, were important for their applications, leading to algebraic quadrature rules in non-standard domains as planar lenses, bubbles, linear blending of elliptical arcs [16], geographical rectangles on the sphere [28], intersection of planar disks [41], and even on spherical polygons [43]. Furthermore, the interpolation sets $\{\theta_j\}$ were used to determine weakly admissible meshes with low cardinality on several domains, even in 3D instances as cones, solids of rotations [26] , geographical rectangles [28]. All this work had several applications, for instance in numerical cubature on polygonal elements with curved sides for the solution of PDEs by Virtual Elements [1], in computational adaptive optics, in regional scale modelling/simulation in spherical and toroidal geometry.

In many of these multivariate formulas, often based on partitioning the domain or suitable tensorial rules, the cardinality was actually too high, still having the properties of being of PI type, i.e. , with points inside the quadrature compact domain $\Omega$ and positive weights. Much work has been devoted to finding rules with fewer nodes and fixed algebraic degree of precision $ADE = n$, the so called *minimal rules*, but in general the integration domains are standard, e.g. the simplex, the disk, the square, and the $n$ are low. It was already known that rules could be extracted from those of PI type by Matlab `backslash` used for solving the moments linear system $V^T w = m$, but unfortunately they could have negative weights, becoming more unstable and less attractive for some applications, e.g. *hyperinterpolation*. In spite of that, Tchakaloff's theorem asserts that a formula of PI type and $ADE = n$ can be still obtained with cardinality at most equal to the dimension of $\mathbb{P}_n(\Omega)$. To this purpose, the key point was observing that the routine `lsqnonneg`, implementing the Lawson-Hanson algorithm, gives a nonnegative solution $w^*$ to the moments linear system, with nonzero entrances of $w^*$ at most equal to the cardinality of the vector $m$, i.e. the dimension of the polynomial space $\mathbb{P}_n(\Omega)$ [40]. The aforementioned built-in code was later improved in a work in collaboration with Monica Dessole and Fabio Marcuzzi [27]. What is interesting is that for many bivariate o trivariate domains $\Omega$, the computation of these rules is in general fast for mild algebraic degrees of precision $n$, see e.g. [2], [42], [43]. All these discoveries found later applications in the construction of near-optimal regression designs [10], [11], in optical system design [2], in numerical cubature on polygonal/polyhedral and curved elements for the discretization of PDEs [48], polynomial optimization [30].

An important point is that, differently from many researchers in our area, Marco has been always deeply convinced that it is necessary to provide open source numerical software to ensure the soundness of the theoretical research [51].

Last but not least, recently he has opened a new collaboration with his old friend Francesco Dell'Accio (and some collaborators), not only on fishing techniques but also on numerical differentiation and local polynomial interpolation [21], all subjects in which he has been interested for years. On this point I should say that Marco's mother once told me that when he was a kid he was not so good with the fishing pole, but I am pretty sure that all the passion he put in this hobby made him competitive even with Francesco, who has a well-recognized reputation [1].

As Stefano wrote in his dedication, Marco likes to work in his office, writing ideas on paper or on the blackboard. In spite of that, a lot of research has also been done talking on the phone, especially when his daughters Elisa, Marianna or his wife Giulia say that he is at home and possibly awake. Indeed, all his friends know how unreliable communication with him is when he is out and about with his old Nokia 3110 mobile (of which our beloved *technological caveman* has several specimens!).

Though it may seem that everything was smooth in his career, as for many of us, he has had to face challenges that were hard to overcome. Being trapped in these situations, after an initial demoralisation, he always fought them with all the possible strength and determination, still remaining upright, being for me a source of inspiration not only in mathematics but also in everyday life.

I like to conclude this dedication with a slight modification of the last words of Calvin and Hobbes, a cartoon that very often hid philosophical insights behind funny gags, *It's a magical world Marco, ol' buddy ... let's go exploring!*

## 3 Friendship, Collaboration and Serendipity

**by Len Bos**

Italian cuisine and lifestyle are of course world famous. Somewhat less known, due I suppose to the language barrier, is the depth of Italian culture. In particular, Italian cinema is wonderful, among the best, if not the very best, in the world. A particular favourite of mine is *Pane, Amore e Fantasia* a particularly Italian example of a sentimental comedy from 1953 and starring Vittorio De Sica and Gina Lollobrigida. The title means *Bread, Love and Fantasy* and was the response given to the director, Luigi Comencini, when he asked a resident of the village how it came to be that he managed to survive in the poverty of post war Italy.

In Mathematics, the answer to how our results come to be is often *Friendship, Collaboration and Serendipity*, and this has definitely been the case with my collaboration with Marco Vianello. I would just like to recount two such cases, from my personal perspective.

The first of which is the Padova Points, which have already been discussed in some detail above. Hence I will be brief.

They arose as an adjustment of the so-called Morrow-Patterson points which are one of the very few explicit examples of a Gaussian type quadrature in two variables, for the product Chebyshev measure

$$d\mu = \frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}}dxdy$$

on the unit square $[-1,1]^2$. It was Tom Bloom of the University of Toronto who informed me that such points existed and he suggested that they would also probably be good for bivariate polynomial interpolation, this being well-known in the univariate case. About the same time I was becoming aware of Yuan Xu's wonderful work on multivariate orthogonal polynomials, in particular of multivariate Christoffel-Darboux formulas. Using this technology it was not difficult to show that Tom's intuition was correct: the Lebesgue constant for the Morrow-Patterson points grows at most polynomially in the degree! Moreover, there was the strong suspicion that a much better result was lurking in the background. This turned out to indeed be the case!

