



UNIVERSITÀ
DEGLI STUDI
DI PADOVA



Sede amministrativa: Università degli Studi di Padova
Dipartimento di Matematica

Scuola di Dottorato in Scienze Matematiche
Indirizzo Matematica
XXIV° ciclo

Tesi di dottorato realizzata nell'ambito di una convenzione di cotutela con l'Université du Littoral Côte d'Opale (ULCO). Laboratoire de Physico-Chimie de l'Atmosphère (LPCA).

Quantum manifestations of the adiabatic chaos of perturbed superintegrable Hamiltonian systems

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Abstract

The abundance, among physical models, of perturbations of superintegrable Hamiltonian systems makes the understanding of their long-term dynamics an important research topic. While from the classical standpoint the situation, at least in many important cases, is well understood through the use of Nekhoroshev stability theorem and of the adiabatic invariants theory, in the quantum framework there is, on the contrary, a lack of precise results.

The purpose of this thesis is to study a perturbed superintegrable quantum system, obtained from a classical counterpart by means of geometric quantization, in order to highlight the presence of indicators of superintegrability analogues to the ones that characterize the classical system, such as the coexistence of regular motions with chaotic ones, due to the effects of resonances, opposed to the regularity in the non resonant regime.

The analysis is carried out by studying the Husimi distributions of chosen quantum states, with particular emphasis on stationary states and evolved coherent states. The computation are performed using both numerical methods and perturbative schemes.

Although this should be considered a preliminary work, the purpose of which is to lay the foundations for future investigations, the results obtained here give interesting insights into quantum dynamics. For instance, it is shown how classical resonances exert a considerable influence on the spectrum of the quantum system and how it is possible, in the quantum behaviour, to find a trace of the classical adiabatic invariance in the resonant regime.

L'abbondanza, fra i modelli fisici, di perturbazioni di sistemi Hamiltoniani superintegrabili rende la comprensione della loro dinamica per tempi lunghi un importante argomento di ricerca. Mentre dal punto di vista classico la situazione, perlomeno in molti casi importanti, è ben compresa grazie all'uso del teorema di stabilità di Nekhoroshev e della teoria degli invarianti adiabatici, nel caso quantistico vi è, al contrario, una mancanza di risultati precisi.

L'obiettivo di questa tesi è di studiare un sistema superintegrabile quantistico, ottenuto partendo da un corrispettivo classico tramite quantizzazione geometrica, al fine di evidenziare la presenza di indicatori di superintegrabilità analoghi a quelli che caratterizzano il sistema classico, come la coesistenza di moti regolari e caotici, dovuta all'effetto delle risonanze, in contrapposizione con la regolarità nel regime non risonante.

L'analisi è condotta studiando le distribuzioni di Husimi di stati quantistici scelti, con particolare enfasi posta sugli stati stazionari e sugli stati coerenti evoluti. I calcoli sono effettuati sia utilizzando tecniche numeriche che schemi perturbativi.

Pur essendo da considerarsi questo un lavoro preliminare, il cui compito è di porre le fondamenta per analisi future, i risultati qui ottenuti offrono interessanti spunti sulla dinamica quantistica. Per esempio è mostrato come le risonanze classiche abbiano un chiaro effetto sullo spettro del sistema quantistico, ed inoltre come sia possibile trovare una traccia, nel comportamento quantistico, dell'invarianza adiabatica classica nel regime risonante.

To my family

Acknowledgements

First and foremost I would like to thank my supervisors prof. Francesco Fassò and prof. Dmitrií Sadoovskí for their patience and constant guidance through this long journey.

I am also sincerely grateful to Dmitrií and his wife Irina for their hospitality and for making me feel truly at home during my staying in Boulogne-sur-Mer.

I am indebted to prof. Boris Zhilinskií for his teachings which introduced me into the fascinating realm of quantum chemistry and to dr. Jonathan Robbins from whom I got the idea of the use of coherent states as a tool for studying the quantum system.

Special thanks are also due to prof. Nicola Sansonetto, who patiently helped through a lengthy process of correction of my thesis and whose expertise on the subject of quantum mechanics and quantization has been a constant source of inspiration.

It is impossible to list here all the colleagues, friends and lovers that contributed, every one in its own way, to determining the person that I am now. To each one of them I would like to express my deepest gratitude.

The last, but most heartfelt, acknowledgements go to my mother Noemi, my father Mauro, and my sister Sara for unconditionally supporting all my choices and for always being at my side in my moments of joy but, more importantly, when I needed them the most. This thesis is dedicated to them.

Daniele Fontanari

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Introduction

The presence of superintegrable systems in physical applications is ubiquitous. When microscopic quantities are involved, for an accurate physical description it is necessary to resort to a quantum, rather than a classical, formalism. In this framework it is often of practical importance the study of a perturbed superintegrable system. For instance this is the case for the hydrogen atom in a small electromagnetic field or for certain simple molecular models.

If, on a classical level, superintegrable systems are regarded simply as Liouville integrable ones, the underlying geometry is lost. However it is exactly this rich geometrical structure that causes, on a classical level, the presence of the characteristic phenomenon of extended resonant chaos. In the quantum framework, where the interest seems to lay more in the study of completely chaotic systems or in a statistical analysis, this aspect might be overlooked. We believe that the field of quantum chaos could greatly benefit if we understand how this classical setting translates into the quantum realm. In the first place this would give a common ground for people that work in the field quantum mechanics and the ones that study classical perturbation theory, since it seems that the former tends to ignore the *dynamical* role that resonances, of the unperturbed system, play in the behaviour of the perturbed system and the study of resonant normal forms is rather performed only on a formal level. On the other hand we have already mentioned the abundance of perturbed superintegrable systems in quantum mechanics. Even if we have chosen a system which is particularly simple (compact phase space, polynomial character of the Hamiltonian), it contains all the characteristics we need to produce a rich phenomenology. It is not difficult (at least conceptually) to extend the following analysis to more complex systems. It is our belief that simple inorganic molecules with rotational-vibrational degrees of freedom, system describing interacting laser beams in a non-linear medium and Rydberg molecules or asymmetric top molecules in a weak electric field are some of the examples that (each with its own limitation) could be described by an adequate extension of the theory presented here. In particular it seems that, in particular conditions, it is possible nowadays to observe a concrete quantum system for a relatively long timescale. It is thus not completely unrealistic to believe that an experimental

apparatus, aimed to support the theory, could be set up.

In the first chapter the classical setting is presented. The mathematical foundations of Hamiltonian theory are given, and we recall the concept of Liouville integrable system. After providing some basic examples and describing the structure of a generic Liouville integrable system (using the Liouville-Arnol'd theorem) we state the main results in perturbation theory of Liouville integrable systems, namely KAM and Nekhoroshev theorem.

We then introduce the concept of superintegrable system, along with some concrete examples. We describe the structure of a generic superintegrable system (using the Mishchenko-Fomenko-Nekhoroshev theorem) with particular emphasis on its geometry. At this point we state the superintegrable versions of KAM and Nekhoroshev theorem. In particular we see how superintegrability predicts chaotic resonant motions along with regular non resonant motions in the case of a superintegrable system with 3 degrees of freedom and 2-dimensional invariant tori.

The second chapter introduces the system we intend to study defined by an Hamiltonian h_ϵ . This is a perturbed superintegrable system defined on $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ with 2-dimensional tori. After applying Nekhoroshev theory, as presented in the previous chapter, to h_ϵ , we describe its motions. In particular we introduce, for resonant motions, the concept of adiabatic invariance and of adiabatic chaos. This is the reason for the presence of both regular and chaotic motions in resonance and leads to a theory that allows to predict for which initial data chaotic behaviour is to be expected. The chapter closes with the analysis of some numerical simulations of the system that confirm the theory presented before and are used to make a comparison with the quantum case.

The third chapter deals with quantum mechanics. After the general theory is presented we introduce the topic of quantization of a classical system and the challenges it poses. We describe the method of geometric quantization, its application to the quantization of the phase space and its extension, the *BKS*-quantization method, which allows to quantize the considered system. In this chapter coherent states and the Husimi distribution of a quantum state, a powerful tools for the comparison between a classical system and its quantum analogue, are introduced.

The fourth chapter deals with the problem of comparing the dynamics of the classical system and its quantum analogue. Preliminarily we will see how the stationary states of the quantum system (which completely encode the dynamics of the system) define a distribution in the action space which highlight the fact that classical resonances have a relevant role also in quantum mechanics. Studying the evolution of coherent states through Husimi distribution we analyze the dynamics on the third sphere, on which classically localized chaos is present. We give preliminary results which suggest that analogous structures might be present in the

quantum case.

The last part is devoted to the analysis of the system through a perturbative approach. The results that such method provides, although approximate, important information on the behaviour of the quantum system. Moreover in this way it is possible to explore the high spin limit.

Chapter 1

Superintegrable classical systems

In this chapter the main results of the theory of perturbation of superintegrable systems are presented.

The first section is devoted to the generalities of symplectic geometry and Hamiltonian theory. Then a particular class of systems, the so called Liouville integrable ones, is analyzed. These systems are, among the general ones, characterized by a particularly simple dynamics, being quasi-periodic with its orbits supported on tori of dimension equal to the number of degrees of freedom of the system. Some concrete examples of Liouville integrable systems are given, and then the two main theorems of the theory of perturbations of Liouville-integrable system, KAM theorem and Nekhoroshev theorem, are stated. These results deal with the long-term (possibly for infinite time scales) behaviour of Liouville integrable systems, when a small perturbation is added. It is possible to show that, under adequate hypotheses on the perturbed system, the dynamics is still regular and closely follows the unperturbed one.

Another class of Hamiltonian systems of particular interest is realized by superintegrable systems, which is closely related to the previous ones. In this case the dynamics is still quasi periodic, but the dimension of the tori that support the orbits is smaller than the number of degrees of freedom of the systems. Concrete examples of superintegrable systems are given by the Kepler problem (periodic orbits in a 3 degrees of freedom system) and by the rigid body with a fixed point (quasi-periodic orbits supported on 2 dimensional tori in a 3 degrees of freedom system). Superintegrability is characterized by a underlying rich geometric structure, which is absent in the Liouville-integrable case, this prevents the immediate applicability of KAM and Nekhoroshev theorem. In the last section of this chapter we will present the main results of perturbation theory of superintegrable systems and how the presence of localized chaos along with regular orbits is possible.

1.1 Hamiltonian systems

This section is devoted to presenting the basic results and notations in symplectic geometry and Hamiltonian mechanics. The experienced user can safely skip this part. References can be found, for example, in [4].

Let $(\mathcal{M}, \omega)$ be a symplectic manifold (a smooth finite dimensional manifold $\mathcal{M}$ equipped with a non-degenerate closed differential 2-form ω). A function h on $\mathcal{M}$ (from now on every function on $\mathcal{M}$ we consider will be, otherwise stated, $\mathbb{R}$ -valued and smooth) defines in a unique way, thanks to the non degeneracy of ω , a vector field $X_h \in \Gamma(T\mathcal{M})$ satisfying $\iota_{X_h}\omega + dh = 0$.

In this context h is called a *Hamiltonian function* (or briefly *Hamiltonian*) while X_h is its *Hamiltonian vector field*. For our purposes we assume that the flow of every considered Hamiltonian vector field X_h , which is denoted by $\Phi_h^t(p)$ is complete.

The *Poisson bracket* of two functions f and g , $\{f, g\}$, is the function on $\mathcal{M}$ defined as

$$\{f, g\}(p) := \iota_{X_g}df(p) = \iota_{X_f}\iota_{X_g}\omega(p).$$

When $\{f, g\} = 0$, f and g are said to be in *involution* and the the Hamiltonian vector field of g is parallel to the level sets of f (and so the value of f is conserved under the Hamiltonian flow of g).

1.2 Liouville integrable systems

An important class of Hamiltonian systems is given by *Liouville integrable* systems. Before giving the definition of Liouville integrability for a Hamiltonian system we will state the fundamental Liouville-Arnol'd theorem:

Theorem (Liouville-Arnol'd [4]). *Let $\mathcal{M}$ be a symplectic manifold of dimension $2n$. Let $f : \mathcal{M} \rightarrow \mathbb{R}^n$ be a submersion with compact and connected fibers such that*

$$\{f_i, f_j\} = 0 \quad \forall i, j = 1 \dots n.$$

Then f defines a fibration, with each fiber diffeomorphic to the n -dimensional torus, and in a neighbourhood of every fiber it is possible to find local coordinates (action-angle coordinates) (I, ϕ) with $I \in \mathcal{A}$, being $\mathcal{A}$ an open subset of $\mathbb{R}^n$, and ϕ angular coordinates on $\mathbb{T}^n$ such that:

- *the fibers are locally defined by $I = \text{const}$.*
- *in these coordinates the local representative of the symplectic form is*

$$\sum_{i=1}^n dI_i \wedge d\phi_i.$$

If moreover h is a function such that $\{f_i, h\} = 0$ for every $i = 1, \dots, n$ then the local representative of h depends only on I and h is called a Liouville integrable Hamiltonian.

Definition. A Hamiltonian h is Liouville integrable if there exist a function f satisfying the hypotheses of Liouville-Arnol'd theorem and such that

$$\{h, f_i\} = 0 \quad i = 1, \dots, n$$

In local action-angle coordinates h depends only on the I_i variables.

Remark. The condition on f being a submersion is crucial if we want to have a fibration instead of a foliation with singularities. It might happen that f is not a submersion on the whole manifold $\mathcal{M}$, when this is the case we still refer to f as a Liouville integrable Hamiltonian, but we tacitly restrict our attention to the open subset where the rank of f is maximal. $\triangle$

If h is Liouville integrable its flow is quasi-periodic on the fibers (referred usually as the *invariant tori* of the system) and motions $t \mapsto (I(t), \phi(t))$ are given by $I(t) = I(0)$, $\phi(t) = \phi(0) + t\omega(I(0))$, where

$$\omega = \frac{\partial h}{\partial I}.$$

1.3 Examples of Liouville integrable systems

Since $\{h, h\} = 0$ it is possible to choose $f_1 = h$, so, in order to prove the Liouville integrability of h , only $n - 1$ functions in involution are needed. This implies that systems with 1-degree of freedom, provided that h has compact level sets, are Liouville integrable.

Another example of Liouville integrable system is the spherical pendulum, a particle subjected to a constant gravitational field and constrained to the surface of a sphere: there are 2 degrees of freedom and both the Hamiltonian and the projection of the angular momentum on the direction of the force field define the integrable fibration.

A last example is given by the n -dimensional isochronous oscillator

$$h(q, p) = \sum_{i=1}^n \omega_i \frac{q_i^2 + p_i^2}{2}.$$

In this case the functions $f_i(q, p) = q_i^2 + p_i^2$, for $i = 1, \dots, n$, define the invariant fibration. It should be observed that, if there exists a non zero $\nu \in \mathbb{Z}^n$ such that

$\omega_1\nu_1 + \dots + \omega_n\nu_n = 0$ (the oscillator in this case is said to be resonant) then there exists a finer fibration which is left invariant by the dynamics, and the system is said to be *superintegrable*. This situation will be analyzed in detail later.

1.4 Perturbations of integrable systems

Let h_0 be a Liouville integrable Hamiltonian on the n -dimensional symplectic manifold $\mathcal{M}$, $h_\epsilon = h_0 + \epsilon f$ for some function f and a real parameter ϵ . If $\epsilon \neq 0$ the constancy of the actions of the so called unperturbed system h_0 is in general lost.

There are two main theorems that, under some suitable assumptions on h_0 and f , give estimates, over long, or possibly infinite, timescales, on the behaviour of the actions.

A first result is given by *KAM* theorem. It is not restrictive to state it in a single chart of action angle coordinates of the unperturbed system, so, in the following, $\mathcal{M} = \mathcal{A} \times \mathbb{T}^n$ with $\mathcal{A}$ open subset of $\mathbb{R}^n$.

Theorem 1 (KAM [1, 7]). *Suppose that, for every ϵ in some neighbourhood of 0, all the Hamiltonians h_ϵ are real analytic on $\text{cl}(\mathcal{A}) \times \mathbb{T}^n$ and h_0 satisfies the Kolmogorov non-degeneracy condition:*

$$\det(h_0''(I)) \neq 0 \quad \forall I \in \mathcal{A}. \quad (1.1)$$

Then there exists a constant $\epsilon_0 > 0$ such that, if $|\epsilon| < \epsilon_0$, there exist:

- an integrable smooth Hamiltonian h' with action-angle coordinates defined on $\mathcal{A}' \times \mathbb{T}^n$ for some open subset $\mathcal{A}'$ of $\mathbb{R}^n$;
- a near the identity symplectic diffeomorphism:

$$w : \mathcal{A}' \times \mathbb{T}^n \rightarrow \mathcal{A} \times \mathbb{T}^n;$$

- a subset $\mathcal{A}'_\epsilon$ of $\mathcal{A}'$ such that the Lebesgue measure of $\mathcal{A}' \setminus \mathcal{A}'_\epsilon$ is of order $\sqrt{\epsilon}$ and such that the function $h_\epsilon \circ w$ is equal, with the equality extending to the derivatives of every order, to h' on $\mathcal{A}'_\epsilon \times \mathbb{T}^n$;
- a positive number $\gamma \sim \epsilon^2$, such that the function

$$I' \rightarrow \frac{\partial h'(I')}{\partial I'}$$

maps $\mathcal{A}'_\epsilon$ into the set of (γ, n) -Diophantine vectors, which are the $v \in \mathbb{R}^n$ such that, for every integer vector ν , $|v \cdot \nu| > \gamma |\nu|^{-n}$, where $|\nu| = |\nu_1| + \dots + |\nu_n|$.

KAM theorem essentially states that there exists a subset $\mathcal{M}_\epsilon$ of $\mathcal{M}$ of *large* measure, defined as the image of $\mathcal{A}'_\epsilon \times \mathbb{T}^n$ under the map w , that is fibered by invariant tori, close to the ones defined by h_0 , supporting quasi periodic motions with the same frequencies as the unperturbed one. On the other hand $\mathcal{A}'_\epsilon$, being homeomorphic to a subsets of the set of (γ, n) -Diophantine points on $\mathbb{R}^n$, is a Cantor-like set with *empty* interior.

In some sense *Nekhoroshev theorem* is complementary to KAM theorem. It deals with very long, but finite, timescales however the results hold on the whole manifold $\mathcal{M}$:

Theorem (Nekhoroshev [7, 37]). *Suppose that, for every ϵ in some neighbourhood of 0, all the Hamiltonians h_ϵ are real analytic on $\text{cl}(\mathcal{A}) \times \mathbb{T}^n$ and h_0 is m -convex for some $m > 0$, that is:*

$$h''_0(I)u \cdot u \geq m\|u\|^2 \quad \forall I \in \mathcal{A}, u \in \mathbb{R}^n.$$

There exist positive constants ϵ_0 , T , R , a , b , independent from ϵ , such that for every ϵ with $|\epsilon| \leq \epsilon_0$ and t satisfying

$$|t| \leq T_N = Te^{\left(\frac{\epsilon_0}{|\epsilon|}\right)^a}$$

then every orbit of h_ϵ satisfies:

$$|I_i(t) - I_i(0)| \leq R \left(\frac{|\epsilon|}{\epsilon_0} \right)^b \quad \forall i = 1 \dots n.$$

A possible choice for the stability exponents a and b is $a = b = \frac{1}{2n}$.

These are not the most general statements for the two theorems, weaker conditions can be imposed. For instance KAM theorem is still valid for smooth Hamiltonians while Nekhoroshev theorem holds, with worse values of the stability exponents, for quasi-convex, or even steep, h_0 (see [34, 37]).

Summarizing, we can say that we have, when the perturbation is small enough, a large, but topologically complicated, set of unperturbed tori that survives, slightly deformed, indefinitely while the others “dissolve” very slowly. For the latter, after the exponential timescale, the so called *Arnol'd diffusion* can take place and no control on the actions is possible.

1.5 Superintegrable systems

Another class of Hamiltonian systems is defined by *superintegrable systems*, which possess a number of first integrals *larger* than the number of degrees of freedom

of the system. This causes the dynamics to take place on tori of dimension *lower* than the one expected from Liouville integrability, in fact Liouville Integrability can be seen as a particular case of superintegrability.

The characterization of superintegrability (sometimes referred to as *non commutative integrability* or *degenerate integrability*), is mainly due to Nekhoroshev ([33]) and to Mishchenko and Fomenko ([35]).

The geometric structure of the fibration by the invariant tori is richer than in the Liouville integrable case [13, 28] and is important for the comprehension of the dynamics of small perturbations of these systems [16]. Here the principal results will be presented; for more details see [20] and references therein.

Theorem (Nekhoroshev-Mishchenko-Fomenko [35]). *Let $\mathcal{M}$ be a $2n$ -dimensional symplectic manifold, $0 \leq m \leq n$ and $f : \mathcal{M} \rightarrow \mathbb{R}^{2n-m}$ be a submersion with compact and connected fibers such that*

$$\{f_i, f_j\} = P_{ij} \circ f$$

for some function $P : f(\mathcal{M}) \rightarrow M_{2n-m}$ (the set of square matrices with real coefficient of dimension $2n - m$) with $\text{rank}(P) = 2n - 2m$ everywhere.

Under these hypotheses f defines a fibration with fibers diffeomorphic to m -dimensional tori $\mathbb{T}^m$ and every fiber has a neighbourhood equipped with local coordinates (generalized action angle coordinates) (I, ϕ, p, q) with $I \in \mathcal{A}$, $\phi \in \mathbb{T}^m$ and $(p, q) \in \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^{2n-2m}$ respectively. In these coordinates:

- *the fibers of f are defined by $(I, p, q) = \text{const}$;*
- *the local representative of the symplectic form is*

$$\sum_{i=1}^m dI_i \wedge d\phi_i + \sum_{i=1}^{n-m} dp_i \wedge dq_i.$$

If moreover h is a Hamiltonian such that $\{f_i, h\} = 0$ for every $i = 1, \dots, 2n - m$ then the local representative of h depends only on I and h is called a superintegrable Hamiltonian.

If h is superintegrable then its flow is quasi-periodic on the fibers of f and its motions in local generalized action-angle coordinates, $t \mapsto (I(t), \phi(t), p(t), q(t))$, are given by $I(t) = I(0)$, $\phi(t) = \phi(0) + t\omega(I(0))$, $p(t) = p(0)$, $q(t) = q(0)$, where

$$\omega = \frac{\partial h}{\partial I}.$$

As mentioned in the Liouville integrable case f might not be a submersion on the whole manifold $\mathcal{M}$. Again, if this is the situation, we still refer to h as to a superintegrable Hamiltonian but we will tacitly restrict ourselves on the submanifold of $\mathcal{M}$ on which the rank of f is maximal.

From the statement of the theorem is clear, as mentioned above, that Liouville integrability is recovered in the case $m = n$.

1.6 Examples of superintegrable systems

An example of superintegrable system is given by a particle in a potential which is invariant under rotations. The Hamiltonian h and the components of the angular momentum are four functions that satisfy the requirements of the theorem. Generic motions (for which the conditions on the rank of f are satisfied) are supported on 2-dimensional tori. If the potential is Keplerian, then the system has an additional independent integral of motion, coming from the conservation of the eccentricity vector; the five integrals of motion satisfy the hypotheses of the theorem with $m = 1$, reflecting the well known periodicity of the (bounded, non collisional) motions.

As already pointed out the harmonic oscillator with resonant frequencies is superintegrable as well. For instance, in the 2 degrees of freedom case with $\omega_1 = \omega_2$,

$$h = \frac{1}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2),$$

the function $f = (q_1 q_2 + p_1 p_2, p_1 q_2 - p_2 q_1, q_1^2 + p_1^2 - q_2^2 - p_2^2)$ defines (when $h \neq 0$) the invariant fibration by periodic orbits (see [20]).

Another fundamental example (see [5, 17]) is represented by the free rigid body with a fixed point. In this case the configuration space is identified (after choosing a reference frame for the body, see the analysis performed in the section devoted to the study of the global geometric structure of the examples for more details) by the rotation group $SO(3)$. The phase space is therefore 6-dimensional. If we choose as f_1, f_2 and f_3 the components of the angular momentum and as f_4 the Hamiltonian we have that f satisfies (when the system is not one of its equilibrium positions or performing a proper rotation around one of the axes of inertia or following a separatrix of the system) the conditions of the Nekhoroshev-Mishchenko-Fomenko theorem and so it defines a $\mathbb{T}^2$ -fibration invariant under the flow of h , which defines therefore a superintegrable system.

The comprehension of this system is quite important because its presence is ubiquitous in the applications. In particular the system we have chosen to analyze, as we will see, shares with this the principal geometrical characteristics (dimension of the phase space and of the invariant tori of the unperturbed system).

1.7 Global structure of superintegrable systems

For the understanding of the dynamics of small perturbations of superintegrable systems it is essential to have a clear picture of the global geometric structure of the fibration in invariant tori, and how it differs from the Liouville integrable case.

In a Liouville integrable system with first integrals in involution $(f_i)_{i=1,\dots,n}$, the map $f = (f_1, \dots, f_n)$ defines a fibration of $\mathcal{M}$ by invariant n -dimensional tori and the base of this fibration is a n -dimensional manifold (*the action manifold*).

For a superintegrable system the situation is different because there is, beside the fibration in invariant tori, a further foliation of $\mathcal{M}$ that should be taken into account and that form, in symplectic geometric terms, what is called a isotropic-coisotropic bifoliation, or dual pair.

1.7.1 Elements of Poisson geometry

In order to describe the geometry of superintegrable systems we need to briefly recall some standard definitions and results on symplectic and Poisson manifolds; for a complete account of Poisson geometry see e.g. [31], in this section we will outline the basic concepts. The experienced reader can safely skip this part.

If $\mathcal{M}$ is a symplectic manifold and $p \in \mathcal{M}$, two tangent vectors $v, w \in T_p\mathcal{M}$ are said to be *symplectically orthogonal* if $\omega_p(v, w) = 0$. If V_p is a subspace of $T_p\mathcal{M}$ then its *symplectic orthogonal complement* $V_p^\perp \subseteq T_p\mathcal{M}$ is defined as the set of elements of $T_p\mathcal{M}$ that are symplectically orthogonal to every element of V_p . If V is a distribution on $\mathcal{M}$ then $V^\perp$ denotes the distribution on $\mathcal{M}$ whose fibers are the symplectic orthogonal complements of the fibers of V . If a distribution V is involutive, and hence generates a foliation $\mathcal{F}$, and if its symplectic orthogonal $V^\perp$ is involutive as well, then $\mathcal{F}$ is said to be *symplectically complete*; the foliation generated by $V^\perp$, denoted $\mathcal{F}^\perp$, is called the *polar* of $\mathcal{F}$. The pair $(\mathcal{F}, \mathcal{F}^\perp)$ is called a *dual pair* [13, 28] or a *bifoliation* or, if both foliations are fibrations, a *bifibration* [20].

If $p \in \mathcal{M}$, a subspace V_p of $T_p\mathcal{M}$ is said to be *isotropic* if $V_p \subseteq V_p^\perp$, *coisotropic* if $V_p \supseteq V_p^\perp$ and *Lagrangian* if $V_p = V_p^\perp$. A submanifold $\mathcal{N}$ of $\mathcal{M}$ is isotropic (or coisotropic) if its tangent spaces are; a foliation is isotropic (or coisotropic) if its leaves are. Note that, if $\mathcal{F}$ is a symplectically complete isotropic foliation, then $\mathcal{F}^\perp$ is coisotropic and its leaves are union of leaves of $\mathcal{F}$.

For example, the orbits of the flow of a Hamiltonian h define an isotropic foliation $\mathcal{F}$ which is symplectically complete: the leaves of $\mathcal{F}^\perp$ are the level sets of h . If h is a Liouville integrable Hamiltonian then the foliation defined by its invariant tori is Lagrangian and coincides with its own polar.

