UNIVERSITÀ DEGLI STUDI DI PADOVA

Università degli Studi di Padova

Dipartimento di Fisica "Galileo Galilei"

Scuola di dottorato in Fisica XXIV Ciclo

Tesi di Dottorato

Supersymmetry-breaking vacua in simple and extended supergravity and flux compactifications

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Abstract

The search for semi-realistic supergravity models, seen as the low-energy limit of superstring theory, strongly motivates the study of gauged supergravities and flux compactifications. Outstanding problems in this context are supersymmetry breaking, moduli stabilization and the generation of a small positive vacuum energy density. Whilst spontaneously broken N=1 supergravity is the only phenomenologically viable possibility in four dimensions, its vast arbitrariness is significantly reduced when considering consistent truncations of extended supergravities or, more generally, effective theories of flux compactifications. In this thesis we explore some aspects of supersymmetry breaking for which the constrained theoretical frameworks of extended supergravities and of flux compactifications play an important role. We begin by reviewing the basic structure of four-dimensional supergravities and of flux compactifications of the higher-dimensional ones. We then describe some original work on Fayet-Iliopoulos (FI) terms and de Sitter (dS) vacua: we introduce a novel distinction between genuine FI terms and impostors; we formulate a simple anomaly-free model with a genuine FI term, a classically stable dS vacuum and no global symmetries; we explore the relations between N=1 FI terms and their counterparts in extended supergravities, by discussing suitable truncations of the latter. We continue with other original work on the relation between M-theory compactifications with geometrical and non-geometrical fluxes, gauged N=8 supergravity in four dimensions and consistent N=1 truncations of the latter. In particular: we discuss the quadratic constraints on general fluxes in M-theory, relating them with the conditions on the embedding tensor that defines the gauged N=8 theory; we identify the fluxes generating four-parameter supersymmetry breaking *à la* Scherk-Schwarz, and comment on the one-loop stability of the resulting Minkowski background.

Sommario

La ricerca di modelli semi-realistici di supergravità, visti come limite di bassa energia delle teorie di superstring, motivano fortemente lo studio delle teorie di supergravità con gruppi di gauge non banali e delle compattificazioni con flussi. In questo contesto, problemi ancora aperti sono la rottura di supersimmetria, la stabilizzazione dei moduli e la generazione di densità di energia del vuoto piccola e positiva. Nonostante la supergravità N=1 con supersimmetria spontaneamente rotta sia l'unica possibilità fenomenologicamente valida in quattro dimensioni, la sua vasta arbitrarietà viene significativamente ridotta quando si considerano troncazioni consistenti di supergravità estese o, più in generale, le teorie efficaci di compattificazioni con flussi. In questa tesi si esplorano alcuni aspetti della rottura di supersimmetria in cui un ruolo importante è giocato dalla struttura teorica vincolata delle supergravità estese e delle compattificazioni con flussi. Si inizia con una breve rassegna sulle strutture basilari delle supergravità in quattro dimensioni e delle compattificazioni con flussi di modelli di supergravità formulati in dimensione più alta. Successivamente si descrive del lavoro originale sui termini di Fayet-Iliopoulos (FI) ed i vuoti di de Sitter (dS): si introduce una nuova distinzione tra termini di FI genuini e 'impostori'; si formula un semplice modello senza anomalie con un termine di FI genuino, un vuoto di dS classicamente stabile e senza simmetrie globali; si analizzano le relazioni tra i termini di FI N=1 e le loro controparti nelle supergravità estese, discutendo opportune troncazioni di queste ultime. Si prosegue poi con dell'altro lavoro originale sulla relazione tra le compattificazioni di M-teoria con flussi geometrici e non-geometrici, le teorie di supergravità N=8 in quattro dimensioni con gruppi di gauge non banali e le troncazioni consistenti di queste ultime a N=1. In particolare: si discutono i vincoli quadratici su tutti i flussi in M-teoria, collegandoli con le condizioni sul tensore di embedding che definisce la teoria di gauge N=8; si identificano i flussi che generano la rottura di supersimmetria *à la* Scherk-Schwarz con quattro parametri, e si commenta sulla stabilità a un loop del risultante vuoto di Minkowski.

Acknowledgments

I would like to thank all people who have helped and inspired me during my doctoral study.

First of all, I would like to thank my advisor, Fabio Zwirner, for his guidance and support during these three years.

I would also thank Gianguido Dall'Agata for useful discussions and Giovanni Villadoro for important suggestions.

I would like to acknowledge the University of Padova and the theoretical group of the INFN for the financial, academic and technical support.

Special thanks to the Ph.D students that I met in Padova and abroad, for their advises and their incredible support, not only as colleagues, but especially as friends.

Of course no acknowledgments would be complete without giving thanks to my parents and my sisters. They have encouraged me in any circumstance.

Contents

Introduction

Simple and extended supergravity theories in four dimensions have been extensively studied in the last decades. There are two ways of studying supergravity, either by constructing it directly from the field content and the symmetries of the action, or by obtaining it from higher dimensions by dimensional reduction. Supergravity theories obtained by dimensional reduction from ten- or eleven-dimensional supergravity can be seen as the effective low-energy limit of some superstring or M-theory compactification. From this point of view, the study of supergravity theories can provide important insight on the fundamental string theory.

The construction of (semi)realistic models becomes, in this context, a fundamental issue. An outstanding problem is supersymmetry breaking. In fact, if supersymmetry is of any relevance in Nature, it must be realized in the broken phase, and, since supergravity is a theory with local supersymmetry, this breaking must be spontaneous. Supergravity models with broken supersymmetry are however more involved than those with exact supersymmetry, because the stability of the critical points is not guaranteed by supersymmetry and has thus to be checked case by case.

In particular, compactifications of superstring theories to four-dimensional supergravities produce a number of scalar fields, namely the moduli of the compactification. Their stabilization is related to the presence of a suitable scalar potential, strongly constrained by the compactification scheme. String compactifications with background fluxes provide a simple framework in which the stabilization of the moduli fields can be discussed in a controlled and natural way.

From a cosmological point of view, it is especially important to understand moduli stabilization at a positive potential energy, either to obtain local de Sitter minima, so as to describe the present accelerated cosmic expansion, or in inflationary potentials. The search for de Sitter vacua has thus become an important problem in supergravity.

Spontaneously broken $\mathcal{N} = 1$ supergravity rightly deserved special attention, since it is the only possibility to construct phenomenologically viable models in four dimensions. However, its vast arbitrariness makes it more difficult to estabilish a precise correspondence with superstring/M-theory and to determine what $\mathcal{N} = 1$ models can eventually find such an embedding. For this reason, it is useful to study also models with $\mathcal{N} > 1$ and their $\mathcal{N} = 1$ truncations, whose more constrained structure makes such a discussion easier.

This thesis is organized as follows. The first chapter is devoted to a brief overview of simple and extended supergravity in four space-time dimensions. It is focused on the scalar manifold, the scalar potential and the gaugings, notions that will be useful for the rest of the thesis.

In the second chapter, we present a summary of some flux compactification scenarios, from type II and M-theory. The fluxes turned on are both geometric and non-geometric, giving the most general setup in this framework.

In chapter three, some original work on Fayet-Iliopoulos (FI) terms and de Sitter vacua is presented [1]. We concentrate on constant FI terms, associated with the gauging of an R-symmetry, and explain how we should distinguish between genuine FI terms and impostors. We then present some simple examples of $\mathcal{N} = 1$ models, with either a genuine FI term or an impostor, paying also attention, in the first case, to the anomaly cancellation conditions. The chapter ends with the analysis of the truncation to $\mathcal{N} = 1$ of the only known extended supergravity models with stable de Sitter vacua, for which one of the necessary ingredients is a FI term.

The fourth chapter contains other original work [2]. Here we analyze the relation between M-theory compactifications with geometric and non-geometric fluxes, gauged $\mathcal{N} = 8$ supergravity in four dimensions and its consistent truncations to $\mathcal{N} = 4$ and $\mathcal{N} = 1$. In particular we discuss the complete set of quadratic constraints on general fluxes in M-theory, or equivalently the Jacobi identities on the embedding tensor that defines the gauged $\mathcal{N} = 8$ theory. We then discuss the embedding in M-theory of some examples analyzed in the recent literature and identify the set of fluxes generating four-parameter supersymmetry breaking *à la* Scherk-Schwarz.

Chapter 1

Four-dimensional supergravities

This chapter is a short review of some features of four-dimensional supergravities. It is not intended to be self-consistent and complete, but it focuses on some aspects that will be useful later in this thesis. Supergravity here is considered in its own right, independently of its rôle in superstring theory, and fields occur in the action up to two derivatives.

Particular attention is given to the scalar fields, which determine the vacua of the theory and define a geometry, the so-called scalar manifold, the structure of which sets a large part of the full action. Emphasis will be given also to the symmetries of the theory, both global and local. The gauge group has to be a subgroup of the global symmetry group and the invariance under this local group, if any, induces further modifications to the supergravity action. Full Lagrangians are not given here, however in this chapter there are some details about the scalar potential, since the latter is fundamental in the study of the vacua of the theory and of supersymmetry breaking.

Four-dimensional supergravities with ascending number of supersymmetries are considered: section 1.1 is devoted to $\mathcal{N} = 1$, section 1.2 to $\mathcal{N} = 2$, section 1.3 to $\mathcal{N} = 4$ and section 1.4 to $\mathcal{N} = 8$ supergravity. The structure of each section is more or less the same: first of all the matter content is specified in terms of supermultiplets, than there are some considerations about global and local symmetries and at the end some details are given about supersymmetry transformations of the fermionic fields, scalar potential and supersymmetry breaking. The formalism used in the literature is different in each theory, but in the following the notation will be unified as much as possible and the links between different formalisms will be emphasized.

1.1 $\mathcal{N} = 1$ supergravity

$\mathcal{N} = 1$ supergravity was initially proposed in 1976 [3, 4] as pure supergravity. In more than 30 years, many developments have occurred, summarized in many books and reviews (see for example [5, 6]).

In supergravity theories there is always at least the supergravity multiplet, possibly coupled to matter multiplets. In $\mathcal{N} = 1$, the former is made up of a spin-2 boson (*graviton*) e_μ^α and a spin-3/2 fermion (*gravitino*) ψ_μ. It can be coupled to chiral or vector multiplets. A chiral multiplet contains a spin-1/2 Weyl fermion χ and a complex scalar field z. A vector multiplet contains a vector boson A_μ and a spin-1/2 fermion (*gaugino*) λ.

Given the number of chiral N_C and vector N_V multiplets, the theory is completely defined once some ingredients are given, namely

- the *Kähler potential* $K(z, \bar{z})$, a real function that determines the geometry of the scalar manifold $\mathcal{M}_{scalar}$;
- the holomorphic *superpotential* $W(z)$, which encodes the self-interactions of the chiral multiplets;
- the holomorphic *gauge kinetic function* $f_{\Lambda\Sigma}(z)$ $(\Lambda, \Sigma = 0, 1, ..., N_V)$, related to the kinetic terms of the vector multiplets;
- the action of the gauge group on $\mathcal{M}_{scalar}$, specified by the holomorphic *Killing vectors* $X_\Lambda^m(z)$ $(\Lambda = 0, 1, ..., N_V,\ m = 1, ..., N_C)$ and the corresponding *Killing prepotentials* $\mathcal{P}_\Lambda(z, \bar{z})$;
- the *Fayet-Iliopoulos terms* ξ_Λ $(\Lambda = 0, 1, ..., N_V)$, which might be non-zero only for Abelian gauge group factors.

In the following, the rôle of these ingredients in constructing the $\mathcal{N} = 1$ supergravity action, and in particular the scalar potential, is explained in more detail. We will use the standard supergravity notation, in which the reduced Planck mass M_P is set equal to one (see Appendix A for more details).

1.1.1 Scalar manifold and Kähler transformations

The N_C complex scalar fields z^m $(m = 1, ..., N_C)$ span a Hodge-Kähler manifold, a complex manifold such that the metric can be defined by the Kähler potential as

$$g_{m\bar{n}} = \partial_m \partial_{\bar{n}} K, \tag{1.1}$$

up to Kähler transformations

$$K(z, \bar{z}) \longrightarrow K(z, \bar{z}) + h(z) + \bar{h}(\bar{z}), \tag{1.2}$$

with $h(z)$ an holomorphic function and $\partial_m = \partial/\partial z^m$, $\partial_{\overline{m}} = \partial/\partial \overline{z^m}$. The total action is invariant under these transformations if accompanied by chiral rotations of the fermions and an appropriate transformation of the superpotential

$$\psi_\mu \to e^{-i(\operatorname{Im} h)\gamma_5/2}\psi_\mu, \quad \lambda^\Sigma \to e^{-i(\operatorname{Im} h)\gamma_5/2}\lambda^\Sigma, \quad \chi^m \to e^{i(\operatorname{Im} h)\gamma_5/2}\chi^m,$$
$$W \to e^{-h}W. \tag{1.3}$$

The bosonic fields, on the other hand, do not transform. Kähler transformations are local $U(1)$ symmetries of the theory with gauge parameters $\theta(x) = \operatorname{Im} h[z(x)]$ and the corresponding gauge field, denoted by $\mathcal{A}_\mu$, is a composite field, which depends on the scalars as

$$\mathcal{A}_\mu = \frac{i}{2}(\partial_\mu z^m \partial_m K - \partial_\mu \bar{z}^{\bar{m}} \partial_{\bar{m}} K). \tag{1.4}$$

We assume, for the moment, the absence of Fayet-Iliopoulos terms, which will be introduced later when gaugings will be switched on.

It is easy to show that $\mathcal{A}_\mu$ transforms with the derivative of the gauge parameter under a Kähler transformation

$$\delta_K \mathcal{A}_\mu = \frac{i}{2}(\partial_\mu z^m \partial_m h - \partial_\mu \bar{z}^{\bar{m}} \partial_{\bar{m}} \bar{h}) = -\partial_\mu (\operatorname{Im} h) = -\partial_\mu \theta(x). \tag{1.5}$$

Moreover, all covariant derivatives for the fermionic fields that appear in the Lagrangian are covariant under the Kähler transformation

$$\hat{\partial}_\mu \psi_\nu = \cdots + \frac{i}{2}\mathcal{A}_\mu \gamma_5 \psi_\nu, \quad \hat{\partial}_\mu \lambda^\Sigma = \cdots + \frac{i}{2}\mathcal{A}_\mu \gamma_5 \lambda^\Sigma, \quad \hat{\partial}_\mu \chi^m = \cdots - \frac{i}{2}\mathcal{A}_\mu \gamma_5 \chi^m. \tag{1.6}$$

1.1.2 Gauge group

For $N_V > 0$, the $\mathcal{N} = 1$ supergravity action is always invariant under a local symmetry group G_0. The trivial case is $G_0 = U(1)^{N_V}$, under which

$$\delta A^\Sigma_\mu = \partial_\mu \epsilon^\Sigma, \tag{1.7}$$

where ϵ^Σ are the corresponding gauge parameters and all the other fields are invariant. $\mathcal{N} = 1$ supergravity with only this trivial local invariance is called *ungauged.* It is possible to switch on a more general gauge group than $U(1)^{N_V}$, and in this case the supergravity is called *gauged.* The gauge group has to be a subgroup of the isometry group of the scalar manifold Iso($\mathcal{M}_{scalar}$), which is the set of the transformations of the scalar fields that leave the metric g_{mn} invariant up to a Kähler transformation. The dimension of the non-Abelian part of G_0 is at most N_V, but it is convenient to include also trivial $U(1)$ transformations, in order to have always $\dim G_0 = N_V$. A^Λ_μ are the vector bosons related to the gauge group generators t_Λ in the fundamental representation of G_0. The action of the gauge group on the scalar fields is given by the Killing vectors $X^m_\Lambda(z)$ and does not mix z^m and $\bar{z}^{\bar{m}}$

$$\delta z^m = X^m_\Sigma \epsilon^\Sigma, \qquad \delta \bar{z}^{\bar{m}} = \overline{X}^{\overline{m}}_\Sigma \epsilon^\Sigma. \tag{1.8}$$

Note that, if $X^m_\Sigma = 0$ for every m, the vector A^Σ_μ is related to a trivial $U(1)$ factor. The Killing vectors are holomorphic functions such that there exist real functions, the Killing prepotentials $\mathcal{P}_\Lambda$, related to the Killing vectors by

$$i g_{m\bar{n}} X^m_\Lambda = \partial_{\bar{n}} \mathcal{P}_\Lambda. \tag{1.9}$$

Gauge transformations should leave the Kähler potential invariant up to a Kähler transformation, *i.e.*

$$\begin{aligned} \delta_\Lambda K &= X^m_\Lambda \partial_m K + X^{\bar{m}}_\Lambda \partial_{\bar{m}} K = r_\Lambda(z) + \bar{r}_\Lambda(\bar{z}), \\ \delta_\Lambda W &= \partial_m W X^m_\Lambda = -r_\Lambda(z) W, \end{aligned} \tag{1.10}$$

with $r_\Lambda(z)$ any holomorphic function. As as consequence of (1.10) and of $g_{m\bar{n}} = \partial_m \partial_{\bar{n}} K$ and $\partial_{\bar{n}} X^m_\Lambda = 0$, the Killing prepotentials defined in (1.9) have the following form

$$\begin{aligned} \mathcal{P}_\Lambda &= i\left[X^m_\Lambda \partial_m K - r_\Lambda(z)\right] = i\left[X^{\bar{m}}_\Lambda \partial_{\bar{m}} K + \bar{r}_\Lambda(\bar{z})\right] \\ &= \frac{i}{2}(X^m_\Lambda \partial_m K - X^{\bar{m}}_\Lambda \partial_{\bar{m}} K - [r_\Lambda(z) + \bar{r}_\Lambda(\bar{z})]). \end{aligned} \tag{1.11}$$

The last equality is due to reality of the prepotential, $\mathcal{P}_\Lambda = \mathcal{P}^*_\Lambda$. If we pick the Kähler potential to be invariant under gauge transformations, then $r_\Lambda + \bar{r}_\Lambda = 0$ and therefore, if W is also gauge invariant, the $\mathcal{P}_\Lambda$ reduce to

$$\mathcal{P}_\Lambda = i X^m_\Lambda \partial_m K. \tag{1.12}$$

For a linearly realized gauge symmetry,

$$i\,\partial_m K\, X^m_\Lambda = -\partial_m K\,(t_\Lambda)^m{}_n z^n\,,$$

and we recover the standard expression of [7] for the D-terms. For example, in the case of canonical Kähler potential, and fields z^m with definite charges q^m with respect to a single U(1) gauge factor

$$K = \sum_m |z^m|^2 \,, \qquad X^m = i\, q^m\, z^m \,, \tag{1.13}$$

and the prepotential (with an implicit lower index) reads

$$\mathcal{P} = -\sum_m q^m\, |z^m|^2 \,. \tag{1.14}$$

For an axionic realization, $X^m_\Lambda = i\, q^m_\Lambda$, where q^m_Λ is a real constant, and we obtain what are often called, with an abuse of language, field-dependent FI terms. A classic example [8], which often arises in string compactifications, is

$$K = -\log(S + \overline{S}) \,, \qquad X^S = i\, q^S \,, \tag{1.15}$$

which leads to

$$\mathcal{P} = \frac{q^S}{S + \overline{S}} \,. \tag{1.16}$$

If W is not gauge invariant, the only freedom left is

$$i\, X^m_\Lambda \frac{\partial_m W}{W} = \xi_\Lambda , \tag{1.17}$$

so that the gauge non-invariance of W can be at most an overall phase with real parameter ξ_Λ, for the Abelian factors $U(1)_\Lambda$, as one can see from the equivariance condition

$$X^m_\Lambda \partial_m \mathcal{P}_\Sigma + X^{\bar{m}}_\Lambda \partial_{\bar{m}} \mathcal{P}_\Sigma = f_{\Lambda\Sigma}{}^\Gamma \mathcal{P}_\Gamma . \tag{1.18}$$

The constants ξ_Λ correspond to gaugings of R-symmetries, and give rise to the supergravity expression for the prepotential [5]:

$$\mathcal{P}_\Lambda = i\, \partial_m K\, X^m_\Lambda + \xi_\Lambda \,. \tag{1.19}$$

The most general definition of the Killing prepotential in $\mathcal{N}=1$ supergravity is

$$\mathcal{P}_\Lambda = i X^m_\Lambda \partial_m K + \xi_\Lambda = i X^m_\Lambda (\partial_m K + \frac{\partial_m W}{W}) = i X^m_\Lambda \frac{D_m W}{W} , \tag{1.20}$$

where $D_m W = \partial_m W + W \partial_m K$. For example, in the linear case of eq. (1.13) and assuming $q^1 = -\xi$, the superpotential

$$W = M^2\, z^1 \tag{1.21}$$

gauges a suitable U(1) R-symmetry, modifying the prepotential into

$$\mathcal{P} = \xi - \sum_m q^m \, |z^m|^2 \,. \tag{1.22}$$

Similarly, in the non-linear case of eq. (1.15) and assuming $q^S = \xi$, the superpotential

$$W = W_0 \, e^{-S} \,, \tag{1.23}$$

where W_0 is a non-vanishing S-independent factor, also gauges a U(1) R-symmetry, and modifies the D-term into

$$D = \xi \left(1 + \frac{1}{S + \overline{S}} \right) \,. \tag{1.24}$$

The presence of Fayet-Iliopoulos terms for $U(1)$ factors also modifies the field (1.4) as

$$\mathcal{A}_\mu = \frac{i}{2}(\partial_\mu z^m \partial_m K - \partial_\mu \bar{z}^{\bar{m}} \partial_{\bar{m}} K) + A_\mu^\Lambda \mathcal{P}_\Lambda. \tag{1.25}$$

1.1.3 Scalar potential

The $\mathcal{N} = 1$ scalar potential is the sum of three different contributions:

$$V = V_F - V_G + V_D, \tag{1.26}$$

where

$$V_F = e^K g^{m\bar{n}} (D_m W)(D_{\bar{n}} \overline{W}) \tag{1.27}$$

is the one that comes from the chiral auxiliary fields,

$$V_D = \frac{1}{2} (Re\, f)^{-1\,\Lambda\Sigma} \mathcal{P}_\Lambda \mathcal{P}_\Sigma \tag{1.28}$$

is the one that comes from the vector auxiliary fields and

$$V_G = 3 e^K |W|^2 \tag{1.29}$$

is related to the gravitino mass.

F- and D-terms give positive semidefinite contributions, while the term related to the gravitino mass is negative semidefinite. It obviously means that, unlike the global supersymmetry case, the scalar potential might be positive, negative or zero. Moreover F- and D-terms are related.

There is another useful way of writing down the potential. It exploits the fact that supersymmetry relates the scalar potential and the supersymmetry transformations of the fermionic fields. This relation is extremely important because it generalizes to higher number of supersymmetries and also to higher dimensions. The supersymmetry transformation rules for the fermions of the theory are

$$\begin{aligned} \delta\psi_{\mu L} &= \partial_\mu \epsilon_L + \gamma_\mu S_{LR}\epsilon_R + \cdots , \\ \delta\chi^m_L &= \mathcal{N}^m \epsilon_L + \cdots , \\ \delta\lambda^\Lambda_L &= N^\Lambda \epsilon_L + \cdots , \end{aligned} \tag{1.30}$$

where ϵ, the supersymmetry transformation parameter, is a Majorana spinor and the dots stand for terms that depend on fermions, scalar derivatives and vector bosons.

Here S_{LR} is the gravitino mass term

$$S_{LR} = \frac{1}{2} e^{K/2} W, \tag{1.31}$$

while the shift of the fermions in the chiral multiplets is

$$\mathcal{N}^m = -\frac{1}{2} g^{m\bar{n}} e^{K/2} \overline{D_n W} \tag{1.32}$$

and the shift of the gaugini is

$$N_\Lambda = \frac{i}{2}\mathcal{P}_\Lambda, \tag{1.33}$$

where vector indices Λ are lowered and raised by the gauge kinetic function and its inverse.

The scalar potential is given by

$$V = -12 S_{LR}\bar{S}^{RL} + 4 g_{m\bar{n}}\mathcal{N}^m \bar{\mathcal{N}}^{\bar{n}} + 2(Re\, f)_{\Lambda\Sigma} N^\Lambda \bar{N}^\Sigma. \tag{1.34}$$

Also from this formula it is evident that matter fields always give positive contributions, while the gravitino variation may give a negative contribution.

Critical points of the potential are those values of the scalar fields for which $\partial_m V = 0$. A critical point is a (meta)stable vacuum if the squared scalar masses are all equal or greater than zero in Minkowski and de Sitter backgrounds, or greater than $\frac{3}{4}\langle V\rangle$ in Anti de Sitter background [9]. At a critical point, and in particular at a vacuum, supersymmetry is preserved if and only if the supersymmetry variations of the fermionic matter and gauge fields vanish

$$\delta\chi^m = \delta\lambda^\Lambda = 0. \tag{1.35}$$

This requires both

$$\mathcal{N}^m = 0 \iff D_m W = 0 \tag{1.36}$$

and

$$N^\Lambda = 0 \iff \mathcal{P}_\Lambda = 0, \tag{1.37}$$

but there is no condition on the gravitino mass S_{LR}. The vacuum expectation value of the scalar potential might be only negative or zero. If $\langle W \rangle = 0$, then $\langle V \rangle = 0$ and the vacuum is Minkowski. The condition of unbroken supersymmetry becomes $\langle W \rangle = \langle \partial_m W \rangle = 0$. If $\langle W \rangle \neq 0$, the supersymmetric vacuum has $\langle V \rangle < 0$ (Anti de Sitter). Supersymmetry is broken if one or both (1.36) and (1.37) are not satisfied in the vacuum. The vacuum expectation value of the potential might be in this case positive (de Sitter), negative (Anti de Sitter) or zero (Minkowski). No conditions as simple as in the supersymmetric case can be given in this case to characterize the vacuum.

1.2 $\mathcal{N} = 2$ supergravity

In this section, some basic features of $\mathcal{N} = 2$ supergravity are given, following more or less the same structure of the previous section. $\mathcal{N} = 2$ supergravity displays a high degree of complexity and very involved geometrical structures, however, for the purpose of this thesis we can focus only on a particular framework, extensively studied in the literature [10] – [13]. The interest in this framework is related to the fact that here the only (meta)stable de Sitter vacua in extended supergravity have been found [14]. For a general and comprehensive review and the full Lagrangian, a useful reference is [15].

The first step is to point out the matter content of the theory. There is the $\mathcal{N} = 2$ supergravity multiplet, made up of a spin-2 boson (*graviton*) e^α_μ, two spin-3/2 fermions (*gravitini*) ψ^a_μ $(a = 1, 2)$ and a spin-1 boson (*graviphoton*) A^0_μ. It can be coupled to vector multiplets and hypermultiplets. An $\mathcal{N} = 2$ vector multiplet contains a spin-1 boson B_μ, two spin-1/2 fermions λ^a and a complex scalar field y. An $\mathcal{N} = 2$ hypermultiplet contains two spin-1/2 fermions ζ^a and four real scalar fields q^u $(u = 1, \dots, 4)$.

Given the number of vector multiplets n_V and hypermultiplets n_H, the supergravity action is completely specified by fixing a suitable scalar manifold and a gauge group.

In the case we are interested in, the scalar manifold is

$$\underbrace{\left(\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n_V-1)}{SO(2)\times SO(n_V-1)}\right)}_{\mathcal{SK}[n_V]} \times \underbrace{\left(\frac{SO(4,n_H)}{SO(4)\times SO(n_H)}\right)}_{\mathcal{QK}[n_H]}, \tag{1.38}$$

where $\mathcal{SK}[n_V]$ is a special Kähler manifold with complex dimension n_V and $\mathcal{QK}[n_H]$ is a quaternionic manifold with real dimension $4n_H$. The description of these geometrical structures would be beyond the aim of the thesis, and the reader interested in this aspect is referred to [15]. The aim of this section is to give the ingredients to write down the Lagrangian and the supersymmetry transformation rules, without explaining their geometrical meaning.

Before proceeding, it is useful to summarize, once and for all, one important feature that characterizes many field theories with vector and scalar fields and has a prominent rôle in extended supergravity, since vector and scalar fields are related by supersymmetry: it is the so-called symplectic covariance.

1.2.1 Symplectic covariance

Symplectic covariance is a covariance of any Abelian theory in four dimensions [16]. We consider here a theory with n vector bosons, arranged in n 1-forms

$$A^\Lambda = A^\Lambda_\mu \mathrm{d}x^\mu, \tag{1.39}$$

m real scalars ϕ^I ($I = 1, ..., m$) that span an m-dimensional scalar manifold endowed with a metric $g_{IJ}(\phi)$, and a kinetic Lagrangian written as

$$\mathcal{L}_{kin} = -\gamma_{\Lambda\Sigma}(\phi)\mathcal{F}^\Lambda_{\mu\nu}\mathcal{F}^{\Sigma\mu\nu} + \frac{1}{2}\theta_{\Lambda\Sigma}(\phi)\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^\Lambda_{\mu\nu}\mathcal{F}^\Sigma_{\rho\sigma} + \frac{1}{2}g_{IJ}(\phi)\partial_\mu\phi^I\partial^m\phi^J, \tag{1.40}$$

where $\mathcal{F}^\Lambda = \frac{1}{2}(\partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu)$ are field strengths and γ, θ are symmetric matrices.

Introducing the dual field strengths

$$\mathcal{G}^{\mu\nu}_\Lambda = \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\mathcal{F}^\Lambda_{\mu\nu}} \tag{1.41}$$

and the (anti)self-dual field strengths

$$\mathcal{F}^\pm = \frac{1}{2}\left(\mathcal{F} \pm \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma}\right), \quad \mathcal{G}^\pm = \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\mathcal{F}^\pm}, \tag{1.42}$$

the vector kinetic Lagrangian becomes

$$\mathcal{L}_{vec} = i\left(\mathcal{F}^{-T}\overline{\mathcal{N}}\mathcal{F}^- - \mathcal{F}^{+T}\mathcal{N}\mathcal{F}^+\right), \tag{1.43}$$

with $\mathcal{N} = \theta - i\gamma$ and $\overline{\mathcal{N}} = \theta + i\gamma$ symmetric matrices, and the equations of motion and Bianchi identities are

$$\partial^\mu \mathrm{Im}\mathcal{F}^{\pm\Lambda}_{\mu\nu} = 0, \qquad \partial^\mu \mathrm{Im}\mathcal{G}^{\pm\Lambda}_{\mu\nu} = 0. \tag{1.44}$$

In terms of the $2n$ column vectors

$$\mathbb{V}^\pm = \begin{pmatrix} \mathcal{F}^\pm \\ \mathcal{G}^\pm \end{pmatrix}, \tag{1.45}$$

with

$$\mathcal{G}^+ = \mathcal{N}\mathcal{F}^+, \quad \mathcal{G}^- = \overline{\mathcal{N}}\mathcal{F}^-, \tag{1.46}$$

the transformation is

$$\mathbb{V}^{\pm\,\prime} = \Lambda\mathbb{V}^\pm, \tag{1.47}$$

with $\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R})$, in order to preserve the relation (1.46) between $\mathcal{G}$ and $\mathcal{F}$.

While the duality rotation (1.47) is performed on the field strengths and their duals, also the scalar fields are transformed by the action of some isomorphism of the scalar manifold $\xi \in \mathrm{Iso}(\mathcal{M}_{scalar})$ and as a consequence also the matrix $\mathcal{N}$ transforms. In other words, we assume that there exists a homomorphism

$$\iota_\delta : \ \mathrm{Iso}(\mathcal{M}_{scalar}) \longrightarrow SL(2n, \mathbb{R}), \tag{1.48}$$

so that for every $\xi \in \mathrm{Iso}(\mathcal{M}_{scalar})$ there exists

$$\iota_\delta(\xi) = \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix} \in Sp(2n, \mathbb{R}). \tag{1.49}$$

Using such a homomorphism, we can define the transformation

$$\xi : \begin{cases} \phi \to \xi(\phi) \\ \mathbb{V}^\pm \to \iota_\delta(\xi)\mathbb{V}^\pm \\ \mathcal{N}(\phi) \to \mathcal{N}'(\xi(\phi)) = (C_\xi + D_\xi\mathcal{N}(\phi))(A_\xi + B_\xi\mathcal{N}(\phi))^{-1} \end{cases} . \tag{1.50}$$

This transformation is not a symmetry of the Lagrangian in the Abelian, ungauged case, but only of the set of field equations and Bianchi identities. When some non-trivial gauge group is switched on and electric charges are assigned, symplectic transformations cease to yield physically equivalent theories. Gaugings break symplectic covariance and the choice of the correct symplectic frame becomes a physical issue.

1.2.2 Scalar manifold and symplectic embedding

The scalar manifold (1.38) we are interested in is the product of two factors, a special Kähler manifold $\mathcal{SK}[n_V]$ and a quaternionic manifold $\mathcal{QK}[n_H]$. The former is spanned by the scalar fields of the vector multiplets, while the latter is spanned by the scalar fields of the hypermultiplets.

The special Kähler manifold $\mathcal{SK}[n_V]$ has a complex coordinate, the axion-dilaton S, which spans the $SU(1,1)/U(1)$ factor, and n_V-1 coordinates y^i $(i=1,...,n_V-1)$, called *Calabi-Vesentini* coordinates, which span the $SO(2,n_V-1)/(SO(2)\times SO(n_V))$ factor. These coordinates can be arranged in a useful way into the holomorphic section

$$\Omega = \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}, \quad (\Lambda = 0,1,\cdots,n_V), \tag{1.51}$$

where

$$X^\Lambda = \begin{pmatrix} \frac{1}{2}(1+y^2) \\ \frac{i}{2}(1-y^2) \\ y^0 \\ y^i \end{pmatrix}, \qquad F_\Lambda = -i\,S\eta^{(2,2)}_{\Lambda\Sigma}X^\Sigma = \begin{pmatrix} -\frac{i}{2}S(1+y^2) \\ \frac{1}{2}S(1-y^2) \\ i\,S\,y^0 \\ i\,S\,y^i \end{pmatrix}, \tag{1.52}$$

with $y^2 = \sum_i (y^i)^2$and $\eta^{(2,2)} = diag(++--)$ metric of $SO(2,2)$. The Kähler potential is then defined by

$$K = -\log(i\left\langle\Omega|\bar\Omega\right\rangle) = -\log(i(\bar X^\Lambda F_\Lambda - \bar F_\Lambda X^\Lambda)), \tag{1.53}$$

where the symbol $i\left\langle|\right\rangle$ stands for the symplectic product

$$i\left\langle A|\bar B\right\rangle = A^T\cdot\mathbb{C}\cdot\bar B, \tag{1.54}$$

where $\mathbb{C} = \begin{pmatrix} \mathbb{O} & \mathbb{I}_{N_V} \\ -\mathbb{I}_{N_V} & \mathbb{O} \end{pmatrix}$ is the metric of $Sp(2N_V+2,\mathbb{R})$. Explicitly:

$$K = K_1(S,\bar S) + K_2(y,\bar y), \tag{1.55}$$

$$K_1 = -\log(S+\bar S), \quad K_2 = -\log\left(\frac{1}{2}Y\right),$$

with $Y = 1-2y^i\bar y^i + |y^iy^i|^2$.

The Kähler potential allows to endow the special Kähler manifold with a metric

$$g = \begin{pmatrix} g_{S\bar S} & \\ & g_{i\bar\jmath} \end{pmatrix}, \quad g_{S\bar S} = \frac{1}{(S+\bar S)^2} \quad \text{and} \quad g_{i\bar\jmath} = \frac{\partial}{\partial y^i}\frac{\partial}{\partial\bar y^{\jmath}}K_2. \tag{1.56}$$

The metric is defined from the Kähler potential up to a Kähler transformation

$$K \longrightarrow K + h + \bar{h}, \tag{1.57}$$

as in $\mathcal{N} = 1$ supergravity. Under Kähler transformations the holomorphic section transforms as

$$\Omega \longrightarrow e^{-h}\Omega. \tag{1.58}$$

We can then define a covariant non-holomorphic section

$$V = \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} = e^{K/2}\Omega \tag{1.59}$$

that satisfies $i \left\langle V|\bar{V}\right\rangle = 1$. The relation between L^Λ and F_Λ,

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma} L^\Sigma, \tag{1.60}$$

defines the so-called *period matrix* $\mathcal{N}_{\Lambda\Sigma}(S, y)$, which is nothing else than the gauge kinetic function.

The quaternionic manifold

$$\mathcal{QK}[n_H] = \frac{SO(4, n_H)}{SO(4) \times SO(n_H)} \tag{1.61}$$

has $4n_H$ coordinates given by the real scalar fields q^u ($u = 1, ..., 4n_H$). Since this is a coset manifold, it is useful to introduce a coset representative $L(q) \in SO(4, n_H)$. ${L^u}_v$ is a $(4+n_H)\times(4+n_H)$ matrix, such that $L^T\eta^{(4,n_H)}L = \eta^{(4,n_H)}$, where the $SO(4, n_H)$ metric is $\eta^{(4,n_H)} = \mathrm{diag}(+1, +1, +1, +1, -1, ..., -1)$, and its indices can be split into $m, n = 1, ..., 4$, vector indices of $SO(4)$ and $s, t = 1, ..., n_H$, vector indices of $SO(n_H)$. In the matrix

$$L^{-1}dL = \begin{pmatrix} \theta^{mn} & E^{mt} \\ (E^T)^{tm} & \Delta^{st} \end{pmatrix}, \tag{1.62}$$

the diagonal blocks are the $SO(4)$ and $SO(N_H)$ connections θ^{mn} and Δ^{st} and the off-diagonal blocks are the vielbein 1-form $E^{mt} = E^{mt}_u dq^u$ and its transpose.

The quaternionic manifold $\mathcal{QK}[n_H]$ is endowed with a metric

$$\mathrm{d}s^2 = h_{uv}\mathrm{d}q^u\mathrm{d}q^v = \frac{1}{2}\sum_{m,t} E^{mt}E^{mt} \tag{1.63}$$

and three complex structures

$$(J^x)_u{}^v = E^{mt}_u (\mathbb{J}^x)_m{}^n E^v_{nt}, \tag{1.64}$$

with E^v_{nt} the inverse vielbein and

$$\mathbb{J}^1=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\quad \mathbb{J}^2=\begin{pmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{pmatrix},\quad \mathbb{J}^3=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}. \tag{1.65}$$

To make the notation more explicit, we can consider a useful example, the quaternionic manifold $\mathcal{QK}[n_H=2]$, parametrized by solvable coordinates $q=(a_1,a_2,a_3,a_4,b_1,b_2,h_1,h_2)$. The coset representative is a 6×6 matrix that can be written in block form as

$$L_{solv}=\begin{pmatrix}A_{solv}&B_{solv}\\C_{solv}&D_{solv}\end{pmatrix}, \tag{1.66}$$

with

$$A_{solv}=\begin{pmatrix}1&0&-a_4e^{-h_1}&-b_2e^{-h_2}\\0&1&-a_3e^{-h_1}&-b_1e^{-h_2}\\a_4+\sqrt{2}a_1b_2&a_3+\sqrt{2}a_1b_1&c+\cosh h_1&\frac{1}{\sqrt{2}}(a_1e^{h_2}-a_2e^{-h_2})-d\\b_2&b_1&\frac{1}{\sqrt{2}}(a_2-a_1)e^{-h_1}&\cosh h_2-\frac{1}{2}b^2e^{-h_2}\end{pmatrix},$$

$$B_{solv}=\begin{pmatrix}a_4e^{-h_1}&b_2e^{-h_2}\\a_3e^{-h_1}&b_1e^{-h_2}\\\sinh h_1-c&\frac{1}{\sqrt{2}}(e^{h_2}a_1+a_2e^{-h_2})+d\\\frac{1}{\sqrt{2}}(a_1-a_2)e^{-h_1}&\frac{1}{2}e^{-h_2}b^2+\sinh h_2\end{pmatrix},$$

$$C^T_{solv}=\begin{pmatrix}a_4+\sqrt{2}a_1b_2&b_2\\a_3+\sqrt{2}a_1b_1&b_1\\\sinh h_1+c&\frac{1}{\sqrt{2}}(a_1+a_2)e^{-h_1}\\\frac{1}{\sqrt{2}}(a_1e^{h_2}-a_2e^{-h_2})-d&\sinh h_2-\frac{1}{2}b^2e^{-h_2}\end{pmatrix},$$

$$D_{solv}=\begin{pmatrix}\cosh h_1-c&d+\frac{1}{\sqrt{2}}(e^{h_2}a_1+a_2e^{-h_2})\\-\frac{1}{\sqrt{2}}(a_1+a_2)e^{-h_1}&\frac{1}{2}e^{-h_2}b^2+\cosh h_2\end{pmatrix}, \tag{1.67}$$

and $b^2=b_1^2+b_2^2$, $c=\frac{1}{2}e^{-h_1}(2a_1a_2-a_3^2-a_4^2)$, $d=e^{-h_2}(a_3b_1+a_4b_2+\frac{1}{\sqrt{2}}a_1b^2)$.

