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Regular Biproduct Decompositions of Objects

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ABSTRACT. This thesis mainly pertains biproduct decompositions of objects in certain additive categories that exhibit a peculiar regular behaviour. More precisely, in certain additive categories, a biproduct of objects $\{X_i\}_{i<r}$ is completely determined up to isomorphism by a list of invariants $([X_i]_{\equiv_\mu})_{i<r,\mu<n}$, where $\{\equiv_\mu\}_{\mu<n}$ are suitable equivalence relations (n-Krull-Schmidt Theorem).

In the first chapter we introduce prerequisite notions that enable us to extend results regarding certain module categories to suitable preadditive categories: The Jacobson radical of a preadditive category and ideals associated to ideals of endomorphism rings (subject of research by Facchini and Příhoda), the universal embedding of a preadditive category into an additive category, and the universal embedding of an additive category into an idempotent-complete additive category. We give a version of the Chinese Remainder Theorem for preadditive categories, extrapolated from results of Facchini and Perone, and generalised, and we provide an improved version of the classical Krull-Schmidt Theorem which is the starting point of later developments. Semilocal rings and categories are reviewed in the second chapter, and their relationship with the notion of dual Goldie dimension is explained. The third chapter also deals with prerequisites, namely, we thereby try to give a careful review of the theory of the Auslander-Bridger transpose.

In the fourth chapter we generalise Warfield's results on finitely presented modules over semiperfect rings to Auslander-Bridger modules, a more general class of modules over arbitrary rings. We show how such modules are characterised by two invariants and such invariants are interchanged by the Auslander-Bridger transpose. The fifth chapter culminates in a criterion for the aforementioned n-Krull-Schmidt Theorem to hold in a given additive category, and we give some concrete examples in the case of categories of modules, such as artinian modules with prescribed heterogeneous socle, and quiver representations. The case $n = 2$ of said theorem has long been known as "Weak Krull-Schmidt Theorem," and has been proved over the years for various classes of modules. One of these, the class of couniformly presented modules, is dealt with in a more elementary way in the sixth chapter.

SOMMARIO. Questa tesi riguarda principalmente le decomposizioni in biprodotti di oggetti di certe categorie additive che esibiscono un comportamento regolare peculiare. Più precisamente, in certe categorie additive, un biprodotto di oggetti $\{X_i\}_{i<r}$ è completamente caratterizzato a meno di isomorfismo da una lista di invarianti $([X_i]_{\equiv_\mu})_{i<r,\mu<n}$, dove $\{\equiv_\mu\}_{\mu<n}$ sono opportune relazioni di equivalenza (n-teorema di Krull-Schmidt).

Nel primo capitolo introduciamo prerequisiti che ci permettono di estendere risultati che riguardano certe categorie di moduli a opportune categorie preadditive: il radicale di Jacobson di una categoria preadditiva e suoi ideali associati ad ideali di anelli di endomorfismi (soggetto di ricerche da parte di Facchini e

Příhoda), l'immersione universale di una categoria preadditiva in una categoria additiva, e l'immersione universale di una categoria additiva in una categoria additiva in cui gli idempotenti si spezzano. Diamo una versione del Teorema Cinese dei Resti per le categorie preadditive, estrapolato da risultati di Facchini e Perone e generalizzato, e forniamo una versione migliorata del teorema classico di Krull-Schmidt che è il punto di partenza di sviluppi seguenti. Gli anelli e le categorie semilocali sono passati in rassegna nel secondo capitolo, in cui viene anche spiegata la loro relazione con la nozione di dimensione duale di Goldie. Il terzo capitolo è pure dedicato ai prerequisiti, precisamente, ivi cerchiamo di passare in attenta rassegna la teoria della trasposta di Auslander-Bridger.

Nel quarto capitolo generalizziamo i risultati di Warfield sui moduli finitamente presentati su anelli semiperfetti ai moduli di Auslander-Bridger, che sono una classe più generale di moduli su anelli arbitrari. Mostriamo come tali moduli sono caratterizzati da due invarianti e come tali invarianti siano scambiati dalla trasposta di Auslander-Bridger. Il quinto capitolo culmina in un criterio per stabilire la validità del sopracitato n-teorema di Krull-Schmidt in una data categoria additiva, a diamo alcuni esempi concreti nel caso di categorie di moduli, come i moduli artiniani con zoccolo eterogeneo prefissato, e nel caso di categorie di rappresentazioni di quiver. Il caso $n = 2$ di detto teorema è noto come "teorema debole di Krull-Schmidt," ed è stato dimostrato negli anni per varie classi di moduli. Una di queste, la classe dei moduli couniformemente presentati, è trattata in un modo più elementare nel sesto capitolo.

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Contents

Introduction

As the title says, this thesis is about "regular biproduct decompositions of objects." If, for the benefit of those readers not acquainted with the language of category theory, we were to restrict our attention to the archetypal case of modules over a ring, we could say that this thesis is about regular finite direct-sum decompositions of modules. Indeed, the notion of finite direct-sum (for modules) is the specialisation of the notion of biproduct (for objects of additive categories) to the category of modules over a ring. Therefore, in this brief introduction, the reader can replace "object" with "module" and "biproduct" with "finite direct sum," if needed. Let us go on to explain what we mean by *regular* biproduct decompositions.

An object X of an additive category may be represented as a biproduct of other objects, a fact which is indicated by notation such as

$$X \cong \bigoplus_{i<n} X_i. \tag{1}$$

In other words, X can be expressed, or decomposed, as the biproduct of the family $\{X_i\}_{i<n}$. The question naturally arises whether this decomposition of X is to some extent regular, whether it must respect some pattern.

The order of the biproduct factors X_i never matters, that is, if equation (1) holds, it is also true that

$$X \cong \bigoplus_{i<n} X_{\sigma(i)},$$

for every permutation σ of the first n natural numbers. In some cases, this is the only alteration possible to the original decomposition, i.e., if

$$X \cong \bigoplus_{i<m} Y_i,$$

then $n = m$ and $X_i \cong Y_{\sigma(i)}$, for all $i < n$, for a suitable permutation σ. This happens, for instance, when every X_i has a local ring of endomorphisms ((Krull-Schmidt) Theorem 1.14), but not only (cf. Theorem 5.12(iii) and the remarks after its proof). Of course, this is the most regular behaviour for a biproduct decomposition of X — no other decompositions of X are possible, except for the obvious ones obtained by reordering the biproduct factors. When this happens we say that the decomposition of X is unique.

In 1975 Warfield proved that every finitely presented module over a serial ring decomposes as a finite direct sum of uniserial modules (= modules M such that, for every two submodules A and B of M, either $A \leqslant B$ or $B \leqslant A$), and he asked whether such decomposition is unique, despite the fact that uniserial modules do not necessarily have local endomorphism rings [War75]. In 1996, besides giving a negative answer to said question, Facchini also discovered a fascinating regularity, which we now explain. To each uniserial module U are attached two invariants, its monogeny class $[U]_m$ and its epigeny class $[U]_e$. His brilliant finding is that two finite direct sums $\bigoplus_{i<n} U_i$ and $\bigoplus_{i<m} V_i$ of uniserial modules are isomorphic if and only if $n = m$ and the invariants $[U_0]_m, \dots, [U_{n-1}]_m$ coincide, counting multiplicities, with the invariants $[V_0]_m, \dots, [V_{n-1}]_m$, and the invariants $[U_0]_e, \dots, [U_{n-1}]_e$ coincide, again counting multiplicities, with the invariants $[V_0]_e, \dots, [V_{n-1}]_e$. In other words, there are two permutations σ and τ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$, for all $i < n$ [Fac96, Theorem 1.9]. In particular, $U \cong V$ if and only if $[U]_m = [V]_m$ and $[U]_e = [V]_e$.

The author of the present work was deeply impressed by the elegance of this result, and it had a major influence on his research. Indeed, the main results in this thesis are generalisations of the mentioned theorem to more general categories and involving possibly more than two invariants/permutations (Theorems 5.8 and 5.10). This sort of regularity of biproduct decompositions is the one alluded to in the title, which we have now hopefully clarified.

Let us now give a summary of the content of the thesis.

After agreeing on some notation and conventions, in Chapter 1, we set about to discuss some notions concerning preadditive categories and their ideals. In particular, we introduce the most important ideal of all, the Jacobson radical, and ideals associated to ideals of endomorphism rings of objects, a key idea from [FP09b]. We show how every preadditive category $\mathbf{C}$ can be naturally embedded in an additive category $\mathrm{Sums}(\mathbf{C})$, thus introducing biproducts when they do not already exist in $\mathbf{C}$. The category $\mathrm{Sums}(\mathbf{C})$ is unique up to equivalence. (This is an idea that goes back to [Kel64].) Also, we exhibit how an additive category $\mathbf{C}$ can be embedded into one in which idempotents split, $\hat{\mathbf{C}}$, also known as an idempotent-complete additive category. Roughly speaking, this means that in the larger category idempotent endomorphisms correspond to biproduct factors. The category $\hat{\mathbf{C}}$ is also unique up to equivalence. (This construction was explained in [Fac07].) Thus a preadditive category, apparently poor in structure, actually *determines* the richer structure of an idempotent-complete additive category, which is the best setting for the study of biproduct decompositions. For instance, idempotent endomorphisms correspond to biproduct factors (= direct summands), as in the case of modules (Lemma 1.11). More importantly, the Krull-Schmidt Theorem holds in these categories. In this chapter we also prove a strong version of the classical Krull-Schmidt theorem due to the author (Theorem 1.18), which has a key role in the proofs of the main results of Chapter 5. Also, we prove a Chinese Remainder Theorem for preadditive categories and ideals, which is implicit in many proofs of [FP09b] and [FP10] and underpins their main results.

After introducing the classical notions of Goldie dimension and dual Goldie dimension, first for lattices, then their specialised versions for modules and rings, and after reviewing some standard material about semisimple and semilocal rings, a brief exposition of results from [FP10] concerning semilocal categories (= categories where every non-zero object has a semilocal ring of endomorphisms, and with at least a non-zero object) follows in Chapter 2. Special attention is given to those semilocal categories in which every non-zero object has a ring of endomorphisms S such that $S/J(S)$ is a finite direct product of division rings, which were studied in [FP09b]. Also, the author introduces a notion of dual Goldie dimension for preadditive categories, and characterises the preadditive categories of finite dual Goldie dimension as those semilocal categories with finitely many objects.

In the third chapter we construct the stable category of modules, which is a quotient of the category of (right) R-modules by a suitable ideal, and discuss the Auslander-Bridger transpose, which is a duality between the stable category of finitely presented left R-modules and the stable category of finitely presented right R-modules.

In Chapter 4 we finally put the machinery previously discussed to good use. This chapter contains the material from a joint paper with A. Facchini. We introduce Auslander-Bridger modules, which are to general rings what finitely presented modules are to semiperfect rings. As the name suggests, these modules behave very nicely under the Auslander-Bridger transpose. The transpose induces a bijection between isomorphism classes of Auslander-Bridger left modules and isomorphism classes of Auslander-Bridger right modules. (This generalises results of [War75] on finitely presented modules over semiperfect rings.) Also, Auslander-Bridger modules are characterised by two invariants, the lower-isomorphism class and the epi-isomorphism class, which are interchanged by the transpose. That is, M and N have the same lower-isomorphism (epi-isomorphism) class if and only if their transposes $\mathrm{Tr}_0(M)$ and $\mathrm{Tr}_0(N)$ have the same epi-isomorphism (lower-isomorphism) class.

Instrumental in defining Auslander-Bridger modules are finite direct sums of couniform projective modules. In Chapter 4 we also take a slight detour to study the analogue of these objects in preadditive categories.

There is a representable contravariant functor $\mathrm{Hom}(-, E)$ which induces a duality between uniform injective modules and couniform projective modules; this induces a duality between Auslander-Bridger modules and the dual notion of dual Auslander-Bridger modules, characterised by the dual invariants, the mono-isomorphism class and the upper-isomorphism class.

The fifth chapter deals with the author's results on classes of objects (in preadditive categories, enlarged if needed to idempotent-complete additive categories) for which biproduct decompositions are regular, in the sense that (*) two biproducts $\bigoplus_{i<n} X_i$ and $\bigoplus_{i<m} Y_i$ are isomorphic if and only if $n = m$ and $X_i \equiv_\mu Y_{\sigma_\mu(i)}$ for $i < n$, and permutations σ_μ, where $\equiv_\mu$ are finitely many suitable equivalence relations indexed by μ. Several examples, both old — biuniform modules, uniserial modules, couniformly presented modules, kernels of morphisms between indecomposable injective modules — and new — artinian modules with heterogeneous socle, noetherian modules with heterogeneous top, quiver representations "of type 1 pointwise" — are given. In the last section of Chapter 5 we show how the question whether the property (*) holds for a certain class of modules $\mathcal{F}$ translates to a combinatorial condition on a hypergraph $H(\mathcal{F})$ canonically associated to $\mathcal{F}$.

Chapter 6 is devoted to couniformly presented modules, a class of modules for which a result like (*) holds with two invariants/permutations, and how these relate to uniserial modules and kernels of morphisms between indecomposable injective modules via suitable dualities. This was the subject of a joint paper with A. Facchini [FG10].

In the mathematical literature there are versions of some of the results in this thesis for infinite direct sums of certain modules. Among the uniserial modules, there are the quasi-small uniserial modules, for which the '96 result by Facchini, which was quoted above, holds also in the case of an infinite direct sum (the two permutations of finite sets become bijections of sets) [DF97]. Unfortunately, there are also uniserial modules that are not quasi-small, as was discovered by Puninski [Pun01]. His methods rely heavily on the model theory of modules. In the final Chapter 7 we explain his example of a non-quasi-small uniserial module avoiding model theory as much as possible, to make it available to a larger audience. It recently came to the author's knowledge, and we promptly duly point out, that Příhoda developed an alternative algebraic route to Puninski's non-quasi-small uniserial module [Pří06].

We close with some annotations in which we justify some constructions in the thesis, which may appear solid at first glance, but actually turn out to need some attention when looked at more closely. The discussion of these relatively small issues make us reflect on what foundations we build all of our theories on, and we find ourselves forced to confront the puzzling idea that our mathematical foundations may not be solid as we tend to believe.

Notation and conventions

All the rings we consider are associative rings with identity. For a ring we require that $1 \neq 0$, with the only exception of the endomorphism rings of zero objects in preadditive categories. All modules considered are unital. Notation such as M_R denotes a right R-module, while ${}_RM$ denotes a left R-module. When no index is used, it either means that the ring and side have been specified earlier or that they are clear from context, or that they are not relevant in the discussion. For instance, in the phrase "M is semisimple if and only if every submodule of M is a direct summand of M," neither the base ring nor the side need to be specified.

The symbol $\mathbb{N}$ stands for the set of non-negative integers, that is, $0 \in \mathbb{N}$.

The role of index sets is covered almost exclusively by ordinal numbers, denoted by Greek letters α, β, γ, $\dots$, except for the natural numbers, which are denoted by roman letters such as n and m. Thus we will encounter notation such as $(M_i)_{i<n}$ and $\prod_{i<\kappa} R_i$, and when the set $\mathbb{N}$ of natural numbers is used as an index set, we denote it by ω.

The symbol $\subseteq$ denotes set inclusion, and $\subset$, or $\subsetneq$ for emphasis, denotes strict set inclusion. When a set inclusion respects some sort of algebraic structure, we prefer to use the symbols $\leqslant$ and $<$.

Sometimes we will use calligraphic letters such as $\mathcal{A}$ and $\mathcal{B}$ to denote sets or classes, if using the corresponding roman letters might cause confusion with other entities, such as objects A and B.

Categories and their ideals are denoted by bold letters, such as $\mathbf{C}$ and $\mathbf{I}$. The set of morphisms between two objects X and Y of $\mathbf{C}$ is denoted by $\mathbf{C}(X, Y)$. Similarly, the subset of morphisms in an ideal $\mathbf{I}$ that are in $\mathbf{C}(X, Y)$ is denoted $\mathbf{I}(X, Y)$. When $X = Y$, we shorten $\mathbf{C}(X, X)$ to $\mathbf{C}(X)$ and $\mathbf{I}(X, X)$ to $\mathbf{I}(X)$. To indicate that g is a morphism from X to Y we will write $g \in \mathbf{C}(X, Y)$, or sometimes $g\colon X \to Y$, if the category is understood or not relevant. If g is a morphism, then $\operatorname{dom}(g)$ is its domain and $\operatorname{codom}(g)$ its codomain, in other words we could write $g\colon \operatorname{dom}(g) \to \operatorname{codom}(g)$.

Chapter 1

Preadditive categories

In this chapter we recall the notion of "ideal" in a preadditive category, and most importantly, the Jacobson radical of a preadditive category and "associated ideals," which are ideals of the category associated to ideals of endomorphism rings of its objects, a very important idea from [FP09b]. We discuss universal embeddings of preadditive categories into additive categories first [Kel64] and into idempotent-complete additive categories later [Fac07]. We also prove a strong version of the classical Krull-Schmidt theorem (Theorem 1.18), which is of fundamental importance in proving the main results of Chapter 5. Eventually, we prove a Chinese Remainder Theorem for preadditive categories (= rings with many objects) and ideals, which is implicitly used in many proofs of [FP09b] and [FP10].

1.1 Ideals in preadditive categories

A category $\mathbf{C}$ is *preadditive* if for every $X, Y \in \mathbf{C}$ the set of morphisms $\mathbf{C}(X, Y)$ is an abelian group, and the composition of morphisms in $\mathbf{C}$ is bilinear over the integers, that is, $f(g_1 + g_2) = fg_1 + fg_2$ and $(f_1 + f_2)g = f_1g + f_2g$ for all morphisms $f, f_1, f_2 \colon Y \to Z$ and $g, g_1, g_2 \colon X \to Y$ and all objects X, Y and Z of $\mathbf{C}$.

In a preadditive category two objects have a product if and only if they have a coproduct, if and only if they have a biproduct [Bor94b, Proposition 1.2.4]. Coproducts are often called "direct sums," although strictly speaking the direct sum is the coproduct in the category of modules. If finite products exist in a preadditive category $\mathbf{C}$, we say that $\mathbf{C}$ is an *additive* category. Finite direct sums are the biproducts in the category of modules.

An *ideal* of a preadditive category $\mathbf{C}$ is a class of morphisms $\mathbf{I}$ of $\mathbf{C}$ such that, for every pair of objects X and Y of $\mathbf{C}$, the set $\mathbf{I}(X, Y) := \mathbf{C}(X, Y) \cap \mathbf{I}$ is a subgroup of $\mathbf{C}(X, Y)$, and such that for every $g \in \mathbf{I}(X, Y)$, every $f \in \mathbf{C}(Y, Y')$

and every $h \in \mathbf{C}(X', X)$, we have $fgh \in \mathbf{I}(X', Y')$. We will abbreviate $\mathbf{I}(X, X)$ to $\mathbf{I}(X)$.

Ideals of preadditive categories satisfy the same basic properties as ideals of rings. A preadditive category $\mathbf{C}$ always has the *zero* ideal, consisting of all the zero morphisms, and the *improper* ideal, consisting of all morphisms of the category.

For any subset or subclass S of morphisms of $\mathbf{C}$ one may consider the ideal of $\mathbf{C}$ *generated* by S, that is, the *smallest* ideal (with respect to inclusion) of $\mathbf{C}$ containing S. Of course, it can be defined as the intersection $\mathbf{S}$ of all ideals containing S, as there is always one such, viz., the improper ideal, and the intersection of a collection of ideals of $\mathbf{C}$ is again an ideal of $\mathbf{C}$. The ideal $\mathbf{S}$ can also be described (as is the case for rings) as the class $\mathbf{S}$ of morphisms of the form $\sum_{i<n} f_i g_i h_i$, where each $g_i \in S$.

As we mentioned, arbitrary intersections of ideals are ideals; similarly, the sum of an arbitrary family $\{\mathbf{I}_i\}_{i<\kappa}$ of ideals of $\mathbf{C}$ can also be defined, letting $\left(\sum_{i<\kappa} \mathbf{I}_i\right)(X, Y) = \sum_{i<\kappa} \mathbf{I}_i(X, Y)$ for every X and Y in $\mathbf{C}$.

The union of a chain of ideals of $\mathbf{C}$ is again an ideal of $\mathbf{C}$. This entails that the Zorn Lemma can be applied to the collection of ideals of $\mathbf{C}$ not containing a given set or class S of non-zero morphisms, thus obtaining a maximal ideal disjoint from S. This does not grant the existence of maximal proper ideals, though, cf. [FP10, Example 4.1].

If we consider the class $\mathrm{Latt}(\mathbf{C})$ of all ideals of $\mathbf{C}$ partially ordered by inclusion, we see that $\mathrm{Latt}(\mathbf{C})$ is a large complete lattice with respect to the operations of intersection and sum, just as is the case for rings.

Ideals are instrumental in defining *factor categories.* If $\mathbf{I}$ is an ideal of a preadditive category $\mathbf{C}$, we may construct a new category $\mathbf{C}/\mathbf{I}$ with the same class of objects as $\mathbf{C}$ and morphisms given by the quotient abelian groups $(\mathbf{C}/\mathbf{I})(X, Y) = \mathbf{C}(X, Y)/\mathbf{I}(X, Y)$, for every pair of objects X, Y of $\mathbf{C}$. The composition rule on the factor category $\mathbf{C}/\mathbf{I}$ is induced by that of $\mathbf{C}$, namely, $(f + \mathbf{I}(Y, Z))(g + \mathbf{I}(X, Y)) = fg + \mathbf{I}(X, Z)$. There is a canonical additive full functor $\mathbf{C} \to \mathbf{C}/\mathbf{I}$, to which we may refer as the "reduction modulo $\mathbf{I}$."

Suppose $G\colon \mathbf{A}_1 \to \mathbf{A}_2$ is an additive functor and $\mathbf{I}$ is an ideal of $\mathbf{A}_2$. Then we define the *inverse image* $G^{-1}(\mathbf{I})$ of $\mathbf{I}$ pointwise, that is, as

$$(G^{-1}\mathbf{I})(M, N) = \{f \in \mathbf{A}_1(M, N) : G(f) \in \mathbf{I}(G(M), G(N))\},$$

for all pairs of objects M, N of $\mathbf{A}_1$. In short, $f \in G^{-1}(\mathbf{I})$ if and only if $G(f) \in \mathbf{I}$, for every morphism f in the category $\mathbf{A}_1$. Thus we obtain an ideal $G^{-1}(\mathbf{I})$ of $\mathbf{A}_1$.

We define the *kernel* of G to be the preimage of the zero ideal of $\mathbf{A}_2$, and we denote it as $\mathbf{K}(G)$.

We have the analogue of the fundamental theorem of homomorphisms of rings, namely, we have that every additive functor $F\colon \mathbf{C} \to \mathbf{D}$ factors as the composition $F = \bar{F}P$ of a full functor P and a faithful functor $\bar{F}$. Precisely, P is the canonical functor $P\colon \mathbf{C} \to \mathbf{C}/\mathbf{K}(F)$ while $\bar{F}\colon \mathbf{C}/\mathbf{K}(F) \to \mathbf{D}$ is induced by F in the obvious way.

Following [FP09c], an ideal $\mathbf{I}$ of a preadditive category $\mathbf{C}$ is called *completely prime* if

(C1) $\mathbf{I}$ contains no non-zero identity morphisms, i.e., $\mathbf{I}(X) \neq \mathbf{C}(X)$ for every non-zero object X of $\mathbf{C}$, and

(C2) whenever a composition fg is in $\mathbf{I}$, then either f or g is in $\mathbf{I}$.

Notice that condition (C1) is stronger than just requiring $\mathbf{I}$ to be a proper ideal.

This definition extends that of a completely prime ideal I of a ring R, which is a proper ideal I satisfying $ab \in I$ if and only if $a \in I$ or $b \in I$ for all $a, b \in R$.

1.1.1 The Jacobson radical of a preadditive category

In this section we define probably the most important ideal, namely, the Jacobson radical of a preadditive category. Cf. [Mit72, page 21].

Lemma-Definition 1.1. *Let $\mathbf{C}$ be a preadditive category. Given any two objects A and B of $\mathbf{C}$, the following sets are all equal:*

$$J_1(A,B) = \{f \in \mathbf{C}(A,B) \mid 1_B - fg \textit{ is right invertible for all } g \in \mathbf{C}(B,A)\},$$
$$J_2(A,B) = \{f \in \mathbf{C}(A,B) \mid 1_B - fg \textit{ is invertible for all } g \in \mathbf{C}(B,A)\},$$
$$J_3(A,B) = \{f \in \mathbf{C}(A,B) \mid 1_A - gf \textit{ is left invertible for all } g \in \mathbf{C}(B,A)\},$$
$$J_4(A,B) = \{f \in \mathbf{C}(A,B) \mid 1_A - gf \textit{ is invertible for all } g \in \mathbf{C}(B,A)\},$$
$$J_5(A,B) = \{f \in \mathbf{C}(A,B) \mid 1_B - fg \textit{ is left invertible for all } g \in \mathbf{C}(B,A)\},$$
$$J_6(A,B) = \{f \in \mathbf{C}(A,B) \mid 1_A - gf \textit{ is right invertible for all } g \in \mathbf{C}(B,A)\}.$$

If $\mathbf{J}(A,B)$ denotes the set described above, then $\mathbf{J}$ is an ideal of $\mathbf{C}$, which we call the Jacobson radical *of $\mathbf{C}$. It readily follows from the above description that $\mathbf{J}(A)$ is the Jacobson radical of the endomorphism ring $\mathbf{C}(A)$ of A in the category $\mathbf{C}$.*

The proof relies on the purely syntactical fact that $z(1 - xy) = 1$ implies $(1 + yzx)(1 - yx) = 1$. Cf. [Mit72, Lemma 4.2].

Proof. It is clear that $J_2 \subseteq J_1$. Conversely, suppose $f \in J_1$. Let $g \in \mathbf{C}(B,A)$ be arbitrary. There exists $h \in \mathbf{C}(B)$ such that $(1_B - fg)h = 1_B$. This implies that $h = 1_B - f(-gh)$, so that h is also right invertible in $\mathbf{C}(B)$. It follows that h is a two-sided inverse for $1_B - fg$ and $f \in J_1$. This proves that $J_1 = J_2$, and similarly one proves that $J_3 = J_4$.

Let $f\colon A \to B$ and $g\colon B \to A$ be morphisms in $\mathbf{C}$. Then $1_A - gf$ is left invertible in $\mathbf{C}(A)$ if and only if $1_B - fg$ is left invertible in $\mathbf{C}(B)$. Indeed, if h is a left inverse of $1_A - gf$, then $1_B + fhg$ is a left inverse for $1_B - fg$. Therefore $J_3 = J_5$.

Similarly, if h is a right inverse for $1_A - gf$ then $1_B + fhg$ is a right inverse of $1_B - fg$. Hence $1_A - gf$ is right invertible in $\mathbf{C}(A)$ if and only if $1_B - fg$ is right invertible in $\mathbf{C}(B)$, so that $J_1 = J_6$.

It also follows that $1_A - gf$ is invertible in $\mathbf{C}(A)$ if and only if $1_B - fg$ is invertible in $\mathbf{C}(B)$, so that $J_2 = J_4$.

To prove that $\mathbf{J}$ is an ideal of $\mathbf{C}$, we are left to show that if $g \in \mathbf{J}(A, B)$, $h \in \mathbf{C}(X, A)$, and $f \in \mathbf{C}(B, Y)$, then $fgh \in \mathbf{J}(X, Y)$. Let $i \in \mathbf{C}(Y, X)$ be arbitrary. Then $1_B - ghif$ has a left inverse, say $\ell \in \mathbf{C}(B)$. It follows that $1_Y + f\ell ghi$ is a left inverse for $1_Y - fghi$, as required. $\square$

The Jacobson radical of a preadditive category may also be defined using maximal right ideals or maximal left ideals as it is done for preadditive categories with one object, i.e., for rings. This approach is taken in [Mit72, page 21]. We will expand on this in Section 2.3, where we characterise preadditive categories of finite dual Goldie dimension.

1.1.2 Ideals of a category associated to ideals of endomorphism rings of its objects

Consider a preadditive category $\mathbf{C}$ and an object X of $\mathbf{C}$. Given an ideal M of the endomorphism ring $\mathbf{C}(X)$, one may construct the ideal $\mathbf{M}$ of $\mathbf{C}$ generated by M, obtaining $\mathbf{M}(X) = M$. There is another way, a more useful way, of constructing another different ideal of $\mathbf{C}$ with this last property. This technique has been introduced in [FP09a], and subsequently adopted in [FP10, FP09b, Gir11a, Gir11b]. Suppose M is an ideal of $\mathbf{C}(X)$. (It is actually enough for M to be an additive subgroup.) Define an ideal $\mathbf{A}_M$ by declaring that $g \in \mathbf{A}_M$ if and only if $fgh \in M$ for every $h \in \mathbf{C}(X, \mathrm{dom}(g))$ and every $f \in \mathbf{C}(\mathrm{codom}(g), X)$. This is easily seen to be an ideal of $\mathbf{C}$. We call $\mathbf{A}_M$ the ideal of $\mathbf{C}$ *associated* to M. An ideal $\mathbf{I}$ of $\mathbf{C}$ of this type will be called an *associated ideal.* The associated ideal $\mathbf{A}_M$ can be characterised as follows:

Lemma 1.2. *Suppose M is an ideal of the endomorphism ring $\mathbf{C}(X)$. The ideal $\mathbf{A}_M$ of $\mathbf{C}$ is the* largest *among the ideals $\mathbf{I}$ of $\mathbf{C}$ such that $\mathbf{I}(X) \subseteq M$. As a matter of fact, $\mathbf{A}_M(X) = M$.*

Proof. For $g \in \mathbf{C}(X)$, we have $g \in \mathbf{A}_M(X)$ if and only if $\mathbf{C}(X)g\mathbf{C}(X) \subseteq M$, if and only if $g \in M$. Hence $\mathbf{A}_M(X) = M$. Suppose $\mathbf{I}$ is an ideal of $\mathbf{C}$ such that $\mathbf{I}(X) \subseteq M$. If $g \in \mathbf{I}(Y, Z)$, then $\mathbf{C}(Z, X)g\mathbf{C}(X, Y) \subseteq \mathbf{I}(X) \subseteq M$, so that $g \in \mathbf{A}_M(X)$. This proves that $\mathbf{I} \subseteq \mathbf{A}_M$. $\square$

Remark 1.3. Notice that $\mathbf{A}_M(A, B)$ depends only on the objects A, B, X and on the morphisms between them. Therefore, if we consider any *full* subcategory $\mathbf{E}$

of $\mathbf{C}$, the ideal of $\mathbf{E}$ associated to M is a restriction of that of $\mathbf{C}$ associated to M. Thus we may unambiguously say that A *is isomorphic to* B *modulo* $\mathbf{A}_M$ if A and B are isomorphic as objects of $\mathbf{E}/\mathbf{A}_M$, where $\mathbf{E}$ is any full subcategory of $\mathbf{C}$ containing the objects A, B, X.

The above characterisation of $\mathbf{A}_M$ recalls one possible description of the Jacobson radical $\mathbf{J}$ of $\mathbf{C}$. Indeed, one may define $\mathbf{J}$ to be the largest ideal $\mathbf{I}$ of $\mathbf{C}$ such that $\mathbf{I}(X) \subseteq J(\mathbf{C}(X))$ for every object X of $\mathbf{C}$ [Kel64, Theorem 1]. Indeed, if $\mathbf{I}$ has this last property, and $g \in \mathbf{I}(Y, X)$, then $fg \in \mathbf{I}(Y)$ for every $f \in \mathbf{C}(X, Y)$. Hence $fg \in J(\mathbf{C}(Y))$ and it follows that $1_Y - fg$ is invertible. Thus $g \in \mathbf{J}(X, Y)$.

The resemblance just remarked is no coincidence: It is possible to describe the Jacobson radical of a preadditive category as the intersection of a very natural family of associated ideals. Recall that an ideal I of a ring R is *right (resp. left) primitive* if it is the annihilator of a simple right (resp. left) R-module, and that the Jacobson radical of R is the intersection of all its primitive right (resp. left) ideals [AF92, Theorem 15.3].

Proposition 1.4. *Let* $\mathbf{C}$ *be a preadditive category and let* $\mathrm{Prim}(\mathbf{C})$ *be the collection of all the ideals of* $\mathbf{C}$ *associated to right primitive ideals of endomorphism rings of objects of* $\mathbf{C}$*. Then*

$$\mathbf{J} = \bigcap \mathrm{Prim}(\mathbf{C}). \tag{1.5}$$

Proof. Let $\mathbf{P}$ be the intersection on the right hand side of (1.5).

Pick any morphism $g \colon A \to B$ in $\mathbf{J}(A, B)$. Consider an ideal $\mathbf{I} \in \mathrm{Prim}(\mathbf{C})$. Then there is an object X of $\mathbf{C}$ and a right primitive ideal P of $\mathbf{C}(X)$ such that $\mathbf{I} = \mathbf{A}_P$. Because $g \in \mathbf{J}(A, B)$, it follows that $\mathbf{C}(B, X) g \mathbf{C}(X, A) \subseteq \mathbf{J}(X)$, which is contained in P because the Jacobson radical of a ring is the intersection of its primitive ideals. Thus $g \in \mathbf{A}_P = \mathbf{I}$. Since g and $\mathbf{I}$ are arbitrary, this shows that $\mathbf{J}$ is contained in $\mathbf{P}$.

Suppose $g \in \mathbf{C}(A, B)$ is not in $\mathbf{J}(A, B)$. Then there exists $f \in \mathbf{C}(B, A)$ such that $1_A - fg$ is not right invertible in $\mathbf{C}(A)$. This implies that $1_A - fg$ is contained in a maximal right ideal M of $\mathbf{C}(A)$. Let P be the right primitive ideal of $\mathbf{C}(A)$ defined by $P = \mathrm{r.ann}_{\mathbf{C}(A)}(\mathbf{C}(A)/M)$. Then $g \notin \mathbf{A}_P(A, B)$. Indeed, if g was an element of $\mathbf{A}_P(A, B)$, then fg would be in $A_P(A) = P \subseteq M$, and this would lead to $1_A \in M$, a contradiction. Thus $g \notin \mathbf{A}_P(A, B)$, hence $g \notin \mathbf{P}(A, B)$. □

1.2 Universal embeddings of preadditive categories

Recall that a preadditive category has finite products if and only if it has finite coproducts, if and only if it has biproducts, and that it is called an *additive* category if these equivalent conditions are satisfied [Bor94b, Proposition 1.2.4]. Not all preadditive categories are additive, yet every one of them embeds canonically as a full subcategory of an additive category, its additive closure (Section 1.2.1).

A preadditive category $\mathbf{C}$ is said to be *idempotent-complete* if every idempotent endomorphism in $\mathbf{C}$ has a kernel. Indeed, the condition that an idempotent endomorphism has a kernel has many equivalents, cf. Lemma 1.10. For every preadditive category $\mathbf{C}$, there is a universal full and faithful functor $\Gamma\colon \mathbf{C} \to \widehat{\mathbf{C}}$ of $\mathbf{C}$ into an idempotent-complete preadditive category $\widehat{\mathbf{C}}$, which is called the *idempotent completion* of $\mathbf{C}$. In particular, given any pair of objects X and Y in $\mathbf{C}$, we have that $X \cong Y$ if and only if $\Gamma(X) \cong \Gamma(Y)$, and, given any morphism g in $\mathbf{C}$, that g is an isomorphism if and only if so is $\Gamma(g)$. If $\mathbf{C}$ is additive, then so is its idempotent completion. Idempotent-complete additive categories provide the best setting for the study of biproduct decompositions of objects—most notably, the classical Krull-Schmidt Theorem holds in these categories (Theorem 1.14). The theorem says that from an isomorphism $g\colon X_1 \oplus \cdots \oplus X_n \to Y_1 \oplus \cdots \oplus Y_n$, where all the X_i and Y_i have local endomorphism rings, it follows that $X_i \cong Y_{\sigma(i)}$ for a suitable permutation σ. In Theorem 1.18, we show that more information can be gleaned from said isomorphism g. Indeed, we find that $g_{\sigma(i),i}\colon X_i \to Y_{\sigma(i)}$ is an isomorphism, for a suitable permutation σ.

1.2.1 Additive closure

The difference between a preadditive and an additive category is that in the former finite products (coproducts, biproducts) may not exist, while they do in the latter. It is nevertheless possible to embed any preadditive category $\mathbf{C}$ as a full subcategory of an additive category $\mathrm{Sums}(\mathbf{C})$, as remarked in [Kel64], that we may call the *additive closure* of $\mathbf{C}$. The objects of $\mathrm{Sums}(\mathbf{C})$ are the finite sequences $(A_i)_{i<n}$ of objects A_i of $\mathbf{C}$. For morphisms, we define

$$(\mathrm{Sums}(\mathbf{C}))((A_i)_{i<n}, (B_i)_{i<m}) \tag{1.6}$$

to be the set of $m \times n$ matrices g such that $g_{i,j} \in \mathbf{C}(A_j, B_i)$ for $i < m$ and $j < n$. The set (1.6) is an additive abelian group with respect to pointwise addition. Matrix multiplication serves as the composition rule. The biproduct of $(A_i)_{i<n}$ and $(B_i)_{i<m}$ is the concatenation of the two sequences, that is, the sequence $(A_0, \ldots, A_{n-1}, B_0, \ldots, B_{m-1})$. The injections

$$(A_i)_{i<n} \to (A_0, \ldots, A_{n-1}, B_0, \ldots, B_{m-1})$$
$$(B_i)_{i<m} \to (A_0, \ldots, A_{n-1}, B_0, \ldots, B_{m-1})$$

are the matrices

$$\begin{pmatrix} 1_{A_0} & & \\ & \ddots & \\ & & 1_{A_{n-1}} \\ \hline & \mathbf{0}_{m\times n} & \end{pmatrix}, \qquad \begin{pmatrix} & \mathbf{0}_{n\times m} & \\ \hline 1_{B_0} & & \\ & \ddots & \\ & & 1_{B_{m-1}} \end{pmatrix} \tag{1.7}$$

and the corresponding projections are their transposes. In particular the sequence $(A_i)_{i<n}$ is the biproduct of the family $\{(A_i)\}_{i<n}$. In some sense, we may think of $(A_i)_{i<n}$ as the "formal biproduct" of the objects $A_0, \dots, A_{n-1}$ of $\mathbf{C}$.

We have a full and faithful functor $(-)\colon \mathbf{C} \to \mathrm{Sums}(\mathbf{C})$ which identifies the objects of $\mathbf{C}$ with the sequences of length one.

The embedding $(-)$ is minimal in the following sense:

Proposition 1.8. *For every additive functor $G\colon \mathbf{C} \to \mathbf{D}$ with $\mathbf{D}$ additive, there is an additive functor $H\colon \mathrm{Sums}(\mathbf{C}) \to \mathbf{D}$ such that $H \circ (-) = G$, and such H is unique up to natural isomorphism. This property is universal and characterises $\mathrm{Sums}(\mathbf{C})$ up to category equivalence.*

Proof. The property is manifestly universal. Indeed, suppose $\Gamma_0\colon \mathbf{C} \to \mathbf{C}_0$ and $\Gamma_1\colon \mathbf{C} \to \mathbf{C}_1$ are full and faithful functors from the preadditive category $\mathbf{C}$ into additive categories, and that both embeddings satisfy the property in the statement. Then there are additive functors H_0 and H_1 such that $\Gamma_0 = H_0\Gamma_1$ and $\Gamma_1 = H_1\Gamma_0$, so that $\Gamma_0 = H_0H_1\Gamma_0$ and $\Gamma_1 = H_1H_0\Gamma_1$. By uniqueness up to natural isomorphism, we have that H_0H_1 and H_1H_0 are naturally isomorphic to the corresponding identity functors, so that $\mathbf{C}_0$ and $\mathbf{C}_1$ are equivalent categories.

We have to show that the full and faithful functor $(-)$ satisfies this universal property. To sketch a proof of that we need to agree on some notation. An object of $\mathrm{Sums}(\mathbf{C})$ will be denoted by $A = (A_i)_{i<n_A}$. That is, the length of the sequence A is n_A and its entries are $A_0, \dots, A_{n_A-1}$. Thus A is the biproduct of the family $\{(A_i)\}_{i<n_A}$, with canonical morphisms given by matrices analogous to those in (1.7). We denote by $\iota_{A,i}$ the injections and by $\pi_{A,i}$ the projections, for $i < n_A$.

For each sequence A choose a biproduct $H(A)$ in the category $\mathbf{D}$ of the family $\{G(A_i)\}_{i<n_A}$, with canonical injections $H(\iota_{A,i})$ and projections $H(\pi_{A,i})$. The definition of H on morphisms follows by the requirement that H extend G. Indeed, suppose $g\colon A \to B$ is a morphism in $\mathrm{Sums}(\mathbf{C})$. Let $g_{j,i} \in \mathbf{C}(A_i, B_j)$ be such that $(g_{j,i}) = \pi_{B,j}g\iota_{A,i}$. Then we have

$$g = \sum_{i<n_A}\sum_{j<n_B} \iota_{B,j}\pi_{B,j}g\iota_{A,i}\pi_{A,i},$$

hence

$$H(g) = \sum_{i<n_A}\sum_{j<n_B} H(\iota_{B,j})G(g_{j,i})H(\pi_{A,i}).$$

It is easy to check that H respects the identities, the composition, and the sum, hence H is an additive functor, and by construction $H \circ (-) = G$.

Suppose that H' is another additive functor such that $H' \circ (-) = G$. For a sequence A, define $\eta_A = \sum_{i<n_A} H'(\iota_{A,i})H(\pi_{A,i})$. It is easy to check that these maps define a natural isomorphism $H \to H'$. □

From the universal property above, it follows that if $\mathbf{C}$ is a full subcategory of an additive category $\mathbf{D}$, then the category $\mathrm{Sums}(\mathbf{C})$ is equivalent to the full subcategory of $\mathbf{D}$ whose objects are all the finite biproducts of objects of $\mathbf{D}$.

Let us record the following useful fact for future reference:

Remark 1.9. An additive functor $F\colon \mathbf{C}_1 \to \mathbf{C}_2$ between preadditive categories canonically extends to an additive functor $\tilde{F}\colon \mathrm{Sums}(\mathbf{C}_1) \to \mathrm{Sums}(\mathbf{C}_2)$ between the additive closures. Moreover, if F is full (resp. faithful) (resp. dense), then so is $\tilde{F}$. In particular, if F is a category equivalence, then so is $\tilde{F}$.

Notice that an ideal $\mathbf{I}_0$ of $\mathbf{C}$ extends canonically to an ideal $\mathbf{I}$ of its additive closure $\mathrm{Sums}(\mathbf{C})$: For a morphism $g\colon (X_i)_{i<n} \to (Y_i)_{i<m}$, we inevitably have that $g \in \mathbf{I}$ if and only if all its entries are in $\mathbf{I}_0$.

1.2.2 Idempotent completion

Let $\mathbf{C}$ be any category and e an *idempotent* endomorphism in $\mathbf{C}$ of some object X, i.e., an element of the ring $\mathbf{C}(X)$ such that $e = e^2$. For an object Y of $\mathbf{C}$, we say that e *splits through* Y if there exist morphisms $f \in \mathbf{C}(X,Y)$ and $g \in \mathbf{C}(Y,X)$ such that $e = gf$ and $fg = 1_Y$. We simply say that e *splits* if it splits through some object of $\mathbf{C}$.

Recall that, for $\varphi, \psi \in \mathbf{C}(X,Y)$, an *equaliser* of φ and ψ is a morphism f such that $\varphi f = \psi f$, and universal with this property, that is, if $\varphi h = \psi h$ for some morphism h, then there exists a morphism g such that $fg = h$, and such morphism g is unique. This implies that the equaliser f of a given pair of morphisms is a *monomorphism,* i.e., left-cancellable, and that it is unique up to isomorphism, that is, if there is another morphism f' with the same universal property, there exist isomorphisms η and η' such that $f' = f\eta$ and $f = f'\eta'$. If $\mathbf{C}$ is preadditive, an equaliser of $f \in \mathbf{C}(X,Y)$ and the zero element of $\mathbf{C}(X,Y)$ is the *kernel* of f. Dually one defines *coequalisers,* which are always *epimorphisms,* i.e., right cancellable, and *cokernels.*

The following lemma characterises idempotents that split. Cf. [Bor94a, Proposition 6.5.4] and [Fac07, Lemma 2.1].

Lemma 1.10. *Let $\mathbf{C}$ be any category, X an object of $\mathbf{C}$ and e an idempotent endomorphism of X in $\mathbf{C}$. The following are equivalent:*

(i) The idempotent e splits, i.e., there exist morphisms f and g in $\mathbf{C}$ such that $e = gf$ with $fg = 1$.

(ii) The pair $(e, 1_X)$ has an equaliser.

(iii) The pair $(e, 1_X)$ *has a coequaliser.*

If $\mathbf{C}$ *is preadditive, then the above conditions are also equivalent to:*

(iv) The (idempotent) endomorphism $1_X - e$ *has a kernel.*

(v) The (idempotent) endomorphism $1_X - e$ *has a cokernel.*

In the notation of (i), g *(resp.* f*) is the equaliser (resp. coequaliser) of* $(e, 1_X)$*, equivalently, when* $\mathbf{C}$ *is preadditive, the kernel (resp. cokernel) of* $1_X - e$*.*

Proof. Suppose e splits as in (i). It follows that $g = eg$. Moreover, if $h = eh$ then $h = g(fh)$, so that h factors through g, and it does so uniquely because g is a monomorphism. Thus g is the equaliser of the pair $(e, 1_X)$, and this shows that (i) implies (ii).

If (ii) holds, let g be the equaliser of the pair $(e, 1_X)$. Since $e \cdot e = 1_X \cdot e$, we have that $e = gf$ for a unique morphism f. Since $g = eg = gfg$ and g is a monomorphism (being an equaliser) it follows that $fg = 1$. This shows that e splits, so (i) holds.

If $\mathbf{C}$ is preadditive, (ii) and (iv) are easily seen to be equivalent.

The remaining equivalences follow by duality, because e splits in $\mathbf{C}$ if and only if it splits in $\mathbf{C}^{op}$. □

As a consequence, we see that an idempotent-complete preadditive category, earlier defined as one in which every idempotent endomorphism has a kernel, is a preadditive category in which every idempotent endomorphism splits.

For objects X and Y of a preadditive category, we say that X is a *biproduct factor* of Y if there exists an object X' of the category such that Y is a biproduct of X and X'. As is the case for modules, we have that:

Lemma 1.11. *For objects* A *and* X *of an idempotent-complete preadditive category,* A *is a biproduct factor of* X *if and only if* 1_A *factors through* X*, equivalently, if and only if there is an idempotent endomorphism of* X *that splits through* A*.*

Proof. One implication is trivially true in every preadditive category. Suppose $1_A = \pi_A \iota_A$ with $\iota_A \in \mathbf{C}(A, X)$ and $\pi_A \in \mathbf{C}(X, A)$. Then $1_X - \iota_A \pi_A$ is an idempotent endomorphism, hence there exist an object B and morphisms $\iota_B \in \mathbf{C}(B, X)$ and $\pi_B \in \mathbf{C}(X, B)$ such that $1_X = \iota_A \pi_A + \iota_B \pi_B$ and $\pi_B \iota_B = 1_B$, and automatically $\pi_B \iota_A = \pi_A \iota_B = 0$, so that $X \cong A \oplus B$. □

Let us turn to the construction of the idempotent completion $\widehat{\mathbf{C}}$ of a preadditive category $\mathbf{C}$. The objects of $\widehat{\mathbf{C}}$ are the pairs (X, e) where X is an object of $\mathbf{C}$ and e is an idempotent endomorphism of X in $\mathbf{C}$. As far as morphisms are concerned, we define $\widehat{\mathbf{C}}((X_1, e_1), (X_2, e_2))$ to be the subgroup of $\mathbf{C}(X_1, X_2)$ consisting of those elements g such that $e_2 g e_1 = g$. In other words, we let

$$\widehat{\mathbf{C}}((X_1, e_1), (X_2, e_2)) = e_2 \mathbf{C}(X_1, X_2) e_1.$$

The composition rule in $\widehat{\mathbf{C}}$ is induced by that of $\mathbf{C}$. It follows that e is the identity morphism of (X, e), and that composition in $\widehat{\mathbf{C}}$ is associative and bilinear over the integers. Thus $\widehat{\mathbf{C}}$ is a preadditive category. More importantly, it is idempotent-complete. Indeed, let g be an idempotent endomorphism of (X, e) in $\widehat{\mathbf{C}}$. Then g can also be regarded as a morphism $g' \colon (X, e) \to (X, g)$ or $g'' \colon (X, g) \to (X, e)$. Then $g = g''g'$ and $g'g'' = 1_{(X,g)}$, hence g splits.

Furthermore, if $\mathbf{C}$ is additive then so is $\widehat{\mathbf{C}}$. To see this, let $\{(X_i, e_i)\}_{i<n}$ be a finite family of objects of $\widehat{\mathbf{C}}$ and let X be the biproduct of the family $\{X_i\}_{i<n}$ in $\mathbf{C}$, with canonical injections $\iota_i \colon X_i \to X$ and projections $\pi_i \colon X \to X_i$. The morphism $\varepsilon = \sum_{i<n} \iota_i e_i \pi_i$ is an idempotent endomorphism of X, and the morphisms

$$\iota_i e_i \colon (X_i, e_i) \to (X, \varepsilon)$$
$$e_i \pi_i \colon (X, \varepsilon) \to (X_i, e_i)$$

make (X, ε) the biproduct of the family $\{(X_i, e_i)\}_{i<n}$ in $\widehat{\mathbf{C}}$.

There is a full and faithful functor $\Gamma \colon \mathbf{C} \to \widehat{\mathbf{C}}$, defined on objects by $X \mapsto (X, 1_X)$ and defined as the identity on morphisms. It is clear that $X \cong Y$ if and only if $\Gamma(X) \cong \Gamma(Y)$, and that $g \in \mathbf{C}(X, Y)$ is an isomorphism if and only if so is $\Gamma(g)$.

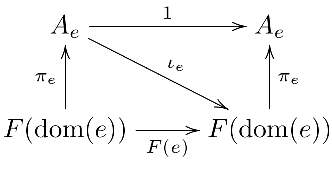
The construction of the idempotent completion is universal, as the following proposition shows.

Proposition 1.12. *Suppose* $\mathbf{L}$ *is an idempotent-complete preadditive category and* $F \colon \mathbf{C} \to \mathbf{L}$ *is an additive functor. Then there exists a functor* $G \colon \widehat{\mathbf{C}} \to \mathbf{L}$ *such that* $G\Gamma = F$ *and such* G *is unique up to natural isomorphism. This property of* Γ *is universal and characterises* $\widehat{\mathbf{C}}$ *up to category equivalence.*

Proof. The proof that the property is universal is standard, cf. the beginning of the proof of Proposition 1.8.

It is left to prove that $\Gamma \colon \mathbf{C} \to \widehat{\mathbf{C}}$ satisfies the universal property. Let us prove the existence of $G \colon \widehat{\mathbf{C}} \to \mathbf{L}$ such that $G\Gamma = F$.

For every idempotent endomorphism e in $\mathbf{C}$ that is not an identity morphism choose a "splitting diagram" in the category $\mathbf{L}$, that is, a commutative diagram like the following:



For identity morphisms, let $A_{1_X} = F(X)$ and $\pi_{1_X} = \iota_{1_X} = F(1_X) = 1_{F(X)}$.

Next define $G(X, e) = A_e$ on objects, and for a morphism $g \colon (X_1, e_1) \to (X_2, e_2)$, let $G(g) = \pi_{e_2} F(g) \iota_{e_1}$. Thus

$$G\Gamma(X) = G(X, 1_X) = A_{1_X} = F(X)$$

and

$$G\Gamma(g\colon X \to Y) = \pi_{1_Y} F(g) \iota_{1_X} = F(g).$$

It is left to prove that G is indeed a functor and that it is additive. Identity morphisms are preserved, because

$$G(1_{(X,e)}) = \pi_e F(e) \iota_e = \pi_e \iota_e \pi_e \iota_e = 1_{A_e} = 1_{G(X,e)}.$$

Consider a morphism $f\colon (X_2, e_2) \to (X_3, e_3)$. Then

$$\begin{aligned} G(fg) &= \pi_{e_3} F(fg) \iota_{e_3} \\ &= \pi_{e_3} F(f e_2 g) \iota_{e_3} \\ &= \pi_{e_3} F(f) F(e_2) F(g) \iota_{e_3} \\ &= (\pi_{e_3} F(f) \iota_{e_2})(\pi_{e_2} F(g) \iota_{e_3}) \\ &= G(f) G(g), \end{aligned}$$

hence G respect the composition rule. Finally, G is clearly additive.