A *Poisson bracket* on a manifold $\mathcal{P}$ is a skew symmetric bilinear map:

$$\{\cdot, \cdot\} : \mathcal{C}^\infty(\mathcal{P}) \times \mathcal{C}^\infty(\mathcal{P}) \rightarrow \mathcal{C}^\infty(\mathcal{P})$$

which satisfies *Leibniz rule* ($\{f, gh\} = g\{f, h\} + \{f, g\}h$) and the *Jacobi identity* ($\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$). A pair $(\mathcal{P}, \{\cdot, \cdot\})$ with $\mathcal{P}$ differentiable manifold and $\{\cdot, \cdot\}$ Poisson bracket on $\mathcal{P}$ is called a *Poisson manifold*.

A symplectic manifold $\mathcal{M}$, equipped with its standard Poisson bracket, is (trivially) a Poisson manifold. An example of Poisson manifold that is not a symplectic manifold is $\mathbb{R}^3$ equipped with Poisson bracket given by the cross product:

$$\{f, g\}(x) = x \cdot \nabla f(x) \times \nabla g(x), \quad x \in \mathbb{R}^3. \quad (1.2)$$

In a Poisson manifold $(\mathcal{P}, \{\cdot, \cdot\})$, the map $g \rightarrow \{g, f\}$ is a derivation. In this way to each function f it is associated a vector field X_f on $\mathcal{P}$ such that $\{g, f\} = \mathcal{L}_{X_f}g$ for every function g on $\mathcal{P}$, where:

$$\mathcal{L}_{X_f}g = \iota_{X_f}dg$$

is the Lie derivative of g in the direction of X_f .

X_f is called the *Hamiltonian vector field* of f . The family of all Hamiltonian vector fields defines an involutive distribution on $\mathcal{P}$. The leaves of the corresponding foliation, called the *symplectic leaves* of $(\mathcal{P}, \{\cdot, \cdot\})$, are endowed with a symplectic structure induced by the Poisson structure of $\mathcal{P}$. The dimension of the symplectic leaf through a point $p \in \mathcal{P}$ is the *rank* of the Poisson manifold at the point p .

For example in the case of $\mathbb{R}^3$, equipped with the Poisson bracket defined in 1.2, the rank of the Poisson manifold is everywhere 2 except at the origin, where it is 0. The symplectic leaves are the origin and the spheres centred at the origin. The symplectic structure on the latter is proportional to the oriented area form induced by the Euclidean structure on $\mathbb{R}^3$.

A *Poisson morphism* between two Poisson manifolds $(\mathcal{P}_1, \{\cdot, \cdot\}_1)$ and $(\mathcal{P}_2, \{\cdot, \cdot\}_2)$ is a map $\phi : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ such that $\{f, g\}_2 \circ \phi = \{f \circ \phi, g \circ \phi\}_1$ for all functions f, g on $\mathcal{P}_2$.

Poisson geometry is relevant to the description of bifoliations because of the following (see [20] for the proof):

Lemma. *A fibration $\pi : \mathcal{M} \rightarrow \mathcal{B}$ of a symplectic manifold $\mathcal{M}$ is symplectically complete if and only if the base manifold $\mathcal{B}$ has a Poisson structure (which is uniquely defined) such that π is a Poisson morphism.*

1.7.2 Geometric formulation of Nekhoroshev-Mishchenko-Fomenko theorem

The geometric content of the Mishchenko-Fomenko theorem has been clarified by Dazord and Delzant [13], Karasev and Maslov [28], Woodhouse [43] and others (see [20] for a more detailed survey) and can be summarized by the following

Proposition 1. *Under the hypotheses, and with the notation, of the Mishchenko-Fomenko theorem:*

1. *the fibration $f : \mathcal{M} \rightarrow f(\mathcal{M}) =: \mathcal{B} \subseteq \mathbb{R}$ has isotropic fibers and is symplectically complete. Its polar (which in general is only a foliation, not a fibration) has coisotropic leaves, of dimension $2n - m$;*
2. *the base $\mathcal{B}$ is a Poisson manifold with symplectic leaves of dimension $2(n - m)$. The preimages under f of the symplectic leaves are the leaves of the foliation polar to $f : \mathcal{M} \rightarrow \mathcal{B}$;*
3. *in any system of local generalized action-angle coordinates (I, ϕ, p, q) :*
 - $\phi \in \mathbb{T}^m$ *are (angular) coordinates on the fibers of f ;*
 - (ϕ, p, q) *are local coordinates on the leaves of the polar, coisotropic foliation, which are parametrized by $I = \text{const}$;*
 - *if we regard (I, p, q) as coordinates on the base $\mathcal{B}$, then $(p, q) \in \mathbb{R}^{2(n-m)}$ are local coordinates on the symplectic leaves of $\mathcal{B}$, which are parametrized by $I = \text{const}$.*

Here we only point out the fact that the existence of the polar to $f : \mathcal{M} \rightarrow \mathcal{B}$ follows from the hypothesis $\{f_i, f_j\} = P_{ij} \circ f$, which implies that the Poisson brackets of the f_i 's are constant on the fibers of f , while the isotropy of the fibers of f is equivalent to the hypothesis on the rank of the matrix P . The relationship between the polar foliation and the symplectic foliation of $\mathcal{B}$ follows from the fact that f is a Poisson morphism.

In general the foliation polar to a symplectically complete fibration needs not to be a fibration, and so, in this case, neither the symplectic foliation of $\mathcal{B}$. If we assume however that the symplectic foliation of $\mathcal{B}$ is a fibration $\sigma : \mathcal{B} \rightarrow \mathcal{A}$, where $\mathcal{A}$ is a manifold, then $\dim(\mathcal{A}) = m$ and, moreover, the coisotropic foliation polar to $f : \mathcal{M} \rightarrow \mathcal{B}$ is a fibration as well, and is given by $\sigma \circ f : \mathcal{M} \rightarrow \mathcal{A}$ as shown in the following commutative diagram:

$$\begin{array}{ccc}
 & \mathcal{M} & \\
 f \swarrow & & \searrow \sigma \circ f \\
 \mathcal{B} & \xrightarrow{\sigma} & \mathcal{A}
 \end{array}$$

The actions I provide local coordinates on the base manifold $\mathcal{A}$ of the coisotropic fibration, which is thus called *action manifold*.

Superintegrable Hamiltonians, that in local coordinates depend only on the actions, are lifts to $\mathcal{M}$ of functions on the action manifold $\mathcal{A}$. The fact that the frequencies of the motions depend on the actions alone has the implication that they are constant on the coisotropic fibers: hence, all tori contained in the same coisotropic fiber (that is, based on a same symplectic leaf of $\mathcal{B}$) carry motions with same frequencies. As such, not only the coisotropic fibration has a *dynamical* meaning, but since the dynamics of small perturbations is heavily influenced by the arithmetic properties of the frequencies, its fibers are the basic “bricks” to consider in any perturbation treatment [20].

This geometric structure can be pictorially represented by an analogy [20], see figure 1.1: if $\mathcal{M}$ is represented by a flowers meadow, then the action manifold is the meadow, the flowers are the coisotropic leaves of the polar foliation, their petals are the fibers of f (the isotropic tori), and their centre are the symplectic leaves.

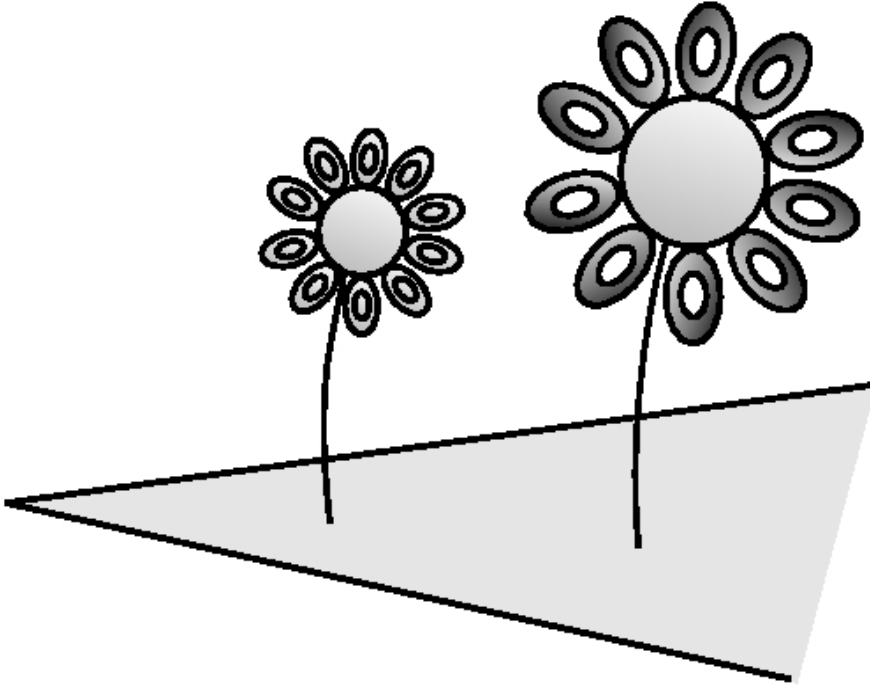


Figure 1.1: pictorial representation of the superintegrable structure, from [20].

The advantage of this global picture over the coordinate description provided by the Nekhoroshev-Mishchenko-Fomenko theorem is that the latter is intrinsically local. If for instance, as it often happens, the symplectic leaves are compact,

then no single system of generalized action-angle coordinates can cover an entire “flower”. The local Nekhoroshev-Mishchenko-Fomenko description fails to detect the basic sets to be considered in presence of perturbations.

In the Liouville integrable case, where $m = n$ and the tori are Lagrangian, the two fibrations coincide.

1.8 Examples of superintegrable systems: global aspects

We now go back to the previous examples of superintegrable systems and study their geometric global structure.

In the case of the resonant isochronous oscillator with two degrees of freedom considered before, the level sets of h (when the origin is excluded) define a (coisotropic) fibration of $\mathbb{R}^4$ with fibers diffeomorphic to $\mathbb{S}^3$ and base $\mathbb{R}^{>0}$ (the action space). On every level set of h , the periodic orbits define a Hopf fibration with fibers diffeomorphic to $\mathbb{S}^1$ (the invariant tori) and base $\mathbb{S}^2$ (the symplectic leaves). The base of the fibration by invariant tori is thus diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}^{>0}$.

Similarly, for the subsystem of Kepler problem defined by periodic and non collisional motions, which have negative energy and non zero angular momentum, the coisotropic fibration is given by h and the action manifold is $\mathbb{R}^{<0}$. The base of the isotropic fibration is a 5-dimensional manifold with 4-dimensional symplectic leaves, diffeomorphic to the product of two spheres $\mathbb{S}^2$ (with a 2-dimensional manifold, parametrizing the collisional motions, removed).

In the case of a particle in a generic central force field the coisotropic foliation is given by the common level sets of the Hamiltonian h and of the norm of the angular momentum vector J . The base manifold is 4-dimensional, with 2-dimensional symplectic leaves which can be identified with spheres of constant $\|J\|$. The action space is 2-dimensional; one action is $\|J\|$, the other a function of h and $\|J\|$.

For the description of the rigid body with a fixed point details can be found in [17], here the main results will be sketched. Let $\mathcal{E}_s^* = (e_x^*, e_y^*, e_z^*)$ and $\mathcal{E}_b = (e_1, e_2, e_3)$ be orthonormal frames with the origin placed in the fixed point of the rigid body: the first one fixed in space, while the second attached to the body and such that the vector e_i is parallel to the principal axis of inertia I_i (for simplicity we consider the case $I_1 > I_2 > I_3 > 0$). The case of the body with equal axes of inertia is actually simpler, see for instance [5, 17]). A displacement of the rigid body in space is uniquely determined by an element $\mathcal{R}$ of $SO(3)$ that transforms the basis $\mathcal{E}_b$ into $\mathcal{E}_s^*$. In this way the configuration space of the rigid body is identified with $SO(3)$. If $m_b = (m_1, m_2, m_3)$ is the expression of the angular momentum of the system in $\mathcal{E}_b$, then the pair $(\mathcal{R}, m_b)$ defines a point in the phase space which can

be thus identified with $\mathcal{M} = SO(3) \times \mathbb{R}^3 = T^*SO(3)$. In these coordinates the Hamiltonian for the system is

$$H(\mathcal{R}, m_b) = \frac{m_1^2}{2I_1} + \frac{m_2^2}{2I_2} + \frac{m_3^2}{2I_3}.$$

We define $m_s(\mathcal{R}, m_b) = \mathcal{R}m_b$ (the components in the frame $\mathcal{E}_s^*$ of the angular momentum of the system) and $f = (H, m_s) : \mathcal{M} \mapsto \mathbb{R} \times \mathbb{R}^3$. The first thing to observe is that, in order for f to be a submersion, we need to restrict our analysis to the submanifold $\mathcal{M}^*$ of $\mathcal{M}$, obtained excluding the points of $\mathcal{M}$ where $m_b = 0$ or $H(\mathcal{R}, m_b) = \frac{\|m_b\|^2}{2I_i}$ for some $i = 1, 2, 3$ (i.e. we must exclude the equilibrium, the stationary rotations and the separatrices connecting them). f defines on $\mathcal{M}^*$ a fibration, whose fibers have two connected component and it is thus possible to find a decomposition of $\mathcal{M}^*$ as $\mathcal{M}_+^* \cup \mathcal{M}_-^*$, with $\mathcal{M}_+^*$ and $\mathcal{M}_-^*$ disjoint, such that the restrictions of f on $\mathcal{M}_+^*$ and $\mathcal{M}_-^*$ separately satisfy the hypotheses of the Nekhoroshev-Mishenko-Fomenko theorem (for example, for $H < \frac{\|m_b\|^2}{2I_2}$, $\mathcal{M}_+^*$ is defined by the condition $m_1 > 0$ while $\mathcal{M}_-^*$ is defined by the condition $m_1 < 0$. Conversely, for $H > \frac{\|m_b\|^2}{2I_2}$, $\mathcal{M}_+^*$ is defined by the condition $m_3 > 0$ while $\mathcal{M}_-^*$ is defined by the condition $m_3 < 0$). We can, in this way, conclude that f determines an invariant fibration in isotropic 2-tori on $\mathcal{M}^*$.

The image of f on $\mathcal{M}_+^*$ is the set:

$$\mathcal{P}_+ = \{(E, \mu) \in \mathbb{R} \times \mathbb{R}^3 \mid \frac{\|\mu\|^2}{2I_1} < E < \frac{\|\mu\|^2}{2I_3}, E \neq \frac{\|\mu\|^2}{2I_2}\}$$

which is a Poisson manifold with bracket defined by

$$\begin{aligned} \{E, \mu_i\} &= 0 \\ \{\mu_i, \mu_j\} &= \epsilon_{ijk} \mu_k. \end{aligned}$$

The symplectic leaves of $\mathcal{P}_+$ (the centre of the flowers) are defined by the level sets of $(E, \mu) \mapsto (E, \|\mu\|^2)$ and are diffeomorphic to spheres. The bifoliation is symplectically complete (and hence it is a bifibration) and the map g from $\mathcal{M}_+^*$ to $\mathbb{R}^2$:

$$g(\mathcal{R}, m_b) = (H(m_b), \|m_b\|^2)$$

defines the polar fibration, with fibers diffeomorphic to $SO(3) \times \mathbb{S}^1$ and base

$$\mathcal{A}_+ = \{(E, G) \in \mathbb{R}^2 \mid 0 < E < \frac{G}{2I_3}, E \neq \frac{G}{2I_1}, \frac{G}{2I_2}\}.$$

Obviously the same analysis can be done in an analogous way for $\mathcal{M}_-^*$, obtaining $\mathcal{P}_- = \mathcal{P}_+$ and $\mathcal{A}_- = \mathcal{A}_+$, and for $\mathcal{M}^*$, obtaining $\mathcal{P} = \mathcal{P}_+ \dot{\cup} \mathcal{P}_-$ and $\mathcal{A} = \mathcal{A}_+ \dot{\cup} \mathcal{A}_-$.

($\dot{\cup}$ indicates the operation of disjoint union of sets). Using the notation adopted previously we have that $g = \sigma \circ f$ with $\sigma(E, \mu) = (E, \|\mu\|^2)$. It is clear then that $\mathcal{A}$ represents the meadow (the action space), while the fibers of g represent the flowers.

From a physical point of view we observe that the points on each symplectic leaf parametrize the direction (in the frame $\mathcal{E}_s^*$) of the angular momentum of the system for a given energy, a quantity which may be of interest to study after the system is perturbed, showing that it is of practical importance the understanding of the theory of perturbation in the superintegrable framework (which differs, as we will see, from the Liouville integrable case exactly in the behaviour on the symplectic leaves).

1.9 Liouville integrability versus superintegrability

It is possible, in principle, to regard a superintegrable system as a Liouville integrable one, introducing action-angle coordinates on the symplectic leaves (see, for more details, [20]).

Let (I, ϕ, p, q) , with p, q defined on some open set $\mathcal{D} \in \mathbb{R}^{2(n-m)}$, be a generalized action-angle coordinate system on the symplectic manifold $\mathcal{M}$ and $h = h(I)$ be a superintegrable Hamiltonian. It is not restrictive (since we are working locally) to consider the case:

$$\mathcal{D} = \{(p, q) \in \mathbb{R}^{n-m} \mid 0 < q_i^2 + p_i^2 < r\}$$

for some $r > 0$. In this way we can define new action angle coordinates (a, α) : $\phi_i = \alpha_i$, $I_i = a_i$, $q_j = -\sqrt{2a_{m+j}} \cos \alpha_{m+j}$, $p_j = \sqrt{2a_{m+j}} \sin \alpha_{m+j}$ ($i = 1, \dots, m$, $j = 1, \dots, n - m$). Obviously, since h does not depend on the variables ϕ , p and q , it does not depends on the new angles α , and the actions a determine (locally) an invariant fibration of $\mathcal{M}$ in Lagrangian tori.

Geometrically this procedure is equivalent to finding a fibration by Lagrangian tori of each symplectic leaf and then lifting it to the symplectic manifold using the superintegrable structure. This define a fibration by n -dimensional tori on each coisotropic leaf, which in turn defines a fibration on the symplectic manifold by invariant n -dimensional tori.

This has two evident disadvantages.

1. The definition of action-angle on the symplectic leaves is not unique. For instance consider the simplest case where the symplectic leaves are diffeomorphic to $\mathbb{T}^2$ and q and p are both angular coordinates on them. It is possible

to consider as action angle coordinates both (I, p, ϕ, q) and $(I, -q, \phi, p)$ obtaining different admissible fibrations.

2. The fibration may not be globally defined. A typical example is when the symplectic leaves are diffeomorphic to $\mathbb{S}^2$ (for example this happen for the rigid body, see [5]). It is possible to resort to the use of cylindrical like coordinates, however in doing so the fibration is not defined globally (In general it is impossible to find a $\mathbb{S}^1$ -fibration on $\mathbb{S}^2$).

This reflects the fact that this fibration is not naturally defined by the dynamics, that detects only the m -dimensional invariant tori (this will be clear especially when dealing with perturbations, where the geometric structure of superintegrable systems plays a fundamental role).

1.10 Perturbations of superintegrable systems

In order to understand the differences between the behaviour, under the effect of a perturbation, of superintegrable systems and Liouville integrable systems it should be noted that in the former case the unperturbed part does not produce motions along the symplectic leaves. In this way the projection onto the symplectic leaves of motions of the system is, to a large extent, determined by the perturbation.

This has the consequence that it is not possible to control the motions along the symplectic leaves using the integrable part alone, without the perturbation. For a short review on the subject, see [20].

In principle it is possible to approach the study of a small perturbation of a superintegrable Hamiltonian, with either KAM or Nekhoroshev theory.

The application of KAM theorem to superintegrable systems is rather standard, dating back to Arnol'd's study of the 3-body problem in his fundamental work [2]. Later applications to, e.g., perturbations of the Euler-Poinsot rigid body can be found in [26, 32]. The theory is local, so the results can be stated in a system of generalized action-angle coordinates $(I, \phi, p, q) \in \mathcal{I} \times \mathbb{T}^m \times \mathcal{D}$ with $\mathcal{I}$ open set in $\mathbb{R}^m$ and $\mathcal{D}$ open set in $\mathbb{R}^{2(n-m)}$.

The Hamiltonian has the form $h_\epsilon = h_0 + \epsilon f$. Assume that h_0 is superintegrable and non degenerate in the Kolmogorov sense (i.e. satisfying condition (1.1) of KAM theorem). Let $\mathcal{I}'$ be the subset of $\mathcal{I}$ consisting of points I such that the frequency vector

$$\omega_0(I) = \frac{\partial h_0}{\partial I}(I)$$

is strongly non resonant, say Diophantine. Then, in a (small) neighbourhood of $\mathcal{I}' \times \mathbb{T}^m \times \mathcal{D}$, it is possible to average the Hamiltonian h_ϵ over the angles ϕ any number $k \geq 1$ of times.

In this way it is possible to find a canonical diffeomorphism Φ_k , ϵ -close to the identity, from $\mathcal{I}' \times \mathbb{T}^m \times \mathcal{D}$ into itself (generated for example by the Lie method) such that if, in coordinates,

$$(I, \phi, p, q) = \Phi_k(I', \phi', p', q')$$

then

$$h_\epsilon \circ \Phi_k(I', \phi', p', q') = h_0(I') + \epsilon \hat{f}_\epsilon(I', p', q') + O(\epsilon^{k+1})$$

for some function $\hat{f}_\epsilon$ independent from the angles ϕ .

If $\hat{h}_\epsilon = h_0 + \epsilon \hat{f}_\epsilon$ is Liouville-integrable and non degenerate and if $|\epsilon|$ is small enough then it is possible to apply KAM theorem to conclude that, in $\mathcal{I}' \times \mathcal{D}$, there exists a set of large measure of invariant n -dimensional tori.

Of course, the hypothesis of Liouville-integrability of $\hat{h}_\epsilon$ is rather strong, and often not fulfilled. A special case where this is true is for example when $n = m + 1$.

It is interesting to interpret this result in terms of the geometry of the superintegrable system h_0 with reference to figure 1.1: each of these flowers that support quasi-periodic motions of h_0 with Diophantine frequencies (they form a set of large measure in the action space, even though with a Cantor-like structure), contains a set of large measure (and yet Cantor-like) of invariant n -dimensional tori of the perturbed system. Each torus is close to the product of an m -dimensional isotropic torus of h_0 (a petal of the flower) and of a $(n - m)$ -dimensional torus built from $\hat{h}_\epsilon$ on the symplectic leaf (the centre).

The KAM construction however does not provide any mechanism to bound the motions starting on flowers with non-Diophantine frequencies (with respect to h_0), nor on those starting on Diophantine flowers but on petals with non-Diophantine frequencies (with respect to $\hat{h}_\epsilon$). All those motions, on a long timescale, might wander away (Arnol'd diffusion).

The application of Nekhoroshev theory to superintegrable systems is somehow different. Nekhoroshev himself proved his theorem for a Hamiltonian h_ϵ , defined on a domain $\mathcal{A} \times \mathbb{T}^m \times \mathcal{D}$, with $\mathcal{A} \subseteq \mathbb{R}^m$ and $\mathcal{D} \subseteq \mathbb{R}^{2(n-m)}$, of the form

$$h(I, \phi, p, q) = h_0(I) + \epsilon f(I, \phi, p, q),$$

which he calls “system with parameters (p, q) ” and showed that, under the standard convexity or steepness assumption on h_0 , the actions remain nearly constant ($|I_t - I_0|$ small with ϵ^b) for times

$$|t| \leq T_N = \min(Te^{-\frac{1}{\epsilon^a}}, T_{esc}),$$

where T_{esc} is the exit time of the coordinates (p, q) from $\mathcal{D}$, T is a positive constant and a, b are the stability exponent of Nekhoroshev theorem. The reason for this

limitation is that Nekhoroshev theorem does not provide any direct control on the coordinates (p, q) .

This formulation of Nekhoroshev theorem can be applied to the perturbation of a superintegrable system within the domain of a system of local generalized action-angle coordinates (I, ϕ, p, q) . If however this domain is not a union of entire flowers, so that (p, q) are not global coordinates on the symplectic leaves, then the absence of control on the motions of (p, q) has the consequence that there are no estimates on T_{esc} , and therefore on T_N . This happens for example when the symplectic leaves are compact (and it will be the case for the system we will analyze). This difficulty was overcome in [16] by showing that normal forms constructed in the proof of the theorem are in fact defined not only locally in a coordinate domain, but “semi-globally” in a neighbourhood of an entire flower, so that projections of motions on the symplectic leaves cannot escape the domain where the normal form is defined. This leads to the following (we deal with the compact case only):

Theorem 2 (Nekhoroshev, superintegrable version [16, 34]). *Let h_0 be a superintegrable convex Hamiltonian defined on the n -dimensional symplectic manifold $\mathcal{M}$ with m -dimensional invariant isotropic tori and compact symplectic leaves. Let f be a function on $\mathcal{M}$ such that the representatives, in generalized action-angle coordinates (I, ϕ, p, q) , of h_0 and f are real analytic.*

Then there exist positive constants ϵ_0, T, R, a, b , independent from ϵ , such that for $|\epsilon| \leq \epsilon_0$ and t satisfying

$$|t| \leq T_N = T e^{\left(\frac{\epsilon_0}{|\epsilon|}\right)^a} \quad (1.3)$$

every motion of h_ϵ satisfies:

$$|I_i(t) - I_i(0)| \leq R \left(\frac{|\epsilon|}{\epsilon_0} \right)^b \quad \forall i = 1 \dots m.$$

A possible choice for a and b is $a = b = 1/(2m)$.

This result alone does not allow to control the motion along the symplectic leaves which in several cases (for example for the perturbed rigid body) conveys important information on the system. Nevertheless, these motions can be studied by means of the normal form constructed within the proof of Nekhoroshev theorem.

A cornerstone of Nekhoroshev theory is the partition of the action manifold, and consequently of the whole phase space, in a finite number of non-overlapping *zones*. A detailed description in the case of this decomposition for $m = 2$ will be presented in the next chapter. On a neighbourhood of each zone (which are sets with non empty interior) there exists a symplectic diffeomorphism Φ , ϵ -close to the identity, such that

$$h_\epsilon \circ \Phi(I', \phi', p', q') = h_0(I') + \epsilon u_\epsilon(I', \phi', p', q') + \epsilon e^{-\left(\frac{|\epsilon|}{\epsilon_*}\right)^a} v_\epsilon(I', \phi', p', q') \quad (1.4)$$

where u_ϵ , v_ϵ and their Hamiltonian vector fields are uniformly bounded by a constant independent from ϵ and the zone on which Φ is defined, and u_ϵ depends on the angles ϕ' only thru certain combinations. For simplicity we will assume u_ϵ depending only from $\phi'_1, \dots, \phi'_k$ for some $k \leq m$, this assumption is not really restrictive since, in the general case, a coordinate change is sufficient to achieve this situation. The dynamics of the actions $I_{k+1}, \dots, I_m$ is determined only by $\epsilon e^{-\left(\frac{|\epsilon|}{\epsilon^*}\right)^a} v_\epsilon$ and so, over the exponential timescale (1.3), they move over a length of order ϵ . A smart choice in the definition of the resonance zones, exploiting convexity of h_0 and conservation of energy, provides the desired confinement of the other actions (and so of the original actions).