It is easy to show that

$$E^{mt}=\begin{pmatrix}e^{-h_1}(\mathrm{d}a_4+\sqrt{2}b_2\mathrm{d}a_1)&e^{-h_2}\mathrm{d}b_2\\e^{-h_1}(\mathrm{d}a_3+\sqrt{2}b_1\mathrm{d}a_1)&e^{-h_1}\mathrm{d}b_1\\\mathrm{d}h_1&\frac{1}{\sqrt{2}}e^{-h_2}(e^{h_1}\mathrm{d}a_1+e^{-h_1}A_2)\\\frac{1}{\sqrt{2}}e^{-h_2}(e^{h_1}\mathrm{d}a_1+e^{-h_1}A_2)&\mathrm{d}h_2\end{pmatrix}, \tag{1.68}$$

with $A_2 = \mathrm{d}a_2 + b^2\mathrm{d}a_1 + \sqrt{2}(b_1 \mathrm{a}_3 + b_2 \mathrm{d}a_4)$. The metric is then given by

$$2h_{uv}\mathrm{d}q^u\mathrm{d}q_v = e^{-2(h_1+h_2)}a_{ij}\mathrm{d}a_i\mathrm{d}a_j + e^{-2h_2}((\mathrm{d}b_1)^2 + (\mathrm{d}b_2)^2) + (\mathrm{d}h_1)^2 + (\mathrm{d}h_2)^2 \quad (1.69)$$

with

$$a_{ij} = \begin{pmatrix} (e^{2h_2}+b^2)^2 & b^2 & \sqrt{2}b_1(e^{2h_2}+b^2) & \sqrt{2}b_2(e^{2h_2}+b^2) \\ b^2 & 1 & \sqrt{2}b_1 & \sqrt{2}b_2 \\ \sqrt{2}b_1(e^{2h_2}+b^2) & \sqrt{2}b_1 & e^{2h_2}+2b_1^2 & 2b_1b_2 \\ \sqrt{2}b_2(e^{2h_2}+b^2) & \sqrt{2}b_2 & 2b_1b_2 & e^{2h_2}+2b_2^2 \end{pmatrix}, \quad (1.70)$$

and also the other objects can be calculated from these vielbeins and coset representatives.

1.2.3 Gauge group

As we have already seen in the $\mathcal{N}=1$ case, also in $\mathcal{N}=2$ supergravity the non-trivial transformations of the scalar fields that leave the action invariant are the isometries of the scalar manifold, *i.e.* those transformations that leave the metric

$$\mathbf{g} = \begin{pmatrix} g & \\ & h \end{pmatrix} \quad (1.71)$$

invariant (up to a possible Kähler transformation in the special sector). The group of local symmetries can be trivially an Abelian $U(1)^{n_V+1}$, which transforms the N_V+1 vector fields as

$$\delta A_\mu^\Sigma = \partial_\mu \epsilon^\Sigma \quad (1.72)$$

and leaves all the other fields invariant.

It is possible to have a gauge group G_0 less trivial than $U(1)^{n_V+1}$. This gauge group has to be a subgroup of the isometry group of the scalar manifold and its dimension is at most n_V+1. We can however incorporate also in this case the trivial $U(1)$ factors to have always $\dim G_0 = n_V+1$. The vector fields A_μ^Λ $(\Lambda = 0,1,...,n_V)$ are associated to the generators t_Λ of G_0. Among these vector fields there is also the graviphoton, whose field strength is

$$T_{\mu\nu} = 2i(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}L^\Sigma F_{\mu\nu}^\Lambda + \cdots, \quad (1.73)$$

where the dots stand for contributions that depend on fermions.

We consider from now on the gauge group analyzed in [14],

$$G_0 = SO(2,1)\times G_1 \times \cdots \times G_r, \quad (1.74)$$

with G_k $SO(3)$ or $U(1)$ factors with dimension d_k such that $\sum_{k=1}^{r} d_k = n_V - 2$. Let t_Λ be the generators of the algebra of G_0, where t_x ($x = 1,2,3$) are the ones of the simple non-compact factor $SO(2,1)$, t_{x+3} are the generators of one $SO(3)$ factor (the generalization to more $SO(3)$'s is easy) or $t_{\Lambda_\odot}$ the generator of a $U(1)$ factor. The structure constants of the gauge algebra are given by

$$[t_\Lambda, t_\Sigma] = f_{\Lambda\Sigma}{}^{\Gamma} t_\Gamma, \tag{1.75}$$

namely

$$[t_x, t_y] = e_0\, \epsilon_{xyz}\, (\eta^{(2,1)})^{zw} t_w, \tag{1.76}$$

where $\eta^{(2,1)} = \text{diag}(+1,+1,-1)$ is the metric of $SO(2,1)$ and e_0 the coupling constant. Also

$$[t_{x+3}, t_{y+3}] = e_1 \epsilon_{xyz} t_{z+3}, \tag{1.77}$$

where e_1 is the coupling constant for this gauge factor. The other commutators are all zero.

The symplectic embedding of the adjoint representation of G_0 in the fundamental representation of $Sp(2n_V+2,\mathbb{R})$ is realized by

$$G_0 \ni t_\Lambda \hookrightarrow T_\Lambda = \begin{pmatrix} t_\Lambda & 0 \\ 0 & -t_\Lambda^T \end{pmatrix} \in Sp(2n_V+2,\mathbb{R}). \tag{1.78}$$

Different factors in the gauge group can be embedded in $Sp(2n_V+2,\mathbb{R})$, rotated with respect to each other by means of some angles θ_k, the so called *de Roo-Wagemans angles*, which define a rotation matrix $R(\theta) \in SL(2n_V+2,\mathbb{R})/(SU(1,1)\times SO(2,n_V-1))$ such that $R(\theta)^{-1} G_0 R(\theta) = G_0$. This matrix rotates the symplectic section V as

$$V \longrightarrow R(\theta)V. \tag{1.79}$$

In the case considered above $G_0 = SO(2,1)\times G_1 \times \cdots \times G_r$, $\text{dim}G_k = d_k$,

$$R(\theta) = \left(\begin{array}{cccc|cccc} \mathbb{1}_3 & & & & & & & \\ & \cos\theta_1 \mathbb{1}_{d_1} & & & & \sin\theta_1 \mathbb{1}_{d_1} & & \\ & & \ddots & & & & \ddots & \\ & & & \cos\theta_r \mathbb{1}_{d_r} & & & & \sin\theta_r \mathbb{1}_{d_r} \\ \hline & & & & \mathbb{1}_3 & & & \\ & -\sin\theta_1 \mathbb{1}_{d_1} & & & & \cos\theta_1 \mathbb{1}_{d_1} & & \\ & & \ddots & & & & \ddots & \\ & & & -\sin\theta_r \mathbb{1}_{d_r} & & & & \cos\theta_r \mathbb{1}_{d_r} \end{array}\right). \tag{1.80}$$

For example for $n_V = 3$ and $G_0 = SO(2,1) \times U(1)$,

$$R(\theta) = \left(\begin{array}{cc|cc} \mathbb{I}_3 & & \mathbb{O}_3 & \\ & \cos\theta & & \sin\theta \\ \hline \mathbb{O}_3 & & \mathbb{I}_3 & \\ & -\sin\theta & & \cos\theta \end{array}\right) \tag{1.81}$$

and X^Λ becomes

$$X^\Lambda = \begin{pmatrix} \frac{1}{2}(1+y^2) \\ \frac{i}{2}(1-y^2) \\ y^0 \\ y^1(\cos\theta - S\sin\theta) \end{pmatrix}. \tag{1.82}$$

To describe the action of the gauge group on the scalar manifold we can write down the Killing vectors such that

$$\delta t^I = \epsilon^\Lambda X^I_\Lambda(t), \quad \delta q^u = \epsilon^\Lambda X^u_\Lambda(q), \tag{1.83}$$

with $t^I = (S, y^a)$ $(I = 1, ..., n_V)$, which are related to the so-called moment maps. Some kind of moment map has been introduced in the previous section and called Killing prepotential. Here there is one moment map for the special Kähler sector

$$\mathcal{P}^0_\Lambda = e^K \left\langle \bar{\Omega} | T_\Lambda \Omega \right\rangle, \tag{1.84}$$

such that

$$X^I_\Lambda = i g^{I\bar{J}} \partial_{\bar{J}} \mathcal{P}^0_\Lambda, \tag{1.85}$$

and three for the quaternionic sector $\mathcal{P}^x_\Lambda$, such that

$$2X^u_\Lambda \Omega^x_{uv} = \partial_v \mathcal{P}^x_\Lambda + \epsilon^{xyz} \omega^y_v \mathcal{P}^z_\Lambda. \tag{1.86}$$

They should also satisfy an equivariance relation,

$$2X^u_\Lambda X^v_\Sigma \Omega^x_{uv} - \epsilon^{xyz} \mathcal{P}^y_\Lambda \mathcal{P}^z_\Lambda = f_{\Lambda\Sigma}{}^\Gamma \mathcal{P}^x_\Gamma, \tag{1.87}$$

and

$$X^u_\Lambda \partial_u X^v_\Sigma - X^u_\Sigma \partial_u X^v_\Lambda = -f_{\Lambda\Sigma}{}^\Gamma X^v_\Gamma. \tag{1.88}$$

In the absence of hypermultiplets, the equivariance condition (1.87) reduces to

$$-\epsilon^{xyz} \mathcal{P}^y_\Lambda \mathcal{P}^z_\Lambda = f_{\Lambda\Sigma}{}^\Gamma \mathcal{P}^x_\Gamma \tag{1.89}$$

and $\mathcal{P}^x_\Lambda$ are constants which are differ from zero only for $SO(3)$

$$\mathcal{P}^x_{y+3} = -e_1 \delta^x_y \tag{1.90}$$

and $U(1)$ factors

$$\mathcal{P}^x_\odot = (a, b, c), \quad a, b, c \in \mathbb{R} \quad \text{and} \quad a^2 + b^2 + c^2 = 1. \tag{1.91}$$

These objects are analogous to the constant Fayet-Iliopoulos terms in $\mathcal{N} = 1$ supergravity.

1.2.4 Scalar potential and supersymmetry breaking

The scalar potential is the sum of three different contributions,

$$\begin{aligned} V_1 &= g_{I\bar{J}} k^I_\Lambda k^{\bar{J}}_\Sigma \bar{L}^\Lambda L^\Sigma, \\ V_2 &= 4 h_{uv} k^u_\Lambda k^v_\Sigma \bar{L}^\Lambda L^\Sigma, \\ V_3 &= (U^{\Lambda\Sigma} - 3\bar{L}^\Lambda L^\Sigma) \mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma, \end{aligned} \tag{1.92}$$

where

$$U^{\Lambda\Sigma} = g^{I\bar{J}} f^\Lambda_I \bar{f}^\Sigma_{\bar{J}} = -\frac{1}{2} (\text{Im}\mathcal{N})^{-1\,\Lambda\Sigma} - \bar{L}^\Lambda L^\Sigma \tag{1.93}$$

and

$$f^\Lambda_I = \left(\partial_I + \frac{1}{2} \partial_I K \right) L^\Lambda. \tag{1.94}$$

By definition, the contributions V_1, V_2 are positive definite and the only term that might involve negative contributions is V_3.

In $\mathcal{N} = 2$ supergravity there is an extension of (1.34). Given the fermionic supersymmetry transformation rules,

$$\begin{aligned} \delta\psi_{\mu\alpha} &= \partial_\mu \epsilon_\alpha + i S_{\alpha\beta} \gamma_\mu \epsilon^\beta + \cdots \\ \delta\lambda^{I\alpha} &= W^{I\alpha\beta} \epsilon_\beta + \cdots \\ \delta\zeta_A &= N^\alpha_A \epsilon_\alpha + \cdots \end{aligned} \tag{1.95}$$

where the dots stand for the terms that depend on fermions, vector bosons and derivatives of the scalar fields, the scalar potential can be written as

$$\delta^\alpha_\beta V = -12 \bar{S}^{\alpha\gamma} S_{\gamma\beta} + g_{I\bar{J}} W^{I\,\alpha\gamma} W^{\bar{J}}_{\beta\gamma} + 2 N^\alpha_A N^B_\beta, \tag{1.96}$$

in terms of the fermionic shifts

$$
\begin{aligned}
S_{\alpha\beta} &= \frac{i}{2}(\sigma_x)_\alpha{}^\gamma \epsilon_{\beta\gamma} \mathcal{P}^x_\Lambda L^\Lambda , \\
W^{I\alpha\beta} &= \epsilon^{\alpha\beta} k^I_\Lambda \bar{L}^\Lambda + i(\sigma_x)_\gamma{}^\beta \epsilon^{\gamma\alpha} \mathcal{P}^x_\Lambda g^{I\bar{J}} \bar{f}^\Lambda_{\bar{J}} , \\
N^\alpha_A &= 2E^\alpha_{Au} k^u_\Lambda \bar{L}^\Lambda ,
\end{aligned}
\tag{1.97}
$$

where σ_x are the Pauli matrices and E^α_{Au} a vielbein analogous to E^{mt}_u, where in this case the vector index u is split in $\alpha = 1, 2$ and $A = 1, ..., 2n_H$.

1.3 $\mathcal{N} = 4$ gauged supergravity

This section is devoted to $\mathcal{N} = 4$ gauged supergravity. The basic reference is [17], where references to earlier works can be found, the notation is fixed and the bosonic Lagrangian is given. Here we summarize the ingredients to write down the scalar potential.

The starting point is again the $\mathcal{N} = 4$ supergravity multiplet, made up of a spin-2 boson (*graviton*) e^α_μ, four spin-3/2 fermions (*gravitini*) ψ^i_μ $(i = 1, 2, 3, 4)$, six spin-1 bosons (*graviphotons*) A^m_μ $(m = 1, ..., 6)$, four spin-1/2 fermions χ^i and a complex scalar field S. It can be coupled to n vector multiplets, each of them made up of a spin-1 boson A_μ, four spin-1/2 fermions λ^i and six real scalar fields z^m.

1.3.1 Scalar manifold

The $6n + 2$ scalar fields of the theory span the scalar manifold

$$
\mathcal{M}_{scalar} = \frac{SU(1,1)}{U(1)} \times \frac{SO(6,n)}{SO(6) \times SO(n)} , \tag{1.98}
$$

which is the product of two coset manifolds. The first has complex dimension one and is spanned by the axion-dilaton S, while the second has real dimension $6n$ and is spanned by the other scalars. As usual in the case of a coset manifold G/H, one can introduce a coset representative $L \in G$ such that $gL = Lh$ for every $g \in G$ and $h \in H$. In this case, for the $SU(1,1)/U(1)$ factor, a coset representative is a complex $SU(1,1)$ vector, namely

$$
L_\alpha = \frac{1}{\sqrt{\mathrm{Im}S}} \begin{pmatrix} S \\ 1 \end{pmatrix} , \quad (\alpha = \pm), \tag{1.99}
$$

from which one can also define a symmetric positive-definite matrix

$$M_{\alpha\beta} = \mathrm{Re}(L_\alpha L_\beta^*) = \frac{1}{\mathrm{Im}S}\begin{pmatrix} |S|^2 & \mathrm{Re}S \\ \mathrm{Re}S & 1 \end{pmatrix}. \tag{1.100}$$

For the $SO(6,n)/(SO(6)\times SO(n))$ factor, the coset representative $L_M{}^{\underline{N}} \in SO(6,n)$, $M, \underline{N} = 1, ..., 6+n$, is a $(6+n)\times(6+n)$ matrix whose row index is related to the adjoint representation of $SO(6,n)$, while the column underlined index is related to the adjoint representation of $SO(6)\times SO(n)$. The column index can be split in the vector index $m = 1, ..., 6$, of $SO(6)$ and the vector index $a = 1, ..., n$ of $SO(n)$, such that

$$L = (L_M^m, L_M^a). \tag{1.101}$$

Since $L \in SO(6,n)$, it has to be $L^T\eta L = \eta$, *i.e.*

$$\eta_{MN} = -L_M^m L_N^m + L_M^a L_N^a, \tag{1.102}$$

where $\eta = \mathrm{diag}(+1,+1,+1,+1,+1,+1,-1,...,-1)$ is the $SO(6,n)$ metric. From the coset representative it is possible to define a symmetric positive-definite scalar metric

$$M_{MN} = LL^T = L_M^a L_N^a + L_M^m L_N^m, \tag{1.103}$$

and a completely antisymmetric tensor

$$M_{MNPQRS} = \epsilon_{mnpqrs} L_M^m L_N^n L_P^p L_Q^q L_R^r L_S^s. \tag{1.104}$$

1.3.2 Gauge group

The global symmetry group is the group of the isometries of the scalar manifold, namely

$$G_{global} = SU(1,1)\times SO(6,n). \tag{1.105}$$

The gauge group G_0 has to be a subgroup of G_{global} with dimension at most $6+n$. It can be parametrized by two real constant tensors $f_{\alpha MNP}$ and $\xi_{\alpha M}$, where $\alpha = \pm$ and $M = 1, ..., 6+n$ are vector indices of $SU(1,1)$ and $SO(6,n)$ respectively and its dimension is at most $6+n$. One should construct from these tensors the structure constants of the algebra of the gauge group

$$\begin{aligned} X_{\mathcal{M}\mathcal{N}}^{\mathcal{P}} &= X_{M\alpha N\beta}^{P\gamma} = \delta_\beta^\gamma f_{\alpha MN}{}^P \\ &+\frac{1}{2}(\delta_M^P\delta_\beta^\gamma\xi_{\alpha N} - \delta_N^P\delta_\alpha^\gamma\xi_{\beta M} - \delta_\beta^\gamma\eta_{MN}\xi_\alpha^P + \epsilon^{\alpha\beta}\delta_N^P\xi_{\delta M}\epsilon^{\delta\gamma}). \end{aligned} \tag{1.106}$$

The curly indices $\mathcal{M}, \mathcal{N}, \mathcal{P}$ are composite indices and stand for a vector index of $SO(6,n)$ and one of $SU(1,1)$, *i.e.* $V_{\mathcal{M}} = V_{M\alpha}$ for every $SU(1,1) \times SO(6,n)$ vector V. These structure constants satisfy the following linear constraints

$$X_{\mathcal{M}[\mathcal{N}}{}^{\mathcal{Q}}\Omega_{\mathcal{P}]\mathcal{Q}} = 0, \quad X_{(\mathcal{M}\mathcal{N}}{}^{\mathcal{Q}}\Omega_{\mathcal{P})\mathcal{Q}} = 0, \tag{1.107}$$

with $\Omega_{\mathcal{M}\mathcal{N}} = \Omega_{M\alpha N\beta} = \eta_{MN}\epsilon_{\alpha\beta}$. We can define the generators of the gauge algebra $(X_{\mathcal{M}})_{\mathcal{N}}{}^{\mathcal{P}} = X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ with the following commutation relations

$$[X_{\mathcal{M}}, X_{\mathcal{N}}] = -X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} X_{\mathcal{P}}, \tag{1.108}$$

and then the Jacobi identities impose quadratic constraints on the structure constants. In terms of f and ξ they are:

$$\begin{aligned}
\xi_\alpha^M \xi_{\beta M} &= 0, \\
\xi_{(\alpha}^P f_{\beta)PMN} &= 0, \\
3 f_{\alpha R[MN} f_{\beta PQ]}{}^R &= 0, \\
\epsilon^{\alpha\beta}(\xi_\alpha^P f_{\beta PMN} + \xi_{\alpha M}\xi_{\beta N}) &= 0, \\
\epsilon^{\alpha\beta}(f_{\alpha MNR} f_{\beta PQ}{}^R - \xi_\alpha^R f_{\beta R[M[P}\eta_{Q]N]} - \xi_{\alpha[M} f_{N][PQ]\beta} + \xi_{\alpha[P} f_{Q][MN]\beta}) &= 0.
\end{aligned} \tag{1.109}$$

One may note that the number of generators $X_{\mathcal{M}}$ is twice what we expect. However, imposing linear (1.107) and quadratic constraints (1.109), there are at most $6+n$ independent generators. Each generator is associated to a vector boson, $A_\mu^{\mathcal{M}} = A_\mu^{M\alpha}$. In the ungauged case, only A_μ^{M+}, the electric ones, appear in the Lagrangian, while the magnetic ones A_μ^{M-} are introduced only from the equations of motion. When some gauge group is switched on, the magnetic vectors can appear into covariant derivatives if the associated generators are non-zero, but they have no kinetic terms and so via their equations of motion they eventually turn out to be dual to the electric vector fields. Thus the number of degrees of freedom remains unchanged as compared to the ungauged theory.

It can be shown that as a consequence of the constraints (1.109), for every consistent gauging one can perform a symplectic rotation such that only the electric vectors fields serve as gauge fields [18]. The pure electric gaugings are those for which $f_{-MNP} = 0$ and $\xi_{\alpha M} = 0$, thus only f_{+MNP} does not vanish. None of the pure electric gaugings admit a ground state with non-vanishing cosmological constant, therefore de Roo and Wagemans introduced a further deformation of the theory [19]. Starting from a semi-simple gauging, they introduced a phase for every simple group factor as additional parameters in the description of the gauging.

To explain this idea, one can consider the case of non zero $f_{\pm MNP}$ and $\xi_{\alpha M}=0$. The gauge group one wants to deal with is product of K factors $G_0 = G_1 \times .... \times G_K$ with $f^{(i)}_{MNP}$ being the structure constants of the i-th factor, each of them satisfying the Jacobi identities separately. The solutions of the constraints (1.109) are then given by

$$f_{\alpha MNP} = \sum_{i=1}^{K} w^{(i)}_{\alpha} f^{(i)}_{MNP}, \qquad w^{(i)}_{\alpha} = (w^{(i)}_{+}, w^{(i)}_{-}) = (\cos\alpha_i, \sin\alpha_i), \tag{1.110}$$

where $w^{(i)}_{\alpha}$ are arbitrary $SU(1,1)$ vectors which we could restrict to have unit length, without loss of generality. The $\alpha_i \in \mathbb{R}$ $(i=1,...,K)$ are the so called *de Roo-Wagemans angles*, analogous to those introduced for $\mathcal{N}=2$ in the previous section. If $K=1$ we find f_{+MNP} and f_{-MNP} to be proportional and this case is equivalent to the case of purely electric gauging.

1.3.3 Scalar potential

The scalar potential depends on the generalized structure constants as follows

$$\begin{aligned} V = \; & \tfrac{1}{16}\left\{ f_{\alpha MNP} f_{\beta QRS} M^{\alpha\beta}\left[\tfrac{1}{3}[M^{MQ}M^{NR}M^{PS} - \left(\tfrac{2}{3}\eta^{MQ} - M^{MQ}\right)\eta^{NR}\eta^{PS}\right] \right. \\ & \left. - \tfrac{4}{9} f_{\alpha MNP} f_{\beta QRS}\epsilon^{\alpha\beta} M^{MNPQRS} + 3\xi^{M}_{\alpha}\xi^{N}_{\beta} M^{\alpha\beta} M_{MN} \right\}. \end{aligned} \tag{1.111}$$

It is also possible to write it by using its relation with the fermionic shifts

$$\begin{aligned} A_1^{ij} &= \epsilon^{\alpha\beta}(L_\alpha)^* L^{M}_{[kl]} L^{[ik]}_{N} L^{[jl]}_{P} f_{\beta M}{}^{NP}, \\ A_2^{ij} &= \epsilon^{\alpha\beta} L_\alpha L^{M}_{[kl]} L^{[ik]}_{N} L^{[jl]}_{P} f_{\beta M}{}^{NP} + \frac{3}{2}\epsilon^{\alpha\beta} L_\alpha L^{ij}_{M}\xi^{M}_{\beta}, \\ A_{2ai}{}^{j} &= \epsilon^{\alpha\beta} L_\alpha L_{Ma} L^{N}_{[ik]} L^{[jk]}_{P} f_{\beta MN}{}^{P} - \frac{1}{4}\delta^{j}_{i} L_\alpha L^{M}_{a}\xi_{\beta M}, \end{aligned} \tag{1.112}$$

which appear in the supersymmetry transformation rules for the fermionic fields:

$$\begin{aligned} \delta\psi^{i}_{\mu} &= \partial_\mu \epsilon^i - \frac{2}{3} A_1^{ij}\gamma_\mu \epsilon_j, \\ \delta\chi^i &= -\frac{4}{3} i A_2^{ji}\epsilon_j, \\ \delta\lambda^{i}_{a} &= 2i A_{2aj}{}^{i}\epsilon^j. \end{aligned} \tag{1.113}$$

The relation between the scalar potential and the fermionic shifts is the following:

$$-\frac{1}{4}\delta^{j}_{i} V = \frac{1}{3} A_1^{ik}\bar{A}_{1jk} - \frac{1}{9} A_2^{ik}\bar{A}_{2jk} - \frac{1}{2} A_{2aj}{}^{k}\bar{A}_{2a}{}^{i}{}_{k}. \tag{1.114}$$

Here the coset representative of $SO(6,n)/(SO(6)\times SO(n))$ is written as $(L_M^{[ij]}, L_M^a)$, where $[ij]$ instead of m as row index of the first element denote the antisymmetric representation of $SU(4)\sim SO(6)$.

1.4 $\mathcal{N}=8$ gauged supergravity

In this section, some elements of the maximal supergravity theory in four dimensions [20, 21] are given, with emphasis on the formalism of the embedding tensor [22], introduced here for the first time in this thesis but useful also for the other gauged supergravity theories described above.

$\mathcal{N}=8$ supergravity in four dimensions has eight supercharges, the maximal number of supercharges allowed in four-dimensional theories with local supersymmetry. The theory has only the supergravity multiplet, made up of a graviton e_μ^α, 8 gravitini ψ_μ^m $(m=1,...,8)$, 28 vector bosons A_μ^Λ $(\Lambda=1,...,28)$, 56 spin-1/2 fermions $\chi^{[mnp]}$ and 70 scalar fields $\phi^{[mnpq]}$:

$$(e_\mu^\alpha, \psi_\mu^m, A_\mu^\Lambda, \chi^{[mnp]}, \phi^{[mnpq]}). \tag{1.115}$$

The 70 scalars span a coset manifold

$$\mathcal{M}_{scalar} = E_{7(7)}/SU(8), \tag{1.116}$$

where $E_{7(7)}$ is the isometry group of the scalar manifold. The indices $m, n, ...$ denote the transformations of the fields under the group $SU(8)$. Vector bosons A_μ^Λ and their magnetic duals $A_{\mu\,\Lambda}$, which do not appear in the ungauged Lagrangian, are in the **56** representation of $E_{7(7)}$.

1.4.1 Gauge group and embedding tensor

Despite the presence of only the supergravity multiplet, the $\mathcal{N}=8$ ungauged Lagrangian is not uniquely defined in four dimensions and alternative Lagrangians, not related by local field redefinitions, can be obtained via electric-magnetic duality relations. These transformations constitute the symplectic group $Sp(56,\mathbb{R})$. Lagrangians related by electric-magnetic duality do not share the same symmetry group and therefore they may allow different gaugings. It means that, once the charges are switched on, the possibility of performing electric-magnetic duality are severely restricted and electric charges cannot be converted into magnetic ones via local field redefinitions. The largest subgroup of $E_{7(7)}$ under which the Lagrangian is invariant and all the fields, including the gauge fields, transform, is called electric duality group $G_e \subset E_{7(7)}$. Examples of possible electric duality group are $SL(8,\mathbb{R})$ and its subgroups $SL(7,\mathbb{R})\times SO(1,1)$, $E_{6(6)}\times SO(1,1)$ and $SL(2,\mathbb{R})\times SO(1,1)\times SL(6,\mathbb{R})$ which is exactly $(E_{6(6)} \times SO(1,1)) \cap SL(8,\mathbb{R})$.

The gauge group G_0 has to be a subset of the electric duality group G_e and then of $E_{7(7)}$. This means that it can be defined by selecting a subset of generators $\{X_M\}_{M=1,\ldots,56}$ within the algebra $\{t_\alpha\}_{\alpha=1,\ldots,133}$ of $E_{7(7)}$, given by

$$X_M = \Theta_M^\alpha t_\alpha. \tag{1.117}$$

The constant 56×133 tensor Θ_M^α is the so-called *embedding tensor*, which describes the explicit embedding of the gauge group G_0 into the global symmetry group $E_{7(7)}$. The index M runs over both electric and magnetic charges, but consistency of the gauging requires the rank of Θ to be not grater than 28. This bound is satisfied because an admissible tensor is subjected to linear and quadratic constraints, which ensure that one is dealing with a proper subgroup of $E_{7(7)}$ (quadratic constraints) and that the supergravity action remains supersymmetric (linear constraints). The linear constraints

$$\begin{aligned} (t_\alpha)_M{}^N \Theta_N^\alpha &= 0, \\ (t_\beta t^\alpha)_M{}^N \Theta_N^\beta &= -\frac{1}{2}\Theta_M^\alpha \end{aligned} \tag{1.118}$$

imply that Θ trasforms under $\mathbf{912} \in \mathbf{56} \times \mathbf{133}$.

The quadratic constraints are obtained by requiring the closure of the algebra

$$[X_M, X_N] = -X_{MN}{}^P X_P, \tag{1.119}$$

where $X_{MN}{}^P = \Theta_M^\alpha (t_\alpha)_N{}^P$ are generalized structure constants and can be written in a useful equivalent way as

$$\Theta_M^\alpha \, \Theta_N^\beta \, \Omega^{MN} = 0, \tag{1.120}$$

with $\Omega_{MN} = \Omega^{MN} = \begin{pmatrix} 0 & \mathbb{I}_{28} \\ -\mathbb{I}_{28} & 0 \end{pmatrix}$. These constraints ensure that there always exists a symplectic rotation acting on the index M as a consequence of which all the vectors associated with the generators $X_{\Lambda=1,\ldots,28}$ are all electric (or all magnetic).

1.4.2 Scalar potential

Also in $\mathcal{N} = 8$ supergravity, gauging some subgroup of the electric duality group induces a scalar potential in the Lagrangian. It can be written in terms of the embedding tensor and the scalar fields arranged in a positive definite 56×56 matrix $\mathcal{M}$, defined by

$$\mathcal{M} = LL^T, \tag{1.121}$$

where L is the coset representative of the scalar manifold. The scalar potential is then

$$V = \left(X_{MN}{}^R X_{PQ}{}^S \mathcal{M}^{MP} \mathcal{M}^{NQ} \mathcal{M}_{RS} + 7 X_{MN}{}^Q X_{PQ}{}^N \mathcal{M}^{MP} \right). \tag{1.122}$$

As in all the supergravity theories seen above, the scalar potential can be also written in terms of the fermionic supersymmetry transformations (see *e.g.* [23])

$$\begin{aligned} \delta\psi^i &= \partial_\mu \epsilon^i + \sqrt{2} A_1^{ij} \gamma_\mu \epsilon_j + \cdots , \\ \delta\chi^{ijk} &= -2 A_{2,l}^{ijk} \epsilon^l + \cdots , \end{aligned} \tag{1.123}$$

as

$$V = -\frac{3}{4} |A_1|^2 + \frac{1}{24} |A_2|^2. \tag{1.124}$$

As usual, the only negative contribution comes from the gravitini.

The search for critical points of the scalar potential is very involved, since the minimization conditions are complicated functions of 70 scalar fields. One interesting approach [24] – [26] that will be useful for later purposes consists in reversing the problem: one can search for what embedding tensor the theory has a minimum at the origin of the scalar manifold, where all the scalar fields are zero. The minimization conditions are now quadratic in the embedding tensor. Of course other difficulties arise, because the embedding tensor has to satisfy the linear and quadratic constraints, but now the problem is computationally very simplified and this approach can be usefull to classify all possible gaugings. On the other hand, this approach may seem very restrictive, since apparently only vacua at the origin of the σ-model can be found. However one can actually have the advantage of the properties of coset spaces and of different ways in which a particular choice of the gauge group can be embedded into $E_{7(7)}$ to recover information on the vacia of all the rest of the scalar manifold.

Chapter 2

Flux compactifications of higher-dimensional supergravities

Supergravity theories are usually regarded as low-energy effective limits of string theory or M-theory. The latter are naturally defined in ten or eleven dimensions, as well as their field-theory limits in supergravity. However, if we want to construct (semi)realistic models, we need a procedure to obtain four-dimensional descriptions. Up to now, there are more possible four-dimensional supergravity realizations than those explained by string theory reduction, but the higher-dimensional origin of four-dimensional supergravity models is especially important if we want to make contact between string theory and phenomenology.

There are several ways of obtaining four-dimensional supergravities from higher-dimensional, string-derived models. Compactifications on group manifolds, or certain coset spaces, give effective four-dimensional actions that not only reproduce the original vacuum around which the theory was expanded, but also take into account certain deformations of the geometry of the internal space in a potential for the scalar fields. In these models, part or all of the gauge group of the effective theory is related to the symmetry group of the internal manifold. Other approaches leading to gauged supergravities are Scherk-Schwarz reductions [27, 28], which can be re-interpreted as compactifications on twisted tori [29], and flux compactifications (see *e.g.* [30, 31, 32]).

In this chapter we summarize some results in flux compactifications that will be useful for later purposes. We start in section 2.1 with a quick overview of ten- and eleven- dimensional maximal supergravity, then we recall in section 2.2 some important aspects of Scherck-Schwarz reductions, which are historically the first attempt to perform dimensional reduction in supergravity with a special non-trivial dependence of the fields on the internal coordinates. Flux compactifications are reviewed both in type II theories (section 2.3) and in M-theory (section 2.4), with emphasis on the effective four-dimensional superpotential that emerges after the compactification when geometric and non-geometric fluxes are turned on.

2.1 Supergravities in ten and eleven dimensions

As shown by Nahm [33], supergravity theories can live at most in eleven dimensions, because in more than eleven dimensions the supermultiplets always contain states with helicity greater than two. In this section, a short summary of the maximal supergravity theories in eleven and ten dimensions is given. In eleven dimensions the maximal supergravity is $\mathcal{N}=1$ and it is summarized in subsection 2.1.1. In ten dimensions one can have either $\mathcal{N}=1$ and $\mathcal{N}=2$ supergravity theories. The former allows for the inclusion of the supergravity multiplet and of vector multiplets. Due to the appearance of several supermultiplets, non-maximal supergravity in ten dimensions is not unique. In the following we will concentrate on the maximal case, which contains two supercharges, which are Majorana-Weyl spinors of certain chirality. This allows for two different theories with spinors of either opposite or the same chirality, type IIA and IIB supergravity respectively. These two cases are inequivalent and are summarized in subsection 2.1.2.

2.1.1 Supergravity in eleven dimensions

Eleven-dimensional supergravity [34] contains only the supergravity multiplet. It is made up of an elfbein e_M^A, a Majorana spin-3/2 fermion ψ_M and a completely antisymmetric tensor A_{MNP}. Here $M, N, ...$ are the eleven-dimensional world indices, while $A, B, ...$ are the eleven-dimensional tangent space indices. The degrees of freedom are 256, equally divided between bosons and fermions as required by supersymmetry, in particular the elfbein has 44 degrees of freedom and the antisymmetric tensor has 84.

The Lagrangian, up to four-fermion terms, is

$$\begin{aligned} e^{-1}\mathcal{L} &= -\frac{1}{4}eR - \frac{i}{2}\bar{\psi}_M\gamma^{MNP}D_N\psi_P - \frac{1}{48}F_{MNPQ}F^{MNPQ} \\ &\quad + \frac{1}{96}(\bar{\psi}_M\gamma^{MNPQRS}\psi_N + 12\bar{\psi}^P\gamma^{QR}\psi^S)F_{PQRS} \\ &\quad + \frac{2}{144^2}\epsilon^{M_1\dots M_4N_1\dots N_4PQR}F_{M_1M_2M_3M_4}F_{N_1N_2N_3N_4}A_{PQR}, \end{aligned} \tag{2.1}$$

where $F_{MNPQ} = 4\partial_{[M}A_{NPQ]}$ and $D_M\psi_N = (\partial_M + \frac{1}{4}\omega_M^{AB}\gamma_{AB})\psi_N$. This Lagrangian is invariant under general coordinate and local Lorentz transformations. It is also invariant under the local supertransformations

$$\delta e_M^A = -i\bar{\epsilon}\gamma^A\psi_M, \quad \delta A_{MNP} = \frac{2}{3}\bar{\epsilon}\gamma_{[MN}\psi_{P]},$$

$$\delta\psi_M = D_M\epsilon + \frac{i}{144}(\gamma^{ABCD}{}_M - 8\gamma^{PQR}\delta_M^N)\epsilon F_{NPQR} + (\text{3-fermion terms}), \tag{2.2}$$

with supersymmetry transformation parameter ϵ, and under the Abelian gauge transformation

$$\delta A_{MNP} = \partial_M\zeta_{NP} + \partial_N\zeta_{PM} + \partial_P\zeta_{MN}, \tag{2.3}$$

where $\zeta_{MN} = -\zeta_{NM}$, which leaves all the other fields invariant.

2.1.2 Type IIA and IIB supergravities

There are two inequivalent extended supergravity theories in ten dimensions. The physical states of the two theories belong to different multiplets of $SO(8)$, the group whose representations label helicities of massless states in 10 dimensions. One of the theories, called IIA, contains two inequivalent gravitino multiplets (opposite chirality) and can be obtained by dimensional reduction of eleven dimensional supergravity [35]. The theory has an overall left-right symmetry as a consequence of the non existence of Weyl spinors in eleven dimensions. The other, called IIB, contains two gravitinos of the same chiraity and cannot be formulated in higher dimensions.

We are interested here in summarizing the matter content of these theories and their Lagrangian. The supergravity multiplet is different in both cases. The common bosonic subsector, called Neveu-Schwarz (NS) bosonic subsector, contains the vielbein e_M^A, a rank-two potential B_{MN} and a dilaton ϕ, with $M = 0, 1, ..., 9$ space-time indices and $A = 1, ..., 10$ flat indices. The remaining bosonic part is called Ramond-Ramond (RR) bosonic subsector and only contains rank-d potentials, where d is odd in IIA and even in IIB. For the fermionic sector, in the IIA case the fermions are real and consist of two minimal spinors of both chiralities, while in the IIB case they are complex and contain two minimal spinors of the same chirality.