Next we remark that every additive functor G such that $F = G\Gamma$ arises in this way. Indeed, for every object (X, e) of $\widehat{\mathbf{C}}$ we have a commutative diagram

$$\begin{array}{ccc} (X,e) & \longrightarrow & (X,e) \\ \uparrow & \searrow & \uparrow \\ (X,1_X) & \longrightarrow & (X,1_X) \end{array}$$

where all arrows are equal to e. Applying G we obtain

$$\begin{array}{ccc} A_e := G(X,e) & \xrightarrow{\;1\;} & G(X,e) \\ {\scriptstyle \pi_e}\uparrow & \searrow{\scriptstyle \iota_e} & \uparrow{\scriptstyle \pi_e} \\ F(X) & \xrightarrow[F(e)]{} & F(X) \end{array}$$

and this gives rise to a choice of objects A_e and morphisms π_e and ι_e as above. Moreover, applying G to the commutative square

$$\begin{array}{ccc} (X_1,e_1) & \xrightarrow{\;g\;} & (X_2,e_2) \\ {\scriptstyle e_1}\downarrow & & \uparrow{\scriptstyle e_2} \\ (X_1,1_{X_1}) & \xrightarrow[\Gamma(g)]{} & (X_2,1_{X_2}) \end{array}$$

we see that $G(g) = \pi_{e_2} F(g) \iota_{e_1}$, as in our definition.

To prove uniqueness of G, then, suppose $\{B_e, \bar{\pi}_e, \bar{\iota}_e\}$, where e ranges over all idempotent endomorphisms of $\mathbf{C}$, is another suitable choice of objects and morphisms. Then $\bar{\pi}_e \iota_e\colon A_e \to B_e$ is an isomorphism, for every idempotent endomorphism e in $\mathbf{C}$. In fact,

$$1_{A_e} = \pi_e \iota_e = (\pi_e \iota_e)^2 = \pi_e F(e) \iota_e = (\pi_e \bar{\iota}_e)(\bar{\pi}_e \iota_e),$$

and similarly $1_{B_e} = (\bar{\pi}_e \iota_e)(\pi_e \bar{\iota}_e)$. Moreover, said isomorphism is natural in e, that is, the square

$$\begin{CD} A_{e_1} @>{\bar{\pi}_{e_1}\iota_{e_1}}>> B_{e_1} \\ @V{\pi_{e_2}F(g)\iota_{e_1}}VV @VV{\pi_{e_2}F(g)\bar{\iota}_{e_1}}V \\ A_{e_2} @>>{\bar{\pi}_{e_2}\iota_{e_2}}> B_{e_2} \end{CD}$$

is commutative. Indeed,

$$\bar{\pi}_{e_2} F(g) \bar{\iota}_{e_1} \bar{\pi}_{e_1} \iota_{e_1} = \bar{\pi}_{e_2} F(g) F(e_1) \iota_{e_1} = \bar{\pi}_{e_2} F(e_2) F(g) \iota_{e_1} = \bar{\pi}_{e_2} \iota_{e_2} \pi_{e_2} F(g) \iota_{e_1},$$

as required. □

For a preadditive category $\mathbf{C}$, we can first embed it in its additive closure $\mathrm{Sums}(\mathbf{C})$, and then in the idempotent completion $\widehat{\mathrm{Sums}(\mathbf{C})}$, which is additive. Hence every preadditive category $\mathbf{C}$ is canonically a full subcategory of an idempotent-complete additive category. (Notice that order matters, that is, $\mathrm{Sums}(\hat{\mathbf{C}})$ may not be idempotent-complete.)

1.3 The Krull-Schmidt Theorem

Lemma 1.13. (Cf. [Ste75, Ch. V, Lemma 5.3].) *Let $\mathbf{C}$ be any preadditive category and X_1, X_2, Y_1, Y_2 arbitrary objects of $\mathbf{C}$. Suppose $g\colon (X_1, X_2) \to (Y_1, Y_2)$ is an isomorphism in $\mathrm{Sums}(\mathbf{C})$ and that $g_{11} \in \mathbf{C}(X_1, Y_1)$ is also an isomorphism. Then $X_2 \cong Y_2$.*

Proof. Replacing g by

$$\begin{pmatrix} 1_{Y_1} & 0 \\ -g_{21}g_{11}^{-1} & 1_{Y_2} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix},$$

where the leftmost matrix is an automorphism of (Y_1, Y_2), we see that we can assume that $g_{21} = 0$. Let $f = g^{-1}$. Then

$$\begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} * & * \\ * & g_{22}f_{22} \end{pmatrix}$$

shows that $g_{22}f_{22} = 1_{Y_2}$. Moreover,

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix} = \begin{pmatrix} * & * \\ f_{21}g_{11} & f_{21}g_{12} + f_{22}g_{22} \end{pmatrix}$$

shows that $f_{21}g_{11} = 0$, hence $f_{21} = 0$ and $f_{22}g_{22} = 1_{X_2}$. This shows that g_{22} is an isomorphism, hence $X_2 \cong Y_2$. □

The classical version of the Krull-Schmidt Theorem is proved in [Ste75, Ch. V] for Grothendieck categories, and stated in [Bas68, p. 20] and in [Fac07, Lemma 2.1] for idempotent-complete additive categories. Our version is similar to the latter, and we include a proof for the sake of completeness.

Theorem 1.14 (Krull-Schmidt Theorem). *Let $\mathbf{C}$ be an idempotent-complete additive category and $\{X_i\}_{i<n}$ a finite family of objects of $\mathbf{C}$ with local endomorphism ring.*

(i) If $A \oplus B \cong \bigoplus_{i<n} X_i$, then there is a partition $I_A \sqcup I_B = \{0, \dots, n-1\}$ such that $A \cong \bigoplus_{i \in I_A} X_i$ and $B \cong \bigoplus_{i \in I_B} X_i$.

(ii) In particular, if $\bigoplus_{i<n} X_i$ is isomorphic to the biproduct of a family $\{Y_i\}_{i<m}$ of indecomposable objects of $\mathbf{C}$, then $n = m$ and there is a permutation σ such that $X_i \cong Y_{\sigma(i)}$ for every $i < n$.

Proof. (i) Suppose $A \oplus B \cong \bigoplus_{i<n} X_i$. Write the second biproduct as $X_0 \oplus X$ with $X \cong \bigoplus_{1 \leqslant i<n} X_i$. Let $\pi_A, \iota_A, \pi_B, \iota_B$ and $\pi_0, \iota_0, \pi_X, \iota_X$ be the canonical morphisms.

Since $1_{X_0} = \pi_0\iota_A\pi_A\iota_0 + \pi_0\iota_B\pi_B\iota_0$ and the endomorphism ring $\mathbf{C}(X_0)$ is local, either $\pi_0\iota_A\pi_A\iota_0$ or $\pi_0\iota_B\pi_B\iota_0$ is an automorphism of X_0. Without loss of generality, assume that $\pi_0\iota_A\pi_A\iota_0$ is an automorphism, with inverse, say, g. Then $1_{X_0} = (g\pi_0\iota_A)(\pi_A\iota_0)$ and $(\pi_A\iota_0)(g\pi_0\iota_A)$ is an idempotent endomorphism of A. It follows that $1_A - (\pi_A\iota_0)(g\pi_0\iota_A)$ is also an idempotent endomorphism of A, hence, by Lemma 1.10, there is an object A' and morphisms $\iota' \in \mathbf{C}(A', A)$ and $\pi' \in \mathbf{C}(A, A')$ such that $\pi'\iota' = 1_{A'}$ and $1_A = (\pi_A\iota_0)(g\pi_0\iota_A) + \iota'\pi'$. In other, words, A is a biproduct of X_0 and A' with canonical injections $\pi_A\iota_0$ and ι', and canonical projections $g\pi_0\iota_A$ and π'. It follows that X is a biproduct of X_0, A' and B with injections $\iota_A\pi_A\iota_0$, $\iota_A\iota'$ and ι_B, and projections $g\pi_0\iota_A\pi_A$, $\pi'\pi_A$, and π_B. The identity morphism of X can be seen as an isomorphism $X_0 \oplus (A' \oplus B) \to X_0 \oplus (X_1 \oplus \cdots \oplus X_{n-1})$, and its top-left entry is $\pi_0(\iota_A\pi_A\iota_0)$, which is an isomorphism. By Lemma 1.13, we have $A' \oplus B \cong X_1 \oplus \cdots \oplus X_{n-1}$, hence (i) follows by induction.

(ii) Applying (i) to $A = Y_0$ and $B = Y_1 \oplus \cdots \oplus Y_{m-1}$, we get that $I_A = \{i_0\}$ is a singleton, because Y_0 is indecomposable, hence $Y_1 \oplus \cdots \oplus Y_{m-1} \cong X_0 \oplus \cdots \oplus \widehat{X_{i_0}} \oplus \cdots \oplus X_{n-1}$ and we conclude by induction. ☐

We conclude this section with a version of the Krull-Schmidt Theorem which is instrumental in the proof of the main results of Chapter 5. First we need to recall a result from combinatorics:

Theorem 1.15 (Hall's Theorem). *Let $\{S_i\}_{i \in I}$ be a finite indexed family of finite sets, and let S be their union. An injective mapping $g \colon I \to S$ such that $g(i) \in S_i$ is called a* transversal *for the family. Such a transversal exists if and only if for every $I_0 \subseteq I$ we have $|I_0| \leqslant \left|\bigcup_{i \in I_0} S_i\right|$.*

Proof. Suppose that a transversal g exists. Since g is injective and $g(I_0) \subseteq \bigcup_{i\in I_0} S_i$, we have $|I_0| \leqslant |\bigcup_{i\in I_0} S_i|$, hence the condition is necessary. We prove that it is sufficient by induction on $|I|$. The base step $|I| = 1$ is trivial. Thus assume $|I| = n \geqslant 2$.

Suppose for every non-empty proper subset I_0 of I the inequality in the statement is strict, that is, that

$$|I_0| + 1 \leqslant \left|\bigcup_{i\in I_0} S_i\right|.$$

Choose $i_0 \in I$ and $x_0 \in S_{i_0}$ arbitrarily. Suppose $I_0 \subseteq I\backslash\{i_0\}$ is non-empty. Then

$$\left|\bigcup_{i\in I_0} S_i\backslash\{x_0\}\right| \geqslant \left|\bigcup_{i\in I_0} S_i\right| - 1 \geqslant |I_0|.$$

By induction, the family $\{S_i\backslash\{x_0\}\}_{i\in I\backslash\{i_0\}}$ has a transversal g. Prolonging g by $g(i_0) = x_0$ gives a transversal for the family $\{S_i\}_{i\in I}$.

Thus we may assume that there is a non-empty proper subset I_0 of I such that

$$|I_0| = \left|\bigcup_{i\in I_0} S_i\right|.$$

By induction $\{S_i\}_{i\in I_0}$ has a transversal, say g. Let $I_1 = I\backslash I_0$. If we prove that $\{S_i\backslash g(I_0)\}_{i\in I_1}$ has a transversal f, then $g \cup f$ is a transversal for $\{S_i\}_{i\in I}$. If $I_2 \subseteq I_1$, we have that

$$\left|\bigcup_{i\in I_2} S_i\backslash g(I_0)\right| \geqslant \left|\bigcup_{i\in I_2} S_i\backslash \bigcup_{i\in I_0} S_i\right| = \left|\bigcup_{i\in I_2\cup I_0} S_i\right| - \left|\bigcup_{i\in I_0} S_i\right| \geqslant |I_2 \cup I_0| - |I_0| = |I_2|,$$

and we conclude by induction that the required transversal f exists. □

Lemma 1.16. *Let* $\mathbf{C}$ *be a preadditive category and* $\mathbf{J}$ *its Jacobson radical, as in Lemma-Definition 1.1.*

(i) Let $A, B \in \mathbf{C}$ *be such that* $B \neq 0$ *and* $\mathbf{C}(A)$ *has only the trivial idempotents. Then a morphism* $f \in \mathbf{C}(A, B)$ *is an isomorphism if and only if it has a right inverse.*

(ii) Let $f = f_1 \cdots f_n$ *be a composition of morphisms in* $\mathbf{C}$ *between non-zero objects whose endomorphism rings have only the trivial idempotents. Then* f *is an isomorphism if and only if* $f_1, \dots, f_n$ *are all isomorphisms.*

(iii) If X, Y *are objects of* $\mathbf{C}$ *such that* $\mathbf{C}(X)$ *is a local ring and* $\mathbf{C}(Y)$ *has only the trivial idempotents, then* $\mathbf{J}(X, Y)$ *is the set of non-isomorphisms.*

Proof. (i) Suppose $f\colon A \to B$ has a right inverse, say $g\colon B \to A$, so that $fg = 1_B$. Then gf is an idempotent endomorphism of A. Since $gf = 0_A$ implies

$1_B = fg = (fg)(fg) = 0_B$, which is false because $B \neq 0$, we must have $gf \neq 0_A$. Then $A \neq 0$ and, as $\mathbf{C}(A)$ has only the trivial idempotents, $gf = 1_A$, so that g and f are mutually inverse isomorphisms in $\mathbf{C}$. In particular, f is an isomorphism. The converse implication is clear.

(ii) A composition of isomorphisms is an isomorphism, so that if all $f_1, \dots, f_n$ are isomorphisms, then $f_1 \cdots f_n$ is an isomorphism. Conversely, suppose that $f_1 \cdots f_n$ is an isomorphism. To prove that $f_1, \dots, f_n$ are all isomorphisms, it suffices to prove the case $n = 2$ and use induction. From $1 = f_1 f_2 (f_1 f_2)^{-1}$ we obtain that f_1 has a right inverse, hence is an isomorphism by (i). It follows that $f_2 = f_1^{-1}(f_1 f_2)$ is also an isomorphism.

(iii) If $f\colon X \to Y$ is an isomorphism, then $1_X - f^{-1} f = 0_X$ is not invertible in $\mathbf{C}(X)$ because $X \neq 0$, thus $f \notin \mathbf{J}(X, Y)$. Conversely, if $f\colon X \to Y$ is not in the Jacobson radical, there exists $g\colon Y \to X$ be such that $1_X - gf$ is not invertible. Since $\mathbf{C}(X)$ is a local ring, gf is an automorphism of X. In particular, g has a right inverse. As $X \neq 0$ and $\mathbf{C}(Y)$ has only trivial idempotents, (i) applies to show that g is an isomorphism. Then $f = g^{-1}(gf)$ is also an isomorphism. □

Notice that, if $\mathbf{C}$ is an idempotent-complete additive category, the condition in Lemma 1.16 that the endomorphism ring of a non-zero object X of $\mathbf{C}$ has only the trivial idempotents amounts to the condition that X be an indecomposable object (Lemma 1.11). In general, for a non-zero object X of $\mathbf{C}$, we only have the implication that if $\mathbf{C}(X)$ has only the trivial idempotents then X is indecomposable.

Notation 1.17. If g is a morphism between two biproducts in some additive category, say

$$g\colon X = X_1 \oplus \cdots \oplus X_n \to Y_1 \oplus \cdots \oplus Y_m = Y,$$

we denote by $g_{ji}\colon X_i \to Y_j$ the morphism $\pi_j g \iota_i$, where $\iota_i\colon X_i \to X$ is the i-th canonical injection of the domain and $\pi_j\colon Y \to Y_j$ is the j-th canonical projection of the codomain of g. This way we do not have to explicitly allocate symbols for the canonical morphisms of the various biproducts in question.

Theorem 1.18 (Krull-Schmidt Theorem, revisited)**.** [Gir11a, Theorem 2.2] *Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ be objects with local endomorphism ring of an additive category $\mathbf{A}$. Suppose that $g\colon X_1 \oplus \cdots \oplus X_n \to Y_1 \oplus \cdots \oplus Y_m$ is an isomorphism. Then $n = m$ and there exists a permutation σ of $\{1, \dots, n\}$ such that each $g_{\sigma(i),i}\colon X_i \to Y_{\sigma(i)}$ is an isomorphism.*

Proof. Let $\mathbf{J}$ be the Jacobson radical of $\mathbf{A}$ and $Q\colon \mathbf{A} \to \mathbf{A}/\mathbf{J}$ be the canonical functor. Let $\Gamma\colon \mathbf{A}/\mathbf{J} \to \widehat{\mathbf{A}/\mathbf{J}}$ be the canonical additive full and faithful functor into the idempotent completion $\widehat{\mathbf{A}/\mathbf{J}}$ of $\mathbf{A}/\mathbf{J}$. For all morphisms α in $\mathbf{A}$ between objects with local endomorphism ring, α is an isomorphism if and only if $Q(\alpha) \neq 0$ by Lemma 1.16(iii), if and only if $\Gamma Q(\alpha) \neq 0$.

By the classical Krull-Schmidt Theorem 1.14 for idempotent-complete additive categories, and because $\Gamma Q(g)$ is an isomorphism, we deduce in particular that $n = m$.

View g as an $n \times n$ matrix, where $g_{ji}\colon X_i \to Y_j$. For each $j = 1, \dots, n$ let S_j be the set of indices $i = 1, \dots, n$ such that g_{ij} is an isomorphism. Consider the collection of sets $\{S_i\}_{i=1,\dots,n}$. We need to pick an element $\sigma(j)$ from S_j for each $j = 1, \dots, n$ in such a way that $\sigma(j) \neq \sigma(k)$ if $j \neq k$. By Hall's Theorem 1.15, this can be done if and only if $|I| \leqslant |\bigcup_{i\in I} S_i|$ for all $I \subseteq \{1, \dots, n\}$. Assume by contradiction that for some $\bar{I}$ we have $|\bar{I}| > |\bigcup_{i\in\bar{I}} S_i|$. Without loss of generality we may assume that $\bar{I} = \{1, \dots, r\}$ and that $\{1, \dots, n\}\backslash(S_1 \cup \cdots \cup S_r) = \{s + 1, \dots, n\}$ with $0 \leqslant s < r \leqslant n$. Thus we can write g in block matrix form as

$$g = \begin{pmatrix} \alpha & * \\ \alpha' & * \end{pmatrix}$$

where $\alpha\colon X_1 \oplus \cdots \oplus X_r \to Y_1 \oplus \cdots \oplus Y_s$ and $\alpha'\colon X_1 \oplus \cdots \oplus X_r \to Y_{s+1} \oplus \cdots \oplus Y_n$ is such that $\Gamma Q(\alpha') = 0$, and similarly we write

$$g^{-1} = \begin{pmatrix} \beta & \beta' \\ * & * \end{pmatrix}$$

where $\beta\colon Y_1 \oplus \cdots \oplus Y_s \to X_1 \oplus \cdots \oplus X_r$ and $\beta'\colon Y_{s+1} \oplus \cdots \oplus Y_n \to X_1 \oplus \cdots \oplus X_r$. Computing the top left entry of $g^{-1}g$ we have $1 = \beta\alpha + \beta'\alpha'$, from which $1 = \Gamma Q(\beta\alpha)$. Hence $\Gamma Q(\beta\alpha)$ is an automorphism of $\Gamma Q(X_1) \oplus \cdots \oplus \Gamma Q(X_r)$ which factors through $\Gamma Q(Y_1) \oplus \cdots \oplus \Gamma Q(Y_s)$. Since idempotents split in $\widehat{\mathbf{A}/\mathbf{J}}$, we conclude by Lemma 1.11 that $\Gamma Q(X_1) \oplus \cdots \oplus \Gamma Q(X_r)$ is a biproduct factor of $\Gamma Q(Y_1) \oplus \cdots \oplus \Gamma Q(Y_s)$. By the classical Krull-Schmidt Theorem 1.14, it follows that $r \leqslant s$, contradiction. □

1.4 The Chinese Remainder Theorem

In this section we develop a generalisation of the Chinese Remainder Theorem to rings with many objects, i.e., to preadditive categories. This result is implicitly used in the proof of the main theorems of [FP10, FP09b].

For the sake of completeness, we include, and begin with, a version of the Chinese Remainder Theorem for non-commutative rings. (The only non-commutative version known to the author is the one sketched in [Hun80, Theorem 2.25], for *rüngs,* i.e., rings that are not required to have an identity element.)

Recall that two ideals A and B of a ring R are called *comaximal* if $A + B = R$.

Theorem 1.19 (C.R.T. I). *Let R be a ring and $\{I_i\}_{i<n}$ a finite collection of ideals of R. Consider the canonical injective ring morphism*

$$p\colon R/\bigcap_{i<n} I_i \to \prod_{i<n} R/I_i,$$

which maps $r \in R$ to the n-tuple $(r + I_i)_{i<n}$. The following are equivalent:

(i) The ideals $I_0, \dots, I_{n-1}$ are pairwise comaximal.

(ii) The map p is an isomorphism.

Proof. If (ii) holds and $i < j < n$, there is an element $r \in R$ such that $r + I_i = 1 + I_i$ and $r + I_j = 0 + I_j$, that is, $r \in I_j$ and $1 - r \in I_i$. Thus $1 \in I_i + I_j$, and (i) holds.

Let us prove that (i) implies (ii) by induction on n. The case $n = 1$ is trivial and the case $n = 2$ follows from the definition of comaximality. To give a proof by induction it then suffices to show that I_1 and $I_2 \cap \cdots \cap I_n$ are comaximal, and use the canonical commutative diagram

$$\begin{array}{ccc} R/(I_1 \cap \cdots \cap I_n) & \xrightarrow{\ p\ } & R/I_1 \times \cdots \times R/I_n \\ \downarrow & & \Vert \\ R/I_1 \times R/(I_2 \cap \cdots \cap I_n) & \longrightarrow & R/I_1 \times \cdots \times R/I_n \end{array}$$

Thus we turn to showing by induction on $i = 2, \dots, n$ that

$$I_1 + (I_2 \cap \cdots \cap I_i) = R$$

The case $i = 2$ is trivial. If $i > 2$, then

$$\begin{aligned} R &= R^2 \\ &= (I_1 + (I_2 \cap \cdots \cap I_{i-1}))(I_1 + I_i) \\ &\subseteq I_1 + (I_2 \cap \cdots \cap I_i) \\ &\subseteq R \end{aligned}$$

from which the conclusion follows. □

Definition 1.20. If $\mathbf{A}$ and $\mathbf{B}$ are ideals of a preadditive category $\mathbf{C}$, we say that they are *comaximal* if $\mathbf{A} + \mathbf{B}$ is the improper ideal of $\mathbf{C}$, equivalently, if $\mathbf{A}(X)$ and $\mathbf{B}(X)$ are comaximal ideals of the ring $\mathbf{C}(X)$, for every object X of $\mathbf{C}$.

Definition 1.21. Recall that, in any preadditive category, when we have an equality $1_A = gf$ where $f\colon A \to B$ and $g\colon B \to A$, we say that A is a *retract* of B, g is a *retraction* of f, and f is a *section* of g [Bor94a].

A functor $F\colon \mathbf{C} \to \mathbf{D}$ is called *isomorphism-reflecting* , or we say that F reflects isomorphisms, if, for every pair of objects X and Y of $\mathbf{C}$, we have that $X \cong Y$ if $F(X) \cong F(Y)$ [Fac07]. Similar notions were introduced already in [Kel64].

The functor F is said to be *retract-reflecting* if, for every pair of objects X and Y of $\mathbf{C}$, we have that X is a retract of Y whenever $F(X)$ is a retract of $F(Y)$. In other words, for every pair of objects X and Y of $\mathbf{C}$, the identity morphism of X factors through Y if the identity morphism of $F(X)$ factors through $F(Y)$. When this happens we may also say that F reflects retracts. This notion is weaker than the notion of a "functor which reflects direct summands," given in [Fac07] for additive categories. Indeed, consider the additive category $\mathbf{C}$ of all vector spaces over a field and let $\mathbf{C}'$ be the full subcategory of $\mathbf{C}$ consisting of all the vector spaces of dimension $\neq 1$, so that $\mathbf{C}'$ is also additive. Then the inclusion of $\mathbf{C}'$ in $\mathbf{C}$ reflects retracts but not direct summands. For instance K^2 and K^3 are objects of $\mathbf{C}'$, and obviously K^2 is a direct summand of K^3 in $\mathbf{C}$, but not in $\mathbf{C}'$.

Theorem 1.22 (C.R.T. II). *Let $\mathbf{C}$ be a preadditive category and $\{\mathbf{I}_i\}_{i<n}$ a finite collection of ideals of $\mathbf{C}$. Then we have a canonical faithful additive functor*

$$F\colon \mathbf{C}/\bigcap_{i<n}\mathbf{I}_i \to \prod_{i<n}\mathbf{C}/\mathbf{I}_i.$$

The following are equivalent:

(i) The ideals $\mathbf{I}_0,\dots,\mathbf{I}_{n-1}$ are pairwise comaximal.

(ii) The functor F is also full.

If these equivalent conditions hold, then F reflects isomorphisms and retracts.

Let us note that the above theorem does not grant, in general, a category equivalence, because the canonical functor F may not be dense. To see an example, please consult Section 7.2.

Proof. If (ii) holds, for each object X of $\mathbf{C}$ we have that the canonical ring morphism

$$\mathbf{C}(X)/\bigcap_{i<n}\mathbf{I}_i(X) \to \prod_{i<n}\mathbf{C}(X)/\mathbf{I}_i(X) \tag{1.23}$$

is an isomorphism, and by Theorem 1.19 this implies that the ideals $\{\mathbf{I}_i(X)\}_{i<n}$ of $\mathbf{C}(X)$ are pairwise comaximal. Since X is arbitrary, this gives (i).

Suppose now that (i) holds. Let X and Y be fixed objects of $\mathbf{C}$. To prove (ii), we need to show that

$$F\colon \mathbf{C}(X,Y) \to \prod_{i<n}\mathbf{C}(X,Y)/\mathbf{I}_i(X,Y)$$

is surjective. By Theorem 1.19, we have that the ring morphism (1.23) is an isomorphism. Hence, for each $i<n$, there is an endomorphism h_i of X such that $P_iF(h_i) = 1_X + \mathbf{I}_i(X)$ and $P_jF(h_i) = 0$ for $j \neq i$. Hence for a given tuple $(g_i + \mathbf{I}_i(X,Y))_{i<n}$ we have that

$$P_iF\left(\sum_{j<n} g_jh_j\right) = \sum_{j<n} P_iF(g_j)P_iF(h_j) = P_iF(g_i) = g_i + \mathbf{I}_i(X,Y),$$

as required.

Assume that the two equivalent conditions hold and that $F(X) \cong F(Y)$. Then there is an isomorphism $\eta\colon F(X) \to F(Y)$. Since F is full, we may choose $f\colon X \to Y$ and $g\colon Y \to X$ such that $F(f) = \eta$ and $F(g) = \eta^{-1}$. Then $F(fg) = 1$ and $F(gf) = 1$, so that $fg = 1$ and $gf = 1$, because F is faithful, hence $X \cong Y$.

Suppose $1_{F(X)} = \beta\alpha$ with $\alpha\colon F(X) \to F(Y)$ and $\beta\colon F(Y) \to F(X)$. Since F is full, $\alpha = F(g)$ and $\beta = F(f)$ for suitable morphisms $g\colon X \to Y$ and $f\colon Y \to X$. Therefore $F(1_X - fg) = 0$, hence $1_X = fg$, because F is faithful. □

Conditions (i) and (ii) of Theorem 1.22 are not equivalent to "F is a category equivalence." Indeed, even if F is faithful and full, it is not necessarily a category equivalence, because it may not be dense. We will give an example later on (page 135) when the necessary notions will have been introduced.

We conclude with an easy generalisation of the Chinese Remainder Theorem for preadditive categories:

Theorem 1.24 (C.R.T. III). *Let $\mathbf{C}$ be a preadditive category and $\{\mathbf{I}_i\}_{i<\kappa}$ a family of pairwise comaximal ideals of $\mathbf{C}$, and suppose that for every object X of $\mathbf{C}$ the set $\operatorname{supp}(X) = \{i < \kappa : \mathbf{I}_i(X)$ is proper $\}$ is finite. Then the canonical faithful additive functor*

$$F\colon \mathbf{C}/\bigcap_{i<\kappa} \mathbf{I}_i \to \prod_{i<\kappa} \mathbf{C}/\mathbf{I}_i$$

is also full. As a consequence, F reflects isomorphisms and retracts.

Proof. This is a straightforward generalisation of the previous version, Theorem 1.22. Indeed, for any two objects X and Y of $\mathbf{C}$, let $S = \operatorname{supp}(X) \cap \operatorname{supp}(Y)$. We have the following canonical commutative diagram

$$\begin{array}{ccc} \mathbf{C}(X,Y)/\bigcap_{i\in S} \mathbf{I}_i(X,Y) & \longrightarrow & \prod_{i\in S} \mathbf{C}(X,Y)/\mathbf{I}_i(X,Y) \\ \| & & \downarrow \\ \mathbf{C}(X,Y)/\bigcap_{i<\kappa} \mathbf{I}_i(X,Y) & \longrightarrow & \prod_{i<\kappa} \mathbf{C}(X,Y)/\mathbf{I}_i(X,Y) \end{array}$$

where the top arrow is an abelian group isomorphism by Theorem 1.22. If $i < \kappa$ is not in S, then either $1_X \in \mathbf{I}_i(X)$ or $1_Y \in \mathbf{I}_i(Y)$, and in both cases $\mathbf{I}_i(X,Y)$ is improper; this justifies the leftmost vertical identification and shows that the rightmost vertical arrow is an isomorphism. It now follows that the bottom arrow is an isomorphism as well. That F reflects isomorphisms and retracts follows as in the proof of Theorem 1.22. □

Chapter 2

Semilocal categories

In this chapter we first recall some notions from ring theory and module theory. More precisely, we define the concepts of Goldie dimension of a module and of dual Goldie dimension of modules and rings, and explain their arithmetical properties. We define semisimple modules and rings, and semilocal rings, and we collect several characterisations from the literature.

The salient point of this chapter is a result of [FP10], which pertains categories $\mathbf{C}$ that have at least a non-zero object, and such that for every non-zero object X of $\mathbf{C}$ the endomorphism ring $\mathbf{C}(X)$ is semilocal. There exists an additive functor $F\colon \mathbf{C}/\mathbf{J} \to \prod_{\mathbf{I}\in \mathrm{Prim}(\mathbf{C})} \mathbf{C}/\mathbf{I}$ with the property that if $F(X) \cong F(Y)$ then $X \cong Y$ (the functor F reflects isomorphisms), and if $1_{F(X)}$ factors through $F(Y)$, then 1_X factors through Y (the functor F reflects retracts). Thus objects of $\mathbf{C}$ have a full class of invariants, their isomorphism classes modulo the ideals $\mathbf{I}$. This is a result that we will use in the study of Auslander-Bridger modules (Chapter 4).

In the last part of the chapter we concentrate on the objects of finite type of a preadditive category $\mathbf{A}$, that is, objects whose endomorphism rings E have the property that $E/J(E)$ is isomorphic to a finite product of division rings. Thus the full subcategory $\mathbf{C}$ of $\mathbf{A}$ whose objects are of finite type is semilocal (provided it has a non-zero object), hence a canonical isomorphism- and retracts-reflecting functor F as above exists. Some more can be said about isomorphism of indecomposable objects of $\mathbf{C}$ in this special case (Corollary 2.25). These results are a straightforward generalisation of those in [FP09b] for categories of modules, and they will be used in the study of biproducts of objects of finite type in Chapter 5.

2.1 Goldie dimensions and semilocal rings

In this section we introduce the arithmetical properties of the Goldie dimension and the dual Goldie dimension of modular lattices and the facts about semilocal rings that are most relevant for this thesis.

The notion of Goldie dimension for modules and rings was introduced in [Gol58, Gol60] and generalised to arbitrary modular lattices in [GP84]. The notion of dual Goldie dimension was introduced in [Var79]. The most important properties of semilocal rings and results about them are originally found in [CD93], [Bas64] and [Eva73]. All this information is collected in [Fac98, 2.6–8] and [Fac98, Ch. 4].

2.1.1 Goldie dimension of a modular lattice

Both the notions of Goldie dimension and dual Goldie dimension of a module are specialisations of the notion of Goldie dimension of a *complete modular lattice* L, that is, a lattice satisfying the modular identity, which is the property

$$c \leqslant a \implies a \wedge (b \vee c) = (a \wedge b) \vee c,$$

and with a greatest element 1 and a smallest element 0. (These can always be added.) For elements a and b of L such that $a \leqslant b$, we denote by $[a, b]$ the *interval* from a to b, that is, the set of elements x of L such that $a \leqslant x \leqslant b$.

The central notion here is that of a *join-independent* subset of non-zero elements of L. For a finite subset $A = \{a_i\}_{i<n}$ of non-zero elements of L, we say that A is join-independent if $a_i \wedge (\bigvee_{i \neq j < n} a_j) = 0$ for every $i < n$. For an arbitrary family A of non-zero elements of L, we say that A is join-independent if all its finite subsets are. Thus, the notion of join-independence is finitary by its very definition. An element a of L is said to be *essential* if it satisfies the property

$$a \wedge x = 0 \implies x = 0,$$

while a is called *uniform* if it is non-zero and satisfies

$$0 \neq x, y \leqslant a \implies x \wedge y \neq 0.$$

In other words, the interval $[0, a]$, seen as a complete modular lattice in its own right, has the property that every non-zero element is essential. Here is the main theorem of this section:

Theorem 2.1. [GP84] *For a complete modular lattice L the following are equivalent:*

(i) There are no join-independent subsets of L of infinite cardinality.

(ii) The cardinality of an arbitrary join-independent subset of L is bounded by some $m < \omega$.

(iii) There is a finite join-independent subset $\{a_i\}_{i<n}$ of L such that $\bigvee_{i<n} a_i$ is an essential element and all the elements a_i are uniform.

When the equivalent conditions of Theorem 2.1 are satisfied, we say that L has *Goldie dimension* n (the integer from condition (iii)), and denote this fact by $\dim(L) = n$. In all other cases we agree that $\dim(L) = \infty$, so that $\dim$ is a function taking values in $\mathbb{N} \cup \{\infty\}$, with the usual conventions that $n \leqslant \infty$ and $n + \infty = \infty + \infty = \infty$ for all $n < \omega$.

2.1.2 Goldie dimension of a module

The Goldie dimension of a module M is defined as the Goldie dimension of its lattice of submodules. Uniform elements translate to non-zero uniform submodules. More explicitly, a module U is *uniform* if it is non-zero and the intersection of non-zero submodules of U is non-zero, equivalently, if all non-zero submodules of U are essential submodules. An essential element translates to an essential submodule. A subset $\{M_i\}_{i<\kappa}$ of non-zero submodules of M is a join-independent subset of the lattice of submodules of M if and only if the sum $\sum_{i<\kappa} M_i$ is in fact direct. Therefore, for a module M, if there exists a non-negative integer n such that M contains an essential submodule of the form $\bigoplus_{i<n} U_i$ where each U_i is uniform, then we say that M has *Goldie dimension* n and denote this fact by $\dim(M) = n$. Such non-negative integer n may not exist, and (*) this is the case precisely when M contains as submodules direct sums of arbitrarily many submodules, as it follows by specialising Theorem 2.1. When this is the case, we say that M has infinite Goldie dimension and write $\dim(M) = \infty$. Thus the dimension $\dim(M)$ is defined for every module M and is an element of the linearly ordered set $\mathbb{N} \cup \{\infty\}$, with the usual convention that $n < \infty$ for every $n < \omega$, and that anything added to ∞ yields ∞. From the definition and property (*) one obtains easily that

Proposition 2.2. *The Goldie dimension enjoys the following properties:*

(i) $\dim(M) = 0$ *if and only if* $M = 0$.

(ii) $\dim(M) = 1$ *if and only if M is uniform.*

(iii) If $M \leqslant_e N$, then $\dim(M) = \dim(N)$.

(iv) $\dim(M) = \dim(E(M))$.

(v) An injective module has finite Goldie dimension n if and only if it is a direct sum of n uniform submodules.

(vi) *A module* M *has finite Goldie dimension* n *if and only if* $E(M) = \bigoplus_{i<n} E_i$ *where each* $E_i \leqslant E$ *is uniform.*

(vii) *If* $M = A \oplus B$*, then* $\dim(M) = \dim(A) + \dim(B)$.

(viii) *If* $M \leqslant N$*, then* $\dim(M) \leqslant \dim(N)$.

Proof. The definition proves (i) and (ii) (a module with a uniform essential submodule is uniform). For (iii), if $\dim(M)$ is finite then $\dim(M) = \dim(N)$ follows by the definition, while if $\dim(M)$ is infinite, it follows from (*). Since $M \leqslant_e E(M)$, (iv) is a special case of (iii).

Let us prove property (v). If E is injective and $\dim(E) = n < \omega$, then there is a direct sum $\bigoplus_{i<n} U_i$ with each U_i uniform which is essential in E. This implies that $E \cong \bigoplus_{i<n} E(U_i)$. An injective module is uniform if and only if it is the injective envelope of a uniform module [Fac98, Lemma 2.24], hence each $E(U_i)$ is uniform. The converse implication holds by the definition.

Property (vi) follows at once from (iv) and (v).

If A or B have infinite dimension, then (vii) follows from (*). If they have finite dimensions a and b respectively, we have that $E(M) \cong E(A) \oplus E(B)$ is the direct sum of $a + b$ uniform modules by (iii) and (v), hence $\dim(M) = a + b$ by (iii).

To prove (viii), notice that if $\dim(M) = \infty$ then $\dim(N) = \infty$ by (*), and if $\dim(N) = \infty$ then the inequality holds trivially. Hence we can assume that $M \leqslant N$ and both M and N have finite dimension m and n respectively. Since $E(M)$ is a direct summand of $E(N)$, we have that $m = \dim(E(M)) \leqslant \dim(E(N)) = n$ by (iii) and (vii). □

As an example of how the Goldie dimension could be used, suppose E is an injective module of finite Goldie dimension, M is any module with $\dim(M) = \dim(E)$, and $\varphi\colon E \to M$ is a morphism. Then if φ is injective, it is an isomorphism. Indeed, since E is injective, φ splits, that is, there is $\psi\colon M \to E$ such that $\psi\varphi = 1$. Therefore $M = \varphi(E) \oplus \ker(\psi)$. Hence $\dim(M) = \dim(E) + \dim(\ker(\psi))$, from which $\ker(\psi) = 0$. In particular:

Lemma 2.3. *An endomorphism of an injective module of finite Goldie dimension is an automorphism if and only if it is injective.*

2.1.3 Dual Goldie dimension of a modular lattice

To any lattice L one may associate its dual lattice L^{op}, obtained by reversing the partial order, exchanging supremum with infimum. If L is a complete modular lattice, its dual is still complete and modular, where maximum and minimum are swapped, that is, $1_{L^{\mathrm{op}}} = 0_L$ and $0_{L^{\mathrm{op}}} = 1_L$. The dual Goldie dimension of a lattice L is defined as the Goldie dimension of the dual lattice L^{op}. Let us be more explicit.

A finite subset $\{a_i\}_{i<n}$ of $L\backslash\{1\}$ is *meet-independent* if $a_i \vee \left(\bigwedge_{i\neq j<n} a_j\right) = 1$ for every $i < n$. An arbitrary subset $A \subseteq L\backslash\{1\}$ is meet-independent if every finite subset of A is. Thus meet-independence is a finitary notion, dual to that of join-independence. An element a of L is *superfluous* if we have

$$a + x = 1 \implies x = 1$$

for every $x \in L$. A lattice L' is *couniform* if

$$x, y \neq 1 \implies x + y \neq 1,$$

for every $x, y \in L'$. Thus "superfluous" is the dual of "essential" and "$[a, 1]$ is couniform" dualises "$[0, a]$ is uniform." We have the dual of Theorem 2.1:

Theorem 2.4. *For a complete modular lattice L, the following are equivalent:*

(i) All meet-independent subsets of L are finite.

(ii) There exists $m < \omega$ such that, for every meet-independent subset A of L, we have that $|A| \leqslant m$.

(iii) There is a finite meet-independent subset $\{a_i\}_{i<n}$ of L such that $\bigwedge_{i<n} a_i$ is superfluous and each $[a_i, 1]$ is couniform.

When the equivalent conditions of Theorem 2.4 are satisfied, we say that L has *dual Goldie dimension* n (the integer from condition (iii)), and denote this fact by $\operatorname{codim}(L) = n$. Otherwise we set $\operatorname{codim}(L) = \infty$ and agree on the usual arithmetic rules as for the dimension.

2.1.4 Dual Goldie dimension of a module

The first specialisation of the general notion of dual Goldie dimension is to lattices of submodules. The dual Goldie dimension of a module M is defined as the dual Goldie dimension of its lattice of submodules. Elements a such that $[a, 1]$ is a couniform lattice translate to submodules A of M such that M/A is couniform (= non-zero and the sum of two proper submodules is a proper submodule). A superfluous element a of the lattice L translates to a superfluous submodule A of M. A subset $\{M_i\}_{i<\kappa}$ of non-zero submodules of M is a meet-independent subset of the lattice of submodules of M if and only if the canonical morphism $M \to \bigoplus_{i<\kappa} M/M_i$ is onto. In the context of modules, we prefer the term *coindependent* subset rather than *meet-independent* subset. Therefore, for a module M, if there exists a non-negative integer n such that M contains a superfluous submodule K such that M/K is a direct sum of n couniform submodules, then we say that M has *dual Goldie dimension* n and denote this fact by $\operatorname{codim}(M) = n$. Such non-negative integer n may not exist, and (**) this is the case precisely when M contains arbitrarily large coindependent subsets of submodules, as it follows by specialising Theorem 2.4. When this is the case, we say that M has infinite dual Goldie dimension and write $\operatorname{codim}(M) = \infty$. The codimension function assigns to a module M an element of the linearly ordered set $\mathbb{N} \cup \{\infty\}$ with the usual arithmetic conventions, as for the dimension. We have the analogue of Proposition 2.2, though both the statement and its proof are slightly different due to the potential lack of projective covers.

Proposition 2.5. *The dual Goldie dimension enjoys the following properties:*

(i) $\operatorname{codim}(M) = 0$ *if and only if* $M = 0$.

(ii) $\operatorname{codim}(M) = 1$ *if and only if* M *is couniform.*

(iii) If $X \leqslant_s M$*, then* $\operatorname{codim}(M/X) = \operatorname{codim}(M)$.

(iv) If M *has a projective cover* P*, then* $\operatorname{codim}(M) = \operatorname{codim}(P)$.

(v) If $M = A \oplus B$*, then* $\operatorname{codim}(M) = \operatorname{codim}(A) + \operatorname{codim}(B)$.

(vi) If $X \leqslant M$*, then* $\operatorname{codim}(M/X) \leqslant \operatorname{codim}(M)$.

Proof. Properties (i) and (ii) follow from the definition (a module M with a superfluous submodule A such that M/A is couniform is couniform).

If $\{N_i/X\}_{i<n}$ is a coindependent family of n submodules of M/X, then $\{N_i\}_{i<n}$ is a coindependent family of n submodules of M. Therefore (vi) follows from (**).

Suppose that $X \leqslant_s M$. If $\operatorname{codim}(M/X)$ is infinite, then $\operatorname{codim}(M)$ is infinite as well, by (vi). Then suppose $\operatorname{codim}(M/X) = n < \omega$. There is a superfluous submodule K/X of M/X such that $(M/X)/(K/X) \cong M/K$ is isomorphic to a direct sum of n couniform modules. From $X \leqslant_s M$ and $K/X \leqslant_s M/X$ we deduce $K \leqslant_s M$, hence also $\operatorname{codim}(M) = n$. This proves (iii).

Inasmuch as it is a particular case of (iii), property (iv) holds.

Let us prove (v). If A or B has infinite codimension, then so has M, by (vi). Then assume A and B have finite codimensions a and b respectively. There are $K_A \leqslant_s A$ and $K_B \leqslant_s B$ such that A/K_A and B/K_B are the direct sum of a and b couniform submodules respectively. Then $M/(K_A \oplus K_B) \cong A/K_A \oplus B/K_B$ is the direct sum of $a + b$ couniform submodules and $K_A \oplus K_B \leqslant_s M$. Thus $\operatorname{codim}(M) = a + b$. □

Let $g\colon M \to P$ be a morphism of M into a projective module P, and suppose $\operatorname{codim}(M) = \operatorname{codim}(P) < \omega$. If g is surjective, then g is an isomorphism. Indeed, if g is surjective, it splits, that is, there is $f\colon P \to M$ such that $gf = 1_P$. Thus $M = f(P) \oplus \ker(g)$. Since f is injective, we have $P \cong f(P)$, hence $\operatorname{codim}(f(P)) = \operatorname{codim}(M)$ and $\operatorname{codim}(\ker(g)) = 0$, so that g is also injective. In particular:

Lemma 2.6. *An endomorphism of a projective module of finite dual Goldie dimension is an automorphism if and only if it is surjective.*

2.1.5 Semilocal rings

Recall that a module M is *simple* if its lattice of submodules is trivial, that is, its only submodules are 0 and M. The module M is called *semisimple* if every submodule of M is a direct summand of M. It is well-known that a module M is semisimple if and only if it is a sum of simple submodules, if and only if it is a direct sum of simple submodules. The direct-sum decomposition into simple submodules of a semisimple module is unique, i.e., if $\{M_i\}_{i\in I}$ and $\{M'_i\}_{i\in I'}$ are sets of simple modules, then the direct sums $\bigoplus_{i\in I} M_i$ and $\bigoplus_{i\in I'} M'_i$ are isomorphic if and only if there is a bijection $\sigma\colon I \to I'$ such that $M_i \cong M_{\sigma(i)}$ for all $i \in I$. Submodules and quotients of semisimple modules are semisimple. Cf. [AF92, §9].

If a ring R is such that R_R is a semisimple module, then R is said to be a *semisimple ring.*[I] There are many characterisations of semisimple rings: R is semisimple if and only if every right R-module is semisimple, if and only if every right R-module is projective, if and only if every right R-module is injective, if and only if every short exact sequence of right R-modules splits, if and only if R is right artinian and $J(R) = 0$. The most important characterisation is due to Wedderburn-Artin: A ring R is semisimple if and only if it is isomorphic to a finite product of matrix rings over division rings, that is, $R \cong \prod_{i<n} M_{n_i}(D_i)$, where each D_i is a division ring, and $n, n_i < \omega$. This last condition is left-right symmetric, hence all the left versions of the conditions above also characterise semisimple rings. For details about semisimple rings, see [AF92, §13].

A ring R is a *semilocal* ring if $R/J(R)$ is a semisimple ring. The connection between semilocal rings and the theory of dimensions is that a ring R is semilocal if and only if $\operatorname{codim}(R_R)$ is finite, if and only if $\operatorname{codim}({}_RR)$ is finite. If R is semilocal, $\operatorname{codim}(R_R) = \operatorname{codim}({}_RR) = \operatorname{codim}({}_RR/J(R)) = \operatorname{codim}(R_R/J(R))$. Cf. [Fac98, Proposition 2.43]. This common codimension is denoted $\operatorname{codim}(R)$.

Lemma 2.7. *The codimension for rings is additive, in the sense that* $\operatorname{codim}(R_1 \times R_2) = \operatorname{codim}(R_1) + \operatorname{codim}(R_2)$. *Moreover, if I is an ideal of an arbitrary ring R, then* $\operatorname{codim}(R/I) \leqslant \operatorname{codim}(R)$, *with equality if* $I \leqslant J(R)$. *Thus the class of semilocal rings is closed by finite products and by quotients.*

Proof. Suppose $R = R_1 \times R_2$. Consider the central orthogonal idempotents e_1 and e_2 such that $1 = e_1 + e_2$, and $R_i \cong e_iRe_i = e_iR$. Then $\operatorname{codim}(R) = \operatorname{codim}(R_R) = \operatorname{codim}(e_1R_R) + \operatorname{codim}(e_2R_R)$. Notice that an additive subgroup

[I] We note that some authors call these rings "semisimple *artinian* rings."

of e_iR is an R-submodule if and only if it is an e_iR-submodule, hence it follows that $\operatorname{codim}(e_iR_R) = \operatorname{codim}(e_iR_{e_iR}) = \operatorname{codim}(e_iR)$.

If I is an ideal of an arbitrary ring R, then $\operatorname{codim}(R/I) = \operatorname{codim}((R/I)_{R/I})$ by definition. Since the lattice of R/I-submodules of R/I coincides with the lattice of R-submodules, $\operatorname{codim}(R/I) = \operatorname{codim}((R/I)_R) \leqslant \operatorname{codim}(R_R) = \operatorname{codim}(R)$, by Proposition 2.5(vi), where the inequality is an equality if $I \leqslant J(R) \leqslant_s R_R$, by Proposition 2.5(iii).

The last assertion of the statement stems from the fact that a ring R is semilocal if and only if $\operatorname{codim}(R)$ is finite, as recalled above from [Fac98, Proposition 2.43]. □

For instance, a matrix ring $R = M_n(D)$ over a division ring D has codimension n. If e_i denotes the square matrix of order n whose only non-zero entry is the i-i entry, one sees that $R_R = e_1R \oplus \cdots \oplus e_nR$, and each e_iR is a simple (hence couniform) right ideal. Thus $\operatorname{codim}(M_n(D)) = n$.

Lemma 2.8. *A semilocal ring R has at most $\operatorname{codim}(R)$ maximal two-sided ideals, and their intersection is $J(R)$.*

Proof. Suppose M is a maximal two-sided ideal of R. Then $M \subseteq N_R$ for some maximal right ideal N_R. Thus $M \subseteq \operatorname{r.ann}_R(R_R/N_R)$, and equality holds by the maximality of M. Since R_R/N_R is simple and $J(R)$ is the intersection of all annihilators of simple right R-modules, $J(R) \subseteq M$. This proves that $J(R)$ is contained in the intersection of all the maximal two-sided ideals of R. Factoring out the Jacobson radical, it now remains to prove that the lemma is true for semisimple rings. Now a semisimple ring S is isomorphic to a finite product of matrix rings over division rings $M_{d_1}(K_1) \times \cdots \times M_{d_n}(K_n)$ say. Each $M_{d_i}(K_i)$ is a simple and artinian ring of dual Goldie dimension d_i, as we calculated above. Thus S has dimension $d_1 + \cdots + d_n$ (Lemma 2.7), has exactly 2^n two-sided ideals, n of which are maximal, and the intersection of the maximal two-sided ideals is zero. This allows us to conclude. □

A ring morphism $g\colon R \to S$ is said to be *local* if, for every $r \in R$, we have $r \in U(R)$ if $g(r) \in U(S)$. Here is a very important result about semilocal rings:

Theorem 2.9 (Camps and Dicks)**.** *Suppose $R \to S$ is a local ring morphism. Then $\operatorname{codim}(R) \leqslant \operatorname{codim}(S)$. Thus if S is semilocal, so is R.* (Cf. [CD93, Theorem 1].)

If $\mathbf{C}$ is any additive category, then the class of objects X of $\mathbf{C}$ such that $\mathbf{C}(X)$ is a semilocal ring is closed under the formation of biproducts. To prove it, we need the following:

Lemma 2.10. [SV79, Theorems 2.3 and 2.5] *Let R be a ring and P a finitely generated projective right R-module. Then $\operatorname{codim}(P) = \operatorname{codim}(\operatorname{End}_R(P))$. In other words, the dual Goldie dimension (as a module) of P equals the dual Goldie dimension (as a ring) of its endomorphism ring.*

Proof. Suppose first that $J(R) = 0$. Assume that P has finite dual Goldie dimension n. Then there is a surjective R-morphism $g\colon P \to \bigoplus_{i<n} C_i$ with each C_i couniform and $\ker(g) \leqslant_s P$. Since $J(R) = 0$, we have $\mathrm{Rad}(P) = PJ(R) = 0$, from which $\ker(g) = 0$, hence g is an isomorphism. Since C_i is couniform, it has at most one maximal submodule. Because C_i is projective, $\mathrm{Rad}(C_i) = C_iJ(R) = 0$. Since the radical of a module is the intersection of its maximal submodules, we conclude that 0 is the maximal submodule of C_i, hence C_i is simple. Thus P is a semisimple module of length n, and rearranging the factors in the direct-sum decomposition of P, we obtain $P \cong \bigoplus_{\mu<m} S_\mu^{n_\mu}$ with $\sum_{\mu<m} n_\mu = n$ and the modules S_μ pairwise non-isomorphic simple modules. Thus $\mathrm{End}_R(P) \cong \prod_{\mu<m} M_{n_\mu}(\mathrm{End}_R(S_\mu))$ is a semisimple ring, whose codimension is n by Lemma 2.7 and the remark before Lemma 2.8.