The dynamics of (p', q') is determined by the perturbation and so it is relevant on a timescale of order ϵ^{-1} , which is much longer than the one determined by the rotations of the angles but it is also much shorter than the exponential timescale of Nekhoroshev theorem. This means that (p', q') (and so the original variables (p, q)) could exhibit chaotic motions, this phenomenon goes under the name of *slow chaos* or *adiabatic chaos*. This different behaviour between non resonant and resonant motions has been numerically demonstrated for perturbations of the Euler-Poinsot rigid body ($n = 3$, $m = 2$) [5] and confirmed by numerical analysis [19]. A similar situation arises in spin-orbit problems [9], figures 1.2 and 1.3 show this behaviour. In the next chapter we will analyze this situation in depth for our case of interest. The purpose of this thesis is to investigate the existence of analogues of these phenomena in quantum systems. To this end, we will consider a classical simple model formed by a superintegrable system with 3 degrees of freedom and 2-dimensional compact symplectic leaves. We will first study a perturbation of it with the theories of Nekhoroshev and of the adiabatic invariants (chapter 2). Then (chapter 3) we will quantize it and analyze its quantum counterpart using the apparatus of the coherent states in order to highlight the features that can be linked to the classical behaviour.

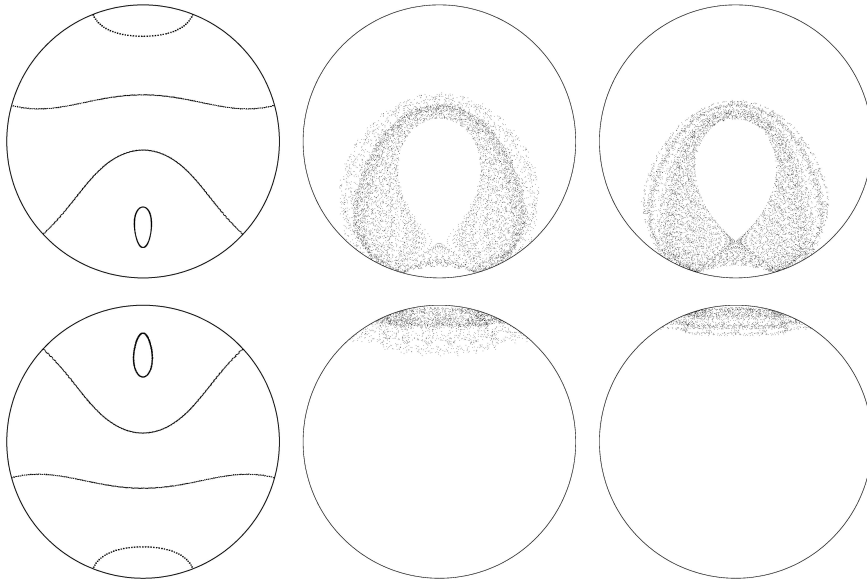


Figure 1.2: nonresonant (left) and resonant (centre and right) motions on the symplectic leaves for the perturbed Euler-Poinsot rigid body. In this case the leaves are spheres, represented here using the stereographic projection from the north and the south pole (respectively first and second row). In the resonant case the projection of the trajectories of the same point for 2 different values of ϵ is depicted, showing that the chaotic zone in both cases are roughly the same (on the appropriate time-scale). Pictures from [19].

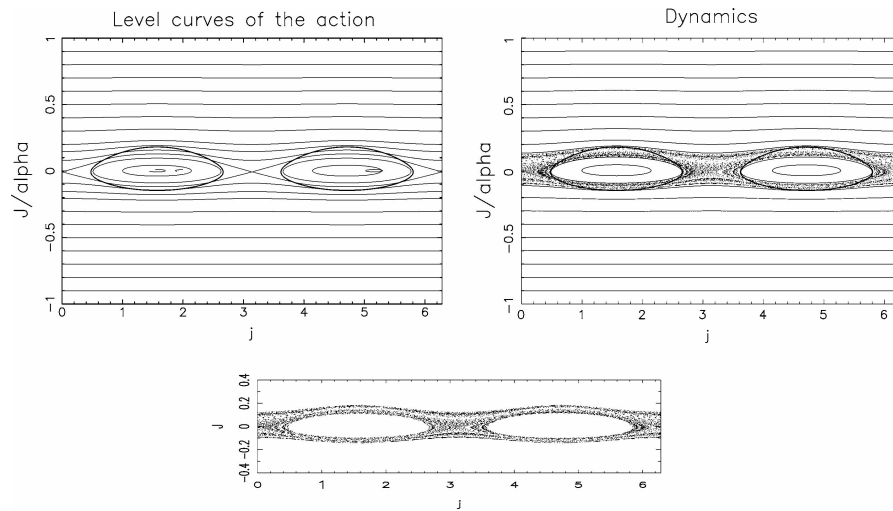


Figure 1.3: resonant motions on the symplectic leaves for the spin-orbit problem. In the picture on the left the level set of the adiabatic invariant and the curve Γ , projected on the symplectic leaves, which are spheres represented using cylindrical coordinates, are shown (The theory of adiabatic invariance is explained in the next chapter). The picture on the bottom shows an actual orbit and the picture on right shows a superimposition of the other two. Pictures from [9].

Chapter 2

Study of the classical system

The present chapter is devoted to the introduction of the classical version of the system that we will analyze. It is defined as a perturbation of a superintegrable system and it shares some common features with the rigid body.

The geometry of resonance zones, specific to this system, is explained and Nekhoroshev theorem is applied to prove long term stability of actions. An analysis of the different normal forms adopted in every resonant (or non-resonant) zone is then performed and in particular the presence of regular and chaotic features of the dynamics is highlighted using the semi rigorous theory of adiabatic invariants adapted to this system.

We conclude with some numerical results that illustrate the theory.

2.1 Definition of the classical system

The system considered here is a small perturbation of a superintegrable one defined on the manifold $\mathcal{M} = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ equipped with the symplectic structure given by the product of standard symplectic structure on $\mathbb{S}^2$ seen as the symplectic leaf of $\mathbb{R}^3$, equipped with the Poisson structure (1.2), described in the previous chapter, of points of unit norm.

If the functions I_1 and I_2 denote the height, with respect to the choice of some arbitrary axis in $\mathbb{R}^3$, on the first two spheres we may consider the Hamiltonian

$$h_0 = \frac{1}{2}(I_1^2 + I_2^2).$$

The dynamics is constant on the third sphere and on the first two spheres is defined by uniform simultaneous rotations on the level sets of I_1 and I_2 and it is represented in figure (2.1). The frequency of rotation on the i^{th} -sphere of a motion is proportional to the value of I_i . h_0 is thus an anisochronous superintegrable system,

with action manifold parametrized by the values of I_1 and I_2 , coisotropic fibers (the flowers) defined by the level sets of I_1 and I_2 and, on them, the invariant tori are parametrized by the points on the third sphere. The situation is summarized in the diagram:

$$\begin{array}{ccc} & \mathcal{M} & \\ i \swarrow & & \searrow c \\ \mathcal{A} \times \mathbb{S}^2 & & \mathcal{A} \end{array}$$

where $\mathcal{A} = [-1, 1] \times [-1, 1]$ is the action manifold and

$$\begin{aligned} c &= (I_1, I_2) : \mathcal{M} \mapsto \mathcal{A} \\ i &= (I_1, I_2, \pi_{\mathbb{S}^2_3}) : \mathcal{M} \mapsto \mathcal{A} \times \mathbb{S}^2, \end{aligned}$$

define respectively the coisotropic and the isotropic foliation (here $\pi_{\mathbb{S}^2_3}$ is the natural projection from $\mathcal{M}$ on the third sphere).

It should be noted that the isotropic foliation is not a fibration: when $|I_1|$ and $|I_2|$ are both different from 1 the invariant tori have dimension 2 but, on the border of $\mathcal{A}$, the tori become circles or points. We can choose to address this problem by removing the singular fibers and consider the manifold $\mathcal{M}^* = \mathbb{S}^2_{\times} \times \mathbb{S}^2_{\times} \times \mathbb{S}^2$, where $\mathbb{S}^2_{\times}$ represents $\mathbb{S}^2$ with the poles removed. $\mathbb{S}^2_{\times}$ is diffeomorphic to a cylinder, and if we adopt cylindrical coordinates (I, ϕ) (I measures the height on the axis containing the removed points) we have that the symplectic structure is written as $dI \wedge d\phi$.

Now we perturb h_0 defining the Hamiltonian

$$h_{\epsilon} = h_0 + \epsilon f$$

where f is an arbitrary (for simplicity ϵ -independent) real valued function on $\mathcal{M}$.

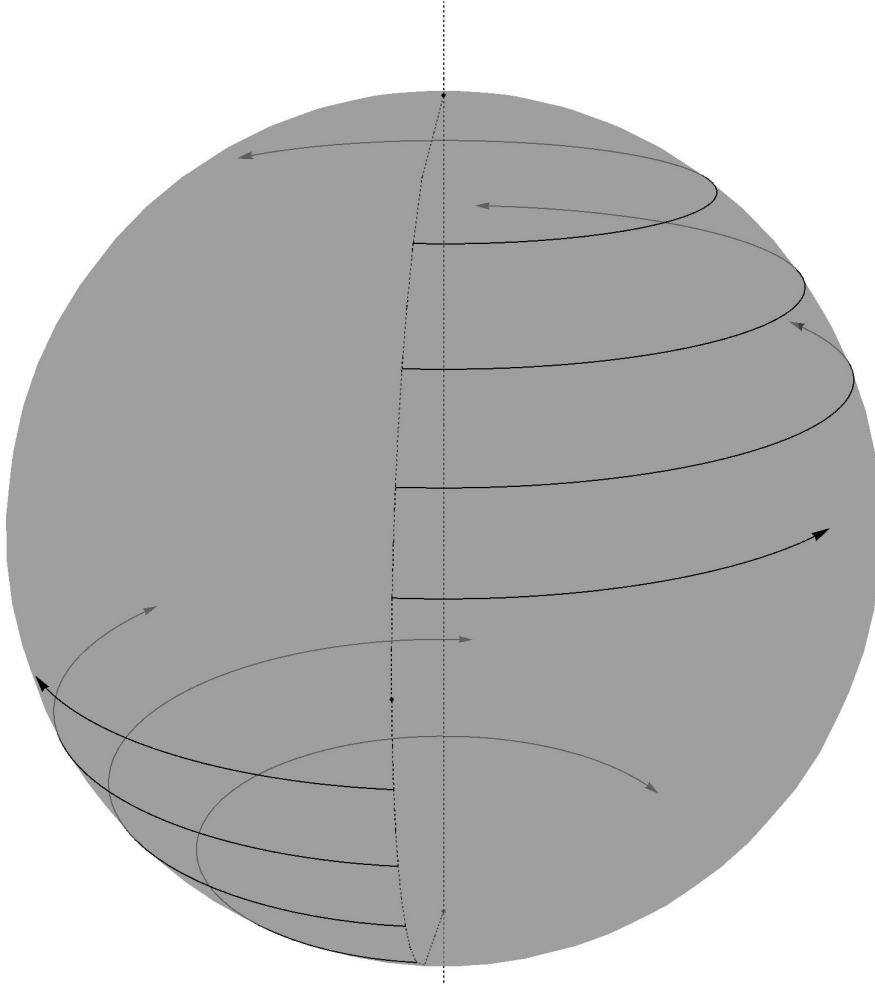


Figure 2.1: projection of the orbits of h_0 on the first (or second) sphere. The length of the arrows is proportional to the speed of the corresponding motion. On the equator and on the poles (the two points joined by a dashed line) the motions are equilibria and thus represented here by points.

2.2 Nekhoroshev estimates

The Hamiltonian h_0 , as a function of the actions I_1 and I_2 , is convex and it is thus possible to apply, as long as we manage to avoid motions that cross the boundary of $\mathcal{A}$, Nekhoroshev theorem:

Proposition 2. *There exist positive constants $\epsilon_0, \delta, T, R, a, b$, independent from ϵ , with $\delta < 1$ such that for $|\epsilon| \leq \epsilon_0$, for t satisfying*

$$|t| \leq T_N = T e^{\left(\frac{\epsilon_0}{|\epsilon|}\right)^a} \quad (2.1)$$

and for every point $x \in \mathcal{M}$ such that $|I_\alpha(x)| \leq 1 - \delta$ for $\alpha = 1, 2$ then $x(t)$, the motion of h_ϵ with initial point x , is well defined at the time t and

$$|I_\alpha(x(t)) - I_\alpha(x)| \leq R \left(\frac{|\epsilon|}{\epsilon_0} \right)^b \quad i = 1, 2.$$

A possible choice for the stability exponents is $a = b = \frac{1}{4}$.

Remark. When dealing with Nekhoroshev theorem we need to remove the sets with $I_\alpha = \pm 1$ for α equal to 1 or 2 and thus consider $\mathcal{M}^*$ as the phase space. In this way, however, the flow of h_ϵ is no longer complete and may even be not defined for the exponential timescale (2.1). Removing a “cap”, around the poles of the first and second sphere, of radius δ ensures the well definition of the flow. This kind of obstruction is common in concrete system (for example in the perturbed rigid body, see [6]).

Choosing worse stability exponents leads to the asymptotic estimate $\delta \rightarrow 0$ as $\epsilon \rightarrow 0$. On the other hand it is possible to modify Nekhoroshev theory to obtain stability results around the points $I_\alpha = \pm 1$ (see [6, 18]). In this case the same estimates holds globally on $\mathcal{M}$ (i.e. the same result holds without requirements on the initial data x). We will not investigate further these cases and we will stick to motions far enough from the poles of the first two spheres. $\triangle$

2.3 Geometry of resonances

The proof of Nekhoroshev theorem for h_ϵ , as previously mentioned, is based on a partition of the action manifold in different zones.

This decomposition depends on the small parameter ϵ .

In every zone h_ϵ is conjugated, via a symplectomorphism Φ close to the identity, to a Hamiltonian of the form (1.4):

$$h_\epsilon \circ \Phi(I', \phi', p', q') = h_0(I') + \epsilon u_\epsilon(I', \phi', p', q') + \epsilon e^{-\left(\frac{|\epsilon|}{\epsilon^*}\right)^a} v_\epsilon(I', \phi', p', q').$$

There are three kind of zones:

- the *non resonant zone*, $\mathcal{R}_0$, where u_ϵ does not depend from the angles but only on the actions and on the coordinates on the third sphere;
- the ν -*resonant zone*, $\mathcal{R}_\nu$, for some $\nu \in \mathbb{Z}^2$, where u_ϵ depends on the angles ϕ' only thru the resonant combination $\sigma = \nu \cdot \phi'$;
- the *fully resonant zone*, $\mathcal{R}_2$, where u_ϵ depends on both ϕ'_1 and ϕ'_2 .

For an accurate description of the zones for general n see [37].

A positive integer N (the *cut-off parameter*) and two real positive constant α and β are fixed, it turns out that a good choice is $N \sim \epsilon^{-\frac{1}{4}}$. The fully resonant zone is defined as

$$\mathcal{R}_2 = \{I \in \mathcal{A} \mid \|I\| \leq \frac{\alpha}{N}\}$$

only those $\nu \in \mathbb{Z}^2$ such that $|\nu_1| + |\nu_2| \leq N$, ν generates a maximal lattice (i.e. $\gcd(\nu_1, \nu_2) = 1$) and ν lies in the first or fourth quadrant are considered. For these ν the ν -resonant zone is defined as

$$\mathcal{R}_\nu = \left\{ I \in \mathcal{A} \mid I \notin \mathcal{R}_2, |I \cdot \nu| \leq \frac{\beta}{N^2} \right\}.$$

Every point in $\mathcal{A}$ that is not in $\mathcal{R}_2$ or in $\mathcal{R}_\nu$, for some ν , belongs to $\mathcal{R}_0$. There exists a good choice of the parameters α and β such that the zones are mutually disjointed and so we have a partition of $\mathcal{A}$.

As ϵ decreases the number of resonant zones increases, since more and more $\nu \in \mathbb{Z}^2$ should be considered. $\mathcal{R}_\nu$ is a strip around the line $I' \cdot \nu = 0$ with width of order $\epsilon^{\frac{1}{2}}$, while $\mathcal{R}_2$ is a disk of radius of order $\epsilon^{\frac{1}{4}}$ around the origin. In the following, if there is no risk of ambiguity, we will use the term fully resonant, ν -resonant and non resonant zone $\mathcal{R}_\nu$ also for the subset of $\mathcal{M}$ defined respectively by $c^{-1}(\mathcal{R}_2)$, $c^{-1}(\mathcal{R}_\nu)$ and $c^{-1}(\mathcal{R}_0)$.

In order to understand the dynamics of the system in the different zones it is useful to consider the approximation of $h_\epsilon \circ \Phi$ at the order ϵ (i.e. neglecting the exponentially small remainder) given by the “truncated normal form”:

$$\hat{h}_\epsilon = h_0 + \epsilon u_\epsilon. \tag{2.2}$$

which give a good approximation of the dynamics of the systems for the Nekhoroshev timescale T_N .

In general, even when the actions are treated as constant parameters, the dynamics of the angles is, in general, strongly coupled with the one of the (p', q') variables via u_ϵ and no control is, as already mentioned, a priori possible and a more deep analysis, which will be performed in the following sections, is needed.

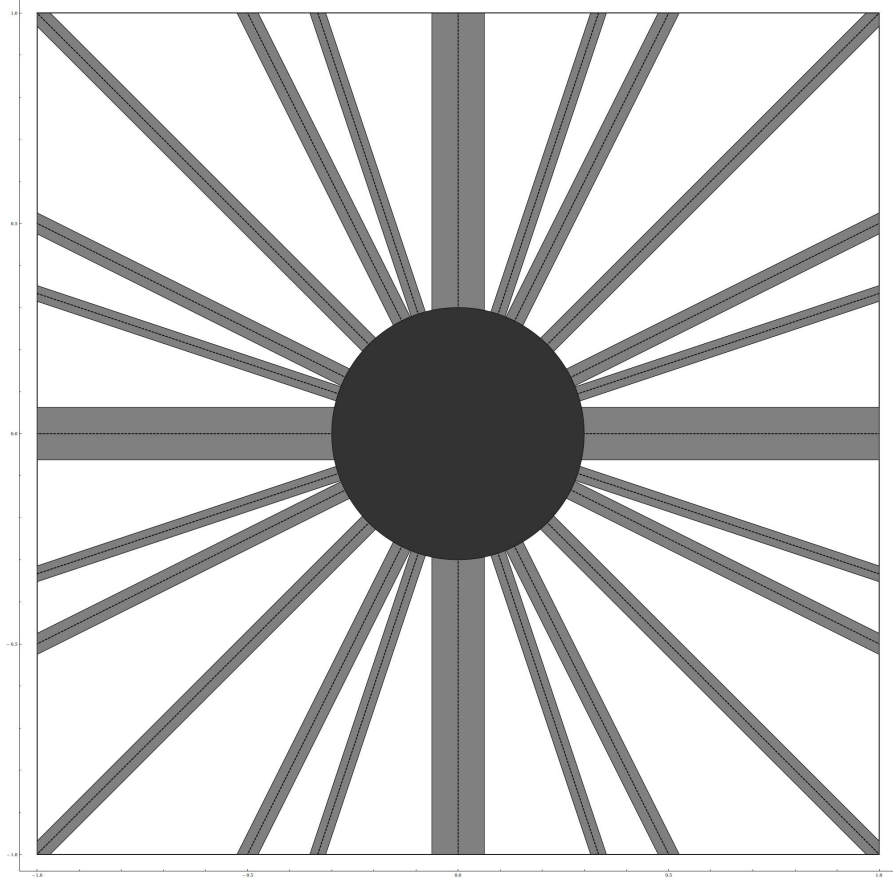


Figure 2.2: geometry of resonances. The white domain represents $\mathcal{R}_0$, the light grey domains represent $\mathcal{R}_\nu$ for different values of ν (up to $|\nu| \leq 4$), the black area covers $\mathcal{R}_2$, the dashed lines represent the resonances, i.e. the sets of points $I \in \mathcal{A}$ such that $I \cdot \nu = 0$ for some $|\nu| \leq 4$.

Remark. While for the proof of the stability of the actions in Nekhoroshev theorem the exact expression of u_ϵ is not needed, it is nevertheless useful when dealing with the description of motions on the third sphere.

In general the perturbation f can be expanded in Fourier series:

$$f(I, \phi, p, q) = \sum_{\nu \in \mathbb{Z}^2} \hat{f}_\nu(I, \nu \cdot \phi, p, q).$$

If we define f_ν as

$$f_\nu = \sum_{k \in \mathbb{Z}} \hat{f}_{k\nu}$$

it is possible to show that in $\mathcal{R}_0$ u_ϵ is $\sqrt{\epsilon}$ -close to $f_{(0,0)}$ while in $\mathcal{R}_\nu$ u_ϵ is $\sqrt{\epsilon}$ -close to f_ν . $\triangle$

2.4 Nonresonant motions

In $\mathcal{R}_0$ the truncated normal form $\hat{h}_\epsilon$ defined in (2.2) does not depend from the angles. For the dynamics defined by this Hamiltonian the functions I'_1 , I'_2 and u_ϵ are constants of motion. In particular, once I'_1 , I'_2 are fixed, it is possible to regard u_ϵ as a function defined on the third sphere invariant under the dynamics of $\hat{h}_\epsilon$ with its level sets defining curves followed by the orbits.

If we consider also the exponential remainder as in (1.4) we have that I'_1 , I'_2 , u_ϵ are no longer conserved. However, since the norm of the Hamiltonian vector field of v is uniformly bounded in the considered domain, in the exponential timescale T_N of Nekhoroshev theorem the variation of these function is of order ϵ . Since the norm of the Hamiltonian vector field of u_ϵ is also uniformly bounded we can conclude that the variation of the function u_ϵ , for a fixed value of I' , is also of order ϵ .

The orbits on the third sphere of the system thus closely follows, at least for an exponential time, the level sets of a smooth function. Moreover this Hamiltonian is obtained from the original one, h_ϵ , by means of a near to the identity transformation, thus motions of the system closely follows, for the considered timescale, the level sets of the function u_ϵ .

As remarked in the previous section, $u_\epsilon - f_{(0,0)}$ is of order $\epsilon^{\frac{1}{4}}$ and so we can conclude that

Proposition 3. *If x is a non resonant point for h_ϵ and $|t| < T_N$, with T_N defined in Proposition 2, then the projection of x_t on the third sphere is $\epsilon^{\frac{1}{4}}$ -close to the level sets of $f_{(0,0)}$ for values of the actions fixed by x .*

See for example [5] for a detailed description of this situation in a concrete example.

2.5 Resonant motions

When considering some resonant zone $\mathcal{R}_\nu$ the truncated normal form $\hat{h}_\epsilon$, defined above, depends from the angles ϕ' only through the linear combination $\nu \cdot \phi'$ and we are in an intermediate situation between chaotic and regular behaviour. A quantitative description of the dynamics on the symplectic leaves in this case can be obtained using the semi-rigorous theory of adiabatic invariants, originally developed by Neidsthardt, see [3, 8, 9] for reference.

It is possible to consider the new coordinates (keeping the same ones on the

third sphere):

$$\begin{aligned} S &= -n \wedge I' \\ F &= \nu \wedge I' \\ \sigma &= \nu \cdot \phi' \\ \psi &= n \cdot \phi' \end{aligned}$$

where n is an element of $\mathbb{Z}^2$ such that $\nu \wedge n := \nu_1 n_2 - \nu_2 n_1 = 1$ (the existence of n is guaranteed by the condition $\gcd(\nu_1, \nu_2) = 1$). While (S, F) are coordinates on $\mathcal{A}$, (σ, ψ) are angular coordinates on the tori of the isotropic fibration, since they are obtained from the old angles ϕ' via a linear transformation belonging to $SL(\mathbb{Z})$. It is immediate to check that this transformation is canonical. In this way we have defined new action-angle coordinates for the system.

In the new coordinates $\hat{h}_\epsilon$ depends only on the angle σ and thus F is a constant of motion. For this reason from now on we consider only motions with a fixed value of F which is treated as a parameter. The unperturbed part of the Hamiltonian becomes:

$$h_0 = \frac{\|\nu\|^2}{2} S^2 + \frac{\|n\|^2}{2} F^2 + \nu \cdot n F S$$

with its global (and unique, by convexity) minimum achieved in

$$S = -\frac{\nu \cdot n}{\|\nu\|^2} F.$$

For this reason it is useful to define $\hat{S}$ as

$$S = \hat{S} - \frac{\nu \cdot n}{\|\nu\|^2} F.$$

As long as F is seen as a parameter, and not as a dynamical variable, this substitution defines a canonical transformation. In the new variables (ignoring constant terms and those depending only from F) the Hamiltonian becomes

$$\hat{h}_\epsilon(\hat{S}, \sigma, p, q) = \frac{\|\nu\|^2}{2} \hat{S}^2 + \epsilon \hat{u}_\epsilon(\hat{S}, \sigma, p, q)$$

where (p, q) are variables on the third sphere and $\hat{u}_\epsilon$ is u_ϵ evaluated in the new coordinate system for the fixed value of F .

$\hat{S}$ measures the distance from the exact resonance $\nu \cdot I = 0$ in the action space. In the new coordinates the section, with fixed F , of $\mathcal{R}_\nu$ is defined by

$$|\hat{S}| \leq \frac{\beta}{\|\nu\|^2 N^2} \sim \sqrt{\epsilon}$$

hence we have:

$$\hat{h}_\epsilon = \tilde{h}_\epsilon + o(\epsilon)$$

where

$$\tilde{h}_\epsilon(\hat{S}, \sigma, p, q) = \frac{\|\nu\|^2}{2} \hat{S}^2 + \epsilon \tilde{u}_\epsilon(\sigma, p, q) \quad (2.3)$$

and

$$\tilde{u}_\epsilon(\sigma, p, q) = \hat{u}_\epsilon(0, \sigma, p, q).$$

As observed before it is sufficient, for our purpose, to consider f_ν in place of u_ϵ .

To understand the dynamics we will consider the approximation given by (2.3) and the particular case (which will be sufficient for the study of the actual system):

$$\tilde{u}_\epsilon(\sigma, p, q) = a_\epsilon(p, q) + b_\epsilon^{(1)}(p, q) \cos \sigma + b_\epsilon^{(2)}(p, q) \sin \sigma. \quad (2.4)$$

Moreover we will suppose $\epsilon > 0$. While $\tilde{h}_\epsilon$ describes a system with 2 degrees of freedom, which can a priori exhibit chaotic behaviours, it is possible to consider the 1-degree of freedom system $\tilde{h}_\epsilon^f$, the so called *frozen system*, obtained from $\tilde{h}_\epsilon$ by simply fixing the values of (p, q) . It should be kept in mind that the dynamics of $\tilde{h}_\epsilon^f$ is not directly related to the one of $\tilde{h}_\epsilon$, since for the latter (p, q) are not, in general, constants of motion.

Defining (from now on any dependence from (p, q) in the frozen system will be neglected):

$$b_\epsilon = \sqrt{\left(b_\epsilon^{(1)}\right)^2 + \left(b_\epsilon^{(2)}\right)^2}$$

it is possible to find σ_0 such that, if $\sigma = \hat{\sigma} + \sigma_0$,

$$\tilde{h}_\epsilon^f(\hat{S}, \hat{\sigma}) = \frac{\|\nu\|^2}{2} \hat{S}^2 - \epsilon b_\epsilon \cos \hat{\sigma} + \epsilon a_\epsilon$$

and so $\tilde{h}_\epsilon^f$ is the Hamiltonian of a pendulum with its stable equilibrium in the origin and the unstable one in $\hat{\sigma} = \pi$. It is thus possible to define action-angle (A, α) variables (except on the separatrix) for the system $\tilde{h}_\epsilon^f$ where the action A depends from its arguments (S, σ) only thru $\tilde{h}_\epsilon^f$.