In the standard formulation, the RR potentials have $d = 1, 3$ for IIA, $d = 0, 2, 4$ for type IIB, and the bosonic field content is

$$\begin{aligned} \text{IIA}: &\quad \{e_M^A, B_{MN}, \phi, C_M^{(1)}, C_{MNP}^{(3)}\}, \\ \text{IIB}: &\quad \{e_M^A, B_{MN}, \phi, C^{(0)}, C_{MN}^{(2)}, C_{MNPQ}^{(4)+}\}. \end{aligned} \tag{2.4}$$

The field strength of the IIB rank-4 potential $C^{(4)+}$ satisfies a self-duality constraint, halving the number of degrees of freedom. There is also a special formulation of IIA and IIB supergravity [36] that emphasizes the equivalence of dual R-R potentials by including all the odd or even R-R potentials. The bosonic field content of IIA and IIB supergravity reads, in this 'democratic' formulation:

$$\begin{aligned} \text{IIA}: &\quad \{e_M^A, B_{MN}, \phi, C_M^{(1)}, C_{MNP}^{(3)}, C_{M_1\ldots M_5}^{(5)}, C_{M_1\ldots M_7}^{(7)}\}, \\ \text{IIB}: &\quad \{e_M^A, B_{MN}, \phi, C^{(0)}, C_{MN}^{(2)}, C_{MNPQ}^{(4)}, C_{M_1\ldots M_6}^{(6)}, C_{M_1\ldots M_8}^{(8)}\}. \end{aligned} \tag{2.5}$$

To get the correct number of degrees of freedom, one must by hand impose duality relations between the field strengths of rank-d and rank-$(8-d)$ potentials, which read

$$G^{(d+1)} = (-1)^{[(d+1)/2]} e^{(d-4)\phi/2} \star G^{(9-d)}, \qquad G^{(d+1)} = dC^{(d)} - H \wedge C^{(d-2)}, \tag{2.6}$$

for vanishing fermions and $H = dB$. The bosonic action is then

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{-\phi} H \cdot H - \sum_d \frac{1}{4} e^{(4-d)\phi/2} G^{(d+1)} \cdot G^{(d+1)}, \tag{2.7}$$

subject to the duality relations (2.6). Note that one can include a a 9-form potential $C^{(9)}$ in (2.5), which carries no degrees of freedom but it is very natural from the point of view of R-R equivalence. The corresponding field strength trivially satisfies the Bianchi identity. Its Hodge dual is a rank-zero field strength, which has no corresponding potential nor a field equation. Its Bianchi identity implies that it is a constant. Thus we have effectively introduced a mass parameter in the theory, given by

$$G^{(0)} = e^{-5\phi/2} \star G^{(10)}. \tag{2.8}$$

The corresponding action is given by (2.7) with the field strengths

$$G^{(d+1)} = dC^{(d)} - H \wedge C^{(d-2)} + \frac{1}{(d+1)/2} G^{(0)} B \wedge \ldots \wedge B. \tag{2.9}$$

Coming back to the standard (and massless) formulation, the bosonic Lagrangian becomes

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{IIA}} = \quad & R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{-\phi} H \cdot H - \textstyle\sum_{d=1,3} \tfrac{1}{2} e^{(4-d)\phi/2} G^{(d+1)} \cdot G^{(d+1)} \\ & - \tfrac{1}{2} \star (dC^{(3)} \wedge dC^{(3)} \wedge B) \end{aligned} \tag{2.10}$$

for type IIA supergravity, and

$$e^{-1}\mathcal{L}_{\rm IIB} = \; R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{-\phi}H\cdot H - \textstyle\sum_{d=0,2,4}\tfrac{1}{2}e^{(4-d)\phi/2}G^{(d+1)}\cdot G^{(d+1)} + \tfrac{1}{2}\star(C^4\wedge dC^{(2)}\wedge H), \qquad (2.11)$$

supplemented with the self-duality relation (2.6) for $d=4$ for type IIB. Note that, with respect to the action in the democratic formalism, the kinetic terms have as canonical coefficient $1/2$ instead of the $1/4$ in (2.7) and there are now explicit Chern-Simons terms (the last line in (2.10) and (2.11)).

2.2 Scherk-Schwarz compactifications

As we have already discussed, if one is interested in (semi)realistic models, the ten- or eleven-dimensional supergravity theories must be reduced to four-dimensional supergravity by means of some dimensional reduction mechanism. An interesting mechanism of coordinate-dependent dimensional reduction in supergravity was introduced by Scherk and Schwarz in the seminal papers [27, 28], where they provide an elegant solution to the problem of spontaneous breaking of supersymmetry in extended supergravities. In fact, for simple supergravity the mechanism for spontaneous supersymmetry breaking could consist of coupling simple supergravity to a scalar multiplet such that spin-1/2 field is absorbed by the gravitino, yielding a super-Higgs effect. This solution is not applicable to the extended supergravities with $\mathcal{N}>4$, for which no matter multiplets exist.

The basic idea of dimensional reduction is to consider the space-time manifold in $D+E$ dimensions $\mathcal{M}^{D+E}$ as the product of a D dimensional space-time $\mathcal{M}^D$ and an E-dimensional internal space $\mathcal{K}_E$

$$\mathcal{M}^{D+E} = \mathcal{M}^D \times \mathcal{K}_E. \qquad (2.12)$$

We are interested in the case with maximally symmetric $\mathcal{M}^D$, which means that it could be only Minkowski, Mkw_D, or de Sitter, dS_D, or Anti-de Sitter, AdS_D. The most general $(D+E)$-dimensional metric consistent with four dimensional maximal symmetry is

$$ds^2 = e^{2A(Y)}\tilde{g}_{\mu\nu}dX^\mu dX^\nu + g_{mn}dY^m dY^n, \qquad (2.13)$$

where X^μ ($\mu=0,1,...,D-1$) are four-dimensional space-time coordinates, Y^m ($m=1,...,E$) are internal coordinates, $A(Y)$ is called *warp factor*, $\tilde{g}_{\mu\nu}(X)$ is a Mkw_D, dS_D or AdS_D metric and $g_{mn}(X)$ is any six internal metric.

One important point in this context is the identification of the compact space. Some of its properties are fixed by requirements on the reduced theory, in particular, requiring a residual local supersymmetry in four dimensions fixes the holonomy group of the manifold. The simplest case is the torus $\mathbb{T}^E$, with trivial holonomy. Other interesting cases are given in this table:

Manifold $\mathcal{K}$	$\dim(\mathcal{K})$	$Hol(\mathcal{K})$
$\mathbb{T}^E$	E	$\mathbb{I}$
CY_n	$2n$	$SU(n)$
$\mathcal{K}_{G_2}$	7	G_2

(2.14)

Dimensional reduction of type II theories or M-theory on $\mathbb{T}^6$ gives rise respectively to $\mathcal{N}=8, D=4$ and $\mathcal{N}=8, D=5$ supergravity; compactifications of type II theories on Calabi-Yau manifolds of complex dimension $n=3$ give $\mathcal{N}=2, D=4$ supergravities and compactifications of M-theory on seven-dimensional manifolds with G_2-holonomy give $\mathcal{N}=1, D=4$ supergravity. For the moment we will consider the case of $\mathcal{N}=8$ four-dimensional effective theories, obtained by compactifying on a torus or a twisted torus, and in the next section we will analyze the effective $\mathcal{N}=1$ theories.

2.2.1 Generalized dimensional reduction

In the simplest case of dimensional reduction, all the fields and transformation laws in the original theory are independent of the internal coordinates Y^m

$$\phi(X^M)=\phi(X^\mu). \tag{2.15}$$

The vielbein e_M^A, where M, A are respectively the $(D+E)$ dimensional space-time and flat space indices, reduces in D dimensions, after an ordinary dimensional reduction, to

$$e_M^A(x,y)=\begin{pmatrix} \delta^{-1/(D-2)}e_\mu^\alpha(x) & 2A_\mu^m(x)\Phi_m^a(x) \\ 0 & \Phi_m^a(x) \end{pmatrix}, \tag{2.16}$$

where $\delta=\det\Phi$ and the space-time indices M split into the D-dimensional space-time indices μ and the internal space indices m, while the flat space indices A splits into $\alpha=1,...,D$ and $a=1,...,E$. The bosonic content of the reduced theory is then the four-dimensional vielbein e_μ^α, E vector fields A_μ^m and E^2 scalar fields Φ_m^a, of which only $\frac{1}{2}E(E+1)$ correspond to propagating particles.

Under general coordinate transformations, in $(D+E)$ dimensions

$$\delta e_M^A=\xi^N\partial_N V_M^A+\partial_M\xi^N V_N^A, \tag{2.17}$$

with ξ^M infinitesimal parameter. Assuming that ξ^M is Y-independent, the transformations in the reduced D-dimensional theory are the usual general coordinate transformations with infinitesimal parameter ξ^μ, while for $\xi^m(X)$

$$\delta e_\mu^\alpha = \delta\Phi_m^a = 0, \qquad \delta A_\mu^m = \frac{1}{2}\partial_\mu \xi^m, \tag{2.18}$$

which is $(U(1))^E$ invariance.

A limitation of this approach is that, starting from a massless theory, the resulting reduced theory is also massless (aside from possible masses due to the Higgs mechanism or topological excitations). Also, it is exactly invariant under the extended supersymmetry algebra, which needs to be broken to describe the real world.

A generalization of this method, formulated by Scherk and Schwarz in [27, 28], makes it possible to obtain a theory with mass parameters and spontaneously broken supersymmetry. The new ingredient is to allow the fields and transformation laws to depend upon the internal coordinates Y in a well-defined fashion and must satisfy a number of criteria. First of all, it should be possible to define a limit in which the fields become Y-independent and the results of the ordinary case are recovered. This implies that the number of physical modes is unchanged. Moreover, the Y-dependence of fields and transformation laws must be of a simple factorisable form that can be factored out of the transformation laws. Finally, the reduced theory should contain no ghost particles.

The generalized reduction is uniquely specified by choosing the Y dependence of the infinitesimal parameter $\xi^M(X, Y)$. The choice should be made in such a way as to describe the general coordinate transformations in D-dimensional space-time and local transformations such that the $A_\mu^m(X)$ are gauge fields for as a general Lie group as possible. If we set

$$\begin{aligned} \xi^\mu(X,Y) &= \xi^\mu(X), \\ \xi^m(X,Y) &= (U^{-1}(Y))^m{}_n \xi^n(X), \end{aligned} \tag{2.19}$$

then we recover the results of the ordinary case in the limit in which the non-singular $E \times E$ matrix U approaches unity.

This translates into a modification of the internal metric $g_{mn}(X)$ as $g'_{m'n'}(X,Y) = U_{m'}{}^m(Y) U_{n'}{}^n(Y) g_{mn}(X)$ and

$$ds^2 = e^{-A(Y)} \tilde{g}_{\mu\nu} dX^\mu dX^\nu + g'_{mn} \eta^m \eta^n, \tag{2.20}$$

with η^m the new vielbein 1-form. These vielbeins describe a space that is a deformation of the original torus $\mathbb{T}^E$ and for this reason this reduction is also known as *compactification on a twisted torus.* The vielbein η^m satisfies the relation

$$d\eta^m = -\frac{1}{2}\omega^m_{np}\eta^n\eta^p, \tag{2.21}$$

with ω^m_{np} constant. This kind of compactification induces a gauge group on the reduced theory, with an algebra defined by

$$[Z_m, Z_n] = \omega^p_{mn} Z_p, \tag{2.22}$$

The constants ω are the structure constants of the gauge group and can be written in terms of the U's as

$$\omega^m_{np} = (U^{-1})^{n'}{}_n (U^{-1})^{p'}{}_p (\partial_{p'} U_{n'}{}^m - \partial_{n'} U_{p'}{}^m). \tag{2.23}$$

The essential point is that the U's must have the property that the Y-dependence cancels in (2.23) to give ω's that are constants. Another condition over the $U's$, related to the request of the invariance of the D-dimensional Lagrangian under the $\xi^m(x, y)$ transformations, essential for decoupling ghost components of the vector fields

$$\partial_m((U^{-1})^m{}_n U) = 0, \tag{2.24}$$

with $U(y) = \det U_m{}^n(y)$. It is easy to see that (2.24) is equivalent to

$$\omega^m_{mn} = 0. \tag{2.25}$$

For all the other D-dimensional fields one has

$$\begin{aligned} e^\alpha_\mu(X, Y) &= e^\alpha_\mu(X), \\ A^m_\mu(X, Y) &= (U^{-1}(Y))^m{}_n A^n_\mu(X), \\ \Phi^a_m(X, Y) &= U_m{}^n(Y)\Phi^a_n(X), \end{aligned} \tag{2.26}$$

and the transformation laws become

$$\begin{aligned} \delta\Phi^a_m(X) &= \omega^p{}_{mn}\xi^n(X)\Phi^a_p(X), \\ \delta\Phi^m_a(X) &= \omega^m{}_{np}\xi^n(X)\Phi^p_a(X), \\ \delta A^m_\mu(X) &= \frac{1}{2}\partial_\mu\xi^m(X) + f^m{}_{np}\xi^n(X)A^p_\mu(X), \\ \delta e^\alpha_\mu(X) &= 0. \end{aligned} \tag{2.27}$$

Thus the $A^m_\mu(X)$ are gauge potentials for the gauge group G with structure constants $\omega^m{}_{np}$.

After the reduction we recover a D-dimensional scalar potential, which depends on the scalar fields via $g_{mn} = \Phi^a_m \delta_{ab} \Phi^b_n$, a positive-definite metric for the internal space. Aside from a positive coefficient, it is explicitly given by

$$V(h) = 2\omega^m_{np}\omega^n_{mq}g^{pq} + \omega^m_{np}\omega^{m'}_{n'p'}g_{mm'}g^{nn'}g^{pp'}. \qquad (2.28)$$

Whenever G is a semisimple group this potential is unbounded from below, and then this possibility has to be abandoned. Requiring that V be a non-negative function which vanishes for $g_{mn} = \eta_{mn}$, we have further conditions for the structure constants

$$\begin{aligned} V_0 &= 2\omega^m_{np}\omega^n_{mp} + \omega^m_{np}\omega^m_{np} = 0, \\ \left.\frac{\partial V}{\partial g^{mn}}\right|_0 &= 2\omega^{m'}_{n'm}\omega^{n'}_{m'n} + 2\omega^{m'}_{n'm}\omega^{m'}_{n'n} - \omega^m_{m'n'}\omega^n_{m'n'} = 0, \\ V(h) &\geq 0. \end{aligned} \qquad (2.29)$$

The groups whose structure constants satisfy these further conditions are called *flat groups*. Examples of flat groups can be generated by a technique that was suggested by Scherk and Schwarz in [27, 28]. In particular, these authors analyzed the case in which eleven-dimensional supergravity is reduced to four-dimensional $\mathcal{N} = 8$ supergravity, The fields and transformation laws depend only on Y^1. Setting

$$U_n{}^m(Y) = (\exp MY^1)^m_n, \qquad (2.30)$$

where M is an antisymmetric traceless 7×7 matrix with zeros in the first row and column. Substituting this expression in (2.23) gives

$$\omega^p_{mn} = M^p_m\delta^1_n - M^p_n\delta^1_m. \qquad (2.31)$$

In other worlds, the algebra is given by

$$[Z_1, Z_m] = -M^n_m Z_n, \qquad [Z_m, Z_n] = 0, \quad m, n \neq 1. \qquad (2.32)$$

The potential expressed in terms of M is now

$$V(h) = 2g^{11}\mathrm{Tr}(M^2 - MgMg^{-1}) - 2M^m_n M^{m'}_{n'} g_{mm'}g^{n1}g^{n'1}. \qquad (2.33)$$

2.2.2 Scherk-Schwarz breaking with three parameters

As a specific example, we consider eleven-dimensional supergravity and its dimensional reduction to four dimensions. The flat group is specified by a simple matrix M having rank 6. By an orthogonal transformation, which represents no restriction on the theory, M can be brought to the canonical form

$$M = \begin{pmatrix} 0 & & 0 & & & & & \\ & \begin{matrix} 0 & m_1 \\ -m_1 & 0 \end{matrix} & & 0 & & & 0 & \\ 0 & 0 & & \begin{matrix} 0 & m_2 \\ -m_2 & 0 \end{matrix} & & & 0 & \\ & 0 & & 0 & & \begin{matrix} 0 & m_3 \\ -m_3 & 0 \end{matrix} & & \end{pmatrix} . \tag{2.34}$$

We can then calculate the spectrum of the theory. Note that all masses have the same field dependence, via a universal scaling factor associated to flat directions of the scalar potential. Qualitatively,

$$M_i(\phi) = \frac{\mu_i}{\phi} , \tag{2.35}$$

where ϕ can be read from the $D = 4$ Lagrangian. In the following, such a field dependence will be always understood, unless otherwise stated. The mass spectrum is given by [27, 28]

spin	mass	degeneracy	Number of real fields
2	0	1	1
$\frac{3}{2}$	$\frac{1}{2}\lvert m_1 \pm m_2 \pm m_3\rvert$	2	8
1	0	4	4
	m_i	4	12
	$\lvert m_i \pm m_j\rvert$	2	12
$\frac{1}{2}$	$\left\lvert \frac{1}{2}\lvert m_1 \pm m_2 \pm m_3\rvert \pm m_i \right\rvert$	2	48
0	0	8	8
	m_i	4	12
	$2m_i$	2	6
	$\lvert m_i \pm m_j\rvert$	2	12
	$\lvert m_1 \pm m_2 \pm m_3\rvert$	2	8

2.2.3 Scherk-Schwarz breaking with four parameters

An interesting improvement of these results is the introduction of a fourth parameter. The basic idea is to start from eleven-dimensional supergravity, first reduce it to five dimensions by ordinary dimensional reduction, then perform a generalized dimensional reduction from five to four dimensions, in such a way that one can introduce four parameters m_i that give masses to the fields. We start with the eleven-dimensional supergravity as in the previous section. The bosonic matter content is

$$E_M^A(X),\ A_{MNP}(X), \tag{2.36}$$

with $M = 0, 1, ..., 10$ space-time index and $A = 1, ..., 11$ flat space index. In the ordinary dimensional reduction the index M splits into the five-dimensional space-time index $\hat{\mu} = 0, 1, ..., 4$ and the internal space index $\hat{\alpha} = 2, ..., 7$, while A splits into $\hat{m} = 1, ..., 5$ and $\hat{a} = 2, ..., 7$. All the fields and transformation laws do not depend on all the eleven space-time coordinates X^M but only on the five-dimensional ones $\hat{x}^{\hat{\mu}}$, while the internal coordinates $\hat{y}^{\hat{\alpha}}$ do not appear in the Lagrangian. The vielbein becomes

$$E_M^A(\hat{x},\hat{y}) = \begin{pmatrix} \hat{\delta}^{-1/3} e_{\hat{\mu}}^{\hat{\alpha}}(\hat{x}) & 2B_{\hat{\mu}}^{\hat{m}}(\hat{x})\Phi_{\hat{m}}^{\hat{a}}(\hat{x}) \\ 0 & \Phi_{\hat{m}}^{\hat{a}}(\hat{x}) \end{pmatrix}, \tag{2.37}$$

where $\hat{\delta} = \det \Phi$, $e_{\hat{\mu}}^{\hat{\alpha}}(\hat{x})$ is the five-dimensional vielbein, $B_{\hat{\mu}}^{\hat{m}}(\hat{x})$ are six gauge fields and $\Phi_{\hat{m}}^{\hat{a}}(\hat{x})$ are 36 scalar fields, of which only 21 are independent. The three form $A_{MNP}(X)$ splits into

$$A_{\hat{\mu}\hat{\nu}\hat{\rho}}(\hat{x}),\ A_{\hat{\mu}\hat{\nu}\hat{p}}(\hat{x}),\ A_{\hat{\mu}\hat{n}\hat{p}}(\hat{x}),\ A_{\hat{m}\hat{n}\hat{p}}(\hat{x}), \tag{2.38}$$

where the last ones are 20 scalar fields, $A_{\hat{\mu}\hat{n}\hat{p}}$ are six gauge fields and $A_{\hat{\mu}\hat{\nu}\hat{\rho}}$, $A_{\hat{\mu}\hat{\nu}\hat{p}}$ are one three-form and six two-forms whose duals are respectively a scalar field $\tilde{A}$ and six gauge fields $\tilde{A}_{\hat{\mu}\hat{m}}$. Summarizing, the five-dimensional theory obtained by dimensional reduction from eleven-dimensional supergravity contains as bosonic fields the vielbein $e_{\hat{\mu}}^{\hat{\alpha}}$, $(6{+}15{+}6) = 27$ gauge fields $(B_{\hat{\mu}}^{\hat{m}}, A_{\hat{\mu}\hat{n}\hat{p}}, \tilde{A}_{\hat{\mu}\hat{m}})$ and $(20{+}21{+}1) = 42$ scalar fields $(\Phi_{\hat{m}}^{\hat{a}}, A_{\hat{m}\hat{n}\hat{p}}, \tilde{A})$. This is the theory constructed by Cremmer, Scherk and Schwarz in [37] and further studied in [38, 39]. One can note that the isometry group of the scalar manifold is $E_{6(6)}$ and its maximal compact subgroup is $Sp(8)$, which has rank 4. We can write all the fields in terms of representations of $Sp(8)$: $e_{\hat{\mu}}^{\hat{\alpha}}$ is a singlet, the eight gravitini $\psi_{\hat{\mu}}^i$ are in an octet, the 27 gauge fields $A_{\hat{\mu}}^{ij}$ are in the **27**, the 48 spin-1/2 fermions χ^{ijk} are in the **48** and the 42 scalars ϕ^{ijkl} are in the **42** representations of $Sp(8)$. They are written as in the four-dimensional case but the number of real fields and the group under which they transform changes. Since $Sp(8)$ is symplectic, all the fields have to be symplectic traceless

$$\Omega_{ij} A_\mu^{ij} = \Omega_{ij}\chi^{ijk} = \Omega_{ij}\phi^{ijkl} = 0, \tag{2.39}$$

where Ω_{ij} is the antisymmetric metric of $Sp(8)$.

In the dimensional reduction from five to four dimensions, the five-dimensional space-time coordinates $\hat{\mu}$ can split into four-dimensional coordinates x^μ ($\mu = 0, 1, 2, 3$) and the internal coordinate y. To perform a generalized dimensional reduction one could define the following element of the group $Sp(8)$

$$U^i_j(y) = (\exp My)^i_j, \tag{2.40}$$

where M is an 8×8 matrix of the $Sp(8)$ algebra given, in a particular basis, by

$$M = \begin{pmatrix} m_1\epsilon & & & \\ & m_2\epsilon & & \\ & & m_3\epsilon & \\ & & & m_4\epsilon \end{pmatrix}, \qquad \text{with} \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.41}$$

Then, the y-dependence of the fields is given by

$$\begin{aligned} e^{\hat{\alpha}}_{\hat{\mu}}(x,y) &= e^{\hat{\alpha}}_{\hat{\mu}}(x), \\ \psi^i_{\hat{\mu}}(x,y) &= U^i_{i'}(y)\psi^{i'}_{\hat{\mu}}(x), \\ A^{ij}_{\hat{\mu}}(x,y) &= U^i_{i'}(y)U^j_{j'}(y)A^{i'j'}_{\hat{\mu}}(x), \\ \chi^{ijk}(x,y) &= U^i_{i'}(y)U^j_{j'}(y)U^k_{k'}(y)\chi^{i'j'k'}(x), \\ \phi^{ijkl}(x,y) &= U^i_{i'}(y)U^j_{j'}(y)U^k_{k'}(y)U^l_{l'}(y)\phi^{i'j'k'l'}(x). \end{aligned} \tag{2.42}$$

We also have that

$$e^{\hat{\alpha}}_{\hat{\mu}} = \begin{pmatrix} e^{-\varphi/\sqrt{3}}e^\alpha_\mu & 2e^{2\varphi/\sqrt{3}}B_\mu \\ 0 & e^{2\varphi/\sqrt{3}} \end{pmatrix}, \tag{2.43}$$

and

$$\psi_{\hat{\mu}} = (\psi^{ij}_\mu, \psi^{ij}_5), \qquad A^{ij}_{\hat{\mu}} = (A^{ij}_\mu, A^{ij}_5). \tag{2.44}$$

Summarizing, the field content of the reduced theory is a vielbein e^α_μ, 8 gravitini ψ^{ij}_μ, $27+1 = 28$ gauge fields $A^{ij}_\mu + B_\mu$, $48+8 = 56$ spin-1/2 fermions $\chi^{ijk} + \psi^i_5$ and $42+27+1 = 70$ scalar fields $\phi^{ijkl} + A^{ij}_5 + \varphi$. The model is a $N = 8, d = 4$ gauged supergravity, which was already described previously. The fields can be rewritten as $e^\alpha_\mu, \psi^I_\mu, A^{IJ}_\mu, \chi^{IJK}, \phi^{IJKL}$, where now the indices $I, J, ... = 1, ..., 8$ are related to representations of $SU(8)$ rather than $Sp(8)$. The algebra of the gauge group is now $U(1) \ltimes \mathcal{T}^{24}$ as described in (2.32).

We can calculate the spectrum of the theory. As in the previous subsection, we will ignore the universal field-dependence of the masses along the flat directions of the potential, unless otherwise stated. Without giving the details, the spectrum of the theory is given by

spin	mass	degeneracy	Number of fields
2	0	1	1
$\frac{3}{2}$	$\lvert m_i\rvert$	2	8
1	0	4	4
	$\lvert m_i \pm m_j\rvert$	2	24
$\frac{1}{2}$	0	(8)	(8)
	$\lvert m_i\rvert$	4	16
	$\lvert m_i \pm m_j \pm m_k\rvert$	2	32
0	0	6(30)	6(30)
	$\lvert m_i \pm m_j\rvert$	4	24
	$\lvert m_1 \pm m_2 \pm m_3 \pm m_4\rvert$	2	16

(2.45)

where here $i < j$ or $i < j < k$ and there are 24 would-be Goldstone bosons associated with the 24 massive vectors, and 8 would-be Goldstinos associated with the 8 massive gravitinos.

2.3 Type II flux compactifications

The ω^p_{mn} constants that define the twisted torus can be reinterpreted as particular fluxes, called geometric, in the more general framework of flux compactifications. The idea here [2] is to consider the most general case of flux compactifications, with geometric and non geometric fluxes, and to find consistent choices of fluxes that may give compactifications a la Scherk-Schwarz. In this section we consider type-II compactifications to $D = 4$ supergravity, paying attention to the reduction to $\mathcal{N} = 1$ supersymmetry.

The key point in string compactification is the breaking of supersymmetry. Although we wish it to be broken, it is important that at least part of it survives at energy lower than the string scale, in order to have supergravity as effective field theory. For these reasons we constrain ourselves to theories which preserve at least one supersymmetry at scales below the string scale. There are several ways to break supersymmetry in string compactifications, by considering more involved structures than simple tori. For example we can think of filling the space with some solitonic objects, namely D-branes and O-planes. Dp-branes are $(p+1)$-dimensional, dynamical, BPS objects, that usually fill the whole four-dimensional space-time and wrap

$(p-3)$-cycles in the internal space. They are also sources for RR $(p+1)$-forms. It means that, since only odd (even)-rank RR forms are present in type IIA (IIB) string, only p even (odd) Dp-branes will present.

We can also consider singular spaces, called orbifolds, obtained from a manifold such as a six-torus $\mathbb{T}^6$ by modding out the action of a non-freely-acting discrete group. The discrete group has a non-trivial action on the internal coordinates and on the supersymmetry charges, with consequent reduction of supersymmetry. Examples of orbifold are $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{T}^6/\mathbb{Z}_3$ and $\mathbb{T}^6/\mathbb{Z}_6$. Orbifold fixed points are singular points, but these singularities have to be understood as a singular limit of a smooth manifold. For instance, many toric orbifold can be seen as a special limit of Calabi-Yau manifolds .

In type II theories it is also possible to have orientifold compactifications. In this case the discrete symmetry Ω which exchanges left- and right-movers, is gauged. Field parities under Ω are

$$\begin{aligned} \Omega = + \ &: \ G_{MN},\, \Phi, \ \ C^{(2)}, C^{(6)}, C^{(3)}, C^{(7)}, \\ \Omega = - \ &: \ \ B_{MN}, \ \ \ C^{(0)}, C^{(4)}, C^{(8)}, C^{(1)}, C^{(5)}, C^{(9)}. \end{aligned} \tag{2.46}$$

In general Ω can be accompanied also by a target-space involution σ_n, which acts as an inversion of n coordinates. The full orientifold parity thus reads

$$\mathcal{O} = \Omega\sigma_n[(-1)^{\mathbf{F}}]^{[n/2]}, \tag{2.47}$$

where $[n/2]$ is the integer part of $n/2$ and $(-1)^{\mathbf{F}}$ is the fermionic counting operator, which is $+1$ for the NSNS sector and -1 for the RR sector.

The orientifold can also be combined with orbifolds. In this case, the orientifold reduction can be visualized geometrically in terms of $(10-n)$-dimensional, non-dynamical objects, called Op-planes ($p = 9-n$), sitting in the fixed planes of the orbifold and having negative tension and RR charge.

For better understanding, we may concentrate on a single example, used in some papers, for example [40]. We consider a six-torus factorized in three sub-tori $\mathbb{T}^2_I$ $(I = 1,2,3)$:

$$\mathbb{T}^6 = \mathbb{T}^2_1 \times \mathbb{T}^2_2 \times \mathbb{T}^2_3, \tag{2.48}$$

each of them with coordinates (y^a, y^i), with $a = 2I-1$ and $i = 2I$. To fix the conventions, we denote with a, b, c the odd indices, with i, j, k the even indices, and (a,i), (b,j), (c,k) denotes the indices of the coordinates of the sub-torus I, J, K.

A basis of closed 3-forms is then

$$\begin{aligned}
\alpha_0 &= dy^1 \wedge dy^3 \wedge dy^5 \quad ; \quad \beta_0 = dy^2 \wedge dy^4 \wedge dy^6 \ , \\
\alpha_1 &= dy^1 \wedge dy^4 \wedge dy^6 \quad ; \quad \beta_1 = dy^2 \wedge dy^3 \wedge dy^5 \ , \\
\alpha_2 &= dy^2 \wedge dy^3 \wedge dy^6 \quad ; \quad \beta_2 = dy^1 \wedge dy^4 \wedge dy^5 \ , \\
\alpha_3 &= dy^2 \wedge dy^4 \wedge dy^5 \quad ; \quad \beta_3 = dy^1 \wedge dy^3 \wedge dy^6 \ ,
\end{aligned} \tag{2.49}$$

with normalization $\int_{\mathbb{T}^6} \alpha_I \wedge \beta_J = -\delta_{IJ}$. The closed 2-forms and their dual 4-forms have basis:

$$\omega_I = -dy^a \wedge dy^i \, ; \quad \tilde{\omega}_I = dy^b \wedge dy^j \wedge dy^c \wedge dy^k \, ; \quad a \neq b \neq c, \ i \neq j \neq k \ . \tag{2.50}$$

Notice that $\int_{\mathbb{T}^6} \omega_I \wedge \tilde{\omega}_J = -\delta_{IJ}$.

The orbifold $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ can be obtained by imposing two discrete $\mathbb{Z}_2$ symmetries, under which the y's have parities

$$\begin{aligned}
\mathbb{Z}_2 :& \quad \{- \ - \ - \ - \ + +\} \, , \\
\mathbb{Z}_2' :& \quad \{+ \ + \ - \ - \ - -\} \, .
\end{aligned} \tag{2.51}$$

We then combine this orbifold with an orientifold

$$\mathcal{O} = \Omega \sigma_n [(-1)^{\mathbf{F}}]^{[n/2]}, \tag{2.52}$$

for which the involution σ_n is

$$\begin{aligned}
\text{IIA} :& \quad \sigma_3 = \{- \ + \ - \ + \ - +\} \, , \\
\text{IIB} :& \quad \sigma_6 = \{- \ - \ - \ - \ - -\} \, .
\end{aligned} \tag{2.53}$$

These cases are allowed to have D3-branes and O3-planes in IIB and D6-branes and O6-planes in IIA. The theory that emerges is $\mathcal{N} = 1$ supergravity.

The moduli fields have a different realization in terms of ten-dimensional degrees of freedom. In type IIA, S is as axion-dilaton, the real parts of T_I are Kähler moduli and the real parts of U_I are complex structure moduli, while in IIB the rôle of U_I and T_I is exchanged. In the ordinary dimensional reduction on a Calabi-Yau manifold they are all massless, however one can consider more involved backgrounds which induce some scalar potential. In particular the internal components of the p-form field strengths may have non-zero vacuum expectation values. Upon integrating the field strength over a $(p+1)$-dimensional manifold Σ_{p+1} without a boundary the charge has to be integer

$$\frac{1}{l_s^p}\int_{\Sigma_{p+1}} F_{p+1} \in \mathbb{Z}. \tag{2.54}$$

This integer is called *flux* and has to satisfy various constraints, already known and summarized in various papers (see, for example, [40] –[44]).

In IIA, H_3, F_2, F_6 are odd under the orbifold involution, while F_0, F_4 are even, thus the allowed fluxes are

$$H_{135} = h_0, \quad H_{ajk} = h_I,$$
$$F_0 = -m, \quad F_{ai} = q_I, \quad F_{aibj} = e_K, \quad \bar{F}_6 = F_{123456} = -m. \tag{2.55}$$

In IIB, the allowed fluxes are

$$H_{135} = h_0, \quad H_{ibc} = -a_I, \quad H_{ajk} = -\bar{a}_I, \quad H_{246} = \bar{h}_0,$$
$$F_{135} = -e_0, \quad F_{ibc} = e_I, \quad F_{ajk} = -q_I, \quad F_{246} = -m, \tag{2.56}$$

where both H_3 and F_3 are odd under involution.

In the type IIA case, the orientifold involution allows also for the so called *geometric fluxes*, which are deformations of the original manifold and appear naturally in the context of Scherk-Schwarz solutions. They are equivalent to compactifications on a twisted torus defined by

$$d\eta^m = -\frac{1}{2}\omega^m_{np}\eta^n \wedge \eta^p, \tag{2.57}$$

where η^m are the tangent 1-forms. The geometric fluxes are the ω^m_{np}, antisymmetric in the lower indices. The algebra of the isometries of the twisted torus is generated by some generators Z_m, such that

$$[Z_m, Z_n] = \omega_{mn}{}^p Z_p, \tag{2.58}$$

and then the geometric fluxes have to satisfy the Jacobi identities

$$\omega^s_{[mn}\omega^q_{p]s} = 0 \tag{2.59}$$

and the linear constraint

$$\omega^{mn}_n = 0. \tag{2.60}$$

The presence of geometric fluxes modifies the differential operator $d \to \tilde{d} = d + \omega$, but the formal identity $\tilde{d}^2 = 0$ continues to hold.

The geometric fluxes are even under the orientifold involution and then in IIA they can be of the form

$$\omega^i_{bc} = a_I, \quad \omega^i_{jk} = -b_{II}, \quad \omega^a_{bk} = b_{IJ}, \tag{2.61}$$

while they are not present in type IIB compactifications.

The NS fluxes in type IIA must satisfy the Bianchi identity

$$\omega \cdot H_3 = 0 \ , \tag{2.62}$$

with the contraction defined by

$$(\omega \cdot X)_{LMN_1 \dots N_{p-1}} = \omega^A_{[LM} X_{N_1 \dots N_{p-1}]A} \ . \tag{2.63}$$

The fluxes also induce RR tadpoles. In this IIA orientifold there are C_7 tadpoles, taking into account the coupling to O6-planes and stacks of D6-branes wrapping factorizable 3-cycles

$$\Pi_a = (n^1_a, m^1_a) \otimes (n^2_a, m^2_a) \otimes (n^3_a, m^3_a) \ , \tag{2.64}$$

and the corresponding orientifold images wrapping $\otimes_i (n^i_a, -m^i_a)$. Here n^i_a (m^i_a) are the wrapping numbers along the x^i (y^i) torus directions. The O6-planes wrap $\otimes_i (1, 0)$. We obtain then

$$\sum_a N_a n^1_a n^2_a n^3_a + F_0 H_{246} + 3F_{x[2}\omega^x_{46]} = 16 \tag{2.65}$$

and

$$\sum_a N_a n^1_a m^2_a m^3_a + F_6 H_{ibc} + 3F_{x[i}\omega^x_{bc]} = 0 \ . \tag{2.66}$$

In type IIB orientifold, there is a C_4 tadpole induced by the fluxes. There are also contributions from O3-planes and a stack of $N_{\mathcal{D}3}$ D3-branes. Substituting the fluxes and including the sources we obtain the tadpole cancellation condition

$$N_{D3} + \frac{1}{2}(F_{135}H_{246} + F_{246}H_{135} + F_{ibc}H_{ajk} + F_{ajk}H_{ibc}) = 16 \ . \tag{2.67}$$

2.3.1 $\mathcal{N} = 1$ effective potential with geometric fluxes

In the $\mathcal{N} = 1, D = 4$ effective theory obtained by the $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ orbifold and orientifold, the Kähler potential for the seven main moduli is given by

$$K = -\log(S + \bar{S}) - \sum_\alpha \log(U_\alpha + \bar{U}_\alpha) - \sum_\alpha \log(T_\alpha + \bar{T}_\alpha), \tag{2.68}$$

both in type IIA and IIB compactifications. Turning on the fluxes above, we recover a superpotential. Its explicit expression is, in type IIA

$$W = P_1(T) + iSP_2(T) + i\sum_{\alpha=1}^{3} U_\alpha P_{3\alpha}(T), \tag{2.69}$$

where

$$\begin{aligned} P_1(T) &= e_0 + i\sum_\alpha e_\alpha T_\alpha - \frac{1}{2}\sum_{\alpha\neq\beta\neq\gamma} q_\alpha T_\beta T_\gamma + imT_1T_2T_3, \\ P_2(T) &= h_0 + i\sum_\alpha a_\alpha T_\alpha, \\ P_{3\alpha} &= -h_\alpha + \sum_{\beta=1}^{3} b_{\beta\alpha} T_\beta. \end{aligned} \tag{2.70}$$

In type IIB it is

$$W = P_1(U) + iSP_2(U), \tag{2.71}$$

where

$$\begin{aligned} P_1(U) &= e_0 + i\sum_\alpha e_\alpha U_\alpha - \frac{1}{2}\sum_{\alpha\neq\beta\neq\gamma} q_\alpha U_\beta U_\gamma + imU_1U_2U_3, \\ P_2(U) &= h_0 + i\sum_\alpha a_\alpha U_\alpha + \frac{1}{2}\sum_{\alpha\neq\beta\neq\gamma} \bar{a}_\alpha U_\beta U_\gamma + i\bar{h}_0 U_1U_2U_3. \end{aligned} \tag{2.72}$$

2.3.2 Non-geometric fluxes

At this point, it is easy to note that the superpotential (2.71) in type IIB compactifications does not stabilize all the moduli, because it does not depend on T_α. This observation suggests that the analysis of flux compactifications cannot stop at this point. New input came from string dualities, which can be invoked to construct an effective four-dimensional theory in which the superpotential depends on all the moduli. To achieve this one can promote T-, S- and U-duality to symmetries of the four dimensional theory.