Conversely, let us assume that $S = \mathrm{End}_R(P)$ has finite dual Goldie dimension n. Recall that $\varphi \in J(S)$ if and only if $\varphi(P) \leqslant_s P$ [AF92, Proposition 18.20], that is, if and only if $\varphi(P) \leqslant PJ(R) = 0$. Thus $J(S) = 0$. Since $\mathrm{codim}(S) = \mathrm{codim}(S_S)$, we have by the same argument as above that S_S is a semisimple module of length n. Suppose $S = \bigoplus_{i<n} e_iS$ is a decomposition into simples, where $\{e_i\}_{i<n}$ is a suitable complete orthogonal set of idempotents of S. Correspondingly we have $P = \bigoplus_{i<n} e_iP$, and the endomorphism ring of e_iP is isomorphic to e_iSe_i, a division ring. This means that e_iP is couniform by Lemma 4.2, hence $\mathrm{codim}(P) = n$.

Now let $J(R)$ be arbitrary. Let $\bar{R} = R/J(R)$ and $\bar{P} = P/PJ(R)$. Since P is finitely generated, $PJ(R) \leqslant_s P$, hence $\mathrm{codim}(P_R) = \mathrm{codim}(\bar{P}_R)$. Since $\bar{P}$ is a finitely generated projective right $\bar{R}$-module and $J(\bar{R}) = 0$, by the already proved part we have $\mathrm{codim}(\bar{P}_{\bar{R}}) = \mathrm{codim}(\mathrm{End}_{\bar{R}}(\bar{P}))$. An additive subgroup of $\bar{P}$ is an R-submodule if and only if it is an $\bar{R}$-submodule, and an additive group endomorphism of $\bar{P}$ is an R-endomorphism if and only if it is an $\bar{R}$-endomorphism. Hence $\mathrm{codim}(P_R) = \mathrm{codim}(\bar{P}_R) = \mathrm{codim}(\bar{P}_{\bar{R}})$, and $\mathrm{codim}(\mathrm{End}_{\bar{R}}(\bar{P})) = \mathrm{codim}(\mathrm{End}_R(\bar{P}))$. Because P is a finitely generated projective R-module, we have $\mathrm{End}_R(\bar{P}) \cong \mathrm{End}_R(P)/J(\mathrm{End}_R(P))$ [AF92, Corollary 17.12], hence $\mathrm{codim}(\mathrm{End}_R(\bar{P})) = \mathrm{codim}(\mathrm{End}_R(P)/J(\mathrm{End}_R(P)))$, which equals $\mathrm{codim}(\mathrm{End}_R(P))$ by Lemma 2.7. □

The next observation was remarked in passing in [FH04]:

Corollary 2.11. *Let* $\mathbf{C}$ *be an additive category and* M_1 *and* M_2 *objects of* $\mathbf{C}$. *The endomorphism rings of* M_1 *and* M_2 *are semilocal if and only if so is the endomorphism ring of* $M = M_1 \oplus M_2$. *More precisely,* $\mathrm{codim}(\mathbf{C}(M)) = \mathrm{codim}(\mathbf{C}(M_1)) + \mathrm{codim}(\mathbf{C}(M_2))$.

Proof. Let $S = \mathbf{C}(M)$ and $e_i = \iota_i\pi_i \in S$. Thus $S_S = e_1S \oplus e_2S$, hence $\mathrm{codim}(S) = \mathrm{codim}(S_S) = \mathrm{codim}(e_1S) \oplus \mathrm{codim}(e_2S)$, and by Lemma 2.10,

$\mathrm{codim}(e_iS) = \mathrm{codim}(e_iSe_i)$. Because $e_iSe_i \cong \mathbf{C}(M_i)$, the conclusion follows. □

The importance of semilocal rings for the study of direct-sum decompositions is that if a module M has semilocal endomorphism ring, then it cancels from direct sums, that is, $M \oplus X \cong M \oplus Y$ implies $X \cong Y$ (Theorem 2.13 (Evans)). This cancellation theorem actually holds in every preadditive category, where the direct sums become (possibly "formal") biproducts (Section 1.2.1). In other words, it is a theorem about matrices of morphisms; no module theory is actually involved. The proof of the cancellation theorem relies on the fact that a semilocal ring has stable range 1:

Theorem 2.12. [Bas64] *A semilocal ring R has* stable range 1, *that is, if $Ra + Rb = R$, then $a + tb \in U(R)$ for some $t \in R$.*

Proof. Suppose first that R is semisimple and that $Ra + Rb = R$. Because the ring is semisimple, Rb is a semisimple module, and $Ra \cap Rb$ is a direct summand of Rb, i.e., $(Ra \cap Rb) \oplus I = Rb$ for some left ideal I of R. It follows that $Ra \oplus I = {}_RR$. Consider the surjective morphism $\mu\colon R \to Ra$ given by $r \mapsto ra$. We have that $\ker(\mu) \oplus C = {}_RR$ for some left ideal C of R, and $\mu|_C\colon C \to Ra$ is an isomorphism. Then ${}_RR \cong \ker(\mu) \oplus Ra \cong Ra \oplus I$ hence $\ker(\mu) \cong I$, because R is semisimple (cf. remarks at the beginning of the section on the uniqueness of the direct-sum decomposition of a semisimple module into simple modules). Fix an isomorphism $f\colon \ker(\mu) \to I$. Thus we have an automorphism $\eta\colon {}_RR = C \oplus \ker(\mu) \to {}_RR = Ra \oplus I$ given by $c + k \mapsto \mu(c) + f(k)$. On the other hand, η must be right multiplication by some element $u \in R$, necessarily invertible. Write $1 = c + k$ with $c \in C$ and $k \in \ker(\mu)$, so that $ka = 0$. Thus $a = ca$, and $u = \eta(1) = \eta(c + k) = \mu(c) + f(k) = ca + f(k) = a + f(k)$. Recall that f takes values in $I \leqslant Rb$, hence $f(k) = tb$ for some $t \in R$. This proves that $u = a + tb \in U(R)$.

Suppose now that R is semilocal, and again that $Ra + Rb = R$. The ring $\bar{R} = R/J(R)$ is semisimple, and $\bar{R}\bar{a} + \bar{R}\bar{b} = \bar{R}$. Then there is $\bar{t} \in \bar{R}$ such that $\bar{a} + \bar{t}\bar{b} \in U(\bar{R})$, which implies $a + tb \in U(R)$. (The canonical ring morphism $R \to \bar{R}$ is local.) □

The following result is known as the *cancellation property* of modules with semilocal endomorphism rings.

Theorem 2.13. (Cf. [Eva73, Theorem 2].) *Let A be an object of a preadditive category whose endomorphism ring has stable range 1. Suppose that there are objects B and C of the category in question such that the biproducts $A \oplus B$ and $A \oplus C$ are isomorphic (possibly in the additive closure of the category in question). Then $B \cong C$.*

Proof. Consider an isomorphism

$$F = \begin{pmatrix} f_{A,A} & f_{A,B} \\ f_{C,A} & f_{C,B} \end{pmatrix} : A \oplus B \to A \oplus C,$$

whose inverse is

$$\begin{pmatrix} g_{A,A} & g_{A,C} \\ g_{B,A} & g_{B,C} \end{pmatrix} : A \oplus C \to A \oplus B.$$

It follows that

$$\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix} = \begin{pmatrix} g_{A,A}f_{A,A} + g_{A,C}f_{C,A} & g_{A,A}f_{A,B} + g_{A,C}f_{C,B} \\ g_{B,A}f_{A,A} + g_{B,C}f_{C,A} & g_{B,A}f_{A,B} + g_{B,C}f_{C,B} \end{pmatrix}.$$

By hypothesis, there is an endomorphism t of A such that

$$u = f_{A,A} + tg_{A,C}f_{C,A}$$

is an automorphism of A. Consider the mapping

$$G = \begin{pmatrix} 1_A & tg_{A,C} \\ g_{B,A} & g_{B,C} \end{pmatrix} : A \oplus C \to A \oplus B.$$

One has that

$$GF = \begin{pmatrix} u & * \\ 0 & 1_B \end{pmatrix} : A \oplus B \to A \oplus B,$$

which is clearly an automorphism of $A \oplus B$, regardless of the top right entry that we did not calculate. Then G is an isomorphism, and its top left entry is 1_A. Then we conclude that $B \cong C$ by applying Lemma 1.13. □

2.2 A full class of invariants for objects of semilocal categories

It is natural to try to use the Chinese Remainder Theorem 1.24 together with Proposition 1.4 to find a canonical functor $\mathbf{C} \to \prod_{\mathbf{I} \in \mathrm{Prim}(\mathbf{C})} \mathbf{C}/\mathbf{I}$ that reflects isomorphisms, thus obtaining a full set (or class) of invariants for the objects of $\mathbf{C}$. In trying to do so, one is hindered by the fact that the ideals in $\mathrm{Prim}(\mathbf{C})$ may not be pairwise comaximal. Another problem is that, for a given object X of $\mathbf{C}$, there may be infinitely many ideals $\mathbf{I} \in \mathrm{Prim}(\mathbf{C})$ such that $\mathbf{I}(X) \neq \mathbf{C}(X)$.

In this section, we will see that a setting in which these obstructions vanish is that of a semilocal category $\mathbf{C}$. In [FP10], Facchini and Perone define a *semilocal category* to be a preadditive category $\mathbf{C}$ with at least one non-zero object, and such that $\mathbf{C}(X)$ is a semilocal ring for every non-zero object X of $\mathbf{C}$. From Lemma 2.7 we immediately deduce that a factor of a semilocal category by a proper ideal is semilocal, and that the product of finitely many semilocal categories is semilocal, while from Theorem 2.9 we deduce that if $\mathbf{C} \to \mathbf{D}$ is an almost local functor (Lemma-Definition 5.1) and $\mathbf{D}$ is semilocal, then so is $\mathbf{C}$. Also, the additive closure and the idempotent completion of a semilocal category are semilocal, because finite direct sums and direct summands of objects which have semilocal endomorphism rings are again semilocal (by Corollary 2.11).

Recall that for a ring R, a *prime* ideal I is an ideal of R such that, for every pair of ideals A and B of R such that $AB \subseteq I$, it is the case that $A \subseteq I$ or $B \subseteq I$. Equivalently, for every pair of elements $a, b \in R$ such that $aRb \subseteq I$, we have $a \in I$ or $b \in I$ [Lam01, p. 165]. We begin this section with a neat result generalising [FP10, Lemma 2.1].

Theorem 2.14. *Let X and Y be non-zero objects of a preadditive category $\mathbf{C}$, and let I be a* prime *ideal of $\mathbf{C}(X)$. Then:*

(i) If $I' = \mathbf{A}_I(Y)$ is a proper ideal of $\mathbf{C}(Y)$, then $\mathbf{A}_{I'} = \mathbf{A}_I$ and I' is a prime ideal of $\mathbf{C}(Y)$.

(ii) If I is a maximal ideal, I' is proper, and $\mathbf{C}(Y)$ is a semilocal ring, then I' is a maximal ideal of $\mathbf{C}(Y)$.

Proof. (i) Since $\mathbf{A}_I(Y) = I'$, the inclusion $\mathbf{A}_I \subseteq \mathbf{A}_{I'}$ follows by Lemma 1.2. Let us prove that if the inclusion is proper, then $I' = \mathbf{C}(Y)$, so that the first part of (i) follows. Then suppose that there exist objects A and B of $\mathbf{C}$ and a morphism $g \in \mathbf{C}(A, B)$ such that $g \in \mathbf{A}_{I'}(A, B)$ but $g \notin \mathbf{A}_I(A, B)$.

Without loss of generality, we assume that $A = B = X$. Indeed, there exist morphisms $\alpha \in \mathbf{C}(X, A)$ and $\beta \in (B, X)$ such that $\beta g \alpha \notin I = \mathbf{A}_I(X)$, while $\beta g \alpha \in \mathbf{A}_{I'}(X)$.

Let $\varphi \in \mathbf{C}(X, Y)$ and $\psi \in \mathbf{C}(Y, X)$ be arbitrary morphisms. Then $\varphi\langle g\rangle\psi \subseteq \mathbf{A}_{I'}(Y) = I' = \mathbf{A}_I(Y)$, hence $\langle\psi\varphi\rangle\langle g\rangle\langle\psi\varphi\rangle \subseteq \mathbf{A}_I(X) = I$. (Here $\langle e\rangle$ indicates the ideal of $\mathbf{C}(X)$ generated by e, where e is any endomorphism of X.) Since I is prime and $\langle g\rangle \nsubseteq I$, it follows that $\psi\varphi \in I$. Because φ and ψ are arbitrary, $1_Y \in \mathbf{A}_I(Y) = I'$ and I' is not proper, as required.

To prove that I' is a prime ideal of $\mathbf{C}(Y)$, suppose $f, g \in \mathbf{C}(Y)$ are such that $f\mathbf{C}(Y)g \subseteq I'$ and $g \notin I' = \mathbf{A}_I(Y)$. There exist morphisms $\alpha_0 \colon X \to Y$ and $\beta_0 \colon Y \to X$ such that $\beta_0 g \alpha_0 \notin I$. Let $\alpha \colon X \to Y$ and $\beta \colon Y \to X$ be arbitrary morphisms. Then $\beta f \alpha \mathbf{C}(X)\beta_0 g \alpha_0 \subseteq \beta f \mathbf{C}(Y) g \alpha_0 \subseteq \mathbf{A}_I(X) = I$, from which $\beta f \alpha \in I$. Thus $f \in \mathbf{A}_I(Y) = I'$, as required.

(ii) Since I is maximal, $\mathbf{J}(X) \subseteq I$, thus $\mathbf{J} \subseteq \mathbf{A}_I$ by Lemma 1.2. In particular, $\mathbf{J}(Y) \subseteq \mathbf{A}_I(Y) = I'$. Thus $\mathbf{C}(Y)/I'$ is isomorphic to a quotient of $\mathbf{C}(Y)/\mathbf{J}(Y)$, hence it is a semisimple ring. By the Artin-Wedderburn Theorem cited earlier, $\mathbf{C}(Y)/I' \cong \prod_{i<n} M_{d_i}(D_i)$ where each d_i is a positive integer and each D_i is a division ring. Suppose $e + I'$ is a central idempotent of $\mathbf{C}(Y)/I'$. Then $eC_Y(1 - e) \subseteq I'$, so that $e \in I'$ or $1 - e \in I'$, because I' is a prime ideal by (i). Thus in

the decomposition above $n = 1$ and $\mathbf{C}(Y)/I' \cong M_{d_0}(D_0)$ is a simple ring, hence I' is a maximal ideal of $\mathbf{C}(Y)$. □

While preadditive categories need not have maximal ideals [FP10, Example 4.1], the situation is much nicer for semilocal categories, as we show next.

Theorem 2.15. [FP10, Proposition 4.3 and Theorem 4.8] *Let* $\mathbf{C}$ *be a semilocal category.*

(i) Every proper ideal of $\mathbf{C}$ *is contained in a maximal ideal. In particular, maximal ideals exist in* $\mathbf{C}$.

(ii) The maximal ideals of $\mathbf{C}$ *are exactly the ideals of* $\mathbf{C}$ *associated to maximal ideals of endomorphism rings of its objects, that is, the ideals in* $\mathrm{Prim}(\mathbf{C})$.

(iii) The intersection of the maximal ideals of $\mathbf{C}$ *is the Jacobson radical* $\mathbf{J}$ *of* $\mathbf{C}$.

(iv) Distinct maximal ideals of $\mathbf{C}$ *are pairwise comaximal.*

(v) For every object X *of* $\mathbf{C}$*, there are only finitely many maximal ideals* $\mathbf{I}$ *of* $\mathbf{C}$ *such that* $\mathbf{I}(X)$ *is a proper ideal of* $\mathbf{C}(X)$.

Proof. Notice that for a semilocal ring S, as it follows from the Wedderburn-Artin decomposition of $S/J(S)$ into matrix rings, an ideal is primitive if and only if it is maximal, and there are finitely many maximal ideals. Thus the family of ideals $\mathrm{Prim}(\mathbf{C})$ consists of those ideals of $\mathbf{C}$ that are associated to maximal ideals of endomorphism rings of its objects.

Let us first prove that every $\mathbf{I} \in \mathrm{Prim}(\mathbf{C})$ is a maximal ideal. Let X be a non-zero object of $\mathbf{C}$ and M a maximal ideal of $\mathbf{C}(X)$ such that $\mathbf{I} = \mathbf{A}_M$. Since $\mathbf{A}_M(X) = M$, the ideal $\mathbf{A}_M$ is proper. Suppose $\mathbf{I}'$ is a proper ideal of $\mathbf{C}$ containing $\mathbf{A}_M$. There exists an object Y such that $\mathbf{I}'(Y)$ is a proper ideal of $\mathbf{C}(Y)$. Since $\mathbf{A}_M(Y) \subseteq \mathbf{I}'(Y)$ is proper, we have that $\mathbf{A}_M(Y)$ is actually maximal by Theorem 2.14, hence $\mathbf{A}_M(Y) = \mathbf{I}'(Y)$. Theorem 2.14 also tells us that $\mathbf{A}_M = \mathbf{A}_{\mathbf{I}'(Y)}$, so by Lemma 1.2, we have $\mathbf{I}' \subseteq \mathbf{A}_M$. Hence $\mathbf{I}' = \mathbf{A}_M = \mathbf{I}$.

Since $\mathbf{C}$ has a non-zero object, the above also proves that $\mathbf{C}$ has a maximal ideal. More generally, suppose $\mathbf{I}$ is a proper ideal of $\mathbf{C}$. Then there exists an object X of $\mathbf{C}$ such that $\mathbf{I}(X) \neq \mathbf{C}(X)$. Then $\mathbf{I}(X) \subseteq M$ for some maximal ideal of $\mathbf{C}$, and, by Lemma 1.2, we have $\mathbf{I} \subseteq \mathbf{A}_M$. Thus every proper ideal is contained in a maximal ideal, and (i) is proved.

The above reasoning can be applied in particular to a maximal ideal $\mathbf{I}$, obtaining $\mathbf{I} = \mathbf{A}_M$. This completes the proof of (ii).

Part (iii) is Proposition 1.4.

Let $\mathbf{I}_1$ and $\mathbf{I}_2$ be distinct ideals in $\mathrm{Prim}(\mathbf{C})$ and X an object of $\mathbf{C}$. If both $M_1 = \mathbf{I}_1(X)$ and $M_2 = \mathbf{I}_2(X)$ are proper ideals of $\mathbf{C}(X)$, then M_1 and M_2 are maximal ideals and $\mathbf{I}_i = \mathbf{A}_{M_i}$ by Theorem 2.14. Since $\mathbf{I}_1 \neq \mathbf{I}_2$, it follows that M_1 and M_2 are distinct maximal ideals of $\mathbf{C}(X)$, hence $\mathbf{I}_1(X) + \mathbf{I}_2(X) = \mathbf{C}(X)$. If either one of $\mathbf{I}_i(X)$ is not proper, the same conclusion follows. Since X is arbitrary, this proves (iv), i.e., that the ideals in $\mathrm{Prim}(\mathbf{C})$ are pairwise comaximal.

Let X be an object of $\mathbf{C}$ and $\mathbf{I} \in \mathrm{Prim}(\mathbf{C})$. If $\mathbf{I}(X)$ is proper, then $\mathbf{I} = \mathbf{A}_{\mathbf{I}(X)}$ and $\mathbf{I}(X)$ is a maximal ideal of $\mathbf{C}(X)$ by Theorem 2.14. Since there are finitely many, it follows that $\mathbf{I}(X)$ is proper for only finitely many $\mathbf{I} \in \mathrm{Prim}(\mathbf{C})$. Since X is arbitrary, (v) holds. □

The previous result provides a class of invariants for a semilocal category, as an application of the Chinese Remainder Theorem 1.24.

Notation 2.16. For a semilocal category $\mathbf{C}$, we denote by $V(\mathbf{C})$ the class of maximal ideals of $\mathbf{C}$. Thus $V(\mathbf{C}) = \mathrm{Prim}(\mathbf{C})$ (Theorem 2.15(ii)). If M is a non-zero object of $\mathbf{C}$, we let $V(\mathbf{C}, M)$ be the subset of $V(\mathbf{C})$ consisting of those maximal ideals of $\mathbf{C}$ associated to maximal ideals of the endomorphism ring $\mathbf{C}(M)$, that is, those $\mathbf{I}$ such that $\mathbf{I}(M) \neq \mathbf{C}(M)$. The set $V(\mathbf{C}, M)$ is finite, because $\mathbf{C}(M)$ has finitely many maximal ideals (Lemma 2.8). We will write $V(M)$ for $V(\mathbf{C}, M)$ if the category is understood. Thus $V(\mathbf{C}) = \bigcup V(\mathbf{C}, M)$, where the union is taken over all non-zero objects M of $\mathbf{C}$.

Theorem 2.17. *Let $\mathbf{C}$ be a semilocal category. The canonical additive functor*

$$\mathbf{C} \to \prod_{\mathbf{I} \in V(\mathbf{C})} \mathbf{C}/\mathbf{I}.$$

is full, and it reflects isomorphisms and retracts.

Proof. The previous result implies, together with Theorem 1.24, that the canonical functor

$$\mathbf{C}/\mathbf{J} \to \prod_{\mathbf{I} \in V(\mathbf{C})} \mathbf{C}/\mathbf{I}$$

is faithful, full, isomorphism- and retracts-reflecting. We claim that the canonical functor $F\colon \mathbf{C} \to \mathbf{C}/\mathbf{J}$ is isomorphism- and retracts-reflecting as well. The statement then follows by composing the two functors.

To prove the claim, suppose $\alpha\colon F(X) \to F(Y)$ is an isomorphism. Since F is full, $\alpha = F(g)$ and $\alpha^{-1} = F(f)$ for some $g\colon X \to Y$ and $f\colon Y \to X$. Thus $F(1_X - fg) = 0$ and $F(1_Y - gf) = 0$, that is, $1_X - fg$ and $1_Y - gf$ are in the Jacobson radical, so that fg and gf are automorphisms, and both f and g are isomorphisms. Finally, assume that $1_{F(X)}$ factors through $F(Y)$, that is, $1_{F(X)} = \beta\alpha$, for some $\alpha\colon F(X) \to F(Y)$ and $\beta\colon F(Y) \to F(X)$. Again, $\alpha = F(g)$ and $\beta = F(f)$. Thus $F(1_X - fg) = 0$ and fg is an automorphism of X. Hence 1_X factors through Y, as required. □

Notice the following elementary fact:

Lemma 2.18. *Suppose X, Y, and $X \oplus Y$ are objects of a semilocal category* $\mathbf{C}$. *Then $V(X \oplus Y) = V(X) \cup V(Y)$.*

A stronger version of the above lemma is Proposition 2.21, and it essentially comes from [FP09b, Corollary 3.5].

Proof. If $\mathbf{I} \in V(X)$, then $\mathbf{I}(X \oplus Y)$ is proper. For instance, $\iota_X \pi_X \notin \mathbf{I}$, otherwise $1_X \in \mathbf{I}(X)$, which is false, because $\mathbf{I}(X)$ is a maximal ideal of $\mathbf{C}(X)$. Thus $\mathbf{I} = \mathbf{A}_{\mathbf{I}(X \oplus Y)}$ by Theorem 2.14, hence $\mathbf{I} \in V(X \oplus Y)$.

Suppose $\mathbf{I} \in V(X \oplus Y)$. Then either $\mathbf{I}(X)$ or $\mathbf{I}(Y)$ is proper, for otherwise $1_{X \oplus Y} = \iota_X \pi_X + \iota_Y \pi_Y \in \mathbf{I}$, which is false. Say $\mathbf{I}(X)$ is proper. Then $\mathbf{I} = \mathbf{A}_{\mathbf{I}(X)}$ by Theorem 2.14, hence $\mathbf{I} \in V(X)$. □

The lemma above shows that when writing $V(\mathbf{C})$ as the union $V(\mathbf{C}) = \bigcup V(X)$, such union can be taken over just the indecomposable objects of $\mathbf{C}$. If $\mathbf{C}$ is semilocal, also the additive closure $\mathrm{Sums}(\mathbf{C})$ of $\mathbf{C}$ is semilocal (Corollary 2.11), and the lemma implies that $V(\mathrm{Sums}(\mathbf{C})) = \bigcup_{X \in \mathbf{C}} V(X)$. Let us state a very simple consequence of these considerations and Theorem 2.17 for later reference:

Lemma 2.19. *Let $\mathbf{C}$ be a semilocal category. Let M and N be biproducts of objects of $\mathbf{C}$, i.e., M and N are supposed to be objects of $\mathrm{Sums}(\mathbf{C})$. The following are equivalent:*

(i) *M and N are isomorphic.*

(ii) *M and N are isomorphic in $\mathrm{Sums}(\mathbf{C})/\mathbf{P}$ for each $\mathbf{P} \in V(\mathrm{Sums}(\mathbf{C}))$.*

(iii) *M and N are isomorphic in $\mathrm{Sums}(\mathbf{C})/\mathbf{P}$ for each $\mathbf{P} \in V(\mathrm{Sums}(\mathbf{C}), X)$ for every object X of $\mathbf{C}$.*

2.2.1 Rings and objects of finite type

In this section, we explain some results from [FP09b] that will be used in Chapter 5. They pertain a special type of semilocal categories.

Recall that a ring morphism $f \colon R \to S$ is said to be *local* if $f(r) \in U(S)$ implies $r \in U(R)$, for every $r \in R$.

Lemma-Definition 2.20. [FP09b, Proposition 2.1] *Let n be a positive integer and R a ring. The following are equivalent:*

(i) *$R/J(R)$ is a product of n division rings.*

(ii) *R admits a local ring morphism into a product of m division rings, and n is the smallest such positive integer m.*

(iii) *R has n maximal right ideals, and they are all two-sided.*

(iv) R has n maximal left ideals, and they are all two-sided.

(v) R has n primitive ideals $I_0, \dots, I_{n-1}$, and R/I_i is a division ring for every $i < n$.

If these equivalent conditions hold, we say that R is a ring of type n. *We also declare that R is a* ring of type 0 *if $|R| = 1$. For an object X of a preadditive category $\mathbf{C}$, we say that X is of type n if its endomorphism ring $\mathbf{C}(X)$ is of type n. For instance, an object of type 0 is a zero object and an object of type 1 is an object with local endomorphism ring.*

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds. Then $R/J(R)$ is isomorphic to a product of n division rings. Since the canonical ring morphism $R \to R/J(R)$ is local, it follows that R has a local morphism into a product of n division rings. Also notice that $\operatorname{codim}(R) = \operatorname{codim}(R/J(R)) = n$, by Lemma 2.7. Suppose there is another local morphism of rings $R \to \prod_{i<m} D_i$, where each D_i is a division ring. By Theorem 2.9 and Lemma 2.7, we have that $n = \operatorname{codim}(R) \leqslant \operatorname{codim}(\prod_{i<m} D_i) = m$. Thus (ii) holds.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $g\colon R \to \prod_{i<n} D_i$ be a local ring morphism and D_i be a division ring for every $i < n$. Let $P_i = \ker(p_i g)$ where $p_i\colon \prod_{i<n} D_i \to D_i$ is the canonical projection. Since R/P_i is a subring of the division ring D_i, each P_i is a completely prime ideal of R. In addition, because the morphism g is local, $\bigcup_{i<n} P_i$ is the set of non-units of R. Thus if M is a maximal right ideal of R, then $M \leqslant P_i$ for some $i < n$ (Lemma 5.4), hence $M = P_i$ and M is two-sided. Hence the set of maximal right ideals is a subset of $\{P_i\}_{i<n}$, say $\{P_{i_\mu}\}_{\mu<m}$, with $m \leqslant n$. Then there is a canonical injective ring morphism $R/J(R) \to \prod_{\mu<m} R/P_{i_\mu}$, which is also surjective by the Chinese Remainder Theorem 1.19, and each R/P_{i_μ} is a division ring, because it has no non-trivial right ideals. By minimality of n, we have $m = n$, hence (iii) holds.

(iii) $\Rightarrow$ (i). Let $\{M_i\}_{i<n}$ be the set of n maximal right ideals of R, and assume each M_i is actually two-sided. Then we have a canonical injective ring morphism $p\colon R/J(R) \to \prod_{i<n} R/M_i$. Since R/M_i has no non-trivial right ideal, it is a division ring. By the Chinese Remainder Theorem 1.19, p is also surjective, hence an isomorphism.

Condition (iv) is also equivalent to (i), (ii), and (iii), because (i) is left-right symmetric.

Here is the only bit of the proof not contained in [FP09b]. This last condition (v) will be useful later.

(i) $\Rightarrow$ (v). If (i) holds and I is a primitive ideal of R, then $J(R) \subseteq I$, and R/I is a simple quotient of $R/J(R) \cong D_0 \times \cdots \times D_{n-1}$, where each D_i is a division ring, hence R/I is also a division ring, and there are n choices for I. Hence (v) holds.

(v) $\Rightarrow$ (i). Conversely, suppose $I_0, \dots, I_{n-1}$ are the finitely many primitive ideals of R, and that R/I_i is a division ring for every $i < n$. The Chinese Remainder Theorem 1.19 implies that the canonical ring morphism $R \to \prod_{i<n} R/I_i$ is surjective, and its kernel is $\bigcap_{i<m} I_i = J(R)$. Thus (v) implies (i). □

What follows was proved in [FP09b, Corollary 3.5] for categories of modules. Here we give a different and somewhat simpler proof, in the case of additive categories.

Proposition 2.21. *Let $\mathbf{C}$ be an additive category and X, Y objects of $\mathbf{C}$ of finite type m and n respectively.*

(i) If $V(X)$ and $V(Y)$ have non-empty intersection, then $X \oplus Y$ is not an object of finite type.

(ii) If $V(X)$ and $V(Y)$ are disjoint, then $X \oplus Y$ is of finite type $m + n$, and $V(X \oplus Y)$ is the disjoint union of $V(X)$ and $V(Y)$.

Proof. (i) Suppose that $\mathbf{I}$ is an ideal in the intersection $V(X) \cap V(Y)$. We claim that

$$M := \begin{pmatrix} \mathbf{C}(X) & \mathbf{C}(Y,X) \\ \mathbf{I}(X,Y) & \mathbf{I}(Y) \end{pmatrix}$$

is a maximal right ideal of $\mathbf{C}(X \oplus Y)$, containing $\mathbf{I}(X \oplus Y)$, but not two-sided; as a consequence, $\mathbf{C}(X \oplus Y)$ is not a ring of finite type.

To show that M is a maximal right ideal, suppose that M' is a right ideal of $\mathbf{C}(X \oplus Y)$ and that $M < M'$. There is an element $g \in M' \backslash M$. Then either $g_{2,2} \notin \mathbf{I}(Y)$ or $g_{2,1} \notin \mathbf{I}(X,Y)$.

In the first case, since $\mathbf{C}(Y)/\mathbf{I}(Y)$ is a division ring, there is $f \in \mathbf{C}(Y)$ such that $1_Y - g_{2,2}f \in \mathbf{I}(Y)$. Then

$$g \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix} + \begin{pmatrix} 0 & -g_{1,2}f \\ 0 & 1_Y - g_{2,2}f \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1_Y \end{pmatrix}$$

is in M', and it easily follows that M' is improper.

In the second case, suppose by contradiction that $g_{2,1}\mathbf{C}(Y,X) \subseteq \mathbf{I}(Y)$. Then $\mathbf{C}(Y)g_{2,1}\mathbf{C}(Y,X) \subseteq \mathbf{I}(Y)$ and $g_{2,1} \in \mathbf{I}(X,Y)$, because $\mathbf{I} = \mathbf{A}_{\mathbf{I}(Y)}$. That is false, hence there exists $f \in \mathbf{C}(Y,X)$ such that $g_{2,1}f \notin \mathbf{I}(Y)$.Thus $g \left(\begin{smallmatrix} 0 & f \\ 0 & 0 \end{smallmatrix}\right) \in M'$, and its bottom-right entry is not in $\mathbf{I}(Y)$, hence we conclude by the first case.

Let us finally show that M is not two-sided. Notice that $\mathbf{I}(X,Y)$ is a proper subgroup of $\mathbf{C}(X,Y)$. If not, we would have $1_X \in \mathbf{A}_{\mathbf{I}(Y)}(X) = \mathbf{I}(X)$, which is false. Hence we can pick a morphism $f \in \mathbf{C}(X,Y) \backslash \mathbf{I}(X,Y)$. Thus, to show that M is not two-sided, just notice that M is not closed by left multiplication by $\left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right)$.

(ii) Assume that $V(X)$ and $V(Y)$ are disjoint. Let I be a primitive ideal of $\mathbf{C}(X \oplus Y)$. Then $\mathbf{A}_I$ is a proper ideal of $\mathbf{C}$, so either $\mathbf{A}_I(X)$ or $\mathbf{A}_I(Y)$ is proper. Without loss of generality, suppose $M = \mathbf{A}_I(X)$ is proper. Then M is a maximal ideal of $\mathbf{C}(X)$ and $\mathbf{A}_I = \mathbf{A}_M$ by Theorem 2.14. It follows that $\mathbf{A}_I(Y) = \mathbf{A}_M(Y)$ is improper, otherwise $\mathbf{A}_M$ would be common to $V(X)$ and $V(Y)$, again by Theorem 2.14. Therefore,

$$I = \mathbf{A}_I(X \oplus Y) = \begin{pmatrix} M & \mathbf{C}(Y,X) \\ \mathbf{C}(X,Y) & \mathbf{C}(Y) \end{pmatrix},$$

which implies that $\mathbf{C}(X \oplus Y)/I \cong \mathbf{C}(X)/M$ is a division ring. The proof also shows that $I \mapsto \mathbf{A}_I$ defines an injective mapping from the set of primitive ideals of $\mathbf{C}(X \oplus Y)$ into $V(X) \sqcup V(Y)$, hence there are finitely many primitive ideals. Thus Lemma 2.20(v) shows that $\mathbf{C}(X \oplus Y)$ is of finite type.

We already know that $V(X \oplus Y) = V(X) \sqcup V(Y)$, by Lemma 2.18. To conclude that the type of $X \oplus Y$ is $m + n$, just notice that the type of an object of finite type Z of $\mathbf{C}$ is simply the cardinality of $V(\mathbf{C}, Z)$. □

Lemma 2.22. *The class of objects of finite type (in any preadditive category) is closed by biproduct factors and the type is additive, i.e., if $X \cong A_1 \oplus A_2$ is of finite type, then both A_1 and A_2 are of finite type, and the type of X is the sum of the type of A_1 and that of A_2.*

Proof. From the decomposition $X \cong A_1 \oplus A_2$ we obtain a local ring morphism $\mathrm{End}(A_i) \to \mathrm{End}(X)$ by sending g to $\iota_i g \pi_i$, and by composing it with the canonical projection $\mathrm{End}(X) \to \mathrm{End}(X)/J(\mathrm{End}(X))$, we obtain a local ring morphism of $\mathrm{End}(A_i)$ into a finite product of division rings. Therefore A_1 and A_2 are objects of finite type. Since X is of finite type, the rest follows from Proposition 2.21. □

Corollary 2.23. *Every non-zero object X of finite type (in any preadditive category) has a decomposition as a biproduct of indecomposable objects of finite type.*

Proof. By induction on the type of X and Lemma 2.22. □

Lemma 2.24. *Let $\mathbf{B}$ be a preadditive category whose objects have finite type. Let $X \in \mathbf{B}$ and $\mathbf{P} \in V(\mathbf{B}, X)$, and let $F \colon \mathbf{B} \to \mathbf{B}/\mathbf{P}$ be the canonical functor. Then $\mathbf{B}/\mathbf{P}$ has only one non-zero object up to isomorphism, and for any object N of $\mathbf{B}$, the following are equivalent:*

(i) $\mathbf{P} \in V(\mathbf{B}, N)$.

(ii) $\mathbf{P}(N)$ *is maximal.*

(iii) $\mathbf{P}(N)$ *is proper.*

(iv) $F(X) \cong F(N)$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial. Suppose (iii) is true, that is, $F(N) \neq 0$. Then $1_N \notin \mathbf{P}$, so there are morphisms $f\colon X \to N$ and $g\colon N \to X$ such that $gf \notin \mathbf{P}(X)$. Thus $gf + \mathbf{P}$ is an automorphism in $\mathbf{B}/\mathbf{P}$, because $\mathbf{B}(X)/\mathbf{P}(X)$ is a division ring. It also follows that $(gf)^2 \notin \mathbf{P}(X)$, that is, $g(fg)f \notin \mathbf{P}(X)$, so that $fg \notin \mathbf{P}(N)$. Since $P(N)$ is proper, it is maximal by Theorem 2.14, and $\mathbf{B}(N)/\mathbf{P}(N)$ is a division ring. Therefore $fg + \mathbf{P}$ is also an automorphism in $\mathbf{B}/P$. Hence $f + \mathbf{P}$ and $g + \mathbf{P}$ are isomorphisms, and $F(X) \cong F(N)$. This shows that (iii) implies (iv), and also that $\mathbf{B}/\mathbf{P}$ has only one non-zero object up to isomorphism. Cf. [FP09b, Lemma 4.5]. If (iv) holds, then $\mathbf{B}(X)/\mathbf{P}(X) \cong \mathbf{B}(N)/\mathbf{P}(N)$, so that $\mathbf{P}(N)$ is maximal. By Theorem 2.14, $\mathbf{P}$ is associated to $\mathbf{P}(N)$, hence (i) holds. □

The following result was proved in [FP09b, Corollary 3.5] for modules of finite type; here is a version for objects of finite type of preadditive categories.

Corollary 2.25. *Let $\mathbf{A}$ be a preadditive category and X and Y objects of finite type of $\mathbf{A}$. Then $X \cong Y$ if and only if $V(\mathbf{C}, X) = V(\mathbf{C}, Y)$, where $\mathbf{C}$ is any full semilocal subcategory of $\mathbf{A}$ containing X and Y. Moreover, X is a retract of Y if and only if $V(\mathbf{C}, X) \subseteq V(\mathbf{C}, Y)$.*

Proof. In both statements one implication is trivially true. Thus suppose that $V(\mathbf{C}, X) = V(\mathbf{C}, Y)$. Let $\mathbf{B}$ be the full subcategory of $\mathbf{C}$ whose only objects are X and Y. By Remark 1.3, $V(\mathbf{B}, X) = V(\mathbf{B}, Y)$. Then $V(\mathbf{B}) = V(\mathbf{B}, X) = V(\mathbf{B}, Y)$. Therefore, if $\mathbf{P}$ is any maximal ideal of $\mathbf{B}$, then X and Y are non-zero objects of $\mathbf{B}/\mathbf{P}$, hence isomorphic. By Theorem 2.17, $X \cong Y$ in $\mathbf{B}$, hence in $\mathbf{C}$.

If $V(X) \subseteq V(Y)$, for $\mathbf{P} \in V(X)$ we have that X and Y are isomorphic modulo $\mathbf{P}$, in particular the identity of X factors through Y in $\mathbf{C}/\mathbf{P}$. If $\mathbf{P}$ is not in $V(X)$, then X is zero modulo $\mathbf{P}$, hence trivially the identity of X factors through Y in $\mathbf{C}/\mathbf{P}$. By Theorem 2.17, the identity of X factors through Y in $\mathbf{C}$. □

Notice that Corollary 2.25 does not hold for objects of a semilocal category. There is a trivial example. Let D be a division ring and $\mathbf{C}$ the category of finite-dimensional right D-vector spaces. Since $\mathrm{End}_D(D^n) \cong M_n(D)$ is a simple artinian ring, we have that $\mathbf{C}$ is a semilocal category. Of course, D^n and D^m are not isomorphic if n and m are distinct. Nevertheless, $V(D^n)$ and $V(D^m)$ coincide. Their only element is zero, and it is the Jacobson radical of $\mathbf{C}$. This example also shows that Lemma 2.24 does not hold if we replace $\mathbf{B}$ with an arbitrary semilocal category.

2.3 Categories of finite dual Goldie dimension

Besides ideals, a preadditive category $\mathbf{C}$ also has one-sided ideals. For instance, a *right ideal* of $\mathbf{C}$ is a collection $\mathbf{I}$ of morphisms of $\mathbf{C}$ such that $\mathbf{I}(X,Y) := \mathbf{I} \cap \mathbf{C}(X,Y)$ is a subgroup of $\mathbf{C}(X,Y)$, for every X and Y in $\mathbf{C}$, and such that for every $f \in \mathbf{I}(Y,Z)$ and $g \in \mathbf{C}(X,Y)$ we have $fg \in \mathbf{I}(X,Z)$. The right ideals of $\mathbf{C}$ form a (large) complete lattice, whose dual Goldie dimension we define to be to be the codimension $\operatorname{codim}(\mathbf{C})$ of the preadditive category $\mathbf{C}$. The prompt objection that the reader will raise at this point is that this ought to be qualified as a "right" codimension. We presently state that this notion of codimension is left-right symmetric, as is the case for rings, and we will see this quite clearly at the end of the section (Remark 2.35).

The main aim of this section is to prove that $\operatorname{codim}(\mathbf{C})$ is finite if and only if $\mathbf{C}$ is a semilocal category with finitely many non-zero objects. This is not too surprising as soon as one understands how right ideals can be partitioned, as we next show, and what the maximal subfunctors of representable functors are. The maximal subfunctors of $\mathbf{C}(-,X)$ are associated to maximal right ideals of the endomorphism ring $\mathbf{C}(X)$, in a way that strongly resembles the situation for the maximal ideals of semilocal categories.

Recall that a *subfunctor* of the representable functor $\mathbf{C}(-,X)$ can be seen as a class of morphisms $\mathbf{M}$ of $\mathbf{C}$ into the object X, such that $\mathbf{M}(A) := \mathbf{M} \cap \mathbf{C}(A,X)$ is a subgroup of $\mathbf{C}(A,X)$, and such that if $g \in \mathbf{M}(A)$ and $f \in \mathbf{C}(B,A)$, then $gf \in \mathbf{M}(B)$, for every A and B in $\mathbf{C}$. It is straightforward to see that:

Lemma 2.26. *A class $\mathbf{I}$ of morphisms of $\mathbf{C}$ is a right ideal if and only if it is a union $\mathbf{I} = \bigcup_{X \in \mathbf{C}} \mathbf{I}_X$ of subfunctors $\mathbf{I}_X$ of $\mathbf{C}(-,X)$.*

If $\mathbf{I}$ is a right ideal and X is an object of $\mathbf{C}$, then $\mathbf{I}_X$ will denote the class of morphisms in $\mathbf{I}$ whose codomain is X.

Corollary 2.27. *A right ideal $\mathbf{I}$ is maximal if and only if there exists an object X such that $\mathbf{I}_X$ is a maximal subfunctor of $\mathbf{C}(-,X)$ and $\mathbf{I}_Y = \mathbf{C}(-,Y)$ for every other object Y.*

Corollary 2.28. *Every proper subfunctor of $\mathbf{C}(-,X)$ is contained in a maximal subfunctor, and every proper right ideal of $\mathbf{C}$ is contained in a maximal right ideal.*

Proof. Standard application of the Zorn Lemma, plus Lemma 2.26. □

It is interesting to notice that the maximal subfunctors of the representable functor $\mathbf{C}(-,X)$ are associated to the maximal right ideals of the endomorphism ring $\mathbf{C}(X)$, in much the same way that the maximal ideals of a semilocal category are associated to the maximal ideals of the endomorphism rings of its objects:

Proposition 2.29. *Let G be a maximal subfunctor of $\mathbf{C}(-, X)$. Then $I := G(X)$ is a maximal right ideal of the endomorphism ring $\mathbf{C}(X)$, and for every object A and every morphism $g\colon A \to X$, we have that*

$$g \in G(A) \iff g\mathbf{C}(X, A) \subseteq I. \tag{2.30}$$

Moreover, given a maximal right ideal I of $\mathbf{C}(X)$, equation (2.30) *defines a maximal subfunctor of $\mathbf{C}(-, X)$.*

Proof. Suppose $I < I_0 \leqslant \mathbf{C}(X)$ for some right ideal I_0. Then $G + I_0\mathbf{C}(-, X) = \mathbf{C}(-, X)$, in particular, $I_0 = I + I_0 = \mathbf{C}(X)$. This shows that I is a maximal right ideal of $\mathbf{C}(X)$.

Let us now prove equation (2.30). Suppose that $g \notin G(A)$. Then $G + g\mathbf{C}(-, A) = \mathbf{C}(-, X)$. In particular, $G(X) + g\mathbf{C}(X, A) = \mathbf{C}(X)$. Since G is a proper subfunctor of $\mathbf{C}(-, X)$, we have that $G(X) \neq \mathbf{C}(X)$, thus $g\mathbf{C}(X, A)$ is not contained in $G(X)$. The other implication is trivial.

Now suppose that I is a maximal right ideal of $\mathbf{C}(X)$ and define G by means of (2.30). It is clear that G is then a subfunctor of $\mathbf{C}(-, X)$. Suppose $G \leqslant G_0 \leqslant \mathbf{C}(-, X)$, for some subfunctor G_0. Then $I \leqslant G_0(X) \leqslant \mathbf{C}(X)$, hence either $G_0(X) = I$ or $G_0(X) = \mathbf{C}(X)$. In the first case, $G_0 = G$ by the already proved part of the proposition, in the second case, $G_0 = \mathbf{C}(-, X)$. This proves that G is in fact a maximal subfunctor. $\square$

Lemma 2.31. *A right ideal $\mathbf{I}$ of $\mathbf{C}$ is superfluous if and only if it is contained in the Jacobson radical $\mathbf{J}$.*

Proof. First notice that $\mathbf{I}$ is a superfluous right ideal of $\mathbf{C}$ if and only if $\mathbf{I}_X$ is a superfluous subfunctor of $\mathbf{C}(-, X)$ for every object X of $\mathbf{C}$, because for right ideals $\mathbf{I}$ and $\mathbf{K}$ we have $(\mathbf{I} + \mathbf{K})_X = \mathbf{I}_X + \mathbf{K}_X$, and because of Lemma 2.26.

Thus we are left to prove that a subfunctor G of $\mathbf{C}(-, X)$ is superfluous if and only if $G \subseteq \mathbf{J}(-, X)$.

First suppose G is superfluous and consider $g \in G(A) \subseteq \mathbf{C}(A, X)$ for an arbitrarily fixed object A of $\mathbf{C}$. Let $f \in \mathbf{C}(X, A)$ also be arbitrary. We have that $1_X - gf$ is not in $G(X)$, otherwise $1_X \in G(X)$ and G is not proper. If H is the subfunctor of $\mathbf{C}(-, X)$ generated by $(1_X - gf)$, that is, $H(B) = (1_X - gf)\mathbf{C}(B, X)$ for every B in $\mathbf{C}$, we have that $G + H = \mathbf{C}(-, X)$. Thus $H = \mathbf{C}(-, X)$, in particular, $1_X \in H(X)$, so that $1_X - gf$ is right invertible. Since f is arbitrary, we conclude that $g \in \mathbf{J}(A, X)$, and because A is also arbitrary, $G \subseteq \mathbf{J}(-, X)$.

To prove the converse, let us show that $\mathbf{J}(-, X)$ is a superfluous subfunctor of $\mathbf{C}(-, X)$. Consider a subfunctor G of $\mathbf{C}(-, X)$ such that $G + \mathbf{J}(-, X) = \mathbf{C}(-, X)$. In particular, $1_X = g + j$ for some $g \in G(X)$ and some $j \in \mathbf{J}(X)$. Thus $g = 1_X - j$ is an automorphism of X, hence $G(X) = \mathbf{C}(X)$, and it follows that $G = \mathbf{C}(-, X)$. $\square$

Lemma 2.32. *The Jacobson radical* $\mathbf{J}$ *of* $\mathbf{C}$ *is the intersection of all maximal right ideals of* $\mathbf{C}$. *(See also* [Mit72, page 21].*)*

Proof. If $g\colon A \to B$ is not in $\mathbf{M}(A, B)$ for some maximal right ideal $\mathbf{M}$ of $\mathbf{C}$, then $\mathbf{M}_B + g\mathbf{C}(-, A) = \mathbf{C}(-, B)$. In particular $1_B = m + gf$ for some $f\colon B \to A$. Thus $1_B - gf \in \mathbf{M}_B$. If this were invertible, $\mathbf{M}$ would contain g, which it does not. Hence $1_B - gf$ is not invertible, and this shows that g is not in $\mathbf{J}$. Thus $\mathbf{J}$ is contained in every maximal right ideal.

Conversely, suppose that $g\colon A \to B$ is contained in every maximal right ideal, and let $f\colon B \to A$ be arbitrary. It follows that $gf\colon B \to B$ is contained in every maximal right ideal of $\mathbf{C}$.

This includes ideals $\mathbf{M}$ thus formed: Let I be a maximal right ideal of $\mathbf{C}(B)$, and let $\mathbf{M}_B$ be a maximal subfunctor of $\mathbf{C}(-, B)$ containing the subfunctor $I\mathbf{C}(-, B)$ (Corollary 2.28). Let $\mathbf{M}_Y = \mathbf{C}(-, Y)$ for every $Y \neq B$. Thus $\mathbf{M}$ is a maximal right ideal of $\mathbf{C}$ (Corollary 2.27). Notice that $\mathbf{M}_B(B) = I$ (Proposition 2.29). Therefore $g \in \mathbf{M}$ implies $gf \in \mathbf{M}$, hence $gf \in I$. Since I is arbitrary, $gf \in \mathbf{J}(B)$, hence $1_B - gf$ is an automorphism of B. Because f is arbitrary, $g \in \mathbf{J}(A, B)$. The reverse inclusion is thus proved. □

Proposition 2.33. *The preadditive category* $\mathbf{C}$ *has finite codimension if and only if its Jacobson radical* $\mathbf{J}$ *is the intersection of finitely many maximal right ideals.*

Proof. If $\mathbf{C}$ has finite codimension then there is a bound n on the cardinality of a family of coindependent right ideals of $\mathbf{C}$. Let $\{\mathbf{M}_i\}_{i<n}$ be a coindependent family of maximal right ideals of $\mathbf{C}$ of greatest cardinality. Since $\mathbf{J}$ is the intersection of all maximal right ideals, $\mathbf{J} \leqslant \bigcap_{i<n} \mathbf{M}_i$. Suppose the inclusion is proper. Then there is a maximal right ideal $\mathbf{M}$ such that $\bigcap_{i<n} \mathbf{M}_i$ is not contained in $\mathbf{M}$. Then (*) $\{\mathbf{M}_i\}_{i<n} \cup \{\mathbf{M}\}$ is a coindependent set of maximal right ideals of cardinality $n + 1$, a contradiction, hence $\mathbf{J}$ is the intersection of n maximal right ideals. To see (*), use the dual of the proof of [Fac98, Proposition 2.31]. Indeed, we have

$$\mathbf{M}_i + \left(\mathbf{M} \cap \bigcap_{i\neq j<n} \mathbf{M}_j\right) = \mathbf{M}_i + \left(\bigcap_{j<n} \mathbf{M}_j + \left(\mathbf{M} \cap \bigcap_{i\neq j<n} \mathbf{M}_j\right)\right) =$$

$$= \mathbf{M}_i + \left(\left(\bigcap_{j<n} \mathbf{M}_j + \mathbf{M}\right) \cap \bigcap_{i\neq j<n} \mathbf{M}_j\right) = \mathbf{M}_i + \bigcap_{i\neq j<n} \mathbf{M}_j = \mathbf{C}$$

Conversely, suppose $\mathbf{J} = \bigcap_{i<n} \mathbf{M}_i$ is the finite intersection of the set of maximal right ideals $\{\mathbf{M}_i\}_{i<n}$. We can assume that n is minimal, so that the family is coindependent. (Indeed, if $\mathbf{M}_i + \bigcap_{i\neq j<n} \mathbf{M}_j$ were proper, we would have that the intersection of $n-1$ maximal right ideals $\bigcap_{i\neq j<n} \mathbf{M}_j$ is contained in $\mathbf{M}_i$, hence it would equal $\mathbf{J}$, against minimality of n.) Moreover, the interval $[\mathbf{M}_i, \mathbf{C}]$ is obviously couniform, and the intersection of the family is $\mathbf{J}$, which is a superfluous right ideal (Lemma 2.31). Therefore the codimension of $\mathbf{C}$ is finite and equal to n. □

Proposition 2.34. *Let $\mathbf{C}$ be a preadditive category with at least a non-zero object. Then $\mathbf{C}$ has finite codimension if and only if $\mathbf{C}$ is a semilocal category with finitely many non-zero objects.*

Proof. Suppose $\mathbf{C}$ is a preadditive category with a non-zero object and finite codimension, and write $\mathbf{J} = \bigcap_{i<n} \mathbf{M}_i$ with each $\mathbf{M}_i$ a maximal right ideal. Then for each non-zero object X we have $\mathbf{J}(X) = \bigcap_{i<n} \mathbf{M}_i(X)$. Each $\mathbf{M}_i(X)$ is either equal to $\mathbf{C}(X)$ or a maximal right ideal of $\mathbf{C}(X)$, by Corollary 2.27 and Proposition 2.29. Thus the Jacobson radical of the endomorphism ring $\mathbf{C}(X)$ is the intersection of finitely many (at most n) maximal right ideals. The injective morphism $\mathbf{C}(X)/\mathbf{J}(X) \to \bigoplus \mathbf{C}(X)/\mathbf{M}_i(X)$ shows that $\mathbf{C}(X)/\mathbf{J}(X)$ is a semisimple $\mathbf{C}(X)/\mathbf{J}(X)$-module, that is, $\mathbf{C}(X)/\mathbf{J}(X)$ is a semisimple ring, hence that $\mathbf{C}(X)$ is semilocal. Thus $\mathbf{C}$ is a semilocal category.