The theory of adiabatic invariants [8, 9] tell us that, as long as we are far from a separatrix of $\tilde{h}_\epsilon^f$, then the action A , which can be regarded as a function in the (S, σ, p, q) variables, is nearly conserved by the dynamics of $\tilde{h}_\epsilon$. If we fix a value for $\tilde{h}_\epsilon$, which is conserved, A defines a function on the third sphere (since for a fixed value of the (p, q) variables the actions for the frozen system depend only on the energy) and so motions of the (p, q) variables follows its level sets. This procedure is no longer valid when motions crosses the separatrix of the frozen system. For a fixed value ϵE of $\tilde{h}_\epsilon$ we have that the point (p, q) corresponds to a separatrix of $\tilde{h}_\epsilon^f$ relative to (p, q) (meaning that $\tilde{h}_\epsilon^f = \epsilon E$ parametrizes a separatrix of the frozen

system) if the maximum attained by $\tilde{u}_\epsilon(\sigma, p, q)$ is equal to ϵE . In this way a curve Γ is defined on the third sphere by:

$$\max_{\sigma} u_\epsilon(\sigma, p, q) = \epsilon E.$$

Summarizing we have that, fixed a value ϵE of the energy of the system $\tilde{h}_\epsilon$, A defines a function on the third sphere whose level sets are followed by the motions of the system. However, when the curve Γ is crossed, the value of A may change and the motions start to follow a different level set. As several crossings accumulate, an extended domain on the third sphere can be visited by the dynamics.

It is possible to compute explicitly the adiabatic invariant and the curve Γ for the Hamiltonian (2.3) when $\tilde{u}_\epsilon$ has the simple form (2.4). the maximum attained by $\tilde{u}_\epsilon$, fixed (p, q) , is equal to $\epsilon(a_\epsilon + b_\epsilon)$ and so, when studying motions relative to the every surface $\tilde{h}_\epsilon = \epsilon E$, we have that Γ is implicitly defined by

$$a_\epsilon(p, q) + b_\epsilon(p, q) = E.$$

It is useful, for notational convenience, to introduce the quantity $\alpha_\epsilon(p, q) = E - a_\epsilon(p, q)$, in this way Γ has equation $\alpha_\epsilon - b_\epsilon = 0$.

For the computation of the adiabatic invariant it is useful to define the function of the angle σ :

$$\tilde{S}(\sigma) = \frac{\sqrt{2\epsilon}}{\|\nu\|} \sqrt{\alpha_\epsilon + b_\epsilon \cos \sigma}.$$

On the orbit of the frozen systems of energy ϵE , for a fixed value of the angle σ , the action S (the hats from the variables names are dropped for notational convenience) can take the values:

$$S = \pm \tilde{S}(\sigma).$$

For this expression to make sense (i.e. $\tilde{S}(\sigma)$ is real) $\alpha_\epsilon + b_\epsilon$ must be non-negative (which means that ϵE should be higher than the ground energy of the frozen system), if $\alpha_\epsilon - b_\epsilon$ is negative, $\tilde{S}(\sigma)$ is defined for $|\sigma| \leq \bar{\sigma} \leq \pi$ where

$$\cos(\bar{\sigma}) + \frac{\alpha_\epsilon}{b_\epsilon} = 0$$

and the orbit parametrized is a *libration*, if it is positive then $\tilde{S}(\sigma)$ is defined for all σ , and the orbit parametrized is a *rotation*. In this case we define $\bar{\sigma} = \pi$.

The corresponding value of the action is given by the area enclosed, in the

(S, σ) plane, by the orbit rescaled by 2π :

$$\begin{aligned}
A &= \frac{2}{\pi} \int_0^{\bar{\sigma}} \tilde{S}(\sigma) d\sigma \\
&= \frac{2\sqrt{2\epsilon}}{\pi \|\nu\|} \int_0^{\bar{\sigma}} \sqrt{\alpha_\epsilon + b_\epsilon \cos \sigma} d\sigma \\
&= \frac{2\sqrt{2\epsilon}}{\pi \|\nu\|} \int_0^{\bar{\sigma}} \sqrt{\alpha_\epsilon + b_\epsilon - 2b_\epsilon \sin^2 \frac{\sigma}{2}} d\sigma \\
&= \frac{2\sqrt{2\epsilon}}{\pi \|\nu\|} \sqrt{\alpha_\epsilon + b_\epsilon} \int_0^{\bar{\sigma}} \sqrt{1 - \frac{2b_\epsilon}{\alpha_\epsilon + b_\epsilon} \sin^2 \frac{\sigma}{2}} d\sigma \\
&= \frac{4\sqrt{2\epsilon}}{\pi \|\nu\|} \sqrt{\alpha_\epsilon + b_\epsilon} \int_0^{\frac{\bar{\sigma}}{2}} \sqrt{1 - \frac{2b_\epsilon}{\alpha_\epsilon + b_\epsilon} \sin^2 \tau} d\tau \\
&= \frac{4\sqrt{2\epsilon}}{\pi \|\nu\|} \sqrt{\alpha_\epsilon + b_\epsilon} E\left(\frac{\bar{\sigma}}{2} \middle| \frac{2b_\epsilon}{\alpha_\epsilon + b_\epsilon}\right)
\end{aligned}$$

Where

$$E(\phi|k) = \int_0^\phi \sqrt{1 - k \sin^2(\theta)} d\theta$$

is the incomplete elliptic integral of the second kind.

We can thus conclude that the quantity

$$\sqrt{\alpha_\epsilon(p, q) + b_\epsilon(p, q)} E\left(\frac{\bar{\sigma}(p, q)}{2} \middle| \frac{2b_\epsilon(p, q)}{\alpha_\epsilon(p, q) + b_\epsilon(p, q)}\right)$$

is almost conserved along the trajectories of the system (2.3) when Γ is not crossed. Figure 2.3 is a contour plot of the adiabatic invariant on the third sphere for a resonant case, along with the Γ curve for the system in study (and whose explicit expression will be given in the next section).

Remark. From the theory of adiabatic invariants it is possible to predict which will be the region where chaotic motions take place, since it is the union of the level sets of the adiabatic invariant that intersect Γ . Moreover it is immediate to check that the expression of both Γ and the level sets of the adiabatic invariant are independent, provided that the energy of the system is rescaled accordingly, from ϵ . This means that, no matter how small ϵ is, chaotic motions will fill the same region. This is in contrast with the behaviour of a regular orbit, which is contained in a strip of decreasing, as ϵ decreases, size around the curve followed by the orbit. $\triangle$

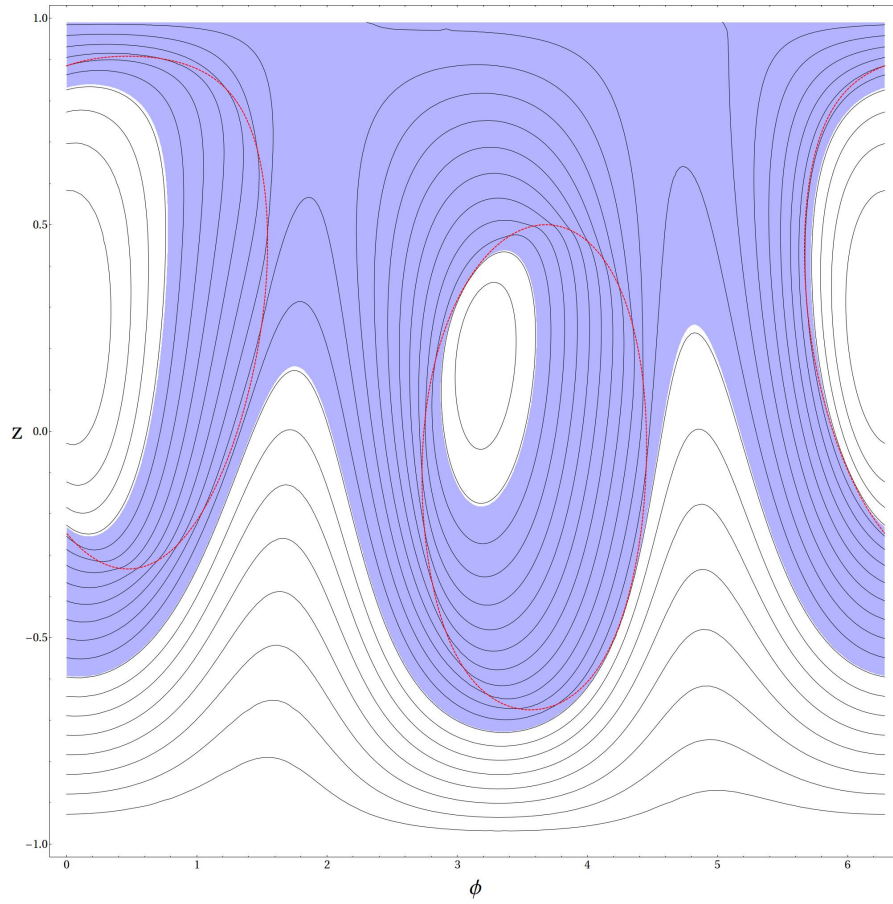


Figure 2.3: the adiabatic invariant (blue lines), the separatrix (red dashed line) and the expected chaotic zone (light blue domain) on the third sphere for $\nu = (0, 1)$ and for value of the fast action $F = -0.525$ and of the normal form $E = 2.5\epsilon$.

2.6 Dynamics in the fully resonant zone

In $\mathcal{R}_2$ we have that $u_\epsilon = f$. In this case the confinement of the actions is provided by the size of the zone itself, which is of order $\epsilon^{\frac{1}{4}}$, however even fixing the value of the (p, q) variables we do not end, in general, with an integrable system and so it is not possible to apply the theory of adiabatic invariants. In this way there is no hope, if not for a very special perturbation f , to control fully resonant motions on the third sphere.

2.7 Explicit expression of the perturbation

While for the qualitative behaviour of the dynamics of the perturbed system the precise expression of the perturbation is not needed, as long as its local representations are analytic, to predict the exact behaviour on the third sphere we need to fix f .

We have already seen how $\mathbb{S}^2$ can be viewed as a symplectic leaf, precisely the one defined by points of unit norm of $\mathbb{R}^3$ equipped by the Poisson structure given by the cross product. The standard global coordinates (x_1, x_2, x_3) on $\mathbb{R}^3$ define, by restriction on $\mathbb{S}^2$, smooth functions on $\mathbb{S}^2$ (still denoted by x_1, x_2, x_3), it is then possible to define a function on the sphere through these functions (actually a stronger results holds: if f is a continuous function on $\mathbb{S}^2$ then it is possible to express f as a function of x_1, x_2, x_3 even if this representation is not unique).

If a Hamiltonian h on $\mathbb{S}^2$ has the form $h(p) = \hat{h}(x_1(p), x_2(p), x_3(p))$, where $p \in \mathbb{S}^2$, it is possible to exploit the simple expression of the Poisson bracket to obtain:

$$\frac{dx_i(p(t))}{dt} = \{x_i(p), h(p)\} = (\nabla \hat{h}(x(p)) \times x(p))_i$$

in this way we recover a simple expression (useful in the numerical computations) for the dynamics of the system (since the values of $x_i(p(t))$ uniquely determine $p(t)$). Another advantage of this approach is that there is no need to work with local charts on the sphere.

The transition from $\mathbb{S}^2$ to $\mathcal{M} = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ seen as a symplectic leaf of $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, is straightforward. Now there are 9 functions on $\mathcal{M}$, denoted by $x_{i\alpha}$, satisfying

$$\{x_{i\alpha}, x_{j\beta}\} = \delta_{ij} \epsilon_{\alpha\beta\gamma} x_{i\gamma}.$$

using these function the unperturbed Hamiltonian of the system can be expressed as

$$h_0 = \frac{1}{2}(x_{13}^2 + x_{23}^2) \tag{2.5}$$

while the perturbation we have chosen takes the form:

$$f = 2x_2 \cdot x_3 + x_3 \cdot x_1 + x_1 \cdot x_2 + x_{31}^2 + x_{21}x_{32} + x_{21}x_{33} + x_{13}^3 \quad (2.6)$$

where $x_i \cdot x_j := \sum_{\alpha} x_{i\alpha} x_{j\alpha}$.

It is immediate to check that, if we use for example cylindrical coordinates on the spheres, the local representatives of f are analytic and so it is possible to apply Nekhoroshev theory. We also have that f has a finite number of Fourier components:

$$f = \hat{f}_{(0,0)} + \hat{f}_{(1,0)} + \hat{f}_{(0,1)} + \hat{f}_{(1,-1)}$$

with

$$\begin{aligned} \hat{f}_{(0,0)} &= 2x_{23}x_{33} + x_{33}x_{13} + x_{13}x_{23} + x_{31}^2 + x_{13}^3 \\ \hat{f}_{(1,0)} &= x_{31}x_{11} + x_{32}x_{12} \\ \hat{f}_{(0,1)} &= x_{21}(2x_{31} + x_{32} + x_{33}) + 2x_{22}x_{32} \\ \hat{f}_{(1,-1)} &= x_{11}x_{21} + x_{12}x_{22}. \end{aligned} \quad (2.7)$$

and so u_{ϵ} can be approximated, as already seen, by $\hat{f}_{(0,0)}$, while in the ν -resonant zone by $\hat{f}_{(0,0)} + \hat{f}_{\nu}$ (if ν is not in the list above f_{ν} is defined equal to zero).

This shows that the analysis described in the previous section should focus only for the resonances $(1, 0)$, $(0, 1)$ and $(1, -1)$, since the effects of any other resonance manifest themselves at higher order of ϵ and may be difficult to study. On the other hand it is possible to give an explicit expression for the functions a_{ϵ} and $b_{\epsilon}^{(i)}$ introduced above (here a_{ν} stands for a_{ϵ} in the ν -resonant case, and similarly for $b_{\nu}^{(i)}$):

$$\begin{aligned} a_{\nu} &= x_{31}^2 + \frac{2\nu_1 - \nu_2}{\|\nu\|^2} F x_{33} \\ b_{(1,0)}^{(1)} &= x_{31} \\ b_{(1,0)}^{(2)} &= x_{32} \\ b_{(0,1)}^{(1)} &= 2x_{31} + x_{32} + x_{33} \\ b_{(0,1)}^{(2)} &= 2x_{32} \\ b_{(1,-1)}^{(1)} &= 1 - \left(\frac{F}{2}\right)^2 \\ b_{(1,-1)}^{(2)} &= 0 \\ b_{\nu}^{(1)} &= b_{\nu}^{(2)} = 0 \quad \nu \neq (1, 0), (0, 1), (1, -1). \end{aligned}$$

Here F is equal, for $\nu = (1, 0)$, $(0, 1)$, $(1, -1)$, respectively to $x_{23}(0)$, $-x_{13}(0)$ and $x_{13}(0) + x_{23}(0)$, where $x(0)$ is the initial point of the considered orbit.

Remark. For points on the separatrices we have that, adopting the notation of the previous section, $\alpha_\epsilon = \beta_\epsilon$ and so the expression of the adiabatic invariant simplifies to

$$A = \frac{8\sqrt{\epsilon}}{\pi\|\nu\|} \sqrt{\beta_\epsilon}$$

In the case of the $(1, -1)$ resonance β_ϵ is constant and so the separatrix corresponds to a level curve of A . In this case no crossing of the separatrices, and thus chaotic motions, are possible. $\triangle$

2.8 Numerical results for the classical system

In this section numerical results for the system are collected.

The numerical integrations have been performed using the standard explicit Runge-Kutta method of order fourth (details on the method can be found in [25]).

Instead of working in local coordinates on $\mathcal{M}$ we consider, as explained when describing the system, a polynomial extension of h_ϵ on $\mathcal{P} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ and its Hamiltonian vector field defined by the Poisson structure on $\mathcal{P}$.

In this way the expression for the vector field and for the numerical approximation of the flow is simple (its component are polynomials) and efficiently implemented. It should be noted that $\mathcal{M}$ is not an invariant set for this map as it should be for a Poisson map from $\mathcal{P}$ to itself (as the sections, at fixed times, of a Hamiltonian flow are); the choice of an integration step small enough allows to control the drift of the orbits from $\mathcal{M}$.

On each orbit we have considered the functions h_ϵ , $C_\alpha = x_{\alpha 1}^2 + x_{\alpha 2}^2 + x_{\alpha 3}^2$ (for $\alpha = 1, 2, 3$) and $I_\alpha = x_{\alpha 3}$ (for $\alpha = 1, 2$). We have that the order of the variations over time of h_ϵ and C_α , which in theory should be constants of motion, is compatible with the numerical error. The variation of the actions on the other hand is compatible with the one predicted by Nekhoroshev theory (rescaling with ϵ if the motion is non resonant or if a fast combination of the actions is considered, with $\sqrt{\epsilon}$ when slow actions are considered).

Now the behaviour on the third sphere is studied. Figure 2.4 shows two non resonant orbits, for a value of ϵ equal to 10^{-3} , projected on the third sphere along with the level curves of the non resonant average. As predicted by the theory the orbits closely follow the level curves and no visible chaos is present.

Figures 2.5, 2.6 and 2.7 shows resonant orbits projected on the third sphere along with the level curves of the adiabatic invariant and the separatrix Γ . It is clear that orbits which start on a level curve of the adiabatic invariant that does not cross Γ stay close to the same level set for the whole considered timescale; on the other hand orbits which eventually cross Γ tend to fill the domain defined by the level sets of the adiabatic invariant crossing Γ .

Figures 2.8, 2.9, 2.10 Shows the same orbits depicted before with the perturbation reduced by a factor of 10 and the timescale accordingly increased by a factor 10. It possible to see how the (resonant and non resonant) orbits are more concentrated around the level curve of the corresponding conserved quantity (the non resonant normal form in the non resonant case and the adiabatic invariant in the resonant case). In the chaotic case however, as predicted by the theory, the orbit still fills the chaotic region.

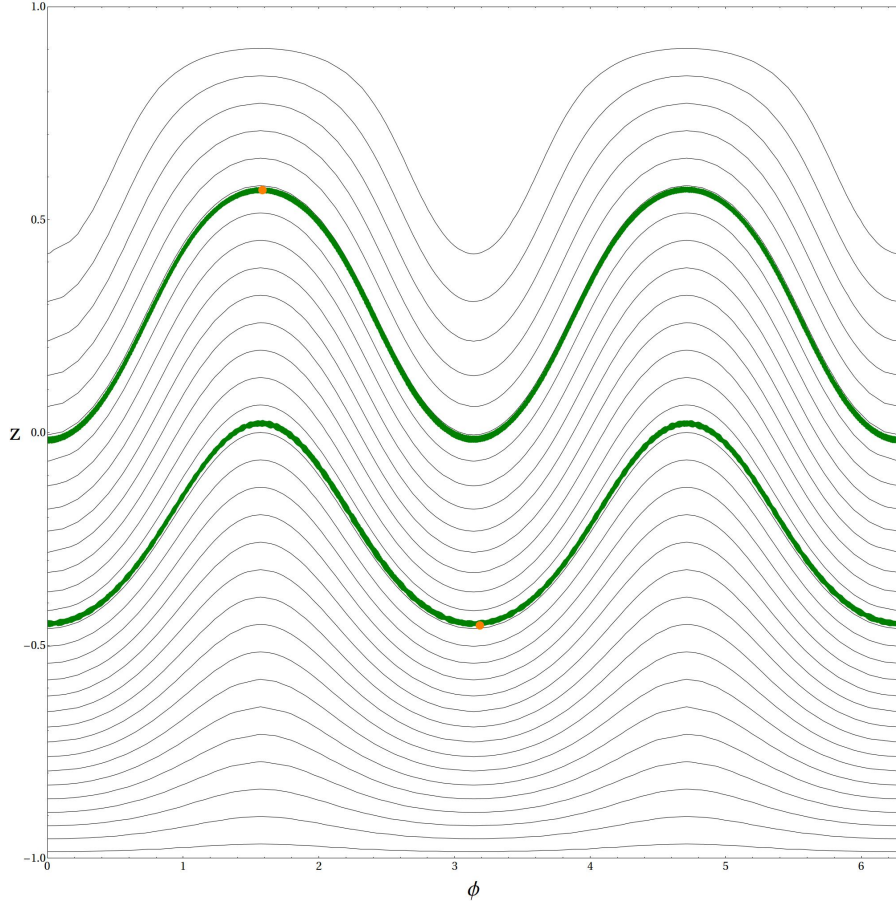


Figure 2.4: non resonant motions for values of actions $I = (\sqrt{0.5}, 0.5)$ and $\epsilon = 10^{-3}$. The projection of the orbits on the third sphere in cylindrical coordinates are represented here for times up to $T = 10^4 \epsilon^{-1}$. The orange dots represents the projection of the initial condition of each orbit.

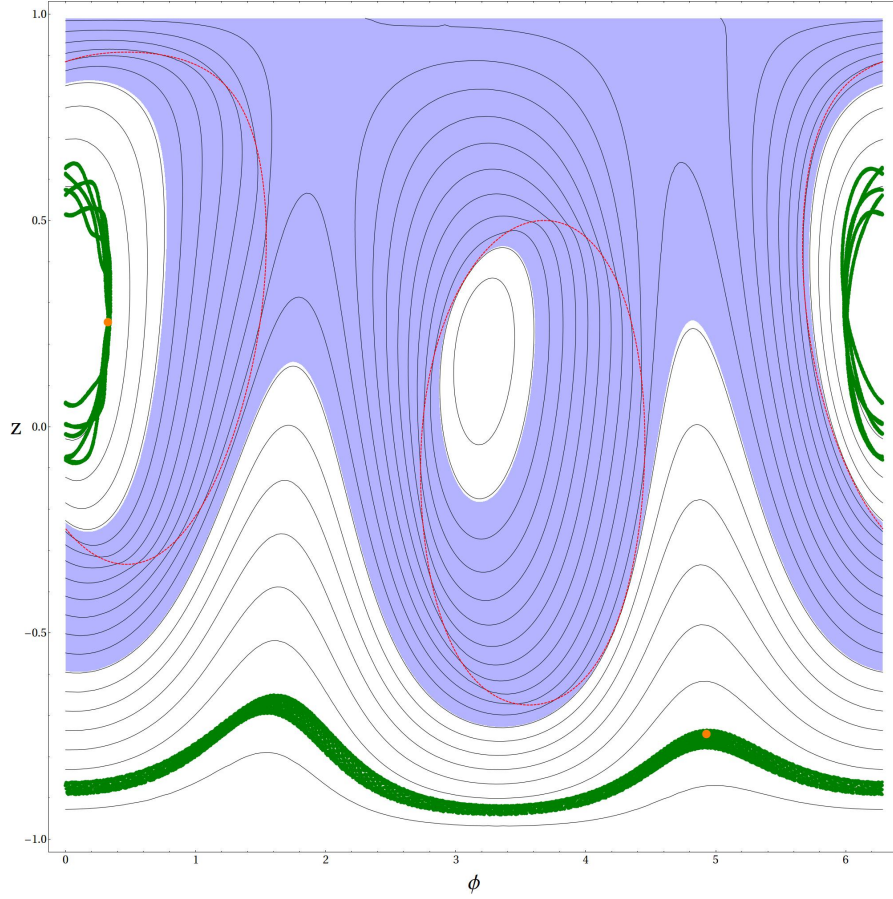


Figure 2.5: resonant regular motions in the $(0, 1)$ resonance for values of the parameters: $\epsilon = 10^{-3}$, $F = -0.525$ and $E = 2.5$. The projection of the orbits on the third sphere in cylindrical coordinates are represented here for times up to $T = 10^4 \epsilon^{-1}$. The orange dots represents the projection of the initial condition of each orbit.

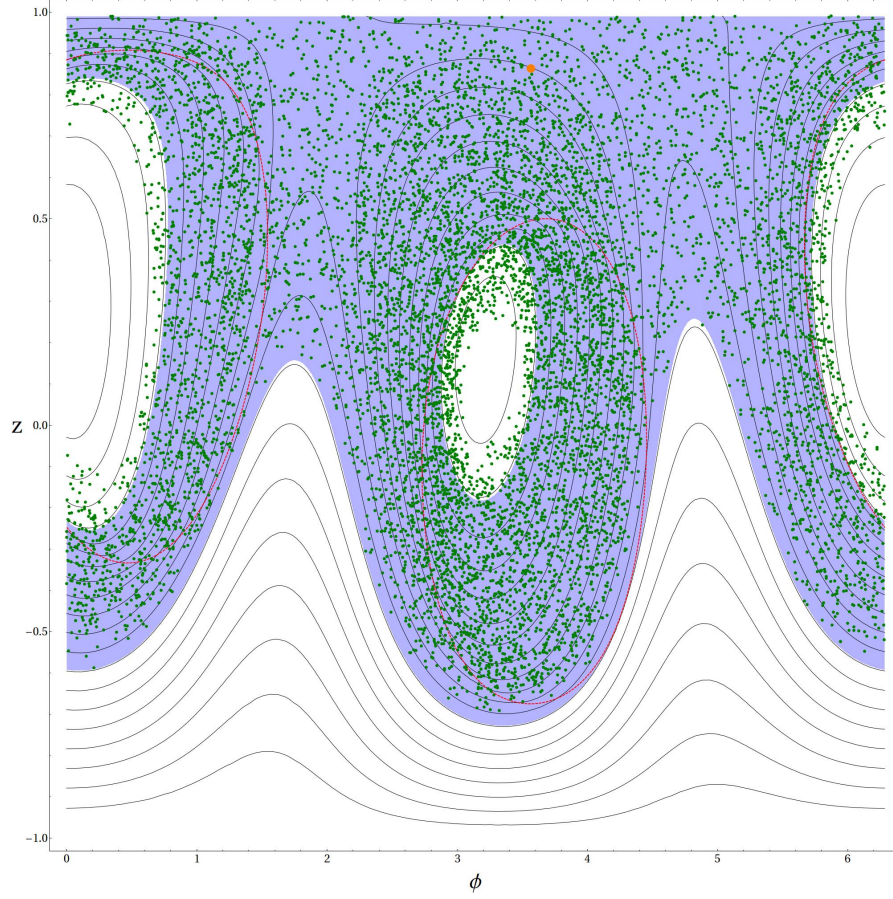


Figure 2.6: a single resonant chaotic motion in the $(0, 1)$ resonance for values of the parameters: $\epsilon = 10^{-3}$, $F = -0.525$ and $E = 2.5$. The projection of the orbit on the third sphere in cylindrical coordinates is represented here for times up to $T = 10^4 \epsilon^{-1}$. The orange dot represent the projection of the initial condition.

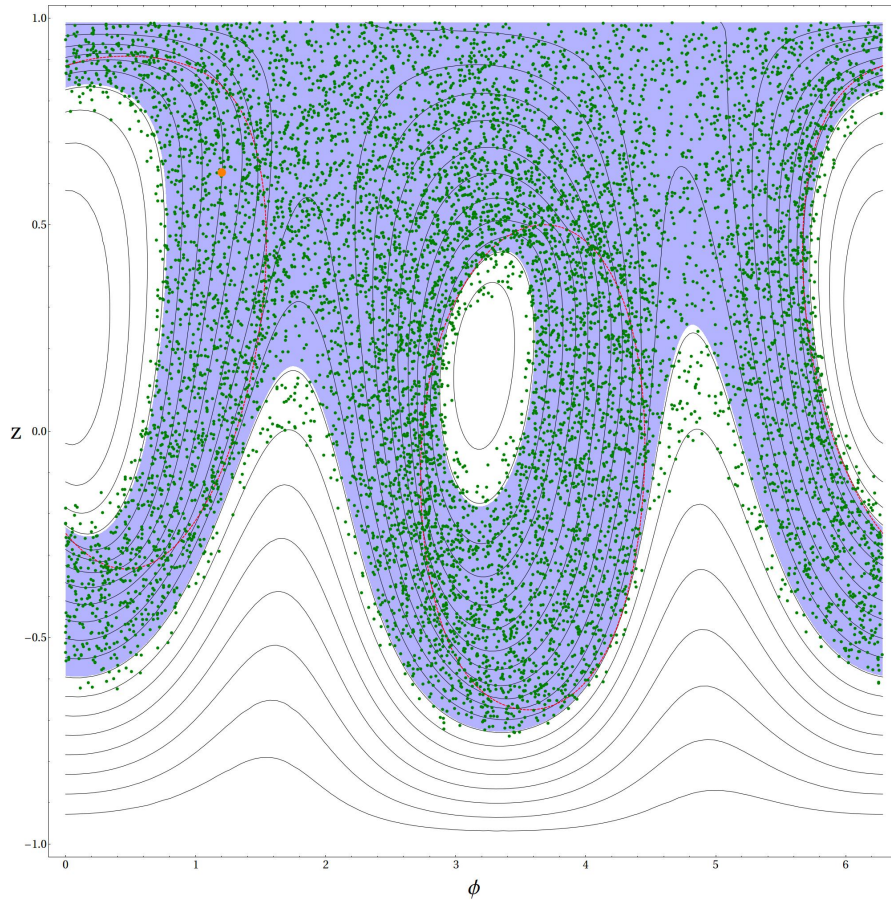


Figure 2.7: a single resonant chaotic motion in the $(0,1)$ resonance. The parameters are the same of figure (2.6), with different initial condition.