T-duality

T-duality in its simplest form identifies a theory compactified on a circle of radius R with a theory compactified on a circle of radius α'/R and exchanges momenta and winding modes. When compactifications are performed in a more general background, T-duality is enhanced to a particular orthogonal group. Without entering in the details, in our particular case of compactifications on the six torus, one can perform a T-duality transformation for each of the six directions. In particular, the duality between type IIA and type IIB orbifold compactifications, the so called *mirror symmetry*, is obtained by performing three T-duality transformations, one for each y-directions of the sub-tori. The effect of this kind of transformations on the moduli is the exchange of U's and T's, while S is left invariant. At the level of fluxes, in the NS sector each T-duality T_m transformation raises (lowers) the m-index and leaves all the others invariant. In the RR sector, the Buscher rules [45] hold

$$F_{mn_1\ldots n_p} \overset{T^m}{\longleftrightarrow} F_{n_1\ldots n_p}. \tag{2.73}$$

It is clear that the IIA and IIB theories described above are dual under the mirror symmetry. In order to restore this symmetry a new set of fluxes were introduced in [46] (for an early attempt along these lines, see also [47]). T-duality was used to connect the H_3 fluxes with geometric and non-geometric fluxes through the chain

$$-H_{mnp} \overset{T^m}{\longleftrightarrow} \omega^m_{np} \overset{T^n}{\longleftrightarrow} Q^{mn}_p \overset{T^p}{\longleftrightarrow} -R^{mnp}. \tag{2.74}$$

Under a single T-duality, the H-flux is mapped into a geometric flux ω, associated to the twist in the torus topology. Another T-duality transformation leads to dual torus, which is locally geometric but which cannot be described globally in terms of a fixed geometry. The last T-duality produces another kind of flux, the highly non-geometric R. Under the involution σ, H, Q fluxes are odd while the ω, R fluxes are even. It means that in type IIB there are only H, Q fluxes, while in type IIA there are H, Q fluxes with a triplet of indices mnp of the type 135, ajk and ω, R fluxes with a triplet of indices mnp that could be 246, abk. Starting from the H and Q fluxes in type IIB, we have also geometric and non-geometric fluxes in type IIA as shown in the following table, where the dualities are emphasized

IIB	IIA	$flux$
H_{135}	H_{135}	h_0
H_{ibc}	$-\omega^i_{bc}$	$-a_\alpha$
H_{ajk}	$-Q^{jk}_a$	$-\bar{a}_\alpha$
H_{246}	R^{246}	$\bar{h}_0$
Q^{jk}_a	$-H_{ajk}$	$-h_\alpha$
Q^{jk}_i	ω^i_{jk}	$-b_{\alpha\alpha}$
Q^{ka}_b	ω^a_{bk}	$b_{\alpha\beta}$
Q^{bc}_a	Q^{bc}_a	$-\bar{b}_{\alpha\alpha}$
Q^{ci}_j	Q^{jc}_i	$\bar{b}_{\alpha\beta}$
Q^{bc}_i	$-R^{ibc}$	$-\bar{h}_\alpha$

(2.75)

and the conventions on the indices are specified above.

The superpotential with these new fluxes becomes, in type IIB notation:

$$W = P_1(U) + iSP_2(U) + \sum_{\alpha=1}^{3} T_\alpha P_{3\alpha}(U), \tag{2.76}$$

with

$$P_1(U) = e_0 + i\sum_\alpha e_\alpha U_\alpha - \frac{1}{2}\sum_{\alpha\neq\beta\neq\gamma} q_\alpha U_\beta U_\gamma + imU_1U_2U_3\,,$$

$$P_2(U) = h_0 + i\sum_\alpha a_\alpha U_\alpha + \frac{1}{2}\sum_{\alpha\neq\beta\neq\gamma} \bar{a}_\alpha U_\beta U_\gamma + i\bar{h}_0 U_1U_2U_3\,,$$

$$P_{3\alpha} = -h_\alpha + i\sum_{\beta=1}^{3} b_{\beta\alpha} U_\beta + \frac{1}{2}\sum_{\beta\neq\gamma\neq\delta} \bar{b}_{\beta\alpha} U_\gamma U_\delta - i\bar{h}_\alpha U_1U_2U_3. \tag{2.77}$$

The same holds in type IIA with $T_i \leftrightarrow U_i$, because the mirror symmetry is restored at the level of the effective four-dimensional theories.

S-duality

There is another duality, S-duality, which we can take in consideration. In particular, type IIB is self-dual under this duality. Here T_α, U_α moduli do not transform under S-duality transformations, while S transform as

$$S \longrightarrow \frac{aS - ib}{icS + d}, \quad ad - bc = 1, \tag{2.78}$$

and the 3-form fluxes rotate as doublets.

$$\begin{pmatrix} F_3 \\ H_3 \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_3 \\ H_3 \end{pmatrix}. \tag{2.79}$$

In the presence of Q fluxes, the S-duality is not preserved, unless one adds new kind of fluxes, denoted by P_m^{np}, which form an $SL(2,\mathbb{Z})$ doublet with Q fluxes

$$\begin{pmatrix} Q \\ P \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}. \tag{2.80}$$

The new P fluxes and their T-duals in IIA are given in the following table

IIB	IIA	flux
P_a^{jk}	X_a^i	$-f_\alpha$
P_i^{jk}	Z^{ii}	$-g_{\alpha\alpha}$
P_b^{ka}	W_b^{jai}	$g_{\alpha\beta}$
P_a^{bc}	θ_{aa}	$-\bar{g}_{\alpha\alpha}$
P_j^{ci}	$f^{j,jck}$	$\bar{g}_{\alpha\beta}$
P_i^{bc}	$\tilde{X}_a^i$	$-\bar{f}_\alpha$

(2.81)

and generate new terms in the type IIB superpotential (and its T-dual in IIA):

$$W_P = -\sum_{\alpha=1}^{3} S T_\alpha P_{4\alpha}(U), \tag{2.82}$$

where

$$P_{4\alpha}(U) = f_\alpha - i\sum_{\beta=1}^{3} g_{\beta\alpha} U_\beta - \frac{1}{2}\sum_{\beta\neq\gamma\neq\delta} \bar{g}_{\beta\alpha} U_\gamma U_\delta + i\bar{f}_\alpha U_1 U_2 U_3. \tag{2.83}$$

U-duality

To conclude, by further demanding the four dimensional effective theory to be invariant under $SL(2,\mathbb{Z})^7$, new *primed* fluxes must be incorporated in type IIB [48]. In fact, the complete underlying theory is invariant under the $SL(2,\mathbb{Z})^7$ transformations corresponding to the seven main moduli. The general flux induced superpotential will then be a polynomial of degree seven on the moduli $M_I = (S, T_\alpha, U_\alpha)$ and at most linear in any of them

$$W = \sum_{N=0}^{7} D^{(N)}_{m_1 \ldots m_N} M_{m_1} \ldots M_{m_N}, \tag{2.84}$$

where $D^{(N)}$ are integer coefficients associated to generalized fluxes. The components of $D^{(N)}$ are $\begin{pmatrix} 7 \\ N \end{pmatrix}$ and all the possible fluxes are $\sum_{N=0}^{7} \begin{pmatrix} 7 \\ N \end{pmatrix} = 2^7$. They are all summarized in type IIA, IIB and M-theory in table 2.1.

The superpotential in type IIB formalism can be obtained from the following table, where in all the entries there are the coefficients (fluxes) of the terms obtained by multiplying the row and column labels. On the right-hand side there is the kind of flux in IIB that generates these terms in the superpotential.

	1	iU_α	$-U_\alpha U_\beta$	$-iU_1U_2U_3$	IIB
1	e_0	e_α	$-q_\gamma$	$-m$	F_3
iS	h_0	a_α	$-\bar{a}_\gamma$	$-\bar{h}_0$	H_3
$iT_{\alpha'}$	$-h_{\alpha'}$	$b_{\alpha\alpha'}$	$-\bar{b}_{\gamma\alpha'}$	$\bar{h}_{\alpha'}$	Q
$-ST_{\alpha'}$	$f_{\alpha'}$	$-g_{\alpha\alpha'}$	$\bar{g}_{\gamma\alpha'}$	$-\bar{f}_{\alpha'}$	P
$-T_{\alpha'}T_{\beta'}$	$-h'_{\gamma'}$	$b'_{\alpha\gamma'}$	$-\bar{b}'_{\gamma\gamma'}$	$-\bar{h}'_{\gamma'}$	Q'
$-iST_{\alpha'}T_{\beta'}$	$f'_{\gamma'}$	$-g'_{\alpha\gamma'}$	$-\bar{g}'_{\gamma\gamma'}$	$-\bar{f}'_{\gamma'}$	P'
$-iT_{\alpha'}T_{\beta'}T_{\gamma'}$	$-e'_0$	$-e'_\alpha$	q'_γ	m'	F'_3
$ST_{\alpha'}T_{\beta'}T_{\gamma'}$	$-h'_0$	a'_α	$\bar{a}'_\gamma$	$\bar{h}'_0$	H'_3

(2.85)

In the above table, $\alpha \neq \beta \neq \gamma$ and $\alpha' \neq \beta' \neq \gamma'$.

The superpotential obtained is now rather generic, but its parameters are strongly constrained by consistency requirements, such as Bianchi identities and anomaly cancellation conditions. By T-duality, it has to be possible to find some fluxes (geometric and non-geometric) which have to turned on in order to recover the same full superpotential (with U and T exchanged) in type IIA. We can read those fluxes from table 2.1.

2.4 M-theory compactifications

The effective theory describing the low-energy dynamics of M-theory on a seven torus $\mathbb{T}^7$ is an ungauged $\mathcal{N} = 8$, $D = 4$ supergravity. The manifest global symmetry $GL(7, \mathbb{R})$ is associated with the $\mathbb{T}^7$ compactification. At the level of the equations of motion and Bianchi identities, the global symmetry group is enhanced to $E_{7(7)}$. The higher-dimensional origin of the various four-dimensional fields can thus be recovered by branching the corresponding $E_{7(7)}$ representation with respect to $GL(7, \mathbb{R}) \sim SL(7, \mathbb{R}) \times O(1, 1)$.

Relevant representation of $E_{7(7)}$ are the fundamental **56** and the adjoint **133** representations, whose branchings are

$$\mathbf{56} \longrightarrow \overline{\mathbf{7}}_{-3} + \mathbf{21}_{-1} + \overline{\mathbf{21}}_{+1} + \mathbf{7}_{+3}, \tag{2.86}$$

$$\mathbf{133} \longrightarrow \overline{\mathbf{7}}_{+4} + \mathbf{35}_{+2} + \mathbf{48}_0 + \mathbf{1}_0 + \overline{\mathbf{35}}_{-2} + \mathbf{7}_{-4}. \tag{2.87}$$

The 28 vector bosons and their duals are in the fundamental representation and the branching (2.86) allows us to identify $\overline{\mathbf{7}}_{-3}$ with the vectors A^m_μ ($m = 1, ..., 7$) coming from the metric, the $\mathbf{21}_{-1}$ with the $A_{\mu mn}$ obtained from the eleven-dimensional

scalar combination	fluxes	IIB	IIA	M-theory
1	e_0	$-F_{135}$	$-F_{123456}$	$-g_7$
iU_α	e_α	F_{ibc}	F_{bjck}	g_{bjck}
$-U_\alpha U_\beta$	$-q_\gamma$	F_{ijc}	F_{ck}	ω^7_{ck}
$-iU_1U_2U_3$	$-m$	F_{246}	F_0	ξ^{77}
iS	h_0	H_{135}	H_{135}	g_{1357}
$-SU_\alpha$	a_α	$-H_{ibc}$	ω^i_{bc}	ω^i_{bc}
$-iSU_\beta U_\gamma$	$-\bar{a}_k$	H_{ijc}	$-Q^{ij}_c$	$-R^{7ij}_c$
$SU_1U_2U_3$	$-\bar{h}_0$	H_{246}	R^{246}	$f^{7,7246}$
iT_α	$-h_\alpha$	Q^{jk}_a	$-H_{ajk}$	g_{ajk7}
$-T_\alpha U_\alpha$	$b_{\alpha\alpha}$	$-Q^{jk}_i$	ω^i_{jk}	ω^i_{jk}
$-T_\alpha U_\beta$	$b_{\beta\alpha}$	Q^{bk}_a	ω^b_{ka}	ω^b_{ka}
$-iT_\alpha U_\alpha U_\beta$	$-\bar{b}_{\gamma\alpha}$	$-Q^{bk}_i$	$-Q^k_{ib}$	$-R^{7ib}_k$
$-iT_\alpha U_\beta U_\gamma$	$-\bar{b}_{\alpha\alpha}$	Q^{bc}_a	Q^a_{bc}	R^{7bc}_a
$T_\alpha U_1U_2U_3$	$\bar{h}_\alpha$	$-Q^{bc}_i$	R^{ibc}	$f^{7,7ibc}$
$-ST_\alpha$	f_α	P^{jk}_a	X^i_a	ω^i_{7a}
$-iST_\alpha U_\alpha$	$-g_{\alpha\alpha}$	P^{jk}_i	Z^{ii}	ξ^{ii}
$-iST_\alpha U_\beta$	$-g_{\beta\alpha}$	P^{bk}_a	W^{ibj}_a	R^{ibj}_a
$ST_\alpha U_\alpha U_\beta$	$\bar{g}_{\gamma\alpha}$	P^{jc}_i	$f^{i,iabk}$	$f^{i,iabk}$
$ST_\alpha U_\beta U_\gamma$	$\bar{g}_{\alpha\alpha}$	$-P^{bc}_a$	θ_{aa}	θ_{aa}
$iST_\alpha U_1U_2U_3$	$-\bar{f}_\alpha$	P^{bc}_i	$\tilde{X}^i_a$	Q^{i7}_a
$-T_\alpha T_\beta$	$-h'_\gamma$	Q'^k_{ab}	X^c_k	ω^c_{7k}
$-iT_\alpha T_\beta U_\beta$	$b'_{\beta\gamma}$	Q'^k_{aj}	W^{bjc}_k	R^{bjc}_k
$-iT_\alpha T_\beta U_\gamma$	$b'_{\gamma\gamma}$	Q'^c_{ab}	Z^{cc}	ξ^{cc}
$T_\alpha T_\beta U_\alpha U_\beta$	$-\bar{b}'_{\gamma\gamma}$	Q'^k_{ij}	θ_{kk}	θ_{kk}
$T_\alpha T_\beta U_\beta U_\gamma$	$-\bar{b}'_{\alpha\gamma}$	Q'^c_{aj}	$f^{c,cbj}$	$f^{c,cbj7}$
$T_\alpha T_\beta U_1U_2U_3$	$\bar{h}'_\gamma$	Q'^c_{ij}	$\tilde{X}^c_k$	Q^{c7}_k
$-iST_\alpha T_\beta$	f'_γ	P'^k_{ab}	$\tilde{R}^{ijc}$	R^{ijc}
$ST_\alpha T_\beta U_\beta$	$-g'_{\beta\gamma}$	P'^k_{aj}	$f^{j,jaib}$	$f^{j,jaib}$
$ST_\alpha T_\beta U_\gamma$	$-g'_{\gamma\gamma}$	P'^c_{ab}	$f^{c,cijk}$	$f^{c,cijk}$
$iST_\alpha T_\beta U_\alpha U_\beta$	$\bar{g}'_{\gamma\gamma}$	P'^k_{ij}	$\tilde{Q}^{jk}_i$	Q^{jk}_i
$iST_\alpha T_\beta U_\beta U_\gamma$	$\bar{g}'_{\alpha\gamma}$	P'^c_{aj}	$\tilde{Q}^{jc}_a$	Q^{jc}_a
$-ST_\alpha T_\beta U_1U_2U_3$	$-\bar{f}'_\gamma$	P'^c_{ij}	$\tilde{H}^{ijc}$	h^{ijc7}
$-iT_1T_2T_3$	$-e'_0$	F'^{246}	$\tilde{R}^{135}$	R^{135}_7
$T_1T_2T_3U_\alpha$	$-e'_\alpha$	F'^{ajk}	$f^{a,aibc}$	$f^{a,aibc}$
$T_1T_2T_3U_\alpha U_\beta$	q'_γ	F'^{abk}	$\tilde{Q}^{ab}_k$	Q^{ab}_k
$T_1T_2T_3U_1U_2U_3$	m'	F'^{135}	$\tilde{H}^{135}$	h^{1357}
$ST_1T_2T_3$	$-h'_0$	H'^{246}	$\tilde{F}_0$	θ_{77}
$ST_1T_2T_3U_\alpha$	a'_α	H'^{ajk}	$\tilde{F}^{ai}$	Q^{ai}_7
$ST_1T_2T_3U_\alpha U_\beta$	$\bar{a}'_\gamma$	H'^{abk}	$\tilde{F}^{aibj}$	h^{aibj}
$ST_1T_2T_3U_1U_2U_3$	$\bar{h}'_0$	H'^{135}	$\tilde{F}_6$	$\tilde{g}_7$

Table 2.1: Relation between fluxes of M-theory, type IIA and type IIB compactifications and superpotential terms if these fluxes are turned on.

$\mathbf{1}_{+7}$	g_7	$(\mathbf{140}+\mathbf{7})_{+3}$	$\tau_{np}^m + \delta_n^m \tau_p$	$\mathbf{28}_{-1}$	$\theta'_{(mn)}$
$\mathbf{1}_{-7}$	$\tilde{g}_7$	$(\overline{\mathbf{140}}+\overline{\mathbf{7}})_{-3}$	$Q_m^{np} + \delta_m^n Q^p$	$\overline{\mathbf{28}}_{+1}$	$\xi'^{(mn)}$
$\mathbf{35}_{-5}$	h^{mnpq}	$\mathbf{224}_{-1}$	f_{npq}^m	$\mathbf{21}_{-1}$	$\theta_{[mn]}$
$\overline{\mathbf{35}}_{+5}$	g_{mnpq}	$\overline{\mathbf{224}}_{+1}$	R_m^{npq}	$\overline{\mathbf{21}}_{+1}$	$\xi^{[mn]}$

Table 2.2: Flux representations under the $GL(7,\mathbb{R})$ decomposition of $E_{7(7)}$.

three-form, and the remaining representations with the corresponding magnetic dual vector potentials.

The generators t_α ($\alpha = 1, ..., 133$) of the algebra $\mathfrak{e}_{7(7)}$ are in the adjoint representation. The branching (2.87) distinguishes between t_m, t^{mnp} in the $\overline{\mathbf{7}}_{+4}$ and $\mathbf{35}_{+2}$ respectively, t^m, t_{mnp} in the $\mathbf{7}_{-4}$ and $\overline{\mathbf{35}}_{-2}$ and $t_m{}^n, t = \sum_m t_m{}^m$, generators of $\mathfrak{gl}(7,\mathbb{R})$, in the $\mathbf{48}_0 + \mathbf{1}_0$. Their commutation relations can be found for example in [49] and are given in Appendix B for completeness.

In this setup, the presence of fluxes induces non-Abelian gauge symmetries in the four-dimensional theory. The four-dimensional Lagrangian has a gauge group G, whose generators can be written in a $E_{7(7)}$ invariant way as $X_{\mathcal{M}}$. According to the branching (2.86), there are Z_M, Z^M generators associated to $\mathbf{7}_{+3}$ and $\overline{\mathbf{7}}_{-3}$ representation respectively and W^{MN}, W_{MN} generators, associated to $\overline{\mathbf{21}}_{+1}$ and $\mathbf{21}_{-1}$. Since G is a subgroup of $E_{7(7)}$, its generators can be expanded in a basis $\{t_\alpha\}_{\alpha=1,...133}$ of $E_{7(7)}$ generators, through an embedding tensor

$$X_{\mathcal{M}} = \Theta_{\mathcal{M}}^\alpha t_\alpha. \tag{2.88}$$

The most general gauging is encoded in the embedding tensor $\Theta_{\mathcal{M}}^\alpha$ transforming in the $\mathbf{912} \in \mathbf{56} \times \mathbf{133}$. Its branching

$$\begin{aligned}\mathbf{912} \quad \longrightarrow \quad & \mathbf{1}_{-7} + \mathbf{1}_{+7} + \mathbf{35}_{-5} + \overline{\mathbf{35}}_{+5} + (\overline{\mathbf{140}} + \overline{\mathbf{7}})_{-3} + (\mathbf{140} + \mathbf{7})_{+3} \\ & + \mathbf{21}_{-1} + \overline{\mathbf{21}}_{+1} + \mathbf{28}_{-1} + \overline{\mathbf{28}}_{+1} + \mathbf{224}_{-1} + \overline{\mathbf{224}}_{+1}\end{aligned} \tag{2.89}$$

distinguishes different M-theory fluxes, associated to each representation of the branching as in the table below [50] The component g_7 is the flux of the 7-form field strength across $\mathbb{T}^7$, while $\tilde{g}_7$ represents the four-dimensional space-time components of the 4-form field strength. The internal flux of the 4-form field strength is g_{ijkl}, while $\tau_{ij}{}^k$ is the twist of the torus. All the components in table 2.2 are part of a single irreducible representation of $E_{7(7)}$ and therefore are mapped into one another by string/M-theory dualities. We can also see that many new "non-geometric" fluxes may appear.

Each representation in the branching of **912** defines a different set of entries of Θ which can be switched on independently of the others and lead to a specific gauging. It is useful to arrange the above representations in a table as follows:

	$\mathbf{7}_{+3}$	$\overline{\mathbf{21}}_{+1}$	$\mathbf{21}_{-1}$	$\overline{\mathbf{7}}_{-3}$
$\overline{\mathbf{7}}_{+4}$	$\mathbf{1}_{+7}$	$\overline{\mathbf{35}}_{+5}$	$(\mathbf{140}+\mathbf{7})_{+3}$	$\overline{\mathbf{28}}_{+1}+\overline{\mathbf{21}}_{+1}$
$\mathbf{35}_{+2}$	$\overline{\mathbf{35}}_{+5}$	$(\mathbf{140}+\mathbf{7})_{+3}$	$\overline{\mathbf{21}}_{+1}+\overline{\mathbf{224}}_{+1}$	$\mathbf{21}_{-1}+\mathbf{224}_{-1}$
$\mathbf{48}_{0}$	$(\mathbf{140}+\mathbf{7})_{+3}$	$\overline{\mathbf{21}}_{+1}+\overline{\mathbf{28}}_{+1}+\overline{\mathbf{224}}_{+1}$	$\mathbf{21}_{-1}+\mathbf{28}_{-1}+\mathbf{224}_{-1}$	$(\overline{\mathbf{140}}+\overline{\mathbf{7}})_{-3}$
$\mathbf{1}_{0}$	$\mathbf{7}_{+3}$	$\overline{\mathbf{21}}_{+1}$	$\mathbf{21}_{-1}$	$\overline{\mathbf{7}}_{-3}$
$\overline{\mathbf{35}}_{-2}$	$\overline{\mathbf{21}}_{+1}+\overline{\mathbf{224}}_{+1}$	$\mathbf{21}_{-1}+\mathbf{224}_{-1}$	$\overline{\mathbf{140}}_{-3}$	$\mathbf{35}_{-5}$
$\mathbf{7}_{-4}$	$\mathbf{21}_{-1}+\mathbf{28}_{-1}$	$(\overline{\mathbf{140}}+\overline{\mathbf{7}})_{-3}$	$\mathbf{35}_{-5}$	$\mathbf{1}_{-7}$

The first row and column contain the representations in the branchings of **56** and **133** respectively, while the bulk contains representations in the branching of **912**. The table specifies the origin of the latter representations in the branching of the product $\mathbf{56}\times\mathbf{133}$.

The embedding tensor also satisfy quadratic constraints, given by (1.120). Taking

$$\Omega_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} \Omega_{m,p} & \Omega_{m,}{}^{p} & \Omega_{m,}{}^{pq} & \Omega_{m,pq} \\ \Omega^{m,}{}_{p} & \Omega^{m,p} & \Omega^{m,pq} & \Omega^{m,}{}_{pq} \\ \Omega^{mn,}{}_{p} & \Omega^{mn,p} & \Omega^{mn,pq} & \Omega^{mn,}{}_{pq} \\ \Omega_{mn,p} & \Omega_{mn,}{}^{p} & \Omega_{mn,}{}^{pq} & \Omega_{mn,pq} \end{pmatrix} = \begin{pmatrix} 0 & \delta^{p}_{m} & 0 & 0 \\ -\delta^{m}_{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta^{mn}_{pq} \\ 0 & 0 & -\delta^{pq}_{mn} & 0 \end{pmatrix}, \tag{2.90}$$

the equations (1.120) become

$$\Theta^{\alpha}_{m}\Theta^{m\beta} - \Theta^{m\alpha}\Theta^{\beta}_{m} + \Theta^{mn\alpha}\Theta^{\beta}_{mn} - \Theta^{\alpha}_{mn}\Theta^{mn\beta} = 0. \tag{2.91}$$

The quadratic constraints on the embedding tensor translate into bilinear equations for the M-theory fluxes. Those equations can be made explicit by writing $\Theta^{\alpha}_{\mathcal{M}}$ as a matrix (see table II) with entries given according to table I. The most general embedding tensor that one can write in M-theory reductions on a seven-torus is then

$\Theta_{\mathcal{M}}^{\alpha}$	Z_m	W^{mn}	W_{mn}	Z^m
t_p	$a_1\, g_7 \delta_m^p$	$b_1\, \epsilon^{mnpr_1\ldots r_4} g_{r_1\ldots r_4}$	$c_5\, \omega_{mn}^{p}$ $+c_5'\, \omega_{[m}\delta_{n]}^{p}$	$d_5\, \xi^{mp}$ $+d_5'\, \xi'^{mp}$
$t^{p_1p_2p_3}$	$a_2\, g_{mp_1p_2p_3}$	$b_2\, \omega_{[p_1p_2}^{[m}\delta_{p_3]}^{n]}$	$c_4\, \xi^{rs}\epsilon_{mnrsp_1p_2p_3}$ $+c_4' R_{[m}^{r_1r_2r_3}\epsilon_{n]r_1r_2r_3p_1p_2p_3}$	$d_4\, \theta_{[p_1p_2}\delta_{p_3]}^{m}$ $+d_4'\, f_{p_1p_2p_3}^{m}$
t_q^p	$a_3\omega_{mp}{}^{q}$ $+a_3'\omega_p\delta_m^q$ $+a_3''\omega_m\delta_p^q$	$b_3\xi^{q[m}\delta_p^{n]}$ $+b_3'\xi'^{q[m}\delta_p^{n]}$ $+b_3'' R_q^{mnp}$ $+b_3'''\xi^{mn}\delta_p^q$	$c_3\theta_{p[m}\delta_{n]}^{q}$ $+c_3'\theta'_{p[m}\delta_{n]}^{q}$ $+c_3'' f_{mnq}^{p}$ $+c_3'''\theta_{mn}\delta_p^q$	$d_3 Q^{mq}{}_{p}$ $+d_3' Q^q\delta_p^m$ $+d_3'' Q^m\delta_p^q$
$t_{p_1p_2p_3}$	$a_4\, \xi^{[p_1p_2}\delta_m^{p_3]}$ $+a_4'\, R_m^{p_1p_2p_3}$	$b_4\, \theta_{rs}\epsilon^{mnrsp_1p_2p_3}$ $+b_4' f_{r_1r_2r_3}^{[m}\epsilon^{n]r_1r_2r_3p_1p_2p_3}$	$c_2\, Q_{[m}^{[p_1p_2}\delta_{n]}^{p_3]}$	$d_2\, h^{mp_1p_2p_3}$
t^p	$a_5\, \theta_{mp}$ $+a_5'\, \theta'_{mp}$	$b_5\, Q_p^{mn}$ $+b_5'\, Q^{[m}\delta_p^{n]}$	$c_1\, \epsilon_{mnpr_1\ldots r_4} h^{r_1\ldots r_4}$	$d_1\, \tilde{g}_7\delta_p^m$

where in the first row and column there are the branchings of the $E_{7(7)}$ and G generators t_α and $X_{\mathcal{M}}$, and in the bulk there are the components of the embedding tensor $\Theta_{\mathcal{M}}^{\alpha}$. The coefficients a, b, c, d's are constants, which in principle can be redefined by a redefinition of fluxes. Here we leave all of them unfixed. The quadratic constraints on fluxes can be read from (1.120). Their explicit form is given in Appendix B.

Chapter 3

Fayet-Iliopoulos terms and de Sitter vacua in simple and extended supergravity

After several decades of extensive studies on simple and extended four -dimensional supergravities, and on the rôle of these theories as effective low-energy theories of superstring or M-theory compactifications, the theoretical status of Fayet-Iliopoulos (FI) terms [51] in supergravity [52] - [55] and its possible ultraviolet completions is still under discussion (for some recent literature, see e.g. [56] - [61]).

Several $\mathcal{N} = 1$ supergravity models with a gauged $U(1)$ R-symmetry, associated with a constant FI term, have been formulated [62] - [72], typically leading to Minkowski or de Sitter (dS) vacua with a massive vector boson associated to the spontaneously broken $U(1)$ R-symmetry. Quantization of the FI parameter in supergravity was inferred in [73] and discussed in more detail in [59] - [61].

In this chapter we clarify some pending issues, keeping the formalism at a minimum and using a number of explicit models for illustration, following our paper [1]. We refer to chapter 1 and Appendix A for the notation.

In section 3.1 we present some novel insight on FI terms. We do not consider the so-called field-dependent FI terms, which do not involve the gauging of an R-symmetry and are nothing else than $U(1)$ D-terms for a gauge symmetry that does not act linearly on the fields. We concentrate instead on constant FI terms, associated with the gauging of an R-symmetry, and explain how we should distinguish between genuine FI terms and impostors. This point is a direct consequence of the general structure of supergravity and may be known to some[1], but to the best of our knowledge an explicit and general supergravity formulation was never given in the literature. We conclude the section by recalling the anomaly cancellation conditions, which are essential for quantum consistency.

In section 3.2 we present three simple examples. First, we display a model with a FI impostor. Second, we present an anomaly-free model with a genuine FI term, a classically stable dS vacuum and no exact global symmetry. Depending on the choice of the parameters, the vector field can be either massless or massive on the vacuum. The scalar potential is positive definite and supersymmetry is always broken, except when the superpotential is trivial. As our last example, we consider a model with a genuine FI term which admits a supersymmetric AdS solution [74].

Objects analogous to the $\mathcal{N}=1$ FI terms do also exist in $\mathcal{N}=2$ [14] and $\mathcal{N}=4$ [17],[75] - [78] supergravity: they play a crucial rôle in the construction of the only known classically stable dS vacua in extended supergravity [14], [79], [80], whilst no stable dS vacuum has been found so far in $\mathcal{N}>2$ supergravity. In section 3.3, we dicuss some consistent $\mathcal{N}=1$ truncations of the $\mathcal{N}=2$ models of [14], adding to the discussion of the first model given in [1] also the discussion of the other two models of [14].

3.1 Known and less known facts on FI terms in supergravity

In chapter 1 we have summarized $\mathcal{N}=1$ supergravity in the standard approach, but it can be also useful to recall some features of the superfield approach. The generic superfield expression of the $\mathcal{N}=1$ supergravity Lagrangian with a gauged $U(1)$ R-symmetry[2] is, in the compensator formalism (see, e.g. [81]):

$$\mathcal{L} = \left[\,\overline{\Phi_0}\Phi_0\, e^{-K/3}\right]_D + \left(\left[\Phi_0^3\, W\right]_F + [f\,\mathcal{W}\,\mathcal{W}]_F + \text{h.c.}\right)\,. \tag{3.1}$$

In the above expression: Φ_0 is the chiral compensator superfield, transforming as $\Phi_0 \to \Phi_0' = \Phi_0\,\exp(\xi\Lambda/3)$; $K = K_0 - \xi\, V$, where K_0 is a real and gauge-invariant function of the chiral superfields Φ, of their conjugates $\overline{\Phi}$ and of the real gauge vector

[1]See, e.g., the example discussed in section 4.5 of [58], which is closely related to the example we will discuss in section 3.1. We thank K. Dienes and B. Thomas for bringing their example to our attention.

[2]We neglect here additional gauge group factors, possible generalized Chern-Simons terms and constant numerical factors that are not important for the present considerations.

superfield V, the latter transforming as $V \to V' = V - \Lambda - \overline{\Lambda}$; W is now an analytic function of the chiral superfields Φ, transforming as $W \to W' = W \exp(-\xi\Lambda)$; f is the gauge kinetic function (with implicit lower indices), analytic in the chiral superfields Φ; $\mathcal{W}$ is the chiral supersymmetric field strength of V. Notice that having a FI term ξ corresponds to giving charge $\xi/3$ to the compensator field Φ_0 under the gauged $U(1)$ R-symmetry.

In most of what follows it will suffice to think of G, f_{ab} and X_a as functions of the complex scalars z^i rather than the superfields Φ^i (as done, for example, in Appendix G of [5]). However, whenever needed we will turn to superfield notation.

3.1.1 Genuine FI terms and impostors

Here we show how, in theories where the gauge boson of the U(1) R-symmetry is massive everywhere in field space, the FI term associated with such vector field is not well defined and can always be redefined away—a genuine FI term, whose value cannot be shifted continuously, exists only when the theory allows the gauged $U(1)$ R-symmetry to be restored at least in one point in field space.

The general argument is quite simple and follows from the supergravity formalism reviewed above. The supermultiplet of a massive vector contains the degrees of freedom of a chiral supermultiplet besides those of a massless vector multiplet. In particular, the superfield of a massive vector V_m can always be decomposed into

$$V_m = V + S + \overline{S}\,, \tag{3.2}$$

with a massless vector superfield V and a chiral superfield S transforming as

$$V \to V' = V - \Lambda - \overline{\Lambda}\,, \qquad S \to S' = S + \Lambda\,. \tag{3.3}$$

The l.h.s. of eq. (3.2) can be thought of as the vector superfield in the unitary gauge $S = 0$, while the r.h.s. is the gauge-invariant combination obtained from the unitary gauge via the Stückelberg trick.

In theories where the vector field is massive everywhere in field space, the field S is globally defined, since it corresponds to the superfield multiplet of the longitudinal mode. On the other hand, in theories where the gauge symmetry is restored somewhere in field space, S is not globally defined. In the points where the vector mass vanishes, the would-be S field is not dynamical (there is no kinetic term) and corresponds to a pure gauge.

Consider a supergravity model with a non-vanishing FI term ξ, associated to a vector superfield V gauging an R-symmetry that is broken everywhere in field space. In this case there exists a globally defined chiral superfield S transforming as in eq. (3.3). We can now perform a Kähler transformation using such field, namely

$$K' = K + \alpha\,(S + \overline{S})\,, \qquad W' = W e^{-\alpha\, S} \tag{3.4}$$

with α an arbitrary real constant[3]. In terms of K', the formula for the D term (1.19) now reads

$$D = iK_i X^i + \xi = iK'_i X^i + \alpha + \xi \equiv iK'_i X^i + \xi'\,, \tag{3.5}$$

from where we can see that in the new frame the FI term has been redefined. If we choose $\alpha = -\xi$, the FI term can be shifted away—in this theory the FI term is not well defined. We obtain the same conclusion by looking at the superpotential. Indeed the charge of the new superpotential W' will be shifted into $-\xi - \alpha$, because of the extra contribution from the S field. When $\alpha = -\xi$, W' will be invariant, which again corresponds to no FI term[4].

Notice that, since this FI term can be shifted by an arbitrary amount, it will not be subjected to quantization conditions. Since theories of this type are equivalent to one with a massive vector not associated to the R-symmetry, in this case we are not allowed to speak of genuine FI terms—such FI terms are impostors.

On the contrary, in theories where there are points in field space where the gauged R-symmetry is restored, we cannot use the field S to redefine the FI term. Only in this case a proper FI term can be defined, we call such terms genuine FI terms.

3.1.2 Anomaly cancellation conditions

For a consistent effective theory, all gauge and gravitational anomalies associated with a gauged $U(1)$ must vanish: in particular, the cubic ($\mathcal{A}_{U(1)^3}$), the gravitational ($\mathcal{A}_{U(1)}$) and the mixed-gauge anomaly ($\mathcal{A}_{U(1)\,\mathcal{G}^2}$) if the full gauge group is $U(1) \times \mathcal{G}$.

To fix the notation, we assign the $U(1)$ charges as

$$Q[\theta] = Q[\lambda^a] = Q[\psi_\mu] = -\xi/2\,, \quad Q[W] = -\xi\,, \quad Q[z^i] = q^i\,, \quad Q[\psi^i] = q^i + \xi/2\,, \tag{3.6}$$

[3] In general, the Kähler transformation may be anomalous. However, we assume that the vector superfield V is associated to an anomaly-free $U(1)$ R-symmetry. Such symmetry can be exploited to redefine the fields and make the full "Kahler transformation + field redefinition" anomaly-free.

[4] In the compensator formalism of eq. (3.1), the Kähler fransformation (3.4) corresponds to the redefinition $\Phi_0 \to \Phi'_0 = \Phi_0\, e^{-\alpha S/3}$, which shifts the compensator charge by $\alpha/3$ and the value of the FI term accordingly.

where θ is the anticommuting coordinate and ψ_μ is the gravitino.

In the above conventions, the fermionic contributions to the cubic and gravitational anomalies are:

$$Tr\, Q^3 = 3\, (Q[\psi_\mu])^3 + \sum_a (Q[\lambda^a])^3 + \sum_i (Q[\psi^i])^3 \,, \tag{3.7}$$

$$Tr\, Q = -21\, Q[\psi_\mu] + \sum_a Q[\lambda^a] + \sum_i Q[\psi^i] \,, \tag{3.8}$$

see [82, 83] for the gravitino contributions. These contributions must either vanish or cancel possible Green-Schwarz (GS) contributions [84] coming from the variation of $\mathrm{Im}\, f_{\Lambda\Sigma}$. All the resulting conditions are model dependent, in particular: all of them depend on the matter content; the GS contribution to $\mathcal{A}_{U(1)}$ depends on higher derivative terms (R^2); $\mathcal{A}_{U(1)\,\mathcal{G}^2}$ depends also on the details of $\mathcal{G}$ and its representations. However, there are in principle strong combined constraints on the possible matter content and on the $U(1)$ charges.

3.2 Examples

3.2.1 A model with a FI impostor

Consider a free massive vector superfield with the usual kinetic F-term and a mass term appearing in the Kähler potential as

$$K = \frac{1}{2}\, M^2\, V_m^2 \qquad \text{or} \qquad K = \frac{1}{2}\, (S + \overline{S} + MV)^2 \,, \tag{3.9}$$

where the first expression refers to the unitary gauge (a discussion of this model, in global supersymmetry and in the unitary gauge, can be found in section 4.5 of [58]), whilst the second uses the Wess-Zumino gauge for V, with the longitudinal components of the massive vector contained in the chiral superfield S. With respect to eq. (3.2), we have reabsorbed the factor M in S, to have the latter superfield canonically normalized. For definiteness, we can take a constant gauge kinetic function $f_0 = 1/g^2$ and a constant superpotential W_0, but we are allowed to take the limit $W_0 \to 0$ at the end of the calculations. Under gauge transformations:

$$V \to V - \Lambda - \overline{\Lambda} \,, \qquad S \to S + M\, \Lambda \,. \tag{3.10}$$

The action described above corresponds to a massive Abelian vector superfield, with no FI term.

We now perform a trivial field redefinition of the chiral superfield S,

$$S = T - \frac{\xi}{2\,M}\,, \tag{3.11}$$

where ξ is a real constant and T transforms as S under gauge transformations. The Kähler potential now reads

$$K = \frac{1}{2}\,(T + \overline{T} + MV)^2 - \frac{\xi}{M}(T + \overline{T} + MV) + \frac{\xi^2}{2\,M^2}\,, \tag{3.12}$$

and after a Kähler transformation we have

$$K = \frac{1}{2}\,(T + \overline{T} + MV)^2 - \xi V\,, \qquad W = W_0\, e^{\frac{\xi^2}{4\,M^2}} e^{-\frac{\xi}{M}\,T}\,. \tag{3.13}$$

This is a theory of a massive vector superfield with a FI term, which consistently appears both as a linear term in V in the Kähler potential and as a gauge non-invariance of the superpotential. As a check of the equivalence of the two theories, we can look at the expressions of the prepotentials in the two frames:

$$\text{(i)} \qquad \mathcal{P} = i\, K_S\, X^S = -M\,(S + \overline{S})\,, \tag{3.14}$$

$$\text{(ii)} \qquad \mathcal{P} = i\, K_T\, X^T + i\,\frac{W_T}{W}\, X^T = \xi - M\,(T + \overline{T})\,, \tag{3.15}$$

which coincide after using eq. (3.11).

Of course the presence of other interactions and charged fields does not affect the proof. The argument above can be run backwards, to show that the FI term can be reabsorbed via a field redefinition and a Kähler transformation. Notice that the FI term generates from the mass term $(1/2)M^2 V_m^2$ in the Kähler potential. Equivalently, a FI term can be reabsorbed by a field redefinition and a Kähler transformation only when such term is present.