By 2004 I had become a regular summer visitor to Italy; mostly of Stefano where ever he happened to be at the time, in Udine, Padova or Verona, but also of Marco, his good friend and hence by the law of associativity of friendship, also mine. By good fortune, Shayne Waldron was also visiting Marco in Padova and we were given to share a small office in the old, not lamented, Math building. I believe that this "office" was actually a re-purposed broom closet; at least it smelled heavily of cleaning fluids. Outside it was very hot and the air was heavily polluted in a way I never experienced before, or subsequently for that matter. Despite these adverse circumstances, Marco, Stefano, Shayne and I managed to still discuss Mathematics, and in particular what to do with this Morrow-Patterson point example. I remember that it was Shayne, while imprisoned in our broom closet, who first suggested adjusting the Morrow-Patterson points to include the boundary giving conception to the Padova points. Numerically this seemed to be a good idea, but we could not, of course, prove anything and so our visit ended and Shayne and I returned to our respective homes, hoping to return the next year to carry on.

In the meantime, Marco, Stefano and Marco Caliari did extensive calculations and confirmed that the new points undoubtedly had Lebesgue constant of minimal growth. This was good news, but even better, Marco Caliari had noticed that the points were always on a curve, which we now know is a Lissajous curve. I once asked him how he discovered this and his answer was "that they looked like points on a curve". This fact turned out to be a crucial observation, but personally, I have to confess that I would never have noticed it!

The next year when I again was visiting Stefano, this time in Verona, we were able to use this to find formulas for the fundamental Lagrange polynomials. But fortune would again smile on us. At the same time Yuan Xu was visiting Marco in Padova

[1] A couple of hours after I wrote this, Marco sent us a picture with a huge smile, holding a fish weighing 1536 grams (very likely caught with the help of his daughter Marianna who shares the same pastime), what a coincidence!

and they were also to give formulas for the Lagrange polynomials using multivariate Christoffel-Darboux formulas. Since Yuan also had a compact form of the formula, we were able to use this to get the growth of the Lebesgue constant.

So that's the story of the Padova points as best I recall. They were conceived in a spartan broom closet and brought to maturity through persistence and a sequence of serendipitous events made possible by collaborations founded in friendship.

My other story is of the so-called Approximate Fekete points, already alluded to above by Alvise. I remember that Marco told me, with great enthusiasm, that they had discovered a great way to find *quadrature* rules. This was very easy to generalize to interpolation functionals and indeed is quite remarkable. Here is the very brief Matlab code that implements his idea for points in the interval $[-1,1]$:

- n=21; % number of interpolation points
- x=linspace(-1,1,m); % discrete model of [-1,1] ($m$ large)
- A = gallery('chebvand',n,x); % vandermonde matrix
- b=rand(n,1); % a random rhs
- y=A\ b; % y is the Matlab solution of Ay=b
- pp=(y∼=0); % indices of the non-zero elements of y
- pts=x(pp); % select the points from x according to index pp

The figure below shows what results, in comparison to some well-known good interpolation points sets. Indeed its hard to tell them apart!

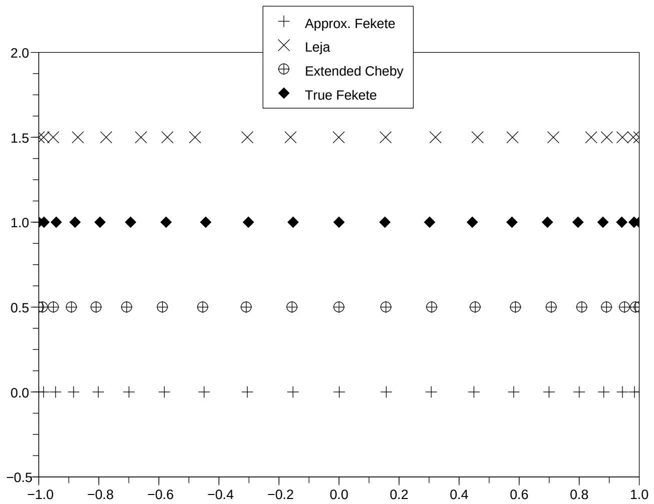


**Figure 2:** Plot of various point sets for $n = 21$

How Marco discovered this I know not, but it appears to me to be a stroke of good fortune!

The secret to its success is how Matlab just happens to implement the backslash command in the case of an *under determined* system. Serendipitously, when applied to the Vandermonde matrix, the selection of $n$ columns is equivalent to the selection of $n$ of the $m \gg n$ points, and Matlab does this by what turns out to be a Greedy Algorithm for maximizing subdeterminants, and hence Fekete points. From this it was, with again good collaboration based on friendship, possible to analyze what was happening in the background.

Finally, I would like to just make a more personal observation. Marco, like all Italians, is a great Football fan, especially of the Italian national team and Inter of Milan. Indeed the old Inter was one of the first proponents of Catenaccio for which Italian football later became famous (or perhaps infamous!). Myself, I cheer for the Netherlands team, and for years now Marco and I wager a pizza on which of our teams will do better in the World Cup and also the European Championships. I won't say what the current score is, but I will say: "Marco you owe me a pizza and I'm coming soon to Padova to collect it!"

All kidding aside, best wishes for your 60th and for many more years in health and mathematical creativity!

## 4 Final notes

- The bibliography that follows are the papers cited in the article. For the complete list of Marco's papers, please refer to his (always updated) web page at the link https://www.math.unipd.it/~marcov/papers.html.
- Marco was always keen and passionate about fishing. A few days ago, he was proud to show that he caught a chub of about two kilos. We can not avoid showing how proud and happy he was. After this hard period for everyone, because of the Covid pandemic, to see a smiling face it is always a positive sign.

**Figure 3:** Happy fisherman

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