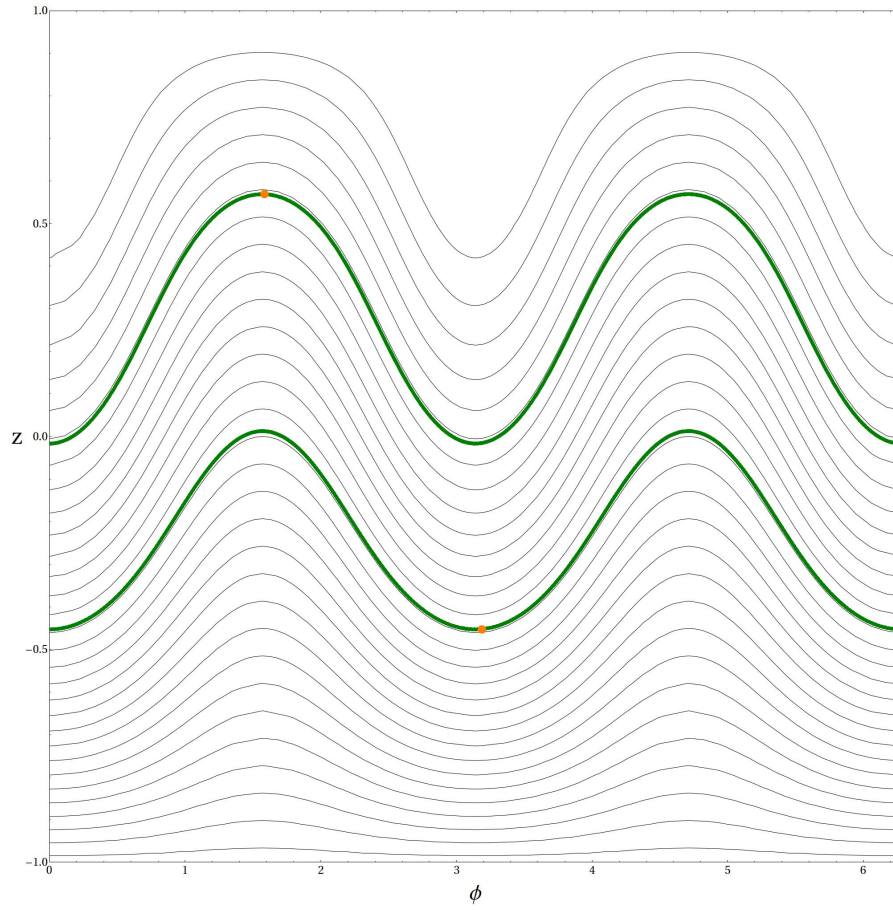


Figure 2.8: the same orbits represented in (2.4) with $\epsilon = 10^{-4}$ and $T = 10^4 \epsilon^{-1}$.

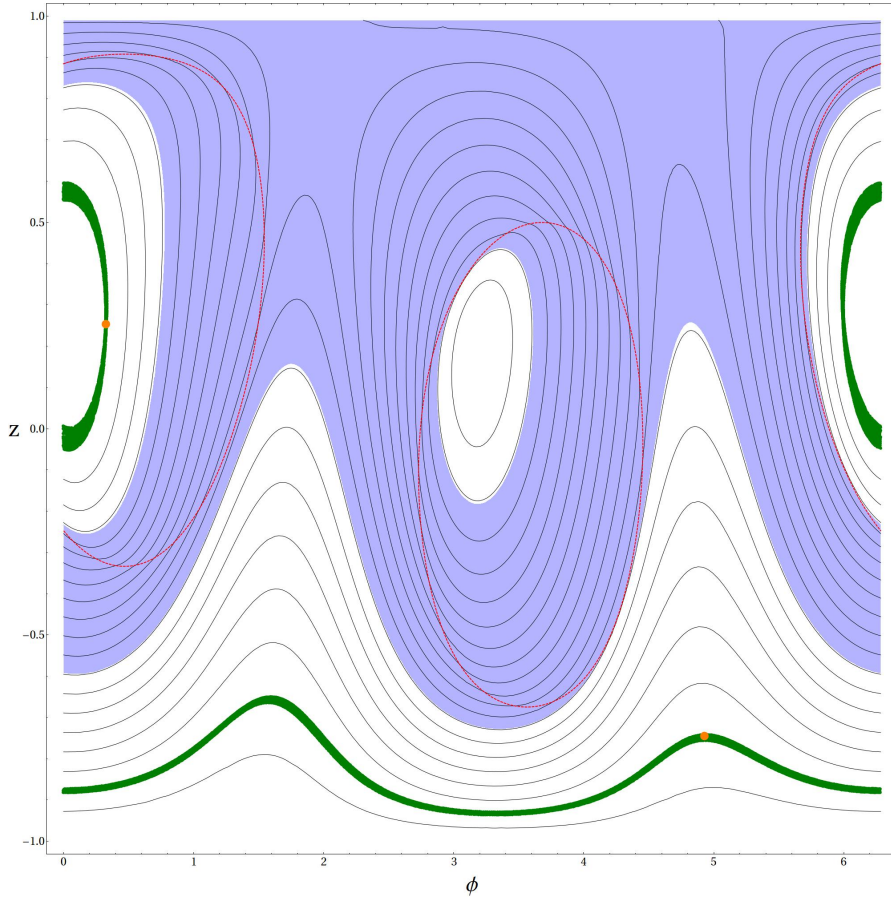


Figure 2.9: the same orbits represented in (2.5) with $\epsilon = 10^{-4}$ and $T = 10^4 \epsilon^{-1}$.

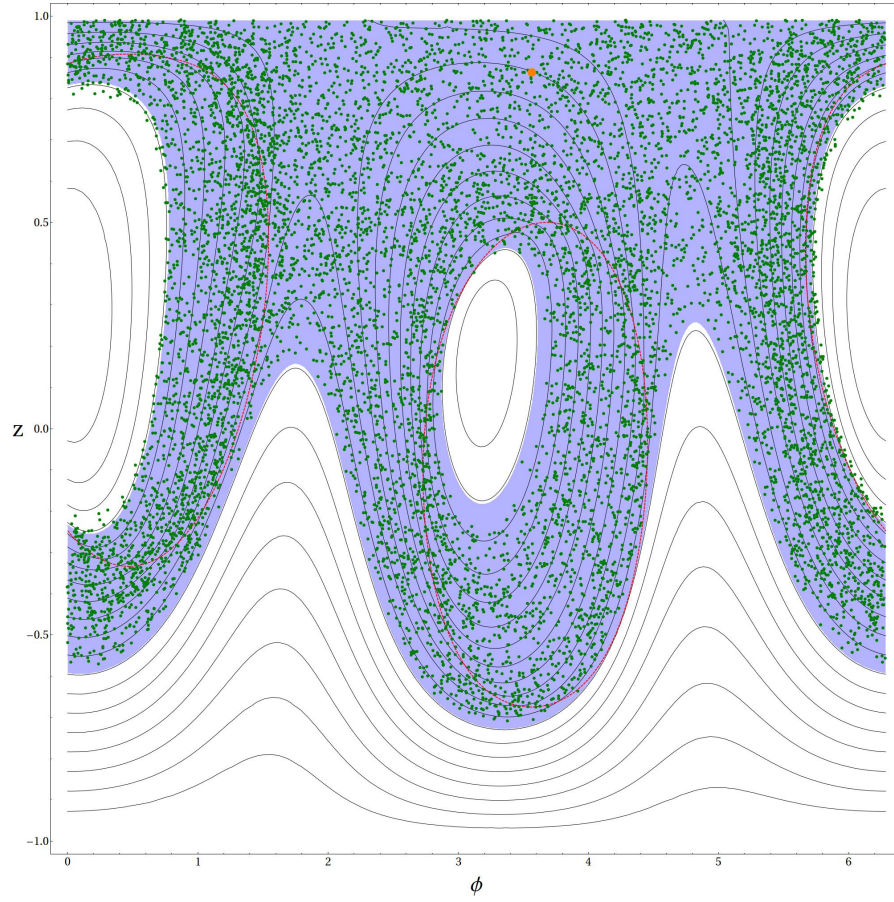


Figure 2.10: the same orbit represented in (2.6) with $\epsilon = 10^{-4}$ and $T = 10^4 \epsilon^{-1}$.

Chapter 3

Quantum mechanics and quantization

In this chapter we turn our attention toward quantum mechanics.

After a brief introduction about the description of a system in the quantum formalism we review the problem of finding a quantum counterpart of a classical system. In particular the method of geometric quantization and the closely related BKS quantization are described. We conclude our analysis introducing the concepts of coherent states, Husimi distribution of a state and covariant/contravariant symbol of an observable relating them to the quantization method and showing how they can be used to perform a quasi “classical” (i.e. on the phase space) analysis of the quantized system.

3.1 Principles of quantum mechanics

In this section we recall the basic concepts, formalism and results from quantum mechanics. Further references can be found for example in [15, 38]. The experienced reader can safely skip this part.

A quantum system can be described by a complex Hilbert space $\mathcal{H}$ and a self-adjoint operator $\mathbf{h}$ acting on $\mathcal{H}$, the *Hamiltonian operator*.

$\mathbf{h}$ defines, by exponentiation, a unitary operator $\mathbf{u}_t$, the *evolution operator*, depending on the real parameter t and satisfying ($\mathbb{I}_{\mathcal{H}}$ represents the identity operator on $\mathcal{H}$):

$$\begin{aligned}\mathbf{u}_0 &= \mathbb{I}_{\mathcal{H}} \\ \frac{d}{dt}\mathbf{u}_t &= \frac{1}{i\hbar}\mathbf{h}\mathbf{u}_t\end{aligned}$$

where $\hbar > 0$ is the (*reduced*) *Planck constant*.

1-dimensional subspaces of $\mathcal{H}$ define the physical states of the system. If $|\psi\rangle$ is an element of $\mathcal{H} \setminus \{0\}$ (this is the well known Dirac “bra-ket” notation) then, for $t \in \mathbb{R}$, the element $|\psi_t\rangle$ of $\mathcal{H}$ defined by

$$|\psi_t\rangle = \mathbf{u}_t |\psi\rangle$$

spans the physical state representing the time evolution at the time t of the state spanned by $|\psi\rangle$ and satisfies *Schrödinger equation*:

$$\begin{aligned} |\psi_0\rangle &= |\psi\rangle \\ \frac{d}{dt} |\psi_t\rangle &= \frac{1}{i\hbar} \mathbf{h} |\psi_t\rangle . \end{aligned}$$

If $|\psi\rangle$ is an eigenvector (called in this context *eigenstate*) of $\mathbf{h}$, with eigenvalue E , then $|\psi_t\rangle = e^{\frac{E t}{i\hbar}} |\psi\rangle$, so $|\psi_t\rangle$ represents, for every t , the same state as $|\psi\rangle$, henceforth the name *stationary state* for $|\psi\rangle$. E is interpreted as the energy of the state represented by $|\psi\rangle$.

Since $\mathbf{h}$ is self-adjoint E must be real and, if $|\psi\rangle$ and $|\psi'\rangle$ are eigenstates relative to different eigenvalues then they must be orthogonal.

Even if it is not the most general case for a quantum system, for us there will exist a countable (possibly finite) orthonormal basis for $\mathcal{H}$ made of stationary states $(|E_i\rangle)_{i \in \mathcal{I}}$ (with $\mathcal{I} \subseteq \mathbb{N}$) such that

$$\mathbf{h} |E_i\rangle = E_i |E_i\rangle \quad \forall i \in \mathcal{I}.$$

The set of energies $\{E_i\}_{i \in \mathcal{I}}$ is the *spectrum* of $\mathbf{h}$.

Every state $|\psi\rangle$ can be expressed, in a unique way, as an infinite linear combination of the basis elements:

$$|\psi\rangle = \sum_{i \in \mathcal{I}} \psi(E_i) |E_i\rangle .$$

Using the orthonormality of the basis elements it results that

$$\psi(E_i) = \langle E_i | \psi \rangle$$

where $\langle E_i |$ is the element of $\mathcal{H}^*$ dual of $|E_i\rangle$ under the natural pairing given by the scalar product on $\mathcal{H}$ (and with $\langle E_i | \psi \rangle$ we denote the value of $\langle E_i |$ in $|\psi\rangle$ or, equivalently, the scalar Hermitian product between $|E_i\rangle$ and $|\psi\rangle$).

By linearity of the evolution operator we have that the expression for the evolution of $|\psi\rangle$ is given by

$$|\psi_t\rangle = \sum_{i \in \mathcal{I}} e^{\frac{E_i t}{i\hbar}} \psi(E_i) |E_i\rangle$$

In particular if $\mathcal{H}$ is finite dimensional then the problem of studying the dynamics in the quantum framework is reduced to the problem of diagonalizing $\mathbf{h}$ (which can be identified, once an orthonormal basis is fixed, by an Hermitian matrix).

Quantum observables of the system (whose classical counterparts are functions on $\mathcal{M}$) are represented by self-adjoint operators on $\mathcal{H}$. The Hamiltonian operator itself is an observable. In the following we will deal, for simplicity, only with time-independent observables. If $\mathbf{a}$ is an observable then it is possible to define the evolution of $\mathbf{a}$ as the family of observables $\mathbf{a}_t$, $t \in \mathbb{R}$, satisfying:

$$\langle \psi' | \mathbf{a}_t | \psi \rangle = \langle \psi'_t | \mathbf{a} | \psi_t \rangle \quad \forall |\psi\rangle, |\psi'\rangle \in \mathcal{H}$$

which leads to:

$$\mathbf{a}_t = \mathbf{u}_t^{-1} \mathbf{a} \mathbf{u}_t.$$

Similarly to the observable $\mathbf{h}$, it is possible to define the spectrum of $\mathbf{a}$ as the family $(a_i)_{i \in \mathcal{I}}$ of its eigenvalues and it is possible to find an orthonormal basis of $\mathcal{H}$ made of eigenstates of $\mathbf{a}$, $(|a_i\rangle)_{i \in \mathcal{I}}$. If a state $|\psi\rangle$ is expressed in this base as:

$$|\psi\rangle = \sum_{i \in \mathcal{I}} \psi(a_i) |a_i\rangle$$

and $|\psi\rangle$ is supposed normalized, then the map $a_i \mapsto |\psi(a_i)|^2$ defines a probability distribution on the spectrum of $\mathbf{a}$.

The spectrum of an observable is interpreted as the possible outcomes of a measurement of that observable and the distribution defined above is the probability distribution of the outcomes of a measurement of the observable when the system is in the state $|\psi\rangle$.

3.2 Quantization of a classical system

The classical description of a physical system may be seen as a macroscopic approximation, hence it is reasonable that classical mechanics should arise as a limit, for $\hbar$ going to 0, of quantum mechanics. The inverse procedure of finding a quantum system that is related to a given classical system goes under the name of quantization.

In the Hamiltonian formalism and in the quantum one there are some striking similarities. In particular in the former the space of functions on the underlying symplectic manifold is a Lie algebra with respect to the pairing given by Poisson bracket, while in the latter the set of observables has a Lie algebra structure when equipped by the commutator rescaled by $(i\hbar)^{-1}$. In both cases these structures play an essential role in the dynamical evolution of the system: we have already

seen that if f is a function on $\mathcal{M}$ and the dynamics is defined by the Hamiltonian h , then if we define the family of function on $\mathcal{M}$ depending smoothly on time:

$$f_t := f \circ \Phi_h^t$$

we have that:

$$\frac{df_t}{dt} = \{f_t, h\}.$$

on the other hand, if $\mathbf{a}$ is an observable on a quantum system with Hamiltonian $\mathbf{h}$, then:

$$\frac{d\mathbf{a}_t}{dt} = \frac{1}{i\hbar}[\mathbf{a}_t, \mathbf{h}] := \frac{1}{i\hbar}(\mathbf{a}_t\mathbf{h} - \mathbf{h}\mathbf{a}_t)$$

this formula descends trivially from the definitions of $\mathbf{a}_t$ and $\mathbf{u}_t$ (and goes under the name of *Heisenberg equation*).

When discussing a quantization procedure (i.e. the construction of a quantum system which models a given classical system) it is natural to require that these Lie algebra structures should be preserved as much as possible.

If $(\mathcal{M}, \omega)$ is a symplectic manifold and $\mathcal{H}$ is a Hilbert space then a map $\mathcal{Q}$ that associates to every $\mathcal{C}^\infty$ function f on $\mathcal{M}$ a self adjoint operator $\mathcal{Q}(f)$ on $\mathcal{H}$ is a *quantization* if it satisfies the *quantization postulates*:

Q1 the map $\mathcal{Q}$ is linear;

Q2 if $1_{\mathcal{M}}$ is the constant function equal to 1, and $\mathbb{I}_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$, then $\mathcal{Q}(1_{\mathcal{M}}) = \mathbb{I}_{\mathcal{H}}$;

Q3 for every f, g , $\mathcal{Q}(\{f, g\}) = (i\hbar)^{-1}[\mathcal{Q}(f), \mathcal{Q}(g)]$;

Q4 $\mathcal{Q}(f)$ is self-adjoint.

To obtain a proper quantization one needs some irreducibility condition on $\mathcal{Q}$. We say that a family $(\mathbf{a}_i)_{i \in \mathcal{I}}$ of observables defines a complete set of compatible observables if:

- $[\mathbf{a}_i, \mathbf{a}_j] = 0$ for every $i, j \in \mathcal{I}$;
- if $|\psi\rangle, |\psi'\rangle \in \mathcal{H}$ such that $\mathbf{a}_i|\psi\rangle = a_i|\psi\rangle$ and $\mathbf{a}_i|\psi'\rangle = a_i|\psi'\rangle$ then $|\psi\rangle$ and $|\psi'\rangle$ are linearly dependent.

The first condition ensures that all the $\mathbf{a}_i$ can be diagonalized simultaneously, while the second means that if $|a\rangle$, with $a = (a_i)_{i \in \mathcal{I}}$, and $\mathbf{a}_i|a\rangle = a_i|a\rangle$, then, for a generic state $|\psi\rangle$, the map $a \mapsto \langle a | \psi \rangle$ uniquely identifies $|\psi\rangle$.

The last quantization condition can, roughly speaking, be expressed as

Q5 there exists a maximal set of classical observables in involution on $\mathcal{M}$, $f_1, \dots, f_n$ (here $2n$ is the dimension of $\mathcal{M}$), with $f = (f_1, \dots, f_n)$ of maximal rank everywhere, such that $\mathcal{Q}(f_1), \dots, \mathcal{Q}(f_n)$ defines a complete set of compatible quantum observables.

When discussing the detailed procedure a more appropriate (i.e. non-global) condition will be imposed.

Groenewold-Van Hove theorem [23, 42] shows that $Q1, \dots, Q5$ are incompatible, so either the procedure of quantization should be restricted only to a subset of the classical observables or the postulates should be relaxed (requiring in particular that $Q3$ must hold only in the limit $\hbar \rightarrow 0$).

In the following sections the procedure of quantization which goes under the name of *geometrical quantization* and the closely related *BKS quantization* are presented. In geometric quantization the map $\mathcal{Q}$ satisfies all quantization postulates, but it is defined only for some functions on $\mathcal{M}$, on the other hand BKS quantization extends $\mathcal{Q}$ to $\mathcal{C}^\infty(\mathcal{M})$ but $Q3$ no longer holds.

While the general theory will be only outlined (see [24, 39, 43] for a detailed description) explicit calculations for the case of $\mathbb{S}^2$ will be made. The case of $\mathcal{M} = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ will follow in a straightforward way.

3.3 Line bundles and prequantization

Geometric quantization is a two-step procedure and prequantization is the first one. From a symplectic manifold $(\mathcal{M}, \omega)$ a Hilbert space $\mathcal{H}$ is defined and a map $\mathcal{Q}$, that to every smooth function on $\mathcal{M}$ associates a linear operator on $\mathcal{H}$ such that $Q1, \dots, Q4$ hold, is constructed.

To define $\mathcal{H}$ the concepts of line bundle and of compatible covariant derivative are needed.

A line bundle over $\mathcal{M}$ is a vector fiber bundle over $\mathcal{M}$ whose fibers are 1-dimensional complex vector space equipped with a Hermitian scalar product. If B is a line bundle over $\mathcal{M}$ a compatible covariant derivative ∇ on B is a map that to a section ψ of B and a vector field X tangent to $\mathcal{M}$ associates a section $\nabla_X \psi$ of B such that:

D1 $\nabla_X \psi$ is linear in both X and ψ ;

D2 if f is any function on $\mathcal{M}$, $\nabla_{fX} \psi = f \nabla_X \psi$;

D3 if f is any function on $\mathcal{M}$, $\nabla_X (f\psi) = (\mathcal{L}_X f)\psi + f \nabla_X \psi$;

D4 if ψ and ψ' are sections of B and (ψ, ψ') denotes the function on $\mathcal{M}$ obtained by taking the scalar product of ψ and ψ' point wise on the fibers of B :

$$\mathcal{L}_X(\psi, \psi') = (\nabla_X \psi, \psi') + (\psi, \nabla_X \psi').$$

(It is understood that if X and ψ are defined only locally on $\mathcal{M}$ then $\nabla_X \psi$ is defined on the common domain of definition). To a covariant derivative ∇ on B it is associated its curvature Ω , the unique 2-form on $\mathcal{M}$ such that

$$\nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi - \nabla_{[X, Y]} \psi = \Omega(X, Y) \psi$$

for every section ψ of B and X, Y vector fields tangent to $\mathcal{M}$.

If B is a line bundle over $\mathcal{M}$ and ψ, ψ' are two smooth global sections of B then (ψ, ψ') defines a smooth function on $\mathcal{M}$. If $\mathcal{M}$ is equipped with a measure μ then it is possible to define the scalar product between sections as:

$$\langle \psi, \psi' \rangle = \int_{\mathcal{M}} (\psi, \psi') d\mu$$

this allows the definition of $L^2(\mathcal{M}, B, \mu)$ as the Hilbert space of square integrable global sections (with respect to the measure μ) of $\mathcal{M}$. In particular, if $\mathcal{M}$ is a $2n$ -dimensional symplectic manifold with symplectic form ω , then $\mathcal{M}$ is equipped with a natural measure given by ω^n and it is natural to define $\mathcal{H} = L^2(\mathcal{M}, B, \omega^n)$. We have the following:

Definition. A prequantization of a $2n$ -dimensional symplectic manifold $(\mathcal{M}, \omega)$ is the choice of a line bundle B over $\mathcal{M}$ equipped with a compatible covariant derivative ∇ with curvature $(i\hbar)^{-1}\omega$. The Hilbert space $\mathcal{H}$ of square integrable, under the measure given by ω^n , global sections of B is the prequantum Hilbert space.

If f is a smooth function on $\mathcal{M}$ then an operator $\mathcal{Q}(f)$ on $\mathcal{H}$ can be defined by

$$\mathcal{Q}(f)\psi = -i\hbar \nabla_{X_f} \psi + f\psi$$

for ψ smooth section of B . Using the definition of curvature it is immediate to check that

Proposition. The map $\mathcal{Q}$ satisfies postulates $Q1, \dots, Q4$.

Proof. Q1 and Q2 are straightforward. To check Q4, let ψ' be another element of $\mathcal{H}$, we have that

$$\begin{aligned}
\langle \psi', \mathcal{Q}(f)\psi \rangle &= \int_{\mathcal{M}} (\psi', -i\hbar \nabla_{X_f} \psi + f\psi) d\omega^n \\
&= \int_{\mathcal{M}} -i\hbar (\psi', \nabla_{X_f} \psi) + (\psi', f\psi) d\omega^n \\
&= \int_{\mathcal{M}} -i\hbar (d(\psi', \psi) - (\nabla_{X_f} \psi', \psi)) + (f\psi', \psi) d\omega^n \\
&= \int_{\mathcal{M}} -i\hbar d(\psi', \psi) + (-i\hbar \nabla_{X_f} \psi' + f\psi', \psi) d\omega^n \\
&= -i\hbar \int_{\mathcal{M}} d((\psi', \psi) \omega^n) + \langle \mathcal{Q}(f)\psi', \psi \rangle \\
&= \langle \mathcal{Q}(f)\psi', \psi \rangle
\end{aligned}$$

where we used the fact that f is real and, by Stokes theorem, the integral over $\mathcal{M}$ of $d((\psi', \psi) \omega^n)$ vanishes. The only property that needs a little effort to be verified is Q3. Let f and g be two function on $\mathcal{M}$, using the definition of $\mathcal{Q}$:

$$\begin{aligned}
\mathcal{Q}(f)\mathcal{Q}(g)\psi &= -\hbar^2 \nabla_{X_f} \nabla_{X_g} \psi - i\hbar \nabla_{X_f} (g\psi) - i\hbar f \nabla_{X_g} \psi + fg\psi \\
&= -\hbar^2 \nabla_{X_f} \nabla_{X_g} \psi - i\hbar g \nabla_{X_f} (\psi) + i\hbar \{f, g\} \psi - i\hbar f \nabla_{X_g} \psi + fg\psi
\end{aligned}$$

from which:

$$\begin{aligned}
[\mathcal{Q}(f), \mathcal{Q}(g)]\psi &= -\hbar^2 (\nabla_{X_f} \nabla_{X_g} \psi - \nabla_{X_g} \nabla_{X_f} \psi) + 2i\hbar \{f, g\} \psi \\
&= -\hbar^2 (\nabla_{[X_f, X_g]} \psi + (i\hbar)^{-1} \omega(X_f, X_g) \psi) + 2i\hbar \{f, g\} \psi \\
&= i\hbar (-i\hbar \nabla_{X_{\{f, g\}}} \psi + \{f, g\} \psi) \\
&= i\hbar \mathcal{Q}(\{f, g\}).
\end{aligned}$$

□

The problem of the existence of a prequantization, given a symplectic manifold, and of its uniqueness (or lack of it) is addressed by the following theorems:

Theorem 3 ([43]). *There exists at least a line bundle B over $\mathcal{M}$ equipped with a compatible covariant derivative ∇ with curvature $\hbar^{-1}\omega$ iff*

$$(2\pi\hbar)^{-1} \int_S \omega \in \mathbb{Z}$$

for every oriented surface without boundary $S \subseteq \mathcal{M}$ (Weil integrality condition).

Theorem 4 ([43]). *If $\mathcal{M}$ is simply connected then there cannot be more than one choice for B and ∇ .*

The application of these results to $\mathcal{M} = \mathbb{S}^2$ is standard. We regard, as usual, $\mathbb{S}^2$ as a sphere of radius R in $\mathbb{R}^3$ equipped with the Poisson structure described by (1.2). As already noted $\mathcal{M}$ is a symplectic leaf.

Proposition 4 ([43]). *There exists a prequantization of $\mathcal{M} = \mathbb{S}^2$ iff $R = \frac{\hbar n}{2}$ for some $n \in \mathbb{N}$ and, when this is the case, it is unique.*

Proof. Since $\mathcal{M}$ is simply connected if there exists a prequantization, by the corollary above, it is unique. The only oriented surface without boundary contained in $\mathcal{M}$ is $\mathcal{M}$ itself and so Weil integrality condition reads:

$$(2\pi\hbar)^{-1} \int_{\mathcal{M}} \omega \in \mathbb{Z},$$

the left hand side of the expression is equal to $2\hbar^{-1}R$ and the result follows. $\square$

It is sometime possible to identify the prequantum Hilbert space with a space of functions over $\mathcal{M}$, thus giving an explicit expression for $\mathcal{Q}$ as a linear differential operator.