3.2.2 An anomaly-free model with genuine FI term

We formulate now an explicit model that provides an existence proof of theories with the following properties: *(i)* presence of a genuine FI term; *(ii)* cancellation of all gauge anomalies; *(iii)* existence of a locally stable vacuum with all scalar field stabilized at tree level; *(iv)* absence of exact global symmetries; *(v)* all physical masses and energy densities parametrically small with respect to the Planck scale, even when the FI term is assumed to be quantized in Planck mass units. The chosen example has also the following features: the vacuum has positive energy; the gauged

$U(1)$ R-symmetry can be chosen to be either broken or unbroken on the vacuum; there exists a limit where also supersymmetry is recovered on the vacuum.

The model contains one vector supermultiplet, associated with the $U(1)$ R-symmetry that generates the constant FI term ξ, and 24 chiral supermultiplets, transforming linearly under the gauged $U(1)$: one (Φ_+) with charge $q_+ = +\xi$ and 23 ($\Phi_-^{i=1..23}$) of charge $q_- = -\xi$. The corresponding fermions have then charges $Q[\psi_+] = 3\,\xi/2$ and $Q[\psi_-^i] = -\xi/2$. It is immediate to check that the anomaly cancellation conditions of eqs. (3.7) and (3.8) are identically satisfied.

We discuss first the model with canonical Kähler potential,

$$K_0 = |z_+|^2 + \sum_{i=1}^{23} |z_-^i|^2 \,, \tag{3.16}$$

'minimal' superpotential with the appropriate charge[5],

$$W_0 = M^2\, z_-^1 \,, \tag{3.17}$$

where M is a real mass parameter, and constant gauge kinetic function,

$$f_0 = \frac{1}{g^2} \,. \tag{3.18}$$

In such a case, the D-term of eq. (1.19) reads

$$D = \xi \left(1 + \sum_{i=1}^{23} |z_-^i|^2 - |z_+|^2 \right) \,. \tag{3.19}$$

The scalar potential has

$$V_F = e^{K_0}\, M^4 \left[1 + |z_-^1|^2 \left(\sum_{i=1}^{23} |z_-^i|^2 + |z_+|^2 - 1 \right) \right] \,, \tag{3.20}$$

and

$$V_D = \frac{g^2\,\xi^2}{2} \left(1 + \sum_{i=1}^{23} |z_-^i|^2 - |z_+|^2 \right)^2 \,. \tag{3.21}$$

Notice that $V_F \geq M^4$ and $D \geq 0$. For $M = 0$, V_F is identically vanishing and V_D can relax to a supersymmetric Minkowski vacuum with unbroken supersymmetry

[5] In the case of a general linear superpotential, $W_0 = \sum_i^{23} M_i^2\, z_-^i$, we can always move to the form of W_0 given in eq. (3.17) by a suitable rotation in the space of the z_-^i fields, which leaves K_0 and f_0 invariant.

and spontaneously broken $U(1)$ gauge symmetry. For $M \neq 0$, the full potential V is strictly positive definite and always admits classically stable dS vacua.

For $g\,\xi < M^2$, the $U(1)$ gauge symmetry is unbroken, $\langle W_0 \rangle = \langle z_+ \rangle = \langle z_-^i \rangle = 0$, the vacuum energy density is $\langle V \rangle = M^4 + g^2\,\xi^2/2$ and the squared masses for the scalar fields $(z_-^1, z_-^{2\ldots 23}, z_+)$ are $(g^2\xi^2,\ M^4 + g^2\xi^2,\ M^4 - g^2\xi^2)$, respectively. They are all positive and of the order of the Hubble scale.

For $g\,\xi > M^2$, the $U(1)$ gauge symmetry is spontaneously broken, the field z_+ develops a VEV v satisfying the equation $M^4\,e^{v^2} = g^2\,\xi^2\,(1 - v^2)$. In this case the vacuum energy is $\langle V \rangle = \frac{1}{2} g^2\xi^2(1-v^2)(3-v^2)$, which is always positive except for $M = 0$ where it vanishes. The squared masses for the scalar fields $(z_-^1, z_-^{2\ldots 23}, \mathrm{Re}\, z_+)$ are all positive and given by $(g^2\xi^2(1-v^2),\ 2g^2\xi^2(1-v^2),\ 2g^2\xi^2 v^2(2-v^2))$, respectively, while $\mathrm{Im}\, z_+$ is eaten by the vector boson, which has a mass $\sqrt{2} g\xi v$. The masses are again of the order of the Hubble scale, except in the supersymmetric limit ($M \to 0$ and $v \to 1$), where the de Sitter curvature goes to zero while the vector superfield remains massive, eating the chiral superfield Φ_+ in a supersymmetric way.

The simple model described above has a large amount of global symmetries, since the canonical Kähler potential K_0 is invariant under $U(24) \times U(1)_R$, gauge interactions break $U(24)$ to $U(23) \times U(1) \times U(1)_R$, superpotential interactions in W_0 break $U(23) \times U(1) \times U(1)_R$ into $U(22) \times U(1) \times U(1)'_R$. However, it is relatively simple to break all the residual global symmetries by introducing higher-dimensional operators into the Kahler potential ($K = K_0 + \Delta K$) and the superpotential ($W = W_0 + \Delta W$), with modifications such as $\Delta K = a_{i\bar{\jmath}}\, z_-^i\, \bar{z}_-^{\bar{\jmath}}\, |z_+|^2$ and $\Delta W = b_{ij}\, z_-^i\, z_-^j z_+$. Since all the scalar fields have positive squared masses at tree level, the presence of the higher-dimensional operators does not destabilize the vacuum if their coefficients are small enough, $a_{i\bar{\jmath}}, b_{ij} \ll 1$.

All the physical masses and energy densities are controlled by the two parameters M^2 and $g\xi$. Even if we assume that the FI term ξ is quantized in units of the Planck mass, by choosing small values for M and g the spectrum of the theory is parametrically below the Planck scale. Interestingly for $\xi \sim O(1)$ in Planck units the relevant mass scale $g\xi$ matches the cut-off scale expected from the weak gravity conjecture (WGC) of [85]. The model above avoids violating the sharp bound from the WGC since it always contains at least one charged particle with mass $m \lesssim g M_{Pl}$. However the absence of an hierarchy between the relevant mass scale of the model and the expected cut-off may signal some deep inconsistency at the quantum gravity level and explain the absence of explicit string theory constructions with genuine FI terms. Alternatively, new physics at the scale $g\xi$ may act as an innocuous spectator, leaving the supergravity Lagrangian with the FI term as a consistent truncation of the whole theory. We are not aware of sharp arguments against any of the two possibilities.

Notice finally that our model is not in conflict[6] with the results of ref. [57], since in the rigid limit none of the matter fields is charged under the gauged $U(1)$: the interactions of the gauged $U(1)$ are a supergravity phenomenon, as in many other examples of gauged supergravity theories.

3.2.3 Model with a genune FI term and supersymmetric AdS vacuum

As our last example, we want to find a supergravity model with a genuine FI term which admits a supersymmetric AdS vacuum. It can be obtained by slightly modifying the model of the previous subsection. To simplify the discussion, we consider only the restriction of the field content to a single z_+ and a single z_-. The Kähler potential, the Killing vectors and the gauge kinetic function are, as in the previous example, and the superpotential is modified as follows:

$$W = M^2\, z_-(1 + \alpha\, z_+\, z_-), \tag{3.22}$$

where α is a constant. The model admits a supersymmetric AdS solution if there is a critical point of the potential, *i.e.* a point such that $\langle \partial_\pm V\rangle = 0$, for which the following equations are satisfied:

$$\begin{aligned}
&\langle W\rangle \neq 0, \\
&\langle D_+ W\rangle = \langle \partial_+ W + W\,\partial_+ K\rangle = \langle M^2 z_-\,(\overline{z_+} + \alpha z_-(1 + |z_+|^2))\rangle = 0, \\
&\langle D_- W\rangle = \langle \partial_- W + W\,\partial_- K\rangle = \langle M^2\,(1 + |z_-|^2 + \alpha z_+ z_-(2 + |z_-|^2))\rangle = 0,
\end{aligned} \tag{3.23}$$

where $\partial_\pm$ means $\partial/\partial z_\pm$. Imposing these conditions, $\langle \partial_\pm V\rangle = 0$ are automatically satisfied. (3.23) are solved by

$$\begin{aligned}
\alpha^2 &= \frac{\rho^2 + 1}{\rho^2(\rho^2 + 2)^2}, \\
\langle z_+\rangle = -\sqrt{\rho^2 + 1}\, e^{i\,\phi}, &\quad \langle z_-\rangle = \rho\, e^{-i\,\phi},
\end{aligned} \tag{3.24}$$

with the modulus of z_-, ρ, greater than zero. It is easy to see that the model has a supersymmetric vacuum and that the energy density at (3.24) is

$$V_0 = -3M^4 e^{2\rho^2+1}\frac{\rho^2}{(\rho^2 + 2)^2} < 0. \tag{3.25}$$

[6] We thank Z. Komargodski for discussions on this point.

It is possible to extend this model by adding all the 23 z_-^i necessary to cancel the anomalies.

3.3 $\mathcal{N}=1$ truncations of $\mathcal{N}=2$ models with classically stable dS vacua

The only known models with extended supersymmetry and classically stable dS vacua are the $\mathcal{N}=2$ models constructed by Fré, Trigiante and Van Proeyen (FTVP) in [14] and some simple $\mathcal{N}=2$ extensions based on the same ingredients [79, 80]. One of their crucial ingredients is the presence of a $\mathcal{N}=2$ FI term, corresponding to an arbitrary constant in the moment map. It is interesting to study the features of the $\mathcal{N}=1$ models obtained from the FTVP models by consistent truncations, to understand the relation between FI terms in $\mathcal{N}=2$ and $\mathcal{N}=1$ supergravity. We will focus on the three FTVP examples, the first one with gauge group $G_0=SO(2,1)\times U(1)$ without hypermultiplets and the other two with gauge group $G_0=SO(2,1)\times SO(3)$ with or without hypermultiplets. In the consistent truncated theories there is always a $U(1)$ factor in the gauge group, which comes from the non compact $SO(2,1)$ factor of the $\mathcal{N}=2$ gauge group and gives a constant FI term. The compact factors of the $\mathcal{N}=2$ gauge group give contributions either to the F-term and D-term factor but are not always associated to FI terms (contrary to what happens in $\mathcal{N}=2$).

3.3.1 $G_0=SO(2,1)\times U(1)$ without hypermultiplets

Following [14] and using the same notation as in section 2.1, we consider $\mathcal{N}=2$ gauged supergravity with three vector multiplets and no hypermultiplets. The three complex scalar fields in the vector multiplets parameterize the special Kähler manifold

$$\frac{SU(1,1)}{U(1)}\times\frac{SO(2,2)}{SO(2)\times SO(2)}\,. \tag{3.26}$$

For a suitable choice of field coordinates (S,y_0,y_1), the Kähler potential reads as in (1.55)

$$K=-\log(S+\overline{S})-\log\left(\frac{Y}{2}\right)\,,\qquad Y=1-2(|y_0|^2+|y_1|^2)+\left|y_0^2+y_1^2\right|^2\,. \tag{3.27}$$

The gauge group is $G_0=SO(2,1)\times U(1)$. We denote by e_0 the coupling constant of the non-compact non-Abelian factor $SO(2,1)$, and by e_1 the parameter controlling the $\mathcal{N}=2$ FI term of the compact $U(1)$ factor: it will not be restrictive to take both of them positive.

Denoting with A_μ^Λ the four vector bosons [$\Lambda=1,2,3$ for $SO(2,1)$, $\Lambda=4$ for $U(1)$], the components of the four Killing vectors along the three complex scalar fields are:

$$X_1^S = X_2^S = X_3^S = X_4^S = 0\,, \tag{3.28}$$

$$X_1^{y_0} = -\frac{i}{2}\,e_0\,(1+y_0^2-y_1^2)\,,\quad X_2^{y_0} = \frac{1}{2}\,e_0\,(1-y_0^2+y_1^2)\,,\quad X_3^{y_0} = i\,e_0\,y_0\,,\quad X_4^{y_0} = 0\,, \tag{3.29}$$

$$X_1^{y_1} = -i\,e_0\,y_0\,y_1\,,\quad X_2^{y_1} = -e_0\,y_0\,y_1\,,\quad X_3^{y_1} = i\,e_0\,y_1\,,\quad X_4^{y_1} = 0\,. \tag{3.30}$$

In $\mathcal{N}=2$ supergravity, the object that plays the rôle of FI term is the triholomorphic momentum map $\mathcal{P}_\Lambda^x$ ($x=1,2,3$), which is constant in the absence of hypermultiplets. In this model, it is zero for $\Lambda=1,2,3$ directions, while $\mathcal{P}_4^x$ is a constant tri-vector with modulus e_1.

In the absence of hypermultiplets, the scalar potential (1.92) can be written as the sum $V=V_1+V_3$, where V_1 and V_3 are related with the square of the supersymmetry transformation of the gauginos and of the gravitinos, respectively. Explicitly, the two contributions to the scalar potential read:

$$V_1 = \frac{e_0^2}{2ReS}\,\frac{P_2^+(y)}{P_2^-(y)}\,,\qquad V_3 = \frac{e_1^2}{2ReS}\,|\cos\theta + i\,S\sin\theta|^2\,, \tag{3.31}$$

where

$$P_2^-(y) = Y\,,\qquad P_2^+(y) = 1-2|y_0|^2+2|y_1|^2+\left|y_0^2+y_1^2\right|^2\,, \tag{3.32}$$

and the angle θ is the de Roo-Wagemans phase, that describes the magnetic rotation of one gauge group factor with respect to the other. In the following, it will not be restrictive to assume $0<\theta<\pi/2$. The potential is minimized for

$$\langle S\rangle = \frac{e_0}{e_1}\frac{1}{\sin\theta} + i\,\cot\theta\,,\quad \langle y_0\rangle \text{ undetermined}\,,\quad \langle y_1\rangle = 0\,, \tag{3.33}$$

and the vacuum energy density is independent of $\langle y_0\rangle$ and given by

$$V_0 \equiv \langle V\rangle = e_0\,e_1\,\sin\theta = e_1^2\,\sin^2\theta\,\langle \mathrm{Re}\,S\rangle > 0\,. \tag{3.34}$$

On the vacuum, two of the four vector fields, associated with the two non-compact generators of $SO(2,1)$, become massive, absorbing two Goldstone degrees of freedom from the scalar field y_0, and the other two vectors remain massless.

The spectrum is most easily computed around the vacuum with $\langle y_0\rangle = 0$. In such a case, the massive vectors are precisely (A_μ^1, A_μ^2), with mass $m_V^2 = V_0/4$, whilst the massless vectors are (A^3_μ, A^4_μ). The two physical complex scalars S and y_1 have masses $m_S^2 = 2\,V_0$ and $m_1^2 = V_0$, and we are in the presence of a classically stable dS vacuum, with completely broken $\mathcal{N}=2$ supersymmetry and vanishing Lagrangian mass terms for the gravitinos.

The rules for consistently truncating gauged $\mathcal{N}=2$ supergravities to $\mathcal{N}=1$ can be found in [86, 87]. We must set to zero one of the two supersymmetry transformation parameters, and project out from the $\mathcal{N}=2$ gravitational multiplet one of the two gravitini and the graviphoton, to obtain the $\mathcal{N}=1$ gravitational multiplet. Each of the three vector multiplets of $\mathcal{N}=2$ contains one vector boson, two spin-1/2 fermions and one complex scalar, and can be truncated to either a $\mathcal{N}=1$ chiral multiplet or to a $\mathcal{N}=1$ vector multiplet.

Starting from three vector multiplets in $\mathcal{N}=2$, and applying the rules of [86], we find that there are two different consistent truncations to $\mathcal{N}=1$: the first with $n_V=1$ vector multiplets and $n_C=2$ chiral multiplets; the second with $n_V=2$ vector multiplets and $n_C=1$ chiral multiplets. Truncations with $n_V=0, n_C=3$ and $n_V=3, n_C=0$ are inconsistent, because the massive vector bosons (including the graviphoton) associated with the two non-compact generators of $SO(2,1)$ must be always truncated away, and their Goldstone degrees of freedom are contained in the complex scalar y_0. We now discuss the two consistent truncations in turn: because of the spontaneously broken non-compact gauge invariance, it will not be restrictive to concentrate for simplicity on the vacuum with $\langle y_0\rangle = 0$. We will see that the FI term of the truncated $\mathcal{N}=1$ theory is associated, in the $\mathcal{N}=2$ theory, to the linear combination of the generator of the compact subgroup of $SO(2,1)$, if gauged, and the component of the $\mathcal{N}=2$ FI terms that survive the truncation, if any. On the other hand, the other components of the $\mathcal{N}=2$ FI terms, if present, produce superpotential terms in the truncated theory.

Truncation with $n_V=1$ and $n_C=2$

The first possibility for a consistent $\mathcal{N}=1$ truncation preserves one $\mathcal{N}=1$ vector multiplet, containing the vector A^3_μ associated with the compact $SO(2)\sim U(1)$ generator inside $SO(2,1)$, and two $\mathcal{N}=1$ chiral multiplets, containing the scalar fields S and y_1. The Kähler potential is obviously the one of eq. (3.27) evaluated for $y_0=0$. The $\mathcal{N}=1$ gauge kinetic function is $f=S$, as can be read directly from the $\mathcal{N}=2$ theory. For consistency, the scalar potential of the $\mathcal{N}=1$ theory must also coincide with the scalar potential of eqs. (3.31) and (3.32), evaluated for $y_0=0$. An interesting feature of the truncated $\mathcal{N}=1$ theory is how such a potential is generated as the sum of an F-term contribution and a D-term contribution,

$$V = V_F + V_D\,, \qquad V_F = V_3\,, \qquad V_D = V_1|_{y_0=0}\,, \tag{3.35}$$

generated by the superpotential (defined as usual up to an irrelevant constant phase factor)

$$W = i\,e_1\,y_1\,(\cos\theta + i\,S\,\sin\theta)\,. \tag{3.36}$$

It is curious that the $\mathcal{N}=2$ FI term associated with the $U(1)$ factor of the $\mathcal{N}=2$ gauge group and with the constant e_1 is mapped into the $\mathcal{N}=1$ F-term potential, whilst the $\mathcal{N}=2$ potential term associated with the non-compact $SO(2,1)$ factor and with the non-Abelian gauge coupling constant e_0 generates a $\mathcal{N}=1$ FI term $\xi = -e_0$ in the prepotential:

$$\mathcal{P}_3 = i\,K_{y_1}\,X_3^{y_1} + i\,\frac{W_{y_1}}{W}\,X_3^{y_1} = -e_0\,\frac{1+|y_1|^2}{1-|y_1|^2}\,. \tag{3.37}$$

As expected, the $\mathcal{N}=2$ vacuum and (truncated) spectrum are reproduced also in the standard $\mathcal{N}=1$ formalism: supersymmetry is broken on a dS background but the $U(1)$ gauge boson has vanishing mass.

Truncation with $n_V = 2$ and $n_C = 1$

The second and last possibility for a consistent $\mathcal{N}=1$ truncation preserves two $\mathcal{N}=1$ vector multiplets, containing A^3_μ and A^4_μ, and only one $\mathcal{N}=1$ chiral multiplet, the one containing S. This time the $\mathcal{N}=1$ Kähler potential is just $K = -\log(S+\overline{S})$, and the $\mathcal{N}=1$ superpotential vanishes, $W=0$. Instead, the gauge kinetic function, which again can be read directly from the $\mathcal{N}=2$ theory, takes the non-trivial form:

$$f_{\Lambda\Sigma} = \begin{pmatrix} S & 0 \\ 0 & \dfrac{S}{\cos\theta\,(\cos\theta + i\,S\,\sin\theta)} \end{pmatrix}\,, \qquad (\Lambda = 3,4)\,. \tag{3.38}$$

Again, the scalar potential of the $\mathcal{N}=1$ theory must coincide with the scalar potential of eqs. (3.31) and (3.32), evaluated for $y_0=0$ and $y_1=0$. This time, the $\mathcal{N}=1$ potential is generated entirely as a D-term contribution:

$$V = V_D = V_3 + V_1|_{y_0=0} = \frac{1}{2ReS}\,\left(e_0^2 + e_1^2\,|\cos\theta + i\,S\,\sin\theta|^2\right)\,, \tag{3.39}$$

thanks to a $\mathcal{N}=1$ FI term associated with each of the two $U(1)$ factors:

$$\mathcal{P}_3 = -e_0\,, \qquad \mathcal{P}_4 = -e_1\,. \tag{3.40}$$

In other words, both the $\mathcal{N}=2$ FI term, associated with $U(1)$ and the constant e_1, and the other $\mathcal{N}=2$ potential term, associated with the non-compact $SO(2,1)$ and the constant e_0, generate constant $\mathcal{N}=1$ FI terms, $\xi_3=-e_0$ and $\xi_4=-e_1$. As required by the consistency of the truncation, the $\mathcal{N}=2$ vacuum and (truncated) spectrum are reproduced also in the standard $\mathcal{N}=1$ formalism, in particular supersymmetry is broken on a dS background but the two $U(1)$ gauge bosons have vanishing masses.

Anomalies in the truncated theory

While the original $\mathcal{N}=2$ theory does not contain chiral fermions, thus all anomaly-cancellation conditions are automatically satisfied, truncating it to $\mathcal{N}=1$ may give rise to an anomalous fermion spectrum. Indeed, we can easily check that this is the case for both truncations considered in the previous subsections.

In the first case, we have a single $U(1)$ and two chiral multiplets (S,y_1) with charges $q^S=0$ and $q^{y_1}=e_0$, thus $Q[\psi^S]=-e_0/2$, $Q[\psi^1]=e_0/2$ and

$$Tr\,Q^3=\frac{1}{2}\,e_0^3\neq 0\,,\qquad Tr\,Q=-10\,e_0\neq 0\,. \tag{3.41}$$

In the second case, the only chiral multiplet S is neutral under both $U(1)$ factors, thus the anomalies are the same for both and are again proportional to those in eq. (3.41).

Inspired by orbifold string constructions, where potential anomalies of the truncated theories are cancelled by twisted sectors localized at the orbifold fixed points, we may think of supplementing the field content of the truncated theories by additional charged multiplets, to achieve anomaly cancellation while keeping the same vacuum. A simple addition that recovers anomaly freedom while keeping the same vacuum consists of $n_2=5$ chiral multiplets Φ_2^i of charge $q_2^i=0$ and $n_3=125$ chiral multiplets Φ_3^i of charge $q_3^i=(3/5)\,e_0$.

3.3.2 $G_0=SO(2,1)\times SO(3)$ without hypermultiplets

We consider now an $\mathcal{N}=2$ gauged supergravity with five vector multiplets and no hypermultiplets. The five complex scalar fields in the vector multiplets parameterize the special Kähler manifold

$$\frac{SU(1,1)}{U(1)}\times\frac{SO(2,4)}{SO(2)\times SO(4)}\,. \tag{3.42}$$

For a suitable choice of field coordinates (S, y_0, y_x), $x = 1, 2, 3$, the Kähler potential reads as in (1.55)

$$K = -\log(S+\overline{S}) - \log\left(\frac{Y}{2}\right)\,, \qquad Y = 1 - 2(|y_0|^2 + \sum_{x=1}^{3}|y_x|^2) + \left|y_0^2 + \sum_{x=1}^{3} y_x^2\right|^2\,. \tag{3.43}$$

The gauge group is $G_0 = SO(2,1)\times SO(3)$. We denote by e_0 the coupling constant of the non-compact non-Abelian factor $SO(2,1)$, and by e_1 the coupling constant of the compact $SO(3)$ factor: it will not be restrictive to take both of them positive.

Denoting with A^Λ_μ the six vector bosons [$\Lambda = 1, 2, 3$ for $SO(2,1)$, $\Lambda = 4, 5, 6$ for $SO(3)$], the components of the six Killing vectors along the five complex scalar fields are:

$$X^S_\Lambda = 0\,, \tag{3.44}$$

$$X^{y_0}_1 = -\frac{i}{2}\,e_0\,(1+y_0^2-y_x^2)\,, \quad X^{y_0}_2 = \frac{1}{2}\,e_0\,(1-y_0^2+y_x^2)\,, \quad X^{y_0}_3 = i\,e_0\,y_0\,, \quad X^{y_0}_{x+3} = 0\,, \tag{3.45}$$

$$X^{y_x}_1 = -i\,e_0\,y_0\,y_x\,, \quad X^{y_x}_2 = -e_0\,y_0\,y_x\,, \quad X^{y_x}_3 = i\,e_0\,y_x\,, \quad X^{y_z}_{x+3} = e_1\epsilon_{xwz}y_w\,. \tag{3.46}$$

In $\mathcal{N}=2$ supergravity, the object that plays the rôle of FI term is the triholomorphic momentum map $\mathcal{P}^x_\Lambda$ $(x = 1, 2, 3)$, which is constant in the absence of hypermultiplets. In this model, it is zero for $\Lambda = 1, 2, 3$ directions, while $\mathcal{P}^x_{z+3} = -e_1\delta^x_z$.

In the absence of hypermultiplets, the scalar potential (1.92) can be written as the sum $V = V_1 + V_3$. The expression is more involved than the previous case

$$V = -\frac{1}{S+\bar{S}}\left(e_1^2|\cos\theta + i\,S\,\sin\theta|^2\frac{P_4^{(1)}(y)}{Y^2} + e_0^2\frac{P_4^{(0)}(y)}{Y^2}\right) \tag{3.47}$$

with $P_4^{(0)}, P_4^{(1)}$ polynomials of holomorphic degree four in y, such that

$$\partial_{y_x}P_4^{(0)}\Big|_{y=0} = \partial_{y_x}P_4^{(1)}\Big|_{y=0}\,, \quad P_4^{(0)}\Big|_{y=0} = 1,\;\; P_4^{(1)}\Big|_{y=0} = 3. \tag{3.48}$$

The angle θ is the de Roo-Wagemans parameter, which describes the magnetic rotation of one gauge group factor with respect to the other. In the following, it will not be restrictive to assume $0 < \theta < \pi/2$. The potential is minimized for

$$\langle S\rangle = \frac{e_0}{e_1}\frac{1}{\sqrt{3}\sin\theta} + i\,\cot\theta\,, \quad \langle y_0\rangle \text{ undetermined}\,, \quad \langle y_x\rangle = 0\,, \tag{3.49}$$

and the vacuum energy density is independent of $\langle y_0\rangle$ and given by

$$V_0 \equiv \langle V\rangle = \sqrt{3}e_0\,e_1\,\sin\theta = e_1^2\,\sin^2\theta\,\langle \mathrm{Re}\,S\rangle > 0\,. \tag{3.50}$$

On the vacuum, two of the four vector fields, associated with the two non-compact generators of $SO(2,1)$, become massive, absorbing two Goldstone degrees of freedom from the scalar field y_0, and the other vectors remain massless.

The spectrum is most easily computed around the vacuum with $\langle y_0 \rangle = 0$. In such a case, the massive vectors are precisely (A^1_μ, A^2_μ), with mass $m^2_V = V_0/4$. The four physical complex scalars S and y_x, $x = 1, 2, 3$ have masses $m^2_S = 2\,V_0$ and $m^2_x = V_0$, and we are again in the presence of a classically stable dS vacuum, with completely broken $\mathcal{N} = 2$ supersymmetry and vanishing Lagrangian mass terms for the gravitinos.

Truncation to $\mathcal{N} = 1$

Starting from five vector multiplets in $\mathcal{N} = 2$, and applying the rules of [86], we find that there is only one consistent truncation to $\mathcal{N} = 1$ with $n_V = 2$ vector multiplets and $n_C = 3$ chiral multiplets.

A consistent $\mathcal{N} = 1$ truncation preserves two $\mathcal{N} = 1$ vector multiplets, one containing the vector A^3_μ associated with the compact $SO(2) \sim U(1)$ generator inside $SO(2,1)$ and one containing the vector A^6_μ associated with the $U(1)$ generator inside $SO(3)$, and three $\mathcal{N} = 1$ chiral multiplets, containing the scalar fields S and y_1, y_2. The Kähler potential is obviously the one of eq. (3.43) evaluated for $y_0 = y_3 = 0$. The $\mathcal{N} = 1$ gauge kinetic function takes the non-trivial form:

$$f_{\Lambda\Sigma} = \begin{pmatrix} S & 0 \\ 0 & \dfrac{S}{\cos\theta\,(\cos\theta + i\,S\,\sin\theta)} \end{pmatrix}, \qquad (\Lambda = 3, 4)\,. \tag{3.51}$$

The superpotential (defined as usual up to an irrelevant constant phase factor) is

$$W = e_1\,(y_1 - i\,y_2)\,(\cos\theta + i\,S\,\sin\theta)\,. \tag{3.52}$$

and the prepotentials along the directions $\Lambda = 3, 6$ are

$$\mathcal{P}_3 = -e_0 \frac{1 - (y_1^2 + y_2^2)(\bar{y}_1^2 + \bar{y}_2^2)}{Y}, \qquad \mathcal{P}_6 = -2i\,e_1 \frac{\bar{y}_1\,y_2 - \bar{y}_2 y_1}{Y} + e_1, \tag{3.53}$$

with $Y = 1 - 2y_1\bar{y}_1 - 2y_2\bar{y}_2 + (y_1^2 + y_2^2)(\bar{y}_1^2 + \bar{y}_2^2)$. Now the $\mathcal{N} = 2$ FI term associated with the $SO(3)$ factor of the $\mathcal{N} = 2$ gauge group and with the constant e_1 is mapped partially into the $\mathcal{N} = 1$ F-term potential and partially into the D-term generated by $\mathcal{P}_6$, whilst the $\mathcal{N} = 2$ potential term associated with the non-compact $SO(2,1)$ factor and with the non-Abelian gauge coupling constant e_0 generates a

$\mathcal{N}=1$ FI term $\xi=-e_0$ in the prepotential, as in the previous cases. As expected, the $\mathcal{N}=2$ vacuum and (truncated) spectrum are reproduced also in the standard $\mathcal{N}=1$ formalism: supersymmetry is broken on a dS background but the $U(1)$ gauge bosons have vanishing masses.

Anomaly cancellation conditions

Also in this case we have an anomalous fermion spectrum. Redefining $Y_1 = y_1 - i\,y_2$ and $Y_2 = y_1 + iy_2$, the Killing vectors are in the diagonal form $X_3^{Y_1} = i\,e_0 Y_1$, $X_6^{Y_1} = i\,e_1 Y_1$ and $X_3^{Y_2} = i\,e_0 Y_2$, $X_6^{Y_2} = -i\,e_1 Y_2$ and the charges for the fermionic fields are

$$Q_3[\psi_\mu] = Q_3[\lambda^3] = Q_3[\lambda^6] = -Q_3[\psi^S] = Q_3[\psi^{Y_1}] = Q_3[\psi^{Y_2}] = e_0/2 \tag{3.54}$$

and

$$Q_6[\psi_\mu] = Q_6[\lambda^3] = Q_6[\lambda^6] = -Q_6[\psi^S] = Q_6[\psi^{Y_1}] = e_1/2,\; Q_6[\psi^{Y_2}] = -3e_1/2. \tag{3.55}$$

The traces for Q_3 and Q_6 charges are then

$$Tr\,Q_3^3 = \frac{3}{4}\,e_0^3 \neq 0\,, \qquad Tr\,Q_3 = -9\,e_0 \neq 0 \tag{3.56}$$

and

$$Tr\,Q_6^3 = -\frac{11}{4}\,e_1^3 \neq 0\,, \qquad Tr\,Q_6 = -11\,e_1 \neq 0\,. \tag{3.57}$$

Again we may think of supplementing the field content of the truncated theories by additional charged multiplets, to achieve anomaly cancellation while keeping the same vacuum. A simple addition that recovers anomaly freedom while keeping the same vacuum consists of $n_3 = 1$ chiral multiplets Φ_2^i of charge $q_3^{3i} = -e_0$, $q_6^{3i} = -23e_1/40$ and $n_4 = 21$ chiral multiplets Φ_4^i of charge $q_3^{4i} = e_0$, $q_6^{4i} = 43e_1/40$.

3.3.3 $G_0 = SO(2,1) \times SO(3)$ with hypermultiplets

In this last case the special sector does not differ from the previous case. What changes is the hypersector and the triholomorphic momentum map. The quaternionic manifold is now

$$\frac{SO(4,2)}{SO(4) \times SO(2)}, \tag{3.58}$$

spanned by eight real scalar fields $q^{u=1,\dots,8}$. A suitable choice of coordinates is the solvable coordinates, introduced in chapter 1. The gauge group is $G_0 = SO(2,1) \times$

$SO(3)$ which is embedded in the $SO(2,4)$ subgroup of both the isometry group of the special Kähler and the quaternionic manifold. Being e_0, e_1 the coupling constants of the non-compact and compact factor of the gauge group respectively, we now couple the quaternionic scalars with factors $r_0 e_0$, $r_1 e_1$, where r_0, r_1 can be 0 or 1, indicating whether the corresponding geometry is gauged or not.

The triholomorphic momentum map here is not a constant but depends on the scalars q^u and can be calculated using the formulas in chapter 1. An important property is that in $q = 0$ it is

$$\mathcal{P}^x_{\Lambda=1,2,3}\Big|_{q=0} = 0, \qquad \mathcal{P}^x_{\Lambda=z+3}\Big|_{q=0} = -r_1 e_1 \delta^x_z, \tag{3.59}$$

which is, if we set $r_1 = 1$, the analogue of the $SO(3)$ FI term which in the model without hypermultiplets was introduced by hand.

The scalar potential is now the sum of three contributions $V = V_1 + V_2 + V_3$, where V_1, V_2, V_3 are the square of the supersymmetry transformations of the gaugini, hyperini and gravitini, respectively. The expression of the potential at the origin of the quaternionic manifold is

$$V_0 = V|_{y=q=0} = -\frac{1}{S+\bar{S}}\left(3r_1^2 e_1^2 |\cos\theta + i\, S \sin\theta|^2 + e_0^2(1+2r_0^2)\right), \tag{3.60}$$

the same as the one found in the second model except that, if the $SO(2,1)$ isometry is gauged there is a rescaling in the $SO(2,1)$ coupling $e_0^2 \to 3\, e_0^2$.

As shown in [14], there is a de Sitter critical point at $q = y = 0$ and $S = S^{(0)} = \frac{e_0\sqrt{1+2r_0^2}}{\sqrt{3} r_1 e_1 \sin\theta} + i\cot\theta$. Note that it requires $r_1 \neq 0$ and then from now on we set $r_1 = 1$. The spectrum of the theory is, in case of $r_0 = 0$,

$$2V_0,\ V_0\,(\times 3),\ \frac{2}{3}V_0(\times 3),\ 0(\times 2), \tag{3.61}$$

where there are two massless complex modes from which one is the Goldstone boson, y_0. In case of $r_0 = 1$ there is a mixing between the scalar of the special Kähler and quaternionic manifold and the sperctrum is

$$2V_0,\ \frac{2}{3}V_0,\ \frac{4}{3}V_0\,(\times 3),\ 0(\times 4). \tag{3.62}$$

with four massless complex modes from which only one is a Goldstone boson.

Truncation to $\mathcal{N} = 1$

The truncation of the special Kähler sector is the same as the second case: there are two vector multiplets and five chiral multiplets. The vector bosons that survive after the truncation are A^3_μ and A^6_μ and the scalar fields are S, y_1, y_2. The truncation of the quaternionic sector requires a separate discussion. First of all, the consistency of the truncation is guaranteed only if $r_0 = 0$ and then the hypermultiplets are neutral under $SO(2,1)$. The truncation of an hypermultiplet may give one or no chiral multiplets. The case with no chiral multiplets is the same as the case without hypermultiplets and so we don't repeat the discussion here. The interesting case is when from two hypermultiplets one obtains two chiral multiplets. The surviving scalar fields are a_3, b_2, h_1, h_2 which can be conveniently arranged into two complex scalar fields $Y_1 = e^{h_1} - i\, a_3$, $Y_2 = e^{h_2} - i\, b_2$. The origin of the quaternionic manifold is then in $Y_1 = Y_2 = 1$.

The Kähler potential is

$$K = -\log(S + \bar{S}) - \log\left(\frac{Y}{2}\right) - 2\log\left(\frac{Y_1 + \bar{Y}_1}{2}\right) - 2\log\left(\frac{Y_2 + \bar{Y}_2}{2}\right) \tag{3.63}$$

and the superpotential is

$$W = e_1\, Y_2 \left(y_1\, Y_1 - \frac{i}{2} y_2 (1 + Y_1^2)\right) (\cos\theta + i\, S\, \sin\theta), \tag{3.64}$$

which becomes (3.52) in the origin of the quaternionic manifold. The Killing vectors are

$$\begin{aligned} X_3^S = X_6^S &= 0, \qquad X_3^{Y_2} = X_6^{Y_2} = 0, \\ X_3^{y_1} &= i e_0 y_1, \qquad X_6^{y_1} = -e_1 y_2, \\ X_3^{y_2} &= i e_0 y_2, \qquad X_6^{y_2} = e_1 y_1, \\ X_3^{Y_1} &= 0, \qquad X_6^{Y_1} = \frac{i}{2} e_1 (1 - Y_1^2), \\ X_3^{Y_2} &= X_6^{Y_2} = 0. \end{aligned} \tag{3.65}$$

Under the transformation along the third direction the Kähler potential is invariant and $W \to W(1 + i\, e_0)$, while under the transformation along the sixth direction

$$K \to K + i\, e_1\, (Y_1 - \bar{Y}_1), \qquad W \to W(1 - i\, e_1 Y_1). \tag{3.66}$$

It means that there is no constant FI term for the second $U(1)$ factor with coupling constant e_1 and we have prepotential

$$\mathcal{P}_6 = -2i\, e_1 \frac{\bar{y}_1\, y_2 - \bar{y}_2 y_1}{Y} + e_1 \frac{2 - Y_1^2 - \bar{Y}_1^2}{2(Y_1 + \bar{Y}_1)} + e_1 \frac{Y_1 + \bar{Y}_1}{2}. \tag{3.67}$$

For the first $U(1)$ factor, conversely, we have a FI $-e_0$, as in all the previous truncated models and the prepotential is

$$\mathcal{P}_3 = -e_0 \frac{1 - (y_1^2 + y_2^2)(\bar{y}_1^2 + \bar{y}_2^2)}{Y}. \tag{3.68}$$

As expected, also in this case the $\mathcal{N} = 2$ vacuum and (truncated) spectrum are reproduced also in the standard $\mathcal{N} = 1$ formalism: supersymmetry is broken on a dS background but the $U(1)$ gauge bosons have vanishing masses.

Chapter 4

On the vacua of M-theory reductions to 4-dimensions

As we have already seen in chapter 2, the effective theory describing the-low energy dynamics of M-theory compactified on a seven-torus with fluxes (but without localized sources) is gauged $\mathcal{N} = 8, D = 4$ supergravity. In spite of the fact that $\mathcal{N} = 8$ models cannot be used to obtain realistic phenomenology, understanding the conditions for the occurrence of interesting vacua with spontaneously broken $\mathcal{N} = 8$ supersymmetry from different points of view (the one of gauged $D = 4$ supergravity and the one of generalized flux compactifications of M-theory) may shed light on the difficulties in reproducing them in the context of string compactifications.

The $\mathcal{N} = 8$ theory can be usually consistently truncated to theories with a lower amount of supersymmetry, where we can write the full scalar potential and study the stabilization of the moduli in a more familiar theoretical framework. Here, we focus on the truncations to $\mathcal{N} = 1, D = 4$ supergravity, for which we give the full superpotential and the quadratic constraints. These theories, while non realistic, are much more constrained than generic $\mathcal{N} = 1$ theories, and can be used as playgrounds for the exploration of part of the moduli space of their 'mother' theories.