Also, to each maximal right ideal $\mathbf{M}_i$ we associate the unique object X_i such that the subfunctor $(\mathbf{M}_i)_{X_i}$ of $\mathbf{C}(-, X_i)$ is proper (Corollary 2.27). If X is a non-zero object, we have $\mathbf{J}(X) \neq \mathbf{C}(X)$, hence $\mathbf{M}_i(X) \neq \mathbf{C}(X)$ for some $i < n$. Thus the mapping $\mathbf{M}_i \mapsto X_i$ is surjective and we have at most n non-zero objects in $\mathbf{C}$.

Conversely, suppose $\mathbf{C}$ is a semilocal category with finitely many objects, say $\{X_i\}_{i<n}$. For each $i < n$ there is a finite set $\{I^{(i,j)}\}_{j<n_i}$ of maximal right ideals of the endomorphism ring $\mathbf{C}(X_i)$ whose intersection is $\mathbf{J}(X_i)$, because $\mathbf{C}(X_i)$ is a semilocal ring.

For every $i < n$ and every $j < n_i$, using Lemma 2.26, Corollary 2.27, and Proposition 2.29, let $\mathbf{M}^{(i,j)}$ be the maximal right ideal of $\mathbf{C}$ defined by

$$\mathbf{M}^{(i,j)}_{X_i}(X_i) = I^{(i,j)}, \text{ and } \mathbf{M}^{(i,j)}_{X_k} = \mathbf{C}(-, X_k) \text{ if } k \neq i.$$

It suffices to prove that the intersection of the maximal right ideals $\mathbf{M}^{(i,j)}$ is the Jacobson radical $\mathbf{J}$ (Proposition 2.33). For every non-zero object X_k of $\mathbf{C}$ we have

$$\begin{aligned}\bigcap_{i<n}\bigcap_{j<n_i} \mathbf{M}^{(i,j)}(X_k) &= \bigcap_{i<n}\bigcap_{j<n_i} \mathbf{M}^{(i,j)}_{X_k}(X_k)\\ &= \bigcap_{j<n_k} \mathbf{M}^{(k,j)}_{X_k}(X_k)\\ &= \bigcap_{j<n_k} I^{(k,j)}\\ &= \mathbf{J}(X_k),\end{aligned}$$

thus said intersection and $\mathbf{J}$ agree on pairs (X_k, X_k). Next notice that $g \in \mathbf{J}(X_h, X_k)$ if and only if $g\mathbf{C}(X_k, X_h) \subseteq \mathbf{J}(X_k)$, so that we conclude by Proposition 2.29. □

Remark 2.35 (Symmetry). The above proof actually shows more, it shows that the notion of codimension for preadditive categories is left-right symmetric, that is, $\mathrm{codim}(\mathbf{C})$ is the same if we use the lattice of left ideals rather than that of right ideals.

To show this, we note that the positive integer n_i can be taken to be the codimension $\mathrm{codim}(\mathbf{C}(X_i))$ of the semilocal ring $\mathbf{C}(X_i)$. Moreover, the $N := \sum_{i<n} n_i$ maximal right ideals $\mathbf{M}^{(i,j)}$ of $\mathbf{C}$ form a coindependent set whose intersection is superfluous (Lemma 2.31), and every interval $[\mathbf{M}^{(i,j)}, \mathbf{C}]$ is trivially couniform. Therefore, we have proved that the codimension $\mathrm{codim}(\mathbf{C}) = N$ of $\mathbf{C}$ is the sum of the codimensions of the endomorphism rings of its non-zero objects. Since for a ring R we have $\mathrm{codim}(R) = \mathrm{codim}(R_R) = \mathrm{codim}({}_RR)$, the conclusion follows.

Chapter 3

The Auslander-Bridger transpose

3.1 The stable category

If $\mathbf{C}$ is a preadditive category and $\mathcal{F}$ is a class of objects of $\mathbf{C}$, there is a canonical factor category $\mathbf{C}/\mathcal{F}$ where all objects in $\mathcal{F}$ become zero objects. This is the factor category $\mathbf{C}/\mathbf{I}_{\mathcal{F}}$ where $\mathbf{I}_{\mathcal{F}}$ is the ideal of $\mathbf{C}$ generated by the class of identity morphisms $\{1_X : X \in \mathcal{F}\}$.

If $\mathbf{C}$ is an additive category and $\mathcal{F}$ is a class of objects of $\mathbf{C}$ closed under biproducts, $\mathbf{I}_{\mathcal{F}}$ can be described as the class of morphisms of $\mathbf{C}$ that factor through an object in $\mathcal{F}$. Indeed, if $g \in \mathbf{I}_{\mathcal{F}}(X, Y)$, then $g = \sum_{i<n} a_i b_i$ for suitable morphisms $a_i \colon F_i \to Y$ and $b_i \colon X \to F_i$, with each F_i in $\mathcal{F}$. Then $g = \left(\sum_{i<n} a_i \pi_i\right)\left(\sum_{j<n} \iota_j b_j\right)$ factors through $\bigoplus_{i<n} F_i$, which is in the family $\mathcal{F}$. The converse is trivial.

If, in addition, $\mathbf{C}$ is idempotent-complete and the class $\mathcal{F}$ is also closed under biproduct factors, then no objects other than those in $\mathcal{F}$ become zero in the quotient $\mathbf{C}/\mathcal{F}$, as we will prove shortly.

Remark 3.1. Suppose $\mathbf{C}$ is additive and $g \in \mathbf{C}(X, Y)$ decomposes as a sum $g = \sum_{i<n} g_i$ of morphisms $g_i \in \mathbf{C}(X, Y)$ such that each g_i factors through some object A_i of $\mathbf{C}$. Then g factors through the biproduct $A = \bigoplus_{i<n} A_i$. To see this, let $\iota_i \colon A_i \to A$ denote the canonical injections and $\pi_i \colon A \to A_i$ the canonical projections of said biproduct. Write g_i as $g_i = \varphi_i \psi_i$ with $\psi_i \colon X \to A_i$ and

$\varphi_i \colon A_i \to Y$. Then we have

$$g = \sum_{i<n} g_i = \sum_{i<n} \varphi_i \psi_i = \sum_{i<n} \varphi_i \pi_i \iota_i \psi_i =$$
$$= \sum_{i<n} \sum_{j<n} \varphi_i \pi_i \iota_j \psi_j = \left(\sum_{i<n} \varphi_i \pi_i \right) \left(\sum_{j<n} \iota_j \psi_j \right),$$

hence g factors through A. In particular, if g is a morphism between two biproducts, we can write g as

$$g = \sum_{i,j} \iota'_i \pi'_i g \iota_j \pi_j = \sum_{i,j} \iota'_i g_{i,j} \pi_j,$$

so that if each entry $g_{i,j}$ of g factors through some $A_{i,j}$, then g factors through $\bigoplus_{i,j} A_{i,j}$.

Lemma 3.2. *Suppose that* $\mathbf{C}$ *is an idempotent-complete additive category and that* $\mathcal{F}$ *is a class of objects of* $\mathbf{C}$ *closed under biproducts and biproduct factors. If* M *and* N *are objects of* $\mathbf{C}$*, the following are equivalent:*

(i) M *and* N *are isomorphic in* $\mathbf{C}/\mathcal{F}$.

(ii) There exist objects X *and* Y *in* $\mathcal{F}$ *such that* $M \oplus X$ *and* $N \oplus Y$ *are isomorphic in* $\mathbf{C}$.

In particular, $\mathcal{F}$ *is the class of objects that become zero objects in* $\mathbf{C}/\mathcal{F}$.

Proof. That (ii) implies (i) follows from the fact that the canonical functor $\mathbf{C} \to \mathbf{C}/\mathcal{F}$ is additive and sends the objects in $\mathcal{F}$ to zero objects. Thus assume (i) that M and N are isomorphic in $\mathbf{C}/\mathcal{F}$. This means that there exist morphisms $f \colon M \to N$ and $g \colon N \to M$ in $\mathbf{C}$ such that $1_M - gf$ and $1_N - fg$ factor through objects in $\mathcal{F}$. Write $1_M - gf = g'f'$ for some $f' \in \mathbf{C}(M, Y)$ and $g' \in \mathbf{C}(Y, M)$, and $Y \in \mathcal{F}$. Then we have

$$1_M = \begin{pmatrix} g & g' \end{pmatrix} \begin{pmatrix} f \\ f' \end{pmatrix}, \qquad e = \begin{pmatrix} f \\ f' \end{pmatrix} \begin{pmatrix} g & g' \end{pmatrix}, \tag{3.3}$$

where e is an idempotent endomorphism of $N \oplus Y$. Since idempotents split in $\mathbf{C}$, we have that

$$1_X = uv, \qquad 1 - e = vu \tag{3.4}$$

for some object X of $\mathbf{C}$ and morphisms $u \in \mathbf{C}(N \oplus Y, X)$ and $v \in \mathbf{C}(X, N \oplus Y)$. Equations 3.3 and 3.4 show that $N \oplus Y$ is the biproduct of M and X in $\mathbf{C}$. Thus it is only left to show that X is in fact in $\mathcal{F}$. Notice that $1 - e$ factors through an object in $\mathcal{F}$. Indeed, in matrix form,

$$1 - e = \begin{pmatrix} 1_N - fg & -fg' \\ -f'g & 1_Y - f'g' \end{pmatrix}$$

and all entries factor through objects in $\mathcal{F}$, hence by Remark 3.1 and the fact that $\mathcal{F}$ is closed under biproducts, $1-e$ factors through an object in $\mathcal{F}$. Therefore we may write $1 - e = u'v'$ with $X' = \mathrm{dom}(u') = \mathrm{codom}(v')$ in the family $\mathcal{F}$. Then

$$1_X = uv = u(vu)v = (uu')(v'v)$$

shows that 1_X factors through an object X' in $\mathcal{F}$, hence X is a biproduct factor of X' (Lemma 1.11), thus X lies in $\mathcal{F}$.

To verify the last assertion, if M becomes a zero object in $\mathbf{C}/\mathcal{F}$, by what has already been proved, $M \oplus X \cong Y$ for some X and Y in $\mathcal{F}$. Since the class $\mathcal{F}$ is closed by biproduct factors, it follows that $M \in \mathcal{F}$. □

Let us specialise the above construction to the category of right R-modules and the class $\mathcal{F}$ of all projective right R-modules, which is closed under arbitrary direct sums and under direct summands. The category $(\mathrm{Mod}\text{-}R)/\mathcal{F}$ is called the *stable category* and is usually denoted by $\underline{\mathrm{Mod}}\text{-}R$. When two modules M and N are isomorphic in the stable category, we also say that M and N are *stably isomorphic,* and it follows from Lemma 3.2 that M and N are stably isomorphic if and only if $M \oplus P \cong N \oplus Q$ for suitable projective modules P and Q.

The full subcategory of the stable category whose objects are the finitely presented right R-modules is denoted $\underline{\mathrm{mod}}\text{-}R$. The analogous constructions for left R-modules are denoted by $R\text{-}\underline{\mathrm{Mod}}$ and $R\text{-}\underline{\mathrm{mod}}$.

3.2 The Auslander-Bridger transpose

The duality after which this subsection is named is a categorical duality from the stable category of finitely presented right R-modules to the stable category of finitely presented left R-modules. Before giving the definition, we recall some general results, that will be useful also later on, and then study the category of morphisms between projective modules.

Recall that if U is an S-R-bimodule, we can consider the U-*dual,* that is, the pair of contravariant additive functors

$$_S\,\mathrm{Hom}_R(-, U) \colon \mathrm{Mod}\text{-}R \to S\text{-}\mathrm{Mod},$$
$$\mathrm{Hom}_S(-, U)_R \colon S\text{-}\mathrm{Mod} \to \mathrm{Mod}\text{-}R.$$

When the bimodule U is clear from the context, we will use $(-)^*$ to denote either of these functors. Recall that M^* is called the U-*dual* of M and M^{**} the U-*double dual,* and similarly for morphisms. For each right R-module or left S-module M, we let

$$\sigma_M(m)(\gamma) = \gamma(m),$$

for $m \in M$ and $\gamma \in M^*$. This defines the *evaluation map* $\sigma_M \colon M \to M^{**}$, which is a module morphism natural in M. A module M is called U-*reflexive* if σ_M is an isomorphism. The class of U-reflexive modules is closed by direct summands and finite direct sums [AF92, Proposition 20.13]. Moreover, if M is U-reflexive, then M^* is also U-reflexive [AF92, Proposition 20.14(3)]. Of particular importance is the following:

Proposition 3.5. [AF92, Proposition 23.1] *The U-dual induces additive categorical dualities between the full subcategory of U-reflexive right R-modules and the full subcategory of U-reflexive left S-modules, indeed, $(-)^{**} \cong 1$ via the evaluation natural isomorphism σ.*

For the details we refer to [AF92, §20 and §23].

In this section, we will focus on the case $U = {}_RR_R$, i.e., the R-dual. The module ${}_RR$ is readily checked to be reflexive, hence all finitely generated free modules ${}_RR^n$ $(n < \omega)$ are reflexive, therefore finitely generated projective modules (= direct summands of R^n for various $n < \omega$) are reflexive modules. Moreover, if P is a finitely generated projective module, then so is its dual P^*. Indeed, since $P \oplus Q \cong {}_RR^n$ for some $n < \omega$, we have $P^* \oplus Q^* \cong R^n_R$. Therefore the categorical duality of Proposition 3.5 restricts to a duality from the category of finitely generated projective right R-modules to that of finitely generated projective left R-modules, and $(-)^{**}$ is naturally isomorphic to the identity functor (via the evaluation map σ).

Let $\mathcal{P}_R$ be the class of finitely generated projective right R-modules, and $\mathrm{Morph}(\mathcal{P}_R)$ the *morphism category* of this class of modules. The objects of this category are the R-module morphisms between objects in the class $\mathcal{P}_R$. We will denote by P the object $\mu_P\colon P_0 \to P_1$. A morphism $u\colon P \to Q$ in the morphism category is a pair of R-module morphisms (u_0, u_1) such that $u_1\mu_P = \mu_Q u_0$.

$$\begin{array}{ccc} P & \quad & P_0 \xrightarrow{\ \mu_P\ } P_1 \\ {\scriptstyle u}\downarrow & & {\scriptstyle u_0}\downarrow \qquad\quad \downarrow{\scriptstyle u_1} \\ Q & & Q_0 \xrightarrow[\ \mu_Q\ ]{} Q_1 \end{array}$$

A morphism (u_0, u_1) in the morphism category is an isomorphism if and only if both u_0 and u_1 are isomorphisms (of R-modules).

From the considerations in the previous paragraphs, it follows that the additive contravariant functor $(-)^* = \mathrm{Hom}(-, R)$ induces an additive duality

$$\mathrm{Morph}(\mathcal{P}_R) \to \mathrm{Morph}({}_R\mathcal{P}), \tag{3.6}$$

and $(-)^{**}$ is naturally isomorphic to the identity functor. Precisely, for any object P of the morphism category, we have that $\sigma_P\colon P \to P^{**}$ is the pair $\sigma_P = (\sigma_{P_0}, \sigma_{P_1})$.

The duality (3.6) is used to define the Auslander-Bridger transpose, a duality between the stable module categories $\underline{\mathrm{mod}}$-R and R-$\underline{\mathrm{mod}}$. The reason is that these stable categories are equivalent to factors of the morphism categories of $\mathcal{P}_R$ and ${}_R\mathcal{P}$ respectively. There exists a canonical additive full and dense functor

$$\mathrm{C}\colon \mathrm{Morph}(\mathcal{P}_R) \to \underline{\mathrm{mod}}\text{-}R$$

which sends an object P to the right R-module $\mathrm{coker}(\mu_P) = P_1/\mu_P(P_0)$. For a morphism $u\colon P \to Q$, we let $\mathrm{C}(u)$ be the equivalence class in $\underline{\mathrm{mod}}$-R of the morphism mapping $x + \mu_P(P_0)$ to $u_1(x) + \mu_Q(Q_0)$.

$$\begin{array}{ccccccc}
P_0 & \xrightarrow{\mu_P} & P_1 & \longrightarrow & P_1/\mu_P(P_0) & \longrightarrow & 0 \\
{\scriptstyle u_0}\downarrow & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle \mathrm{C}(u)} & & \\
Q_0 & \xrightarrow[\mu_Q]{} & Q_1 & \longrightarrow & Q_1/\mu_Q(Q_0) & \longrightarrow & 0
\end{array}$$

Every finitely presented module is (by definition) isomorphic to the cokernel of a morphism between finitely generated projective modules, thus C is dense. By the lifting property of projective modules, C is full.

Lemma 3.7. *The kernel $\mathbf{K}_R$ of the functor* C *consists of those morphisms $u\colon P \to Q$ such that there is a morphism $f\colon P_1 \to Q_0$ such that $u_1\mu_P = \mu_Q f \mu_P$.*

Proof. If $\mathrm{C}(u)$ factors through a projective module, then it also factors through the epimorphism π_Q, say $\mathrm{C}(u) = \pi_Q g$.

$$\begin{array}{ccccccc}
P_0 & \xrightarrow{\mu_P} & P_1 & \xrightarrow{\pi_P} & \mathrm{C}(P) & \longrightarrow & 0 \\
{\scriptstyle u_0}\downarrow & \swarrow{\scriptstyle f} & \downarrow{\scriptstyle u_1} & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle \mathrm{C}(u)} & & \\
Q_0 & \xrightarrow[\mu_Q]{} & Q_1 & \xrightarrow[\pi_Q]{} & \mathrm{C}(Q) & \longrightarrow & 0
\end{array}$$

Then the image of $u_1 - g\pi_P$ sits inside the kernel of π_Q, hence $u_1 - g\pi_P$ factors through μ_Q, say $u_1 = g\pi_P + \mu_Q f$. Then $u_1\mu_P = \mu_Q f \mu_P$.

Conversely, suppose there exists an $f\colon P_1 \to Q_0$ such that $u_1\mu_P = \mu_Q f \mu_P$. Define $g\colon \mathrm{C}(P) \to Q_1$ by $g(x + \mu_P(P_0)) = (u_1 - \mu_Q f)(x)$. Then g is well-defined and $\pi_Q g = \mathrm{C}(u)$. Since Q_1 is projective, $\mathrm{C}(u) = 0$. □

The property that allows one to define the Auslander-Bridger transpose is the following:

Lemma 3.8. *For every morphism $u\colon P \to Q$ in the morphism category we have that $u \in \mathbf{K}_R$ if and only if $u^* \in {}_R\mathbf{K}$.*

Proof. One verifies directly that if u is in $\mathbf{K}_R$ then u^* is in ${}_R\mathbf{K}$. In the same way, if u^* is in ${}_R\mathbf{K}$, then u^{**} is in $\mathbf{K}_R$. Using the "evaluation" natural isomorphism σ one sees that this is equivalent to u being in $\mathbf{K}_R$. □

Therefore we can define the Auslander-Bridger transpose to be *a* functor Tr completing the following commutative square:

$$\begin{array}{ccc} \mathrm{Morph}(\mathcal{P}_R)/\mathbf{K}_R & \xrightarrow{(-)^*} & \mathrm{Morph}(_R\mathcal{P})/_R\mathbf{K} \\ \downarrow{\scriptstyle \mathrm{C}} & & \downarrow{\scriptstyle \mathrm{C}} \\ \underline{\mathrm{mod}}\text{-}R & \underset{\mathrm{Tr}}{\dashrightarrow} & R\text{-}\underline{\mathrm{mod}} \end{array} \tag{3.9}$$

The duality Tr is *not* canonically defined, as it depends on the choice of a quasi-inverse of the leftmost additive equivalence C. Also notice that Tr^2 is naturally isomorphic to the identity functor, because $(-)^{**}$ is.

In practice, suppose that M is a finitely presented left R-module. Then we have an exact sequence

$$_RR^m \xrightarrow{-\times A} {_RR^n} \longrightarrow M \longrightarrow 0, \tag{3.10}$$

where A is a suitable $m \times n$ matrix. Taking the dual yields the following exact sequence of right R-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^* & \longrightarrow & (R^n)^* & \xrightarrow{(-\times A)^*} & (R^m)^* & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & R^n & \xrightarrow{A\times -} & R^m & \longrightarrow & \mathrm{Tr}_0(M) \longrightarrow 0 \end{array}$$

The vertical arrows are canonical isomorphism, and $\mathrm{Tr}_0(M)$ is defined to be the cokernel of $A \times -$, and is also often called the Auslander-Bridger *transpose* of M. Then one lets $\mathrm{Tr}(M) = \mathrm{Tr}_0(M)$, except for the fact that $\mathrm{Tr}_0(M)$ is a module while $\mathrm{Tr}(M)$ is an object of $\underline{\mathrm{mod}}$-R. Notice that different choices of the presentation (3.10) may yield non-isomorphic possible choices for $\mathrm{Tr}_0(M)$, although (3.9) assures that all possible choices for $\mathrm{Tr}(M)$ are isomorphic, i.e., the possible choices for $\mathrm{Tr}_0(M)$ are all stably isomorphic.

In some cases though, we have canonical representatives (up to isomorphism) for the stable-isomorphism classes of M and $\mathrm{Tr}_0(M)$, for instance, when the ring R is semiperfect.

Recall that a ring R is *semiperfect* if every finitely generated right R-module has a projective cover, if and only if every simple right R-module has a projective cover, if and only if there is a decomposition $R = \bigoplus_{i<n} e_iR$ where each e_i is a *local idempotent,* i.e., $e_i^2 = e_i$ and e_iRe_i is a local ring [Bas60, Theorem 2.1].

If the base ring is semiperfect, a finitely presented module M decomposes as a direct sum $M = M_0 \oplus P$ with P projective and M_0 without non-zero projective summands, and if $M = M_1 \oplus Q$ is another such decomposition, then $P \cong Q$ and $M_0 \cong M_1$ [War75, Theorem 1.4]. It follows that M_0 is the canonical representative (up to isomorphism) of the stable-isomorphism class of M [War75, Corollary 1.5]. Moreover, M_0 admits a minimal projective presentation, that is, M_0 is the cokernel of a morphism $\mu_P \colon P_0 \to P_1$ such that both $\ker(\mu_P) \leqslant_s P_0$ and $\mu_P(P_0) \leqslant_s P_1$, and the dual morphism P^* has the same properties, that is, both its kernel and its image are superfluous submodules, and $\operatorname{coker}(\mu_P^*)$ has no non-zero projective summands, and is thus the canonical representative for the stable-isomorphism class of $\operatorname{Tr}_0(M)$ [War75, Lemma 2.3].

All these results involving the Auslander-Bridger duality also hold for a more general class of modules [FG11] that will be treated in Chapter 4.

Chapter 4

Auslander-Bridger modules

The Auslander-Bridger duality finds its best applications in the study of finitely presented modules over semiperfect rings, as we have seen at the end of Section 3.2. We will prove that all those nice results about finitely presented modules over a semiperfect ring also hold for the modules M that are cokernels of morphisms between projective modules whose endomorphism rings are semiperfect, equivalently, between two finite direct sums of couniform projective modules (Proposition 4.7). In other words, we will drop all hypotheses on the ring but require a bit more from the presentations, and the same results will hold. Among the cokernels M mentioned above, we consider those that have no non-zero projective summands. We call these modules *Auslander-Bridger modules*. Besides extending the well-known theory of finitely presented modules over semiperfect rings, we will see that an Auslander-Bridger module M is characterised up to isomorphism by two invariants, namely, its epi-isomorphism class $[M]_{\cong_e}$ and its lower-isomorphism class $[M]_{\cong_\ell}$. In addition, the Auslander-Bridger transpose preserves Auslander-Bridger modules and exchanges the invariants, that is, $[M]_{\cong_e} = [N]_{\cong_e}$ if and only if $[\mathrm{Tr}_0(M)]_{\cong_\ell} = [\mathrm{Tr}_0(N)]_{\cong_\ell}$ and $[M]_{\cong_\ell} = [N]_{\cong_\ell}$ if and only if $[\mathrm{Tr}_0(M)]_{\cong_e} = [\mathrm{Tr}_0(N)]_{\cong_e}$. Via a suitable duality, we also study the class of modules M that have no non-zero injective summands, have finite Goldie dimension, and such that $E(M)/M$ also has finite Goldie dimension. These will be called *dual Auslander-Bridger modules.*

Notation 4.1. conceptdual Auslander-Bridger modules

4.1 Couniform projective modules

In this section we set about to investigate the peculiar class of projective modules that are finite direct sums of couniform submodules.

Recall that a module M is called *local* if it has a largest proper submodule, that is, if M is a cyclic module with a unique maximal submodule. An idempotent e of a ring R is called *local* if eRe is a local ring.

Couniform projective modules are characterised by many equivalent conditions, cf. [AAF08, Lemma 8.7] and [FG10, Lemma 2.2].

Lemma 4.2. *Let R be an arbitrary ring. For a projective right R-module P the following are equivalent:*

(i) P is couniform.

(ii) P is the projective cover of a simple module.

(iii) P is the projective cover of a couniform module.

(iv) P is local.

(v) $\mathrm{End}_R(P)$ is a local ring.

(vi) There exists a local idempotent e of R such that $P \cong eR$.

Moreover, if these equivalent conditions hold, then $\mathrm{Hom}(P, R_R) \cong Re$ is a couniform projective left R-module.

Proof. ((i) $\Rightarrow$ (ii)) Suppose P is couniform. Since P is a non-zero projective module, it has a maximal submodule M. Thus P/M is simple, and the canonical epimorphism $P \to P/M$ is a projective cover, for $M \leqslant_s P$ because P is couniform.

Since a simple module is trivially couniform, (ii) implies (iii).

((iii) $\Rightarrow$ (i)) Let $g\colon P \to C$ be a projective cover with C couniform. Let K be the kernel of g. Suppose A_1 and A_2 are submodules of P such that $A_1 + A_2 = P$. Then $(A_1+K)/K+(A_2+K)/K = P/K$, thus either $(A_i+K)/K = P/K$ because $P/K \cong C$ is couniform. Then $A_i + K = P$ and $A_i = P$ because $K \leqslant_s P$. This proves that P is couniform.

((i) $\Leftrightarrow$ (iv)) Suppose P is couniform. Since all proper submodules are superfluous, they are all contained in $\mathrm{Rad}(P)$. The latter is a proper submodule because P is a non-zero projective module. Thus $\mathrm{Rad}(P)$ is the largest proper submodule of P. For the converse, notice that any local module is couniform.

((i) $\Rightarrow$ (v)) Since P is couniform, P is indecomposable. Because P is indecomposable and projective, an endomorphism of P is an automorphism if and only if it is surjective. Since P is couniform, the set of non-surjective endomorphisms of P is an ideal of $\mathrm{End}_R(P)$. (Cf. page 105.) This proves that $\mathrm{End}_R(P)$ is local.

((v) $\Rightarrow$ (i)) Suppose $A+B = P$. Define a surjective morphism $\sigma\colon A\oplus B \to P$ by $(a, b) \mapsto a + b$. Since P is projective, there exists $\tau\colon P \to A \oplus B$ such that $1_P = \sigma\tau = \sigma\iota_A\pi_A\tau + \sigma\iota_B\pi_B\tau$. Because $\mathrm{End}_R(P)$ is a local ring, either term is an automorphism of P, say $\sigma\iota_A\pi_A\tau$. Thus $\sigma\iota_A\colon A \to P$, which is the set inclusion of A in P, is surjective, i.e., $A = P$. This proves that P is couniform.

((vi) ⇒ (v)) If $P \cong eR$ and e is a local idempotent of R, then $eRe \cong \mathrm{End}_R(P)$ is local.

((ii) ⇒ (vi)) Suppose P is the projective cover of a simple module S. Recall that S is also a homomorphic image of R, hence P is isomorphic to a direct summand of R, by the fundamental lemma of projective covers. Thus $P \cong eR$ for some idempotent e of R, and $eRe \cong \mathrm{End}_R(P)$ is local by the already proved equivalence of (ii) and (v). Thus e is a local idempotent.

Of course the conditions (i–v) plus the dual of (vi) are all equivalent for a projective left R-module P. The last assertion of the statement then follows from the fact that $\mathrm{Hom}_R(eR, R) \cong Re$ for every idempotent e of R, by the isomorphism $g \mapsto g(e)$. □

Trivially, a projective module P satisfying the equivalent conditions of the previous lemma has the property that every quotient of P has a projective cover. We will see that the projective modules with this property are precisely the projective lifting modules.

Lemma-Definition 4.3. *A module M is a* lifting module *if, for every submodule U of M, there exists a direct summand K of M contained in U such that $U/K \leqslant_s M/K$. This last condition is equivalent to $U \cap H \leqslant_s H$ for some (and for every) complement H of K in M.*

Proof. Suppose K is a direct summand of M below U. Let H be any complement of K in M. In the following commutative square, all morphisms are canonical.

$$\begin{array}{ccc} U/K & \longrightarrow & M/K \\ \uparrow & & \uparrow \\ U \cap H & \longrightarrow & H \end{array}$$

The vertical canonical epimorphisms are both isomorphisms, because $M = H \oplus K$, and, by the modular law, $U = (U \cap H) \oplus K$. Therefore, $U/K \leqslant_s M/K$ is equivalent to $U \cap H \leqslant_s H$. □

Remark 4.4. Notice that for a submodule U of a lifting module M, we have $U \leqslant_s M$ if and only if U contains no non-zero summands of M.

Suppose $U \leqslant_s M$ and that $M = M' \oplus M''$ with $M' \leqslant U$. Since $U + M'' = M$, we have $M'' = M$ and $M' = 0$. Thus U contains no non-zero summands of M. Conversely, suppose the latter holds. Since M is lifting, we can write $M = K \oplus H$ with $K \leqslant U$ and $U \cap H \leqslant_s H$. By hypothesis $K = 0$, hence $H = M$ and $U \leqslant_s M$.

Lemma 4.5. *A projective module P is a lifting module if and only if every quotient of P has a projective cover.*

Proof. Suppose that every quotient of P has a projective cover. Let U be any submodule of P. Then there is a projective cover $g\colon Q \to P/U$, and of course we have the canonical epimorphism $\pi\colon P \to P/U$. By the fundamental lemma of projective covers [AF92, Lemma 17.17], P has a direct-sum decomposition $P = H \oplus K$ with $H \cong Q$, $K \leqslant \ker(\pi) = U$, and such that the restriction $\pi|_H\colon H \to P/U$ of π is a projective cover for P/U. In particular, $\ker(\pi|_H) = U \cap H \leqslant_s H$. Thus P is lifting.

Conversely, suppose P is a projective lifting module. Let M be a quotient of P, which means that $M = P/U$ for some submodule U of P. Decompose P as $P = H \oplus K$ with $K \leqslant U$ and $U \cap H \leqslant_s H$. Then the canonical morphism $H \to P/U$ is a projective cover. □

A subclass of projective lifting modules is the class of projective modules that are direct sums of finitely many couniform submodules. Before characterising said class of projectives, let us include here a result from [Rou76], for the sake of completeness.[I] We give a simpler proof using the dual Goldie dimension.

Lemma 4.6. [Rou76, Corollaire 1.2] *Suppose P is a projective module that is the direct sum of n couniform submodules, say $P = \bigoplus_{i<n} P_i$, and suppose that L is a couniform submodule of P not contained in $\mathrm{Rad}(P)$. Then L is a direct summand of P.*

Proof. For some $i < n$, the restriction $\pi_i|_L\colon L \to P_i$ is surjective. If not, $\pi_i(L) \leqslant \mathrm{Rad}(P_i)$ for all $i < n$, hence $\pi_i(L) \leqslant \mathrm{Rad}(P)$ for all $i < n$, from which $L \leqslant \mathrm{Rad}(P)$, which is false. It follows that $P = L + \sum_{j<n, j\neq i} P_j$, hence that there is a canonical epimorphism $g\colon L \oplus P_0 \oplus \cdots \oplus \widehat{P_i} \oplus \cdots \oplus P_{n-1} \to P$. Since the domain and the codomain of g have the same dual Goldie dimension and P is projective, g is in fact an isomorphism (cf. discussion before Lemma 2.6). This means that the sum $L + \sum_{j<n, j\neq i} P_j$ is direct, therefore L is a direct summand of P. □

Proposition 4.7. *The following conditions are equivalent for a projective right module $P \neq 0$ over an arbitrary ring R:*

(i) *P is a direct sum of finitely many couniform submodules.*

(ii) *P is a finitely generated lifting module and $P/PJ(R)$ is semisimple.*

(iii) *P is the projective cover of a semisimple module T of finite length and every direct summand of T has a projective cover.*

(iv) *$\mathrm{End}_R(P)$ is a semiperfect ring.*

[I] Also, it seems to the author that that paper is scarcely available.

Proof. ((i) $\Rightarrow$ (ii)) Suppose P is a finite direct sum $P = \bigoplus_{i<n} P_i$ of couniform submodules. Since each P_i is a couniform projective module, each P_i is local by Lemma 4.2((i) $\Leftrightarrow$ (iv)), hence cyclic, therefore P is finitely generated. Since $P/PJ(R) \cong \bigoplus_{i<n} P_i/P_iJ(R)$ and each $P_i/P_iJ(R)$ is simple (because $P_iJ(R) = \mathrm{Rad}(P_i)$ is the unique maximal submodule of P_i), we have that $P/PJ(R)$ is a semisimple module. It is left to prove that P is a lifting module, equivalently (Lemma 4.5) that every quotient of P has a projective cover. We do this by induction on $n \geqslant 1$. The case $n = 1$ holds by Lemma 4.2, as it was remarked before Definition 4.3. Assume $n > 1$ and fix a submodule M of P. Recall that $PJ(R)$ is superfluous in P because P is finitely generated, by Nakayama's Lemma. Therefore, if $M \leqslant PJ(R)$, then M is superfluous in P, so that the canonical epimorphism $P \to P/M$ is a projective cover. Hence we can assume that $M \nleqslant PJ(R)$. It follows that $(M + PJ(R))/PJ(R) \cong M/(M \cap PJ(R))$ is a non-zero submodule of the semisimple module $P/PJ(R)$. Thus $M/(M \cap PJ(R))$ contains a simple submodule isomorphic to $P_i/P_iJ(R)$, for some $i < n$. Hence there is a non-zero morphism $P_i \to M/(M \cap PJ(R))$, which lifts to a non-zero morphism $g\colon P_i \to M$, because P_i is projective. Since P_i is local and $g \neq 0$, the image of g is a local submodule L of M, and L is not contained in $PJ(R)$. By Lemma 4.6, L is a direct summand of P. Thus $P = L \oplus Q$ for some $Q \leqslant P$, and $M = L \oplus (M \cap Q)$. Since Q is a direct sum of $n - 1$ couniform projective modules by the Krull-Schmidt Theorem 1.14, the inductive hypothesis implies that $P/M \cong Q/(M \cap Q)$ has a projective cover, as required.

((ii) $\Rightarrow$ (iii)) If (ii) holds, simply let $T = P/PJ(R)$. Then T is semisimple and the canonical epimorphism $P \to T$ is a projective cover, because $PJ(R)$ is superfluous in P by Nakayama's Lemma. Moreover, a direct summand of T is isomorphic to a quotient of P, hence every direct summand of T has a projective cover.

((iii) $\Rightarrow$ (i)) Decompose T as $T = \bigoplus_{i<n} S_i$ with each S_i simple. By (iii), each S_i has a projective cover P_i, therefore $\bigoplus_{i<n} P_i$ is a projective cover of T, hence we must have $P \cong \bigoplus_{i<n} P_i$ by the fundamental lemma of projective covers. To conclude, notice that each P_i is couniform by Lemma 4.2((i) $\Leftrightarrow$ (iii)).

((i) $\Leftrightarrow$ (iv)) Let $S = \mathrm{End}_R(P)$. P is a finite direct sum of couniform projective modules if and only if, Lemma 4.2((i) $\Leftrightarrow$ (ii)), P is a finite direct sum of modules with local endomorphism ring, if and only if S has complete orthogonal set of local idempotents (recall that $\mathrm{End}_R(eP) \cong eSe$ for $e \in S$ idempotent), that is S is a semiperfect ring (cf. page 62). □

Recall the duality between the category of finitely generated projective right R-modules and the category of finitely generated projective left R-modules induced by $(-)^* = \mathrm{Hom}_R(-, R)$ in Section 3.2. Said duality restricts to one between finite direct sums of couniform projective right R-modules and finite direct sums of couniform projective left R-modules, as it follows from the isomorphism $(eR)^* \cong Re$ and by additivity of $(-)^*$. Here are some useful features of this duality:

Lemma 4.8. *Let $g\colon Q \to P$ be a morphism between finite direct sums of couniform projective modules. Then:*

(i) $g(Q)$ is superfluous in P if and only if $g^*(P^*)$ is superfluous in Q^*.

(ii) $\ker(g^*)$ is superfluous in P^* if and only if $g(Q)$ is not contained in a proper direct summand of P.

Proof. If $g(Q)$ is not superfluous in P, there exists a non-zero direct summand A of P such that $A \leqslant g(Q)$ (Remark 4.4). Let $\pi\colon P \to A$ be an epimorphism that is the identity on A. Now A is projective and $\pi g\colon Q \to A$ is onto, so that there exists $\alpha\colon A \to Q$ with $\pi g \alpha = 1_A$. Then $1_{A*} = \alpha^* g^* \pi^*$, so that $g^*\pi^*(A^*)$ is a non-zero direct summand of Q^* contained in $g^*(P^*)$. Therefore $g^*(P^*)$ is not superfluous in Q^*.

Now suppose $g^*(P^*)$ not superfluous in Q^*. (The modules P^* and Q^* are also finite direct sums of couniform submodules, as remarked above.) By what has just been shown, $g^{**}(Q^{**})$ is not superfluous in P^{**}. By applying the "evaluation" natural isomorphism σ (Section 3.2), we see that this means that $g(Q)$ is not superfluous in P.

Assume that $g(Q)$ is contained in a proper direct summand of P. Thus there is a decomposition $P = A \oplus B$ with $B \neq 0$ and $g(Q) \leqslant A$. Then $\pi_B g = 0$, from which $g^*\pi_B^* = 0$. Thus $\ker(g^*)$ contains a non-zero direct summand of P^*, isomorphic to B^*. Therefore, $\ker(g^*)$ is not superfluous in P^*.

Conversely, suppose that $\ker(g^*)$ not superfluous in P^*. Thus there is a decomposition $P^* = A \oplus B$ with $A \neq 0$ and $g^*\iota_A = 0$. It follows that $g^{**} = \pi_B^*\iota_B^* g^{**}$, hence that the image of g^{**} is contained in the direct summand $\pi_B^*(B^*) \cong B^*$ of P^{**}, and such direct summand is proper because $A^* \neq 0$. Hence $g^{**}(Q^{**})$ is contained in a proper direct summand of P^{**}. Applying the "evaluation" natural isomorphism σ, we see that $g(Q)$ is contained in a proper direct summand of P. □

4.1.1 Couniform projective objects

In this section we will show how several facts about couniform projective modules and their finite direct sums extend, curiously, to the context of preadditive categories.

Throughout this section we will work inside a preadditive category. We will occasionally expand our environment to the additive closure in order to consider (formal) biproducts, as notation such as (X, Y) will suggest.

Definition 4.9. We say that a morphism $f\colon A \to X$ has *superfluous image* if, whenever $(f, g)\colon (A, B) \to X$ is an epimorphism, then $g\colon B \to X$ is an epimorphism.

It is easy to see that if we assume that the objects are modules and the morphisms are module morphisms, then this notion coincides with the usual one.

Here is the analogue of [AF92, Proposition 17.11] for projective objects of preadditive categories.

Proposition 4.10. *Let P be a projective object and S its endomorphism ring. For an endomorphism f of P, the following are equivalent:*

(i) For every $g\colon X \to P$, if $(f, g)\colon (P, X) \to P$ is an epimorphism, then g is an epimorphism; that is, f has superfluous image.

(ii) The endomorphism f is in the Jacobson radical of S.

Proof. Suppose (i) holds and suppose that $fS + I = S_S$. Then $fs + g = 1_P$ for some $s \in S$ and some $g \in I$. Then $(f, g)\colon (P, P) \to P$ is an epimorphism. Indeed, $h(f, g) = 0$ if and only if $hf = 0$ and $hg = 0$, which implies $h = h(fs + g) = 0$. Therefore $g\colon P \to P$ is an epimorphism. Since P is projective, there is $g' \in S$ such that $gg' = 1_P$, hence $I = S_S$. This shows that fS is a superfluous submodule of S_S, hence contained in $J(S)$.

Assume that (ii) holds and let $(f, g)\colon (P, X) \to P$ be an epimorphism. Let $(f', g')^T\colon P \to (P, X)$ be such that $(f, g)(f', g')^T = 1_P$, that is, $gg' = 1_P - ff'$. Since $f \in J(S)$, it follows that gg' is an invertible element of S, hence g is an epimorphism. □

Definition 4.11. A non-zero object C is called *couniform* if, whenever a morphism of the form $(f, g)\colon (X, Y) \to C$ is an epimorphism, either $f\colon X \to C$ or $g\colon Y \to C$ is an epimorphism. In other words, all morphisms into C that are not epimorphisms have superfluous image.

The definition is coherent with the usual notion of a couniform right R-module, that is, a couniform right R-module is precisely a couniform object of the category of right R-modules.

As in the case of modules (Lemma 4.2), we have that:

Proposition 4.12. *For a projective object P, the following are equivalent:*

(i) P is couniform.

(ii) P has local endomorphism ring.

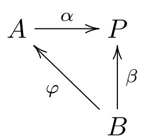
Proof. Suppose P has local endomorphism ring and $(\varphi_A, \varphi_B)\colon (A, B) \to P$ is an epimorphism. Since P is projective, there exists $(\psi_A, \psi_B)^T : P \to (A, B)$ such that $1_P = \varphi_A\psi_A + \varphi_B\psi_B$. Since the endomorphism ring of P is local, either of the two terms is an automorphism of P, say $\varphi_A\psi_A$, and from this it follows that φ_A is an epimorphism. This shows that P is couniform.

Conversely, suppose P is couniform. Let f and g be endomorphisms of P such that $f + g$ is an automorphism of P. We show that either f or g is an automorphism of P, hence the sum of two non-automorphisms is a non-automorphism and the endomorphism ring of P is local.

Since $f + g$ is an automorphism of P, we have that $(f, g)\colon (P, P) \to P$ is an epimorphism, so that either f or g is an epimorphism. Without loss of generality, assume that f is an epimorphism. Then there is an endomorphism f' of P such that $ff' = 1_P$, because P is projective. If, by contradiction, f' is not an epimorphism, then $f'f$ is also not an epimorphism. Since $(f'f, 1_P - f'f)\colon (P, P) \to P$ is an epimorphism and P is couniform, necessarily $1_P - f'f$ is an epimorphism. Then $f(1_P - f'f) = 0$ implies $f = 0$ and $1_P = ff' = 0$, a contradiction. Therefore f' is necessarily an epimorphism. Thus $ff' = 1_P$ implies $(f'f - 1_P)f' = 0$ which implies also $f'f = 1_P$, hence f is an automorphism of P, as required. □

The characterisation of superfluous submodules of a finite direct sum of couniform projective modules (Proposition 4.7 and Remark 4.4) carries over to biproducts of couniform projective objects.

Proposition 4.13. *Suppose $P = (P_i)_{i<n}$ is a biproduct of couniform projective objects. Then a morphism $\alpha\colon A \to P$ has superfluous image if and only if no non-zero splitting monomorphisms $\beta\colon B \to P$ factor through α, that is, whenever there is a commutative triangle*



(4.14)

with β a splitting monomorphism, we have $\beta = 0$.

Proof. Suppose the morphism α has superfluous image. Since $\beta = \alpha\varphi$ is a splitting monomorphism, there is $\beta'\colon P \to B$ such that $\beta'\beta = 1_B$. It is easy to see that $(\alpha, 1_P - \beta\beta')\colon (A, P) \to P$ is an epimorphism. Then $1_P - \beta\beta'$ is an epimorphism, because α has superfluous image. Now $\beta'(1_P - \beta\beta') = 0$, hence $\beta' = 0$, and then $\beta = \beta(\beta'\beta) = 0$.

Next assume that α does not have superfluous image, so that there exists $\beta\colon B \to P$ such that $(\alpha, \beta)\colon (A, B) \to P$ is an epimorphism but β is not. Then $\pi_0(\alpha, \beta)\colon (A, B) \to P_0$ is also an epimorphism, and since P_0 is projective, there is $(\psi_A, \psi_B)^T : P_0 \to (A, B)$ such that $1_{P_0} = \pi_0\alpha\psi_A + \pi_0\beta\psi_B$. Since P_0 has local endomorphism ring (Proposition 4.12), either term is an automorphism of P_0. Without loss of generality we assume that $\pi_0\alpha\psi_A$ is an automorphism of P_0, with inverse, say, ϑ. Then we have the commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & P \\ & {\scriptstyle \psi_A\vartheta}\nwarrow & \uparrow{\scriptstyle \alpha\psi_A\vartheta} \\ & & P_0 \end{array}$$

where $\alpha\psi_A\vartheta$ is a non-zero splitting monomorphism factoring through α. □

4.2 Auslander-Bridger modules

We are ready to give the most important definitions of this chapter.

Definition 4.15. Over any ring R, let $\mathcal{P}^c$ denote the class of projective right R-modules that satisfy the equivalent conditions of Proposition 4.7. (The symbol $\mathcal{P}$ has already been used earlier for finitely generated projective modules.) A right R-module M is said to be $\mathcal{P}^c$-finitely presented if there exists a short exact sequence

$$P_1 \xrightarrow{g} P_0 \xrightarrow{f} M \longrightarrow 0\ , \qquad (4.16)$$

with $P_0, P_1 \in \mathcal{P}^c$, which we call a *presentation* of M. Such presentation is said to be *minimal* if both $\ker(g) \leqslant_s P_1$ and $g(P_1) \leqslant_s P_0$. An *Auslander-Bridger module* is a $\mathcal{P}$-finitely presented module with no non-zero projective direct summands. The full subcategory of $\text{Mod-}R$ whose objects are the Auslander-Bridger modules is denoted $\mathbf{AB}$.

Of course we have analogous notions for left R-modules. We will use a left or right index $_R$ to make the side clear when needed.

Let us first point out that we can assume that (4.16) is a minimal presentation:

Lemma 4.17. *Every $\mathcal{P}^c$-finitely presented module has a minimal presentation.*

Proof. Start with a presentation (4.16). Since M is isomorphic to a quotient of P_0 and P_0 is a projective lifting module (Proposition 4.7), M has a projective cover. By the fundamental lemma of projective covers, there is a decomposition $P_0 = Q_0 \oplus C$ such that $f|_{Q_0} \colon Q_0 \to M$ is a projective cover and $C \leqslant \ker(f)$. Notice that Q_0 is in $\mathcal{P}^c$, because $\mathcal{P}^c$ is closed by direct summands. Also, we have that $g(P_1) = \ker(f) = C \oplus (Q_0 \cap g(P_1)) = C \oplus \ker(f|_{Q_0})$ by the modular law, hence $\ker(f|_{Q_0})$ is isomorphic to a quotient of P_1. As above, we see that $\ker(f|_{Q_0})$ has a projective cover $Q_1 \to \ker(f|_{Q_0})$ and that Q_1 is a direct summand of P_1, hence in $\mathcal{P}^c$. The composite $Q_1 \to \ker(f|_{Q_0}) \leqslant Q_0 \to M \to 0$ is the minimal presentation required. □

Lemma 4.18. *Let X be a homomorphic image of a projective module in $\mathcal{P}^c$. Then X decomposes as $X = N \oplus P$ where P is in $\mathcal{P}^c$ and N has no non-zero projective summands. Moreover, if $X = N' \oplus P'$ is another such decomposition of X, then $N \cong N'$ and $P \cong P'$.*

Proof. Notice that the dual Goldie dimension $\operatorname{codim}(X)$ is finite. Indeed, since X is isomorphic to a quotient of a projective Q in $\mathcal{P}^c$, it has a projective cover Q', isomorphic to a direct summand of Q, hence in $\mathcal{P}^c$. Thus $\operatorname{codim}(X) = \operatorname{codim}(Q')$, which exists and is finite (Proposition 2.5).

Then we can prove the existence of the decomposition by induction on $\operatorname{codim}(X)$. If X has no non-zero projective summands, then we just let $N = X$ and $P = 0$. (This includes the base step of the induction.) Otherwise, $X = P \oplus Y$ with P a non-zero projective. Notice that P is isomorphic to a direct summand of Q', hence P lies in $\mathcal{P}^c$. Moreover, Y is a homomorphic image of Q' and $\operatorname{codim}(Y) < \operatorname{codim}(X)$, so by the inductive hypothesis the required decomposition exists for Y, hence for X.

Again by induction on $\operatorname{codim}(X)$, we prove the essential uniqueness of the above decomposition. Suppose $X = N \oplus P = N' \oplus P'$. If $P = 0$, then also $P' = 0$, because N has no non-zero projective summands, hence $N = N'$. (This includes the base step of the induction.) If $P \neq 0$, it has a couniform direct summand C, hence $P = C \oplus Q$ for some complement $Q \leqslant P$. Since C has local endomorphism ring, it is isomorphic to a direct summand of either N' or P', hence necessarily of P'. Therefore $P' \cong C \oplus Q'$, and cancelling out C (Theorem 2.13) yields $N \oplus Q \cong N' \oplus Q'$. The conclusion follows by the inductive hypothesis. □

Lemma 4.19. *Both the class of $\mathcal{P}^c$-finitely presented modules and the class of Auslander-Bridger modules are closed by finite direct sums and direct summands.*

Proof. It is clear that finite direct sums of $\mathcal{P}^c$-finitely presented modules are $\mathcal{P}^c$-finitely presented. Suppose M is $\mathcal{P}^c$-finitely presented with a minimal presentation (4.16), and suppose that M decomposes as $M = A \oplus B$. Since A and B are isomorphic to quotients of P_0 and P_0 is a projective lifting module, both A and B have a projective cover, say $f_A \colon P_A \to A$ and $f_B \colon P_B \to B$ respectively. By the fundamental lemma of projective covers, we have a commutative diagram

$$\begin{array}{ccccccc} P_1 & \xrightarrow{g} & P_0 & \xrightarrow{f} & M & \longrightarrow & 0\,, \\ & & \downarrow{\scriptstyle \psi} & & \| & & \\ & & P_A \oplus P_B & \xrightarrow[f_A \oplus f_B]{} & A \oplus B & & \end{array}$$

where ψ is an isomorphism. It follows that P_A and P_B are in $\mathcal{P}^c$. Now $\psi g(P_1) = \ker(f_A \oplus f_B) = \ker(f_A) \oplus \ker(f_B)$, so that $\ker(f_A)$ and $\ker(f_B)$ are isomorphic to quotients of P_1, hence each of $\ker(f_A)$ and $\ker(f_B)$ has a projective cover, and it lies in $\mathcal{P}^c$. Thus A and B are $\mathcal{P}^c$-finitely presented.