A trivializing section of B is a smooth section s of B which is non-zero everywhere. If s is a trivializing section, for every $x \in \mathcal{M}$ $s(x)$ spans the fiber of B over x , in this way any smooth section ψ of B can be uniquely written as λs for some smooth function λ on $\mathcal{M}$. In particular if X is a tangent vector field on $\mathcal{M}$, then $\nabla_X s$ is a section of B . In this way there exists a function λ_X such that $\nabla_X s = \lambda_X s$. Since the dependence from X in $\nabla_X s$ is linear we have that there must also exist a 1-form θ such that $\lambda_X = (\imath\hbar)^{-1} \iota_X \theta$. Using the definition of curvature we can see that $\omega = d\theta$. The 1-form $(\imath\hbar)^{-1} \iota_X \theta$ is called a connection 1-form for B (see [43]).

We have that

Proposition. *If s is a trivializing section of B then the map from the space $\mathcal{H}^s = L^2(\mathcal{M}, (s, s)\omega^n)$ of the complex-valued square integrable functions on $\mathcal{M}$ with respect to the measure $(s, s)\omega^n$, to $\mathcal{H}$, the prequantum Hilbert space, defined by*

$$\tau : \alpha \mapsto \alpha s$$

is an isomorphism of Hilbert spaces.

If $(\imath\hbar)^{-1}\theta$ is the connection 1-form associated to s then the operator

$$\tilde{\mathcal{Q}}(f) = \tau^{-1} \circ \mathcal{Q}(f) \circ \tau$$

on $\mathcal{H}^s$ can be written explicitly as:

$$\tilde{\mathcal{Q}}(f)\alpha = -\imath\hbar\{\alpha, f\} + (f - \iota_{X_f}\theta)\alpha$$

Proof. the map τ is obviously linear. If α and β belong to $\mathcal{H}^s$ then

$$\langle \tau(\alpha), \tau(\beta) \rangle = \int_{\mathcal{M}} (\alpha s, \beta s) \omega^n = \int_{\mathcal{M}} \bar{\alpha} \beta (s, s) \omega^n$$

this shows that τ preserves the scalar product and so it is an isometric embedding of $\mathcal{H}^s$ in $\mathcal{H}$. If ψ is an element of $\mathcal{H}$, it is possible to define the function on $\mathcal{M}$:

$$\alpha = \frac{(s, \psi)}{\sqrt{(s, s)}}$$

and it results that $\tau\alpha = \psi$ on $\mathcal{M}$, hence proving that τ is an isometry.

The statement on $\tilde{\mathcal{Q}}(f)$ follows trivially from the property of the covariant derivative:

$$\nabla_{X_f}(\alpha s) = (\mathcal{L}_{X_f}\alpha)s + \nabla_{X_f}s = \{\alpha, f\}s + (\imath\hbar)^{-1}\iota_{X_f}\theta s$$

□

The condition that B possesses a trivializing section is strong. For example in the $\mathbb{S}^2$ case analyzed above such a trivializing section does not exist. This is easily verified observing that ω is not, in this case, exact and so there is no 1-form θ on $\mathcal{M}$ such that $d\theta = \omega$. In this case it is however possible to find a section s which is non-zero *almost everywhere*. In this case the map τ is still an isometry, since it is compatible with the scalar products on $\mathcal{H}$ and $\mathcal{H}^s$. If ψ is moreover a square integrable section of B , then there exists an α , square integrable function defined almost everywhere on $\mathcal{M}$, such that $\tau(\alpha) = \psi$ *almost everywhere*. In this way $\tau(\alpha)$ and ψ identify the same element in $\mathcal{H}$.

From now on we will restrict our analysis to the case where a trivializing section s (defined everywhere or almost everywhere) exists (this will be for instance the case of $\mathbb{S}^2$) and is fixed. $\mathcal{H}^s$ will be identified with $\mathcal{H}$ (and $\tilde{\mathcal{Q}}$ with $\mathcal{Q}$). It should be noted that this is perfectly valid when s is non-zero everywhere (in this case B is the trivial bundle $\mathcal{M} \times \mathbb{C}$, a section of B is a complex valued function f and

$$\nabla_X f = \mathcal{L}_X f - \imath\hbar \iota_X \theta f$$

is a good covariant derivative with the desired properties). When s is non-zero only almost everywhere, on the other hand, this represents a slight abuse of notation.

3.4 Geometric quantization

As explained in [43] the space $\mathcal{H}$ is, in some sense, “too big” and, as anticipated in the introduction, we need to modify the construction of $\mathcal{Q}$ (and of $\mathcal{H}$) to obtain

a map that satisfies also the postulate Q5. This is done by considering a suitable proper subspace $\mathcal{H}_q$ of $\mathcal{H}$ as the space of quantum states and redefining $\mathcal{Q}(f)$ restricting it on $\mathcal{H}_q$. As mentioned in the introduction the postulates Q1, ..., Q5 are incompatible, this is reflected by the fact that there is no guarantee that, for a generic function f , the image of $\mathcal{Q}(f)$ lies in $\mathcal{H}_q$. It is thus necessary to restrict the space of functions for which $\mathcal{Q}$ is defined.

To define $\mathcal{H}_q$ we introduce the concept of *polarization* (see [43]). A (real) polarization of $\mathcal{M}$ is simply a Lagrangian foliation of $\mathcal{M}$ (see the section on Poisson geometry for the definition of Lagrangian foliation). Fixed a polarization P of $\mathcal{M}$, a *P-polarized section* of B is a section ψ of B such that $\nabla_X \psi = 0$ for every X tangent to P . $\mathcal{H}_q$ is defined as the subspace of $\mathcal{H}$ of P -polarized square integrable sections. For now we will not address the issue that, for a general f , $\mathcal{Q}(f)(\mathcal{H}_q) \not\subseteq \mathcal{H}_q$.

If f is constant along the leaves of the polarization P then X_f is tangent to P and so:

$$\mathcal{Q}(f)\psi = f\psi.$$

If s is a polarized trivializing section and θ its associated 1-form it is easy to check that $\nabla_X s = \iota_X \theta = 0$ for every X tangent to the polarization P . It follows immediately that, for X tangent to P ,

$$\nabla_X(\alpha s) = \mathcal{L}_X \alpha s + \alpha \iota_X \theta s = \mathcal{L}_X \alpha s$$

and so $\psi = \alpha s$ is polarized iff α is constant along the leaves of P .

If $q_1, \dots, q_n, p_1, \dots, p_n$ are global (canonical) coordinates on $\mathcal{M} = \mathbb{R}^{2n}$ and the leaves of P are spanned by $X_{q_1}, \dots, X_{q_n}$ then such an α depends only on $q_1, \dots, q_n$, $\mathcal{Q}(q_1), \dots, \mathcal{Q}(q_n)$ are compatible and form a complete set of observables and so in this case Q5 is satisfied, while in general the requirement of the presence of a polarization is a weaker condition than Q5.

When considering the choice of a polarization we face three problems:

- there might exist different polarizations on $\mathcal{M}$. It is not clear which should be the “right” one (or if their choice is equivalent from the physical point of view);
- $\mathcal{H}$ might not contain polarized sections other than the 0 section (for example in the last example if α is non-zero then αs can not be square integrable under ω^n);
- there might not exist a real polarization on $\mathcal{M}$.

The first two issues are addressed by introducing the concept of a metaplectic structure on $\mathcal{M}$. This leads to a theory that goes beyond the introductory purpose of this work and will not be discussed here (see for instance [24, 39, 43] for a detailed exposition).

It is easy to see that the last issue is concrete: for example the choice of a real polarization on $\mathbb{S}^2$ would be equivalent to the choice of a non-vanishing tangent vector field on $\mathbb{S}^2$ but, according to the hairy ball theorem, such a vector field cannot exist. A solution is found by considering a *complex polarization*.

3.5 Elements of complex geometry

To develop further the analysis in this case we need some basic definitions and results in complex geometry. References for the contents of this section are found in [27, 30], the experienced reader can safely skip this part.

The tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ can be complexified, in a standard way, to obtain $(T\mathcal{M})^{\mathbb{C}}$, the *complexified* tangent bundle. Its elements are of the form $v + iw$ with $v, w \in T_x\mathcal{M}$, for some $x \in \mathcal{M}$. If $u = v + iw$ is an element of $(T_x\mathcal{M})^{\mathbb{C}}$ we define the conjugate of u , $\bar{u}$, as $\bar{u} = v - iw$. If U is a subset of $(T\mathcal{M})^{\mathbb{C}}$ then $\bar{U}$ is the image, under conjugation, of U . For every point $x \in \mathcal{M}$, $T_x\mathcal{M}$ can be seen as the subset of $(T_x\mathcal{M})^{\mathbb{C}}$ defined by those u such that $u = \bar{u}$.

Any linear map ϕ defined on $T_x\mathcal{M}$ and taking values in a vector space V , can be extended uniquely to a linear map on $(T_x\mathcal{M})^{\mathbb{C}}$ taking value in $V^{\mathbb{C}}$, the complexification of V , in the following way:

$$\phi(v_x + iw_x) = \phi(v_x) + i\phi(w_x).$$

In particular ω can be extended to a complex-value closed and non degenerate 2-form on $(T\mathcal{M})^{\mathbb{C}}$ (which we still denote, with a slight abuse of notation, by ω), in this way $(T_x\mathcal{M})^{\mathbb{C}}$ is a complex symplectic vector space. A subspace P_x of $(T_x\mathcal{M})^{\mathbb{C}}$ is Lagrangian if ω restricted to P_x vanishes and P_x is maximal (i.e. it has complex dimension n , where $2n$ is the real dimension of $\mathcal{M}$).

Definition. A *complex polarization* is a family $P = (P_x)_{x \in \mathcal{M}}$ of Lagrangian subspaces $P_x \subset (T_x\mathcal{M})^{\mathbb{C}}$ depending smoothly from $x \in \mathcal{M}$ and satisfying:

- the real (coisotropic) distribution $E = (P + \bar{P}) \cap T\mathcal{M}$ is integrable;
- the leaves of the real (isotropic) distribution $D = P \cap \bar{P} \cap T\mathcal{M}$ (which is integrable) has constant dimension;
- $\mathcal{M}/E$ and $\mathcal{M}/D$ are Hausdorff spaces.

In analogy to the real case, a P -polarized section is a section ψ of B such that $\nabla_X \psi = 0$ for every X vector field in P (with the trivial extension of ∇ to complex vector fields). $\mathcal{H}_q$ is defined accordingly.

A real polarization P on $\mathcal{M}$ can be seen as a complex polarization where $P = \bar{P}$, on the other hand a complex polarization P on $\mathcal{M}$ for which $P \cap \bar{P} = 0$ is called a

Kähler polarization (it should be noted that this last condition alone implies that the distribution P is a polarization). While there are other types of complex polarizations, we will be interested in the study of Kähler polarizations only because geometric quantization is simpler when they are involved (and actually sufficient for our problem).

A n -dimensional complex manifold $\mathcal{M}$ is a manifold with charts taking values in $\mathbb{C}^n$ and holomorphic transition functions. A n -dimensional complex manifold can be identified, in a trivial way, with a $2n$ -dimensional real manifold equipped with a complex structure J , which is a fiber bundle homomorphism $J : T\mathcal{M} \mapsto T\mathcal{M}$ that project to the identity on $\mathcal{M}$ and such that $J^2 = -id_{T\mathcal{M}}$. In this identification of $\mathcal{M}$ with a real manifold the action of J corresponds to the multiplication by ι .

On the complexification of $T\mathcal{M}$, $(T\mathcal{M})^{\mathbb{C}}$ (whose fibers are $2n$ -dimensional vector *complex* spaces), J can be diagonalized (since on every fiber the minimal polynomial of J is $\mu(z) = z^2 + 1$ which has distinct roots in $\mathbb{C}$). In this way two fiber bundles $T^{(1,0)}\mathcal{M}$ and $T^{(0,1)}\mathcal{M}$ of $(T\mathcal{M})^{\mathbb{C}}$ are defined such that:

- $(T\mathcal{M})^{\mathbb{C}} = T^{(1,0)}\mathcal{M} \oplus T^{(0,1)}\mathcal{M}$, in particular $T^{(1,0)}\mathcal{M} \cap T^{(0,1)}\mathcal{M} = 0$;
- $\overline{T^{(1,0)}\mathcal{M}} = T^{(0,1)}\mathcal{M}$;
- on $T^{(1,0)}\mathcal{M}$, J operates as the multiplication by ι ;
- on $T^{(0,1)}\mathcal{M}$, J operates as the multiplication by $-\iota$.

A particular class of complex manifolds is given by Kähler manifolds:

Definition. A Kähler manifold is a complex manifold equipped with a symplectic form ω compatible with the complex structure (i.e. if $v, w \in T_x\mathcal{M}$ then $\omega_x(Jv, Jw)$ is equal to $\omega_x(v, w)$).

The compatibility condition between ω and J implies that restriction (of the complexification) of ω vanishes on both $T^{(1,0)}\mathcal{M}$ and $T^{(0,1)}\mathcal{M}$, since if v, w are elements of $T^{(1,0)}\mathcal{M}$ then $Jv = \iota v$ and $Jw = \iota w$ and so

$$\omega(Jv, Jw) = \omega(\iota v, \iota w) = \iota^2 \omega(v, w) = -\omega(v, w),$$

similarly for v, w in $T^{(0,1)}\mathcal{M}$. Since the dimension of $T^{(0,1)}\mathcal{M}$ and $T^{(1,0)}\mathcal{M}$ is maximal then they both define complex polarizations on $\mathcal{M}$. If we set $P = T^{(0,1)}\mathcal{M}$, we have that $P \cap \bar{P} = 0$ and so P is a Kähler polarization (hence the name). It is possible to prove (see [43]) that the converse is also true: if P is a Kähler polarization, then it is possible to find a Kähler structure on $\mathcal{M}$ (a complex structure J compatible with the given ω) such that $P = T^{(0,1)}\mathcal{M}$.

If $\mathcal{M}$ is a complex manifold, and $z^1, \dots, z^n \in \mathbb{C}^n$ are local complex coordinates, defining $z^i = x^i + iy^i$, we have that x^i and y^i are real local coordinates for $\mathcal{M}$ (seen as a $2n$ -dimensional real manifold). If

$$\begin{aligned}\frac{\partial}{\partial z^i} &= \frac{1}{2} \left(\frac{\partial}{\partial x^i} - i \frac{\partial}{\partial y^i} \right) \\ \frac{\partial}{\partial \bar{z}^i} &= \frac{1}{2} \left(\frac{\partial}{\partial x^i} + i \frac{\partial}{\partial y^i} \right)\end{aligned}$$

we have that the set $\left(\frac{\partial}{\partial z^i}\right)_{i=1,\dots,n}$ spans $T^{(1,0)}\mathcal{M}$ while $\left(\frac{\partial}{\partial \bar{z}^i}\right)_{i=1,\dots,n}$ spans $T^{(0,1)}\mathcal{M}$. If $\mathcal{M}$ is a Kähler manifold then the symplectic form has local representative of the form:

$$\sum_{i,j=1}^n \omega_{ij}(z) \frac{i}{2} dz^i \wedge d\bar{z}^j$$

where $\omega_{ij}(z)$ are functions on $\mathcal{M}$ such that $\omega_{ji} = \bar{\omega}_{ij}$.

It is known that there always exists (at least locally) a function K on $\mathcal{M}$, a *Kähler potential*, such that

$$\omega_{ij} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^j}.$$

We have that (recalling that an holomorphic function on a complex manifold is a function α satisfying $\frac{\partial \alpha}{\partial \bar{z}^k} = 0$ for $k = 1, \dots, n$):

Proposition 5. *If K is defined almost everywhere on the Kähler manifold $\mathcal{M}$, then there exists a polarized (with respect to the Kähler polarization) trivializing section s such that*

- $(s, s) = \exp(-\frac{K}{2\hbar})$;
- if α is a function on $\mathcal{M}$, then αs is a polarized section iff α is holomorphic.

In this way $\mathcal{H}$ can be identified with $L^2(\mathcal{M}; \exp(-\frac{K}{2\hbar})\omega^n)$ and, in this identification:

$$\mathcal{H}^q = \{\alpha \in L^2(\mathcal{M}; \exp(-\frac{K}{2\hbar})\omega^n) \mid \alpha \text{ is holomorphic}\}.$$

Proof. If we consider the 1-form $\theta = -\frac{i}{2} \frac{\partial K}{\partial z^k} dz^k$ we have that $\omega = d\theta$. If s is a trivializing section on B relative to the connection 1-form θ then

$$d(s, s) = (\nabla s, s) + (s, \nabla s) = (i\hbar)^{-1}(\theta - \bar{\theta})(s, s) = d\left(-\frac{K}{2\hbar}\right)(s, s).$$

This equation is solved by $(s, s) = \lambda \exp(-\frac{K}{2\hbar})$ for some positive constant λ (which can be freely chosen rescaling s). Since K is defined almost everywhere this

shows that s is non zero almost everywhere and the identification of $\mathcal{H}$ with $L^2(\mathcal{M}; \exp(-\frac{K}{2\hbar})\omega)$ is proved (the value of λ is not important in this case).

If $X \in T^{(0,1)}\mathcal{M}$, it is immediate to check that $\iota_X\theta = 0$, if a section ψ of B satisfies $\psi = \alpha s$ almost everywhere for some function α , then ψ is polarized iff $\frac{\partial \alpha}{\partial \bar{z}^k} = 0$ for $k = 1, \dots, n$. $\square$

3.6 Geometric quantization of $\mathbb{S}^2$

The geometric quantization of the sphere is standard, see for instance [43].

When $\mathcal{M} = \mathbb{S}^2$ we have already seen that there are no real polarizations and if P is a complex polarization on $\mathcal{M}$, since $\dim \mathcal{M} = 2$, it has to be Kähler. Let $\mathcal{M}^N$ be the sphere (of radius R embedded in $\mathbb{R}^3$ and with the usual symplectic structure) with the south pole removed and consider $X^N : \mathcal{M}^N \rightarrow \mathbb{R}^2$ defined by:

$$\begin{aligned} X_1^N &= \frac{x_1}{R + x_3} \\ X_2^N &= -\frac{x_2}{R + x_3} \end{aligned}$$

Similarly let $\mathcal{M}^S$ be the sphere with the north pole removed and $X^S : \mathcal{M}^S \rightarrow \mathbb{R}^2$ defined by:

$$\begin{aligned} X_1^S &= \frac{x_1}{R - x_3} \\ X_2^S &= \frac{x_2}{R - x_3} \end{aligned}$$

X^N and X^S define an atlas (stereographic coordinates) for $\mathcal{M}$. The transition function takes the form

$$\begin{aligned} X_1^S &= \frac{X_1^N}{\|X^N\|^2} \\ X_2^S &= -\frac{X_2^N}{\|X^N\|^2} \end{aligned}$$

and the symplectic form has local representatives:

$$\frac{4R}{(1 + \|X^N\|^2)^2} dX_1^N \wedge dX_2^N \quad \text{and} \quad \frac{4R}{(1 + \|X^S\|^2)^2} dX_1^S \wedge dX_2^S.$$

If we define, for $\lambda = N, S$, $z^\lambda : \mathcal{M}^\lambda \rightarrow \mathbb{C}$ as $z^\lambda = X_1^\lambda + \iota X_2^\lambda$, we have that

$$z^S = (z^N)^{-1}$$

and the local representatives of the symplectic form become

$$\frac{4R}{(1 + z^N \bar{z}^N)^2} \frac{i}{2} dz^N \wedge d\bar{z}^N \quad \text{and} \quad \frac{4R}{(1 + z^S \bar{z}^S)^2} \frac{i}{2} dz^S \wedge d\bar{z}^S$$

which shows that $\mathcal{M}$, has a structure of a Kähler manifold.

Proposition. *In the quantizable case, that is $R = \frac{\hbar n}{2}$ for some $n \in \mathbb{N}^{>0}$, the quantum Hilbert space for the sphere of radius R can be identified with the space $\mathcal{H}^q$ of complex polynomial in one variable of degree at most n . An orthonormal basis for $\mathcal{H}^q$ is given by*

$$e_k = \sqrt{\binom{n}{k}} z^k \quad k = 0, \dots, n.$$

Proof. If we restrict our attention to the chart covering $\mathcal{M}^S$ (or $\mathcal{M}^N$) only and identify it with $\mathbb{C}$, as shown above, we have that a Kähler potential is given by

$$K = 2\hbar n \ln(1 + z^S \bar{z}^S).$$

K is defined almost everywhere on the sphere and so Proposition 5 holds. We thus have

$$\begin{aligned} \theta &= -\frac{i\hbar n \bar{z}^S}{1 + \bar{z}^S z^S} dz^S \\ (s, s) &= \lambda(1 + z^S \bar{z}^S)^{-n} \\ \lambda \exp\left(-\frac{K}{2\hbar}\right) \omega &= i\hbar n \lambda (1 + z^S \bar{z}^S)^{-(2+n)} dz^S \wedge d\bar{z}^S. \end{aligned}$$

A simple calculation shows that only polynomials of degree at most n are square integrable under this measure and

$$\langle e_i, e_j \rangle = \delta_{ij} \frac{2\pi\hbar n}{1 + n} \lambda.$$

A suitable choice of λ allows to conclude. □

In this case $\mathcal{H}^q$ is a non trivial closed subspace of $\mathcal{H}$, and in principle no metaplectic structure is needed to solve the problem of an empty $\mathcal{H}^q$, however the so-called *metaplectic correction* to the quantization procedure is needed to obtain the right physical results.

Without going into too much detail, it can be proved (see for example [39, 40]) that this amounts to rescaling (s, s) (and thus the measure of integration on $\mathcal{H}^s$)

by multiplying it by $(\det \omega_{ij})^{-\frac{1}{2}}$ and to redefine $\mathcal{Q}$ in the following way: for any function f we have that

$$[X_f, X_{z^i}] = a_j^i(f)X_{z^j} + b_j^i(f)X_{\bar{z}^j}$$

for some functions $a_j^i(f)$ and $b_j^i(f)$ defined on $\mathcal{M}$. If we consider the function $a(f)$ on $\mathcal{M}$, $a(f) = \sum_i a_i^i(f)$, then $\mathcal{Q}(f)\alpha$ is obtained from the previous definition by adding the *metaplectic correction term* $-\frac{i\hbar}{2}a(f)\alpha$. Explicitly the extra term reads (ω^{ij} denotes the inverse of the matrix ω_{ij}):

$$-\frac{i\hbar}{2}a(f) = \hbar \frac{\partial}{\partial z^j} \left(\omega^{ij} \frac{\partial f}{\partial \bar{z}^i} \right)$$

and we obtain

$$\mathcal{Q}(f)\alpha = 2\hbar\omega^{ij} \frac{\partial f}{\partial \bar{z}^i} \frac{\partial \alpha}{\partial z^j} + (f + \hbar \frac{\partial}{\partial z^j} \left(\omega^{ij} \frac{\partial f}{\partial \bar{z}^i} \right) - \omega^{ij} \frac{\partial f}{\partial \bar{z}^i} \frac{\partial K}{\partial z^j})\alpha$$

Taking in consideration the metaplectic correction we have:

Proposition. *The quantum Hilbert space for the sphere of radius $\frac{\hbar n}{2}$ can be identified with the space $\mathcal{H}^n$ of complex polynomial in one variable of degree at most $n-1$. An orthonormal basis for $\mathcal{H}^n$ is given by*

$$e_k = \sqrt{\binom{n-1}{k}} z^k \quad k = 0, \dots, n-1$$

and the metaplectic correction is

$$-\frac{i\hbar}{2}a(f) = \frac{1}{2n} \frac{\partial}{\partial z} \left((1 + z\bar{z})^2 \frac{\partial f}{\partial \bar{z}} \right)$$

which results in

$$\mathcal{Q}(f)\alpha = \frac{(1 + z\bar{z})^2}{n} \frac{\partial f}{\partial \bar{z}} \frac{\partial \alpha}{\partial z} + (f + \frac{1}{2n} \frac{\partial}{\partial z} \left((1 + z\bar{z})^2 \frac{\partial f}{\partial \bar{z}} \right) - \bar{z}(1 + \bar{z}z) \frac{\partial f}{\partial \bar{z}})\alpha.$$

As already mentioned there are no guarantees that $\mathcal{Q}(f)$ is a well-defined operator on $\mathcal{H}^n$. This actually happens only when X_f preserves the polarization, and this happens precisely when $b_j^i(f) = 0$. For example it is easy to check that, in the case of the sphere, f should be a linear combination of x_1, x_2, x_3 and 1 (the latter being the constant function on the sphere equal to 1). This is expressed, for a general $\mathcal{M}$, by the celebrated Groenewold-van Hove theorem Which states that this situation is unavoidable (i.e. $Q_1, \dots, Q_5$ are incompatible). For the case of $\mathbb{S}^2$ see [22] and references therein for the case $\mathbb{R}^{2n}$. In the next section we will see how it is possible to find a procedure that allows the quantization of more general observables at the expense of require weaker quantization postulates.

3.7 BKS quantization

When the quantization of more general observables is required it is possible to resort to the so called *BKS*-quantization (standing for Blattner-Kostant-Sternberg, see for example [24]). For a general polarization this procedure is rather intricate (and again related to the definition of a metaplectic structure on $\mathcal{M}$). In the case of a Kähler polarization however $\mathcal{H}^q$ is a closed subspace of $\mathcal{H}$. In this way there exists an orthogonal projection Π from $\mathcal{H}$ onto $\mathcal{H}^q$. Π can be used to define the quantization of a generic observable f by:

$$\mathcal{Q}_{BKS}(f) = \Pi \mathcal{Q}(f).$$

An equivalent way to describe $\mathcal{Q}_{BKS}(f)$ is to defining it as the (unique) operator on $\mathcal{H}^q$ satisfying

$$\langle \alpha', \mathcal{Q}_{BKS}(f)\alpha \rangle = \langle \alpha', \mathcal{Q}(f)\alpha \rangle \quad \forall \alpha, \alpha' \in \mathcal{H}^q.$$

When f preserves the polarization then obviously $\mathcal{Q}_{BKS}(f) = \mathcal{Q}(f)$ and it satisfies $Q1, \dots, Q5$. In general however we have that

$$\mathcal{Q}_{BKS}(\{f, g\}) - (i\hbar)^{-1}[\mathcal{Q}_{BKS}(f), \mathcal{Q}_{BKS}(g)] \neq 0.$$

If we consider as $\mathcal{M}$ the sphere of radius $\frac{\hbar m}{2}$, for $n \in \mathbb{N}^{>0}$, (μ here is the measure on $\mathcal{M}$ defined by $\mu = (s, s)\omega$) it is possible to give an explicit expression for Π :

$$\begin{aligned} (\Pi\alpha)(z) &= \int_{\mathbb{C}} \left(\sum_k e_k(z) \bar{e}_k(w) \right) \alpha(w) \mu(w) \\ &= \int_{\mathbb{C}} (1 + \bar{z}w)^{N-1} \alpha(w) \mu(w) \\ &= \int_{\mathbb{C}} \frac{n}{\pi} \frac{(1 + z\bar{w})^{n-1}}{(1 + w\bar{w})^{n+1}} \alpha(w) \frac{i}{2} dw \wedge d\bar{w} \end{aligned}$$

3.8 Coherent states

The concept of coherent state, which naturally arises in the context of geometric quantization, is central in quantum physics, for references see for example [21, 36]. A coherent state is, roughly speaking, a quantum state that most closely resemble a classical state, namely a point on $\mathcal{M}$, and so it is essential for providing a link between the quantum and the classical world. Coherent states allow to define the so called coherent state quantization of a function on $\mathcal{M}$, which is a procedure that associates an operator on the quantum Hilbert space to the function. BKS quantization of a function can be seen as a coherent state quantization of another

(but “close”) function (see [41]). We will show how coherent states can be used to analyze generic quantum states, such as eigenstates of Hamiltonian operators, from a classical perspective by means of the so called Husimi distributions. This will allow us to develop a technique useful in the numerical analysis of the system we are studying.