This chapter is organized as follows. In section 4.1 we analyze a definite truncation of $\mathcal{N} = 8$ gauged supergravity to $\mathcal{N} = 1$ supergravity, seen as a special case (the singular case) of a G_2 reduction of the eleven-dimensional theory and formulated in terms of just seven chiral $\mathcal{N} = 1$ superfields, corresponding to the seven main moduli of the internal space. We also write down explicitly all the constraints on the fluxes that survive after the truncation. In section 4.2 we give the form of the full superpotential corresponding to the most general set of geometric and non-geometric fluxes compatible with the $\mathcal{N} = 1$ truncation. In section 4.3 and section 4.4 we discuss the embedding in M-theory of examples of $\mathcal{N}=8$ gauged supergravity and their truncation to $\mathcal{N}=1$. In particular in section 4.3 we discuss the models studied in [26] and in section 4.4 we consider the case of flat gaugings and we find a way to recover the Scherk-Schwarz models with four parameters by turning on geometric and non-geometric fluxes.

4.1 The $N=1$ truncation

Our starting point is the singular limit of compact G_2-holonomy[1] manifolds, corresponding to toroidal orbifolds of the form $X_7=\mathbb{T}^7/(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2)$, following the construction of [88].

The $(\mathbb{Z}_2)^3$ action on the coordinates y^I $(I=1,\ldots,7)$ on $\mathbb{T}^7$ is

$$\begin{aligned}\mathbb{Z}_2(y^I) &= \{-y^1,-y^2,-y^3,-y^4,c+y^5,y^6,y^7\},\\ \mathbb{Z}_2'(y^I) &= \{a_1-y^1,a_2-y^2,y^3,y^4,c-y^5,c-y^6,c+y^7\},\\ \mathbb{Z}_2''(y^I) &= \{a_3-y^1,y^2,a_4-y^3,y^4,a_5-y^5,c+y^6,-y^7\},\end{aligned}\tag{4.1}$$

where the coefficients a_i and c can be either 0 or 1/2. This orbifold has singularities that can be either blown-up or eliminated by turning on c and a_4 simultaneously. Then the orbifold action becomes free and produces a smooth 7-manifold. We work here with the case where the a_i's and c are all zero.

The orbifold action (4.1) preserves a G_2-structure, given by the G_2 invariant combination of the 7 surviving 3-forms. In terms of vielbeins $e^I=R_I dy^I$ $(I=1,\ldots,7)$, with R_I the radii of the 7 circles composing $\mathbb{T}^7$, the associated 3-form Φ reads

$$\Phi=e^1\wedge e^2\wedge e^7-e^3\wedge e^4\wedge e^7-e^5\wedge e^6\wedge e^7+e^1\wedge e^3\wedge e^6-e^2\wedge e^3\wedge e^5+e^1\wedge e^4\wedge e^5+e^2\wedge e^4\wedge e^6.\tag{4.2}$$

The effective 4-dimensional theory obtained by compactification on X_7 can be recast in the form of an $\mathcal{N}=1$ supergravity coupled to matter. The effective theory for the bulk fields contains only the $\mathcal{N}=1$ graviton multiplet and 7 chiral multiplets T_I $(I=1,...,7)$, while there are no vector multiplets.

The chiral multiplets contain some of the moduli of the theory . The imaginary parts τ_I of these moduli come from the reduction of the M-theory 3-form potential

[1] To be precise, these orbifolds have a discrete holonomy group $\mathbb{Z}_2\times\mathbb{Z}_2'\times\mathbb{Z}_2''\subset G_2$. Only the manifolds obtained after blowing up the singularities have G_2-holonomy.

on the internal manifold:

$$C = \tau_I(x)\phi^I(y), \tag{4.3}$$

where $\phi^I(y)$ is a basis of closed three forms

$$\phi^1 = -dy^1 \wedge dy^2 \wedge dy^7, \quad \phi^2 = dy^3 \wedge dy^4 \wedge dy^7, \quad \phi^3 = dy^5 \wedge dy^6 \wedge dy^7,$$

$$\begin{aligned} \phi^4 = dy^2 \wedge dy^3 \wedge dy^5, \qquad \phi^5 = -dy^2 \wedge dy^4 \wedge dy^6, \\ \phi^6 = -dy^1 \wedge dy^3 \wedge dy^6, \qquad \phi^7 = -dy^1 \wedge dy^4 \wedge dy^5. \end{aligned} \tag{4.4}$$

The real parts t_I are associated with the internal metric components and parametrize the volumes of the seven surviving 3-cycles:

$$\Phi = t_I(x)\phi^I(y). \tag{4.5}$$

The holomorphic combinations $T_I = t_I + i\,\tau_I$ can be thought of as expansion parameters of a "complexified" G_2-form

$$C + i\,\Phi = i\,T_I(x)\,\phi^I(y). \tag{4.6}$$

This is analogous to the complexification of Kähler moduli in type II string theories. In our notation, $T_{1,2,3}$ correspond to $U_{1,2,3}$ in IIB, $T_{4,7,6}$ correspond to $T_{1,2,3}$ in IIB and T_5 is the S modulus in IIB.

In terms of the the radii R_I we have

$$\begin{aligned} |t_1| &= R_1R_2R_7, \quad |t_2| = R_3R_4R_7, \quad |t_3| = R_5R_6R_7, \quad |t_4| = R_2R_3R_5, \\ |t_5| &= R_2R_4R_6, \quad |t_6| = R_1R_3R_6, \quad |t_7| = R_1R_4R_5. \end{aligned} \tag{4.7}$$

Obviously not all the fluxes survive under the (4.1) identifications. We denote with a_I, b_I, c_I the indices that appears in ϕ^I,

$$(a_I, b_I, c_I) = \{(1,2,7),\,(3,4,7),\,(5,6,7),\,(2,3,5),\,(2,4,6),\,(1,3,6),\,(1,4,5)\}, \tag{4.8}$$

and by i_I, j_I, k_I, l_I the dual indices,

$$\begin{aligned} (i_I, j_I, k_I, j_I) = \ & \{(3,4,5,6),\,(1,2,5,6),\,(1,2,3,4), \\ & (1,4,6,7),\,(1,3,5,7),\,(2,4,5,7),\,(2,3,6,7)\}. \end{aligned} \tag{4.9}$$

The fluxes which survive are those in table 4.1.

The seven moduli parametrize the coset $[SU(1,1)/U(1)]^7$. This follows from the reduction of the generators of the duality group $E_{7(7)}$: of the 133 generators only

Representation	flux name	independent components
$\mathbf{1}_{+7}$	$g_{1234567}$	1
$\overline{\mathbf{35}}_{+5}$	$g_{i_I j_I k_I l_I}$	7
$\mathbf{28}_{+3}$	$\omega_{a_I b_I}^{c_I}, \omega_{b_I c_I}^{a_I}, \omega_{c_I a_I}^{b_I}$	21
$\mathbf{7}_{+3}$	–	0
$\overline{\mathbf{28}}_{+1}$	$\xi'^{mm}, \quad m = 1, \ldots, 7$	7
$\overline{\mathbf{21}}_{+1}$	–	0
$\overline{\mathbf{224}}_{+1}$	$R_{i_I}^{j_I k_I l_I}$, $R_{j_I}^{k_I l_I i_I}$, $R_{k_I}^{j_I l_I i_I}$,$R_{l_I}^{i_I j_I k_I}$	28
$\mathbf{28}_{-1}$	$\theta'_{mm}, \quad m = 1, \ldots 7$	7
$\mathbf{21}_{-1}$	–	0
$\mathbf{224}_{-1}$	$f_{j_i k_I l_I}^{i_I}$, $f_{i_I k_I l_I}^{j_I}$, $f_{i_I j_I l_I}^{k_I}$, $f_{i_I j_I k_I}^{l_I}$	28
$\overline{\mathbf{140}}_{-3}$	$Q_{c_I}^{a_I b_I}$, $Q_{a_I}^{c_I b_I}$, $Q_{b_I}^{a_I c_I}$	21
$\overline{\mathbf{7}}_{-3}$	–	0
$\mathbf{35}_{-5}$	$h^{i_I j_I k_I l_I}$	7
$\mathbf{1}_{-7}$	$\tilde{g}^{1234567}$	1

Table 4.1: M-theory fluxes in the truncated $\mathcal{N} = 1$ theory.

21 survive after the truncation, namely $t_{a_I b_I c_I}, t_m{}^m, t^{a_I b_I c_I}$. We can define $E_I = 2t_{a_I}{}^{a_I} + 2t_{b_I}{}^{b_I} + 2t_{c_I}{}^{c_I} - \frac{24}{7}t$ in order to have the $\mathfrak{sl}(2)$ algebra

$$[E_I, t_{a_I b_I c_I}] = \frac{60}{7}\, t_{a_I b_I c_I}, \quad \left[E_I, t^{a_I b_I c_I}\right] = -\frac{60}{7}\, t^{a_I b_I c_I}, \quad \left[t_{a_I b_I c_I}, t^{a_I b_I c_I}\right] = E_I. \tag{4.10}$$

There are no vector bosons and thus no gauge group surviving after the truncation. However there are some quadratic constraints that survive after the truncation and which have to be taken into account in order to consider models compatible with M-theory compactifications. These constraints can be read from those in $\mathcal{N} = 8$ and, considering the fluxes that survive after the truncation as in table 4.1, we can identify which constraints are non-trivial. A discussion can be found at the end of Appendix B.

4.2 The $N = 1$ superpotential

The reduced $\mathcal{N} = 1$ theory inherits a scalar potential from the parent $\mathcal{N} = 8$ supergravity. Since in $\mathcal{N} = 1$ there is no gauge group, the residual scalar potential does not have a D-term component and can be written in terms of a Kähler potential and a superpotential. The Kähler potential is the usual

$$K = -\sum_I \log(T_I + \bar{T}_I). \tag{4.11}$$

The form of the superpotential could be inferred directly from the scalar potential, but it would be an involved calculation. However, we can construct the superpotential also from the integration of suitable combinations of the fluxes of table 4.1 and the G_2-invariant form Φ and the M-theory 3-form potential C over the internal space. The geometric fluxes appear from

$$W_{geom} = \int_{X_7} \left(g_7 + (C + i\Phi) \wedge g + \frac{1}{2}(C + i\Phi) \wedge \omega \cdot (C + i\Phi) \right), \tag{4.12}$$

where ω acts as a differential

$$\omega \cdot \phi^I = \omega \cdot dy^a \wedge dy^b \wedge dy^c \, \phi^I_{abc} = \omega^a_{[de}\phi^I_{bc]a} \, dy^d \wedge dy^e \wedge dy^b \wedge dy^c. \tag{4.13}$$

Explicitly, the superpotential depending only on geometric fluxes is

$$\begin{aligned} W_{geom} = g_7 + i\,(&-T_1 g_{3456} + T_2 g_{1256} + T_3 g_{1234} + T_4 g_{1647} \\ &-T_5 g_{1357} - T_6 g_{2457} + i\,T_7 g_{2367}) - (-\omega^7_{56} T_1 T_2 - \omega^7_{34} T_1 T_3 \\ &+\omega^7_{12} T_2 T_3 + \omega^6_{13} T_3 T_5 + \omega^6_{24} T_3 T_6 - \omega^6_{75} T_5 T_6 + \omega^5_{23} T_3 T_7 + \omega^5_{14} T_3 T_4 \\ &-\omega^5_{67} T_4 T_7 + \omega^4_{51} T_2 T_5 + \omega^4_{62} T_2 T_7 - \omega^4_{73} T_5 T_7 + \omega^3_{61} T_2 T_4 - \omega^3_{52} T_2 T_6 \\ &-\omega^3_{47} T_4 T_6 + \omega^2_{46} T_1 T_4 - \omega^2_{35} T_1 T_5 + \omega^2_{71} T_4 T_7 + -\omega^1_{45} T_1 T_6 + -\omega^1_{36} T_1 T_7 + -\omega^1_{27} T_6 T_7). \end{aligned} \tag{4.14}$$

We can construct a similar action for the other fluxes. Each of the fluxes contributes to the superpotential by contracting $(C + i\,\Phi)^{n/2}$, where n is given by 7 minus the charge under the $O(1,1)$ classifying the representations. It depends on the fact that the superpotential has charge 7 under $O(1,1)$ and the moduli have charge 2. The geometric superpotential W_{geom} is at most quadratic in the moduli. The non-geometric fluxes on the other hand give contributions up to order three in the moduli fields. For instance, ξ' and R fluxes have charge +1 under $O(1,1)$ and then we expect they generate some terms in superpotential of order three in the moduli. Explicitly,

$$
\begin{aligned}
W_3 = -i\,(& R_1^{357}T_2T_3T_4 - R_1^{467}T_2T_3T_5 - R_1^{234}T_2T_4T_5 - R_1^{256}T_3T_4T_5 + R_2^{367}T_2T_3T_6 \\
&+R_2^{457}T_2T_3T_7 - R_2^{143}T_2T_6T_7 - R_2^{165}T_3T_6T_7 + R_3^{276}T_1T_3T_5 + R_3^{517}T_1T_3T_7 \\
&-R_3^{124}T_1T_5T_7 + R_3^{456}T_3T_5T_7 + R_4^{275}T_1T_3T_4 - R_4^{176}T_1T_3T_6 - R_4^{213}T_1T_4T_6 \\
&+R_4^{653}T_3T_5T_6 + R_5^{247}T_1T_2T_5 + R_5^{137}T_1T_2T_6 - R_5^{126}T_1T_5T_6 + R_5^{346}T_2T_5T_6 \\
&+R_6^{237}T_1T_2T_4 - R_6^{147}T_1T_2T_7 - R_6^{152}T_1T_4T_7 + R_6^{354}T_2T_4T_7 + R_7^{326}T_4T_5T_6 \\
&+R_7^{254}T_4T_5T_7 + R_7^{315}T_4T_6T_7 - R_7^{264}T_5T_6T_7) + i\,(\xi'^{11}T_1T_6T_7 + \xi'^{22}T_1T_4T_5 \\
&+\xi'^{33}T_2T_4T_6 + \xi'^{44}T_2T_5T_7 + \xi'^{55}T_3T_4T_7 + \xi'^{66}T_3T_5T_6 + \xi'^{77}T_1T_2T_3)
\end{aligned}
\tag{4.15}
$$

The quartic terms come from θ'- and f-fluxes:

$$
\begin{aligned}
W_4 = -(&\theta'_{11}T_2T_3T_4T_5 + \theta'_{22}T_2T_3T_6T_7 + \theta'_{33}T_1T_3T_5T_7 \\
&+\theta'_{44}T_1T_2T_4T_6 + \theta'_{55}T_1T_2T_5T_6 + \theta'_{66}T_1T_2T_5T_7 + \theta_{77}T_4T_5T_6T_7) \\
+(&f^1_{357}T_1T_5T_6T_7 + f^1_{647}T_1T_4T_6T_7 + f^1_{234}T_1T_3T_6T_7 + f^1_{256}T_1T_2T_6T_7 \\
&+f^2_{637}T_1T_4T_5T_7 + f^2_{547}T_1T_4T_5T_6 + f^2_{143}T_1T_3T_4T_5 + f^2_{165}T_1T_2T_4T_5 \\
&+f^3_{267}T_2T_4T_6T_7 + f^3_{517}T_2T_4T_5T_6 + f^3_{124}T_2T_3T_4T_6 + f^3_{456}T_1T_2T_4T_6 \\
&+f^4_{257}T_2T_5T_6T_7 + f^4_{167}T_2T_4T_5T_7 + f^4_{213}T_2T_3T_5T_7 + f^4_{653}T_1T_2T_4T_7 \\
&+f^5_{427}T_3T_4T_6T_7 + f^5_{137}T_3T_4T_5T_7 + f^5_{126}T_2T_3T_4T_7 + f^5_{346}T_1T_3T_4T_7 \\
&+f^6_{327}T_3T_5T_6T_7 + f^6_{417}T_3T_4T_5T_6 + f^6_{521}T_2T_3T_5T_6 + f^6_{543}T_1T_3T_5T_6 \\
&+f^7_{326}T_1T_2T_3T_7 + f^7_{245}T_1T_2T_3T_6 + f^7_{315}T_1T_2T_3T_5 + f^7_{146}T_1T_2T_3T_4)
\end{aligned}
\tag{4.16}
$$

Finally, the terms of order 5, 6 and 7 are obtained from Q-, h- and $\tilde{g}_7$-fluxes, which are the duals of g_7, g and τ fluxes. Their contribution to the superpotential is given by

$$
\begin{aligned}
W_{567} = & -i\tilde{g}_7 T_1T_2T_3T_4T_5T_6T_7 - (h^{3456}T_2T_3T_4T_5T_6 + h^{1256}T_1T_3T_4T_5T_6T_7 \\
& + h^{1234}T_1T_2T_4T_5T_6T_7 + h^{1647}T_1T_2T_3T_5T_6T_7 + h^{1357}T_1T_2T_3T_4T_6T_7 \\
& + h^{2457}T_1T_2T_3T_4T_5T_7 + h^{2367}T_1T_2T_3T_4T_5T_6) + i\,(Q_7^{56}T_3T_4T_5T_6T_7 \\
& + Q_7^{34}T_2T_4T_5T_6T_7 + Q_7^{12}T_1T_4T_5T_6T_7 + Q_6^{13}T_1T_2T_4T_6T_7 + Q_6^{24}T_1T_2T_4T_5T_7 \\
& + Q_6^{75}T_1T_2T_3T_4T_7 + Q_5^{32}T_1T_2T_5T_6T_7 + Q_5^{14}T_1T_2T_4T_5T_6 + Q_5^{67}T_1T_2T_3T_5T_6 \\
& + Q_4^{51}T_1T_3T_4T_6T_7 + Q_4^{62}T_1T_3T_4T_5T_6 + Q_4^{73}T_1T_2T_3T_4T_6 + Q_3^{61}T_1T_3T_5T_6T_7 \\
& + Q_3^{25}T_1T_3T_4T_5T_7 + Q_3^{47}T_1T_2T_3T_5T_7 + Q_2^{46}T_2T_3T_5T_6T_7 + Q_2^{53}T_2T_3T_4T_6T_7 \\
& + Q_2^{71}T_1T_2T_3T_5T_6 + Q_1^{45}T_2T_3T_4T_5T_7 + Q_1^{36}T_2T_3T_4T_5T_6 + Q_1^{27}T_1T_2T_3T_4T_5)
\end{aligned} \tag{4.17}
$$

The full superpotential is then given by

$$
W = W_{geom} + W_3 + W_4 + W_{567}. \tag{4.18}
$$

We summarize in the following table how the fluxes appear in the superpotential. Note that $I \neq J \neq K \neq ...$ is always understood.

Scalar combination	flux
1	$g_{1234567}$
T_I	$g_{i_I j_I k_I l_I}$
$T_J T_K$	${\omega_{a_I b_I}}^{c_I}$, where c_I appears in ϕ^J and ϕ^K
$T_I T_J T_K$	ξ'^{ii}, where i is the common index between ϕ^I, ϕ^J, ϕ^K
$T_I T_J T_K$	$R_{i_L}^{j_L k_L l_L}$, associated to $\phi^I_{j_L k_L \bullet} \phi^J_{l_L k_L \bullet} \phi^K_{j_L l_L \bullet}$
$T_I T_J T_K T_L$	θ'_{ii}, associated to all ϕ's without the index i
$T_I T_J T_K T_L$	$f^{i_I}_{j_I k_I l_I}$ and J, K, L don't contain the index i_I
T^5	$Q^{a_I b_I}_{c_I}$, where we select T^I and those without the index c_I
T^6	$h^{i_I j_I k_I l_I}$, all but T_I
T^7	$\tilde{g}^{1234567}$

4.3 Some new vacua

What we have pointed out in the previous section allows us to determine the M-theory embedding of various examples of $\mathcal{N}=1$ supergravity with seven main moduli. In this section we consider some examples of $\mathcal{N}=8, D=4$ vacua discussed in [26] and we give their embedding in M-theory and their truncation to $\mathcal{N}=1$.

4.3.1 Choice of a symplectic frame

As we have already seen in chapter 1, in 4 dimensions vector fields are dual to other vector fields. This means that in maximal 4-dimensional supergravity the same gauge group could be gauged by minimal electric couplings to the 28 vector fields in the Lagrangian before gauging or also by (some of) the dual magnetic fields. The choice of symplectic frame (i.e. which vectors are electric and which are magnetic) provides important information on the description of the gauging and therefore also on the vacua of the theory. The split of the vectors into electric and magnetic ones is related to the splitting of the fundamental representation of $\mathrm{E}_{7(7)}$, **56**, into two distinct sets of 28 vector fields: $A_\mu^{\mathcal{M}} = (A_\mu^\Lambda, A_{\mu\,\Lambda})$ ($\mathcal{M}=1,...,56$, $\Lambda = 1,...,28$), where A_μ^Λ transform in some definite representation of the group G_0 of global symmetries of the Lagrangian. For any choice of the gauge group G_g, there is at least one electromagnetic frame in which G_g (modulo Abelian ideals) is realized as a subgroup of G_0. This is the electric frame, where we can consistently set $\Theta^{\Lambda\alpha} = 0$ and the quadratic constraint (1.120) is automatically satisfied. Imposing $\Theta^{\Lambda\alpha} = 0$ or any other specific choice, however, would generally be a too strong restriction.

A natural choice for the symplectic frame is one that allows for a direct comparison of the results with already known ones, in particular those related to the vacua of the CSO(p,q,r) gaugings. We can use the so-called SL(8,$\mathbb{R}$) basis, where the $\mathrm{E}_{7(7)}$ generators are decomposed according to the embedding of one of its SL(8,$\mathbb{R}$) subgroups. As usual it is useful to consider the branching of relevant representations of $E_{7(7)}$, here with respect to $SL(8,\mathbb{R})$, in particular

$$\mathbf{56} \longrightarrow \mathbf{28} + \mathbf{28}', \tag{4.19}$$

$$\mathbf{133} \longrightarrow \mathbf{63} + \mathbf{70}, \tag{4.20}$$

$$\mathbf{912} \longrightarrow \mathbf{36} + \mathbf{420} + \mathbf{36}' + \mathbf{420}'. \tag{4.21}$$

The **56** electric and magnetic vector fields split into **28** electric A_μ^{AB} and **28**$'$ magnetic fields $A_{\mu\,AB}$ ($A, B, ... = 1,...,8$) and the **133** generators of $E_{7(7)}$ split into **63** generators $t_A{}^B$ and **70** generators t^{ABCD}. Finally, the **912** embedding tensor components, given by the tensor product of the **56** and **133** representations of $E_{7(7)}$, are decomposed as

	28	**28′**
63	**36 + 420**	**36′ + 420′**
70	**420′**	**420**

Since only one **420** representation appears in the branching of the **912**, the two **420** representations in the table must coincide, and so must the **420′** representations. This implies that, if $\Theta_{\mathcal{M}}{}^{\alpha}$ had a contribution in the **420**, it would describe a coupling of the gauge fields to the generators in the **70**, but also, at the same time, induce a coupling of the dual gauge fields to the generators in the **63** of SL(8,ℝ). In the case of purely electric gaugings, this analysis restricts the allowed embedding tensor to just the $\Theta_{AB}{}^{C}{}_{D}$ components. Actually, electric gaugings are described by a symmetric tensor in the **36′** representation of SL(8,ℝ), which is called θ_{AB} [89]. Also, the embedding tensor is completely specified by

$$\Theta_{AB}{}^{C}{}_{D} = \delta^{C}_{[A}\,\theta_{B]D}, \tag{4.22}$$

while all the other components are set to zero. The quadratic constraint (1.120) is obviously identically satisfied for such gaugings, hence any symmetric real matrix θ_{AB} defines a consistent gauging, even in the case of a non-invertible matrix.

While all entries could be non-vanishing, the gauge groups resulting from $\theta \neq 0$ depend only on the number of positive (p), negative (q) and zero (r) eigenvalues. In the search for extrema of the scalar potential in this framework, it should be possible to restrict our analysis to the case of diagonal θ. In fact, the condition to get an extremum of the scalar potential is invariant under similarity transformations $\theta \to P\theta P^{-1}$. Moreover, it is also independent on rescalings, so that we are left with 7 parameters:

$$P\theta P^{-1} \propto \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, 1)\,. \tag{4.23}$$

The electric gaugings in such a frame are restricted to the SO(p,q) groups when $r = 0$, defined as the finite SL(8) transformations g that leave θ invariant: $g\theta g^T = \theta$. When $r \neq 0$ one obtains the group contractions CSO(p,q,r) [90, 91]. The actual gauge group in this case is $SO(p,q) \ltimes T^{r(8-r)}$, with $T^{r(8-r)}$ the group of $r(8-r)$ translations. The total number of inequivalent gaugings is 24, which is calculated taking into account that the gauge group does not change if we just permute the eigenvalues between them, nor if we change an overall sign (which would correspond to exchanging p and q). The permutations modify the embedding of the gauge group in $\mathrm{E}_{7(7)}$, but in a completely trivial way that does not affect in any way our discussion of the stationary points.

In our analysis we will also consider gaugings where the **36** representation is turned on. We will describe the corresponding gauging parameters by the symmetric tensor ξ^{AB} [26]. Following once more the decomposition of the embedding tensor representations presented above, these parameters turn on the magnetic components of the embedding tensor:

$$\Theta^{ABC}{}_D = \delta_D^{[A}\, \xi^{B]C}. \tag{4.24}$$

Choosing $\xi \neq 0$ while $\theta = 0$ is just a relabeling of the vector fields and brings us back to the previous discussion. However, when both θ and ξ are turned on, the quadratic constraint is not identically satisfied anymore and one gets new and interesting situations. The quadratic constraint gives the relation

$$\delta_E^D\, \xi^{AB}\theta_{BC} = \xi^{DB}\theta_{BE}\, \delta_C^A, \tag{4.25}$$

from which also $\xi^{AB}\theta_{BC} = \frac{1}{8}\,\mathrm{Tr}(\theta\xi)\,\delta_C^A$ follows. From this we see that we have only two options. Whenever the matrix θ is invertible, the constraint is solved by

$$\xi = c\,\theta^{-1}, \quad c \in \mathbb{R}, \quad \text{for} \quad \det\theta \neq 0. \tag{4.26}$$

On the other hand, if θ is not invertible it has a non-trivial kernel. In this case the setup solving (4.25) is given by taking ξ non-zero only in the subspace defined by the kernel of θ.

4.3.2 M-theory embedding and reduction to $\mathcal{N} = 1$

The models introduced above are $\mathcal{N} = 8, D = 4$ supergravity theories with global symmetry group $SL(8,\mathbb{R})$. We may ask if they could be embedded in M-theory. The M-theory embedding is specified by the branching of relevant transformations of $E_{7(7)}$ with respect to the global symmetry group $GL(7,\mathbb{R}) \sim SL(7,\mathbb{R}) \times O(1,1)$. Since it is a subgroup of $SL(8,\mathbb{R})$, it is simple to find the embedding of the gaugings in M-theory in this basis, by considering the branching of relevant representations of $SL(8,\mathbb{R})$ with respect to $SL(7,\mathbb{R}) \times O(1,1)$.

For instance, the **28** electric vector fields A_μ^{AB} split into $\overline{\mathbf{21}}$ vector fields A_μ^{mn} and $\overline{\mathbf{7}}$ $A_\mu^{m8} = A_\mu^m$, where the index $A = 1, ..., 8$ index has been split in $m = 1, ..., 7$ and 8.

The same is true for the $\mathbf{28}'$ dual vectors, which split into $\mathbf{21}$ $A_{\mu mn}$ and $\mathbf{7}$ $A_{\mu m}$. The generators of $E_{7(7)}$ are $t_A{}^B$, that split in $t_m{}^n$, $t_m{}^8 = t_m$ and $t_8{}^m = t^m$, and t^{ABCD}, that split in $t^{mnp8} = t^{mnp}$ and $t^{mnpq} = \epsilon^{mnpqr_1r_2r_38}t_{r_1r_2r_3}$.

The embedding tensor components we are interested in are those in the $\mathbf{36}'$ and $\mathbf{36}$. The branching of the former with respect to $SL(7,\mathbb{R}) \times O(1,1)$ is

$$\mathbf{36}' \longrightarrow \mathbf{28}_{-1} + \mathbf{7}_{+3} + \mathbf{1}_{+7}, \tag{4.27}$$

which means that θ_{AB} gives the $\mathbf{28}$ $\theta_{mn} = \theta'_{mn}$, the $\mathbf{7}$ $\theta_{m8} = \omega_m$ and the singlet $\theta_{88} = g_7$. The same happens for the $\mathbf{36}$ ξ^{AB}:

$$\mathbf{36} \longrightarrow \overline{\mathbf{28}} + \overline{\mathbf{7}} + \mathbf{1}, \tag{4.28}$$

where $\overline{\mathbf{28}}$ correspond to $\xi'^{(mn)}$, $\overline{\mathbf{7}}$ to $\xi^{m8} = Q^m$ and $\mathbf{1}$ to $\xi^{88} = \tilde{g}_7$. The symmetric matrices θ_{AB}, ξ^{AB} can be arranged as

$$\theta = \begin{pmatrix} \theta'_{mn} & \omega_m \\ \omega_n & g_7 \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi'^{mn} & Q^m \\ Q^n & \tilde{g}_7 \end{pmatrix}. \tag{4.29}$$

The quadratic constraints on the fluxes can be read from (4.25), or equivalently from the explicit equations in Appendix B. If $\theta \neq 0$ and $\xi \neq 0$, they are

$$\begin{gathered} g_7\tilde{g}_7\delta^p_q - \theta'_{mq}\xi'^{mp} + Q^m\delta^q_p\omega_m - Q^p\omega_p = 0, \\ d'_3g_7Q^q\delta^p_r + d''_3g_7Q^p\delta^q_r - (a'_3 - \tfrac{1}{2}b'_3)\omega_r\xi'^{qp} - (a''_3 + \tfrac{1}{2}b'_3)\omega_m\xi'^{mp}\delta^q_r = 0, \\ a'_3\tilde{g}_7\omega_q\delta^r_p + a''_3\tilde{g}_7\omega_p\delta^r_q - (d'_3 - \tfrac{1}{2}c'_3)Q^r\theta'_{qp} - (d''_3 + \tfrac{1}{2}c'_3)Q^m\theta'_{mp}\delta^r_q = 0, \\ a'_3d'_3(\omega_pQ^s\delta^q_r - \omega_rQ^q\delta^s_p) + (a'_3d''_3 - d'_3a''_3)(\omega_pQ^q\delta^s_r - \omega_rQ^s\delta^q_p) \\ +\tfrac{1}{2}b'_3c'_3(\xi'^{qm}\delta^s_p\theta'_{rm} - \xi'^{sm}\delta^q_r\theta'_{pm}) = 0. \end{gathered} \tag{4.30}$$

We have set some of the coefficients to one by a suitable redefinition of the fluxes. Of course, if $\xi = 0$ or $\theta = 0$ all the equations are identically satisfied.

Following section 4.1 we can reduce this model to $\mathcal{N} = 1$. With this choice of fluxes, the reduction is very simple. The fluxes that survive after the truncation are $\theta_{(mm)}, \xi^{(mm)}$ and $g_7, \tilde{g}_7$. The quadratic constraints are now only

$$g_7\tilde{g}_7 - \theta_{mm}\xi^{mm} = 0. \tag{4.31}$$

As we expect, no vector bosons survive after the reduction and the scalar potential is only F-term. The superpotential has the following general form:

$$\begin{aligned} W = g_7 &+ \theta_{11} T_2 T_3 T_4 T_5 + \theta_{22} T_2 T_3 T_6 T_7 + \theta_{33} T_1 T_3 T_5 T_7 + \theta_{44} T_1 T_3 T_4 T_6 \\ &+\theta_{55} T_1 T_2 T_5 T_6 + \theta_{66} T_1 T_2 T_4 T_7 + \theta_{77} T_4 T_5 T_6 T_7 \\ &+i \left(\tilde{g}_7 T_1 T_2 T_3 T_4 T_5 T_6 T_7 + \xi^{11} T_1 T_6 T_7 + \xi^{22} T_1 T_4 T_5 + \xi^{33} T_2 T_4 T_6 \right. \\ &\left. +\xi^{44} T_2 T_5 T_7 + \xi^{55} T_3 T_4 T_7 + \xi^{66} T_3 T_5 T_6 + \xi^{77} T_1 T_2 T_3 \right). \end{aligned} \tag{4.32}$$

4.3.3 $\theta \neq 0$ and $\xi = 0$

The first scenario we consider is the simple instance where the gauge group of the $\mathcal{N} = 8$ theory is contained in the SL(8,$\mathbb{R}$) electric frame, which means that we are considering vacua of the CSO(p, q, r) gaugings. The fluxes we turn on are thus $\theta'_{(mn)}, \tau_m, g_7$. In $\mathcal{N} = 1$ supergravity, we automatically restrict our analysis to diagonal θ matrices, *i.e.* we work with only θ'_{mm} and g_7 fluxes.

G_g	$\vec{\lambda}$	Λ
SO(8)	$(1, 1, 1, 1, 1, 1, 1, 1)$	AdS
	$(5, 1, 1, 1, 1, 1, 1, 1)$	AdS
SO(3,5)	$(-3, -3, -3, 1, 1, 1, 1, 1)$	dS
SO(4,4)	$(-1, -1, -1, -1, 1, 1, 1, 1)$	dS
CSO(2,0,6)	$(1, 1, 0, 0, 0, 0, 0, 0)$	Mink.

Table 4.2: Possible $\theta = \mathrm{diag}(\lambda_1, \ldots, \lambda_8)$, up to normalization, which give rise to a critical point of the scalar potential at the origin of the moduli space. In the third column we indicate the type of vacuum arising from these solutions and in the first column the gauge group of the $\mathcal{N} = 8$ model.

The $\mathcal{N} = 1$ superpotential in this case is

$$\begin{aligned} W = g_7 &+ \theta'_{11} T_2 T_3 T_4 T_5 + \theta'_{22} T_2 T_3 T_6 T_7 + \theta'_{33} T_1 T_3 T_5 T_7 + \theta'_{44} T_1 T_3 T_4 T_6 \\ &+\theta'_{55} T_1 T_2 T_5 T_6 + \theta'_{66} T_1 T_2 T_4 T_7 + \theta'_{77} T_4 T_5 T_6 T_7 \end{aligned} \tag{4.33}$$

and we can assign in principle arbitrary values $\vec{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_8)$ to the fluxes. We want to find the values of $\vec{\lambda}$ such that there is a critical point at the origin of the scalar manifold, *i.e* for $T_i = 1$. We do not care about which of those values are assigned to g_7 or θ'_{mm} because the minimization conditions are invariant under the exchange of any two of these λ's.

For the simple case $\vec{\lambda} = (a, 1, 1, 1, 1, 1, 1, 1)$, with a real constant, we find that a could be only 1 or 5. All the λ's are in this case positive, thus these models correspond to an $SO(8)$ gauging in $\mathcal{N} = 8$. The value of the scalar potential is negative in both cases, so we have AdS extrema. The first case, in particular, is a supersymmetric AdS vacuum. All the squared scalar masses are $-\frac{2}{3}|V_0|$ and the Breitenlohner–Freedman bound $|m^2/V_0| \leq 3/4$ is of course satisfied, thus the vacuum is stable. The second case is not supersymmetric and there are six directions in the moduli space that do not satisfy the Breitenlohner–Freedman bound $|m^2/V_0| = \frac{4}{5} > \frac{3}{4}$.

In case of $\vec{\lambda} = (a, b, 1, 1, 1, 1, 1, 1)$ we found only already known solutions. Leaving three parameters free, we found that there is another critical point if $\vec{\lambda} = (-3, -3, -3, 1, 1, 1, 1, 1)$, which corresponds in $\mathcal{N} = 8$ to an $SO(3, 5)$ gauging. It has a positive scalar potential at the origin of the moduli space (dS vacuum), but it is not stable because there are two negative squared scalar masses, $-\frac{2}{3}|V_0|$ and $-2|V_0|$.

To conclude the cases with all the λ's different from zero, we consider $\vec{\lambda} = (a, b, c, d, 1, 1, 1, 1)$. In this case we recover all the solutions previously found and one more given by $a = b = c = d = -1$, which corresponds in $\mathcal{N} = 8$ to an $SO(4, 4)$ gauging. The vacuum for these values of the fluxes is again dS and again it is unstable, because there is a scalar field with squared scalar mass given by $-|V_0|$.

Finally, we can consider the cases in which r of the λ's are zero, which correspond to $CSO(p, q, r)$ gaugings in $\mathcal{N} = 8$. We find that there is a vacuum only in the case $r = 6$, which corresponds to the gauging $CSO(2, 0, 6)$ in $\mathcal{N} = 8$. This last case gives a stable Minkowski vacuum and its spectrum coincides with the one of the Scherk-Schwarz model with the four mass parameters which are equal.

In table 4.3 we give the masses for the scalar fields both in the $\mathcal{N} = 1$ truncated theory and in the underlying $\mathcal{N} = 8$ supergravity [26]. It is easy to note that all the masses in $\mathcal{N} = 1$ are also present in the spectrum of $\mathcal{N} = 8$, as we expect. Moreover the flat directions of $\mathcal{N} = 8$ are all projected out in $\mathcal{N} = 1$, except for the last case.

4.3.4 $\theta \neq 0$ and $\xi \neq 0$

There is another similar setup that not only allows us to reproduce some other $\mathcal{N} = 8$ vacua of the SO(8) theory, but also to reveal new stationary points and new gauge groups embedded in SL(8).

G_g in $\mathcal{N}=8$	$m^2_{(\text{multiplicity})}$ in $\mathcal{N}=1$	$m^2_{(\text{multiplicity})}$ in $\mathcal{N}=8$
SO(8)	$-\frac{2}{3}(14)$	$-\frac{2}{3}(70)$
SO(8)	$2_{(1)},\ -\frac{4}{5}(6),\ -\frac{2}{5}(7)$	$0_{(7)},\ 2_{(1)},\ -\frac{4}{5}(27),\ -\frac{2}{5}(35)$
SO(3,5)	$4_{(2)},\ -2_{(1)},\ 2_{(6)},\ \frac{4}{3}(4),\ -\frac{2}{3}(1)$	$0_{(15)},\ 4_{(5)},\ -2_{(1)},\ 2_{(30)},\ \frac{4}{3}(14),\ -\frac{2}{3}(5),$
SO(4,4)	$1_{(4)},\ 2_{(9)},\ -2_{(1)}$	$0_{(16)},\ 1_{(16)},\ 2_{(36)},\ -2_{(2)}$
CSO(2,0,6)	$0_{(9)},\ \frac{1}{2}(4),\ 2_{(1)}$	$0_{(48)},\ \frac{1}{2}{(20)},\ 2_{(2)}$

Table 4.3: Values of the scalar masses in units of the cosmological constant (except in the Minkowski case), both in $\mathcal{N}=1$ and $\mathcal{N}=8$ models.

A priori we are considering a setup where 72 parameters of the embedding tensor, namely the fluxes $g_7, \tilde{g}_7, \theta'_{mn}, \xi'^{mn}, Q^m, \tau_M$ are turned on, although only half of them are independent after we impose the quadratic constraints (4.30). The fact that we can restrict to diagonal θ and ξ without loss of generality is therefore an important simplification. Unfortunately, even after such simplifications, and even if we work in $\mathcal{N}=1$ supergravity, the equation determining the vacua is not anymore quadratic in the θ parameters, because of the presence of the inverse matrix $\xi \propto \theta^{-1}$. Therefore, we cannot find analytic solutions for the general case. However, we can still find analytic solutions by varying 1 or 2 eigenvalues. Denoting again the g_7 and θ'_{mm} values with $\vec{\lambda}=(\lambda_1,\ldots,\lambda_8)$, we can chose

$$\vec{\lambda}=(r\,s)^{-1/8}\mathrm{diag}(r,s,1,1,1,1,1,1). \tag{4.34}$$

In addition to $r=s=1$, which corresponds to the $\mathcal{N}=8$ AdS supersymmetric vacuum, we found two more solutions

i. $r=1$, $s=(7-3\sqrt{5})/2$, which corresponds to the $\mathrm{SO}(7)_-$ vacuum[2].

ii. $r=3+2\sqrt{2}$, $s=1/r$, which corresponds to the $\mathcal{N}=8$ $SU(4)_- \simeq SO(6)$ vacuum found in [92], with $\Lambda/g^2=-8$.