Now, if M is an Auslander-Bridger module and N is a direct summand of M, then N is $\mathcal{P}^c$-finitely presented. Since M has no non-zero projective summands, neither has N, thus N is an Auslander-Bridger module.

Finally, if M and N are Auslander-Bridger modules, then the direct sum $M \oplus N$ is $\mathcal{P}$-finitely presented. It is left to prove that it has no non-zero projective summand. Suppose P is a non-zero projective direct summand of $M \oplus N$. Then P is isomorphic to a direct summand of $P_M \oplus P_N$, where P_M is a projective cover of M and P_N is a projective cover of N. It follows that P is in $\mathcal{P}^c$. Then P, hence $M \oplus N$, has a direct summand C that is couniform, and as such C has local endomorphism ring. The identity of C factors through $M \oplus N$, say $1_C = fg$ for some $g\colon C \to M \oplus N$ and a suitable $f\colon M \oplus N \to C$. Then $1_C = f\iota_M\pi_M g + f\iota_N\pi_N g$, and since the endomorphism ring of C is local, one of the two terms is an automorphism of C. It follows that the identity of C factors through either M or through N, hence that C is isomorphic to a couniform projective direct summand of either M or N, which is not possible. □

Lemma 4.20. *Let M and N be Auslander-Bridger modules. Then M and N are stably isomorphic if and only if they are isomorphic.*

Proof. If M and N are stably isomorphic, then $M \oplus P \cong N \oplus Q$ for suitable projective modules P and Q, cf. Section 3.1. If we prove that we can assume that P and Q are in $\mathcal{P}^c$, we conclude by Lemma 4.18.

It is easy to see that a morphism $M \to N$ between Auslander-Bridger modules factors through a projective if and only if it factors through a projective in $\mathcal{P}^c$, namely, through the projective cover of N. Thus M and N are isomorphic in the stable category if and only of they are isomorphic in $(\text{Mod-}R)/\mathcal{P}^c$. Thus by Lemma 3.2 we can assume that P and Q above are in fact in $\mathcal{P}^c$. □

Let M be a $\mathcal{P}^c$-finitely presented module. By Lemma 4.18 $M = N \oplus P$ with P in $\mathcal{P}^c$ and N with no non-zero projective summands. Since N is $\mathcal{P}^c$-finitely presented by Lemma 4.19, it is an Auslander-Bridger module. Hence M is stably isomorphic to N, that is, the stable isomorphism class of M is represented by the Auslander-Bridger module N. Lemma 4.20 tells us that such N is unique up to isomorphism. Hence Auslander-Bridger modules are canonical representatives of stable isomorphism classes of $\mathcal{P}$-finitely presented modules.

For every Auslander-Bridger module M fix a minimal presentation

$$Q_M \xrightarrow{\vartheta_M} P_M \xrightarrow{\pi_M} M \longrightarrow 0\,. \tag{4.21}$$

Applying $\text{Hom}(-, R)$ to ϑ_M and taking the cokernel we get the exact sequence

$$P_M^* \xrightarrow{\vartheta_M^*} Q_M^* \longrightarrow Q_M^*/\vartheta_M^*(P_M^*) \longrightarrow 0\,. \tag{4.22}$$

Lemma 4.8 tells us that $Q_M^*/\vartheta_M^*(P_M^*)$ is an Auslander-Bridger module and (4.22) is a minimal presentation. (This generalises [War75, Lemma 2.3].) As we have seen at the end of Section 3.2, the module $\mathrm{Tr}_0(M) := Q_M^*/\vartheta_M^*(P_M^*)$ is an eligible choice for the Auslander-Bridger transpose of M, and is actually the best possible choice, since $\mathrm{Tr}_0(M)$ is the canonical representative (up to isomorphism) of its stable isomorphism class, i.e., of the isomorphism class of the object $\mathrm{Tr}(M)$ of the stable category. Cf. Section 3.2. (Here by $\mathrm{Tr}_0(M)$ we mean the module $Q_M^*/\vartheta_M^*(P^*)$, the Auslander-Bridger *transpose* of M, while $\mathrm{Tr}(M)$ stands for the same module but seen as an object of the stable category, again called the Auslander-Bridger *transpose* of M.)

We now have the following extension of [War75, Theorem 2.4].

Theorem 4.23. *Let M and N be Auslander-Bridger right modules. Then:*

(i) $M \cong N$ *if and only if* $\mathrm{Tr}_0(M) \cong \mathrm{Tr}_0(N)$.

(ii) $\mathrm{Tr}_0(\mathrm{Tr}_0(M)) \cong M$.

(iii) $\mathrm{Tr}_0(M \oplus N) \cong \mathrm{Tr}_0(M) \oplus \mathrm{Tr}_0(N)$.

Proof. (i) If M and N are Auslander-Bridger right modules, we have that $M \cong N$ if and only if M and N are stably isomorphic (Lemma 4.20). This happens if and only if $\mathrm{Tr}(M) \cong \mathrm{Tr}(N)$, that is, if and only if $\mathrm{Tr}_0(M)$ and $\mathrm{Tr}_0(N)$ are stably isomorphic, because Tr is a duality. As we have remarked above, both $\mathrm{Tr}_0(M)$ and $\mathrm{Tr}_0(N)$ are Auslander-Bridger modules, hence $\mathrm{Tr}_0(M)$ and $\mathrm{Tr}_0(N)$ are stably isomorphic if and only if $\mathrm{Tr}_0(M) \cong \mathrm{Tr}_0(N)$ (again by Lemma 4.20).

(ii) Since Tr^2 is naturally isomorphic to the identity functor, we know that $\mathrm{Tr}_0(\mathrm{Tr}_0(M))$ is stably isomorphic to M for every finitely presented module M. If M is an Auslander-Bridger module, then so is $\mathrm{Tr}_0(\mathrm{Tr}_0(M))$, from which it follows that $\mathrm{Tr}_0(\mathrm{Tr}_0(M)) \cong M$.

(iii) Since Tr is an additive functor, $\mathrm{Tr}(M \oplus N) \cong \mathrm{Tr}(M) \oplus \mathrm{Tr}(N)$, that is, $\mathrm{Tr}_0(M \oplus N)$ and $\mathrm{Tr}_0(M) \oplus \mathrm{Tr}_0(N)$ are stably isomorphic. Both of them are Auslander-Bridger modules, so that $\mathrm{Tr}_0(M \oplus N) \cong \mathrm{Tr}_0(M) \oplus \mathrm{Tr}_0(N)$. □

The study of biproduct decompositions in an additive category is connected with the study of factorisations in commutative monoids. Let us explain briefly what this means. To every additive category $\mathbf{C}$ we can associate a commutative monoid (possibly large) $\mathrm{Mon}(\mathbf{C})$.[(II)] The underlying class of $\mathrm{Mon}(\mathbf{C})$ is a full class of representatives of objects of $\mathbf{C}$ up to isomorphism. For every object X, we denote by $\langle X \rangle$ its representative in $\mathrm{Mon}(\mathbf{C})$. The operation on the monoid is induced by the construction of biproducts in $\mathbf{C}$, viz., $\langle X \rangle + \langle Y \rangle = \langle X \oplus Y \rangle$. This addition is clearly well-defined, commutative, and there is the identity

[(II)] In the literature, this monoid is usually denoted $V(\mathbf{C})$, but this clashes with the notation we use (and is also typically adopted) for a certain set of ideals of the category $\mathbf{C}$, cf. Section 2.2.

element $\langle 0\rangle$. Moreover, $\mathrm{Mon}(\mathbf{C})$ is *reduced,* that is, no element is invertible, for $\langle X\rangle + \langle Y\rangle = \langle 0\rangle$ means that $X \oplus Y \cong 0$, from which necessarily $X \cong Y \cong 0$, i.e., $\langle X\rangle = \langle Y\rangle = \langle 0\rangle$. Biproduct decompositions of objects of $\mathbf{C}$ translate into decompositions of elements into sums in $\mathrm{Mon}(\mathbf{C})$, or, if we use the multiplicative notation on $\mathrm{Mon}(\mathbf{C})$, into factorisations of elements. For instance, X is a biproduct factor of Y if and only if $\langle X\rangle \leqslant \langle Y\rangle$, that is, there exists $\langle Z\rangle$ such that $\langle X\rangle + \langle Z\rangle = \langle Y\rangle$. Thinking of the operation as multiplication, we would say that $\langle X\rangle$ divides $\langle Y\rangle$. Indecomposable objects correspond to *atoms,* i.e., non-zero elements that cannot be written as a sum of two non-zero elements. Hence a biproduct decomposition into indecomposables corresponds to a factorisation into atoms.

For more information about the connection between commutative monoids and direct-sum decompositions we refer to the survey paper [WW09].

In terms of commutative monoids, Theorem 4.23 can be rephrased as follows:

Corollary 4.24. *The mapping* $\eta\colon \mathrm{Mon}(\mathbf{AB}_R) \to \mathrm{Mon}({}_R\mathbf{AB})$ *defined by the position* $\eta\colon \langle X\rangle \mapsto \langle \mathrm{Tr}_0(X)\rangle$ *is a monoid isomorphism, and* $\eta^2 = \mathrm{id}$.

To be precise, when we write η^2 we mean the composition of the mapping $\eta\colon \mathrm{Mon}(\mathbf{AB}_R) \to \mathrm{Mon}({}_R\mathbf{AB})$ with the mapping $\mathrm{Mon}({}_R\mathbf{AB}) \to \mathrm{Mon}(\mathbf{AB}_R)$ defined analogously.

Proof. From the statement of Theorem 4.23 we glean that η is well-defined and injective (property (i)), and that $\eta^2 = \mathrm{id}$ (property (ii)). In particular, η is surjective. Finally, η respects the operation by property (iii). □

4.3 Epi-isomorphism and lower-isomorphism

Let us turn our attention to the morphism category of $\mathcal{P}^c$ for a moment. Recall that here $\mathcal{P}^c$ denotes the class of finite direct sums of couniform projective modules, i.e., the projective modules of Proposition 4.7. The zero module, that is, the direct sum of the empty family, is a member of $\mathcal{P}^c$. Our interest in $\mathrm{Morph}(\mathcal{P}^c)$ is justified by the fact that among its objects we find the minimal presentations of Auslander-Bridger modules. We will tacitly assume that $\mathcal{P}^c$ contains some non-zero module, otherwise there is little to talk about, that is, no non-zero Auslander-Bridger modules. Because of this assumption, the category of morphisms in question has a non-zero object.

Lemma 4.25. *The morphism category* $\mathrm{Morph}(\mathcal{P}^c)$ *is semilocal.*

Proof. By assumption, it has a non-zero object. Consider any non-zero object P of $\mathrm{Morph}(\mathcal{P}^c)$. By construction, its endomorphism ring E is a subring of the product ring $S := \mathrm{End}_R(P_1) \times \mathrm{End}_R(P_0)$. It consists of those pairs (g_1, g_0) such that (*) $g_0\mu_P = \mu_P g_1$. Now (g_1, g_0) is invertible in S if and only if g_1 and g_0 are both invertible. When this is the case, we obtain from (*) that $\mu_P g_1^{-1} = g_0^{-1}\mu_P$, hence (g_1^{-1}, g_0^{-1}) belongs to E. This shows that the inclusion of rings $\iota\colon E \to S$ is a local ring morphism. Since both $\mathrm{End}_R(P_i)$ are semilocal rings, so is S. Because ι is a local ring morphism, E is also semilocal (Theorem 2.9), as required. □

Since $\mathrm{Morph}(\mathcal{P}^c)$ is a semilocal category, all the machinery of Section 2.2 is available to its study.

In the category of right R-modules, the class of morphisms with superfluous image is an ideal, say $\mathbf{K}$, because the sum of two superfluous submodules is superfluous and superfluous submodules are preserved by module morphisms. It is then clear that the class $\mathbf{K}_0$ (resp. $\mathbf{K}_1$) of morphisms $u\colon P \to Q$ in the category $\mathrm{Morph}(\mathcal{P}^c)$ such that $u_0\colon P_0 \to Q_0$ is in $\mathbf{K}$ (resp. $u_1\colon P_1 \to Q_1$ is in $\mathbf{K}$) is an ideal. (Cf. page 14, preimages of ideals.)

Lemma 4.26. *Every maximal ideal of the semilocal category* $\mathrm{Morph}(\mathcal{P}^c)$ *contains either* $\mathbf{K}_0$ *or* $\mathbf{K}_1$.

Proof. Let $\mathbf{A}_I$ be a maximal ideal of $\mathrm{Morph}(\mathcal{P}^c)$, where I is a maximal ideal of the endomorphism ring E of a non-zero object X of $\mathrm{Morph}(\mathcal{P}^c)$.

We claim that $\mathbf{K}_0(X) \cap \mathbf{K}_1(X) \leqslant J(E)$. Suppose that f_1 is in the intersection, and that f_2 is an arbitrary endomorphism of X. Let $g = f_1 f_2$ and let us show that $1 - g$ is invertible. For each $i < 2$ we have

$$\begin{aligned} X_i = (1_X)_i(X_i) &= ((1-g)+g)_i(X_i) = \\ &= ((1-g)_i + g_i)(X_i) \leqslant (1-g)_i(X_i) + g_i(X_i) \leqslant X_i, \end{aligned} \tag{4.27}$$

hence $(1-g)_i(X_i) + g_i(X_i) = X_i$. Since $g_i(X_i) \leqslant_s X_i$, it follows that $(1-g)_i$ is surjective. Since P_i is a projective module of finite dual Goldie dimension, $(1-g)_i$ is actually an automorphism (Lemma 2.6). Therefore $(1-g)_0$ and $(1-g)_1$ are automorphisms, hence so is $1-g$. This proves our claim.

Recall that $J(E)$ is the intersection of all maximal ideals of E, because E is a semilocal ring (Lemma 2.8). Then $J(E) \leqslant I$. By the claim,

$$\mathbf{K}_0(X)\mathbf{K}_1(X) \leqslant \mathbf{K}_0(X) \cap \mathbf{K}_1(X) \leqslant J(E) \leqslant I.$$

Since a maximal ideal is prime, we have that $\mathbf{K}_i(X) \leqslant I$ for some $i < 2$. Then $\mathbf{K}_i \leqslant \mathbf{A}_I$ by Lemma 1.2, and this completes the proof. □

Theorem 4.28. *The canonical functor*

$$\mathrm{Morph}(\mathcal{P}^c) \to \mathrm{Morph}(\mathcal{P}^c)/\mathbf{K}_0 \times \mathrm{Morph}(\mathcal{P}^c)/\mathbf{K}_1$$

reflects isomorphisms.

Proof. Suppose P and Q are objects of the morphism category in question such that P and Q are isomorphic modulo both $\mathbf{K}_0$ and $\mathbf{K}_1$. Recall that, since $\mathrm{Morph}(\mathcal{P}^c)$ is a semilocal category, Theorem 2.17 grants us a canonical isomorphism reflecting functor

$$\mathrm{Morph}(\mathcal{P}^c) \to \prod_{\mathbf{I} \in V(\mathrm{Morph}(\mathcal{P}^c))} \mathrm{Morph}(\mathcal{P}^c)/\mathbf{I}.$$

By Lemma 4.26, the objects P and Q are isomorphic modulo every maximal ideal $\mathbf{I}$ of $\mathrm{Morph}(\mathcal{P}^c)$, hence P and Q are isomorphic in the morphism category. □

Although we do not need such generality, we note that Lemma 4.26 and Theorem 4.28 hold when $\mathcal{P}^c$ is replaced by the more inclusive class of projective modules of finite dual Goldie dimension, with exactly the same proofs.

To an Auslander-Bridger module M we associate its fixed minimal presentation (4.21), viewed as an object ϑ_M of $\mathrm{Morph}(\mathcal{P}^c)$. A morphism $f\colon M \to N$ between Auslander-Bridger modules lifts to a morphism between their presentations, $(f_1, f_0)\colon \vartheta_M \to \vartheta_N$, thanks to the lifting property of projective modules.

$$\begin{array}{ccccccc} Q_M & \xrightarrow{\vartheta_M} & P_M & \xrightarrow{\pi_M} & M & \longrightarrow & 0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ Q_N & \xrightarrow[\vartheta_N]{} & P_N & \xrightarrow[\pi_N]{} & N & \longrightarrow & 0 \end{array} \tag{4.29}$$

Conversely, a morphism $(f_1, f_0)\colon \vartheta_M \to \vartheta_N$ induces a morphism $f\colon M \to N$. The rule $M \mapsto \vartheta_M$ and $f \mapsto (f_1, f_0)$ does *not* define a functor $\mathbf{AB} \to \mathrm{Morph}(\mathcal{P}^c)$, because multiple choices are possible for the liftings f_1 and f_0. Nevertheless, (f_1, f_0) induce the zero morphism $M \to N$ if and only if there is a morphism $g\colon P_M \to Q_N$ such that $f_0 = \vartheta_N g$, that is, if and only if f_0 factors through ϑ_N. The collection of such morphisms is an ideal $\mathbf{H}$ of $\mathrm{Morph}(\mathcal{P}^c)$, and we obtain a well-defined additive functor $\mathbf{AB} \to \mathrm{Morph}(\mathcal{P}^c)/\mathbf{H}$. The ideal $\mathbf{H}$ is related to, but not quite the same as, the ideal of null-homotopic chain mappings. Indeed, the presentation of M (resp. N) embeds in a projective resolution of M (resp. N).

The important feature of diagram (4.29) is that

Lemma 4.30. *The morphism $f\colon M \to N$ is an isomorphism if and only if both $f_0\colon P_M \to P_N$ and $f_1\colon Q_M \to Q_N$ are isomorphisms. As a consequence, M and N are isomorphic if and only if ϑ_M and ϑ_N are.*

Proof. The lemma holds because the exact rows in (4.29) are the beginning of minimal projective resolutions of M and N respectively. Let us prove it for completeness. If (f_0, f_1) is an isomorphism, i.e., both f_0 and f_1 are isomorphisms, then $f\pi_M = \pi_N f_0$ is surjective, hence f is surjective. A bit of diagram-chasing shows that f is also injective. (Alternatively, one can use the Snake Lemma.) Conversely, if f is an isomorphism, then $\pi_N f_0 = f\pi_M$ is surjective. Because π_N has superfluous kernel, f_0 is surjective as well. Thus f_0 is a splitting epimorphism, so that $\ker(f_0)$ is a direct summand of P_M. Since $\ker(f_0) \leqslant \ker(\pi_M)$ is also superfluous in P_M, it follows that $\ker(f_0) = 0$, so that f_0 is an isomorphism. It also follows that f_0 restricts to an isomorphism $\ker(\pi_M) = \vartheta_M(Q_M) \to \ker(\pi_N) = \vartheta_N(Q_N)$, hence we conclude that f_1 is an isomorphism by repeating the same argument. □

If M and N are Auslander-Bridger right R-modules, we say that M and N are *epi-isomorphic,* or that they have the same *epi-isomorphism class,* and write $M \cong_e N$, if ϑ_M and ϑ_N are isomorphic objects in $\mathrm{Morph}(\mathcal{P}^c)/\mathbf{K}_0$. We also say that M and N are *lower-isomorphic,* or that they have the same *lower-isomorphism class,* and write $M \cong_\ell N$, if ϑ_M and ϑ_N are isomorphic objects of the category $\mathrm{Morph}(\mathcal{P}^c)/\mathbf{K}_1$. Notice that these definitions do not depend on the choice of the minimal presentations ϑ_M and ϑ_N because they are unique up to isomorphism.

The notions of epi-isomorphism and lower-isomorphism just given are equivalent to those introduced in [FG11], as is easily seen. For instance, $M \cong_e N$ if and only if, by definition, ϑ_M and ϑ_N are isomorphic in $\mathrm{Morph}(\mathcal{P}^c)/\mathbf{K}_0$, i.e., if and only if there are morphisms $(f_1, f_0)\colon \vartheta_M \to \vartheta_N$ and $(g_1, g_0)\colon \vartheta_N \to \vartheta_M$ such that $(1_{Q_M} - g_1 f_1, 1_{P_M} - g_0 f_0) \in \mathbf{K}_0$ and $(1_{Q_N} - f_1 g_1, 1_{P_N} - f_0 g_0) \in \mathbf{K}_0$, that is, both $1_{P_M} - g_0 f_0$ and $1_{P_N} - f_0 g_0$ have superfluous image. Thus $M \cong_e N$ if and only if there are morphisms $f\colon M \to N$ and $g\colon N \to M$ such that $1_{P_M} - g_0 f_0$ and $1_{P_N} - f_0 g_0$ have superfluous image, which is the definition of M and N being epi-isomorphic according to [FG11]. Similarly for the notion of lower-isomorphism.

Epi-isomorphism class and lower-isomorphism class characterise Auslander-Bridger modules up to isomorphism:

Proposition 4.31. *If M and N are Auslander-Bridger modules, then $M \cong N$ if and only if M and N are both epi-isomorphic and lower-isomorphic.*

Proof. By Lemma 4.30, M and N are isomorphic if and only if ϑ_M and ϑ_N are isomorphic, and Theorem 4.28 tells us that this happens if and only if they are isomorphic both modulo $\mathbf{K}_0$ and modulo $\mathbf{K}_1$, which means that M and N are both epi-isomorphic and lower-isomorphic. □

Although the positions $M \mapsto \vartheta_M$ and $f \mapsto (f_1, f_0)$ do not define a functor, we have that:

Proposition 4.32. *There is an additive local and isomorphism-reflecting functor*

$$G\colon \mathbf{AB} \to \mathrm{Morph}(\mathcal{P}^c)/\mathbf{K}_0 \times \mathrm{Morph}(\mathcal{P}^c)/\mathbf{K}_1 \tag{4.33}$$

defined by $M \mapsto (\vartheta_M, \vartheta_M)$ *and* $f \mapsto ((f_1, f_0) + \mathbf{K}_0, (f_1, f_0) + \mathbf{K}_1)$*, that is, the reductions of* (f_1, f_0) *modulo* $\mathbf{K}_0$ *and* $\mathbf{K}_1$ *are well-defined. In particular,* $\mathbf{AB}$ *is a semilocal category.*

Proof. Suppose for the moment that the functor is well-defined. If $G(M) \cong G(N)$, then ϑ_M and ϑ_N are isomorphic in $\mathrm{Morph}(\mathcal{P}^c)$ by Theorem 4.28. If $(g_1, g_0)\colon \vartheta_M \to \vartheta_N$ is an isomorphism, then the morphism $M \to N$ it induces is an isomorphism (Lemma 4.30). To see that G is local, suppose $g\colon M \to N$ is such that $G(g)$ is an isomorphism. Hence there are morphisms (f_1, f_0) and (h_1, h_0) such that $1 - (g_1, g_0)(h_1, h_0) \in \mathbf{K}_1$, $1 - (h_1, h_0)(g_1, g_0) \in \mathbf{K}_1$, $1 - (f_1, f_0)(g_1, g_0) \in \mathbf{K}_0$, and $1 - (g_1, g_0)(f_1, f_0) \in \mathbf{K}_0$. These four conditions imply that g_1, h_1, f_0, and g_0 are all surjective. (Otherwise, one can argue as in (4.27) and reach a contradiction.) These surjective morphisms imply that $\mathrm{codim}(P_M) = \mathrm{codim}(P_N)$ and $\mathrm{codim}(Q_M) = \mathrm{codim}(Q_N)$, and that the epimorphisms g_1 and g_0 are in fact isomorphisms. By the remarks before this proposition, g is an isomorphism.

The only thing that remains to prove is that if $f = 0$, necessarily (f_1, f_0) is in both $\mathbf{K}_0$ and $\mathbf{K}_1$, that is, the images of both f_1 and f_0 are superfluous submodules. The minimal projective presentation (4.21) of M yields the short exact sequence

$$0 \longrightarrow Q_M/\ker(\vartheta_M) \xrightarrow{\bar{\vartheta}_M} P_M \xrightarrow{\pi_M} M \longrightarrow 0.$$

Applying the functor $- \otimes \bar{R} = - \otimes_R R/J(R)$, we get an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^R(M, \bar{R}) \xrightarrow{\delta_M} \frac{Q_M}{\ker(\vartheta_M)} \otimes \bar{R} \xrightarrow{\bar{\vartheta}_M \otimes \bar{R}}$$

$$\xrightarrow{\bar{\vartheta}_M \otimes \bar{R}} P_M \otimes \bar{R} \xrightarrow{\pi_M \otimes \bar{R}} M \otimes \bar{R} \longrightarrow 0.$$

The arrow $\bar{\vartheta}_M \otimes \bar{R}$ is the zero morphism, because the image of $\bar{\vartheta}_M$, which is equal to the image of ϑ_M, is superfluous in P_M, hence contained in $P_M J(R)$. Thus the natural (connecting) morphisms δ_M and $\pi_M \otimes \bar{R}$ are isomorphisms. The naturality of these two isomorphisms implies that the morphism $f\colon M \to N$ yields two commutative squares

$$\begin{array}{ccc} \mathrm{Tor}_1^R(M, \bar{R}) & \xrightarrow{\delta_M} & \frac{Q_M}{\ker(\vartheta_M)} \otimes \bar{R} \\ \downarrow{\scriptstyle \mathrm{Tor}_1^R(f, \bar{R})} & & \downarrow{\scriptstyle \bar{f}_1 \otimes \bar{R}} \\ \mathrm{Tor}_1^R(N, \bar{R}) & \xrightarrow[\delta_N]{} & \frac{Q_N}{\ker(\vartheta_N)} \otimes \bar{R} \end{array}$$

and

$$\begin{array}{ccc} P_M \otimes \bar{R} & \xrightarrow{\pi_M \otimes \bar{R}} & M \otimes \bar{R} \\ \downarrow{\scriptstyle f_0 \otimes \bar{R}} & & \downarrow{\scriptstyle f \otimes \bar{R}} \\ P_N \otimes \bar{R} & \xrightarrow[\pi_N \otimes \bar{R}]{} & N \otimes \bar{R} \end{array}$$

Therefore, if $f = 0$, then $f_0 \otimes \bar{R}$ and $\bar{f}_1 \otimes \bar{R}$ are both zero. Recall that $- \otimes_R R/J(R)$ is naturally isomorphic to $-/-J(R)$, hence $f_0(P_M) \leqslant P_M J(R)$ and $f_1(Q_M/\ker(\vartheta_M)) \leqslant (Q_M/\ker(\vartheta_M))J(R)$, which implies $f_1(Q_M) \leqslant Q_M J(R)$. By Nakayama's Lemma, (f_1, f_0) is in both $\mathbf{K}_0$ and $\mathbf{K}_1$.

The fact that $\mathbf{AB}$ is a semilocal category follows from the fact that the class of semilocal rings is closed by quotients, finite products, and that the domain of a local ring morphism into a semilocal ring is again semilocal (Section 2.1.5). □

Recall that the functor $(-)^*$ induces an additive duality $\mathrm{Morph}(\mathcal{P}_R^c) \to \mathrm{Morph}({}_R\mathcal{P}^c)$. Lemma 4.8(i) tells us that, for a morphism $g\colon Q \to P$ in the morphism category of $\mathcal{P}_R^c$, we have that $g \in \mathbf{K}_0$ if and only if $g^* \in \mathbf{K}_1$ and that $g \in \mathbf{K}_1$ if and only if $g^* \in \mathbf{K}_0$. Therefore $(-)^*$ canonically induces additive dualities

$$\mathrm{Morph}(\mathcal{P}_R^c)/\mathbf{K}_0 \to \mathrm{Morph}(\mathcal{P}_R^c)/\mathbf{K}_1,$$
$$\mathrm{Morph}(\mathcal{P}_R^c)/\mathbf{K}_1 \to \mathrm{Morph}(\mathcal{P}_R^c)/\mathbf{K}_0.$$

Also recall how we defined the transpose $\mathrm{Tr}_0(M)$ for an Auslander-Bridger module M: We applied $(-)^* = \mathrm{Hom}_R(-, R)$ to the minimal presentation ϑ_M of M and let $\mathrm{Tr}_0(M) = \mathrm{coker}(\vartheta_M^*)$. In this way, $\mathrm{Tr}_0(M)$ is an Auslander-Bridger left R-module and ϑ_M^* is a minimal presentation of $\mathrm{Tr}_0(M)$ (also by Lemma 4.8). It is then natural to stipulate that the fixed minimal presentation of $\mathrm{Tr}_0(M)$ is $\vartheta_{\mathrm{Tr}_0(M)} = \vartheta_M^*$. From all these considerations it is pretty straightforward to see that the Auslander-Bridger transpose between right and left Auslander-Bridger modules swaps the invariants, i.e., that:

Proposition 4.34. *For Auslander-Bridger modules M and N,*

(i) $M \cong_e N$ *if and only if* $\mathrm{Tr}_0(M) \cong_\ell \mathrm{Tr}_0(N)$*, and*

(ii) $M \cong_\ell N$ *if and only if* $\mathrm{Tr}_0(M) \cong_e \mathrm{Tr}_0(N)$.

Proof. We have that $M \cong_e N$ if and only if ϑ_M and ϑ_N are isomorphic modulo $\mathbf{K}_0$, and this happens if and only if ϑ_M^* and ϑ_N^* are isomorphic modulo $\mathbf{K}_1$, that is, if and only if $\mathrm{Tr}_0(M) \cong_\ell \mathrm{Tr}_0(N)$. Similarly for the other equivalence. □

Theorem 4.35. [FG11, Theorem 5.6] *Let M be a non-zero Auslander-Bridger module and $E = \mathrm{End}_R(M)$. Let*

$$\mathfrak{l} = \{f \in E : \mathrm{Tor}_1^R(f, \bar{R}) = 0\},$$
$$\mathfrak{e} = \{f \in E : f \otimes \bar{R} = 0\}.$$

Then $\mathfrak{l}$ and $\mathfrak{e}$ are proper two-sided ideals of E, the canonical morphism $E \to E/\mathfrak{l} \times E/\mathfrak{e}$ is a local morphism, $\mathfrak{l} \cap \mathfrak{e} \leqslant J(E)$, and every maximal right ideal, every maximal left ideal, every maximal two-sided ideal of E, contains either $\mathfrak{l}$ or $\mathfrak{e}$.

Proof. Everything follows from Proposition 4.32. The local functor G induces a local ring morphism

$$\eta\colon \operatorname{End}_R(M) \to \operatorname{Morph}(\mathcal{P}^c)(\vartheta_M)/\mathbf{K}_0(\vartheta_M) \times \operatorname{Morph}(\mathcal{P}^c)(\vartheta_M)/\mathbf{K}_1(\vartheta_M)$$

In the course of the proof of Proposition 4.32 we have seen that $\operatorname{Tor}_1^R(f, \bar{R}) = 0$ if and only if $f_1(Q_M) \subseteq Q_M J(R)$, i.e., if and only if $(f_1, f_0) \in \mathbf{K}_1$, and that $f \otimes \bar{R} = 0$ if and only if $f_0(P_M) \subseteq P_M J(R)$, i.e., if and only if $(f_1, f_0) \in \mathbf{K}_0$. Hence $\mathfrak{l}$ and $\mathfrak{e}$ are the kernels of the ring morphisms $\pi_0\eta$ and $\pi_1\eta$, thus they are proper two-sided ideals.

Also, η factors as $\eta'\pi$ where $\pi\colon E \to E/\mathfrak{l} \times E/\mathfrak{e}$ is the canonical mapping. Then if $\pi(g)$ is invertible, so is $\eta(g)$, hence so is g, because η is a local morphism. This proves that π is also a local morphism. From this it also follows that $\ker(\pi) = \mathfrak{l} \cap \mathfrak{e}$ has to be contained in $J(E)$. (If $\pi(g) = 0$, for every $f \in E$, we have $\pi(1 - fg) = 1$, so that $1 - fg$ has to be invertible.)

Recall that E is semilocal (Proposition 4.32), hence $J(E)$ is the intersection of all its maximal ideals M. Thus $\mathfrak{l}\mathfrak{e} \leqslant \mathfrak{l} \cap \mathfrak{e} \leqslant J(E) \leqslant M$ implies that $\mathfrak{l} \leqslant M$ or $\mathfrak{e} \leqslant M$, because M is prime. If M is a maximal right (resp. left) ideal of E, notice that M contains the maximal two-sided ideal $\operatorname{r.ann}_E(E/M)$ (resp. $\operatorname{l.ann}_E(E/M)$). ☐

It follows that we have exactly one of the following two conditions:

(i) *The ideals $\mathfrak{l}$ and $\mathfrak{e}$ are comparable.* Suppose for instance $\mathfrak{l} \subseteq \mathfrak{e}$. In this case, every maximal right, left or two-sided ideal of E contains $\mathfrak{l}$, and $J(E)$ is the ideal of E containing $\mathfrak{l}$ such that $J(E)/\mathfrak{l} = J(E/\mathfrak{l})$. Similarly when $\mathfrak{l} \supseteq \mathfrak{e}$.

(ii) *The ideals $\mathfrak{l}$ and $\mathfrak{e}$ are not comparable.* In this case, if $J_\mathfrak{l} \supseteq \mathfrak{l}$ and $J_\mathfrak{e} \supseteq \mathfrak{e}$ are the ideals of E such that $J_\mathfrak{l}/\mathfrak{l} = J(E/\mathfrak{l})$ and $J_\mathfrak{e}/\mathfrak{e} = J(E/\mathfrak{e})$, then $J(E) = J_\mathfrak{l} \cap J_\mathfrak{e}$. By Lemma 2.8, $J_\mathfrak{l}$ is the intersection of the maximal two-sided ideals of E that contain $\mathfrak{l}$ and $J_\mathfrak{e}$ is the intersection of the maximal two-sided ideals of E that contain $\mathfrak{e}$.

We can then rephrase the notions of epi- and lower-isomorphism class as follows. For M and N Auslander-Bridger modules, we have $M \cong_e N$ if and only if there are morphisms $f\colon M \to N$ and $g\colon N \to M$ such that $1_M - gf \in \mathfrak{e}_M$ and $1_N - fg \in \mathfrak{e}_N$. Similarly, we have that $M \cong_\ell N$ if and only if there are morphisms $f\colon M \to N$ and $g\colon N \to M$ such that $1_M - gf \in \mathfrak{l}_M$ and $1_N - fg \in \mathfrak{l}_N$.

As the terminology suggests, there is a connection with the concepts of epigeny class and lower part for couniformly presented modules, that we will study thoroughly in Chapter 6.

Recall that two modules M and N are said to have the same *epigeny class* if there exist two surjective morphisms $M \to N$ and $N \to M$ [Fac96]. Also, if both M and N have a projective cover, then we have the notion of *lower part*. Suppose $p_M : P_M \to M$ and $p_N : P_N \to N$ are projective covers. Then we say that M and N have the same lower part if there are morphisms $f: M \to N$ and $g: N \to M$ such that any two liftings $f_0 : P_M \to P_N$ and $g_0 : P_N \to P_M$ satisfy $f_0(\ker(p_M)) = \ker(p_N)$ and $g_0(\ker(p_N)) = \ker(p_M)$. It is easy to see that the notion is well-defined, that is, that it does not depend on the choice of the liftings f_0 and g_0, or on the choice of projective covers. Indeed, this stems from the fact that if h_0 is a morphism $P_M \to P_N$ such that $\pi_N h_0 = 0$, then $h_0(P_M) \leqslant \ker(p_N) \leqslant_s P_N$. (The notion of lower part was introduced for cyclically presented modules over a local ring [AAF08] and for couniformly presented modules [FG10].)

If M and N are Auslander-Bridger modules, we see that M and N have the same lower part if and only if there exist two morphisms $f: M \to N$ and $g: N \to M$ such that $f_1 : Q_M \to Q_N$ and $g_1 : Q_N \to Q_M$ are surjective.

If M and N are Auslander-Bridger modules, then $M \equiv_e N$ implies that M and N have the same epigeny class. Indeed, suppose that $f: M \to N$ and $g: N \to M$ are such that $1_M - gf \in \mathfrak{e}_M$ and $1_N - fg \in \mathfrak{e}_N$. The image of $1_M - gf$ is equal to that of $(1_M - gf)\pi_M = \pi_M(1_{P_M} - g_0 f_0)$ which is equal to $\pi_M(1_{P_M} - g_0 f_0)(P_M) \leqslant \pi_M(P_M J(R)) \leqslant MJ(R)$. Then we have that $M = 1_M(M) \leqslant gf(M) + MJ(R) \leqslant M$ from which $g(N) = M$, by Nakayama's Lemma. In the same way one shows that $f(M) = N$. Similarly, $M \equiv_\ell N$ implies that M and N have the same lower part. The following result gives a converse to this for the modules of Chapter 6, studied in [FG10].

Lemma 4.36. *Let M and N be non-zero Auslander-Bridger right R-modules.*

(i) Suppose that P_M is couniform and $\mathfrak{e}_M$ is a maximal right ideal (equivalently, a maximal left ideal) of $\mathrm{End}_R(M)$. Then M and N are epi-isomorphic if and only if they have the same epigeny class.

(ii) Assume that Q_M is couniform and $\mathfrak{l}_M$ is a maximal right ideal (equivalently, a maximal left ideal) of $\mathrm{End}_R(M)$. Then M and N are lower-isomorphic if and only if they have the same lower part.

Proof. Both assertions are proved in the same way, hence we only prove (i). Assume that M and N have the same epigeny class. Let $f: M \to N$ and $g: N \to M$ be epimorphisms, so that N and P_N are also couniform modules. Then $g_0 f_0 : P_M \to P_M$ is also surjective, hence $gf \notin \mathfrak{e}_M$, which implies that $gf + \mathfrak{e}_M$ is an invertible element of the division ring $\mathrm{End}_R(M)/\mathfrak{e}_M$, so there exists $h: M \to$

M such that $1_M - hgf \in \mathfrak{e}_M$, thus $1_M - hgf$ is not surjective. It follows that $f(1_M - hgf) = (1_N - fhg)f$ is not surjective, hence $1_N - fhg$ is not surjective. (It is easy to see that a composition ab of morphisms between couniform modules is surjective if and only if both a and b are, cf. [Fac98, Lemma 6.26].) Thus $(1_N - fhg)_0 \colon P_N \to P_N$ is not surjective, so that its image is contained in the unique maximal submodule $P_N J(R)$ of P_N, from which $1_N - fhg \in \mathfrak{e}_N$. Now $f \colon M \to N$ and $hg \colon N \to M$ show that M and N are epi-isomorphic. □

4.4 Application: duals of Auslander-Bridger modules

It is natural to ask if, upon replacing minimal projective presentations by minimal injective copresentations, we obtain similar results. Not only is this the case, but there is a suitable categorical duality that acts as a bridge between the two settings. Through the use of this duality we study what we call *dual Auslander-Bridger modules,* which are defined as the kernels M of morphisms between injective modules of finite Goldie dimension, and such that M has no non-zero injective summands. The category of dual Auslander-Bridger right R-modules will be denoted $\mathbf{DAB}_R$, while ${}_R\mathbf{DAB}$ denotes the category of dual Auslander-Bridger left R-modules.

4.4.1 Duality between uniform injectives and couniform projectives

Recall that a non-zero module is *uniform* if any two non-zero submodules have non-zero intersection, and that a non-zero module is *couniform* if the sum of two proper submodules is a proper submodule. In other words, in a uniform module every non-zero submodule is essential, whereas in a couniform module every proper submodule is superfluous.

Recall that an injective module is uniform if and only if it is the injective envelope of a uniform module, if and only if it is indecomposable, if and only if it has local endomorphism ring, cf. [Fac98, Lemmas 2.24 and 2.25]. Notice the imperfect symmetry with the notion of couniform projective module (Lemma 4.2). We cannot say, for instance, that a uniform injective module is the injective envelope of a simple module. This may or may not happen. On the one hand, $\mathbb{Q}$ as a $\mathbb{Z}$-module is a uniform divisible module and has no simple submodules, because $\mathbb{Q}$ is torsion-free. On the other hand, the Prüfer group $\mathbb{Z}(p^\infty)$ is a uniserial divisible $\mathbb{Z}$-module and it is the injective envelope of $\mathbb{Z}/p\mathbb{Z}$. Indeed, the socle of the Prüfer p-group is the subgroup generated by $1/p + \mathbb{Z}$. (Here we view $\mathbb{Z}(p^\infty)$ as the p-primary part of $\mathbb{Q}/\mathbb{Z}$.) Another imperfection of the symmetry is that an indecomposable projective module may not be couniform (e.g., $\mathbb{Z}_{\mathbb{Z}}$).

Couniform projective modules underpin much of the theory developed in this chapter so far (and in Chapter 6). There is a very natural duality (in the categorical sense) between uniform injective modules over a ring R and the couniform projective modules over another suitable ring S. This provides a bridge for translating our theorems in the context of couniform projective modules to the dual context involving uniform injective modules.

Consider a set of representatives up to isomorphism of uniform injective right R-modules, say $\{E_i\}_{i\in I}$. Here R is a fixed arbitrary ring. (There indeed is a *set* of representatives of isomorphism classes of uniform injective right R-modules, because to the uniform injective module U we can associate (not canonically) a right ideal $I(U)$ of R such that $U \cong E(R/I(U))$. For instance, let $I(U)$ be the right annihilator in R of any non-zero element of U. Then the collection of uniform injective modules modulo isomorphism embeds in the lattice of right ideals of R.)

Let E be the injective envelope of the direct sum $\bigoplus_{i\in I} E_i$, and let $S = \mathrm{End}(E_R)$, so that E is an S-R-bimodule. Then we can consider the E-dual, cf. Section 3.2. For each index $i \in I$, we have

$$E_R = E\left(E_i \oplus \bigoplus_{i'\in I\setminus\{i\}} E_{i'}\right) \cong E_i \oplus E\left(\bigoplus_{i'\in I\setminus\{i\}} E_{i'}\right),$$

hence E_i is isomorphic to a direct summand of E_R. Choose a monomorphism $\iota_i\colon E_i \to E_R$ and an epimorphism $\pi_i\colon E_R \to E_i$ such that $\pi_i\iota_i = 1_{E_i}$. Also let $e_i := \iota_i\pi_i \in S$ be the corresponding idempotent endomorphism of E_R.

Proposition 4.37. *The E-dual enjoys the following properties:*

(i) Each E_i is E-reflexive.

(ii) The dual of a uniform injective right R-module is a couniform projective left S-module, and conversely, that is, the dual of couniform projective left S-module is a uniform injective right R-module.

(iii) The dual of an injective module of finite Goldie dimension is a projective left S-module with semiperfect endomorphism ring, and conversely.

(iv) Injective right R-modules of finite Goldie dimension and projective left S-modules with semiperfect endomorphism ring are E-reflexive.

Proof. (i) Notice that $\iota_i \in \mathrm{Hom}_R(E_i, E) = E_i^*$. Then if $\sigma_{E_i}(x) = 0$, it follows that $\sigma_{E_i}(x)(\iota_i) = \iota_i(x) = 0$, from which, $x = \pi_i\iota_i(x) = 0$. Therefore σ_M is injective. Let g be any element of E_i^{**}, that is, any left S-module morphism

$g\colon E_i^* = \mathrm{Hom}_R(E_i, E) \to E$. For $\gamma \in E_i^*$, we have $g(\gamma) = g(\gamma\pi_i\iota_i) = \gamma\pi_i g(\iota_i) = \sigma_{E_i}(\pi_i g(\iota_i))(\gamma)$. Therefore $g = \sigma_{E_i}(\pi_i g(\iota_i))$, and σ_{E_i} is also surjective.

(ii) Since a uniform injective right R-module is isomorphic to E_i for some $i \in I$, it suffices to prove that E_i^* is projective and couniform. We have $E_i^* = {}_S\mathrm{Hom}_R(E_i, E_R) \cong Se_i$ by the isomorphism $g \mapsto g\pi_i$. Moreover, the local ring $\mathrm{End}_R(E_i)$ is isomorphic to e_iSe_i, by the isomorphism $g \mapsto \iota_i g\pi_i$, therefore e_i is a local idempotent of S. This shows that E_i^* is a couniform projective left S-module, by Lemma 4.2((i) ⇔ (vi)).

Suppose that P is a couniform projective left S-module. Then there is a local idempotent e of S such that $P \cong Se$, by Lemma 4.2((i) ⇔ (vi)). There are R-module morphisms ι and π such that $\iota\pi = e$ and $\pi\iota = 1_A$, where A is some right R-module. Then $P^* \cong A$, because of the R-isomorphism $\mathrm{Hom}_S(Se, E) \to A$ given by $g \mapsto \pi g(e)$. Moreover, $\mathrm{End}_R(A) \cong eSe$ by the ring isomorphism $g \mapsto \iota g\pi$. Hence A is isomorphic to a direct summand of E and it has local endomorphism ring, hence it is a uniform injective module.

(iii) An injective module of finite Goldie dimension is simply a finite direct sum of uniform injective modules, and a projective module whose endomorphism ring is semiperfect is just a finite direct sum of couniform projective modules (Lemma 4.7), hence (iii) follows from (ii) by additivity.

(iv) By (i), every uniform injective right R-module F is E-reflexive. (Precisely, one has to consider an isomorphism $F \to E_i$ and use the fact that σ_M is natural in M.) As recalled at the beginning of Section 3.2, the class of E-reflexive modules is closed by finite direct sums, hence injective right R-modules of finite Goldie dimension are E-reflexive. Recall that if M is E-reflexive, then M^* is [AF92, Proposition 20.14]. Thus we get from (iii) that projective left S-modules whose endomorphism rings are semiperfect are also E-reflexive. □

Remark 4.38. The correspondences in (ii) and (iii) can actually be viewed as "mutually inverse" dualities between the corresponding full subcategories of modules, by Proposition 3.5 and (iv).

It is well-known that an endomorphism g of an injective (resp. projective) module is in the Jacobson radical (of its endomorphism ring) if and only if its kernel is essential (its image is superfluous) [AF92, Propositions 17.11 and 18.20]. It is true also when domain and codomain differ, so long as the projective modules involved are lifting:

Lemma 4.39. *A morphism $g\colon M \to N$ between two injective (resp. lifting projective) modules is in the Jacobson radical if and only if $\ker(g) \leqslant_e M$ (resp. $g(M) \leqslant_s N$).*

Proof. Let M and N be injective. Suppose that $\ker(g) \leqslant_e M$. Let $f\colon N \to M$ be an arbitrary morphism. Since $\ker(g) \leqslant \ker(fg)$, we have $\ker(fg) \leqslant_e M$. Thus $fg \in \mathbf{J}(M)$, hence $1_M - fg$ is invertible. This proves that $g \in \mathbf{J}(M, N)$.

Conversely, suppose that $g \in \mathbf{J}(M, N)$. The module M has a decomposition $M = A \oplus B$ such that $\ker(g) \leqslant_e A$. (The submodule A is simply a copy of the injective envelope of $\ker(g)$ contained in M.) Thus, $g|_B : B \to N$ is injective. Therefore, it splits, i.e., there exists $h : N \to B$ such that $hg|_B = 1_B$. Now B is injective and $1_B \in \mathbf{J}(B)$ forces $0 = \ker(1_B) \leqslant_e B$, hence $B = 0$. Therefore $\ker(g) \leqslant_e A = M$, as required.

Let now M and N be lifting projective modules. (For instance, projective modules whose endomorphism rings are semiperfect.) If $g(M) \leqslant_s N$, then for every $f : N \to M$ we have $fg(M) \leqslant_s M$, hence $fg \in \mathbf{J}(M)$. Thus $1_M - fg$ is invertible. This proves that $g \in \mathbf{J}(M, N)$. Conversely, assume $g \in \mathbf{J}(M, N)$. There exists a direct-sum decomposition $N = A \oplus B$ such that $A \leqslant g(M)$ and $B \cap g(M) \leqslant_s B$. There is a surjective morphism $\pi : N \to A$ such that $\pi|_A = 1_A$. Thus $\pi g : M \to A$ is also surjective, hence there exists $\alpha : A \to M$ such that $1_A = \pi g \alpha$. Thus implies that $1_A \in \mathbf{J}(A)$, which means that $A \leqslant_s A$, which happens if and only of $A = 0$. Hence $N = B$ e $g(M) \leqslant_s N$, as required. □

Proposition 4.40. *For a morphism $g : M \to N$ between injective right R-modules of finite Goldie dimension, we have*

(i) $\ker(g) \leqslant_e M$ *if and only if* $g^*(N^*) \leqslant_s M^*$.

(ii) $\operatorname{im}(g) \leqslant_e N$ *if and only if* $\ker(g^*) \leqslant_s N^*$.

Proof. (i) Thanks to the dualities of Remark 4.38, we have that $g \in \mathbf{J}(M, N)$ if and only if $g^* \in \mathbf{J}(N^*, M^*)$ (also see Lemma 1.1). Then (i) follows by Lemma 4.39.

(ii) Suppose that $g(M) \leqslant_e N$, equivalently, that $g^{**}(M^{**}) \leqslant_e N^{**}$. Since N^* is a lifting module, there is a decomposition $N^* = A \oplus B$ with $A \leqslant \ker(g^*)$ and $B \cap \ker(g^*) \leqslant_s B$. From $g^* \iota_A = 0$ we obtain $g^{**} = \pi_B^* \iota_B^* g^{**}$, so that $g^{**}(M^{**})$ is contained in the direct summand $\pi_B^* \iota_B^*(N^{**}) \cong B^*$ of N^{**}. Since $\operatorname{im}(g^{**}) \leqslant_e N^{**}$, necessarily $\pi_A^* \iota_A^* = 0$, so that $A^* = 0$, hence $A = 0$, $B = N^*$, and $\ker(g^*) \leqslant_s N^*$.

Assume now that $\ker(g^*) \leqslant_s N^*$. There is a decomposition $N = A \oplus B$ with $g(M) \leqslant_e A$. Thus $\pi_B g = 0$. It follows that $g^* \pi_B^* \iota_B^* = 0$, so that $\ker(g^*)$ contains a direct summand of N^* isomorphic to B^*. But $\ker(g^*) \leqslant_s N^*$, so that $B^* = 0$, thus $B = 0$, hence $A = N$ and $g(M) \leqslant_e N$. □

4.4.2 Dual Auslander-Bridger modules

A right R-module M is a *dual Auslander-Bridger module* if it has no non-zero injective summands and embeds in an exact sequence of the form

$$0 \longrightarrow M \xrightarrow{\epsilon_M} E_0(M) \xrightarrow{\rho_M} E_1(M) \tag{4.41}$$

where both ϵ_M and $\bar{\rho}_M : E_0(M)/\epsilon_M(M) \to E_1(M)$ are injective envelopes and both $E_0(M)$ and $E_1(M)$ have finite Goldie dimension. As the notation suggests, for every dual Auslander-Bridger module M we fix such an exact sequence, and we call it its *minimal copresentation.* Such sequence is in fact unique up to isomorphism, as it is the beginning of a minimal injective resolution of M.