Let $z \in \mathcal{M}$. The linear map from $\mathcal{H}^q$ to $\mathbb{C}$ that to a function α associates $\alpha(z)$ is continuous. Hence, by Riesz representation theorem, there exists a unique $c_z \in \mathcal{H}^q$ such that $\alpha(z) = \langle c_z, \alpha \rangle$ for every $\alpha \in \mathcal{H}^q$. Using the fact that the projection Π , defined above, restricted to $\mathcal{H}^q$ is the identity, we have that, if e_k are the elements of an orthonormal basis of $\mathcal{H}^q$,

$$\alpha(z) = \Pi\alpha(z) = \int_{\mathbb{C}} \sum_k e_k(z) \bar{e}_k(w) \alpha(w) \mu(w)$$

so this allows us to conclude that

$$c_z(w) = \sum_k e_k(z) \bar{e}_k(w).$$

For the sphere of radius $\frac{\hbar n}{2}$:

$$c_z(w) = (1 + \bar{z}w)^{n-1}. \quad (3.1)$$

for a generic $\alpha \in \mathcal{H}$ we have that $\langle c_z, \alpha \rangle = \Pi\alpha(z)$. The state c_z is called a (non-normalized) *coherent state* relative to the point z . For the sphere it satisfies:

$$\begin{aligned} c_z &= \sum_k \sqrt{\binom{n-1}{k}} \bar{z}^k e_k \\ \langle c_z, c_w \rangle &= c_w(z) = (1 + \bar{w}z)^{n-1} \\ \Pi\psi &= \int_{\mathbb{C}} c_z \langle c_z, \psi \rangle \mu(z) = \int_{\mathbb{C}} \frac{c_z}{\|c_z\|} \left\langle \frac{c_z}{\|c_z\|}, \psi \right\rangle (2\pi\hbar)^{-1} \omega(z) \end{aligned}$$

and

$$\mathcal{Q}_{BKS}(f)\psi = \int_{\mathbb{C}} c_z \langle c_z, \mathcal{Q}(f)\psi \rangle \mu(z) = \int_{\mathbb{C}} c_z \langle \mathcal{Q}(f)c_z, \psi \rangle \mu(z).$$

To give a physical interpretation of the meaning of the state c_z we can think of $\mathcal{H}$ as a subspace of the space $\mathcal{H}'$ of distributions. Π can be extended to the orthogonal projection from $\mathcal{H}'$ onto $\mathcal{H}^q$. If $\delta_z \in \mathcal{H}'$ is the Dirac delta centred in z then we have

$$c_z = \Pi(\delta_z)$$

so, in some sense, we can consider c_z as the polarized state which is the most (as long as it is allowed by the uncertainty principle) “peaked” in z . Another way to characterize c_z is to define it as the element of $\mathcal{H}^q$ that maximize, for a fixed norm, its absolute value in z (more on the characterization of coherent states can be found in [43]).

3.9 Covariant and contravariant symbols

Coherent states can be used to introduce the notion of covariant and contravariant symbol of an observable. If $\mathbf{f}$ is an observable on $\mathcal{H}^q$, the Hilbert quantum space obtained from the symplectic manifold $\mathcal{M}$ via geometric quantization, then its *covariant symbol* is a function on $\mathcal{M}$, denoted by $\hat{f}$, while if $\check{f}$ is a function on $\mathcal{M}$ then it is possible to associate to it an operator $\mathbf{f}$ on $\mathcal{H}^q$. $\check{f}$ is called the *contravariant symbol* of $\mathbf{f}$. In this section we will define the covariant and contravariant maps $\mathbf{f} \mapsto \hat{f}$ and $\check{f} \mapsto \mathbf{f}$ and we will study their basic properties and their use in the analysis of the relationships between the classical and the quantum behaviour of a physical system.

References for this (and for further generalizations), on which the following is loosely based, can be found in [10, 11, 36].

If $\mathbf{f}$ is an operator on $\mathcal{H}^q$ it is possible to define the function on $\mathcal{M} \times \mathcal{M}$:

$$\tilde{f}(z, w) := \frac{\langle c_z, \mathbf{f}c_w \rangle}{\langle c_z, c_w \rangle}$$

$\hat{f}$ is holomorphic in z and antiholomorphic in w , hence $\tilde{f}$ is completely determined by its value at $w = z$, so it is possible to characterize $\tilde{f}$ using the *covariant symbol* of $\mathbf{f}$:

$$\hat{f}(z) := \tilde{f}(z, z) = \frac{\langle c_z, \mathbf{f}c_z \rangle}{\langle c_z, c_z \rangle}.$$

The map that to an observable A associates its covariant symbol is (trivially) a rings homomorphism if the space of functions on $\mathcal{M}$ is equipped with the product $\star$ defined as

$$(\hat{f} \star \hat{g})(z) := \int_{\mathcal{M}} \tilde{f}(z, w) \tilde{g}(w, z) K(z, w) \mu(w) = \hat{h}(z)$$

where $\mathbf{h} = \mathbf{f}\mathbf{g}$ and

$$K(z, w) = \frac{\langle c_z, c_w \rangle \langle c_w, c_z \rangle}{\langle c_z, c_z \rangle}.$$

The product $\star$ is not commutative and it is possible to consider the commutator between two function as:

$$[\hat{f}, \hat{g}]_{\star} = \hat{f} \star \hat{g} - \hat{g} \star \hat{f}.$$

Since $\star$ is the image through the covariant map of the product of operators we have that $[\cdot, \cdot]_{\star}$ is the image of the commutator of observables, i.e. if $[\mathbf{f}, \mathbf{g}] = \mathbf{h}$, then $[\hat{f}, \hat{g}]_{\star} = \hat{h}$.

Since $\check{f}$ completely determines $\mathbf{f}$, if $\hat{f} = 0$ then $\check{f} = 0$ which is equivalent to

$$\langle c_z, \mathbf{f}c_w \rangle = 0 \quad \forall z, w \in \mathcal{M}.$$

Multiplying both sides of this expression by c_z and c_w^* (the dual of c_w under the pairing given by the scalar product) and integrating in z and w over $\mathcal{M}$ we obtain

$$\Pi \mathbf{f} \Pi = 0$$

which is equivalent to $\mathbf{f} = 0$. This means that the covariant map $\hat{\cdot} : \mathbf{f} \mapsto \hat{f}$ is injective and so the space of quantum observables can be identified by the ring of covariant symbols on $\mathcal{M}$ (function on $\mathcal{M}$ which are of the form $\hat{f}$ for some observable $\mathbf{f}$) equipped with the product defined by $\star$.

Related to the concept of covariant symbol of an observable, and of great importance for our purpose, is the *Husimi distribution* of a state

Definition. If ψ is an element of $\mathcal{H}^q$ then the Husimi distribution H_ψ of ψ is the real valued analytic function defined on $\mathcal{M}$ by the covariant symbol of the orthogonal projection operator on the subspace spanned by ψ . An explicit expression of H_ψ is given by:

$$H_\psi(z) = \frac{|\psi(z)|^2}{\langle c_z, c_z \rangle \langle \psi, \psi \rangle}.$$

If $\mathbf{f}$ is an operator on $\mathcal{H}^q$ and there exists a function $\mathring{f}$ on $\mathcal{M}$ such that

$$\mathbf{f}\psi = \int_{\mathcal{M}} c_z \mathring{f}(z) \psi(z) \mu(z).$$

then $\mathring{f}$ is called the *contravariant symbol* of $\mathbf{f}$, and $\mathbf{f}$ is said to be obtained by coherent-state or Berezin quantization of $\mathring{f}$ (see [10, 11, 40, 41]). If we define the operator $\mathbf{f}'$ on $\mathcal{H}$ (the prequantum Hilbert space) as

$$(\mathbf{f}'\psi)(z) = \mathring{f}(z)\psi(z)$$

we have that $\mathbf{f} = \Pi \mathbf{f}'|_{\mathcal{H}^q}$.

The contravariant symbol $\mathring{f}$ of an operator $\mathbf{f}$ uniquely determines $\mathbf{f}$ and an operator $\mathbf{f}$ uniquely defines its covariant symbol $\hat{f}$. If $\mathring{f}$ is the contravariant symbol of $\mathbf{f}$ then it is possible to verify that:

$$\hat{f}(z) = \int_{\mathcal{M}} \mathring{f}(w) K(z, w) \mu(w)$$

where K has been defined above.

From now on we will only consider the case of the sphere with unitary radius. Using the explicit formulae for the coherent states given in (3.1):

$$K(z, w) = \left(\frac{|1 + \bar{w}z|^2}{1 + \bar{z}z} \right)^{n-1}.$$

Proposition 6. *If $\psi \in \mathcal{H}^q$ is an eigenstate of the observable $\mathbf{h}$ with eigenvalue λ , and $\mathbf{h}$ admits contravariant symbol $\mathring{h}$ then*

- *ψ is concentrated around the λ -level set of $\mathring{h}$, more precisely:*

$$(\mathring{h} - \lambda)\psi = O(\hbar);$$

- *the Husimi distribution H_ψ of ψ is left almost invariant by the flow of $\mathring{h}$, more precisely:*

$$\mathcal{L}_{X_{\mathring{h}}} H_\psi = \{H_\psi, \mathring{h}\} = O(\hbar).$$

To prove these statements we need some results that show the relationships between covariant and contravariant symbols and between the natural Poisson structures on $\mathcal{M}$ and on the space of observables on $\mathcal{H}^q$:

Proposition 7. *If $\mathbf{f}$ and $\mathbf{g}$ are observables of $\mathcal{H}^q$ then*

$$\begin{aligned} \hat{f} \star \hat{g} &= \hat{f}\hat{g} + O(\hbar) \\ (\imath\hbar)^{-1}[\hat{f}, \hat{g}]_\star &= \{\hat{f}, \hat{g}\} + O(\hbar). \end{aligned}$$

If $\mathbf{g}$ admits contravariant symbol $\mathring{g}$ then

$$\begin{aligned} \hat{g} &= \mathring{g} + O(\hbar) \\ (\imath\hbar)^{-1}[\hat{f}, \hat{g}]_\star &= \{\hat{f}, \mathring{g}\} + O(\hbar). \end{aligned}$$

Proof. Following [36] to study the relations between the contravariant and the covariant symbols of an operator and the behaviour of the product $\star$ of covariant symbols, we introduce the linear operator T acting on the space of functions on $\mathcal{M}$ as:

$$Tf(z) = \int_{\mathcal{M}} f(w)K(z, w)\mu(w)$$

so that

$$\hat{f} = T\mathring{f}$$

and, defining $\tilde{f}_z(w) = \tilde{f}'_w(z) = \tilde{f}(z, w)$,

$$(\hat{f} \star \hat{g})(z) = T(\tilde{f}_z \tilde{g}'_z).$$

Since $\int_{\mathcal{M}} c_w c_w^* d\mu(w) = \Pi$ then, for every $z \in \mathcal{M}$:

$$\int_{\mathcal{M}} K(z, w)\mu(w) = 1$$

This implies that if f is constant then $Tf = f$.

Let f be a function on $\mathcal{M}$, there exist (even if they are not unique) functions α and β such that f can be written as

$$f(z) = f(0) + \frac{\partial f}{\partial z}(0)z + \frac{\partial f}{\partial \bar{z}}(0)\bar{z} + \frac{\partial^2 f}{\partial z \partial \bar{z}}(0)z\bar{z} + z^2\alpha(z) + \bar{z}^2\beta(z).$$

Since $K(0, w) = 1$ for every w :

$$Tf(0) = \int_{\mathcal{M}} f(w)\mu(w).$$

For a generic function g :

$$\begin{aligned} wg(w)d\mu(w) &= \frac{1}{n-1} \frac{\partial g}{\partial \bar{w}}(w)d\mu'(w) + d\left(\frac{wg(w)}{2\pi(1+w\bar{w})^n}d\bar{w}\right) \\ \bar{w}g(w)d\mu(w) &= \frac{1}{n-1} \frac{\partial g}{\partial w}(w)d\mu'(w) - d\left(\frac{wg(w)}{2\pi(1+w\bar{w})^n}dw\right) \end{aligned}$$

where μ' is the measure obtained replacing n with $n-1$ in the definition of μ . Integrating over $\mathcal{M}$ these expressions, by Stokes theorem:

$$\begin{aligned} \int_{\mathcal{M}} wg(w)d\mu(w) &= \frac{1}{n-1} \int_{\mathcal{M}} \frac{\partial g}{\partial \bar{w}}(w)d\mu'(w) \\ \int_{\mathcal{M}} \bar{w}g(w)d\mu(w) &= \frac{1}{n-1} \int_{\mathcal{M}} \frac{\partial g}{\partial w}(w)d\mu'(w) \end{aligned}$$

Since $\hbar n = 2$ we conclude that

$$Tf(0) = f(0) + \frac{1}{n-1} \frac{\partial^2 f}{\partial z \partial \bar{z}}(0) + O(\hbar^2) = (f + \frac{\hbar}{2}\Delta f)(0) + O(\hbar^2).$$

If ρ is an isometry (with respect to the Riemannian metric induced by ω on the sphere) and we define $g^\rho = g \circ \rho$, for a generic function g , then $Tg^\rho = Tg \circ \rho$:

$$\begin{aligned} Tg^\rho(z) &= \int_{\mathcal{M}} g^\rho(w)K(z, w)d\mu(w) \\ &= \int_{\mathcal{M}} g(\rho(w))K(z, w)d\mu(w) \\ &= \int_{\mathcal{M}} g(\rho(w))K(z, \rho^{-1}(\rho(w)))d\mu(\rho(w)) \\ &= \int_{\mathcal{M}} g(w')K(z, \rho^{-1}(w'))d\mu(w') \quad (w' = \rho(w)) \\ &= \int_{\mathcal{M}} g(w')K(\rho(z), w')d\mu(w') = Tg(\rho(z)). \end{aligned}$$

In particular, fixed $z \in \mathcal{M}$, there always exists an isometry ρ of $\mathbb{S}^2$ such that $\rho(0) = z$ and so

$$Tf(z) = Tf(\rho(0)) = Tf^\rho(0) = f^\rho(0) + \frac{\hbar}{2}\Delta f^\rho(0) + O(\hbar^2).$$

Since the Laplace-Beltrami operator is invariant under isometries we have that $\Delta f^\rho(0) = \Delta f(\rho(0))$ which allows us to conclude that

$$Tf(z) = f(z) + \frac{\hbar}{2}\Delta f(z) + O(\hbar^2).$$

This shows that, in the limit $\hbar \rightarrow 0$, $Tf - f = O(\hbar)$, but this implies that

$$\mathring{f} - \hat{f} = O(\hbar). \quad (3.2)$$

The formula for Tf can also be used to give a first order approximation of $\hat{f} \star \hat{g}$ in term of $\hat{f}$ and $\hat{g}$:

$$\hat{f} \star \hat{g} = \hat{f}\hat{g} + 2\hbar\omega_{11}^{-1} \frac{\partial f}{\partial \bar{z}} \frac{\partial g}{\partial z} + O(\hbar^2)$$

from which we have that

$$\begin{aligned} \hat{f} \star \hat{g} - \hat{f}\hat{g} &= O(\hbar) \\ (\imath\hbar)^{-1}[\hat{f}, \hat{g}]_\star - \{\hat{f}, \hat{g}\} &= O(\hbar). \end{aligned}$$

It should also be noted that, if X is a Killing vector field of $\mathcal{M}$ (that is a vector field that preserves the Riemannian metric on $\mathcal{M}$, in our case X is the vector field generating a rotation on the sphere) then $\mathcal{L}_X Tf = T(\mathcal{L}_X f)$. If v is a tangent vector of $\mathcal{M}$ at p then there always exists a Killing vector field X such that $X(p) = v$, in particular this implies that

$$\frac{\partial Tf}{\partial v} = T \frac{\partial f}{\partial v}.$$

This means that the convergence properties of T hold in the $\mathcal{C}^1$ sense and so we have, using (3.2), that

$$(\imath\hbar)^{-1}[\hat{f}, \hat{g}]_\star - \{\hat{f}, \mathring{g}\} = O(\hbar).$$

□

Now we can prove proposition 6:

Proof. The first statement is a straightforward calculation:

$$\begin{aligned}
\lambda\psi(z) &= (\mathbf{h}\psi)(z) \\
&= \int_{\mathcal{M}} \mathring{h}(w)\psi(w)\langle c_z, c_w \rangle \mu(w) \\
&= \int_{\mathcal{M}} \mathring{h}(w)\psi(w) \frac{n}{\pi} \frac{(1 + \bar{w}z)^{n-1}}{(1 + \bar{w}w)^{n+1}} \frac{i}{2} dw \wedge d\bar{w} \\
&= \mathring{h}(z)\psi(z) + O(\hbar).
\end{aligned}$$

If Π_ψ is the orthogonal projector onto the space spanned by ψ then

$$[\Pi_\psi, \mathbf{h}] = 0,$$

but this (recalling that, by definition, $H_\psi = \hat{\Pi}_\psi$) is equivalent to

$$[H_\psi, \hat{h}]_\star = 0.$$

The previous results allow to conclude. $\square$

As a side result we see that, since $\hat{h}$ and $\mathring{h}$ are close, we have that H_ψ is also almost conserved by the flow of $\hat{h}$ and its support is concentrated on the λ -level set of $\hat{h}$.

We can conclude then that eigenstates of $\mathbf{h}$, with eigenvalues close to λ , as $\hbar \rightarrow 0$ concentrate their support on classical orbits of $\mathring{h}$ of energy λ . The next chapter will be devoted to the study of the relationship between a function h and the contravariant symbol of $\mathcal{Q}_{BKS}(h)$.

3.10 BKS quantization and contravariant symbols

In the previous section we have defined the coherent state, or Berezin, quantization, a procedure that associates a quantum observable on $\mathcal{H}^q$ to a function on $\mathcal{M}$. Since there is a strong relationship between the classical dynamics driven by the contravariant symbol of an operator and the quantum characteristics (i.e. the Husimi distributions of the eigenstates) of the operator itself, it is desirable to determine if BKS-quantization can be restated in these terms. If f is a function on $\mathcal{M}$ and $\mathbf{g} = \mathcal{Q}_{BKS}(f)$ we wish to investigate, following the calculations in [40, 41], the relationships between f and $\mathring{g}$. An interesting alternative coordinate-free approach can be found in [12].

Here $\mathcal{M}$ will be a generic symplectic compact manifold with symplectic form ω , and $\mathcal{H}^q$ the quantum space obtained by choosing a Kähler polarization on $\mathcal{M}$. The notations will be the same as the previous sections.

Proposition ([41]). *Let Δ be the Laplace-Beltrami operator defined by the Riemannian structure on $\mathcal{M}$ induced by ω , that is*

$$\Delta f(z) = 4\omega^{ij} \frac{\partial^2 f(z)}{\partial \bar{z}^i \partial z^j}.$$

Let f be a function on $\mathcal{M}$ then $\mathcal{Q}_{BKS}(f)$ admits contravariant symbol

$$f - \frac{\hbar}{4} \Delta f.$$

Explicitly this reads:

$$\mathcal{Q}_{BKS}(f)\psi = \int_{\mathcal{M}} \left(f(z) - \frac{\hbar}{4} \Delta f(z) \right) \psi(z) c_z d\mu(z)$$

Proof. Let $\psi, \psi' \in \mathcal{H}^q$. Using Leibniz rule, Jacobi's formula for the derivative of the determinant of a matrix and the fact that ω is closed it is possible to show that:

$$\frac{\partial \omega^{ij}}{\partial z^j} = -\omega^{ij} \frac{\partial \ln \|\omega\|}{\partial z^j}$$

where $\|\omega\| = \det \omega_{ij}$. We can thus write

$$\begin{aligned} \mathcal{Q}(f)\psi &= \left(f - \hbar \frac{\partial}{\partial z^j} \left(\omega^{ij} \frac{\partial f}{\partial \bar{z}^i} \right) - \omega^{ij} \frac{\partial f}{\partial \bar{z}^i} \frac{\partial K}{\partial z^j} \right) \psi + \frac{\partial}{\partial z^j} \left(2\hbar \omega^{ij} \frac{\partial f}{\partial \bar{z}^i} \psi \right) \\ &= \left(f - \hbar \omega^{ij} \frac{\partial^2 f}{\partial \bar{z}^i \partial z^j} \right) \psi + 2\hbar \omega^{ij} \frac{\partial f}{\partial \bar{z}^i} \psi \frac{\partial \ln(\sqrt{\|\omega\|} e^{-\frac{K}{2\hbar}})}{\partial z^j} + \frac{\partial}{\partial z^j} \left(2\hbar \omega^{ij} \frac{\partial f}{\partial \bar{z}^i} \psi \right) \\ &= \left(f - \hbar \omega^{ij} \frac{\partial^2 f}{\partial \bar{z}^i \partial z^j} \right) \psi + \\ &\quad + \left(2\hbar \omega^{ij} \frac{\partial f}{\partial \bar{z}^i} \psi \frac{\partial}{\partial z^j} \left(\sqrt{\|\omega\|} e^{-\frac{K}{2\hbar}} \right) + \frac{\partial}{\partial z^j} \left(2\hbar \omega^{ij} \frac{\partial f}{\partial \bar{z}^i} \psi \right) \sqrt{\|\omega\|} e^{-\frac{K}{2\hbar}} \right) \omega^{-\frac{1}{2}} e^{\frac{K}{2\hbar}} \\ &= \left(f - \hbar \omega^{ij} \frac{\partial^2 f}{\partial \bar{z}^i \partial z^j} \right) \psi + \omega^{-\frac{1}{2}} \frac{\partial}{\partial z^j} \left(\sqrt{\|\omega\|} 2\hbar \omega^{ij} \frac{\partial f}{\partial \bar{z}^i} \psi e^{-\frac{K}{2\hbar}} \right) e^{\frac{K}{2\hbar}}. \end{aligned}$$

Using the fact that $\frac{\partial \bar{\psi}'}{\partial z^i} = 0$ for $i = 1, \dots, n$, defining $\tilde{f}\psi = \left(f - \hbar \omega^{ij} \frac{\partial^2 f}{\partial \bar{z}^i \partial z^j} \right) \psi$ and

$$\gamma = 2\hbar(\psi', \psi) \omega^{ij} \frac{\partial f}{\partial \bar{z}^i} \frac{\partial}{\partial z^j}$$

we have that:

$$\langle \psi', \mathcal{Q}(f)\psi \rangle = \langle \psi', \tilde{f}\psi \rangle + \int_{\mathcal{M}} \operatorname{div}(\gamma) \sqrt{\|\omega\|} \operatorname{Leb}^{2n}$$

($\text{div}(\gamma)$ indicates the divergence of the vector field γ with respect to the volume form $\sqrt{\|\omega\|}\text{Leb}^{2n}$, here Leb^{2n} is the standard $2n$ -dimensional Lebesgue measure). The first term reads:

$$\tilde{f}h = (f - \frac{\hbar}{4}\Delta f)h$$

The second integral, as long as $\mathcal{M}$ is compact, vanishes by Stokes' theorem and this allows to conclude that the contravariant symbol of $\mathcal{Q}_{BKS}(f)$ is $(1 - \frac{\hbar}{4}\Delta)f$. $\square$

In the case of the sphere Δ is the spherical Laplacian rescaled by $\frac{\hbar n}{2}$.

3.11 Quantization of polynomials on the sphere

We wish now to apply the techniques explained in the previous sections to the particular case of $\mathcal{M} = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$. The extension from the case $\mathcal{M} = \mathbb{S}^2$ is completely straightforward by considering the natural (product) Kähler structure on $\mathcal{M}$. In this way we obtain the Hilbert space $\mathcal{H}'$ of the system defined as the tensor product of three copies of $\mathcal{H}^q$ as above for a sphere of unit radius (in this way $\hbar = \frac{2}{n}$). The quantization of an observable f on $\mathcal{M}$ is now defined as before with the (rescaled) spherical Laplacian replaced by the Laplacian defined by the natural Riemannian structure on $\mathcal{M}$.

We consider the positive half-integer $j = \frac{n-1}{2}$, re-label (using the bra-ket notation) the basis elements of $\mathcal{H}^q$ as

$$|m, j\rangle = e_{j+m} \quad m = -j, -j+1, \dots, j-1, j$$

and define the observables $\mathbf{x}_i$, $i = 1, 2, 3$, acting on $\mathcal{H}^q$ as:

$$\begin{aligned} \mathbf{x}_1|m, j\rangle &= \frac{1}{2}(\alpha(j, m)|m+1, j\rangle + \alpha(j, m-1)|m-1, j\rangle) \\ \mathbf{x}_2|m, j\rangle &= \frac{1}{2i}(\alpha(j, m)|m+1, j\rangle - \alpha(j, m-1)|m-1, j\rangle) \\ \mathbf{x}_3|m, j\rangle &= \hbar m|m, j\rangle \end{aligned}$$

where $\alpha(j, m) = \hbar\sqrt{(j+m+1)(j-m)}$. The generalization to $\mathcal{H}'$ is straightforward and leads to the definition of the operators $\mathbf{x}_{\alpha i}$ with $\alpha, i = 1, 2, 3$ as:

$$\begin{aligned} \mathbf{x}_{1i} &= \mathbf{x}_i \otimes \mathbb{I} \otimes \mathbb{I} \\ \mathbf{x}_{2i} &= \mathbb{I} \otimes \mathbf{x}_i \otimes \mathbb{I} \\ \mathbf{x}_{3i} &= \mathbb{I} \otimes \mathbb{I} \otimes \mathbf{x}_i. \end{aligned}$$

We have that:

Proposition 8. Let h_0 , $\hat{f}_\nu$ and f defined in (2.5), (2.6) and (2.7) and $h_\epsilon = h_0 + \epsilon f$, then:

$$\begin{aligned}
\mathcal{Q}_{BKS}(h_0) &= \frac{1}{n(2+n)} + \frac{n(3+n)}{2(1+n)(2+n)}(\mathbf{x}_{13}^2 + \mathbf{x}_{23}^2) \\
\mathcal{Q}_{BKS}(\hat{f}_{(0,0)}) &= \frac{1}{n(2+n)} + \frac{4(3+2n)}{(3+n)(2+n)(1+n)}\mathbf{x}_{13} + \\
&\quad \frac{n(2+n)}{(1+n)^2}(2\mathbf{x}_{23}\mathbf{x}_{33} + \mathbf{x}_{33}\mathbf{x}_{13} + \mathbf{x}_{13}\mathbf{x}_{23}) + \\
&\quad \frac{n(3+n)}{(2+n)(1+n)}\mathbf{x}_{31}^2 + \frac{(6+n)n^2}{(3+n)(2+n)(1+n)}\mathbf{x}_{13}^3 \\
\mathcal{Q}_{BKS}(\hat{f}_{(1,0)}) &= \frac{n(2+n)}{(1+n)^2}(\mathbf{x}_{31}\mathbf{x}_{11} + \mathbf{x}_{32}\mathbf{x}_{12}) \\
\mathcal{Q}_{BKS}(\hat{f}_{(0,1)}) &= \frac{n(2+n)}{(1+n)^2}(\mathbf{x}_{21}(2\mathbf{x}_{31} + \mathbf{x}_{32} + \mathbf{x}_{33}) + 2\mathbf{x}_{22}\mathbf{x}_{32}) \\
\mathcal{Q}_{BKS}(\hat{f}_{(1,-1)}) &= \frac{n(2+n)}{(1+n)^2}(\mathbf{x}_{11}\mathbf{x}_{21} + \mathbf{x}_{12}\mathbf{x}_{22}) \\
\mathcal{Q}_{BKS}(h_\epsilon) &= \frac{1+\epsilon}{n(2+n)} + \frac{n(3+n)}{2(1+n)(2+n)}(\mathbf{x}_{13}^2 + \mathbf{x}_{23}^2) + \\
&\quad \epsilon \left(\frac{n(2+n)}{(1+n)^2}(\mathbf{x}_1 \cdot \mathbf{x}_2 + 2\mathbf{x}_1 \cdot \mathbf{x}_3 + \mathbf{x}_3 \cdot \mathbf{x}_1 + \mathbf{x}_{21}\mathbf{x}_{32} + \mathbf{x}_{21}\mathbf{x}_{33}) + \right. \\
&\quad \frac{n^2(6+n)}{(1+n)(2+n)(3+n)}\mathbf{x}_{13}^3 + \frac{n(3+n)}{(1+n)(2+n)}\mathbf{x}_{31}^2 + \\
&\quad \left. \frac{4(3+2n)}{(1+n)(2+n)(3+n)}\mathbf{x}_{13} \right).
\end{aligned}$$

where

$$\mathbf{x}_\alpha \cdot \mathbf{x}_\beta := \mathbf{x}_{\alpha 1}\mathbf{x}_{\beta 1} + \mathbf{x}_{\alpha 2}\mathbf{x}_{\beta 2} + \mathbf{x}_{\alpha 3}\mathbf{x}_{\beta 3}$$

Proof. Since $f = \hat{f}_{(0,0)} + \hat{f}_{(1,0)} + \hat{f}_{(0,1)} + \hat{f}_{(1,-1)}$ we have that

$$\mathcal{Q}_{BKS}(h_\epsilon) = \mathcal{Q}_{BKS}(h_0) + \epsilon(\mathcal{Q}_{BKS}(\hat{f}_{(0,0)}) + \mathcal{Q}_{BKS}(\hat{f}_{(1,0)}) + \mathcal{Q}_{BKS}(\hat{f}_{(0,1)}) + \mathcal{Q}_{BKS}(\hat{f}_{(1,-1)}))$$

so it is sufficient to verify the first five expressions.