By relaxing the conditions on θ and ξ, other vacua and other gauge theories are identified. We therefore consider the general solution of the quadratic constraint for

[2]The $\pm$ symbol differentiates the two SO(7) vacua found so far in the $N=8$ theory. These are related by a $\mathbb{Z}_2$ symmetry.

#	Gauging	$\Theta_{\mathcal{M}}{}^{\alpha}$	Λ
i	SO(8)	$\theta = \xi^{-1} = a \oplus \mathbb{1}_7$	AdS
ii	SO(8)	$\theta = \xi^{-1} = (a, 1/a) \oplus \mathbb{1}_6$	AdS
iii	SO(7,1)	$\theta = \xi = -1 \oplus \mathbb{1}_7$	AdS
iv	SO(7,1)	$\theta = \xi^{-1} = (b_1, b_2) \oplus \mathbb{1}_6$	AdS
v	SO(7,1)	$\theta = \xi^{-1} = -1 \oplus \mathbb{1}_5 \oplus (c, 1/c)$	AdS
vi	SO(6,2) $\simeq$ SO*(8)	$\theta = \xi = (-\mathbb{1}_2 \oplus \mathbb{1}_6)/2\sqrt{2}$	Mink
vii	SO(5,3)	$\theta = \xi = -\mathbb{1}_3 \oplus \mathbb{1}_5$	dS
viii	SO(7) $\ltimes T^7$	$\theta = 0 \oplus \mathbb{1}_7,\ \xi = \sqrt{5} \oplus \mathbb{0}_7$	AdS
ix	SO(7) $\ltimes T^7$	$\theta = 4 \oplus \mathbb{1}_6 \oplus 0,\ \xi = \mathbb{0}_7 \oplus 2\sqrt{2}$	AdS
x	SO(6) $\times$ SO(1,1) $\ltimes T^{12}$	$\theta = \mathbb{1}_6 \oplus \mathbb{0}_2,\ \ \xi = \mathbb{0}_6 \oplus (\sqrt{2}, -\sqrt{2})$	AdS
xi	SO(6) $\times$ SO(1,1) $\ltimes T^{12}$	$\theta = 3 \oplus \mathbb{1}_5 \oplus \mathbb{0}_2,\ \ \xi = \mathbb{0}_6 \oplus (\sqrt{3}, -\sqrt{3})$	AdS
xii	SO(4) $\times$ SO(2,2) $\ltimes T^{16}$	$\theta = \mathbb{1}_4 \oplus \mathbb{0}_4,\ \ \xi = \mathbb{0}_4 \oplus \mathbb{1}_2 \oplus -\mathbb{1}_2$	Mink
xiii	SO(2)^2 $\ltimes T^{20}$	$\theta = (\mathbb{1}_2 \oplus \mathbb{0}_6)\sqrt{2},\ \ \xi = (\mathbb{0}_2 \oplus \mathbb{1}_2 \oplus \mathbb{0}_4)\sqrt{2}$	Mink

Table 4.4: $\mathcal{N} = 8$ vacuum solutions found for $\theta, \xi \neq 0$. The SO(8) case is also included here, although with a different normalization. Minkowski solutions are normalized to easily compare their mass spectra (see Table 4.5). $a = 3 + 2\sqrt{2}$; $b_1 = 1/2(-1 + \sqrt{2})(-1 + \sqrt{5})$, $b_2 = -1/2(1 + \sqrt{2})(-1 + \sqrt{5})$; $c = 2 + \sqrt{3}$.

case	$m^2{}_{(\text{multiplicity})}$ in $\mathcal{N}=1$	$m^2{}_{(\text{multiplicity})}$ in $\mathcal{N}=8$
i	$2_{(1)},\ -\frac{4}{5}{}_{(6)},\ -\frac{2}{5}{}_{(7)}$	$2_{(1)}, -\frac{4}{5}{}_{(27)}, -\frac{2}{5}{}_{(35)}, 0_{(7)}$
ii	$2_{(2)},\ -1_{(5)},\ -\frac{1}{2}{}_{(4)},\ 0_{(3)}$	$2_{(2)}, -1_{(20)}, -\frac{1}{4}{}_{(20)}, 0_{(28)}$
iii	$2_{(1)},\ -\frac{4}{5}{}_{(6)},\ -\frac{2}{5}{}_{(7)}$	$2_{(1)}, -\frac{4}{5}{}_{(27)}, -\frac{2}{5}{}_{(35)}, 0_{(7)}$
iv	$2_{(2)},\ -1_{(5)},\ -\frac{1}{2}{}_{(4)},\ 0_{(3)}$	$2_{(2)}, -1_{(20)}, -\frac{1}{4}{}_{(20)}, 0_{(28)}$
v	$2_{(3)},\ -\frac{4}{3}{}_{(4)},\ \frac{2}{3}{}_{(1)},\ 0_{(6)}$	$2^{(3)}, -4/3^{(14)}, 2/3^{(5)}, 0^{(48)}$
vi	$2_{(1)},\ \frac{1}{2}{}_{(4)},\ 0_{(9)}$	$2_{(2)}, \frac{1}{2}{}_{(20)}, 0_{(48)}$
vii	$-2_{(1)},\ 4_{(2)},\ 2_{(6)},\ \frac{4}{3}{}_{(4)},\ -\frac{2}{3}{}_{(1)}$	$-2^{(1)}, 4^{(5)}, 2^{(30)}, 4/3^{(14)}, -2/3^{(5)}, 0^{(15)}$
viii	$2_{(1)},\ -\frac{4}{5}{}_{(6)},\ -\frac{2}{5}{}_{(7)}$	$2_{(1)}, -\frac{4}{5}{}_{(27)}, -\frac{2}{5}{}_{(35)}, 0_{(7)}$
ix	$2_{(2)},\ -1_{(5)},\ -\frac{1}{2}{}_{(4)},\ 0_{(3)}$	$2_{(2)}, -1_{(20)}, -\frac{1}{4}{}_{(20)}, 0_{(28)}$
x	$2_{(2)},\ -1_{(5)},\ -\frac{1}{2}{}_{(4)},\ 0_{(3)}$	$2_{(2)}, -1_{(20)}, -\frac{1}{4}{}_{(20)}, 0_{(28)}$
xi	$2_{(3)},\ -\frac{4}{3}{}_{(4)},\ \frac{2}{3}{}_{(1)},\ 0_{(6)}$	$2^{(3)}, -4/3^{(14)}, 2/3^{(5)}, 0^{(48)}$
xii	$4_{(2)},\ 2_{(2)},\ 1_{(4)},\ 0_{(6)}$	$4^{(4)}, 2^{(12)}, 1^{(16)}, 0^{(38)}$
xiii	$4_{(2)},\ 2_{(2)},\ 1_{(4)},\ 0_{(6)}$	$4^{(4)}, 2^{(12)}, 1^{(16)}, 0^{(38)}$

Table 4.5: Mass spectra for the new vacua both in the $\mathcal{N}=1$ and in the $\mathcal{N}=8$ case. When $\Lambda \neq 0$, masses are normalized with respect to it.

these fluxes, namely

$$\tilde{g}_7 = \frac{1}{g_7},\ \xi'^{mm} = \frac{1}{\theta'_{mm}} \quad \text{or} \quad g_7\tilde{g}_7 - \theta'\,\xi' = 0. \tag{4.35}$$

In the first case, the resulting $\mathcal{N}=8$ gauge algebra is still $\mathfrak{so}(p,q)$, with $p+q=8$, however the gauge connection now also involves the "magnetic" vector fields $A_{\mu\,AB}$. Contractions to $\mathfrak{cso}(p,q,r)$ are no more allowed, because they would violate the quadratic constraint. We can, however, consider the case $g_7\tilde{g}_7 + \theta'\,\xi' = 0$. Both $\theta = (\theta', g_7)$ and $\xi = (\xi', \tilde{g}_7)$ must have a non-empty kernel, which corresponds in $\mathcal{N}=8$ supergravity to gauge groups that can be considered as "superpositions" of two $\mathfrak{cso}(p,q,r)$ ones, where the two semisimple factors commute with each other (because of the quadratic constraint), while some of the nilpotent generators are in common between the groups and others add up to form a bigger abelian algebra.

The generic form of these gaugings is

$$(\mathrm{SO}(p,q) \times \mathrm{SO}(p',q')) \ltimes T^{(8-r)r+(8-r')(r'+r-8)} \tag{4.36}$$

where $p+q+r = p'+q'+r' = 8$. They can be defined as different contractions of $\mathrm{SO}(p+p', 8-p-p')$.

Again, finding all solutions analytically is difficult, but several new results can already be obtained by varying only a subset of the eigenvalues of θ and ξ. The results of [26] are summarized in Table 4.4.

Inspecting Table 4.4, we see that θ and ξ have no fixed normalization. This contrasts with the analysis presented previously. When only $\theta \neq 0$, we can obviously change the values of θ to move on the parameter space associated to the moduli space, but we can also rescale the gauge coupling constant to define equivalence classes of gaugings. In particular, we have seen that once we normalize the value of the cosmological constant for a specific vacuum of a given gauging, we can keep consistent normalizations by using rescalings that do not change the determinant of θ. On the other hand, when $\xi = c\theta^{-1}$ for $c \neq 0$, there are three parameters that can be tuned: the proportionality constant c, the determinant of θ, and the gauge coupling constant g. Moreover, a rescaling of the coupling constant acts in the same way on θ and ξ, while a rescaling of $\det\theta$ acts inversely on ξ. Introducing equivalence classes of gaugings for different parameterization is therefore more subtle. For instance, in addition to the vacua in Table 4.4, there is also an SO(4,4) vacuum in this class, whose cosmological constant can be normalized to the same value as the one of the vacuum obtained with only $\theta \neq 0$. In fact we can show that this new vacuum constructed with both $\theta \neq 0$ and $\xi \neq 0$ is equivalent to the previous one.

In table 4.5 we display the mass spectra of the scalar fields for the new vacua, both in $\mathcal{N} = 1$ and $\mathcal{N} = 8$. Note that there are some cases in which the mass spectrum coincides, not only at the level of $\mathcal{N} = 1$, but also in $\mathcal{N} = 8$. A possible explaination of these facts is given in [26].

4.4 Flat gaugings and flux interpretation

4.4.1 Flat gaugings

We want here to find realization in terms of M-theory fluxes of the theory formulated by Cremmer-Scherk-Schwarz [37] via generalized dimensional reduction from five dimensions, already discussed in chapter 2. The Scherk-Schwarz twist is chosen in the global symmetry group $E_{6(6)}$ of the five dimensional theory, so, in order to discuss these gaugings, we have to chose an electric subgroup of $E_{7(7)}$ which contains $E_{6(6)}$. In the context of maximally supersymmetric supergravity theories, the resulting 4-dimensional models are $N = 8$ gauged supergravities with a gauge group that has the following structure:

$$\begin{aligned} [X_0, X_u] &= M_u{}^v X_v, \\ [X_u, X_v] &= 0. \end{aligned} \tag{4.37}$$

Here $u, v = 1, \ldots 27$ are related to the **27** dimensional representation of $\mathrm{E}_{6(6)}$ and in fact, the electric frame for such models is the one that follows from the decomposition of $\mathrm{E}_{7(7)}$ under $\mathrm{E}_{6(6)}$:

$$\mathfrak{e}_{7(7)} = \mathfrak{e}_{6(6)} + \mathfrak{so}(1,1) + \mathbf{27_{-2}} + \mathbf{27'_{+2}}, \tag{4.38}$$

where $\mathbf{p_q}$ denotes the corresponding representations of $\mathfrak{e}_{6(6)} + \mathfrak{so}(1,1)$. In the explicit representation of the gauge algebra above, X_λ is in the $\mathbf{27'_{+2}}$ and X_0 is a generic Cartan generator of $\mathfrak{usp}(8) \subset \mathfrak{e}_{6(6)}$. The gauge group is then a semidirect product of two abelian factors

$$G = U(1) \ltimes T^n, \qquad n \leq 24 \tag{4.39}$$

and the matrix $M_u{}^v$ gives a representation of the $\mathrm{U}(1) \subset \mathrm{USp}(8) \subset \mathrm{E}_{6(6)}$.

The branchings of the relevant $\mathrm{E}_{7(7)}$ representations with respect to $\mathrm{E}_{6(6)} \times \mathrm{O}(1,1)$ are given below (the subscript refers to the $\mathrm{SO}(1,1)$ weight),

$$\begin{aligned} \mathbf{56} &= \overline{\mathbf{27}}_{-1} + \mathbf{1}_{-3} + \mathbf{27}_{+1} + \mathbf{1}_{+3}\,, \\ \mathbf{133} &= \mathbf{78}_0 + \overline{\mathbf{27}}_{+2} + \mathbf{27}_{-2} + \mathbf{1}_0\,, \\ \mathbf{912} &= \overline{\mathbf{351}}_{-1} + \mathbf{351}_{+1} + \overline{\mathbf{27}}_{-1} + \mathbf{27}_{+1} + \mathbf{78}_{-3} + \mathbf{78}_{+3}\,. \end{aligned} \tag{4.40}$$

In this basis the 28 electric vectors fields are $A_\mu^\Lambda = (A_\mu^u, A_\mu^0)$, where A_μ^u $(u = 1, .., 27)$ are the dimensionally reduced five-dimensional vectors in the $\mathbf{27}_{-1}$ of $E_{6(6)} \times O(1,1)$ and A_μ^0 is the Kaluza-Klein vector in the $\mathbf{1}_{-3}$ of the same group, the embedding tensor has just electric components Θ_Λ^α and the gauge generators read:

$$\begin{aligned} X_\Lambda &= \begin{cases} X_0 = \Theta_{0,u}{}^v t_v{}^u \\ X_u = \Theta_u{}^v t_v \end{cases}, \\ \Theta_{0,u}{}^v &= \Theta_u{}^v = M_u{}^v \in E_{6(6)}, \end{aligned} \tag{4.41}$$

where $M_u{}^v$ is the twist matrix depending in general on 78 parameters, $t_u{}^v$ are the $E_{6(6)}$ generators, and t_u are $E_{7(7)}$ generators in the $\overline{\mathbf{27}}_{+2}$, according to the branching of **133**. Generically the matrix M has three zero eigenvalues and 24 eigenvalues equal to the linear combinations $\pm m_i \pm m_j$ with $i > j$ taking the values $1, \ldots, 4$. Just as before, we can conveniently summarize the branchings of $(\mathbf{56} \times \mathbf{133}) \cap \mathbf{912}$ in a table,

	$\overline{\mathbf{27}}_{-1}$	$\mathbf{1}_{-3}$	$\mathbf{27}_{+1}$	$\mathbf{1}_{+3}$
$\mathbf{78}_0$	$\overline{\mathbf{351}}_{-1} + \overline{\mathbf{27}}_{-1}$	$\mathbf{78}_{-3}$	$\mathbf{351}_{+1} + \mathbf{27}_{+1}$	$\mathbf{78}_{+3}$
$\mathbf{27}_{-2}$	$\mathbf{78}_{-3}$		$\overline{\mathbf{351}}_{-1} + \overline{\mathbf{27}}_{-1}$	$\mathbf{27}_{+1}$
$\overline{\mathbf{27}}_{+2}$	$\mathbf{351}_{+1} + \mathbf{27}_{+1}$	$\overline{\mathbf{27}}_{-1}$	$\mathbf{78}_{+3}$	
$\mathbf{1}_0$	$\overline{\mathbf{27}}_{-1}$		$\mathbf{27}_{+1}$	

(4.42)

Again equivalent representations in this table must coincide, since the $\mathbf{912}$ contains just a single copy of each. With a similar reasoning as above, it follows that viable gaugings involve the gauge fields (in the $\overline{\mathbf{27}}_{-1} + \mathbf{1}_{-3}$ representation) coupling to $E_{7(7)}$ generators belonging to the $\mathbf{78}_0 + \overline{\mathbf{27}}_{+2}$ representation and the corresponding embedding matrix is contained the $\mathbf{78}_{+3}$ representation (we recall that the embedding matrix is assigned to the representation that is conjugate with respect to one to which the gauge fields have been assigned).

This completely determines all possible gaugings in this basis. The gauge field in the $\mathbf{1}_{-3}$ representation couples to an element of the adjoint representation of $E_{6(6)}$, whereas (part of) the gauge fields in the $\overline{\mathbf{27}}_{-1}$ representation couple to generators in the $\overline{\mathbf{27}}_{+2}$ representation of $\mathrm{E}_{6(6)} \times \mathrm{SO}(1,1)$. This gauging is not new and has an interpretation as a Scherk-Schwarz reduction from $d = 5$ maximal supergravity [89].

Also this gauging can be represented in the SL(8,ℝ) frame upon an appropriate identification of the common representations. In order to do this, we should compare this with the $\mathbf{56} \to \mathbf{28} + \mathbf{28}'$ decomposition under SL(8,ℝ). We can do it by considering the common GL(6, ℝ) subgroup. The SL(8,ℝ) representations decompose as

$$\mathbf{28} \to \mathbf{15} + 2 \times \mathbf{6} + \mathbf{1}, \qquad \mathbf{28}' \to \mathbf{15}' + 2 \times \mathbf{6}' + \mathbf{1}, \tag{4.43}$$

while the $\mathrm{E}_{6(6)}$ ones branch as

$$\mathbf{27} \to \mathbf{15}' + 2 \times \mathbf{6}, \qquad \overline{\mathbf{27}} \to \mathbf{15} + 2 \times \mathbf{6}'. \tag{4.44}$$

This means that in order to represent the Scherk–Schwarz gauging in the SL(8,ℝ) electric frame one needs to turn on 13 electric couplings $\Theta_{78}{}^{\alpha}$, $\Theta_{I7}{}^{\alpha}$, $\Theta_{I8}{}^{\alpha}$ and 15 magnetic ones $\Theta^{IJ\,\alpha}$. Here we split the SL(8) index as $A = I, 7, 8$, where $I = 1, \ldots, 6$ is the GL(6) index.

By performing our analysis in the SL(8) frame we obviously recovered the known results on the vacua. However, as we will show, our analysis also provided some additional insight on how to perform the partial supersymmetry breaking described by the Scherk–Schwarz mechanism. We then used this insight to discuss the possible embedding of Scherk–Schwarz reductions with 4 parameters in M-theory [26].

As shown in (4.37), the Scherk–Schwarz gaugings depend on 4 parameters embedded in the U(1) charge matrix M. This means that there is a large class of models that show a Minkowski vacuum at the origin of the moduli space in the proper embedding. In order to recover all such models we proceed in steps. The first step is in $\mathcal{N} = 8$ theory. First of all we analyzed simpler instances where only some parameters are turned on, looking at vacua that also preserved some supersymmetry. We then superimpose such solutions to obtain the most general supersymmetry-breaking solution in our parameterization. In fact, we can show that one can superimpose the U(1) charge matrices corresponding to different Scherk–Schwarz vacua and still obtain a consistent Minkowski vacuum. In order to simplify the discussion let us turn back to the $E_{6(6)} \times SO(1,1) \ltimes T^{27}$ electric frame. In this frame the 28 generators of the flat gauge group are

$$X_u = \begin{pmatrix} \mathbb{0}_{27\times 27} & 0 & M_u{}^v d_{vst} & 0 \\ -(M_u)^t & 0 & 0\cdots 0 & 0 \\ \mathbb{0}_{27\times 27} & 0 & \mathbb{0}_{27\times 27} & (M_u)^s \\ 0\cdots 0 & 0 & 0\cdots 0 & 0 \end{pmatrix}, \quad X_0 = \begin{pmatrix} M_s{}^t & 0 & \mathbb{0}_{27\times 27} & 0 \\ 0 & 0 & 0 & 0 \\ \mathbb{0}_{27\times 27} & 0 & -M_t{}^s & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.45)$$

Here u, v, s, t are indices in the $\mathbf{27}$ or $\mathbf{27}'$ of $E_{6(6)}$ and M is an element of the $\mathfrak{usp}(8) \subset e_{6(6)}$ algebra that specifies the action of the U(1) generator on the $\mathbf{27}$. This form of the embedding tensor automatically satisfies the vacuum conditions, for *any* $M \in \mathfrak{usp}(8)$. This means that, given two flat gaugings with a Minkowski vacuum at the origin, parametrizd by M and M' as above, it is automatically guaranteed that $M'' = M + M' \in \mathfrak{u}sp(8)$ also defines a flat gauging with a Minkowski vacuum at the origin. Of course M'' defines a different flat gauging and may break a different amount of supersymmetry with respect to M or M'. Considering the mass matrices A_1, A_1' of the gravitini, the mass matrix of the gravitini when we have M'' is the sum of these two: $A_1'' = A_1 + A_1'$. If we find two $\mathcal{N} = 2$ solutions given by

$$M \to A_1 = \mathrm{diag}(m_1\epsilon, m_2\epsilon, m_3\epsilon, 0, 0), \quad M' \to A_1' = \mathrm{diag}(0, 0, m_2'\epsilon, m_3'\epsilon, m_4'\epsilon), \quad (4.46)$$

where $\epsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ then A_1'' has in general no vanishing eigenvalues and the gauging defined by M'' gives a $\mathcal{N} = 0$ Minkowski vacuum.

Following [24, 26], in order to find the minima of the potential, rather than fixing the gauging and then performing a scan of all possible critical points of the scalar potential, one can simply solve a set of quadratic conditions on the embedding tensor and then read the resulting values of Θ which define at the same time the original gauge group, the value of the cosmological constant and the masses at the critical point. The scalar potential at the origin is proportional to

$$V \propto \mathrm{Tr}(X_M X_M^T) + 7\,\mathrm{Tr}(X_M X_M). \tag{4.47}$$

By taking the above expressions for $M \in \mathfrak{e}_6$ one finds [3] that the potential vanishes if and only if $M \in \mathfrak{usp}(8)$, namely:

$$V \propto \mathrm{Tr}[M(M + M^T)] = 0 \quad \Leftrightarrow \quad M \in \mathfrak{usp}(8). \tag{4.48}$$

We can also check that this point is a critical point of the potential, by computing the derivative of the scalar potential in the origin of the scalar manifold:

$$V' \propto \mathrm{Tr}(\delta\, X_M X_M^T) + 7\mathrm{Tr}(\delta\, X_M X_M), \tag{4.49}$$

where $\delta\, X_M$ denotes the infinitesimal variation under some isometry- $E_{6(6)} \times SO(1,1) \ltimes T^{27}$ has exactly 70 non-compact isometries, therefore cover the whole scalar manifold. Variations with respect to T^{27} turn out to vanish because of the invariance of $d_{\lambda\sigma\gamma}$: $M_{(\lambda}{}^{\rho} d_{\sigma\gamma)\rho} = 0$. With respect to SO$(1,1)$, the embedding tensor has grading $+3$ as one can see from the form of the SO(1,1) generator in the E$_6$ basis

$$t_{SO(1,1)M}{}^{N} = \mathrm{diag}(+\mathbb{1}_{27}, +3, -\mathbb{1}_{27}, -3), \tag{4.50}$$

and therefore, as we will explain more in detail later, its variation is proportional to the potential itself: $\delta_{SO(1,1)} V \propto V$. Finally variations with respect to $E_{6(6)}$ can be taken to act directly on $M_\lambda{}^\sigma$:

$$\delta_{E_{6(6)}} V \propto \mathrm{Tr}\left([t_{E_{6(6)}}, M](M + M^T) \right) \tag{4.51}$$

and it also vanishes if $M \in \mathfrak{usp}(8)$. In conclusion, given two flat gaugings with a Minkowski vacuum at the origin, parametrized by M and M', it is automatically guaranteed that $M'' = M + M' \in \mathfrak{usp}(8)$ also defines a flat gauging with a Minkowski vacuum at the origin and since M'' defines a different flat gauging than the previous ones, generically it will break a different amount of supersymmetry with respect to M or M'. Actually, the mass matrix of the gravitini in the new gauging is also

[3] The normalization of the $E_{6(6)}$ cubic invariant is fixed so that $d_{usr} = d^{usr}$, $d_{ust} d^{vst} = 10\delta_u^v$.

the superposition of the ones in the previous embeddings: $A_1'' = A_1 + A_1'$. Hence, supposing that we find two $\mathcal{N} = 2$ solutions given by

$$M \to A_1 = \mathrm{diag}(m_1\epsilon, m_2\epsilon, m_3\epsilon, 0, 0), \quad M' \to A_1' = \mathrm{diag}(0, 0, m_2'\epsilon, m_3'\epsilon, m_4'\epsilon), \tag{4.52}$$

where $\epsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ then A_1'' has in general no vanishing eigenvalues and the gauging defined by M'' gives a $\mathcal{N} = 0$ Minkowski vacuum.

Let us conclude this discussion with the observation that the analysis we performed at the origin of the moduli space is exhaustive for such gaugings. We can actually see that there are no other vacua for flat gaugings than the one connected to the origin of the moduli space. As already noted previously, $E_{6(6)} \times SO(1,1) \ltimes T^{27}$ contains all the 70 non-compact isometries needed to cover the whole scalar manifold. Therefore we can compute the whole orbit of the embedding tensor and show explicitly that there are no other vacua. The action of $\mathrm{SO}(1,1)$ at most rescales the embedding tensor and hence we can neglect it. $E_{6(6)}$ sends $M \to t \in \mathfrak{e}_{6(6)}$ which in general will not belong to the $\mathfrak{usp}(8)$ subalgebra anymore and therefore would introduce a positive contribution to the scalar potential, which is not allowed at the vacuum because of the $\mathrm{SO}(1,1)$ grading argument. Finally, the action of the T^{27} also leaves $X_{MN}{}^P$ invariant, as can be seen explicitly by computing a finite transformation parametrized by x^μ:

$$T = \begin{pmatrix} \mathbb{1}_{27\times 27} & 0 & x^\sigma d_{\sigma\mu\nu} & x^\sigma x^\lambda d_{\sigma\lambda\mu}/2 \\ -x^\nu & 1 & x^\sigma x^\lambda d_{\sigma\lambda\nu}/2 & x^\sigma x^\lambda x^\rho d_{\sigma\lambda\rho}/6 \\ \mathbb{0}_{27\times 27} & 0 & \mathbb{1}_{27\times 27} & x^\mu \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.53}$$

and applying it to $X_{MN}{}^P$. In conclusion, we have a single vacuum connected to the origin of the scalar manifold for gaugings parametrized by (4.45), with at least 28 flat directions (the T^{27}, some of which correspond to Goldstone bosons, and the SO(1,1)).

4.4.2 One-loop stability

The spectrum of the four-parameter gauging of [37] was discussed in section 2.2.3, we recall here its main features. The spin-2 graviton is of course massless. The masses of the 8 spin-3/2 gravitini are $|\pm m_i|$, thus the residual supersymmetry can be $N = 0, 2, 4, 6, 8$, depending on the choice of the 4 parameters. In the following, we will be interested in the $N = 0$ case, corresponding to $m_i \neq 0$ for $i = 1, 2, 3, 4$.

The 8 massive gravitinos absorb 8 spin-1/2 fermions (which are otherwise massless) via the super-Higgs mechanism. Of the remaining 48 physical spin-1/2 fermions, 16 have masses $|\pm m_i|$ and 32 have masses $|\pm m_i \pm m_j \pm m_k|$ $(i < j < k)$. Of the 28 spin-1 bosons, 24 have masses $|\pm m_i \pm m_j|$ $(i < j)$, whilst 4 have zero mass. Finally, of the 70 spin-0 bosons, up to 24 are absorbed by the massive vectors, otherwise they have zero mass. Of the remaining 46 scalars, 24 have masses $|\pm m_i \pm m_j|$ $(i < j)$, 16 have masses $|\pm m_i \pm m_j \pm m_k \pm m_l|$ $(i < j < k < l)$, and the remaining 6 are massless.

The stability of the Minkowski vacuum with fully broken supersymmetry can be examined by considering the one-loop effective potential, as a function of the 3 complex moduli. It is known that the one-loop effective potential can be expressed in terms of the supertraces of the field-dependent mass matrices,

$$Str\ \mathcal{M}^{2k} \equiv \sum_i (-1)^{2J_i} (2J_i + 1)\, M_i^{2k}(\phi)\,. \tag{4.54}$$

It was already observed in [37] that in the model under considerations $Str\ \mathcal{M}^{2k} = 0$ for $k = 0, 1, 2, 3$: this holds true for any background value of the 3 complex moduli of the classical vacuum. The one-loop contribution to the effective potential is then automatically finite (but field-dependent, a point overlooked in [37] but correctly identified in [93]) and reads

$$\Delta V_1 = \frac{1}{64\,\pi^2} \sum_i (-1)^{2J_i} (2J_i + 1)\, M_i^4(\phi)\, \log M_i^2(\phi) \equiv \frac{1}{\phi^4}\, f(m_1, m_2, m_3, m_4)\,. \tag{4.55}$$

It is easy to check that the function $f(m_1, m_2, m_3, m_4)$, explicitly defined by the above equation and independent of the 3 complex moduli of the classical vacuum, vanishes for any of its four arguments going to zero, in agreement with the fact that all supertraces identically vanish if there is at least one unbroken supersymmetry. The other intriguing feature of $f(m_1, m_2, m_3, m_4)$, which emerges from numerical inspection but we were unable to prove analytically, is the fact that it is negative semi-definite, and vanishes *only* in the supersymmetric limit discussed above.

4.4.3 Flux interpretation and $\mathcal{N} = 1$ truncation

We are now ready to identify the flux configurations that must be switched on to reproduce the four-parameter flat gauging, and to specify the corresponding truncated $\mathcal{N} = 1$ theory. The most geometric configuration we can find corresponds to the following fluxes:

$$\begin{aligned} \omega^2_{71} &= -\omega^1_{72} = M_1, \quad \omega^4_{73} = -\omega^3_{74} = M_2, \\ \omega^6_{75} &= -\omega^5_{76} = M_3, \quad\ \ g_7 = -\theta'_{77} = M_4. \end{aligned} \tag{4.56}$$

Note that in this case we have turned on the ω^p_{mn} geometric fluxes, which generate the original Scherk–Schwarz models with only three mass parameters. The new result is that the fourth parameter is related to g_7 and to the non-geometric flux θ'_{77}.

With this choice of fluxes, the components of the embedding tensor different from zero are

$$\Theta_m{}^m = g_7 = M_4, \quad m = 1, ..., 7, \qquad \Theta_{7,7} = \theta'_{77} = -M_4, \tag{4.57}$$

some of the $\Theta_{M,N}{}^P = \omega^P_{MN}$, explicitly

$$\begin{aligned} \Theta_{1,7}{}^2 &= -\Theta_{2,7}{}^1 = \Theta_{7,2}{}^1 = -\Theta_{7,1}{}^2 = -M_1, \\ \Theta_{3,7}{}^4 &= -\Theta_{4,7}{}^3 = \Theta_{7,4}{}^3 = -\Theta_{7,3}{}^4 = -M_2, \\ \Theta_{5,7}{}^6 &= -\Theta_{6,7}{}^5 = \Theta_{7,6}{}^5 = -\Theta_{7,5}{}^6 = -M_3, \end{aligned} \tag{4.58}$$

some of the $\Theta^{MN,}{}_{PQR} = 2\omega^{[M}_{[PQ}\delta^{N]}_{R]}$,

$$\begin{aligned} \Theta^{1m,}{}_{72m} &= -\Theta^{2m}{}_{71m} = -\alpha\, M_1 \quad \text{for } m = 3, 4, 5, 6, \\ \Theta^{3m,}{}_{74m} &= -\Theta^{4m}{}_{73m} = -\alpha\, M_2 \quad \text{for } m = 1, 2, 5, 6, \\ \Theta^{5m,}{}_{76m} &= -\Theta^{6m}{}_{75m} = -\alpha M_3 \quad \text{for } m = 1, 2, 3, 4, \end{aligned} \tag{4.59}$$

some of the $\Theta_{MN,}{}^P = \omega^P_{MN}$, explicitly

$$\begin{aligned} \Theta_{17,}{}^2 &= -\Theta_{27,}{}^1 = -M_1, \\ \Theta_{37,}{}^4 &= -\Theta_{47,}{}^3 = -M_2, \\ \Theta_{57,}{}^6 &= -\Theta_{67,}{}^5 = -M_3, \end{aligned} \tag{4.60}$$

and

$$\Theta_{7m,7}{}^m = M_4, \quad m = 1, ..., 7. \tag{4.61}$$

Here The quadratic constraints are all identically satisfied in this case and thus we have that the four parameters are all independent.

The superpotential we obtain by performing this compactifications is

$$W = M_1(ST_1+T_2T_3)+M_2(ST_2+T_1T_3)+M_3(ST_3+T_1T_2)+M_4(1+ST_1T_2T_3). \tag{4.62}$$

The model is of course a no-scale model and it has flat directions along U_1, U_2, U_3, so we have an extremum of the potential in $S = T_i = 1$ for every value of $U_i = u_i + i\,\nu_i$.

The mass of the scalar fields are thus zero for the U's moduli. The other scalar fields have masses for both the dilatons (s, t_1, t_2, t_3) and the axions $(\sigma, \tau_1, \tau_2, \tau_3)$. For the dilatons we have

$$\begin{aligned}
&\frac{1}{2}M_1^2 \quad \text{for } (s + t_1 - t_2 - t_3),\\
&\frac{1}{2}M_2^2 \quad \text{for } (s - t_1 + t_2 - t_3),\\
&\frac{1}{2}M_3^2 \quad \text{for } (s - t_1 - t_2 + t_3),\\
&\frac{1}{2}M_4^2 \quad \text{for } (s + t_1 + t_2 + t_3),
\end{aligned} \tag{4.63}$$

while for the axions

$$\begin{aligned}
&\frac{1}{8}(M_1 + M_2 + M_3 - M_4)^2 \quad \text{for } (\sigma + \tau_1 + \tau_2 + \tau_3),\\
&\frac{1}{8}(M_1 + M_2 - M_3 + M_4)^2 \quad \text{for } (\sigma - \tau_1 - \tau_2 + \tau_3),\\
&\frac{1}{8}(M_1 - M_2 + M_3 + M_4)^2 \quad \text{for } (\sigma - \tau_1 + \tau_2 - \tau_3),\\
&\frac{1}{8}(-M_1 + M_2 + M_3 + M_4)^2 \quad \text{for } (\sigma + \tau_1 - \tau_2 - \tau_3).
\end{aligned} \tag{4.64}$$

The gravitino has squared mass

$$m_{3/2}^2 = \frac{1}{32}(M_1 + M_2 + M_3 + M_4)^2, \tag{4.65}$$

and the goldstino

$$\zeta = -\frac{1}{2}(\chi^{U_1} + \chi^{U_2} + \chi^{U_3}) \tag{4.66}$$

is of course removed from the physical spectrum, since it provides the $\pm 1/2$ helicity components to the massive gravitino. The other two orthogonal combinations of $\chi^{U^1}, \chi^{U^2}, \chi^{U^3}$ have squared masses equal to

$$\frac{1}{32}(M_1 + M_2 + M_3 + M_4)^2. \tag{4.67}$$

The spin-1/2 fields associated to S, T_i have squared masses given by

$$\begin{aligned}
\frac{1}{32}(M_1+M_2+M_3-3M_4)^2 \quad &\text{for } (\chi^S+\chi^{T_1}+\chi^{T_2}+\chi^{T_3}),\\
\frac{1}{32}(-3M_1+M_2+M_3+M_4)^2 \quad &\text{for } (\chi^S+\chi^{T_1}-\chi^{T_2}-\chi^{T_3}),\\
\frac{1}{32}(M_1-3M_2+M_3+M_4)^2 \quad &\text{for } (\chi^S-\chi^{T_1}+\chi^{T_2}-\chi^{T_3}),\\
\frac{1}{32}(M_1+M_2-3M_3+M_4)^2 \quad &\text{for } (\chi^S-\chi^{T_1}-\chi^{T_2}+\chi^{T_3}).
\end{aligned} \tag{4.68}$$

In terms of the parameters M_a, the gravitino masses of section 2.2.3 are

$$m_1 = \frac{-M_1+M_2+M_3-M_4}{4\sqrt{2}}, \quad m_2 = \frac{M_1-M_2+M_3-M_4}{4\sqrt{2}},$$

$$m_3 = \frac{M_1+M_2-M_3-M_4}{4\sqrt{2}}, \quad m_4 = \frac{M_1+M_2+M_3+M_4}{4\sqrt{2}}$$

where m_4 is the mass of the gravitino that survives in the $\mathcal{N}=1$ truncation.

4.4.4 Other duality frames

The choice of fluxes described in the previous subsection is the most geometric one we can perform in order to reproduce the four-parameter flat gauging. However there are other possible choices that one can make and which give again a no-scale model with four independent mass parameters. At first sight we could think that all possible choices of the type

$$\begin{aligned}
\tilde{W}(\phi) = M_1(1+\phi_a\phi_b\phi_c\phi_d) + M_2(\phi_a\phi_g+\phi_c\phi_d)\\
+M_3(\phi_a\phi_c+\phi_b\phi_d) + M_4(\phi_a\phi_d+\phi_b\phi_c),
\end{aligned} \tag{4.69}$$

with $\phi_a = (S, T_i, U_i)$ and $a \neq b \neq c \neq d$, are equivalent to the one seen above. However it is not true, because some of these combinations are related to fluxes which have non-trivial quadratic constraints, and only seven possible configurations allow for no-scale models with four mass parameters.

They are those with a superpotential given by

$$\begin{aligned}
&i.\ W = \tilde{W}(S,T_1,U_2,U_3), \quad ii.\ W = \tilde{W}(S,T_2,U_3,U_1), \quad iii.\ W = \tilde{W}(S,T_3,U_1,U_2),\\
&iv.\ W = \tilde{W}(T_2,T_3,U_2,U_3), \quad v.\ W = \tilde{W}(T_1,T_3,U_1,U_3), \quad vi.\ W = \tilde{W}(T_1,T_2,U_1,U_2),
\end{aligned} \tag{4.70}$$

and of course the case already analyzed $o.\ W = \tilde{W}(S,T_1,T_2,T_3)$.

The fluxes in all these cases are

	M_1	$-M_1$	M_2	$-M_2$	M_3	$-M_3$	M_4	$-M_4$
$o.$	g_7	θ'_{77}	ω^2_{71}	ω^1_{72}	ω^4_{73}	ω^3_{74}	ω^6_{75}	ω^5_{76}
$i.$	g_7	θ'_{11}	ω^2_{71}	ω^7_{12}	ω^4_{51}	ω^5_{41}	ω^6_{13}	ω^3_{16}
$ii.$	g_7	θ'_{33}	ω^4_{73}	ω^7_{34}	ω^6_{13}	ω^1_{63}	ω^2_{35}	ω^5_{32}
$iii.$	g_7	θ'_{55}	ω^6_{75}	ω^7_{56}	ω^2_{35}	ω^3_{25}	ω^4_{51}	ω^1_{54}
$iv.$	g_7	θ'_{22}	ω^7_{12}	ω^1_{72}	ω^3_{62}	ω^6_{42}	ω^5_{23}	ω^3_{25}
$v.$	g_7	θ'_{44}	ω^7_{34}	ω^3_{74}	ω^6_{24}	ω^2_{64}	ω^1_{45}	ω^5_{41}
$vi.$	g_7	θ'_{66}	ω^7_{56}	ω^5_{76}	ω^2_{46}	ω^4_{26}	ω^2_{61}	ω^1_{63}

and they evidently satisfy identically the quadratic constraints

$$\begin{aligned} &\omega^{i_I}_{m[j_I}\omega^m_{k_I l_I]} = 0, \\ &\omega^{[q}_{[p_1 p_2}\delta^{r]}_{p_3]}\theta'_{qq} = 0. \end{aligned} \tag{4.71}$$

Chapter 5

Conclusions and outlook

In this thesis we addressed some problems that arise in the search for interesting supersymmetry-breaking vacua in simple and extended four-dimensional supergravities, seen as the low-energy effective theories of flux compactifications of higher-dimensional supergravities, in turn related to superstring theory.