Since E_R is injective, the additive functor $(-, E_R)$ is exact, hence when we take the dual of (4.41) we obtain an exact sequence of left S-modules, namely

$$E_1(M)^* \longrightarrow E_0(M)^* \longrightarrow M^* \longrightarrow 0\ . \tag{4.42}$$

The additive functor $(-, {}_S E)$ is left exact, hence we have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & M & \xrightarrow{\epsilon_M} & E_0(M) & \xrightarrow{\rho_M} & E_1(M) \\ & \downarrow \sigma & & \downarrow \sigma & & \downarrow \sigma \\ 0 \longrightarrow & M^{**} & \xrightarrow{\epsilon_M^{**}} & E_0(M)^{**} & \xrightarrow{\rho_M^{**}} & E_1(M)^{**} \end{array}$$

where both rows are exact and the vertical arrows are given by the “evaluation” natural morphism. Since we know that the second and third vertical arrows are isomorphisms by Proposition 4.37(iv), it follows easily that $\sigma_M : M \to M^{**}$ is an isomorphism. Hence M is E-reflexive. In equation (4.42) we see that M^* is the cokernel of a morphism between projective modules that are finite direct sums of couniform submodules (Proposition 4.37(iii)). Thanks to Proposition 4.40 we know much more, viz., that (4.42) is a minimal presentation of M^*. Moreover, M^* has no non-zero projective summands. If it had one, such summand would in turn have a couniform projective summand, and $M \cong M^{**}$ would have a uniform injective summand (Proposition 4.37), which it has not. This shows that M^* is an Auslander-Bridger left S-module. To sum up:

Proposition 4.43. *Dual Auslander-Bridger right R-modules are E-reflexive, and the dual of a dual Auslander-Bridger right R-module is an Auslander-Bridger left S-module.* □

In a similar fashion one proves that:

Proposition 4.44. *Auslander-Bridger left S-modules are E-reflexive, and the dual of an Auslander-Bridger left S-module is a dual Auslander-Bridger right R-module.* □

Therefore, we have:

Theorem 4.45. *The E-dual establishes an additive categorical duality between Auslander-Bridger left S-modules and dual Auslander-Bridger right R-modules.*

Using the above duality we can describe the endomorphism ring of a dual Auslander-Bridger module. Notice that a morphism $g : M \to N$ between dual

Auslander-Bridger right R-modules extends to a morphism $(g_0, g_1)\colon \rho_M \to \rho_N$, and that g is an isomorphism if and only if (g_0, g_1) is, that is, if and only if both g_0 and g_1 are isomorphisms. (The proof is analogous to that of Lemma 4.30.)

$$\begin{CD}
0 @>>> M @>{\epsilon_M}>> E_0(M) @>{\rho_M}>> E_1(M) \\
@. @V{g}VV @VV{g_0}V @VV{g_1}V \\
0 @>>> N @>{\epsilon_N}>> E_0(N) @>{\rho_N}>> E_1(N)
\end{CD}$$

With this notation we state:

Theorem 4.46. *Let M be a non-zero dual Auslander-Bridger right R-module and $T = \operatorname{End}_R(M)$. Let*

$$\mathfrak{u} = \{g \in T : \ker(g_1) \leqslant_e E_1(M)\},$$
$$\mathfrak{m} = \{g \in T : \ker(g_0) \leqslant_e E_0(M)\}.$$

Then $\mathfrak{u}$ and $\mathfrak{m}$ are proper two-sided ideals of T, the canonical morphism $T \to T/\mathfrak{u} \times T/\mathfrak{m}$ is a local morphism, $\mathfrak{u} \cap \mathfrak{m} \leqslant J(T)$, and every maximal right ideal, every maximal left ideal, every maximal two-sided ideal of T, contains either $\mathfrak{u}$ or $\mathfrak{m}$.

Proof. The duality gives us a ring anti-isomorphism $T \to \operatorname{End}_S(M^*)$. For $g \in T$ we have that $g \in \mathfrak{u}$ if and only if $\ker(g_1) \leqslant_e E_1(M)$, if and only if $g_1^*(E_1(M)^*) \leqslant_s E_1(M)^*$ (Proposition 4.40). Since (4.42) is a minimal presentation (as remarked earlier), this is equivalent to $g^* \in \mathfrak{l}$ (with the notation of Theorem 4.35). (This, in particular, shows that the definition of $\mathfrak{u}$ depends on neither the choice of the minimal copresentation nor on the choice of the extensions f_0 and f_1.) In the same way one sees that $g \in \mathfrak{m}$ if and only if $g^* \in \mathfrak{e}$. Therefore we have a commutative square

$$\begin{CD}
T @>>> \operatorname{End}_S(M^*) \\
@VVV @VVV \\
T/\mathfrak{u} \times T/\mathfrak{m} @>>> \operatorname{End}_S(M^*)/\mathfrak{l} \times \operatorname{End}_S(M^*)/\mathfrak{e}
\end{CD}$$

where the vertical ring morphisms are the canonical ones and the horizontal ones are anti-isomorphisms induced by $(-)^* = (-, E)$. Everything now follows from Theorem 4.35. □

If M and N are dual Auslander-Bridger right R-modules, we say that M and N have the same *upper-isomorphism class,* or that they are *upper-isomorphic,* and write $M \equiv_u N$, if there are morphisms $f\colon M \to N$ and $g\colon N \to M$ such that $1_M - gf \in \mathfrak{u}_M$ and $1_N - fg \in \mathfrak{u}_M$. Similarly, we say that M and N have the same *mono-isomorphism class,* or that they are *mono-isomorphic,* and write $M \equiv_m N$, if there are morphisms $f\colon M \to N$ and $g\colon N \to M$ such that $1_M - gf \in \mathfrak{m}_M$ and $1_N - fg \in \mathfrak{m}_M$.

It follows from the proof of the previous theorem and from Proposition 4.31 that:

Proposition 4.47. *For dual Auslander-Bridger modules M and N,*

(i) $M \equiv_u N$ *if and only if* $M^* \equiv_\ell N^*$*,*

(ii) $M \equiv_m N$ *if and only if* $M^* \equiv_e N^*$*, and*

(iii) $M \cong N$ *if and only if* $M \equiv_u N$ *and* $M \equiv_m N$*.*

Chapter 5

The n-Krull Schmidt Theorem

We have seen in Theorem 2.17 that the objects of a semilocal category $\mathbf{C}$ have a full class of invariants, namely, if X and Y are objects of $\mathbf{C}$, we have that

$$X \cong Y \text{ if and only if } \mathrm{Red}_{\mathbf{P}}(X) \cong \mathrm{Red}_{\mathbf{P}}(Y) \text{ for every } \mathbf{P} \in V(\mathbf{C}),$$

where $\mathrm{Red}_{\mathbf{P}} \colon \mathbf{C} \to \mathbf{C}/\mathbf{P}$ is the canonical functor. As a result, for biproducts (switching to the additive closure of $\mathbf{C}$ if necessary), we have

$$\bigoplus_{i<n} X_i \cong \bigoplus_{i<n} Y_i \text{ if and only if } \bigoplus_{i<n} \mathrm{Red}_{\mathbf{P}}(X_i) \cong \bigoplus_{i<n} \mathrm{Red}_{\mathbf{P}}(Y_i),$$

for every $\mathbf{P} \in V(\mathbf{C})$, but does this imply that $\mathrm{Red}_{\mathbf{P}}(X_i) \cong \mathrm{Red}_{\mathbf{P}}(Y_{\sigma_{\mathbf{P}}(i)})$ for each $i < n$, for a suitable reordering $\sigma_{\mathbf{P}}$ of the terms? In some categories whose objects are of finite type this is exactly what happens [Gir11a]. This is the central result of this section, Theorem 5.10.

5.1 A criterion for the n-Krull-Schmidt Theorem to hold

An additive functor $F \colon \mathbf{A} \to \mathbf{B}$ is *local* if whenever $F(f)$ is an isomorphism, then so is f [Fac07]. This generalises the notion of local ring morphism (see page 47). Indeed, if $F \colon \mathbf{A} \to \mathbf{B}$ is a local functor, then the ring morphism $\mathbf{A}(X) \to \mathbf{B}(F(X))$ induced by F is local, for every object X of $\mathbf{A}$. Thus F is simply a ring morphism when $\mathbf{A}$ and $\mathbf{B}$ have one and only one object.

For instance, if $\mathbf{A}$ is a preadditive category and $\mathbf{J}$ is its Jacobson radical, then $\mathbf{A} \to \mathbf{A}/\mathbf{J}$ is a local functor. Indeed, suppose f is a morphism in $\mathbf{A}$ invertible modulo $\mathbf{J}$. Then there is a morphism g such that $1 - fg$ and $1 - gf$ are in $\mathbf{J}$. It follows that fg and gf are automorphisms, hence f is both left and right invertible, thus invertible.

For our purposes, a slightly weaker notion suffices.

Lemma-Definition 5.1. *Let $\mathbf{A}$ and $\mathbf{B}$ be preadditive categories and let $F\colon \mathbf{A} \to \mathbf{B}$ be an additive functor. The following conditions are equivalent:*

(i) If $f\colon M \to N$ and $g\colon N \to M$ are morphisms in $\mathbf{A}$ such that $F(f)$ and $F(g)$ are isomorphisms, then f and g are isomorphisms.

(ii) For each object M of $\mathbf{A}$, the ring morphism $\mathbf{A}(M) \to \mathbf{B}(F(M))$ is a local morphism.

We say that F is almost local *if it satisfies the above equivalent conditions.*

Proof. It is trivial that (i) implies (ii). Assume (ii) holds and suppose the hypotheses of (i) hold. Then $F(fg)$ and $F(gf)$ are automorphisms of $F(N)$ and $F(M)$ respectively. Thus fg and gf are automorphisms of N and M respectively. It follows that f and g are both right and left invertible, hence isomorphisms. □

Recall that if $\mathbf{A}$ is a preadditive category, by $\mathrm{Sums}(\mathbf{A})$ we denote its additive closure, cf. 1.2.1. Also notice that if $\mathbf{A}$ is a full subcategory of an additive category $\mathbf{C}$, then $\mathrm{Sums}(\mathbf{A})$ is equivalent to the full subcategory of $\mathbf{C}$ whose objects are the biproducts of objects of $\mathbf{A}$.

Setting 5.2. Let us describe the working environment for almost all that follows. Let $\mathbf{A}$ be a preadditive category with no zero objects. Let n be a positive integer. We assume we have an additive functor $T\colon \mathrm{Sums}(\mathbf{A}) \to \prod_{i<n} \mathbf{A}_i$, where each $\mathbf{A}_i$ is a preadditive category, such that:

(S1) For each $i < n$ and each object X of $\mathbf{A}$, the object $T_i(X)$ of $\mathbf{A}_i$ is of type $\leqslant 1$, where $T_i = P_i T$ and $P_i\colon \prod_{i<n} \mathbf{A}_i \to \mathbf{A}_i$ is the canonical projection functor;

(S2) The restriction of T to $\mathbf{A}$ yields an almost local functor.

For each $i < n$, we let $\mathbf{P}_i$ be the inverse image of the Jacobson radical $\mathbf{J}_i$ of $\mathbf{A}_i$ along the additive functor T_i.

In some cases, we will impose a condition stronger than (S1), namely

(S1′) For each $i < n$ and each object X of $\mathbf{A}$, the object $T_i(X)$ of $\mathbf{A}_i$ is of type 1, i.e., has local endomorphism ring.

We will always point out explicitly when we assume condition (S1′).

The restriction of $\mathbf{P}_i$ to $\mathbf{A}$ fails to be a completely prime ideal (cf. page 15) because it may happen for some object X of $\mathbf{A}$ that $T_i(X) = 0$, and in that case $\mathbf{P}_i(X)$ is not a proper ideal of $\mathbf{A}(X)$. Nevertheless, we have the following:

Lemma 5.3. *Condition (C2) on page 15 holds for the restriction of $\mathbf{P}_i$ to $\mathbf{A}$.*

If (S1′) holds, then condition (C1) also holds, i.e., the restriction of $\mathbf{P}_i$ to $\mathbf{A}$ is a completely prime ideal.

(See page 15 for conditions (C1) and (C2).)

Proof. Let $f \in \mathbf{A}(X, Y)$ and $g \in \mathbf{A}(Y, Z)$. If $T_i(X)$, $T_i(Y)$, $T_i(Z)$ are all non-zero, by Lemma 1.16, we have $gf \in \mathbf{P}_i$ if and only if $T_i(gf) \in \mathbf{J}_i$, if and only if $T_i(gf)$ is not an isomorphism, if and only if either $T_i(g)$ or $T_i(f)$ is not an isomorphism, if and only if either $T_i(g) \in \mathbf{J}_i$ or $T_i(f) \in \mathbf{J}_i$, if and only if either $g \in \mathbf{P}_i$ or $f \in \mathbf{P}_i$. The case in which one of $T_i(X)$, $T_i(Y)$, $T_i(Z)$ is zero is trivial.

If (S1′) holds, then we have that $1_A \notin \mathbf{P}_i$ for every object A of $\mathbf{A}$, hence every $\mathbf{P}_i(A)$ is proper and condition (C1) is satisfied. □

The following is a slight generalisation of [AM69, Proposition 1.11(i)], that we include for the sake of completeness. Recall that a *completely prime ideal I* of a ring R is a proper ideal such that $ab \in I$ implies $a \in I$ or $b \in I$, as it also follows by specialising the definition on page 15 to preadditive categories with one object.

Lemma 5.4. *Let R be a ring and $P_0, \dots, P_{n-1}$ completely prime ideals of R. Let A be a multiplicatively closed additive subgroup of R. If $A \subseteq \bigcup_{i<n} P_i$, then $A \subseteq P_i$ for some $i < n$.*

Proof. Consider the set of natural numbers for which the statement is not true. If, by contradiction, it is non-empty, then it has a least element n, and necessarily $n \geqslant 2$. Choose completely prime ideals $P_0, \dots, P_{n-1}$ and a subset A of R such that $A \subseteq \bigcup_{i<n} P_i$ but $A \nsubseteq P_i$ for every $i < n$. By minimality of n, it follows that $A \nsubseteq \bigcup_{j<n, j \neq i} P_j$ for every $i < n$. Hence, for each $i < n$, there exists $a_i \in A$ such that $a_i \in P_i \backslash \bigcup_{j<n, j\neq i} P_j$. Let $x_i = a_0 \cdots \widehat{a}_i \cdots a_{n-1} \in A$. Then $x_i \in \bigcap_{j<n, j\neq i} P_j \backslash P_i$. Let $x = \sum_{i<n} x_i \in A$. Now $x \notin P_i$, for every $i < n$, contradicting $A \nsubseteq \bigcup_{i<n} P_i$. □

Proposition 5.5. *Let M be an object of $\mathbf{A}$. Then:*

(i) For each $i < n$, either $\mathbf{P}_i(M) = \mathbf{A}(M)$ or $\mathbf{P}_i(M)$ is a completely prime two-sided ideal of $\mathbf{A}(M)$.

(ii) There exist indices $i_0, \dots, i_{t-1} < n$ such that $\{\mathbf{P}_{i_\ell}(M)\}_{\ell<t}$ is the set of maximal right ideals of $\mathbf{A}(M)$. Since they are all two-sided ideals, $\mathbf{A}(M)$ is a ring of type $t \leqslant n$.

(iii) The canonical ring morphism $p \colon \mathbf{A}(M)/J(\mathbf{A}(M)) \to \prod_{\ell<t} \mathbf{A}(M)/\mathbf{P}_{i_\ell}(M)$ is an isomorphism.

Proof. (i) When $\mathbf{P}_i(M)$ is proper, it is completely prime by Lemma 5.3.

(ii) By (S2) we have a local morphism $\mathbf{A}(M) \to \prod_{i<n} \mathbf{A}_i((T_i(M))$ induced by T, from which we obtain the local morphism

$$\mathbf{A}(M) \to \prod_{i \text{ s.t. } T_i(M) \neq 0} \mathbf{A}_i(T_i(M))/J(\mathbf{A}_i(T_i(M)))$$

whose codomain is a product of division rings. Notice that the product is non-empty, because $M \neq 0$. Then the set of non-units of $\mathbf{A}(M)$ is the union $\bigcup_{i \text{ s.t. } T_i(M) \neq 0} \mathbf{P}_i(M)$. Insofar as every ideal of this union is completely prime by (i), any proper right or left ideal of $\mathbf{A}(M)$ is contained in some $\mathbf{P}_i(M)$, by Lemma 5.4. This is in particular true for the maximal right ideals, thus (ii) follows.

(iii) By the Chinese Remainder Theorem 1.19, p is an isomorphism. □

For each $i < n$ and each object M of $\mathbf{A}$, let $\mathbf{Q}_{i,M}$ be the ideal of $\mathrm{Sums}(\mathbf{A})$ associated to $\mathbf{P}_i(M)$.

We now define the equivalence relations that will control biproducts of objects in $\mathbf{A}$. For each $i < n$ we define a preorder $\preceq_i$ on the class of objects of $\mathbf{A}$. For each pair of objects M and N we let $M \preceq_i N$ if there exists $f \in \mathbf{A}(M, N)$ such that $T_i(f) \in \mathbf{A}_i(T_i(M), T_i(N))$ is an isomorphism. In view of Lemma 1.16, this amounts to $f \notin \mathbf{P}_i$ when $T_i(M)$ and $T_i(N)$ are non-zero. We let $\equiv_i$ be the equivalence relation defined by $M \equiv_i N$ if and only if $M \preceq_i N$ and $N \preceq_i M$.

Let us show the connection between these equivalence relations and the ideals $\mathbf{Q}_{i,M}$.

Lemma 5.6. *Let $i < n$ and $M, N \in \mathbf{A}$. Then $M \equiv_i N$ if and only if $\mathbf{Q}_{i,M} = \mathbf{Q}_{i,N}$. When this is the case, $\mathbf{P}_i(M)$ is maximal if and only if $\mathbf{P}_i(N)$ is maximal.*

Proof. Suppose $M \equiv_i N$. Then let $f\colon M \to N$ and $g\colon N \to M$ be morphisms in $\mathbf{A}$ such that $T_i(f)$ and $T_i(g)$ are isomorphisms. If $T_i(M) = 0$, then also $T_i(N) = 0$, hence $\mathbf{P}_i(M)$ and $\mathbf{P}_i(N)$ are both improper, and $\mathbf{Q}_{i,M} = \mathbf{Q}_{i,N}$ is the improper ideal of $\mathrm{Sums}(\mathbf{A})$. Thus we can assume that $T_i(M)$ and $T_i(N)$ are non-zero. As a consequence, by Lemma 1.16, f and g are not in $\mathbf{P}_i$. Suppose $b\colon B_1 \to B_2$ is a morphism in $\mathrm{Sums}(\mathbf{A})$ such that $b \in \mathbf{Q}_{i,M}(B_1, B_2)$. To prove that $b \in \mathbf{Q}_{i,N}$, we need to show that for each $\alpha\colon N \to B_1$ and each $\beta\colon B_2 \to N$ we have $\beta b\alpha \in \mathbf{P}_i(N)$. We have $g(\beta b\alpha)f \in \mathbf{P}_i(M)$ because $b \in \mathbf{Q}_{i,M}$. In view of (C2) and of the fact that $f, g \notin \mathbf{P}_i$, it follows that $\beta b\alpha \in \mathbf{P}_i(N)$, as required. This proves that $\mathbf{Q}_{i,M} \subseteq \mathbf{Q}_{i,N}$ and the reverse inclusion follows by symmetry.

Now assume that $\mathbf{Q} = \mathbf{Q}_{i,M} = \mathbf{Q}_{i,N}$. If this is the improper ideal of $\mathrm{Sums}(\mathbf{A})$, then $\mathbf{P}_i(M)$ and $\mathbf{P}_i(N)$ are improper. This implies that $T_i(M) = T_i(N) = 0$, so that $T_i(0\colon M \to N)$ and $T_i(0\colon N \to M)$ are isomorphisms, and $M \equiv_i N$. We can now suppose that $\mathbf{Q}$ is proper. This implies that $1_N \notin \mathbf{P}_i(N) = \mathbf{Q}_{i,M}(N)$, therefore there exist morphisms $f\colon M \to N$ and $g\colon N \to M$ in $\mathbf{A}$ such that $gf \notin \mathbf{P}_i(M)$. Thus both g and f are not in $\mathbf{P}_i$ and, by Lemma 1.16, both $T_i(f)$ and $T_i(g)$ are isomorphisms, so that $M \equiv_i N$.

The last assertion follows from Theorem 2.14, because the endomorphism rings of M and N are of finite type $\leqslant n$, hence semilocal. $\square$

Lemma 5.7. *Let X be an object of $\mathbf{A}$ such that $\mathbf{P}_i(X)$ is maximal, and let $F\colon \mathbf{A} \to \mathbf{A}/\mathbf{Q}_{i,X}$ be the canonical functor. Let N be any object of $\mathbf{A}$. If $X \equiv_i N$, then $F(X) \cong F(N)$, while if $X \not\equiv_i N$, then $F(N) = 0$.*

Proof. If $X \equiv_i N$, we have that $\mathbf{P}_i(N) = \mathbf{Q}_{i,X}(N)$ is maximal, hence $F(X) \cong F(N)$ by Lemma 2.24. Suppose now that $X \not\equiv_i N$. If $T_i(N) = 0$, then $\mathbf{P}_i(N)$ is improper and, since it is contained in $\mathbf{Q}_{i,X}(N)$, $F(N) = 0$. Thus assume $T_i(N) \neq 0$. Note that also $T_i(X) \neq 0$, so that we may apply Lemma 1.16 as follows: For any pair of morphisms $f\colon X \to N$ and $g\colon N \to X$, either $T_i(f)$ or $T_i(g)$ is not an isomorphism, so that $T_i(gf)$ is not an isomorphism, i.e., $T_i(gf) \in \mathbf{J}_i$, hence $gf \in \mathbf{P}_i(X)$. This shows that $1_N \in \mathbf{Q}_{i,X}(N)$, thus $F(N) = 0$. $\square$

Finally we give the main result of this chapter:

Theorem 5.8. *Consider the objects $X = \bigoplus_{\mu<r} X_\mu$ and $Y = \bigoplus_{\mu<s} Y_\mu$ of the additive closure $\mathrm{Sums}(\mathbf{A})$, where $X_0, \dots, X_{r-1}$ and $Y_0, \dots, Y_{s-1}$ are objects of the preadditive category $\mathbf{A}$. For each $i < n$, define $\mathcal{X}_i = \{\mu < r : T_i(X_\mu) \neq 0\}$ and $\mathcal{Y}_i = \{\mu < s : T_i(Y_\mu) \neq 0\}$. Then $X \cong Y$ if and only if there exist bijections $\{\sigma_i\colon \mathcal{X}_i \to \mathcal{Y}_i\}_{i<n}$ such that $X_\mu \equiv_i Y_{\sigma_i(\mu)}$ for each $i < n$ and each $\mu \in \mathcal{X}_i$.*

Proof. Assume that the bijections exist. To show that $X \cong Y$, by Lemma 2.19, we must show that X and Y are isomorphic in $\mathrm{Sums}(\mathbf{A})/\mathbf{Q}$ for each $\mathbf{Q} \in V(\mathrm{Sums}(\mathbf{A}), M)$ for every $M \in \mathbf{A}$. By Theorem 5.5, we then have $\mathbf{Q} = \mathbf{Q}_{i,M}$ for some $i < n$ such that $\mathbf{P}_i(M)$ maximal. The mapping σ_i induces a bijection

$$\{\mu \in \mathcal{X}_i : X_\mu \equiv_i M\} \to \{\mu \in \mathcal{Y}_i : Y_\mu \equiv_i M\}.$$

Let $k \geqslant 0$ be the common cardinality of the two sets. Note that if $\mu < r$ is not in $\mathcal{X}_i$, i.e., $T_i(X_\mu) = 0$, then $\mathbf{P}_i(X_\mu)$ is the improper ideal. Since $\mathbf{P}_i(X_\mu) \subseteq \mathbf{Q}_{i,M}(X_\mu)$, it follows that $F(X_\mu) = 0$. Therefore, $F(X) \cong \bigoplus_{\mu\in\mathcal{X}_i} F(X_\mu) \cong F(M)^k$, where the last isomorphism holds by Lemma 5.7. Since the same holds for Y, it follows that $F(X) \cong F(Y)$.

For the converse implication, assume that $f\colon X \to Y$ and $g\colon Y \to X$ are mutually inverse isomorphisms. Then $T_i(f)\colon T_i(X) \to T_i(Y)$ and $T_i(g)\colon T_i(Y) \to T_i(X)$ are mutually inverse isomorphisms in $\mathbf{A}_i$. By Theorem 1.18, we obtain a bijection $\sigma_i\colon \mathcal{X}_i \to \mathcal{Y}_i$ such that $T_i(f_{\sigma_i(\mu),\mu}) = (T_i(f))_{\sigma_i(\mu),\mu}\colon T_i(X_\mu) \to T_i(Y_{\sigma_i(\mu)})$ is an isomorphism for all $\mu \in \mathcal{X}_i$. Therefore $X_\mu \preceq_i Y_{\sigma_i(\mu)}$ for all $\mu \in \mathcal{X}_i$.

Reasoning in the same way with g, we obtain a bijection $\tau_i\colon \mathcal{Y}_i \to \mathcal{X}_i$ such that $Y_\mu \preceq_i X_{\tau_i(\mu)}$ for all $\mu \in \mathcal{Y}_i$.

Therefore $X_\mu \preceq_i Y_{\sigma_i(\mu)} \preceq_i X_{\tau_i\sigma_i(\mu)}$. Continuing inductively we have $X_\mu \preceq_i Y_{\sigma_i(\mu)} \preceq_i X_{(\tau_i\sigma_i)^k(\mu)}$ for all integers $k \geqslant 1$. Since there exists some $k \geqslant 1$

such that $(\tau_i\sigma_i)^k = 1$ (the symmetric group of the finite set $\mathcal{X}_i$ is finite, hence all its elements have finite order), we have $X_\mu \equiv_i Y_{\sigma_i(\mu)}$ for each $\mu \in \mathcal{X}_i$, as required. □

Corollary 5.9. *Let $X, Y \in \mathbf{A}$. Then $X \cong Y$ if and only if $X \equiv_i Y$ for all $i < n$.*

It is easy to see that if (S1′) holds, the statement becomes more elegant:

Theorem 5.10. *Suppose (S1′) holds. In the notation of Theorem 5.8, we have that $\bigoplus_{\mu<r} X_\mu \cong \bigoplus_{\mu<s} Y_\mu$ if, and only if, $r = s$ and there exist permutations $\{\sigma_i\}_{i<n}$ such that $X_\mu \equiv_i Y_{\sigma_i(\mu)}$ for each $i < n$ and each $\mu < r$.*

Definition 5.11. For a preadditive category $\mathbf{C}$, we say that the n-Krull-Schmidt Theorem holds for $\mathbf{C}$ if there are equivalence relations $\{\sigma_i\}_{i<n}$ on the class of objects of $\mathbf{C}$ such that Theorem 5.10 holds.

5.2 Examples

5.2.1 DCP modules over rings of finite type.

Let R be a ring. A DCP module is a direct summand of a cyclically presented module, i.e., a direct summand of a module isomorphic to R/xR for some $x \in R$. The DCP modules over rings R of finite type have been studied in [AAF09]. Via a suitable duality, the kernels of morphisms between heterogeneous injective modules of finite Goldie dimension, i.e., between finite direct sums of pairwise non-isomorphic indecomposable injective modules, were also studied in that paper [AAF09, §6].

The setting of [AAF09] is a particular instance of Setting 5.2. Namely, let R be a ring of finite type, with maximal ideals $M_1, \ldots, M_n$. We denote R/M_i by K_i when we view it as a division ring, and by S_i when we view it as a simple right (or simple left) R-module. Inasmuch as S_i is an R-K_i-bimodule, we have the additive functors $T_{2i-1} := \mathrm{Tor}_1^R(-, S_i)$ and $T_{2i} := - \otimes S_i$, both $\mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}K_i$. Let $T = T_1 \times \cdots \times T_{2n}$ and $\mathbf{A}$ be the full subcategory of $\mathrm{Mod}\text{-}R$ whose objects are the non-zero DCP right R-modules. At the end of [AAF09, §2], it is proved that $T_i(A_R)$ is of type $\leqslant 1$ for any DCP module A_R, hence (S1) is satisfied. Moreover, (S2) is satisfied by the proof of [AAF09, Theorem 3.2]. It is easy to see that the equivalence relations $[-]_{\otimes,i}$ and $[-]_{T,i}$ introduced in [AAF09, page 3] are specialisations of our equivalence relations $\equiv_i$, and that [AAF09, Theorem 5.3] is a specialisation of Theorem 5.8.

5.2.2 Artinian modules with heterogeneous socle

An artinian module M whose socle is *heterogeneous,* i.e., is a finite direct sum of pairwise non-isomorphic simple modules, is known to be a module of finite type [FP09b, Section 5]. Indeed, suppose M is artinian and $\mathrm{Soc}(M) = \bigoplus_{i<n} S_i$, where the simple modules $\{S_i\}_{i<n}$ are pairwise non-isomorphic. Then $\mathrm{End}(\mathrm{Soc}(M)) \cong \prod_{i<n} \mathrm{End}(S_i)$ is a finite direct product of division rings. The canonical ring morphism $\rho\colon \mathrm{End}(M) \to \mathrm{End}(\mathrm{Soc}(M))$ given by restriction is local, for if $g\colon M \to M$ is not an automorphism, then g is not injective [Fac98, Lemma 2.16], hence $\ker(g) \cap \mathrm{Soc}(M) \neq 0$, because $\mathrm{Soc}(M) \leqslant_e M$. (The socle of an artinian module M is an essential submodule, because every artinian module contains a simple submodule, hence $0 \neq \mathrm{Soc}(A) \leqslant \mathrm{Soc}(M) \cap A$ for every $0 \neq A \leqslant M$.) Therefore $\rho(g)$ is not injective hence not invertible. Thus ρ shows that M is of finite type $\leqslant n$. For a proof that uses the injective envelope of M, see [FP09b, Section 5].

If we restrict our attention to the category $\mathbf{A}$ of Artinian modules whose socle is a fixed heterogeneous semisimple module $\bigoplus_{i<n} S_i$, we find that a version of Theorem 5.10 holds for direct sums of modules in $\mathbf{A}$.

Let us explain this in more detail. Notice that here $\mathrm{Sums}(\mathbf{A})$ is realised as a full subcategory of $\mathrm{Mod}\text{-}R$. For $i < n$ and M in $\mathrm{Sums}(\mathbf{A})$, let $T_i(M)$ be the *trace* of S_i in M, i.e., the largest submodule of M generated by S_i [AF92, p. 109]. For each module morphism $f\colon M \to N$ in $\mathbf{A}$ let $T_i(f)\colon T_i(M) \to T_i(N)$ be the restriction and corestriction of f. (Recall that the trace is preserved by module morphisms, ibid.) Consider the product functor

$$T\colon \mathrm{Sums}(\mathbf{A}) \to \prod_{i<n} \mathrm{Mod}\text{-}R.$$

As a matter of fact, if M is in $\mathbf{A}$, then $T_i(M) \cong S_i$ is the isomorphic copy of S_i in the socle of M, hence the endomorphism ring of $T_i(M)$ is a division ring. This shows that the condition (S1′) of Setting 5.2 holds. Suppose now that g is an endomorphism of M and that $T_i(g)$ is an automorphism for each $i < n$. Let $K = \ker(g) \cap \mathrm{Soc}(M)$. Then K is isomorphic to a submodule of $\bigoplus_{i<n} S_i$. If K is non-zero, it then contains a simple submodule isomorphic to S_i, for some $i < n$. But this implies that $T_i(g) = 0$, which is false. Hence $K = 0$ and, since $\mathrm{Soc}(M) \leqslant_e M$ because M is artinian, we have that g is injective. An injective endomorphism of an artinian module is an automorphism [Fac98, Lemma 2.16(b)], therefore g is an automorphism. This proves that our functor T also satisfies (S2).

Here is the form that the equivalence relations $\equiv_i$ assume in this context: For M and N in $\mathbf{A}$, we have that $M \equiv_i N$ if and only if there are module morphisms $f\colon M \to N$ and $g\colon N \to M$ such that $f(T_i(M)) = T_i(N)$ and $g(T_i(N)) = T_i(M)$. With respect to the equivalence relations $\equiv_i$, the n-Krull-Schmidt Theorem holds for $\mathbf{A}$. In other words, we have an instance of Theorem 5.10 for the finite direct sums of artinian modules with the prescribed heterogeneous socle $\bigoplus_{i<n} S_i$.

5.2.3 Noetherian modules with heterogeneous top

This class of modules is dual to the previous one. Let $\{S_i\}_{i<n}$ be a finite set of pairwise non-isomorphic simple modules. Let $\mathbf{N}$ be the full subcategory of $\mathrm{Mod}\text{-}R$ whose objects are the non-zero noetherian modules N such that the top of N, i.e., $N/\mathrm{Rad}(N)$, is isomorphic to $\bigoplus_{i<n} S_i$.

Recall that, if $\mathcal{U}$ is a family of right R-modules, for each right R-module X, the *reject* of $\mathcal{U}$ in X is the smallest submodule X' of X such that X/X' is cogenerated by $\mathcal{U}$, and it is denoted by $\mathrm{Rej}_X(\mathcal{U})$ [AF92, p. 109]. Any morphism $f\colon X \to Y$ preserves the reject, i.e., $f(\mathrm{Rej}_X(\mathcal{U})) \subseteq \mathrm{Rej}_Y(\mathcal{U})$.

For each $i < n$ and X in $\mathbf{N}$, define $T_i(X) = N/\mathrm{Rej}_X(S_i)$. For a morphism $f\colon X \to Y$ let $T_i(f)\colon X/\mathrm{Rej}_X(S_i) \to Y/\mathrm{Rej}_Y(S_i)$ be the morphism induced by f. Thus T_i is an additive functor into the category of right R-modules. Then we consider the product functor

$$T\colon \mathrm{Sums}(\mathbf{N}) \to \prod_{i<n} \mathrm{Mod}\text{-}R.$$

Let $i < n$ and $N \in \mathbf{N}$. Let us show that $\mathrm{Rej}_N(S_i)$ is a maximal submodule of N, that is, that $T_i(N)$ is a simple module, so that $\mathrm{End}_R(T_i(N))$ is a division ring and (S1′) is satisfied. Suppose $\mathrm{Rej}_N(S_i)$ is a proper submodule of N. On the one hand, $N/\mathrm{Rej}_N(S_i)$ is isomorphic to a non-zero quotient of the heterogeneous semisimple module $\bigoplus_{\ell<n} S_\ell$, hence $N/\mathrm{Rej}_N(S_i)$ is isomorphic to $\bigoplus_{\ell\in F} S_\ell$, for some non-empty $F \subseteq \{\ell < n\}$. (The radical $\mathrm{Rad}(N)$ is the reject in N of the class of all simple modules, hence $\mathrm{Rad}(N) \subseteq \mathrm{Rej}_N(S_i)$.) On the other hand, $N/\mathrm{Rej}_N(S_i)$ is cogenerated by S_i, thus F can only be the singleton $\{i\}$. Therefore, $T_i(N) = N/\mathrm{Rej}_N(S_i) \cong S_i$, as claimed.

Notice also that if M is a maximal submodule of N, it is necessarily equal to $\mathrm{Rej}_N(S_i)$ for some $i < n$. Indeed, N/M is a simple quotient of $\bigoplus_{i<n} S_i$, hence $N/M \cong S_i$ for some $i < n$. (The radical $\mathrm{Rad}(N)$ is the intersection of all maximal submodules, hence $\mathrm{Rad}(N) \subseteq M$.) In particular, N/M is cogenerated by S_i, hence $\mathrm{Rej}_N(S_i) \subseteq M$. We already know that $\mathrm{Rej}_N(S_i)$ is maximal, therefore equality holds.

Suppose now that g is an endomorphism of N and that $T_i(g)$ is an isomorphism for each $i < n$. In particular, $g(N) + \mathrm{Rej}_N(S_i) = N$ for each $i < n$. Thus $g(N) = N$, for if $g(N)$ was proper, then it would be contained in some maximal submodule $\mathrm{Rej}_N(S_i)$ of N. In view of the fact that a surjective endomorphism of a noetherian module is an automorphism [Fac98, Lemma 2.17(b)], we conclude that g is an automorphism. We have therefore proved that the product functor T satisfies also condition (S2).

If M and N are in $\mathbf{N}$, we have that $M \equiv_i N$ means that there exist morphisms $f\colon M \to N$ and $g\colon N \to M$ such that $f(M) \nsubseteq \mathrm{Rej}_N(S_i)$ and $g(N) \nsubseteq \mathrm{Rej}_M(S_i)$. With these equivalence relations $\equiv_i$, the n-Krull-Schmidt Theorem holds for $\mathbf{N}$.

5.2.4 Representations of type 1 pointwise

The example that we now treat was the object of study of [Gir11b].

Let $Q = (Q_0, Q_1)$ be a finite *quiver,* that is, a directed graph with a finite set of vertices Q_0 and a finite set of arrows Q_1. Each arrow a has a *tail* $t(a)$ and a *head* $h(a)$, hence we write $a\colon t(a) \to h(a)$.

For a ring R, $\mathrm{Rep}_R(Q)$ is the category of representations of Q by right R-modules and R-homomorphisms. An object of this category is a sequence $M = (M_i)_{i\in Q_0}$ of right R-modules indexed by the vertices of Q, together with a sequence $(M_a)_{a\in Q_1}$ of R-module morphisms indexed by the arrows of Q, with the requirement that $\mathrm{dom}(M_a) = M_{t(a)}$ and $\mathrm{codom}(M_a) = M_{h(a)}$. A morphism $g\colon M \to N$ of representations, i.e., a morphism in the category $\mathrm{Rep}_R(Q)$, is a sequence of R-module morphisms $g = (g_i)_{i\in Q_0}$ subject to the condition that for every arrow $a \in Q_1$ we have $g_{h(a)}M_a = N_a g_{t(a)}$.

In other words, the category of representations is the category of functors from Q to $\mathrm{Mod}\text{-}R$, where we regard the quiver as a category with Q_0 as objects and the directed paths of Q as the morphisms (plus the identities, or paths of length zero), with composition given by juxtaposition of paths.

Another natural way of seeing $\mathrm{Rep}_R(Q)$ is as a subcategory (definitely not full) of the product category $\prod_{i\in Q_0} \mathrm{Mod}\text{-}R$. Moreover, the faithful functor $\mathrm{Rep}_R(Q) \to \prod_{i\in Q_0} \mathrm{Mod}\text{-}R$ is local. (A morphism of representations g is invertible if and only if each g_i is invertible.)

The example we are going to present is now obvious: Let $\mathbf{P}$ be the full subcategory of $\mathrm{Rep}_R(Q)$ whose objects are the representations X such that X_i is a module with local endomorphism ring for every vertex $i \in Q_0$. Then the restriction T to $\mathrm{Sums}(\mathbf{P})$ of the above local faithful functor satisfies (S1′) and (S2), so that an instance of Theorem 5.10 holds for finite direct sums of representations in $\mathbf{P}$.

Notice that in Theorem 5.10 the objects of which we consider direct sums may not be indecomposable. In particular, the representations in the object class of $\mathbf{P}$ may not be indecomposable. In this particular case, though, we will show that they have a unique decomposition into indecomposable representations.

Let $\mathcal{C}$ be the class of representations M such that, for all $i \in Q_0$, either $M_i = 0$ or M_i is indecomposable. Among these we have the objects of $\mathbf{P}$.

Let $M \in \mathcal{C}$. Let $G(M) = (V(M), E(M), \psi)$ be the non-directed graph whose vertices and edges are defined by

$$V(M) = \{i \in Q_0 \mid M_i \neq 0\} \subseteq Q_0$$
$$E(M) = \{a \in Q_1 \mid M_a \neq 0\} \subseteq Q_1$$

and whose incidence function ψ is defined by $\psi(a) = \{t(a), h(a)\}$ for all $a \in E(M)$. Notice that the incidence function ψ does not depend on the representation considered. Thus two representations in $\mathcal{C}$ have the same associated graph if and only if they have the same sets of vertices and edges. Also notice that isomorphic representations have the same associated graph. In other words, the graph $G(M)$ is obtained from Q by deleting the vertices and arrows on which M vanishes, and then forgetting the direction of the arrows.

Let us recall some notions from [BM76]. Let $G = (V, E, \psi)$ be a non-directed graph. If $W \subseteq V$, the subgraph of G *induced* by W is the subgraph $G[W]$ with vertex set W and whose edges are all the edges of G whose endpoints are both in W. A subgraph of G induced by some subset W of V is called an *induced subgraph*. In other words, H is an induced subgraph of G if and only if, for each edge e of G whose endpoints are vertices of H, e is also an edge of H. By a *component* of G we mean a subgraph induced by a maximal connected subset $C \subseteq V$.

If $V \subseteq V(M)$, we define a representation $M(V)$ as follows. We let $M(V)_i = M_i$ for all $i \in V$ and $M(V)_i = 0$ if $i \in Q_0 \backslash V$. If $a \in Q_1$ is an arrow whose endpoints are both in V, then we let $M(V)_a = M_a$, while we let $M(V)_a = 0$ otherwise. It is easy to see that $G(M(V))$ coincides with the subgraph of $G(M)$ induced by V. Also notice that $M(V(M)) = M$.

We now characterise direct summands of a non-zero representation in $\mathcal{C}$ and prove a Krull-Schmidt-type Theorem for such a representation. The information on the decompositions of a representation $M \in \mathcal{C}$ is completely determined by the graph $G(M)$.

Theorem 5.12. *Let $0 \neq M \in \mathcal{C}$. Let $C_1, \ldots, C_r$ be the maximal connected subsets of $V(M)$.*

(i) Let N be a representation. Then N is a direct summand of M if and only if $N \cong M(C_{i_1} \sqcup \cdots \sqcup C_{i_t})$ for some subset $\{i_1, \ldots, i_t\}$ of $\{1, \ldots, r\}$.

(ii) M is indecomposable if and only if $G(M)$ is connected.

(iii) M has a decomposition into indecomposable representations, unique up to order and isomorphism of the factors, namely $M \cong M(C_1) \oplus \cdots \oplus M(C_r)$.

Proof. Step 1. We prove the "only if" part of (i). Suppose N is a direct summand of M. Let $V = V(N)$ for short. Let $f\colon M \to N \oplus N'$ be an isomorphism. We will construct from f an isomorphism $g\colon M(V) \to N$.

For all $i \in Q_0$, let $\varepsilon_i, \varepsilon_i'$ and π_i, π_i' denote the canonical injections and projections of the codomain $N_i \oplus N_i'$ of f_i. If $i \in V$, then $N_i \neq 0$. Therefore M_i is indecomposable, hence $N_i' = 0$. It follows that $\pi_i f_i\colon M(V)_i = M_i \to N_i$ is an isomorphism. We then let $g_i = \pi_i f_i$ for all $i \in V$. If $i \notin V$, both $M(V)_i$ and N_i are zero, thus the zero morphism $g_i = 0$ is an isomorphism. To show that $g\colon M(V) \to N$ is an isomorphism we are left to check that $N_a g_{i(a)} = g_{t(a)} M(V)_a$ for all arrows $a \in Q_1$. Let $i = i(a)$ and $j = t(a)$.

If $\{i,j\} \nsubseteq V$, then $M(V)_a = 0$. Moreover, $N_a = 0$ as well, so that the condition is trivially satisfied. (Indeed, if $N_a \neq 0$, then N_i and N_j are non-zero modules, hence $i,j \in V$.) Therefore we may suppose $\{i,j\} \subseteq V$. From $f_j M_a = (N \oplus N')_a f_i$ we obtain $\pi_j f_j M_a = N_a \pi_i f_i$. The conclusion follows from $\pi_i f_i = g_i$, $\pi_j f_j = g_j$ and $M_a = M(V)_a$.

It is left to prove that V is a union of maximal connected subsets. It suffices to prove that if $i \in V$ and j is adjacent to i in $G(M)$, then $j \in V$. To see this, pick $a \in E(M)$ such that $\psi(a) = \{i,j\}$. Since $i \in V$, then $N_i \neq 0$, so that $N'_i = 0$ and $N'_a = 0$. Suppose by contradiction $j \notin V$, i.e., $N_j = 0$. Then $N_a = 0$, so that $(N \oplus N')_a = 0$, from which $M_a = 0$, contradiction.

Step 2. We prove the "if" part of (ii). So assume $G(M)$ is connected. Suppose that $M \cong N \oplus N'$. By Step 1, $N \cong M(V)$ for some V which is a union of maximal connected subsets of $V(M)$. But $V(M)$ is the only maximal connected subset because $G(M)$ is connected, so $V = \varnothing$ or $V = V(M)$, i.e., $N = 0$ or $N \cong M(V(M)) = M$, in which case $N' = 0$, as it is required.

Step 3. Here we prove existence of the decomposition in (iii). By Step 2, $M(C_\nu)$ is indecomposable for each $\nu = 1, \ldots, r$ because $G(M(C_\nu))$ is a component of $G(M)$.

Let us now write an isomorphism $f\colon M \to M(C_1) \oplus \cdots \oplus M(C_r)$. Let $\iota_1, \ldots, \iota_r$ and $\pi_1, \ldots, \pi_r$ denote the canonical injections and projections of the codomain. If $i \in Q_0 \backslash V(M)$, then $i \notin C_\nu$ and $M(C_\nu)_i = 0$ for all $\nu = 1, \ldots, r$. Therefore $f_i = 0$ is an isomorphism. If $i \in V(M)$, then there exists a unique $\nu = 1, \ldots, r$ such that $i \in C_\nu$. Therefore, $M(C_\nu)_i = M_i$ and $M(C_\mu)_i = 0$ for all $\mu \neq \nu$. Thus we may define the isomorphism f_i by letting $f_i = \iota_{\nu,i}$.

Let $a \in Q_1$, $i = i(a)$ and $j = t(a)$. It is left to check that $f_j M_a = (M(C_1) \oplus \cdots \oplus M(C_r))_a f_i$. If $\{i,j\} \nsubseteq C_\nu$ for all $\nu = 1, \ldots, r$, then $M(C_\nu)_a = 0$ for all $\nu = 1, \ldots, r$ thus $(M(C_1) \oplus \cdots \oplus M(C_r))_a = 0$. Moreover, $M_a = 0$ because i and j are not adjacent. Thus the commutativity condition holds trivially in this case. Therefore we may assume that there exists $\nu = 1, \ldots, r$ such that $i,j \in C_\nu$. It is enough to check that $\pi_{\mu,j} f_j M_a = \pi_{\mu,j}(M(C_1) \oplus \cdots \oplus M(C_r))_a f_i$ for $\mu = 1, \ldots, r$. Since $f_i = \iota_{\nu,i}$ and $f_j = \iota_{\nu,j}$, we must have $\pi_{\mu,j}\iota_{\nu,j} M_a = \pi_{\mu,j}(M(C_1) \oplus \cdots \oplus M(C_r))_a \iota_{\nu,i}$, which is true. Thus we have the required commutativity and f is an isomorphism.

Step 4. By the previous step, the "if" part of (i) and the "only if" part of (ii) follow, so that (i) and (ii) are proved.

Step 5. We now turn to uniqueness in (iii). Suppose $M = N_1 \oplus \cdots \oplus N_t$ is an arbitrary decomposition of M into indecomposable representations $N_1, \ldots, N_t$, necessarily members of $\mathcal{C}$. By (i), $N_\nu \cong M(V(N_\nu))$ and $V(N_\nu)$ is a union of maximal connected subsets. By (ii), N_ν indecomposable implies $V(N_\nu)$ is a connected subset hence $V(N_\nu) = C_{\sigma(\nu)}$ for some $\sigma(\nu) = 1, \ldots, r$. Hence $N_\nu \cong M(C_{\sigma(\nu)})$. It remains to show that σ is a bijection.

Notice that $V(M) = V(N_1) \sqcup \cdots \sqcup V(N_t)$. Indeed, let $i \in V(M)$. Since $M_i \cong N_{1,i} \oplus \cdots \oplus N_{t,i}$ and M_i is indecomposable, there exists a unique index $\nu = 1, \ldots, t$ such that $N_{\nu,i} \neq 0$, i.e., such that $i \in V(N_\nu)$. Therefore $C_1 \sqcup \cdots \sqcup C_r = V(M) = C_{\sigma(1)} \sqcup \cdots \sqcup C_{\sigma(t)}$ implies that σ is onto. If $\mu \neq \nu$ we have $V(N_\mu) \cap V(N_\nu) = \varnothing$, i.e., $C_{\sigma(\mu)} \cap C_{\sigma(\nu)} = \varnothing$ so that $\sigma(\mu) \neq \sigma(\nu)$ and σ is also injective. $\square$

Note that the indecomposable representations which appear in the previous decomposition theorem need not have local endomorphism ring. (So it does not follow from Theorem 1.14.) Trivially, fix an indecomposable right R-module X and a vertex $i \in Q_0$. Define $M_i = X$ and $M_j = 0$ if $i \neq j \in Q_0$, and $M_a = 0$ for all $a \in Q_1$. Then the endomorphism ring of M in $\mathrm{Rep}_R(Q)$ is isomorphic to that of X in $\mathrm{Mod}\text{-}R$.

5.3 A second look at old results

In his 1996 paper [Fac96] Facchini proved that the Krull-Schmidt Theorem fails for the class of *uniserial* modules (their lattices of submodules are linearly ordered), that is, it may happen that $\bigoplus_{i<n} U_i \cong \bigoplus_{i<n} V_i$, with U_i and V_i uniserial for every $i < n$, but that *no* permutation σ exists such that $U_i \cong V_{\sigma(i)}$ for every $i < n$. (The number of direct factors has to be the same, as follows by equalling the Goldie dimensions of the two isomorphic direct sums.) Nevertheless, he proved that the 2-Krull-Schmidt Theorem (Definition 5.11) holds for the class of uniserial modules. In his book [Fac98], Facchini generalised the 2-Krull-Schmidt Theorem to the class of biuniform modules. After ten years more classes of modules exhibiting the same behaviour have been found: cyclically presented modules over local rings [AAF08], couniformly presented modules [FG10], kernels of morphisms between uniform injective modules [FEK10]. In this section we briefly show how all these classes of modules satisfy the conditions of Setting 5.2, hence how the known 2-Krull-Schmidt Theorems are essentially special cases of Theorem 5.10.

5.3.1 Biuniform and uniserial modules

Recall that a module U is *uniform* if and only if the set of non-zero submodules of U is closed by finite intersections. On the full subcategory of uniform modules it is possible to define the completely prime ideal $\mathbf{I}$ of non-injective morphisms. Uniform modules are non-zero, therefore zero morphisms are non-injective. If f and g are non-injective, then

$$0 \neq \ker(f) \cap \ker(g) \leqslant \ker(f - g)$$

shows that $f - g$ is non-injective. If fg is injective, then g is obviously injective, and also f is injective, because $\ker(f) \cap \mathrm{im}(g) = 0$ and $\mathrm{im}(g) \neq 0$ implies $\ker(f) = 0$. This shows that $fg \in \mathbf{I}$ if and only if either f or g is in $\mathbf{I}$.

Recall that a module C is *couniform* if and only if the set of proper submodules of C is closed by finite sums. In a dual manner, on the full subcategory of couniform modules it is possible to define the completely prime ideal $\mathbf{S}$ of non-surjective morphisms. Since couniform modules are non-zero, all zero morphisms are non-surjective. If f and g are non-surjective, then

$$\mathrm{im}(f - g) \leqslant \mathrm{im}(f) + \mathrm{im}(g)$$

and the right side is a proper submodule of the codomain of f and g, thus $f - g$ is non-surjective. If fg is surjective, then f is obviously surjective, and also g is, because $\mathrm{im}(g) + \ker(f) = \mathrm{dom}(f)$ and $\ker(f)$ is proper, so $\mathrm{im}(g) = \mathrm{dom}(f)$. Therefore if $fg \in \mathbf{S}$ if and only if $f \in \mathbf{S}$ or $g \in \mathbf{S}$.

In the category $\mathbf{B}$ of *biuniform* modules, i.e., those that are both uniform and couniform, it is possible to define both completely prime ideals $\mathbf{I}$ and $\mathbf{S}$. These extend uniquely to ideals of $\mathrm{Sums}(\mathbf{B})$. (Cf. end of Section 1.2.1.) Consider the canonical product functor

$$T\colon\ \mathrm{Sums}(\mathbf{B}) \to \mathrm{Sums}(\mathbf{B})/\mathbf{I} \times \mathrm{Sums}(\mathbf{B})/\mathbf{S}.$$

Let g be an endomorphism of a biuniform module X and suppose $T(g)$ is an automorphism. Then g is injective and surjective, hence an automorphism. Therefore T satisfies the condition (S2) of Setting 5.2.