We first presented some original results on Fayet-Iliopoulos terms in four-dimensional $\mathcal{N}=1$ supergravity. We pointed out that there is a substantial difference between FI terms in models where the restoration of the gauged $U(1)$ R-symmetry never happens in field space, which can be redefined away (we called them 'impostors'), and FI-terms in models where such gauge symmetry is restored at least at one point in field space, which have a gauge-invariant meaning. We produced some representative examples to illustrate the point and to better understand the range of validity of some 'folklore theorems' on FI terms. In particular, we constructed a model with a genuine FI-term, a classically stable de Sitter vacuum, no anomalies and no exact global symmetries. We also discussed the consistent $\mathcal{N}=1$ truncations of the only known models of four-dimensional extended supergravity with stable de Sitter vacua, which happen to be $\mathcal{N}=2$ models with $\mathcal{N}=2$ FI terms. We pointed out that the $\mathcal{N}=2$ FI terms do not necessarily become FI terms in the $\mathcal{N}=1$ truncated theory, and viceversa.

The other original results presented in this thesis concern the gauged $\mathcal{N}=8$ supergravity theories obtained from M-theory compactifications with geometrical and non-geometrical fluxes, and their $\mathcal{N}=1$ truncations. We found the configurations of fluxes that describe the uplift to M-theory of known examples of gauged $\mathcal{N}=8$ supergravities, and the corresponding $\mathcal{N}=1$ truncations. In particular, we were able to identify the fluxes that reproduce the $\mathcal{N}=8$ no-scale model with Minkowski vacuum, partially or totally broken supersymmetry and four independent mass parameters, observing that it can be reproduced, in its most geometrical description, by a compactification of eleven-dimensional supergravity that involves only fluxes with a well-understood geometrical interpretation, apart from a non-geometrical flux θ'_{77}. Also in this case, we obtained a consistent $\mathcal{N} = 1$ truncation. We further studied the one-loop stability of the Minkowski vacuum with totally broken supersymmetry, noticing the curious fact that the one-loop corrections to the effective potential always lead to unstable configurations with negative vacuum energy.

While the work on FI terms is complete, the work on $\mathcal{N} = 8$ gauged supergravities, their origin from flux compactifications and their truncations to a lower number of supersymmetries is still in progress. One of the open questions is whether at least some locally geometrical interpretation of the four-parameter model with supersymmetry-breaking Minkowski vacuum could be given. Also, we are planning to study further $\mathcal{N} = 4$ truncations and additional $\mathcal{N} = 8$ models with broken supersymetry on a flat background.

Appendix A

Notation and Conventions

Maximal symmetric spaces

The supergravity theories in D dimensions we are interested in have a maximal symmetric space-time, endowed with a metric $g_{\mu\nu}(x)$ such that $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ ($\mu, \nu = 0, 1, ..., D-1$ space-time indices). The maximal symmetric space could be Minkowski

$$\mathrm{Mkw}_D: \quad ds^2 = -(dx^0)^2 + \sum_{i=1}^{D-1}(dx^i)^2, \tag{A.1}$$

for which the metric is the flat metric $\eta_{\mu\nu} = diag(-1, 1, ..., 1)$, de Sitter ($\mathrm{dS}_D$) or Anti de Sitter ($\mathrm{AdS}_D$)

$$ds^2 = -(dx^0)^2 + \sum_{i=1}^{D}(dx^i)^2, \tag{A.2}$$

where $-(dx^0)^2 + \sum_{i=1}^{D}(dx^i)^2 = \alpha^2$ in the de Sitter case and $-(dx^0)^2 + \sum_{i=1}^{D}(dx^i)^2 = -\alpha^2$ in Anti de Sitter, with α real constant.

Planck mass

The reduced mass Planck is defined by

$$M_P = \sqrt{\frac{\hbar c}{8\pi G}} = 2.44 \cdot 10^{18} GeV, \tag{A.3}$$

where c is the light velocity, $\hbar$ is the reduced Planck constant, and G is the Newton constant. Apart from when it is explicitly specified, we set M_P to 1.

Dimensions	Spinors	Components
2, 10	Majorana-Weyl	$2^{D/2-1}$
3, 9, 11	Majorana	$2^{(D-1)/2}$
4, 8, 12	Majorana or Weyl	$2^{D/2}$
5, 7	Dirac	$2^{(D+1)/2}$
6	Weyl	8

Table A.1: Fermions in diverse dimensions and their minimum number of components.

Spinors

The spinor fields that appear in supergravity theories in D dimensions transform under the fermionic representation of the Lorentz group $SO(1, D-1)$. It is the Dirac representation and its generators are given by $[\Gamma_\mu, \Gamma_\nu]$, where the gamma matrices Γ_μ satisfy the Clifford algebra

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}. \tag{A.4}$$

The dimension of this representation of the Clifford algebra is $2^{[D/2]+1}$, where $[D/2]$ is the integer part of $D/2$.

In certain dimensions the Dirac representation is reducible, allowing one to impose conditions preserved under Lorentz symmetry. For instance, in even dimensions one can impose the following chirality condition to define a left or right handed Weyl spinor:

$$\Gamma_* \psi_{L,R} = \pm \psi_{L,R}, \tag{A.5}$$

with $\Gamma_* = i^{D/2+1}\Gamma_0\Gamma_1...\Gamma_{D-1}$. It could be also possible in some cases to impose a reality condition:

$$\psi^* = \alpha B \psi \tag{A.6}$$

where α is an arbitrary phase and B is a matrix such that

$$\Gamma^*_\mu = \eta B \Gamma_a B^{-1} \tag{A.7}$$

for both $\eta = \pm$ in even dimensions, for only $\eta = (-1)^{\frac{D+1}{2}}$ in odd dimensions. A Majorana spinor thus furnishes a complete representation of the Lorentz algebra and has only half as many degrees of freedom as an unconstrained complex Dirac spinor. In some dimensions, the Majorana and the Weyl condition can be imposed simultaneously. This reduces the number of independent degrees of freedom to one quarter relative to an unconstrained Dirac spinor.

Appendix B

Useful formulae in $\mathcal{N} = 8$ supergravity

$E_{7(7)}$ group

The group of the isometries in $\mathcal{N} = 8$ supergravity is $E_{7(7)}$. Its algebra, $\mathfrak{e}_{7(7)}$ is one of the semisimple algebras. The subscript 7 in parenthesis denotes the difference between the 70 non compact generators t^{ABCD} and the 63 compact ones t_A^B $(A, B = 1, ..., 8)$. Following [21], the 133 generators of the algebra $\mathfrak{e}_{7(7)}$ in the fundamental **56** representation are matrices

$$T = \begin{pmatrix} \delta^{[C}_{[A} t_{B]}{}^{D]} & t_{ABEF} \\ t^{CDGH} & -\delta^{[G}_{[E} t_{F]}{}^{H]} \end{pmatrix}, \tag{B.1}$$

with

$$t_{ABCD} = \frac{1}{24} \epsilon_{ABCDEFGH} t^{EFGH}. \tag{B.2}$$

We can rewrite the generators and the corresponding algebra using a $SL(7, \mathbb{R}) \times O(1, 1)$ basis. In this case, the compact generators are $t_m{}^n, t = \sum_m t_m{}^m, t_m, t^m$, and the non compact ones are t_{mnp}, t^{mnp}. The commutation relations then read

$$[t_m{}^n,\, t_p{}^q] = \delta^n_p\, t_m{}^q - \delta^q_m\, t_p{}^n\,,$$
$$[t_m{}^n,\, t^{p_1p_2p_3}] = -3\,\delta^{[p_1}_m\, t^{p_2p_3]n} + \frac{5}{7}\,\delta^n_m\, t^{p_1p_2p_3}\,,$$
$$[t_m{}^n,\, t_p] = \delta^n_p\, t_m + \frac{3}{7}\,\delta^n_m\, t_p\,,$$

$$
\begin{aligned}
&[t^{n_1n_2n_3},\, t^{p_1p_2p_3}] = \epsilon^{n_1n_2n_3p_1p_2p_3q}\, t_q\,, \\
&[t_m{}^n,\, t_{p_1p_2p_3}] = 3\,\delta^n_{[p_1}\, t_{p_2p_3]m} - \frac{5}{7}\,\delta^n_m\, t_{p_1p_2p_3}\,, \\
&[t_m{}^n,\, t^p] = -\delta^p_m\, t^n - \frac{3}{7}\,\delta^n_m\, t^p\,, \\
&[t_{n_1n_2n_3},\, t_{p_1p_2p_3}] = \epsilon_{n_1n_2n_3p_1p_2p_3q}\, t^q\,, \\
&[t^n,\, t_m] = t_m{}^n + \frac{1}{7}\,\delta^n_m\, t\,, \\
&[t^m,\, t^{n_1n_2n_3}] = -\frac{1}{6}\,\epsilon^{mn_1n_2n_3p_1p_2p_3}\, t_{p_1p_2p_3}\,, \\
&[t_m,\, t_{n_1n_2n_3}] = -\frac{1}{6}\,\epsilon_{mn_1n_2n_3p_1p_2p_3}\, t^{p_1p_2p_3} \\
&[t_{m_1m_2m_3},\, t^{n_1n_2n_3}] = 18\,\delta^{[n_1n_2}_{[m_1m_2}\, t_{m_3]}{}^{n_3]} - \frac{24}{7}\,\delta^{n_1n_2n_3}_{m_1m_2m_3}\, t\,,
\end{aligned}
\tag{B.3}
$$

where $t \equiv t_m{}^m$.

$\mathcal{N} = 8$ quadratic constraints

The quadratic constraints on fluxes can be read from (1.120). Here we will give all of them exlicit.

- $\epsilon_{mnpqrs_1s_2}\Theta^{s_1}_{\mathcal{M}}\,\Theta^{s_2}_{\mathcal{N}}\,\Omega^{\mathcal{M}\mathcal{N}} = 0$:

$$
a_1d_5g_7\xi^{s_1s_2}\epsilon_{mnpqrs_1s_2} + 4\cdot 5!\,b_1c_5\tau^s_{[mn}g_{pqr]s} + 2\cdot 5!\,b_1c_5^{'}\tau_{[m}g_{npqr]} = 0 \tag{B.4}
$$

- $\epsilon^{mnpqrs_1s_2}\Theta_{\mathcal{M}s_1}\,\Theta_{\mathcal{N}s_2}\,\Omega^{\mathcal{M}\mathcal{N}} = 0$:

$$
a_5d_1\tilde{g}_7\theta_{s_1s_2}\epsilon^{mnpqrs_1s_2} + 4\cdot 5!\,b_5c_1Q^{[mn}_{[s}h^{pqr]s} + 2\cdot 5!\,b_5'c_1Q^{[m}h^{npqr]} = 0 \tag{B.5}
$$

- $\Theta^p_{\mathcal{M}}\,\Theta_{\mathcal{N}p_1p_2p_3}\,\Omega^{\mathcal{M}\mathcal{N}} = 0$:

$$
\begin{aligned}
&a_1d_4g_7\theta_{[p_1p_2}\delta^p_{p_3]} + a_1d_4^{'}g_7f^p_{p_1p_2p_3} + 2^4\cdot 3^2\,b_1c_4\xi^{mn}g_{mn[p_1p_2}\delta^p_{p_3]} \\
&-a_2d_5^{'}g_{mp_1p_2p_3}\xi^{'(mp)} - b_2c_5\tau^p_{m[p_1}\tau^m_{p_2p_3]} - b_2c_5'\tau_m\tau^{[m}_{[p_1p_2}\delta^{p]}_{p_3]} \\
&+2^3\cdot 3^2\,b_1c_4'(2R^{m_1m_2m_3}_{[p_1}\delta^p_{p_2}g_{p_3]m_1m_2m_3} + 3R^{pmn}_{[p_1}g_{p_2p_3]mn}) \\
&-(a_2d_5 + 2^5\cdot 3\,b_1c_4)g_{mp_1p_2p_3}\xi^{[mp]} = 0
\end{aligned}
\tag{B.6}
$$

- $\Theta_{\mathcal{M}p}\Theta_{\mathcal{N}}^{p_1p_2p_3}\ \Omega^{\mathcal{M}\mathcal{N}}=0$:

$$\begin{aligned}
&a_4 d_1 \tilde{g}_7 \xi^{[p_1p_2}\delta_p^{p_3]} + a_4' d_1 \tilde{g}_7 R_p^{p_1p_2p_3} + 2^4\cdot 3^2\, b_4 c_1 \theta_{mn} h^{mn[p_1p_2}\delta_p^{p_3]}\\
&-a_5' d_2 h^{mp_1p_2p_3}\theta'_{(mp)} - b_5 c_2 Q_p^{m[p_1} Q_m^{p_2p_3]} - b_5' c_2 Q^m Q_{[m}^{[p_1p_2}\delta_{p]}^{p_3]}\\
&+2^3\cdot 3^2\, b_4' c_1 (2 f_{m_1m_2m_3}^{[p_1}\delta_p^{p_2} h^{p_3]m_1m_2m_3} + 3 f_{pmn}^{[p_1} h^{p_2p_3]mn})\\
&-(a_5 d_2 + 2^5\cdot 3\, b_4 c_1) h^{mp_1p_2p_3}\theta_{[mp]} = 0
\end{aligned} \tag{B.7}$$

- $\Theta_{\mathcal{M}}^{p}\ \Theta_{\mathcal{N}q}^{r}\ \Omega^{\mathcal{M}\mathcal{N}}=0$:

$$\begin{aligned}
&a_1 d_3 g_7 Q_q^{pr} + a_1 d_3' g_7 \delta_q^p Q^r + a_1 d_3'' g_7 \delta_q^r Q^p\\
&-a_3 d_5 \xi^{mp}\tau_{mq}^r - a_3' d_5 \tau_q \xi^{rp} - a_3'' d_5 \tau_m \xi^{mp}\delta_q^r\\
&-a_3 d_5' \xi'^{mp}\tau_{mq}^r - a_3' d_5' \tau_q \xi'^{rp} - a_3'' d_5' \tau_m \xi'^{mp}\delta_q^r\\
&-b_3 c_5 \xi^{rm}\tau_{mq}^p - b_3' c_5 \xi'^{rm}\tau_{mq}^p - b_3'' c_5 R_q^{mnr}\tau_{mn}^p - b_3''' c_5 \xi^{mn}\tau_{mn}^p\delta_r^q\\
&-b_3 c_5' \xi^{r[m}\delta_q^{p]}\tau_m - b_3' c_5' \xi'^{r[m}\delta_q^{p]}\tau_m - b_3'' c_5' R_q^{mpr}\tau_m - b_3''' c_5' \xi^{mp}\tau_m \delta_q^r\\
&-b_1 c_3 \epsilon^{mprabcd} g_{abcd}\theta_{qm} - b_1 c_3' \epsilon^{mprabcd} g_{abcd}\theta'_{qm}\\
&+b_1 c_3'' \epsilon^{mnpabcd} g_{abcd} f_{mnq}^r + b_1 c_3''' \epsilon^{mnpabcd} g_{abcd}\theta_{mn}\delta_q^r = 0
\end{aligned} \tag{B.8}$$

- $\Theta_{\mathcal{M}p}\ \Theta_{\mathcal{N}r}^{q}\ \Omega^{\mathcal{M}\mathcal{N}}=0$:

$$\begin{aligned}
&a_3 d_1 \tilde{g}_7 \tau_{pr}^q + a_3' d_1 \tilde{g}_7 \delta_p^q \tau_r + a_3'' d_1 \tilde{g}_7 \delta_r^q \tau_p\\
&-a_5 d_3 \theta_{mp} Q_r^{mq} - a_5 d_3' Q^q \theta_{rp} - a_5 d_3'' Q^m \theta_{mp}\delta_r^q\\
&-a_5' d_3 \theta'_{mp} Q_r^{mq} - a_5' d_3' Q^q \theta'_{rp} - a_5' d_3'' Q^m \theta'_{mp}\delta_r^q\\
&-b_5 c_3 \theta_{rm} Q_p^{mq} - b_5 c_3' \theta'_{rm} Q_p^{mq} - b_5 c_3'' f_{mnr}^q Q_p^{mn} - b_5 c_3''' \theta_{mn} Q_p^{mn}\delta_q^r\\
&-b_5' c_3 \theta_{r[m}\delta_{p]}^q Q^m - b_5' c_3' \theta'_{r[m}\delta_{p]}^q Q^m - b_5' c_3'' f_{mpr}^q Q^m - b_5' c_3''' \theta_{mp} Q^m \delta_r^q\\
&-b_3 c_1 \epsilon_{mprabcd} h^{abcd}\xi^{qm} - b_3' c_1 \epsilon_{mprabcd} h^{abcd}\xi'^{qm}\\
&+b_3'' c_1 \epsilon_{mnpabcd} h^{abcd} R_r^{mnq} + b_3''' c_1 \epsilon_{mnpabcd} h^{abcd}\xi^{mn}\delta_r^q = 0
\end{aligned} \tag{B.9}$$

- $\Theta^{p}_{\mathcal{M}}\ \Theta^{p_1p_2p_3}_{\mathcal{N}}\ \Omega^{\mathcal{M}\mathcal{N}}=0$:

$$\begin{aligned}&a_1d_2g_7h^{pp_1p_2p_3}-b_1c_2Q^{[p_1p_2}_{m}\epsilon^{p_3]mpabcd}g_{abcd}\\&-a_4d_5\xi^{[p_1p_2}\xi^{p_3]p}-a_4d'_5\xi^{[p_1p_2}\xi'^{p_3]p}\\&-a'_4d_5R^{p_1p_2p_3}_{m}\xi^{mp}-a'_4d'_5R^{p_1p_2p_3}_{m}\xi'^{mp}\\&-b_4c_5\epsilon^{mnabp_1p_2p_3}\theta_{ab}\tau^{p}_{mn}-b'_4c_5f^{m}_{abc}\epsilon^{nabcp_1p_2p_3}\tau^{p}_{mn}\\&-b_4c'_5\theta_{ab}\tau_m\epsilon^{mpp_1p_2p_3ab}-b'_4c'_5f^{[m}_{abc}\epsilon^{p]abcp_1p_2p_3}\tau_m=0\end{aligned}\tag{B.10}$$

- $\Theta_{\mathcal{M}p}\ \Theta_{\mathcal{M}p_1p_2p_3}\ \Omega^{\mathcal{M}\mathcal{N}}=0$:

$$\begin{aligned}&a_2d_1\tilde{g}_7g_{pp_1p_2p_3}-b_2c_1\tau^{m}_{[p_1p_2}\epsilon_{p_3]mpabcd}h^{abcd}\\&-a_5d_4\theta_{[p_1p_2}\theta_{p_3]p}-a'_5d_4\theta_{[p_1p_2}\theta'_{p_3]p}\\&-a_5d'_4f^{m}_{p_1p_2p_3}\theta_{mp}-a'_5d'_4f^{m}_{p_1p_2p_3}\theta'_{mp}\\&-b_5c_4\epsilon_{mnabp_1p_2p_3}\xi^{ab}Q^{mn}_{p}-b_5c'_4R^{abc}_{m}\epsilon_{nabcp_1p_2p_3}Q^{mn}_{p}\\&-b'_5c_4\xi^{ab}Q^{m}\epsilon_{mpp_1p_2p_3ab}-b'_5c'_4R^{abc}_{[m}\epsilon_{p]abcp_1p_2p_3}Q^{m}=0\end{aligned}\tag{B.11}$$

- $\Theta^{p}_{\mathcal{M}}\ \Theta_{\mathcal{N}q}\ \Omega^{\mathcal{M}\mathcal{N}}=0$:

$$\begin{aligned}&a_1d_1g_7\tilde{g}_7\delta^{p}_{q}+2^4\cdot 3\,b_1c_1(g_{abcd}h^{abcd}\delta^{p}_{q}-4g_{abcq}h^{abcp})\\&-a_5d_5\theta_{mq}\xi^{mp}-a_5d'_5\theta_{mq}\xi'^{mp}-a'_5d_5\theta'_{mq}\xi^{mp}-a'_5d'_5\theta'_{mq}\xi'^{mp}\\&-b_5c_5Q^{mn}_{q}\tau^{p}_{mn}-b_5c'_5Q^{mp}_{q}\tau_m-b'_5c_5Q^{m}\tau^{p}_{mq}-b'_5c'_5Q^{[m}\tau_m\delta^{p]}_{q}=0\end{aligned}\tag{B.12}$$

- $\Theta_{\mathcal{M}p_1p_2p_3}\ \Theta_{\mathcal{N}q_1q_2q_3}\ \Omega^{\mathcal{M}\mathcal{N}}=0$:

$$\begin{aligned}&a_2d_4\theta_{[q_1q_2}g_{q_3]p_1p_2p_3}+a_2d'_4g_{mp_1p_2p_3}f^{m}_{q_1q_2q_3}-b_2c_4\tau^{m}_{[p_1p_2}\epsilon_{p_3]mabq_1q_2q_3}\xi^{ab}\\&-b_2c'_4\tau^{m}_{[p_1p_2}\delta^{n}_{p_3}R^{abc}_{[m}\epsilon_{n]abcq_1q_2q_3}-(p_1p_2p_3\leftrightarrow q_1q_2q_3)=0\end{aligned}\tag{B.13}$$

- $\Theta^{p_1p_2p_3}_{\mathcal{M}}\ \Theta^{q_1q_2q_3}_{\mathcal{N}}\ \Omega^{\mathcal{M}\mathcal{N}}=0$:

$$\begin{aligned}&a_4d_2\xi^{[q_1q_2}h^{q_3]p_1p_2p_3}+a'_4d_2h^{mp_1p_2p_3}R^{q_1q_2q_3}_{m}-b_4c_2Q^{[p_1p_2}_{m}\epsilon^{p_3]mabq_1q_2q_3}\theta_{ab}\\&-b'_4c_2Q^{[p_1p_2}_{m}\delta^{p_3}_{n}f^{[m}_{abc}\epsilon^{n]abcq_1q_2q_3}-(p_1p_2p_3\leftrightarrow q_1q_2q_3)=0\end{aligned}\tag{B.14}$$

- $\Theta_{\mathcal{M}P_1P_2P_3}\,\Theta_{\mathcal{N}Q}^{R}\,\Omega^{\mathcal{M}\mathcal{N}}=0$:

$$\begin{aligned}
&a_2d_3g_{mp_1p_2p_3}Q_q^{mr}+a_2d_3'g_{qp_1p_2p_3}Q^r+a_2d_3''g_{mp_1p_2p_3}Q^m\delta_q^r\\
&-a_3d_4\theta_{[p_1p_2}\tau_{p_3]q}^{r}-a_3'd_4\tau_q\theta_{[p_1p_2}\delta_{p_3]}^{r}-a_3''d_4\tau_{[p_1p_3}^{p_3]}\delta_q^r\\
&-a_3d_4'\tau_{mq}^{r}f_{p_1p_2p_3}^{m}-a_3'd_4'\tau_q f_{p_1p_2p_3}^{r}-a_3''d_4'\tau_m f_{p_1p_2p_3}^{m}\delta_q^r\\
&+b_2c_3\tau_{[p_1p_2}^{[m}\delta_{p_3]}^{r]}\theta_{qm}+b_2c_3'\tau_{[p_1p_2}^{[m}\delta_{p_3]}^{r]}\theta_{qm}'\\
&+b_2c_3''\tau_{[p_1p_2}^{m}f_{p_3]qm}^{r}+b_2c_3'''\theta_{m[p_1}\tau_{p_2p_3]}^{m}\delta_q^r\\
&-b_3c_4\xi^{rm}\xi^{ab}\epsilon_{mabqp_1p_2p_3}-b_3c_4'\xi^{rm}R_{[m}^{abc}\epsilon_{q]abcp_1p_2p_3}\\
&-b_3'c_4\xi'^{rm}\xi^{ab}\epsilon_{mabqp_1p_2p_3}-b_3'c_4'\xi'^{rm}R_{[m}^{abc}\epsilon_{q]abcp_1p_2p_3}\\
&-b_3''c_4R_q^{mnr}\xi^{ab}\epsilon_{mnabp_1p_2p_3}-b_3''c_4'R_q^{mnr}R_m^{abc}\epsilon_{nabcp_1p_2p_3}\\
&-b_3'''c_4\xi^{mn}\delta_q^r\xi^{ab}\epsilon_{mnabp_1p_2p_3}-b_3'''c_4'\xi^{mn}\delta_q^rR_m^{abc}\epsilon_{nabcp_1p_2p_3}=0
\end{aligned}\qquad\text{(B.15)}$$

- $\Theta_{\mathcal{M}}^{P_1P_2P_3}\,\Theta_{\mathcal{N}R}^{Q}\,\Omega^{\mathcal{M}\mathcal{N}}=0$:

$$\begin{aligned}
&a_3d_2h^{mp_1p_2p_3}\tau_{mr}^{q}+a_3'd_2h^{qp_1p_2p_3}\tau_r+a_3''d_2h^{mp_1p_2p_3}\tau_m\delta_r^q\\
&-a_4d_3\xi^{[p_1p_2}Q_r^{p_3]q}-a_4d_3'Q^q\xi^{[p_1p_2}\delta_r^{p_3]}-a_4d_3''\xi^{[p_1p_3}Q^{p_3]}\delta_r^q\\
&-a_4'd_3Q_r^{mq}R_m^{p_1p_2p_3}-a_4'd_3'Q^qR_r^{p_1p_2p_3}-a_4'd_3''Q^mR_m^{p_1p_2p_3}\delta_r^q\\
&+b_3c_2Q_{[m}^{[p_1p_2}\delta_{r]}^{p_3]}\xi^{qm}+b_3'c_2Q_{[m}^{[p_1p_2}\delta_{r]}^{p_3]}\xi'^{qm}\\
&+b_3''c_2Q_m^{[p_1p_2}r_r^{p_3]qm}+b_3'''c_2\xi^{m[p_1}Q_m^{p_2p_3]}\delta_r^q\\
&-b_4c_3\theta_{rm}\theta_{ab}\epsilon^{mabqp_1p_2p_3}-b_4c_3'\theta_{rm}'\theta_{ab}\epsilon^{mabqp_1p_2p_3}\\
&-b_4'c_3\theta_{rm}f_{abc}^{[m}\epsilon^{q]abcp_1p_2p_3}-b_4'c_3'\theta_{rm}'f_{abc}^{[m}\epsilon^{q]abcp_1p_2p_3}\\
&-b_4c_3''f_{mnr}^{q}\theta_{ab}\epsilon^{mnabp_1p_2p_3}-b_4'c_3''f_{mnr}^{q}f_{abc}^{m}\epsilon^{nabcp_1p_2p_3}\\
&-b_4c_3'''\theta_{mn}\delta_r^q\theta_{ab}\epsilon^{mnabp_1p_2p_3}-b_4'c_3'''\theta_{mn}\delta_r^qf_{abc}^{m}\epsilon^{nabcp_1p_2p_3}=0
\end{aligned}\qquad\text{(B.16)}$$

- $\Theta_{\mathcal{M}p_1p_2p_3}\ \Theta_{\mathcal{N}}^{q_1q_2q_3}\ \Omega^{\mathcal{M}\mathcal{N}} = 0$:

$$\begin{aligned}
&a_2 d_2 g_{mp_1p_2p_3} h^{mq_1q_2q_3} + b_2 c_2 \omega^m_{[p_1p_2} \delta^n_{p_3]} Q^{[q_1q_2}_{[m} \delta^{q_3]}_{n]} \\
&-a_4 d_4 \xi^{[q_1q_2} \delta^{q_3]}_{[p_1} \theta_{p_2p_3]} - a_4 d'_4 \xi^{[q_1q_2} f^{q_3]}_{p_1p_2p_3} \\
&-a'_4 d_4 R^{q_1q_2q_3}_{[p_1} \theta_{p_2p_3]} - a'_4 d'_4 R^{q_1q_2q_3}_{m} f^m_{p_1p_2p_3} \\
&-2^3 \cdot 3\, b_4 c_4 (\theta_{ab} \xi^{ab} \delta^{q_1q_2q_3}_{p_1p_2p_3} + 6\theta_{a[p_1} \delta^{[q_1q_2}_{p_2p_3]} \xi^{q_3]a} + 3\theta_{[p_1p_2} \delta^{[q_1}_{p_3]} \xi^{q_2q_3]}) \\
&+2^2 \cdot 3^2\, b_4 c'_4 (R^{q_1q_2q_3}_{[p_1} \theta_{p_2p_3]} + 3\theta_{ab} \delta^{[q_1q_2}_{[p_1p_2} R^{q_3]ab}_{p_3]} + 6\theta_{a[p_1} \delta^{[q_1}_{p_2} R^{q_2q_3]a}_{p_3]}) \\
&-2^2 \cdot 3^2\, b'_4 c_4 (f^{[q_1}_{p_1p_2p_3} \xi^{q_2q_3]} + 3\xi^{ab} \delta^{[q_1q_2}_{[p_1p_2} f^{q_3]}_{p_3]ab} + 6\xi^{a[q_1} \delta^{q_2}_{[p_1} f^{q_3]}_{p_2p_3]a}) \\
&-2 \cdot 3^3 b'_4 c'_4 (f^{[q_1}_{abc} \delta^{q_2q_3]}_{[p_1p_2} R^{abc}_{p_3]} + 6 f^{[q_1}_{ab[p_1} \delta^{q_2}_{p_2} R^{q_3]ab}_{p_3]} + 3 f^{[q_1}_{a[p_1p_2} R^{q_2q_3]a}_{p_3]}) = 0
\end{aligned} \tag{B.17}$$

- $\Theta^q_{\mathcal{M}p}\ \Theta^s_{\mathcal{N}r}\ \Omega^{\mathcal{M}\mathcal{N}} = 0$:

$$\begin{aligned}
&a_3 d_3 \omega^q_{mp} Q^{ms}_r + a_3 d'_3 \omega^q_{rp} Q^s + a_3 d''_3 \omega^q_{mp} Q^m \delta^s_r \\
&+a'_3 d_3 \omega_p Q^{qs}_r + a'_3 d'_3 \omega_p Q^s \delta^q_r + a'_3 d''_3 Q^q \omega_p \delta^s_r \\
&+a''_3 d_3 \omega_m Q^{ms}_r \delta^q_p + a''_3 d'_3 \omega_r \delta^q_p Q^s + a''_3 d''_3 \omega_m Q^m \delta^q_p \delta^s_r \\
&+b_3 c_3 \xi^{q[m} \delta^{s]}_p \theta_{rm} + b_3 c'_3 \xi^{q[m} \delta^{s]}_p \theta'_{rm} + b_3 c''_3 \xi^{qm} f^s_{mpr} + b_3 c'''_3 \xi^{qm} \theta_{mp} \delta^s_r \\
&+b'_3 c_3 \xi'^{q[m} \delta^{s]}_p \theta_{rm} + b'_3 c'_3 \xi'^{q[m} \delta^{s]}_p \theta'_{rm} + b'_3 c''_3 \xi'^{qm} f^s_{mpr} + b'_3 c'''_3 \xi'^{qm} \theta_{mp} \delta^s_r \\
&+b''_3 c_3 R^{msq}_p \theta_{rm} + b''_3 c'_3 R^{msq}_p \theta'_{rm} + b''_3 c''_3 R^{mnq}_p f^s_{mnr} + b''_3 c'''_3 R^{mnq}_p \theta_{mn} \delta^s_r \\
&+b'''_3 c_3 \xi^{ms} \theta_{rm} \delta^q_p + b'''_3 c'_3 \xi^{ms} \theta'_{rm} \delta^q_p + b'''_3 c''_3 \xi^{mn} f^s_{mnr} \delta^q_p + b'''_3 c'''_3 \xi^{mn} \theta_{mn} \delta^s_r \delta^q_p \\
&-(pq \leftrightarrow rs) = 0
\end{aligned} \tag{B.18}$$

Truncation to $\mathcal{N} = 1$

We conclude by commenting on the identification of the constraints that remain non-trivial after the truncation to $\mathcal{N} = 1$ discussed in section 4.1. The fluxes that survive after the truncation can be read from table 4.1. We can see that equations (B.4) and (B.5) are identically satisfied, whilst there are non-trivial constraints such as (B.12) when the two free indices are equal. The constraints (B.8) and (B.9) are not identically satisfied only if the free indices p, q, r correspond to one of the combinations in (4.8). The equations (B.6,B.7,B.10,B.11) that are non-trivial in the reduced case are those in which the four free indices p, p_1, p_2, p_3 correspond to one of the combinations in (4.9). From (B.18) only those equations with the four free indices p, q and r, s such that m, p, q and m, r, s are two combinations from (4.8) for some $m = 1, ..., 7$ survive. Finally, the constraints (B.11,B.14,B.17) are non-trivial if the two triplets of free indices p_1, p_2, p_3 and q_1, q_2, q_3 together with the same $m = 1, ..., 7$ correspond to two distinct combinations of (4.9); (B.15,B.16) are non-trivial if q, r and p_1, p_2, p_3 together with some $m = 1, ..., 7$ form respectively a combination of those in (4.8) and one of those in (4.9) .

Explicitly, the quadratic constraints in the reduced theory are

$$a_1 d'_4 g_7 f^{i_I}_{j_I k_I l_I} - a_2 d'_5 g_{i_I j_I k_I l_I} \xi'^{i_I i_I} + 2^3 \cdot 3^3\, b_1 c'_4 R^{i_I mn}_{[j_I} g_{k_I l_I]mn} - b_2 c_5 \omega^{i_I}_{m[j_I} \omega^m_{k_I l_I]} = 0, \quad \text{(B.19)}$$

$$a'_4 d_1 \tilde{g}_7 R^{j_I k_I l_I}_{i_I} - a'_5 d_2 h^{i_I j_I k_I l_I} \theta'_{i_I i_I} + 2^3 \cdot 3^3\, b'_4 c_1 f^{[j_I}_{i_I mn} h^{k_I l_I]mn} - b_5 c_2 Q^{m[j_I}_{i_I} Q^{k_I l_I]}_m = 0, \quad \text{(B.20)}$$

$$\begin{aligned} & a_1 d_3 g_7 Q^{b_I c_I}_{a_I} - a_3 d'_5 \xi'^{b_I b_I} \omega^{c_I}_{b_I a_I} + b_1 c'_3 \epsilon^{a_I b_I c_I mnpq} g_{mnpq} \theta'_{a_I a_I} \\ & + b_1 c''_3 \epsilon^{b_I mnpqrs} g_{pqrs} f^{c_I}_{mna_I} - b'_3 c_5 \xi'^{c_I c_I} \omega^{b_I}_{c_I a_I} - b''_3 c_5 R^{mnc_I}_{a_I} \omega^{b_I}_{mn} = 0, \end{aligned} \quad \text{(B.21)}$$

$$\begin{aligned} & a_3 d_1 \tilde{g}_7 \tau^{a_I}_{b_I c_I} - a'_5 d_3 \theta'_{b_I b_I} Q^{b_I a_I}_{c_I} + b'_3 c_1 \epsilon_{a_I b_I c_I mnpq} h^{mnpq} \xi'^{a_I a_I} \\ & + b''_3 c_1 \epsilon_{b_I mnpqrs} h^{pqrs} R^{mna_I}_{c_I} - b_5 c'_3 \theta'_{c_I c_I} Q^{c_I a_I}_{b_I} - b_5 c''_3 f^{a_I}_{mnc_I} Q^{mn}_{b_I} = 0, \end{aligned} \quad \text{(B.22)}$$

$$\begin{aligned} & a_2 d_1 g_7 h^{i_I j_I k_I l_I} + b_2 c_1 Q^{j_I k_I}_m \epsilon^{l_I] i_I mabcd} g_{abcd} \\ & - a'_5 d'_4 R^{j_I k_I l_I}_{i_I} \xi'^{i_I i_I} - b_5 c'_4 f^m_{abc} \epsilon^{nabcj_i k_I l_I} \omega^{i_I}_{mn} = 0, \end{aligned} \quad \text{(B.23)}$$

$$\begin{aligned} & a_1 d_2 \tilde{g}_7 g_{i_I j_I k_I l_I} + b_1 c_2 \omega^m_{j_I k_I} \epsilon_{l_I] i_I mabcd} h^{abcd} \\ & - a'_4 d'_5 f^{i_I}_{j_I k_I l_I} \theta'_{i_I i_I} - b'_4 c_5 R^{abc}_m \epsilon_{nabcj_i k_I l_I} Q^{mn}_{i_I} = 0, \end{aligned} \quad \text{(B.24)}$$

$$a_1 d_1 g_7 \tilde{g}_7 - a'_5 d'_5 \theta'_{mm} \xi'^{mm} - b_5 c_5 Q^{np}_m \omega^m_{np} + 2^4 \cdot 3 b_1 c_1 (g_{abcd} h^{abcd} - 4 g_{mabc} h^{mabc}) = 0, \quad \text{(B.25)}$$

$$a_2 d'_4 g_{mp_1p_2p_3} f^m_{q_1q_2q_3} - b_2 c'_4 \omega^m_{[p_1p_2} \delta^n_{p_3]} R^{abc}_{[m} \epsilon_{n]abcq_1q_2q_3} - (p_1p_2p_3 \leftrightarrow q_1q_2q_3) = 0, \quad \text{(B.26)}$$

$$a'_4 d_2 h^{mp_1p_2p_3} R^{q_1q_2q_3}_m - b'_4 c_2 Q^{[p_1p_2}_m \delta^{p_3]}_n f^{[m}_{abc} \epsilon^{n]abcq_1q_2q_3} - (p_1p_2p_3 \leftrightarrow q_1q_2q_3) = 0, \quad \text{(B.27)}$$

$$a_2 d_2 g_{mp_1p_2p_3} h^{mq_1q_2q_3} - a_4' d_4' R_m^{q_1q_2q_3} R_m^{p_1p_2p_3} + b_2 c_2 \omega_{[p_1p_2}^{[m} \delta_{p_3]}^{n]} Q_m^{[q_1q_2} \delta_n^{q_3]}$$
$$-2 \cdot 3^2\, b_4' c_4' f_{abc}^{[q_1} \delta_{[p_1p_2}^{q_2q_3]} R_{p_3]}^{abc} + 3 f_{ab[p_1}^{[q_1} \delta_{p_2}^{q_2} R_{p_3]}^{q_3]ab} + 3 f_{m[p_1p_2}^{[q_1} R_{p_3]}^{q_2q_3]m} = 0, \tag{B.28}$$

$$a_2 d_3 g_{mp_1p_2p_3} Q_q^{mr} - a_3 d_4' \omega_{mq}^r f_{p_1p_2p_3}^m + b_2 c_3' \omega_{[p_1p_2}^{[q} \delta_{p_3]}^{r]} \theta_{qq}' + b_2 c_3'' \omega_{[p_1p_2}^m f_{p_3]qm}^r$$
$$-b_3' c_4' \xi'^{rr} R_{[r}^{abc} \epsilon_{q]abcp_1p_2p_3} - b_3'' c_4' R_q^{mnr} R_m^{abc} \epsilon_{nabcp_1p_2p_3} = 0, \tag{B.29}$$

$$a_3 d_2 h^{mp_1p_2p_3} \omega_{mr}^q - a_4' d_3 Q_r^{mq} R_m^{p_1p_2p_3} + b_3' c_2 Q_{[q}^{[p_1p_2} \delta_{r]}^{p_3]} \xi'^{qq}$$
$$+b_3'' c_2 Q_m^{[p_1p_2} R_r^{p_3]qm} - b_4' c_3' \theta_{rr}' f_{abc}^{[r} \epsilon^{q]abcp_1p_2p_3} - b_4' c_3'' f_{mnr}^q f_{abc}^m \epsilon^{nabcp_1p_2p_3} = 0, \tag{B.30}$$

$$a_3 d_3 \omega_{mp}^q Q_r^{ms} + b_3' c_3' \xi'^{r[q} \delta_p^{s]} \theta_{rr}' + b_3' c_3'' \xi'^{qq} f_{qpr}^s + b_3'' c_3' R_p^{rsq} \theta_{rr}' + b_3'' c_3'' R_p^{mnq} f_{mnr}^s - (pq \leftrightarrow rs) = 0. \tag{B.31}$$

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