Since a non-invertible endomorphism either fails to be injective or fails to be surjective, we have that $\mathbf{I}(X) \cup \mathbf{S}(X)$ is the set of non-invertible endomorphisms of $\mathbf{B}(X)$. Moreover, as we noted, they are completely prime ideals. Then we deduce from Lemma 5.4 that every proper ideal of $\mathbf{B}(X)$ is contained in either $\mathbf{I}(X)$ or $\mathbf{S}(X)$. It follows that X is an object with local endomorphism ring in both $\mathbf{B}/\mathbf{I}$ and $\mathbf{B}/\mathbf{S}$. Therefore (S1′) is also satisfied.

For X and Y biuniform, define $X \equiv_{\mathbf{I}} Y$ if X and Y are isomorphic modulo $\mathbf{I}$ and $X \equiv_{\mathbf{S}} Y$ if X and Y are isomorphic modulo $\mathbf{S}$. Then Theorem 5.10 says that, for $\{X_i\}_{i<n}$ and $\{Y_i\}_{i<m}$ biuniform, we have $\bigoplus_{i<n} X_i \cong \bigoplus_{i<m} Y_i$ if and only if $n = m$ and $X_i \equiv_{\mathbf{I}} Y_{\sigma(i)}$ and $X_i \equiv_{\mathbf{S}} Y_{\tau(i)}$ for suitable permutations σ and τ. This is essentially [Fac98, Theorem 9.13] and [Fac96, Theorem 1.9], the first results of this kind.

"Essentially," because $\equiv_{\mathbf{I}}$ and $\equiv_{\mathbf{P}}$ do not correspond precisely to the notions of monogeny class and epigeny class. Recall that X and Y have the same *monogeny* class, denoted $[X]_m = [Y]_m$, if they are isomorphic to submodules of each other, while they have the same *epigeny* class, denoted $[X]_e = [Y]_e$, if they are isomorphic to quotients of each other. We have:

Lemma 5.13. *Let X and Y be biuniform modules. Then:*

(i) $X \equiv_{\mathbf{I}} Y \implies [X]_m = [Y]_m$.

(ii) $X \equiv_{\mathbf{S}} Y \implies [X]_e = [Y]_e$.

(iii) If X or Y has type 2, then the reverse implications hold in (i) and (ii).

Proof. (i) Let $f\colon X \to Y$ and $g\colon Y \to X$ be such that $1_X - gf \in \mathbf{I}(X)$ and $1_Y - fg \in \mathbf{I}(Y)$ are non-injective. Then $gf \notin \mathbf{I}(X)$, hence g and f are not in $\mathbf{I}$, that is, f and g are both injective. Thus $[X]_m = [Y]_m$.

(ii) Exactly as in (i).

(iii) Suppose that X has type 2. If $f\colon X \to Y$ and $g\colon Y \to X$ are injective morphisms, then gf is an injective endomorphism of X, that is, $gf \notin \mathbf{I}(X)$. Since $\mathbf{B}(X)/\mathbf{I}(X)$ is a division ring, there is an endomorphism g of X such that $1_x - hgf \in \mathbf{I}(X)$. Hence $f(1_X - hgf) \in \mathbf{I}(Y)$, and then Now $(1_Y - fhg)f \in \mathbf{I}$. Since $\mathbf{I}$ is completely prime and $f \notin \mathbf{I}$, we have $1_Y - fhg \in \mathbf{I}$. Thus f and hg show that $X \equiv_{\mathbf{I}} Y$. The proof that the implication in (ii) can be reversed is analogous. □

5.3.2 Couniformly presented modules

A module M is *couniformly presented* if it is isomorphic to P/K for some couniform projective module P and some non-trivial couniform submodule K of P. These modules were the main object of study of [FG10], and will be treated in Chapter 6. There is a subclass of couniformly presented modules, though, which is also a subclass of Auslander-Bridger modules, that fits Setting 5.2. Let $\mathbf{C}$ be the category of non-zero non-projective modules M with a (fixed) presentation

$$Q_M \xrightarrow{\vartheta_M} P_M \xrightarrow{\pi_M} M \longrightarrow 0 \tag{5.14}$$

where both Q_M and P_M are couniform projective modules. (We are using the same notation of Chapter 4.) The additive local and isomorphism-reflecting functor G of Proposition 4.32 restricts to an additive local and isomorphism-reflecting functor defined on $\mathrm{Sums}(\mathbf{C})$. Since it is local, it satisfies (S2) of Setting 5.2. It also satisfies (S1′), because if M is as above, the endomorphism ring of $G_0(M)$ is $\mathrm{End}_R(P_M)$ (which is a local ring) modulo the ideal of non-surjective endomorphisms (its Jacobson radical), and that of $G_1(M)$ is $\mathrm{End}_R(Q_M)/J(\mathrm{End}_R(Q_M))$, again a division ring. The equivalence relations $\equiv_1$ and $\equiv_0$ on the class of objects of $\mathbf{C}$ correspond respectively to "lower-isomorphism" and "epi-isomorphism" as in Chapter 4. The correspondence with the notions of "lower part" and "epigeny" of Chapter 6 is instead imperfect, cf. Lemma 4.36.

5.3.3 Kernels of morphisms between indecomposable injectives

This class of modules is the class of dual Auslander-Bridger modules that are the E-duals of the class of couniformly presented modules of the previous example. The minimal projective presentation (5.14) is sent by $\mathrm{Hom}(-, E)$ to a minimal injective copresentation $0 \to M^* \to P_M^* \to Q_M^*$, with P_M^* and Q_M^* indecomposable injectives. The class $\mathcal{F}$ of modules of the type M^*, or rather, its closure by isomorphic copies, satisfies via $\mathrm{Hom}(-, E)$ and by the previous example, Setting 5.2. The equivalence relations $\equiv_0$ and $\equiv_1$ are the "mono-isomorphism" and "upper-isomorphism" we defined when we studied dual Auslander-Bridger modules in Section 4.4.2. For the modules in $\mathcal{F}$ that do *not* have a local ring of endomorphisms, the notion of "mono-isomorphism" coincides with the notion of "monogeny" and the notion of "upper-isomorphism" coincides with the notion of "upper part." Therefore, for such modules, the specialisation of Theorem 5.10 and [FEK10, Theorem 2.7] agree.

5.4 The associated hypergraph

The aim of this section is to establish when the n-Krull-Schmidt Theorem holds for a preadditive category **C** (see Definition 5.11), in terms of a *hypergraph* associated to **C**. The results of this section are thus a generalisation of some results of [FP09c], and they provide a geometrical interpretation of Theorem 5.10.

Let us recall some combinatorial notions. By a *hypergraph* we mean a class of vertices V together with a class E of non-empty finite subsets of V, which are the edges of the hypergraph, such that the union of the class E is V. The original definition of hypergraph as given in [Ber89] is way too restrictive for our purposes, in that it allows only finite sets of vertices and edges, although it allows edges to be repeated.

We denote by $H = (V, E)$ a hypergraph whose class of vertices is V and whose class of edges is E. We say that H is *n-uniform* if all its edges have n elements, and it is called *simple* if there are no inclusion relations between its edges. Also recall that a *partial hypergraph* is obtained from H by selecting a subclass F of the class of edges E, and is denoted $H[F]$. The class of vertices of $H[F]$ is (necessarily) the union of the class F.

Let $H = (V, E)$ be a hypergraph. Let $\mathbb{N}^{(V)}$ be the (large) free commutative monoid with free basis V. (Notice that the fact that V may be a proper class implies that the construction of the above free commutative monoid cannot be done in the usual manner, cf. Appendix A.) Thus the element v of V, when seen as an element of $\mathbb{N}^{(V)}$, is the function $V \to \mathbb{N}$ which maps v to 1 and everything else to 0. If $e \in E$, denote by $\chi(e)$ the characteristic function of e, i.e., $\chi(e) = \sum_{v \in e} v$. To the hypergraph H we associate the submonoid $\mathrm{Mon}(H)$ of $\mathbb{N}^{(V)}$ generated by the characteristic functions of edges.

Definition 5.15. Let $H = (V, E)$ be a hypergraph and n a positive integer. We say that the *n-Krull-Schmidt Theorem* holds for H if there exist equivalence relations $\{\sim_i\}_{i<n}$ on the class of edges E such that, given finite sets of edges $\{e_\mu\}_{\mu<r}$ and $\{f_\mu\}_{\mu<s}$, the equality

$$\sum_{\mu<r} \chi(e_\mu) = \sum_{\mu<s} \chi(f_\mu)$$

holds in the monoid $\mathrm{Mon}(H)$ if, and only if, $r = s$ and there exist permutations $\{\sigma_i\}_{i<n}$ such that $e_\mu \sim_i f_{\sigma_i(\mu)}$ for all $i < n$ and $\mu < r$.

To a preadditive category $\mathbf{C}$ whose objects are of finite type we associate a hypergraph $H(\mathbf{C})$. This hypergraph has the class $V(\mathbf{C})$ of maximal ideals of $\mathbf{C}$ as its class of vertices and its edges are the finite sets $V(X) = V(\mathbf{C}, X)$ where X is an object of the semilocal category $\mathbf{C}$ (cf. Section 2.2). The following dictionary between $\mathbf{C}$ and its hypergraph $H(\mathbf{C})$ justifies turning our attention to hypergraphs and their associated monoids.

Lemma 5.16. *The n-Krull-Schmidt Theorem holds for $\mathbf{C}$ if and only if it holds for $H(\mathbf{C})$.*

Proof. Let $\{X_\mu\}_{\mu<r}$ and $\{Y_\mu\}_{\mu<s}$ be finite sets of objects of $\mathbf{C}$. We claim that

$$\sum_{\mu<r} \chi(V(X_\mu)) = \sum_{\mu<s} \chi(V(Y_\mu)) \tag{5.17}$$

holds in $\mathrm{Mon}(H(\mathbf{C}))$ if and only if we have an isomorphism of direct sums in $\mathrm{Sums}(\mathbf{C})$

$$\bigoplus_{\mu<r} X_\mu \cong \bigoplus_{\mu<s} Y_\mu. \tag{5.18}$$

Indeed, equation (5.17) holds if and only if, for each $\mathbf{P} \in V(\mathbf{C})$, the number of indices i such that $\mathbf{P} \in V(X_i)$ is equal to the corresponding number of indices j such that $\mathbf{P} \in V(Y_j)$. By an application of Lemma 2.24, this is equivalent to $\bigoplus_{\mu<r} X_\mu$ and $\bigoplus_{\mu<s} Y_\mu$ being isomorphic in $\mathrm{Sums}(\mathbf{C})/\mathbf{P}$, for all $\mathbf{P} \in V(\mathrm{Sums}(\mathbf{C}), M)$ and for all non-zero objects M of $\mathbf{C}$, which is equivalent to equation (5.18) by Lemma 2.19. This proves the claim.

Suppose that the n-Krull-Schmidt Theorem holds for $\mathbf{C}$ relatively to the equivalence relations $\{\equiv_i\}_{i<n}$ on the class of objects of $\mathbf{C}$. Let $V(X)$ and $V(Y)$ be edges of $H(\mathbf{C})$, where X and Y are objects of $\mathbf{C}$. Let $V(X) \sim_i V(Y)$ if and only if $X \equiv_i Y$. The definition of $\sim_i$ does not depend on the choice of X and Y, because $X \cong Y$ if and only if $V(X) = V(Y)$ (Corollary 2.25). It is easy to see that the n-Krull-Schmidt Theorem holds for $\mathrm{Mon}(H(\mathbf{C}))$ relatively to the relations $\{\sim_i\}_{i<n}$. A similar argument shows the converse. $\square$

From now on, let $n \geqslant 2$ be a fixed integer and let $\mathbf{C}$ be a preadditive category whose non-zero objects are indecomposable and of finite type at most n. The condition that the objects of $\mathbf{C}$ be indecomposable is equivalent to requiring that there are no inclusion relations between the edges of $H(\mathbf{C})$ (Corollary 2.25), i.e., that $H(\mathbf{C})$ is a *simple* hypergraph.

For each family of pairwise disjoint classes $\{X_i\}_{i<n}$, we define the *n-partite complete hypergraph on* $\bigsqcup_{i<n} X_i$ to be the hypergraph $P(X_0, \dots, X_{n-1})$ with class of vertices $\bigsqcup_{i<n} X_i$ and whose class of edges $E(X_0, \dots, X_{n-1})$ consists of all n-element subsets of vertices which have exactly one vertex from each X_i [Ber89, pg. 19]. Thus $P(X_0, \dots, X_{n-1})$ is simple and n-uniform.

A hypergraph $H = (V, E)$ is *n-partite* if V is a disjoint union $V = \bigsqcup_{i<n} U_i$ such that each U_i is not empty, and such that for each $e \in E$ and each $i < n$, the set $e \cap U_i$ has at most one element. Clearly, a partial hypergraph of an n-partite complete hypergraph is n-partite.

Recall that in a commutative monoid M, an element x is an *atom* if it is non-zero and, for all $a, b \in M$, the equality $x = a + b$ implies $a = 0$ or $b = 0$. The following generalises [FP09c, Proposition 3.5]. (Also see page 76 for the connection between biproduct decomposition in additive categories and factorisations in reduced commutative monoids.)

Theorem 5.19. *Let $H = (V, E)$ be a simple hypergraph. The following are equivalent:*

(i) The n-Krull Schmidt Theorem holds for H.

(ii) There exists an injective morphism $\varphi\colon \operatorname{Mon}(H) \to \operatorname{Mon}(P(X_0, \dots, X_{n-1}))$ of monoids which sends atoms to atoms, where $X_0, \dots, X_{n-1}$ are suitable pairwise disjoint classes.

(iii) There exists an injective mapping $\eta\colon E \to E(X_0, \dots, X_{n-1})$, where $\{X_i\}_{i<n}$ are suitable pairwise disjoint classes, such that

$$\sum_{\mu<r} \chi(e_\mu) = \sum_{\mu<s} \chi(f_\mu) \tag{5.20}$$

holds in $\operatorname{Mon}(H)$ if and only if

$$\sum_{\mu<r} \chi(\eta(e_\mu)) = \sum_{\mu<s} \chi(\eta(f_\mu)) \tag{5.21}$$

holds in $\operatorname{Mon}(P(X_0, \dots, X_{n-1}))$.

Proof. Suppose (i) holds, with respect to a suitable choice of equivalence relations $\{\sim_i\}_{i<n}$ on the class of edges E. Let $\pi_i\colon E \to E/{\sim_i}$ be the canonical projection.

Let $X_i = (E/{\sim_i}) \times \{i\}$ and let $p_i\colon E \to X_i$ be defined by $p_i(e) = (\pi_i(e), i)$. This makes $X_0, \dots, X_{n-1}$ pairwise disjoint classes. The mapping p_i induces a monoid homomorphism $\widetilde{p}_i\colon \operatorname{Mon}(H) \to \mathbb{N}^{(X_i)}$ as follows: If $\sum_{\mu<r} \chi(e_\mu)$ is an arbitrary element of $\operatorname{Mon}(H)$, let

$$\widetilde{p}_i\left(\sum_{\mu<r} \chi(e_\mu)\right) = \sum_{\mu<r} p_i(e_\mu).$$

To show that $\tilde{p}_i$ is well-defined, suppose (5.20) holds in $\operatorname{Mon}(H)$. Then $r = s$ and there exists a permutation $\sigma \in S_r$ such that $e_\mu \sim_i f_{\sigma(\mu)}$ for all $\mu < r$. Therefore,

$$\sum_{\mu<r} p_i(e_\mu) = \sum_{\mu<r} p_i(f_{\sigma(\mu)}) = \sum_{\mu<r} p_i(f_\mu),$$

hence $\widetilde{p}_i$ is well-defined. Furthermore, we have an injective monoid homomorphism

$$p = \prod_{i<n} \widetilde{p}_i\colon \operatorname{Mon}(H) \to \prod_{i<n} \mathbb{N}^{(X_i)}.$$

To show injectivity, suppose that

$$\sum_{\mu<r} p_i(e_\mu) = \sum_{\mu<s} p_i(f_\mu)$$

holds in $\mathbb{N}^{(X_i)}$ for each $i < n$. Then $r = s$ and there exists a permutation $\sigma_i \in S_r$ such that $p_i(e_\mu) = p_i(f_{\sigma_i(\mu)})$, i.e., such that $e_\mu \sim_i f_{\sigma_i(\mu)}$ for each $\mu < r$. This implies (5.20), hence p is injective.

In the following diagram, α is the isomorphism defined by $\alpha\colon (g_i)_{i<n} \mapsto \sum_{i<n} g_i$, while the bottom morphism is set inclusion.

$$\begin{array}{ccc}
\operatorname{Mon}(H) & \xrightarrow{\quad p \quad} & \prod_{i<n} \mathbb{N}^{(X_i)} \\
{\scriptstyle\varphi}\,\vdots & & \downarrow{\scriptstyle\alpha} \\
\operatorname{Mon}(P(X_0,\dots,X_{n-1})) & \longrightarrow & \mathbb{N}^{(\bigsqcup_{i<n} X_i)}
\end{array}$$

For each $e \in E$, we have $\alpha p(\chi(e)) = \sum_{i<n} p_i(e)$. Thus $\varepsilon = \{p_i(e)\}_{i<n}$ is an edge in $E(X_0,\dots,X_{n-1})$, and $\chi(\varepsilon) = \alpha p(\chi(e)) \in \operatorname{Mon}(P(X_0,\dots,X_{n-1}))$. Inasmuch as $\operatorname{Mon}(H)$ is generated by $\{\chi(e) : e \in E\}$, we conclude that the image of αp is contained in $\operatorname{Mon}(P(X_0,\dots,X_{n-1}))$, thus we can complete the diagram to a commutative square by adding φ, necessarily injective. Since the atoms of $\operatorname{Mon}(H')$ are the characteristic functions of edges for every simple hypergraph H', the equality $\varphi(\chi(e)) = \chi(\varepsilon)$ also implies that φ sends atoms to atoms. We have thus proved that (ii) holds.

Now assume (ii). For each $e \in E$, $\chi(e)$ is an atom of $\operatorname{Mon}(H)$, thus $\varphi(\chi(e))$ is an atom of $\operatorname{Mon}(P(X_0,\dots,X_{n-1}))$, hence it is equal to $\chi(\eta(e))$ for some $\eta(e) \in E(X_0,\dots,X_{n-1})$. Since $\eta(e)$ is uniquely determined, we have a mapping $\eta\colon E \to E(X_0,\dots,X_{n-1})$. If (5.20) holds, applying φ to it we obtain that (5.21) holds. If the latter holds, the former holds by injectivity of φ. In particular, $\eta(e) = \eta(f)$ implies that $\chi(\eta(e)) = \chi(\eta(f))$, which is equivalent to $\chi(e) = \chi(f)$, which implies $e = f$. Hence η is injective, and this completes the proof that (ii) implies (iii).

Eventually, assume (iii). For each $i < n$ and each pair $e, f \in E$, define $e \sim_i f$ if $\eta(e) \cap X_i = \eta(f) \cap X_i$. This is an equivalence relation on E. Assume (5.20) holds, so that (5.21) holds. Inasmuch as $P(X_0, \dots, X_{n-1})$ is n-uniform, we must have $r = s$.

Write $\chi(\eta(e_\mu)) = \sum_{i<n} e_{\mu,i}$, where $e_{\mu,i} \in X_i$ for all $i < n$, and write $\chi(\eta(f_\mu))$ accordingly. Thus

$$\sum_{\mu<r} \sum_{i<n} e_{\mu,i} = \sum_{\mu<r} \sum_{i<n} f_{\mu,i},$$

and in view of the fact that the classes $X_0, \dots, X_{n-1}$ are pairwise disjoint, it follows that

$$\sum_{\mu<r} e_{\mu,i} = \sum_{\mu<r} f_{\mu,i}$$

for each $i < n$. Thus there exist permutations $\{\sigma_i\}_{i<n} \subseteq S_r$ such that $e_{\mu,i} = f_{\sigma_i(\mu),i}$, i.e., $\eta(e_\mu) \cap X_i = \eta(f_{\sigma_i(\mu)}) \cap X_i$, uniformly in $\mu < r$ and $i < n$. Hence $e_\mu \sim_i f_{\sigma_i(\mu)}$, for $i < n$ and $\mu < r$. Conversely, if $r = s$ and such permutations exist, then (5.21) holds, hence (5.20) also holds. This proves that (i) holds with respect to the equivalence relations $\{\sim_i\}_{i<n}$. □

Corollary 5.22. *If the n-Krull-Schmidt Theorem holds for a simple hypergraph $H = (V, E)$, then it also holds for any partial hypergraph of H.*

Proof. Let F be a subclass of E and consider the partial hypergraph $H[F]$. There is a canonical injective monoid homomorphism $\iota\colon \mathrm{Mon}(H[F]) \to \mathrm{Mon}(H)$ that sends atoms to atoms. Thus if φ is as in Theorem 5.19(ii), then $\varphi\iota$ shows that the relations of $\mathrm{Mon}(H[F])$ are controlled by n permutations. □

Consider the *intersection graph* G of the edges E of H, i.e., the simple graph having E as its class of vertices, and such that two elements of E are adjacent in G whenever their intersection is non-empty. Partition E as the disjoint union $E = \bigcup_{i\in I} E_i$ of the maximal connected subclasses of vertices of G. For each $i \in I$, let $H_i = H[E_i]$, i.e., let H_i be the partial hypergraph of H on the subclass of edges E_i, and denote by V_i its class of vertices. Note that V is the disjoint union $V = \bigcup_{i\in I} V_i$. We refer to the hypergraphs H_i as the *connected components* of H.

Lemma 5.23. *The n-Krull-Schmidt Theorem holds for $H = (V, E)$ if and only if it holds for each connected component of H.*

Proof. There is a canonical isomorphism of monoids $\bigoplus_{i\in I} \mathbb{N}^{(V_i)} \to \mathbb{N}^{(V)}$, more precisely, the one which sends $(g_i\colon V_i \to \mathbb{N})_{i\in I}$ to the function $g\colon V \to \mathbb{N}$ obtained by $g(x) = g_i(x)$ for $x \in V_i$. It is easy to see that it induces an isomorphism $q\colon \bigoplus_{i\in I} \mathrm{Mon}(H_i) \to \mathrm{Mon}(H)$.

One implication of the lemma follows at once from the previous corollary. For the other implication, assume that the n-Krull-Schmidt Theorem holds for each H_i, so that there is an injective monoid homomorphism $\varphi_i\colon \operatorname{Mon}(H_i) \to \operatorname{Mon}(P(X_{i,0}, \dots, X_{i,n-1}))$ which sends atoms to atoms, for each $i \in I$. Without loss of generality, suppose that the classes $X_{i,j}$ are pairwise disjoint. Therefore, once we define $X_j = \bigcup_{i\in I} X_{i,j}$, we obtain that $X_0, \dots, X_{n-1}$ are pairwise disjoint. Let $\iota_i\colon \operatorname{Mon}(P(X_{i,0}, \dots, X_{i,n-1})) \to \operatorname{Mon}(P(X_0, \dots, X_{n-1}))$ be the canonical embedding of monoids, for each $i \in I$. Define $\varphi\colon \bigoplus_{i\in I} \operatorname{Mon}(H_i) \to \operatorname{Mon}(P(X_0, \dots, X_{n-1}))$ by $\varphi((g_i)_{i\in I}) = \sum_{i\in I} \iota_i\varphi_i(g_i)$. It is easy to check that φ is injective and sends atoms to atoms, hence the n-Krull-Schmidt Theorem holds for H. □

For integers r and n such that $1 \leqslant r \leqslant n$, let K_n^r denote the *r-uniform complete hypergraph of order n*, i.e., the hypergraph whose vertices are the elements of a set X of cardinality n and whose edges are all the r-element subsets of X [Ber89, pg. 5]. Thus the number of edges of K_n^r is $\binom{n}{r}$. Recall that in a hypergraph the *degree* of a vertex v, denoted by $d(v)$, is the number of edges e such that $v \in e$.

The following extends [FP09c, Proposition 3.9].

Proposition 5.24. *Let $n \geqslant 2$ be an integer. If a simple hypergraph $H = (V, E)$ admits K_{2n}^n as a partial hypergraph, then the n-Krull-Schmidt Theorem does not hold for H.*

Proof. In view of Corollary 5.22, we may assume $H = K_{2n}^n$. Assume that the n-Krull-Schmidt Theorem holds for K_{2n}^n, and let φ, η, $X_0, \dots, X_{n-1}$ be as in Theorem 5.19. We are going to show that the partial hypergraph $C = P(X_0, \dots, X_{n-1})[\eta(E)]$ is a copy of K_{2n}^n and that the latter is not n-partite, which contradicts C being a partial hypergraph of an n-partite hypergraph.

By a construction by induction, it is possible to write E as a disjoint union

$$E = \{e_1, \dots, e_m\} \sqcup \{V\backslash e_1, \dots, V\backslash e_m\}.$$

Necessarily, $m = |E|/2$. The element $s = \sum_{v\in V} v$ of $\operatorname{Mon}(K_{2n}^n)$ can be written as $s = \chi(e_i) + \chi(V\backslash e_i)$ for any $i = 1, \dots, m$.

Let u be a vertex of C. Then $u \in \eta(e_i)$ or $u \in \eta(V\backslash e_i)$ for some i. Since $\varphi(s) = \chi(\eta(e_i)) + \chi(\eta(V\backslash e_i))$, it follows that the coefficient of u in $\varphi(s)$ is strictly positive. This implies that $u \in \eta(e_i)$ or $u \in \eta(V\backslash e_i)$, now for *all* indices $i = 1, \dots, m$. Since η is injective, it follows that the degree $d_C(u)$ of u in C is at least m. Let U be the set of vertices of C. Then

$$m|U| \leqslant \sum_{u\in U} d_C(u) = n|\eta(E)| = n|E| = 2mn,$$

from which $|U| \leqslant 2n$. Since C is n-uniform on $|U|$ vertices, we must have $|\eta(E)| \leqslant \binom{|U|}{n}$. But η is injective, hence $|\eta(E)| = |E| = \binom{2n}{n}$, so that $|U| \geqslant 2n$.

Thus $|U| = 2n$, and it follows that C is the complete n-uniform hypergraph on $2n$ vertices.

To reach the required contradiction, let us finally show that C is not n-partite. Suppose it is n-partite. Then write U as a disjoint union $U = \bigsqcup_{i<n} U_i$ in such a way that for each $\varepsilon \in \eta(E)$, the set $\varepsilon \cap U_i$ has at most one element. Insofar as $2n = \sum_{i<n} |U_i|$, there exists $i < n$ such that U_i has at least two elements. Pick an n-element subset ε of U with two elements from U_i. Then this is an edge of C by completeness, contradiction. □

Chapter 6

Couniformly presented modules and dualities

In Chapters 4 and 5 we have already mentioned a subclass of couniformly presented modules, namely, those modules of the form $\operatorname{coker}(\varphi)$ where φ is a morphism between couniform projective modules. In general a *couniformly presented module* is a module $M \neq 0$ that embeds in a short exact sequence

$$0 \longrightarrow C \xrightarrow{\iota} P \longrightarrow M \longrightarrow 0 \tag{6.1}$$

where P is projective, and both P and C are couniform. We will assume without loss of generality that ι is set inclusion. We say that (6.1) is a *couniform presentation* of M. Notice that $P \to M$ is necessarily a projective cover, because $M \neq 0$ implies $C < P$ hence $C \leqslant_s P$.

If C has a projective cover, then this is a module of the type already studied in the previous chapters, but this may not be the case. For instance, if R is a valuation domain that is not a principal ideal domain, R/I is couniformly presented even when I is not finitely generated. A concrete example is the domain R of Puiseux series $R = \bigcup_{1 \leqslant n < \omega} K\left[\!\left[x^{1/n}\right]\!\right]$, modulo the ideal $I = J(R)$.

Cyclically presented modules over a local ring R [AAF08] are either couniformly presented or isomorphic to 0 or R. Over a right chain ring R, that is, a ring R with R_R uniserial, a right module is couniformly presented if and only if it is cyclic but not projective. In particular, couniformly presented right modules over right chain rings are uniserial.

In this chapter we study couniformly presented modules in full generality [FG10]. In particular we will prove a version of Theorem 5.10 (with $n = 2$) for this class of modules.

It would be possible to study these modules using the general machinery of Chapter 5, although the choice of ideals has to be done with care, cf. page 123.

Here we prefer to give elementary proofs instead, thus illustrating the techniques used in [Fac96] and then later in [AAF08, FG10, FEK10] to prove several instances of the case $n = 2$ of Theorem 5.10.

Recall that a module M is said to be *couniform* (or *hollow*) if it has dual Goldie dimension one, that is, if it is non-zero and the sum of any two proper submodules of M is a proper submodule of M. Equivalently, a non-zero module is couniform if and only if all its proper submodules are superfluous, if and only if all its non-zero homomorphic images are indecomposable modules. For instance, every non-zero uniserial module, that is, every non-zero module whose lattice of submodules is linearly ordered under inclusion, is couniform.

Projective couniform modules have been characterised in several equivalent ways (Lemma 4.2). In particular, a couniform projective right R-module is isomorphic to eR for some local idempotent r of R, "local" meaning that eRe is a local ring, and eR is the projective cover of the simple module $eR/eJ(R)$. Thus there a injective mapping $\langle eR\rangle \to \langle eR/eJ(R)\rangle$ from the family of isomorphism classes of couniform projective right R-modules into the family of simple right R-modules. This mapping is a bijection if and only if the ring R is semiperfect [Bas60, Theorem 2.1] [Fac98, Theorem 3.6(d)].

Given any couniformly presented module M with couniform presentation (6.1), every endomorphism f of M lifts to an endomorphism f_0 of the projective cover P, which in turn induces by restriction/corestriction an endomorphism f_1 of C. Hence we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \xrightarrow{\iota} & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & C & \xrightarrow{\iota} & P & \longrightarrow & M & \longrightarrow & 0 \end{array} \tag{6.2}$$

The morphisms f_0 and f_1 that complete diagram (6.2) are not uniquely determined by f. Nevertheless, it is easily seen that $f\colon M \to M$ is an epimorphism if and only if $f_0\colon P \to P$ is an epimorphism, if and only if f_0 is an automorphism. It follows that if we substitute f_0 and f_1 with two other morphisms f_0' and f_1' making the diagram analogous to diagram (6.2) commute, then $f_0\colon P \to P$ is an epimorphism if and only if $f_0'\colon P \to P$ is an epimorphism. In this notation, let us show that the same holds for C, i.e., that

Lemma 6.3. *The endomorphism $f_1\colon C \to C$ is surjective if and only if $f_1'\colon C \to C$ is surjective.*

Proof. The commutativity of the two diagrams (6.2), one relative to f_0 and f_1, the other relative to f'_0 and f'_1, gives, by subtraction, a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \xrightarrow{\iota} & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1-f'_1} & & \downarrow{\scriptstyle f_0-f'_0} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & C & \xrightarrow{\iota} & P & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Hence $(f_0 - f'_0)(P) \subseteq C$. Since C is superfluous in P, it follows that $(f_0 - f'_0)(C)$ is superfluous in $(f_0 - f'_0)(P)$, so that $(f_0 - f'_0)(C) = (f_1 - f'_1)(C)$ is a proper submodule of C. Thus $f_1 - f'_1$ is not an epimorphism. This and the fact that C is couniform yield that $f_1 \colon C \to C$ is an epimorphism if and only if $f'_1 \colon C \to C$ is an epimorphism. Indeed, $f_1(C) \leqslant (f_1 - f'_1)(C) + f'_1(C) \leqslant C$, so if f_1 is onto then so is f'_1. □

Our proof of Lemma 6.3 is essentially the same as the proof of [FH06, Lemma 7.1].

Notice that, in the proof of Lemma 6.3, we have seen that, for every morphism $g \colon P \to C$ (where $C < P$ are couniform modules and P is projective), $g(C)$ is properly contained in C.

It is easy to see that for every couniform right R-module U, the endomorphism ring $\mathrm{End}_R(U)$ has a proper completely prime ideal K_U consisting of all the non-surjective endomorphisms of U. Cf. [Fac98, Lemma 6.26]).

The ring $\mathrm{End}_R(U)/K_U$ is thus an integral domain, but it is not a division ring in general. For instance, take as U the Prüfer group $\mathbb{Z}(p^\infty)$ viewed as a $\mathbb{Z}$-module. Since it is a uniserial divisible module, it is a uniform injective module, hence its endomorphism ring $S \cong \mathbb{Z}_p$ is local and its Jacobson radical consists of those endomorphisms that are not injective. Multiplication by p induces a non-injective endomorphism that is surjective, hence $p \in J(S) \backslash K_{\mathbb{Z}(p^\infty)}$, hence $E/K_{\mathbb{Z}(p^\infty)}$ is a local ring but not a division ring.

Our proof of Lemma 6.3 also shows that for every couniformly presented right R-module M with couniform presentation (6.1), there is a well-defined ring morphism $\mathrm{End}_R(M) \to \mathrm{End}_R(C)/K_C$, defined by $f \mapsto f_1 + K_C$.

Similarly to [FH06, Section 7], by Lemma 6.3, we can consider the ring morphism

$$\Phi \colon \mathrm{End}_R(M) \to \mathrm{End}_R(M)/K_M \times \mathrm{End}_R(C)/K_C$$

defined by $\Phi(f) = (f + K_M, f_1 + K_C)$ for every $f \in \mathrm{End}_R(M)$. Recall that a ring morphism $\varphi \colon S \to S'$ is said to be *local* if, for every $s \in S$, $\varphi(s) \in U(S')$ implies $s \in U(S)$.

Lemma 6.4. *Let M be a couniformly presented right R-module with a couniform presentation* (6.1). *Then the ring morphism Φ is local.*

Proof. Let $f \in \mathrm{End}_R(M)$ be an endomorphism with $\Phi(f)$ invertible. Consider the commutative diagram (6.2). Then $f + K_M$ and $f_1 + K_C$ are invertible in

$\mathrm{End}_R(M)/K_M$ and $\mathrm{End}_R(C)/K_C$ respectively, so that, in particular, $f \notin K_M$ and $f_1 \notin K_C$, that is, the morphisms f and f_1 are epimorphisms. Thus f_0 also is an epimorphism, hence an automorphism of P because P is projective and indecomposable. By the Snake Lemma applied to diagram (6.2), f_0 isomorphism and f_1 epimorphism imply f monomorphism. □

The next result describes the endomorphism ring of a couniformly presented module.

Theorem 6.5. *Let M be a couniformly presented module with a couniform presentation* (6.1) *and endomorphism ring S. Let $\mathfrak{e} := \{\, f \in S : f$ is not surjective$\,\}$ and $\mathfrak{l} := \{\, f \in S : f_1 : C \to C$ is not surjective$\,\}$. Then $\mathfrak{e}$ and $\mathfrak{l}$ are completely prime two-sided ideals of S, the union $\mathfrak{e} \cup \mathfrak{l}$ is the set of all non-invertible elements of S, and every proper right ideal of S and every proper left ideal of S is contained either in $\mathfrak{e}$ or in $\mathfrak{l}$. Moreover, one of the following two conditions holds:*

(i) Either the ideals $\mathfrak{e}$ and $\mathfrak{l}$ are comparable, so that S is a local ring with maximal ideal the greatest ideal among $\mathfrak{e}$ and $\mathfrak{l}$, or

(ii) $\mathfrak{e}$ and $\mathfrak{l}$ are not comparable, $J(S) = \mathfrak{e} \cap \mathfrak{l}$, and $S/J(S)$ is canonically isomorphic to the product of the two division rings $S/\mathfrak{e}$ and $S/\mathfrak{l}$.

Proof. Let π_1 and π_2 be the canonical projections of $S/K_M \times \mathrm{End}_R(C)/K_C$ onto S/K_M and $\mathrm{End}_R(C)/K_C$, respectively. We already know that $\mathfrak{e} = K_M$ is a completely prime ideal of $\mathrm{End}_R(M)$. Notice that $\mathfrak{l}$ is the kernel of the composite morphism $\pi_2\Phi\colon S \to \mathrm{End}_R(C)/K_C$. As $\mathrm{End}_R(C)/K_C$ is an integral domain, it follows that $\mathfrak{l}$ is a completely prime ideal of S.

As the ideals $\mathfrak{e}$ and $\mathfrak{l}$ are proper, it follows that $\mathfrak{e} \cup \mathfrak{l} \subseteq S \backslash U(S)$. Conversely, if $f \in S$ is non-invertible, it is not an automorphism, so that it is either non-surjective or non-injective. If f is not surjective, then $f \in \mathfrak{e}$. If f is surjective but not injective, then in diagram (6.2) we have that f_0 is surjective, so that f_0 is an automorphism of P. By the Snake Lemma applied to (6.2), we have that f_0 automorphism of P and f non-injective imply f_1 non-surjective. Thus $f \in \mathfrak{l}$.

Every proper right or left ideal L of S is contained in $\mathfrak{e} \cup \mathfrak{l}$. If there exist $x \in L\backslash\mathfrak{e}$ and $y \in L\backslash\mathfrak{l}$, then $x + y \in L$, $x \in \mathfrak{l}$ and $y \in \mathfrak{e}$. Hence $x + y \notin \mathfrak{e}$ and $x + y \notin \mathfrak{l}$. Thus $x + y \notin \mathfrak{e} \cup \mathfrak{l}$, so that $x + y \in L$ and is an invertible element of S, a contradiction. This proves that L is contained either in $\mathfrak{e}$ or in $\mathfrak{l}$. In particular, the unique maximal right ideals of S are at most $\mathfrak{e}$ and $\mathfrak{l}$. Similarly, the unique maximal left ideals of S are at most $\mathfrak{e}$ and $\mathfrak{l}$.

If $\mathfrak{e}$ and $\mathfrak{l}$ are comparable, then (i) clearly holds. If $\mathfrak{e}$ and $\mathfrak{l}$ are not comparable, the ring S has exactly two maximal right ideals $\mathfrak{e}$ and $\mathfrak{l}$, so that $J(S) = \mathfrak{e} \cap \mathfrak{l}$, $S/\mathfrak{e}$ and $S/\mathfrak{l}$ are division rings, and there is a canonical injective ring homomorphism $\pi\colon S/J(S) \to S/\mathfrak{e} \times S/\mathfrak{l}$. But $\mathfrak{e} + \mathfrak{l} = S$ because $\mathfrak{e}$ and $\mathfrak{l}$ are incomparable maximal right ideals of S, hence π is surjective by the Chinese Remainder Theorem 1.19. □

Remark 6.6. The ideal $\mathfrak{l}$ in the statement of Theorem 6.5 does not depend on the couniform presentation (6.1) of M. Suppose $0 \to C \to P \to M \to 0$ and $0 \to C' \to P' \to M \to 0$ are two couniform presentations of M. Let f be an endomorphism of M, and consider a diagram (6.2) relative to f for each of the two couniform presentations. We need to show that f_1 is an epimorphism if and only if f_1' is an epimorphism. Construct another diagram (6.2) as follows. The identity of M lifts to an isomorphism $g_0 \colon P \to P'$ between the two projective covers of M, and g_0 restricts to a morphism $g_1 \colon C \to C'$, which is an isomorphism as well. By Lemma 6.3, we then have that f_1 is an epimorphism if and only if $g_1^{-1} f_1' g_1$ is an epimorphism, and this is an epimorphism if and only if f_1' is an epimorphism.

By Theorem 6.5, couniformly presented modules have semilocal endomorphism ring, hence cancel from direct sums (Theorem 2.13).

Remark 6.7. Choose a local idempotent e such that $P \cong eR$. When the base ring R is commutative, the endomorphism ring of the cyclic R-module $M \cong eR/C$ is isomorphic to the ring $eR/C = eRe/C$, via the isomorphism $g \mapsto g(e + C)$. The endomorphism ring eR/C is a local ring with maximal ideal $eJ(R)e/C = eJ(R)/C$. From the isomorphism above it is easy to see that the maximal ideal $eJ(R)/C$ corresponds to $\mathfrak{e}$, the ideal of non-surjective endomorphisms. Therefore, in this case, $\mathfrak{l} \subseteq \mathfrak{e}$. This inclusion can be proper. For instance, let R be a commutative valuation domain of Krull dimension $\geqslant 2$, that is, a valuation domain with at least three prime ideals $0 \subset P \subset J(R)$, and consider the couniformly presented module R/P, whose endomorphism is isomorphic to R/P as above. If $r \in J(R) \backslash P$, then $r + P \in \mathfrak{e} = J(R)/P$, but $r + P \notin \mathfrak{l}$ because $rP = P$. (For every $p \in P$, we have $pR \leqslant P \leqslant rR$, so that $p = rs$ for some $s \in R$. We have that $s \in P$ because $p \in P$ and $r \notin P$, and P is a prime ideal.)

6.1 Epigeny class and lower part

Recall that if A and B are two modules, we say that A and B have the same *epigeny class*, and write $[A]_e = [B]_e$, if there exist an epimorphism $A \to B$ and an epimorphism $B \to A$; cf. [Fac96]. If M and M' are two couniformly presented modules with couniform presentations $0 \to C \to P \to M \to 0$ and $0 \to C' \to P' \to M' \to 0$, we say that M and M' have the same *lower part,* and we write $[M]_\ell = [M']_\ell$, if there are two homomorphisms $f \colon P \to P'$ and $f_0' \colon P' \to P$ such that $f_0(C) = C'$ and $f_0'(C') = C$. In particular, if M and M' have the same lower part, then C and C' have the same epigeny class.

Notice the duality between this notion of having the same lower part, and the definition of having the same upper part given in [FEK10]. For any right R-module A, let $E(A)$ denote the injective envelope of A. Two modules A and B are said to have the same *upper part* if there exist a homomorphism $f_0 \colon E(A) \to$

$E(B)$ and a homomorphism $f_0' \colon E(B) \to E(A)$ such that $f_0^{-1}(B) = A$ and $f_0'^{-1}(A) = B$. We write $[A]_u = [B]_u$ if A and B have the same upper part.

Also notice that if M and M' are couniformly presented right R-modules with couniform presentations $0 \to C \to P \to M \to 0$ and $0 \to C' \to P' \to M' \to 0$, then there are local idempotents $e, e' \in R$ with $P \cong eR$ and $P' \cong e'R$. If we assume $P = eR$ and $P' = e'R$, C, C' right ideals of R contained in $eR, e'R$ respectively, and $M = eR/C, M' = e'R/C'$, then M and M' have the same lower part if and only if there exist $r, s \in R$ such that $rC = C'$ and $sC' = C$. Also notice that our definition of having the same lower part for arbitrary couniformly presented modules over arbitrary rings extends the definition of having the same lower part given in [AAF08] for cyclically presented modules over local rings.

Remark 6.8. Let $\mathfrak{l}$ and $\mathfrak{e}$ be the completely prime ideals of $\mathrm{End}_R(M)$ defined in the statement of Theorem 6.5.

Let M and M' be couniformly presented modules. It is easily seen that M and M' have the same lower part if and only if there exists an endomorphism $f \in \mathrm{End}_R(M)\backslash\mathfrak{l}$ of M that factors through M'. In particular, since the ideal $\mathfrak{l}$ does not depend on the couniform presentation of M (Remark 6.6), our notion of having the same lower part is well defined.

Similarly, M and M' have the same epigeny class if and only if there exists an endomorphism $f \in \mathrm{End}_R(M)\backslash\mathfrak{e}$ of M that factors through M'.

Epigeny class and lower part characterise a couniformly presented module up to isomorphism:

Lemma 6.9. *Let M and N be couniformly presented modules. Then $M \cong N$ if and only if $[M]_\ell = [N]_\ell$ and $[M]_e = [N]_e$.*

Proof. For the non-trivial implication, let $E := \mathrm{End}_R(M)$ and let $\mathfrak{l}$ and $\mathfrak{e}$ be the ideals of E as in Theorem 6.5. Assume that M and M' have the same epigeny class and the same lower part. If M has local endomorphism ring, then either $[M]_e = [M']_e$ or $[M]_\ell = [M']_\ell$ implies that an automorphism of M factors through M' (Remark 6.8). In that case, M is isomorphic to a non-zero direct summand of M', and because M' is indecomposable, $M \cong M'$. Hence we can assume that M has non-local endomorphism ring, hence $\mathfrak{l} + \mathfrak{e} = \mathrm{End}_R(M)$ and $1 = i + k$ with $i \in \mathfrak{l}\backslash\mathfrak{e}$ and $k \in \mathfrak{e}\backslash\mathfrak{l}$. Let $f \colon M \to M'$ and $f' \colon M' \to M$ be such that $f(C) = C'$ and $f'(C') = C$, and let $g \colon M \to M'$ and $g' \colon M' \to M$ be epimorphisms. If any of these morphisms is an isomorphism, we are done. Hence we can assume that none of them is an isomorphism. We claim that $\eta = (fk + gi)(kf' + ig') = fk^2f' + fkig' + gikf' + gi^2g'$ is an automorphism of M'. Then M is isomorphic to a non-zero direct summand of M', hence $M \cong M'$ as above. To prove the claim, suppose that η is not an automorphism of M'. Then $\eta \in \mathfrak{l}'$ or $\eta \in \mathfrak{e}'$. Since $i_1(C) \neq C$, we have that $fkig'$, $gikf'$, and

gi^2g' all belong to $\mathfrak{l}'$. Hence $fk^2f' \in \mathfrak{l}$, which is false. In the same way $\eta \in \mathfrak{e}'$ leads to a contradiction. □

6.2 2-Krull-Schmidt Theorem for couniformly presented modules

Lemma 6.10. *Let $M, N_0, \dots, N_{n-1}$ be couniformly presented modules. Suppose that M is a direct summand of $\bigoplus_{i<n} N_i$ and that $M \not\cong N_i$ for all $i < n$. Then there are distinct indices $i, j < n$ such that $[M]_\ell = [N_i]_\ell$ and $[M]_e = [N_j]_e$.*

Proof. Assume that M is a direct summand of $\bigoplus_{i<n} N_i$ and that M is not isomorphic to N_i, for every $i < n$. In particular, $n > 2$. With the obvious notation for the canonical mappings, we have $1_M = \pi_M \iota_M = \sum_{k<n} \pi_M \iota_k \pi_k \iota_M$. Let $E = \mathrm{End}_R(M)$ be the endomorphism ring of M and let $\mathfrak{l}$ and $\mathfrak{e}$ be the ideals of E as in Theorem 6.5. There exist indices i and j such that $\pi_M \iota_i \pi_i \iota_M \in E \backslash \mathfrak{l}$ and $\pi_M \iota_j \pi_j \iota_M \in E \backslash \mathfrak{e}$. This implies that $[M]_\ell = [N_i]_\ell$ and $[M]_e = [N_j]_e$ (Remark 6.8). Moreover, $i \neq j$, otherwise M would be isomorphic to $N_i = N_j$ (Lemma 6.9), which it is not. □

Lemma 6.11. *Let M, M', M'' be couniformly presented modules such that $[M]_\ell = [M']_\ell$ and $[M]_e = [M'']_e$. Then:*

(i) $M \oplus D \cong M' \oplus M''$ for some module D.

(ii) The module D in (i) is unique up to isomorphism and is couniformly presented.

(iii) $[D]_\ell = [M'']_\ell$ and $[D]_e = [M']_e$.

Proof. (i) Let $E = \mathrm{End}_R(M)$ and let $\mathfrak{l}$ and $\mathfrak{e}$ be the ideals of E as in Theorem 6.5. There exist an endomorphism $f \in E \backslash \mathfrak{l}$ which factors through M' and an endomorphism $g \in E \backslash \mathfrak{e}$ which factors through M'' (Remark 6.8). If either f or g is an automorphism, then $M \cong M'$ or $M \cong M''$, thus (i) clearly holds with $D = M''$ and $D = M'$ respectively. We can thus assume $f \in \mathfrak{e} \backslash \mathfrak{l}$ and $g \in \mathfrak{l} \backslash \mathfrak{e}$. It then follows that $f + g$ is an automorphism of M which factors through $M' \oplus M''$, thus (i) holds also in this case.

(ii) If $M \oplus D \cong M' \oplus M''$ and $M \oplus D' \cong M' \oplus M''$, then $M \oplus D \cong M \oplus D'$, so that $D \cong D'$ because the endomorphism ring of M is semilocal, hence M cancels from direct sums (Theorem 2.13). This shows that the complement D is unique up to isomorphism.

Taking the dual Goldie dimension of both sides of $S := M \oplus D \cong M' \oplus M''$, we get that D is a couniform module. Let π_D be the canonical projection of S onto D. Then, $D = \pi_D(S) = \pi_D(M' + M'') \leqslant \pi_D(M') + \pi_D(M'') \leqslant D$, hence

$D = \pi_D(M') + \pi_D(M'')$. Since D is couniform, either $\pi_D(M') = D$ or $\pi_D(M'') = D$. Without loss of generality we can assume that D is a homomorphic image of M', thus it is a homomorphic image of the projective cover P' of M'. We then have a short exact sequence $0 \to A \to P' \to D \to 0$, which is a couniform presentation of D provided that we prove that A is couniform. With the usual notation for the couniform presentations of M, M', M'', consider the two short exact sequences $0 \to C \oplus A \to P \oplus P' \to M \oplus D \cong S \to 0$ and $0 \to C' \oplus C'' \to P' \oplus P'' \to M' \oplus M'' \cong S \to 0$. By Schanuel's Lemma [AF92, Ex. 18.9, page 214] we have $C \oplus A \oplus P' \oplus P'' \cong C' \oplus C'' \oplus P \oplus P'$. Taking the dual Goldie dimension of both sides, we see that A is couniform.

(iii) If $D \cong M'$, then $M \cong M''$ by cancellation, so that $[D]_e = [M']_e$ and $[D]_\ell = [M']_\ell = [M]_\ell = [M'']_\ell$, as required. The case $D \cong M''$ is exactly the same. So we can assume that $D \not\cong M'$ and $D \not\cong M''$. By Proposition 6.10, either $[D]_\ell = [M']_\ell$ and $[D]_e = [M'']_e$ or $[D]_\ell = [M'']_\ell$ and $[D]_e = [M']_e$. In the first case, $D \cong M$ so that D, M, M', M'' are all isomorphic, which they are not. Thus the second case holds, as required. □

Here is the 2-Krull-Schmidt Theorem for couniformly presented modules.

Theorem 6.12. *[FG10] Let $M_0, \dots, M_{n-1}, N_0, \dots, N_{m-1}$ be couniformly presented modules. Then the direct sums $\bigoplus_{i<n} M_i$ and $\bigoplus_{i<m} N_i$ are isomorphic if and only if $n = m$ and there are two permutations σ, τ such that $[M_i]_\ell = [N_{\sigma(i)}]_\ell$ and $[M_i]_e = [N_{\tau(i)}]_e$ for all $i < n$.*

Proof. Assume that the two direct sums are isomorphic. Thus they have the same dual Goldie dimension, hence $n = m$.

We will prove by induction on n the existence of the permutations σ and τ, the case $n = 1$ being trivial. Suppose $M_i \cong N_j$ for suitable indices $i, j < n$. Since the endomorphism ring of $M_i \cong N_j$ is semilocal, we can cancel out M_i and N_j (by the cancellation property, Theorem 2.13) and obtain $\bigoplus_{k<n, k\neq i} M_k \cong \bigoplus_{k<n, k\neq j} N_k$. By the inductive hypothesis there are bijections $\sigma, \tau \colon \{k < n, k \neq i\} \to \{k < n, k \neq j\}$ such that $[M_k]_\ell = [N_{\sigma(k)}]_\ell$ and $[M_k]_e = [N_{\tau(k)}]_e$ for $k < n, k \neq i$. To conclude, prolong these bijections to permutations of $\{k < n\}$ by $\sigma(i) = \tau(i) = j$. Therefore we may assume that $M_i \not\cong N_j$ for all indices $i, j < n$.

Since M_0 is isomorphic to a direct summand of $\bigoplus_{k<n} N_k$, but to $M_0 \not\cong N_k$ for every $k < n$, Proposition 6.10 implies the existence of two distinct indices $i, j < n$ such that $[M_0]_\ell = [N_i]_\ell$ and $[M_0]_e = [N_j]_e$. By Lemma 6.11 applied to the three couniformly presented modules M_0, N_i, N_j, we can find a couniformly presented module N_n, unique up to isomorphism, such that $M_0 \oplus N_n \cong N_i \oplus N_j$, $[N_n]_\ell = [N_j]_\ell$ and $[N_n]_e = [N_i]_e$. Thus $\bigoplus_{k<n} M_k \cong \bigoplus_{k<n} N_k \cong M_0 \oplus \bigoplus_{k\leqslant n, k\neq i,j} N_k$. Cancelling out M_0, we get that $\bigoplus_{1\leqslant k<n} M_k$ is isomorphic to $\bigoplus_{k\leqslant n, k\neq i,j} N_k$. By the inductive hypothesis, there exist bijections $\sigma', \tau' \colon \{1 \leqslant$

$k < n\} \to \{k \leqslant n, k \neq i, j\}$ such that $[M_k]_\ell = [N_{\sigma'(k)}]_\ell$ and $[M_k]_e = [N_{\tau'(k)}]_e$ for $1 \leqslant k < n$. Let k' be such that $\sigma'(k') = n$ and k'' such that $\tau'(k'') = n$. To conclude, prolong and modify σ and τ by $\sigma(0) = i$, $\sigma(k') = j$, and $\tau(0) = j$, $\tau(k'') = i$.

The converse implication is trivial for $n = m = 1$ by Lemma 6.9, and we proceed by induction again to prove the converse in general. Assume thus that $M_0, N_0, \dots, M_{n-1}, N_{n-1}$ are couniformly presented and that there are two permutations σ, τ such that $[M_i]_\ell = [N_{\sigma(i)}]_\ell$ and $[M_i]_e = [N_{\tau(i)}]_e$ for every $i < n$. If $\sigma(0) = \tau(0)$, then $M_0 \cong N_{\sigma(0)}$. Thus σ and τ induce two bijections $\{1, 2, \dots, n-1\} \to \{0, 1, \dots, n-1\} \setminus \{\sigma(0)\}$, with the same properties as σ and τ, so that, by induction, $M_1 \oplus \cdots \oplus M_{n-1}$ is isomorphic to the direct sum $\bigoplus_{k<n, k\neq\sigma(0)} N_k$, from which it clearly follows that $\bigoplus_{i<n} M_i \cong \bigoplus_{i<n} N_i$.

Thus we can suppose $\sigma(0) \neq \tau(0)$. By Lemma 6.11, there exists a couniformly presented module M', unique up to isomorphism, such that $M' \oplus M_0 \cong N_{\sigma(0)} \oplus N_{\tau(0)}$, $[M']_\ell = [N_{\tau(0)}]_\ell$ and $[M']_e = [N_{\sigma(0)}]_e$. Therefore, the modules M', M_0 and the modules $N_{\sigma(0)}, N_{\tau(0)}$ have the same lower parts and the same epigeny classes, counting multiplicities. The modules $M', M_0, \dots, M_{n-1}$ and the modules M', $N_0, \dots, N_{n-1}$ have the same lower parts and the same epigeny classes as well, so that the modules M_1, M_2, $\dots$, M_{n-1} and the modules M', N_0, $\dots$, $\widehat{N_{\sigma(0)}}$, $\dots$, $\widehat{N_{\tau(0)}}$, $\dots$, N_{n-1} have the same lower parts and the same epigeny classes. By the inductive hypothesis, $M_1 \oplus M_2 \oplus \cdots \oplus M_{n-1}$ and the direct sum of the modules M' and N_j with j different from $\sigma(0)$ and $\tau(0)$ are isomorphic. Thus $M' \oplus N_0 \oplus \cdots \oplus N_{n-1} \cong M_1 \oplus \cdots \oplus M_{n-1} \oplus N_{\sigma(0)} \oplus N_{\tau(0)} \cong M' \oplus M_0 \oplus M_1 \oplus \cdots \oplus M_{n-1}$. Cancelling the module M', we obtain that $N_0 \oplus N_1 \oplus \cdots \oplus N_{n-1} \cong M_0 \oplus M_1 \oplus \cdots \oplus M_{n-1}$, as desired. □

As noted in the introduction of this chapter, our results on the category $\mathbf{C}$ of couniformly presented modules could also be seen as an application of the theory developed in Chapter 5. Let $\mathbf{E}$ be the ideal of $\mathbf{C}$ consisting of all morphisms f such that the image of f is a superfluous submodule of the codomain $\mathrm{codom}(f)$, cf. page 78. With reference to diagram 6.2, let $\mathbf{L}$ be the class of all morphisms f between couniformly presented modules such that f_1 has superfluous image. This class is indeed a well-defined ideal of $\mathbf{C}$. The product functor $\mathbf{C} \to \mathbf{C}/\mathbf{L} \times \mathbf{C}/\mathbf{E}$ satisfies the conditions (S1′) and (S2) of Setting 5.2. Specialising the theorems of Chapter 5 gives many of the results of this chapter. The equivalence relations involved in Theorem 5.10 are precisely the epigeny class and the lower part involved in Theorem 6.12.

6.3 Kernels of morphisms between indecomposable injective modules

The main results of [FEK10] concern the category $\mathbf{K}$ of kernels of morphisms between indecomposable injective modules, a full subcategory of the category of right or left modules over some fixed ring. (An index on the right or on the left of $\mathbf{K}$ will clarify whether we are considering right or left modules and over which ring, e.g., $\mathbf{K}_R$ or ${}_R\mathbf{K}$.) Namely, it is proved that the endomorphism ring of a module in $\mathbf{K}$ has a structure analogous to that of the endomorphism ring of a couniformly presented module [FEK10, Theorem 2.1]. Moreover, the 2-Krull-Schmidt Theorem holds for $\mathbf{K}$ [FEK10, Theorem 2.7]. In this section we set about to deduce those results from our theory of couniformly presented modules by means of the E-dual, that is, the duality $(-)^* = \mathrm{Hom}(-, E)$ of Section 4.4.1.

Notice that $\mathbf{K}_R$ is a full subcategory of the category of dual Auslander-Bridger right R-modules (Section 4.4.2), as the indecomposable injective modules are exactly the uniform ones, i.e., those of Goldie dimension one.

The duality $\mathbf{DAB}_R \to {}_S\mathbf{AB}$ induced by $(-)^* = (-, E)$ (Theorem 4.45) restricts to a duality $\mathbf{K}_R \to {}_S\mathbf{C}$, where ${}_S\mathbf{C}$ denotes the full subcategory of couniformly presented left S-modules with a minimal projective presentation, i.e., those that are cokernels of morphisms between couniform projective modules (cf. Section 5.3.2). This follows from Proposition 4.37(ii).

The structure of the endomorphism ring of a module M in $\mathbf{K}_R$ follows from Theorem 6.5 and the duality $(-)^*$, in a manner similar to the proof of Theorem 4.46. Indeed, recall that we have a commutative square

$$\begin{array}{ccc} \mathrm{End}(M_R) & \longrightarrow & \mathrm{End}({}_SM^*) \\ \downarrow & & \downarrow \\ \mathrm{End}(M_R)/\mathfrak{u} \times \mathrm{End}(M_R)/\mathfrak{m} & \longrightarrow & \mathrm{End}({}_SM^*)/\mathfrak{l} \times \mathrm{End}({}_SM^*)/\mathfrak{e} \end{array}$$

where the vertical morphisms are canonical and the horizontal ones are canonical anti-isomorphisms induced by $(-)^*$. In this particular case, we have that the right vertical morphism is surjective with kernel $J(\mathrm{End}({}_SM^*))$, because $\mathfrak{l}$ and $\mathfrak{e}$ are the ideals of Theorem 6.5. Indeed, for $f \in \mathrm{End}({}_SM^*)$, we have that $f \in \mathfrak{e}$ if and only if the endomorphism $f_0 \colon E_0(M)^* \to E_0(M)^*$ of the couniform projective left S-module $E_0(M)^*$ has superfluous image, if and only if f_0 is not surjective, if and only if f is not surjective, if and only if $f \in K$. Similarly, $f \in \mathfrak{l}$ if and only if $f_1 \colon E_1(M)^* \to E_1(M)^*$ has superfluous image, that is, if and only if f_1 is not surjective, if and only if $f_0(\ker(\epsilon_M^*)) < \ker(\epsilon_M^*)$, if and only if $f \in \mathfrak{l}$. The following theorem on $\mathrm{End}(M_R)$ now follows easily:

Theorem 6.13. (Cf. [FEK10, Theorem 2.1].) *The ideals* $\mathfrak{u}$ *and* $\mathfrak{m}$ *of* $\mathrm{End}(M_R)$ *are completely prime proper ideals, their union is the set of all non-automorphisms of* M_R*, and every proper right or left ideal of* $\mathrm{End}(M_R)$ *is contained either in* $\mathfrak{u}$ *or in* $\mathfrak{m}$*. Moreover, one of the following two conditions holds:*

(i) Either the ideals $\mathfrak{u}$ *and* $\mathfrak{m}$ *are comparable, so that* $\mathrm{End}(M_R)$ *is a local ring with maximal ideal the greatest ideal between* $\mathfrak{u}$ *and* $\mathfrak{m}$*, or*

(ii) the ideals $\mathfrak{u}$ *and* $\mathfrak{m}$ *are not comparable, their intersection is the Jacobson radical* $J(\mathrm{End}(M_R))$*, and* $\mathrm{End}(M_R)/J(\mathrm{End}(M_R))$ *is canonically isomorphic to the product of the two division rings* $\mathrm{End}(M_R)/\mathfrak{u}$ *and* $\mathrm{End}(M_R)/\mathfrak{m}$*.*

Exactly as couniformly presented modules are determined up to isomorphism by their lower part and epigeny class, modules in $\mathbf{K}$ are determined up to isomorphism by upper part and monogeny class.

Recall that two arbitrary modules A and B have the same *monogeny class* if they are isomorphic to submodules of each other, that is, there are injective morphisms $A \to B$ and $B \to A$. If A and B have the same monogeny class we write $[A]_m = [B]_m$. We say that A and B have the same *upper part,* and write $[A]_u = [B]_u$, if there are morphisms $f_0 \colon E(A) \to E(B)$ and $g_0 \colon E(B) \to E(A)$ such that $f_0^{-1}(B) = A$ and $g_0^{-1}(A) = B$. If A and B are modules in the category $\mathbf{K}$, it is easy to see that $[A]_m = [B]_m$ if and only if there is an endomorphism of A not in $\mathfrak{m}_A$ that factors through B, and $[A]_u = [B]_u$ if and only if there is an endomorphism of A not in $\mathfrak{u}_A$ that factors through B. Cf. Remark 6.8. Therefore we have:

Proposition 6.14. [FG10] *For modules* A *and* B *in the category* $\mathbf{K}$*, we have that*

(i) $[A]_m = [B]_m$ *if and only if* $[A^*]_e = [B^*]_e$*, and*

(ii) $[A]_u = [B]_u$ *if and only if* $[A^*]_\ell = [B^*]_\ell$*.*

The reader certainly noticed the similarity with Proposition 4.47. There is a slight difference though between the notions "mono-isomorphic" and "in the same monogeny class", and between the notions "upper-isomorphic" and "with the same lower part". The two are equivalent for the modules M in $\mathbf{K}$ whose endomorphism ring is not local, so that $\mathrm{End}(M_R)/\mathfrak{m}_M$ and $\mathrm{End}(M_R)/\mathfrak{u}_M$ are division rings.

Finally, thanks to Proposition 6.14 and the duality $(-)^* = (-, E)$, we easily obtain the 2-Krull-Schmidt Theorem for $\mathbf{K}$:

Theorem 6.15. (Cf. [FEK10, Theorem 2.7].) *Let* $M_0, \dots, M_{n-1}, N_0, \dots, N_{m-1}$ *be modules in* $\mathbf{K}_R$*. Then the direct sums* $\bigoplus_{i<n} M_i$ *and* $\bigoplus_{i<m} N_i$ *are isomorphic if and only if* $n = m$ *and there are two permutations* σ, τ *such that* $[M_i]_u = [N_{\sigma(i)}]_u$ *and* $[M_i]_m = [N_{\tau(i)}]_m$ *for all* $i < n$*.*

6.4 A further duality between epigeny class and monogeny class

In Section 6.3, we saw that monogeny class and epigeny class (and lower part and upper part) are related by a duality between suitable categories of modules: the category of kernels of morphisms between uniform injective modules and the category of cokernels of morphisms between couniform projective modules. In [AAF08, Proposition 7.1] it was shown that, for cyclically presented modules over local rings, lower part and epigeny class are related by the Auslander-Bridger transpose, which also can be seen as a duality between suitable categories. More generally, the Auslander-Bridger transpose relates lower-isomorphism and epi-isomorphism in the context of Auslander-Bridger modules (Proposition 4.34). In this section, we will show that there is a similar relation between monogeny class and epigeny class in the case of suitable categories of uniserial modules.

Recall that if ${}_SA$ and ${}_SB$ are left modules over a ring S, ${}_SA$ is said to be *cogenerated* by ${}_SB$ if ${}_SA$ is isomorphic to a submodule of a product of copies of ${}_SB$. Equivalently, if for every non-zero $a \in {}_SA$ there exists a morphism $\varphi\colon {}_SA \to {}_SB$ such that $\varphi(a) \neq 0$. If ${}_SX$ generates every left S-module, then we say that ${}_SX$ is a *cogenerator.* Cf. [AF92, §18].

Let R be a ring. Fix a set $\{Q_\lambda\}_{\lambda\in\Lambda}$ of representatives up to isomorphism of all injective right R-modules that are injective envelopes of non-zero uniserial R-modules. Let Q_R be the injective envelope of $\bigoplus_{\lambda\in\Lambda} Q_\lambda$. It is easy to see that an injective module is a generator if and only if it contains an isomorphic copy of every simple module [AF92, Proposition 18.15]. Since simple modules are uniserial, it follows that Q_R is a cogenerator, i.e., it cogenerates all right R-modules.

Let $S := \mathrm{End}(Q_R)$. Then ${}_SQ_R$ is an S-R-bimodule and we can consider the Q-dual, that is, the pair of additive contravariant functors

$$
{}_S\mathrm{Hom}_R(-,Q)\colon \text{Mod-}R \to S\text{-Mod},
$$
$$
\mathrm{Hom}_S(-,Q)_R\colon S\text{-Mod} \to \text{Mod-}R,
$$

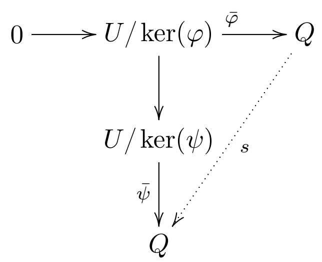
as in Sections 3.2 and 4.4.1. For every uniserial module U_R, its injective envelope is isomorphic to some Q_λ and is Q-reflexive.

Let $\mathbf{C}_R$ denote the full subcategory of Mod-R whose objects are all serial right R-modules of finite Goldie dimension. Let ${}_S\mathbf{C}'$ be the full subcategory of S-Mod whose objects are all finite direct sums of uniserial left S-modules with a projective cover and cogenerated by ${}_SQ$. Notice that if a non-zero uniserial module U has a projective cover P, then P is a couniform module (Lemma 4.2), so that, in particular, P, hence U, are cyclic modules.

Proposition 6.16. *The functor* ${}_S\operatorname{Hom}_R(-, Q)\colon \text{Mod-}R \to S\text{-Mod}$ *induces a categorical duality between* $\mathbf{C}_R$ *and* ${}_S\mathbf{C}'$.

Proof. Thanks to Remark 1.9, it suffices to show that $\operatorname{Hom}(-, {}_SQ_R)$ induces a duality between the category of uniserial right R-modules and the category of uniserial left S-modules with a projective cover and cogenerated by ${}_SQ$.

Suppose U is a non-zero uniserial right R-module. To prove that ${}_SU^*$ is uniserial it suffices to show that its cyclic submodules are comparable. Thus let φ and ψ be R-morphisms $U \to Q$. Their kernels are submodules of U, hence we may assume without loss of generality that $\ker(\varphi) \leqslant \ker(\psi)$. Because Q is an injective right R-module, we have an endomorphism s of Q such that the following diagram commutes:



It follows that $\psi = s\varphi$. Thus $S\psi \leqslant S\varphi$. This proves that ${}_SU^*$ is uniserial.

The injective envelope of U can be chosen to be an injective R-morphism $g\colon U \to Q_\lambda$ into some Q_λ. Since Q is an injective right R-module, the S-morphism $g^*\colon {}_SQ_\lambda^* \to {}_SU^*$ is surjective. Notice that Q_λ is isomorphic to a direct summand of Q, hence there are morphisms $\pi\colon Q \to Q_\lambda$ and $\iota\colon Q_\lambda \to Q$ such that $\pi\iota = 1$. Therefore ${}_SQ_\lambda^* = S\iota \cong S\iota\pi$ by $\varphi \mapsto \varphi\pi$. Moreover, the idempotent $\iota\pi$ of S is local, because $\operatorname{End}_R(Q_\lambda) \cong \iota\pi S\iota\pi$ by $\varphi \mapsto \iota\varphi\pi$ and the former ring is local as Q_λ is an indecomposable injective module. Thus ${}_SQ_\lambda^*$ is a couniform projective module and, consequently, the non-zero surjective morphism g^* is a projective cover.

Finally, suppose φ is a non-zero element of ${}_SU^*$, i.e., a non-zero R-morphism $U \to Q$. Then there is $u \in U$ such that $\varphi(u)$ is a non-zero element of Q. The rule $\psi \mapsto \psi(u)$ defines and S-morphism ${}_SU^* \to {}_SQ$, and it is non-zero on φ. In other words, the mapping $\psi \mapsto (\psi(u))_{u\in U}$ defines an S-embedding of ${}_SU^*$ into the Cartesian power ${}_SQ^U$. This proves that ${}_SU^*$ is cogenerated by ${}_SQ$.

Conversely, let us prove that every uniserial left S-module with a projective cover and cogenerated by ${}_SQ$ is isomorphic to ${}_SU^*$ for some uniserial right R-module U.

The projective cover of a uniserial module is a couniform projective module, hence the left S-module in question can be assumed to be of the form Se/T with e a local idempotent of S (Lemma 4.2). Now, define $U = eQ \cap \operatorname{r.ann}_Q(T)$. That is, U is the set of elements of Q of the form $e(x)$ for some $x \in Q$, and such that

$te(x) = 0$ for every $t \in T$. Thus U is an R-submodule of Q. Let us prove that it is uniserial. Let x and y be any two elements of U. Then $\mathrm{l.ann}_S(x)$ and $\mathrm{l.ann}_S(y)$ are two left ideals of S that contain $1-e$ and T. Thus $\mathrm{l.ann}_S(x)/(S(1-e) \oplus T)$ and $\mathrm{l.ann}_S(y)/(S(1-e) \oplus T)$ are two submodules of $S/(S(1-e) \oplus T) \cong Se/T$, which is uniserial. It follows that the left ideals $\mathrm{l.ann}_S(x)$ and $\mathrm{l.ann}_S(y)$ are comparable, say (*) $\mathrm{l.ann}_S(x) \leqslant \mathrm{l.ann}_S(y)$. Let us prove that this implies $yR \leqslant xR$. Assume by contradiction that $(yR + xR)/xR$ is a non-zero right R-module. Recall that Q_R is a cogenerator (see the considerations before this theorem). Therefore, there exists a morphism $\varphi\colon (yR + xR)/xR \to Q$ such that $\varphi(y + xR) \neq 0$. Thus there exists a morphism $\psi\colon yR + xR \to Q$ such that $\psi(y) \neq 0$ and $\psi(x) = 0$. Because Q is an injective right R-module, ψ extends to an endomorphism s of Q. Since $sx = 0$, we have $sy = 0$ by (*), and this is a contradiction. Therefore $yR \leqslant xR$ as required and U is uniserial.

Finally, we prove that ${}_SU^* \cong Se/T$. Notice that U is contained in eQ, hence we have a surjective morphism $\mathrm{res}\colon {}_S(eQ, Q) \to {}_S(U, Q)$ given by restriction. If $\varphi\colon eQ \to Q$ is an R-morphism, let s_φ be the element of S obtained prolonging φ with zero on $(1-e)Q$. Thus $\varphi \mapsto s_\varphi$ yields an isomorphism ${}_S(eQ, Q) \cong Se$. It is left to prove that $\mathrm{res}(\varphi) = 0$ if and only if $s_\varphi \in T$. Recall that $U = eQ \cap \mathrm{r.ann}_Q(T)$, hence if $s_\varphi \in T$, then $s_\varphi U = 0$, and since $U \leqslant eQ$, this means that $\varphi(U) = 0$, that is, $\mathrm{res}(\varphi) = 0$. Conversely, suppose $s_\varphi \notin T$. Then $s_\varphi + T$ is a non-zero element of Se/T. Because this left S-module is cogenerated by ${}_SQ$, there is an S-morphism $f\colon Se/T \to {}_SQ$ such that $f(s_\varphi + T) \neq 0$. Thus there is an S-morphism $g\colon Se \to {}_SQ$ such that $g(s_\varphi) \neq 0$ and $g(T) = 0$. Notice that $g(e) = g(e^2) = eg(e)$ hence $g(e) \in eQ$. Moreover, $g(x) = xg(e)$ for every $x \in Se$. Thus $g(T) = 0$ means $Tg(e) = 0$. Hence $g(e) \in U$, by definition. Lastly, $g(s_\varphi) \neq 0$ means $\varphi(g(e)) \neq 0$. Thus $\mathrm{res}(\varphi) \neq 0$.

It is left to prove that the functor in question is full and faithful. Let $\iota_1\colon U_1 \to Q_{\lambda_1}$ and $\iota_2\colon U_2 \to Q_{\lambda_2}$ be injective envelopes of the uniserial right R-modules U_1 and U_2. Recall that $Q^*_{\lambda_i}$ is a couniform projective module, hence the surjective morphism ι_i^* is a projective cover. Any S-morphism $f\colon {}_SU_1^* \to {}_SU_2^*$ lifts to an S-morphism g between the projective covers.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & U_1 & \xrightarrow{\iota_1} & Q_{\lambda_1} & \qquad & Q^*_{\lambda_1} & \xrightarrow{\iota_1^*} & {}_SU_1^* & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle k} & & \uparrow{\scriptstyle h} & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & U_2 & \xrightarrow[\iota_2]{} & Q_{\lambda_2} & \qquad & Q^*_{\lambda_2} & \xrightarrow[\iota_2^*]{} & {}_SU_2^* & \longrightarrow & 0 \end{array} \tag{6.17}$$

Since each Q_{λ_i} is Q-reflexive, we have that $g = h^*$ for some $h\colon Q_{\lambda_2} \to Q_{\lambda_1}$. We claim that $h(\iota_2(U_2)) \leqslant \iota_1(U_1)$. Assume the contrary. Because Q_R is a cogenerator, there exists a morphism $\varphi\colon Q_{\lambda_1} \to Q_R$ such that $\varphi(\iota_1(U_1)) = 0$ and $\varphi(h(\iota_2(U_2))) \neq 0$. Then we have $\iota_2^* h^* \varphi = \varphi h \iota_2 \neq 0$ and at the same time $\iota_2^* h^* \varphi = f(\iota_1^*(\varphi)) = f(0) = 0$. This contradiction shows that $h(\iota_2(U_2)) \leqslant$

$\iota_1(U_1)$, hence that there is a morphism $k\colon U_2 \to U_1$ such that $h\iota_2 = \iota_1 k$. It follows that $k^* = f$.

Eventually, to show faithfulness, suppose $k \neq 0$. Because Q_R is a cogenerator, there is a morphism $\varphi\colon U_1 \to Q_R$ such $\varphi k \neq 0$, that is, $k^*(\varphi) \neq 0$, so that $k^* \neq 0$. □

Proposition 6.18. *For uniserial right R-modules U_1 and U_2, we have that:*

(i) $[U_1]_m = [U_2]_m$ *if and only if* $[U_1^*]_e = [U_2^*]_e$, *and*

(ii) $[U_1]_e = [U_2]_e$ *if and only if* $[U_1^*]_m = [U_2^*]_m$.

Proof. It suffices to prove that in the group isomorphism

$$_S(-, Q_R)\colon \operatorname{Hom}_R(U_1, U_2) \to \operatorname{Hom}_S(U_2^*, U_1^*)$$

given by Proposition 6.16, k is injective (resp. surjective) if and only if k^* is surjective (resp. injective).

It is true for every contravariant Hom-functor that surjective morphisms are sent to injective ones. Since Q_R is an injective module, it also sends injective R-morphisms to surjective S-morphisms.

Suppose $k\colon U_1 \to U_2$ is not injective. Then there is $0 \neq u \in U_1$ such that $k(u) = 0$. Since Q_R is a cogenerator, there is $\varphi\colon U_1 \to Q_R$, that is, $\varphi \in U_1^*$, such that $\varphi(u) \neq 0$. It easily follows that φ is not in the image of k^*, hence that k^* is not surjective.

Similarly, if k is not surjective, there is $u \in U_2 \backslash k(U_1)$. Again using the fact that Q_R is a cogenerator, there is an element φ of U_2^* such that $\varphi k = 0$ and $\varphi(u) \neq 0$. Then $\varphi \neq 0$ but $k^*(\varphi) = 0$, hence k^* is not injective. □

Chapter 7

A couple of examples

7.1 On a uniserial module that is not quasi-small

The class of uniserial modules (modules whose lattices of submodules are linearly ordered) was the first one for which a result like Theorem 6.12, called "Weak Krull-Schmidt Theorem," was proved [Fac96]. In [DF97], said result was extended to infinite direct sums of quasi-small uniserial modules. Recall that a module M is *quasi-small* if, whenever M is isomorphic to a direct summand of a direct sum $\bigoplus_{i\in I} M_i$, then M is isomorphic to a direct summand of $\bigoplus_{i\in F} M_i$ for some finite subset F of I. In his book [Fac98], Facchini asked whether a uniserial non-quasi small module existed. Of course, if all uniserial modules were quasi-small, then the Weak Krull-Schmidt Theorem would hold for infinite direct sums of uniserial modules. The question was answered by Puninski in [Pun01], where he proves the existence of a uniserial module that is not quasi-small. His proofs rely on model-theoretical methods and results. In this brief final chapter our aim is to explain Puninski's example giving purely algebraic proofs wherever possible.

Příhoda also studied uniserial modules that are not quasi-small, providing, in particular, non-model-theoretical proofs of Puninski's example, and following a wholly different approach [Pří06].

Lemma-Definition 7.1. *For a surjective R-module morphism $g\colon B_R \to C_R$, the following are equivalent:*

(i) The module $A_R := \ker(g)$ is a pure submodule *of B_R, i.e., every system of equations*

$$\sum_{i=1}^{n} x_i r_{ij} = a_j, \text{ for all } j = 1, \dots, m \tag{S1}$$

with each $r_{ij} \in R$ and each $a_j \in A_R$, which has a solution in B_R, also has a solution in A_R. Cf. [Fac98, Section 1.4].

(ii) Given any system of equations

$$\sum_{i=1}^{n} x_i r_{ij} = 0, \text{ for all } j = 1, \dots, m \tag{S2}$$

where each $r_{ij} \in R$, whenever the system has a solution $(c_1, \dots, c_n)$ in C_R, it also has a solution $(b_1, \dots, b_n)$ in B_R such that $g(b_i) = c_i$ for each i. In other words, the solutions of (S2) lift along g.

A surjective morphism of R-modules $g\colon B_R \to C_R$ satisfying the above equivalent conditions is called a pure epimorphism. *A module M is called* pure-projective *if it is projective with respect to pure epimorphisms, i.e., if $\operatorname{Hom}_R(M, g)$ is an epimorphism whenever g is a pure epimorphism.*

Proof. Suppose (1) holds. Consider a system of equations (S2) and suppose that $(c_1, \dots, c_n)$ is a tuple of elements from C_R satisfying (S2). First choose $b_1, \dots, b_r \in B_R$ such that $g(b_i) = c_i$ for each i. For each j,

$$g\left(\sum_{i=1}^{n} b_i r_{ij}\right) = 0,$$

hence there are element $a_1, \dots, a_m \in A_R$ such that $(b_1, \dots, b_n)$ solves the system (S1). Since A_R is a pure submodule of B_R, we have that (S1) also has a solution, say $(a'_1, \dots, a'_n)$, in A_R. Thus the tuple $(b_1 - a_1, \dots, b_n - a'_n)$ solves (S2) and is mapped to the tuple $(c_1, \cdots, c_n)$ by g. This proves that (2) holds.

Assume now that (2) holds. Consider a system of equations (S1) with a solution $(b_1, \dots, b_n)$ in B_R. Therefore $(g(b_1), \dots, g(b_n))$ solves the system (S2). By (2), there are elements $b'_1, \dots, b'_n$ in B_R such that $g(b'_i) = g(b_i)$ for each i and $(b'_1, \dots, b'_n)$ solves (S2). It follows that $b_i - b'_i \in A_R$ and that $(b_1 - b'_1, \dots, b_n - b'_n)$ solves (S1). This proves that A_R is a pure submodule of B_R, that is, that (1) holds. □

It is easy to see that finitely presented modules are pure-projective. Indeed, a finitely presented module A_R is generated by finitely many elements $a_1, \dots, a_n$ subject to a finite number of relations, say $\sum_{i=1}^{n} a_i r_{ij} = 0$, for $1 \leqslant j \leqslant m$. A morphism $\varphi\colon A_R \to C_R$ is given by the elements $\varphi(a_1), \dots, \varphi(a_n)$ of C, that is, by a choice of elements $c_1, \dots, c_n$ in C such that $\sum_{i=1}^{n} c_i r_{ij} = 0$, for all $1 \leqslant j \leqslant m$. Is is now clear that if $g\colon B_R \to C_R$ is a pure epimorphism, there are elements $b_1, \dots, b_n$ in B such that $\sum_{i=1}^{n} b_i r_{ij} = 0$, for all j, and such that $g(b_i) = c_i$ for all i. Hence we can define a morphism $\psi\colon A_R \to B_R$ such that $\psi(a_i) = b_i$ for all i, so that $\varphi = g\psi$.

From the definition it also follows easily that the class of pure-projective modules is closed under direct sums and direct summands. Therefore, direct summands of direct sums of finitely presented modules are pure-projective. It is possible to construct, for every given module X, a pure epimorphism

$g\colon \bigoplus_{i\in I} M_i \to X$, where each M_i is a finitely presented module [Fac98, Proposition 1.23]. If X is pure-projective, such epimorphism splits, hence we deduce that:

Proposition 7.2. [Fac98, Proposition 1.24] *A module M is pure-projective if and only if it is a direct summand of a direct sum of finitely presented modules, if and only if every pure-epimorphism g such that $\operatorname{codom}(g) = M$ splits.*

As a consequence, a quasi-small pure projective module is finitely generated, or, which is the same, a projective module that is not finitely generated is not quasi-small. We will see that Puninski's non-quasi small uniserial module is a non-finitely generated pure-projective module.

Definition 7.3. A uniserial domain R is called *nearly simple* if $J(R)$ is the unique non-zero proper two-sided ideal of R. In other words, R has exactly three two-sided ideals.

Dubrovin proved that nearly simple uniserial domains do exist [Dub80].

What follows is the algebraic equivalent of [Pun01, Lemma 5.4]. The only proof known at the time of writing employs methods from the model theory of modules. In particular, it involves the classification of the indecomposable pure-injective modules over a serial ring up to isomorphism [EH95].

Lemma 7.4. [Pun01, Lemma 5.4] *Let R be a nearly simple uniserial domain. If a, b, and c are non-zero elements in the Jacobson radical $J(R)$ of R, then $a \in Rca + abR$.*

Let us proceed to the explanation of Puninski's example, cf. [Pun01, Section 8]. Let a be an arbitrary non-zero non-invertible element of the nearly simple uniserial domain R. For $1 \leqslant i \leqslant j < \omega$, let $\mu_{i,j}\colon R/a^iR \to R/a^jR$ be the morphism given by multiplication by a^{j-i} on the left. Let U be the direct limit of this system, and let $\mu_i\colon R/a^iR \to U$, for $1 \leqslant i < \omega$, be the canonical injections.

Theorem 7.5. *The module U is uniserial, not finitely generated, countably generated, and pure-projective. In particular, it is not quasi small.*

Proof. The fact that U is uniserial follows from the fact that U is a direct limit of uniserial modules. To see that U is uniserial, if suffices to show that cyclic submodules of U are comparable. Thus let x_1 and x_2 be two elements of U. There is an index $1 \leqslant i < \omega$ such that both x_1 and x_2 are in the image of μ_i. Such image is uniserial because the domain of μ_i is the uniserial module R/a^iR. Hence x_1R and x_2R are comparable, as required.

If U were finitely generated, it would be cyclic. There would then be an index $i < \omega$ such that μ_i is surjective. Then there is an element $r \in R$ such that

$$\mu_{i+1}(\bar{1}) = \mu_i(\bar{r}) = \mu_{i+1}\mu_{i,i+1}(\bar{r}) = \mu_{i+1}(\bar{a}\bar{r}),$$

from which $\mu_{i+1}(\bar{1} - \bar{a}\bar{r}) = 0$. This means that, for some $j < \omega$, we have that $a^j(1 - ar) \in a^j a^{i+1}R$, but this means that $1 - ar \in a^{i+1}R \leqslant aR$ to begin with, because R is a domain, which leads to $1 \in aR \leqslant J(R)$, a contradiction. Hence U is not finitely generated. It is, of course, countably generated though, e.g., by the set $\{\mu_i(\bar{1})\}_{1 \leqslant i < \omega}$.

To show that U is pure-projective, let $g\colon M \to U$ be an arbitrary pure epimorphism, and let us show that g splits (then we apply Proposition 7.2). To show that g splits, we define inductively morphisms $f_i\colon R/a^iR \to M$ such that $gf_i = \mu_i$, and such that $f_{i+1}\mu_{i-1,i+1} = f_i\mu_{i-1,i}$, for all $2 \leqslant i < \omega$. Once this is accomplished, the family of morphisms $\{\varphi_i\}_{1 \leqslant i < \omega}$, defined by $\varphi_i = f_{i+1}\mu_{i,i+1}$, is compatible with the direct system, hence it induces a morphism $\varphi\colon U \to M$ such that $\varphi\mu_i = \varphi_i$. It follows that $g\varphi\mu_i = \mu_i$ for all $i < \omega$, hence that $g\varphi = 1$, as we need.

Since $\mu_2(\bar{1})a^2 = 0$ and g is a pure epimorphism, there is an element $m \in M$ such that $g(m) = \mu_2(\bar{1})$ and $ma^2 = 0$. Thus $f_2\colon \bar{1} \mapsto m$ gives a well-defined morphism $R/a^2R \to M$ such that $gf_2 = \mu_2$. This is the base step of the construction.

Assume now that $f_2, \dots, f_i$ have been constructed. By Lemma 7.4, we have that $a \in Ra^2 + a^iR$. Hence there exist $s, t \in R$ such that

$$ta^2 = a + a^i s. \tag{7.6}$$

From this it follows that

$$(\mu_i(\bar{1})t - \mu_{i+1}(\bar{1}))a^2 = 0.$$

Since g is a pure epimorphism, there is an element $m \in M$ such that:

$$\begin{cases} g(m) = (\mu_i(\bar{1})t - \mu_{i+1}(\bar{1})) \\ ma^2 = 0 \end{cases}. \tag{7.7}$$

Define $f_{i+1}\colon R/a^{i+1}R \to M$ by the rule

$$f_{i+1}(\bar{1}) = f_i(\bar{1})t - m. \tag{7.8}$$

The morphism is well-defined, because $ma^{i+1} = 0$ (since $i \geqslant 2$) and

$$\begin{aligned} f_i(\bar{1})ta^{i+1} &= f_i(\bar{1})ta^2a^{i-1} \\ &= f_i(\bar{1})(a + a^i s)a^{i-1} \qquad \text{by (7.6)} \\ &= f_i(\bar{1})(a^i + a^i s a^{i-1}) \\ &= 0. \end{aligned}$$

Moreover, we have that $gf_{i+1} = \mu_{i+1}$, because $gf_{i+1}(\bar{1}) = g(f_i(\bar{1})t - m) = \mu_i(\bar{1})t - g(m) = \mu_{i+1}(\bar{1})$, by (7.7). Lastly, we need to verify that $f_{i+1}\mu_{i-1,i+1} =$

$f_i\mu_{i-1,i}$. We have the following chain of equalities:

$$\begin{aligned} f_{i+1}\mu_{i-1,i+1}(\bar{1}) &= f_{i+1}(\bar{1})a^2 \\ &= (f_i(\bar{1})t - m)a^2 \quad \text{by (7.8)} \\ &= f_i(\bar{1})ta^2 - ma^2 \\ &= f_i(\bar{1})(a + a^i s) \quad \text{by (7.6)} \\ &= f_i(\bar{1})a \\ &= f_i\mu_{i-1,i}(\bar{1}), \end{aligned}$$

and this completes the proof. □

7.2 An example showing that the Chinese Remainder Theorem does not provide a category equivalence

When discussing the Chinese Remainder Theorem for preadditive categories, or rings with many objects, we pointed out that the canonical functor provided by Theorem 1.22 is not in general an equivalence, owing to the fact that it may not be dense. In this section we illustrate this with an example.

Fix a ring R and consider the full subcategory $\mathbf{C}$ of $R\text{-Mod}$ whose objects are the uniserial left R-modules ${}_RU$ that are not strongly indecomposable, that is, such that $\text{End}({}_RU)$ is not a local ring. Thus $\text{End}({}_RU)$ has exactly two maximal right ideals, which are necessarily two-sided, the ideal $\mathfrak{m}_U$ of non-injective endomorphism and the ideal $\mathfrak{e}_U$ of non-surjective endomorphisms. The category $\mathbf{C}$ has two ideals $\mathbf{M}$ and $\mathbf{E}$, consisting of non-injective morphisms and non-surjective morphisms respectively, and clearly $\mathbf{M}(U) = \mathfrak{m}_U$ and $\mathbf{E}(U) = \mathfrak{e}_U$ for every U in $\mathbf{C}$. (For all this we refer back to Section 5.3.1, where we discussed biuniform modules.) Moreover, $\mathbf{M}$ and $\mathbf{E}$ are comaximal ideals, and their intersection is the Jacobson radical $\mathbf{J}$ of $\mathbf{C}$. By the Chinese Remainder Theorem 1.22 we have a canonical faithful and full functor

$$\mathbf{C}/\mathbf{J} \to \mathbf{C}/\mathbf{M} \times \mathbf{C}/\mathbf{E}.$$

It is easy to see that a pair of uniserial modules in $\mathbf{C}$ are isomorphic in $\mathbf{C}/\mathbf{M}$ if and only if they have the same monogeny class, and that they are isomorphic in $\mathbf{C}/\mathbf{E}$ if and only if they have the same epigeny class. Therefore the above canonical functor is dense if and only if, given any monogeny class $[U]_m$ and any epigeny class $[V]_e$, with U and V left R-modules that are not strongly indecomposable, there is a third uniserial left R-module W such that $[U]_m = [W]_m$ and $[V]_e = [W]_e$. Let us show with an example that this may not be the case, thus showing as promised earlier (on page 31) that the Chinese Remainder Theorem 1.22 does *not* necessarily grant us with a category equivalence.

Before giving the example, we need a special case of [Fac84, Theorem 1].

Recall that a lattice L is called *complete* if every subset of L has a supremum and an infimum in L. In particular, L has a greatest element $1 = \sup(L)$ and a smallest element $0 = \inf(L)$.

Theorem 7.9. (Cf. [Fac84, Theorem 1].) *The following are equivalent for a linearly ordered complete lattice L:*

(i) There is a ring R and a uniserial left R-module M such that $\mathcal{L}({}_RM)$ is isomorphic to L.

(ii) For every $a < b$ in L, there are a_1 and b_1 in L such that $a \leqslant a_1 < b_1 \leqslant b$ and no element of L is between a_1 and b_1.

Proof. Suppose M is a uniserial module and that $A < B$ are submodules of M, and consider the non-zero quotient B/A. This has a non-zero cyclic submodule, say B_1/A. Since B_1/A is in particular finitely generated, it has a maximal submodule, say A_1/A. Now $A \leqslant A_1 < B_1 \leqslant B$ and there are no submodules of M between A_1 and B_1. Thus it follows that condition (i) implies condition (ii).

Now let L be a linearly ordered complete lattice satisfying condition (ii). Let C be the set of elements b_1 of L such that there exists $a_1 \in L$ such that $a_1 < b_1$ and no element of L is between a_1 and b_1. In other words, C is the set of elements of L that have an "immediate predecessor," or the set of "immediate successors." Thus $0 \notin C$, while 1 may or may not be in C.

Let k be any field, and M a k-vector space having C as a basis. For $x \in L$, let M_x be the k-subspace of M generated by $\{a \in C : a \leqslant x\}$. Let now R be the set of k-endomorphisms g of M such that $g(M_x) \leqslant M_x$ for every $x \in L$. Thus R is a k-subalgebra of the endomorphism ring $\mathrm{End}_k(M)$ and M is canonically a left R-module, i.e., we let $r.m = r(m)$ for $r \in R$ and $m \in M$. Also, each M_i is by construction an R-submodule of ${}_RM$.

We claim that (*) the cyclic submodules of ${}_RM$ are precisely the subspaces M_x with $x \in C$, and (**) the submodules of ${}_RM$ are precisely the subspaces M_x with $x \in L$. Since $M_x \leqslant M_y$ if and only if $x \leqslant y$, we then conclude that L is isomorphic to the lattice of submodules of ${}_RM$, thus proving (i). Hence let us turn to proving the claim.

First notice that, for $x \in C$, the R-submodule M_x of ${}_RM$ is the cyclic submodule generated by the basis element x. Indeed, $x \in M_x$ hence $Rx \leqslant M_x$. If $a \in C$ and $a \leqslant x$, consider the k-endomorphism g of M defined by $g(x) = a$ and $g(y) = 0$ for all other basis vectors y, i.e., for all $y \in C\backslash\{x\}$. It follows that $g \in R$, so that $a \in Rx$. Thus Rx contains a set of generators for M_x, so that $M_x \leqslant Rx$, and ultimately $M_x = Rx$.

Let now $m \in M$. Then m is a k-linear combination of finitely many elements from C, say $m = \sum_{i=1}^{n} x_i\lambda_i$, with all $x_i \in C$ and $0 \neq \lambda_i \in k$. We can choose the indices in such a way that $x_1 < \cdots < x_n$. Thus $Rm \leqslant M_{x_n}$. On the other hand, there is a k-endomorphism g of M such that $g(x_n) = x_n$ and $g(y) = 0$ for $y \in C \backslash \{x_n\}$. Clearly $g \in R$ and $g(m)\lambda_n^{-1} = x_n \in Rm$, hence $M_{x_n} = Rx_n \leqslant Rm$, which provides the missing inclusion. This proves (*).

If x is an arbitrary element of L, it follows from the definition that M_x is the union of the chain $\{M_a : a \in C,\ a \leqslant x\}$ of k-vector spaces, hence of cyclic submodules of $_RM$, so that M_x is also a submodule of $_RM$. It is left to prove that an arbitrary submodule N of $_RM$ is of this shape.

Suppose N is an arbitrary submodule of $_RM$. Let $C_0 \subseteq C$ be the set of $a \in C$ such that $M_a \leqslant {}_RN$. The subset C_0 of L has a supremum x in L, because L is complete. We claim that $M_x = {}_RN$.

First we prove the inclusion $M_x \leqslant {}_RN$. Suppose $a \in C$ is such that $a \leqslant x$. If $a < x$, then a is not an upper bound of C_0 in L. Hence there exists $b \in C_0$ such that $a < b$. Thus $M_a \leqslant M_b \leqslant {}_RN$. On the other hand, suppose that $a = x$. Since $a \in C$, there is $b \in L$ such that $b < a$ and no element of L is between b and a. Since a is the supremum of C_0, we have that b is not an upper bound of C_0 in L. Hence there is $c \in C_0$ such that $b < c$. Since $c < a$ is not possible, we must have $a \leqslant c$, and $c \leqslant a$ also holds, hence $a = c \in C_0$, and again $M_a \leqslant {}_RN$. Thus, so far we have proved that $M_x \leqslant {}_RN$.

Let us turn to the reverse inclusion. Suppose $n \in {}_RN$. Since Rn is a cyclic submodule of $_RM$, we have that $Rn = M_a$ for some $a \in C$. Since $M_a \subseteq {}_RN$ we have that $a \in C_0$, hence $a \leqslant x$ and $n \in M_a \leqslant M_x$. Because n is arbitrary, $_RN \leqslant M_x$, and this concludes the proof. □

With Theorem 7.9 at hand, we can proceed to the promised example. Suppose U and V are uniserial modules such that the lattice of submodules $\mathcal{L}(U)$ is countable and the lattice $\mathcal{L}(V)$ is uncountable. There does *not* exist a uniserial module W such that $[W]_m = [U]_m$ and $[W]_e = [V]_e$. If this were the case, W would be a submodule of U and V a quotient of W, thus V would be a subquotient of U. Therefore $\mathcal{L}(V)$ would embed in $\mathcal{L}(U)$, which is impossible.

By Theorem 7.9, there is a ring R and a uniserial module $_RU$ such that $\mathcal{L}(_RU)$ is countable, say isomorphic to the linearly ordered complete sublattice of the real line $\{\pm 1/n : 1 \leqslant n < \omega\} \cup \{0\}$, and a ring S and a uniserial module $_SV$ such that $\mathcal{L}(_SV)$ is isomorphic to the linearly ordered complete lattice $\mathbb{R} \cup \{\pm\infty\}$. Then U and V are canonically left modules over $T = R \times S$, $\mathcal{L}(_TU) = \mathcal{L}(_RU)$ and $\mathcal{L}(_TV) = \mathcal{L}(_SV)$, and $\mathrm{End}(_TU) = \mathrm{End}(_RU)$ and $\mathrm{End}(_TV) = \mathrm{End}(_SV)$.

To complete the example, one has to show that U and V are not strongly indecomposable, in other words, that $\mathfrak{m}_U$ and $\mathfrak{e}_U$ are not comparable, and similarly that $\mathfrak{m}_V$ and $\mathfrak{e}_V$ are not comparable. To see this we have to keep in mind the structure of $_RU$ and $_SV$ as constructed in Theorem 7.9. As far as $_RU$ is concerned, the positions

$$\begin{cases} \dfrac{1}{-n} \mapsto \dfrac{1}{-n} \quad (n \geqslant 1) \\ 0 \mapsto 0 \\ \dfrac{1}{n} \mapsto \dfrac{1}{n+1} \quad (n \geqslant 1) \end{cases}$$

define an injective non-surjective endomorphism of ${}_RU$, while the positions

$$\begin{cases} -1 \mapsto -1 \\ \dfrac{1}{-n} \mapsto \dfrac{1}{-n+1} \quad (n \geqslant 2) \\ 0 \mapsto 0 \\ \dfrac{1}{n} \mapsto \dfrac{1}{n} \quad (n \geqslant 1) \end{cases}$$

define a surjective non-injective endomorphism of ${}_RU$. Similar endomorphisms are defined for ${}_SV$, by mapping each remaining basis vector to itself. This concludes the example.

Appendix A

Foundational issues

There are some mathematical constructions in this thesis that are not entirely correct from the formal standpoint, but that admit equivalent and formally valid alternatives, though sometimes a bit cumbersome or unusual. Because of this, we preferred to avoid detours about foundational issues in the exposition and postpone their discussion to this appendix.

Because different authors give slightly different definitions and choose different foundations for category theory, it is virtually impossible to justify all of our constructions in a manner that rigorously fits all systems. It is the author's hope that while not all readers may completely agree on the formal correctness of our constructions, most of them will concur on their reasonableness.

Let us consider first the collection $V(\mathbf{C})$ of ideals of a semilocal category $\mathbf{C}$ associated to maximal ideals of endomorphism rings of objects of $\mathbf{C}$ (Section 2.2). An ideal is usually defined as a subclass of the class of all morphisms of $\mathbf{C}$, or as a doubly-indexed collection of subgroups $\mathbf{I}(X, Y)$ of $\mathbf{C}(X, Y)$, for X and Y objects of $\mathbf{C}$. In the first case, if $\mathbf{C}$ is not small, that is, if it has a proper class of objects, then an ideal is a proper class, because of the injective mapping $X \mapsto 0_{\mathbf{C}(X)}$ from the proper class of objects of $\mathbf{C}$ into any ideal $\mathbf{I}$ of $\mathbf{C}$. In the second case, we have an injective mapping $X \mapsto ((X, X), \mathbf{I}(X))$, and again the ideal is a proper class. In either case $\mathbf{I}$ is a proper class, hence $\mathbf{I}$ cannot be a member of any set or class $V(\mathbf{C})$. It is possible to obviate this problem by replacing the (possibly proper) class $\mathbf{I}$ with a suitable set, as in [FP10]. Let $\mathcal{S}$ be the class of all pairs (X, P) where X is an object of $\mathbf{C}$ and P is a maximal ideal of $\mathbf{C}(X)$. Introduce an equivalence relation on $\mathcal{S}$ by declaring $(X_1, P_1) \sim (X_2, P_2)$ if the ideal of $\mathbf{C}$ associated to P_1 is the same as the ideal of $\mathbf{C}$ associated to P_2. Thus we may identify $V(\mathbf{C})$ with a class of representatives of $\mathcal{S}$ modulo the equivalence relation $\sim$. (We note that to choose these representatives we need a strong version of the Axiom of Choice.) We may then identify $V(\mathbf{C}, X)$ with the subclass of $V(\mathbf{C})$ consisting of those elements (X_2, P_2) such that $(X_2, P_2) \sim (X, P)$ for some maximal ideal P of $\mathbf{C}(X)$. Thus $V(\mathbf{D}, X)$ is a set.

Recall that we defined a hypergraph to be a pair (V, E) where V is a class and E is a suitable class of finite subsets of V (Section 5.4). The definition is sound because it is possible to consider classes of subsets of a class, whereas it is not possible to form the collection of all subclasses of a proper class.

Sometimes we have an equivalence relation $\sim$ on a proper class C and we wish to consider the quotient $C/\sim$. Of course the equivalence classes of C modulo $\sim$ may be proper classes hence not members of a class $C/\sim$. In this case, we let $C/\sim$ be a class of representatives of elements of C modulo $\sim$, using a strong version of the Axiom of Choice. This was employed in Section 5.4 when we considered quotients of a class of edges E modulo the equivalence relations $\sim_i$.

In Section 5.4 we also considered monoids $\mathbb{N}^{(V)}$ where V is allowed to be a proper class. Traditionally $\mathbb{N}^{(V)}$ is a (large) monoid whose underlying set (class) is the set (class) of functions $V \to \mathbb{N}$ whose support is finite. When V is a proper class, this construction is not possible, because, for instance, the zero element of $\mathbb{N}^{(V)}$ is a proper class, thus it cannot be a member. Indeed, the zero element of $\mathbb{N}^{(V)}$ is the proper class $\{(v, 0) : v \in V\}$. Here is a slight modification of the usual definition of $\mathbb{N}^{(V)}$. Consider the class $\mathcal{S}$ of functions from finite subsets of V to $\mathbb{N}\backslash\{0\}$. An element of $\mathcal{S}$ can be extended with zeroes to a function $V \to \mathbb{N}$ with finite support, and conversely, starting with a function $g \colon V \to \mathbb{N}$ with finite support S, the restriction $g|_S$ is an element of $\mathcal{S}$. These mutually inverse correspondences allow us to identify the class $\mathcal{S}$ with the "collection" $\mathbb{N}^{(V)}$. As for the operation, if $f_1, f_2 \in \mathcal{S}$, say $f_i \colon S_i \to \mathbb{N}\backslash\{0\}$, we define $f_1 + f_2 \colon S_1 \cup S_2 \to \mathbb{N}\backslash\{0\}$ by letting $(f_1 + f_2)(s) = f_1(s) + f_2(s)$ for $s \in S_1 \cap S_2$, and $(f_1 + f_2)(s) = f_i(s)$ when s belongs only to S_i. Note in particular that the zero element is the empty set.

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