It is immediate to check that for $\mathbb{S}^2$:

$$\begin{aligned}
\mathcal{Q}_{BKS}(x_i^2) &= \frac{1}{n(2+n)} + \frac{n(3+n)}{(2+n)(1+n)}\mathbf{x}_i^2 \\
\mathcal{Q}_{BKS}(x_i^3) &= \frac{4(3+2n)}{(3+n)(2+n)(1+n)}\mathbf{x}_i + \frac{(6+n)n^2}{(3+n)(2+n)(1+n)}\mathbf{x}_i^3
\end{aligned}$$

from these computations it follows almost immediately that for $\mathcal{M}$

$$\begin{aligned}\mathcal{Q}_{BKS}(x_{\alpha i}^2) &= \frac{1}{n(2+n)} + \frac{n(3+n)}{(2+n)(1+n)} \mathbf{x}_{\alpha i}^2 \\ \mathcal{Q}_{BKS}(x_{\alpha i}^3) &= \frac{4(3+2n)}{(3+n)(2+n)(1+n)} \mathbf{x}_{\alpha i}^3 + \frac{(6+n)n^2}{(3+n)(2+n)(1+n)} \mathbf{x}_{\alpha i}^3\end{aligned}$$

A little more effort is needed for computing the following (in these formulae it is assumed $\alpha \neq \beta$):

$$\begin{aligned}\mathcal{Q}_{BKS}(x_{\alpha i} x_{\beta j}) &= \frac{n(2+n)}{(1+n)^2} \mathbf{x}_{\alpha i} \mathbf{x}_{\beta j} \\ \mathcal{Q}_{BKS}(x_{\alpha} \cdot x_{\beta}) &= \frac{n(2+n)}{(1+n)^2} \mathbf{x}_{\alpha} \cdot \mathbf{x}_{\beta}.\end{aligned}$$

By linearity the quantization of h_{ϵ} immediately follows from these expressions. $\square$

As already mentioned a good approximation of the truncated normal form (2.2) is given by the (resonant or non resonant) average of h_{ϵ} defined by $\bar{h}_{\epsilon}^{(0)} = h_0 + \epsilon \hat{f}_{(0,0)}$ in the non resonant case and by $\bar{h}_{\epsilon}^{\nu} = h_0 + \epsilon(\hat{f}_{(0,0)} + \hat{f}_{\nu})$ in the ν resonant case. By linearity, using the expressions given above, it is straightforward to calculate $\mathcal{Q}_{BKS}(\bar{h}_{\epsilon}^{(0)})$ and $\mathcal{Q}_{BKS}(\bar{h}_{\epsilon}^{\nu})$.

Remark. The definition of $\mathbf{x}_{\alpha i}$ is motivated by observing that $\mathcal{Q}_{BKS}(x_{\alpha i}) = \mathbf{x}_{\alpha i}$. In general if P is a polynomial in the $x_{\alpha i}$ variables, even if in general $\mathcal{Q}_{BKS}(P)$ is not equal to $P(\mathbf{x}_{\alpha i})$, we have that $\mathcal{Q}_{BKS}(P) = P(\mathbf{x}_{\alpha i}) + O(\hbar)$ (where different choices in the ordering of the operators $\mathbf{x}_{\alpha i}$ inside P result in differences of order $\hbar$). $\triangle$

Chapter 4

Study of the quantum system

Using the techniques developed in the previous chapters we will study now the properties of the quantum system with Hamiltonian $\mathbf{h}_\epsilon := \mathcal{Q}_{BKS}(h_\epsilon)$, corresponding to the classical system with Hamiltonian h_ϵ defined before. In particular we plan to investigate the existence of quantum behaviours that reflect the differences between regular and chaotic motions of the classical system. To this end, we will focus on two aspects:

1. the distribution of the expectation values of the actions for the stationary states of the perturbed system;
2. the evolution of the coherent states in the phase space through their Husimi distributions.

The analysis of the eigenstates of the quantum system highlights characteristics of the spectrum of $\mathbf{h}_\epsilon$ that can be arguably linked to the differences existing between resonant and non resonant motions, but it does not allow that finer analysis needed to detect the richer structure that characterizes a superintegrable system compared to the Liouville integrable case.

To investigate the quantum phenomena corresponding to the difference between regular and chaotic resonant motions (which is typical of superintegrability) we will explore the quantum dynamical behaviour of coherent states since they are, in some sense, the most localized states on $\mathcal{M}$. It is natural to analyze them through their Husimi distribution, which allows to describe the system on the phase space.

From now on we will consider, otherwise specified, the perturbative parameter ϵ and the Plank constant $\hbar$ fixed. We recall that the Hilbert space $\mathcal{H}$, on which $\mathbf{h}_\epsilon$ is defined, has dimension n^3 , with $n = 2\hbar^{-1} \in \mathbb{N}$.

4.1 Analysis of stationary states in the action space

The dynamics of $\mathbf{h}_\epsilon$ is driven by Schrödinger equation which, as previously pointed out, is trivially solved once the eigenstates and eigenvalues of $\mathbf{h}_\epsilon$ are known. In this part of our analysis we will focus on the study of these eigenstates. In particular we are interested in characterizing each eigenstate by its resonance properties, since classically the presence of chaotic behaviour in a motion is strongly related to them.

We have that the quantum analogue of the classical actions is represented by $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$ defined in the previous section. To give the aforementioned characterization of each stationary state it is possible to consider the expectation value of $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$ in the state.

The spectra of $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$ coincide and are defined by:

$$\text{Sp}(x_{\alpha 3}) = \{\lambda_0, \dots, \lambda_{n-1}\}$$

where

$$\lambda_k := \frac{1-n}{n} + \frac{2}{n}k$$

(using the standard notation, defining the spin $j = \frac{n-1}{2}$ and $m = k - j = -j, \dots, j$ we have the well known condition $\lambda_k = \hbar m$). Since $[\mathbf{x}_{13}, \mathbf{x}_{23}] = 0$ the two observables can be diagonalized simultaneously and therefore $\mathcal{H}$ can be expressed as an orthogonal direct sum of their n -dimensional eigenspaces:

$$\mathcal{H} = \bigoplus_{k_1, k_2} \mathcal{H}_{k_1, k_2}$$

where $\mathcal{H}_{k_1, k_2}$ is defined by the condition

$$|\psi\rangle \in \mathcal{H}_{k_1, k_2} \Leftrightarrow \mathbf{x}_{\alpha 3} |\psi\rangle = \lambda_{k_\alpha} |\psi\rangle, \quad \alpha = 1, 2.$$

If $|\psi\rangle \in \mathcal{H}$ is a generic (normalized) state, then there exists a unique decomposition of $|\psi\rangle$:

$$|\psi\rangle = \sum_{k_1, k_2} |\psi_{k_1, k_2}\rangle$$

with $|\psi_{k_1, k_2}\rangle \in \mathcal{H}_{k_1, k_2}$. The value $\langle \psi_{k_1, k_2} | \psi_{k_1, k_2} \rangle$ can be interpreted as the probability that simultaneous measurements of the actions $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$, when the system is in the state $|\psi\rangle$, return the values λ_{k_1} and λ_{k_2} respectively.

We define $I_\psi \in \mathbb{R}^2$ as

$$I_\psi^\alpha := \langle \psi | \mathbf{x}_{\alpha 3} | \psi \rangle = \sum_{k_1, k_2} \lambda_{k_\alpha} \langle \psi_{k_1, k_2} | \psi_{k_1, k_2} \rangle$$

i.e. I_ψ^α is the expectation value of $\mathbf{x}_{\alpha 3}$ in the state $|\psi\rangle$. I_ψ is contained in the square defined by

$$|I_\psi^1|, |I_\psi^2| \leq 1 - \frac{1}{n}.$$

If $|\psi\rangle$ is an eigenstate of both $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$ then there exist k_1 and k_2 such that $|\psi\rangle \in \mathcal{H}_{k_1, k_2}$ and thus $I_\psi^\alpha = \lambda_{k_\alpha}$. This means that the values of I_ψ , as $|\psi\rangle$ range over the eigenstates of both $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$, define a grid of n^2 equispaced points in $\mathbb{R}^2$ which is represented in figure 4.1. In figures 4.2, 4.3 and 4.4 we have represented the values of I_ψ for every eigenstate $|\psi\rangle$ of $\mathbf{h}_\epsilon$ for different values of ϵ . In these plot two different kinds of behaviour are present. There is a class of states that cluster near the points of the grid. These are actually so close to the grid that in the plot each cluster appears as a single point. In particular we see that, for ϵ small these points are the one far from the resonances which are present in the classical system. As ϵ increases, however, resonances of higher order disrupt this ordered displacement.

On these resonances (i.e. near the resonant lines, but far from the origin) points are distributed along the fast drift lines (i.e. the direction in the action space, perpendicular to the resonant line, on which the classical resonant actions evolve). As ϵ grows larger the points are scattered farther from the resonance.

Points that are close to the origin (and thus are fully resonant) are scattered in various directions and this “fully chaotic” zone grows as ϵ increases.

This means that the eigenstates of $\mathbf{h}_\epsilon$ are either close to some $\mathcal{H}_{k_1, k_2}$, when the corresponding point is non-resonant (by “close” we mean that if $|\psi\rangle$ is the state, then there exists a state $|\psi'\rangle \in \mathcal{H}_{k_1, k_2}$ such that the norm of $|\psi\rangle - |\psi'\rangle$ is very small), while in the resonant case the eigenstate is close to $\bigoplus_{k_1, k_2} \mathcal{H}_{k_1, k_2}$ where the sum is taken over pairs (k_1, k_2) that represent points which are transverse to a resonance but close to each other.

Remark. While the spectrum of $\mathbf{h}_0$ is degenerate, we have numerically verified that, for the values of ϵ and n we considered, the degeneracy is completely removed by the perturbation. i.e. $\mathbf{h}_\epsilon$ has n^3 *distinct* eigenvalues. This is an essential condition for the results we will obtain in this chapter. For this reason non-degeneracy of the perturbed system will be always assumed. It is well known that, while this situation is typical, there exist particular values for ϵ and n for which the spectrum of $\mathbf{h}_\epsilon$ might contain degeneracies (which are however *accidental*, i.e. not determined by symmetries of the system). When studying the system it is thus necessary to check the non degeneracy of the spectrum case by case. We will discuss further this issue when we will consider a perturbative approach. $\triangle$

As mentioned in the introduction, the previous preliminary analysis, while highlighting the fact that resonances play a relevant role also in quantum dynamics, does not take into account the contribution coming from the third sphere and

so it does not consider the superintegrable character of the unperturbed part of the problem. A similar analysis, performed on a perturbed Liouville integrable system, provides an analogous difference between non-resonant eigenstates near the points of the grid and resonant eigenstates displaced transversely to the resonance along the fast drift line, see for instance figures 4.5, 4.6 and 4.7, the only notable difference being that now every non-resonant point is covered by only one eigenstate.

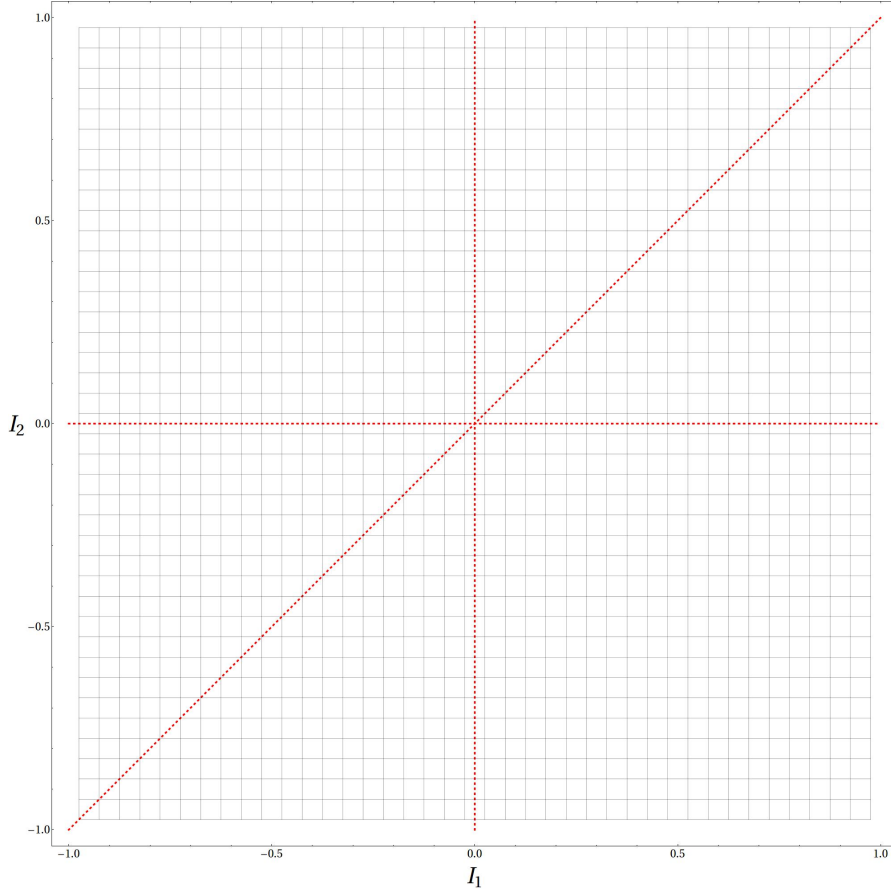


Figure 4.1: the joint spectrum of $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$, for $n = 40$ is plotted as a subset of $\mathbb{R}^2$. If $|\psi\rangle$ is an element of $\mathcal{H}_{k_1, k_2}$, then its expectation value is represented as a point on the grid with coordinates $((2k_1 - n + 1)n^{-1}, (2k_2 - n + 1)n^{-1})$. The dashed lines indicates the classical resonances which are present at the lower order in the classical perturbation.

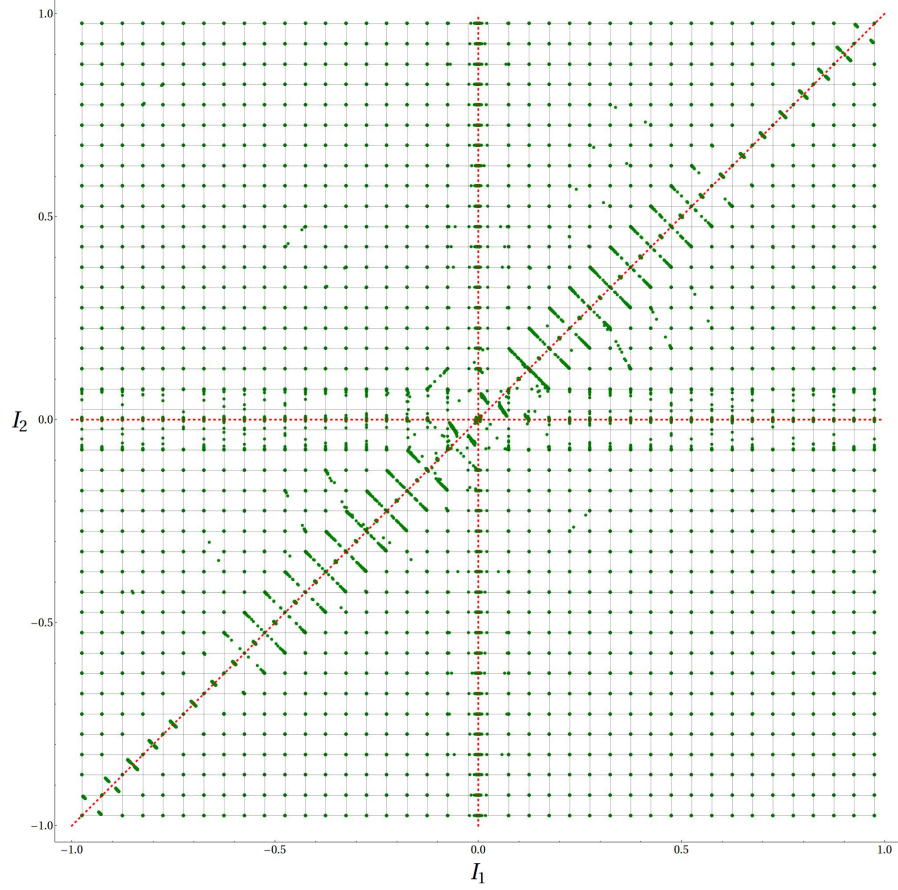


Figure 4.2: expectation values of $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$, plotted as points $I_\psi \in \mathbb{R}^2$, for every eigenstate $|\psi\rangle$ of $\mathbf{h}_\epsilon$, for $n = 40$ and $\epsilon = 10^{-4}$. As reference the grid defined in figure 4.1 is plotted. While to every state is associated a point, several states may have expectation values close to each other and thus appear as a single point in the figure.

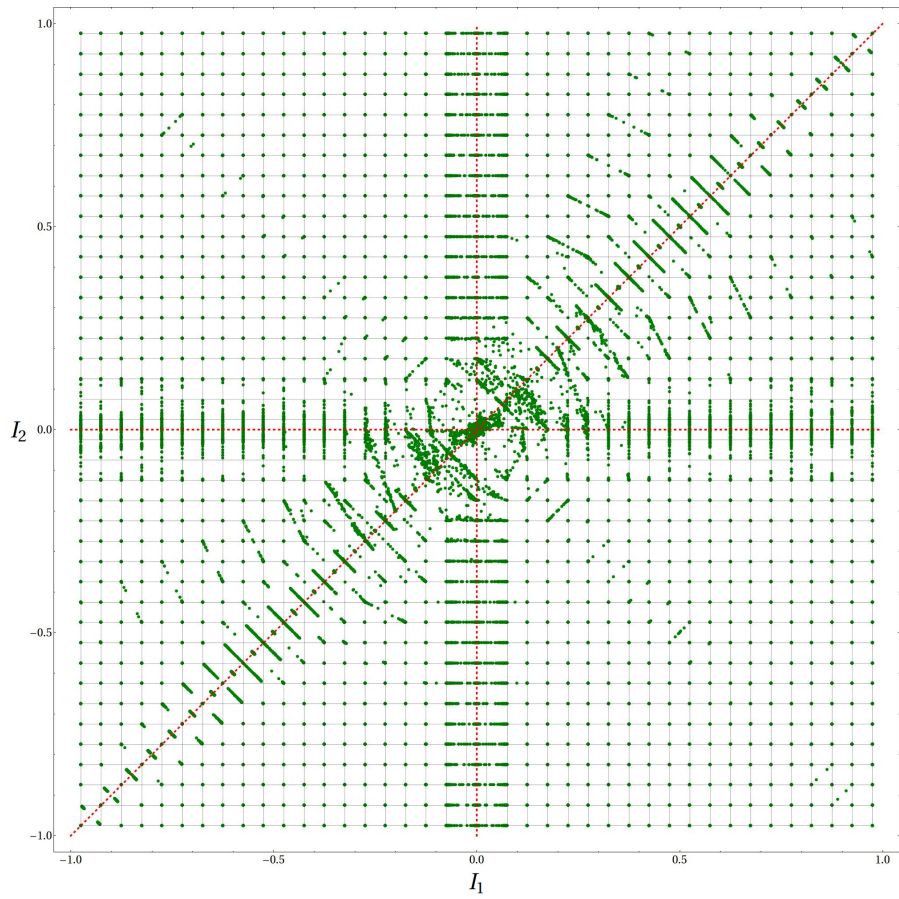


Figure 4.3: the same as figure 4.2, but with $\epsilon = 10^{-3}$.

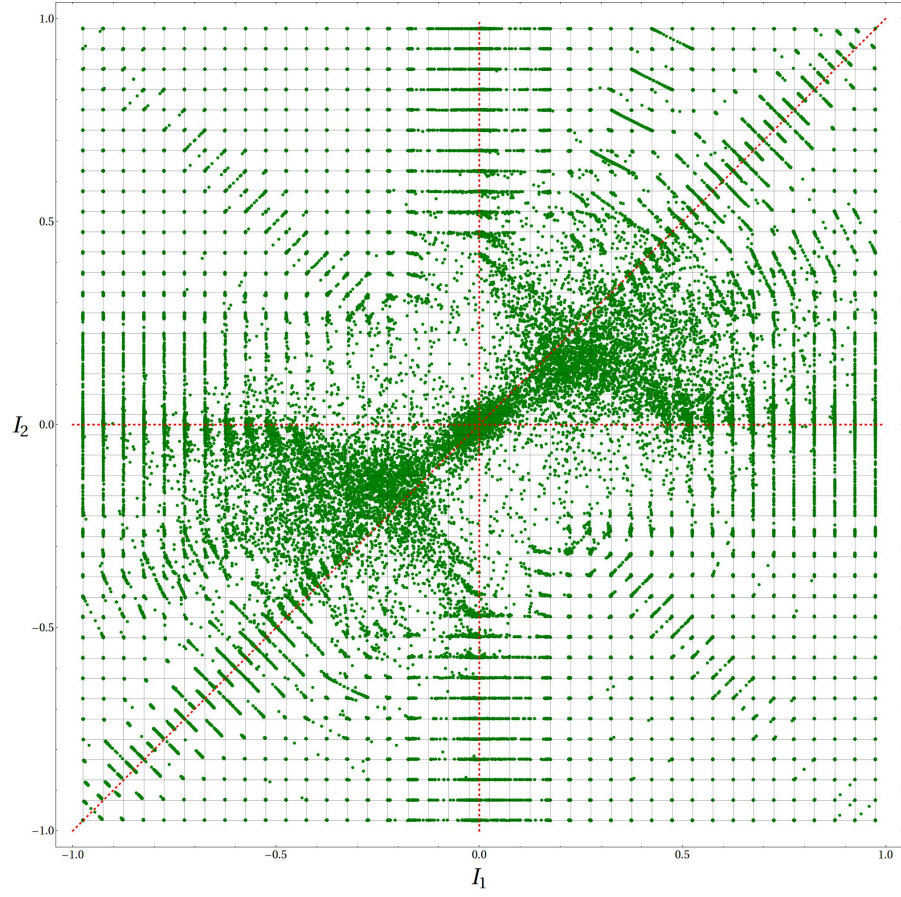


Figure 4.4: the same as figure 4.2 and 4.3, but with $\epsilon = 10^{-2}$.

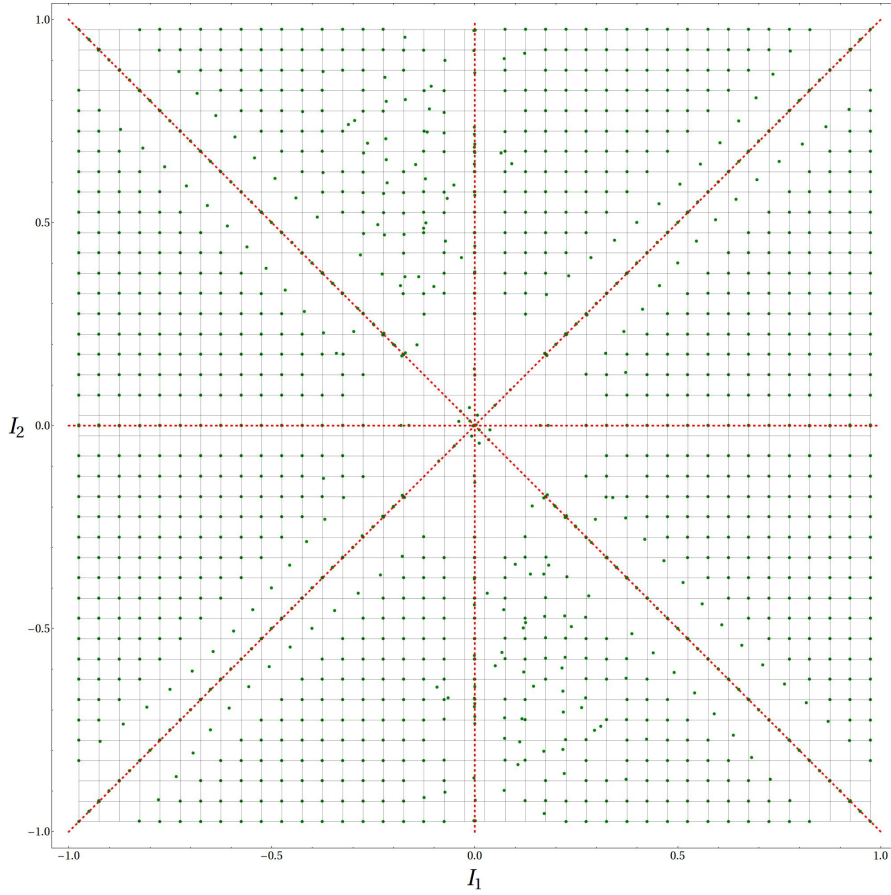


Figure 4.5: expectation values of $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$, plotted as points $I_\psi \in \mathbb{R}^2$, of every eigenstate $|\psi\rangle$ of the quantization of a perturbed Liouville integrable Hamiltonian $h_\epsilon = h_0 + \epsilon f$ defined on $\mathbb{S}^2 \times \mathbb{S}^2$. In this case $n = 40$ and the perturbative parameter is set equal to $\epsilon = 10^{-4}$. The integrable part is $h_0 = \frac{1}{2}(x_{13}^2 + x_{23}^2)$, while the perturbation is $f = x_1 \cdot x_2 + x_{11} - 2x_{21} + 3x_{11}x_{22}$ (and thus it contains 4 resonances: $(1, 0)$, $(0, 1)$, $(1, -1)$ and $(1, 1)$). As reference the grid defined by the spectrum of $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$ is also plotted with the resonances present in the Hamiltonian (red thick dashed lines). In this case in the proximity of every non-resonant point of the grid lies the image of only one eigenstate of the Hamiltonian.

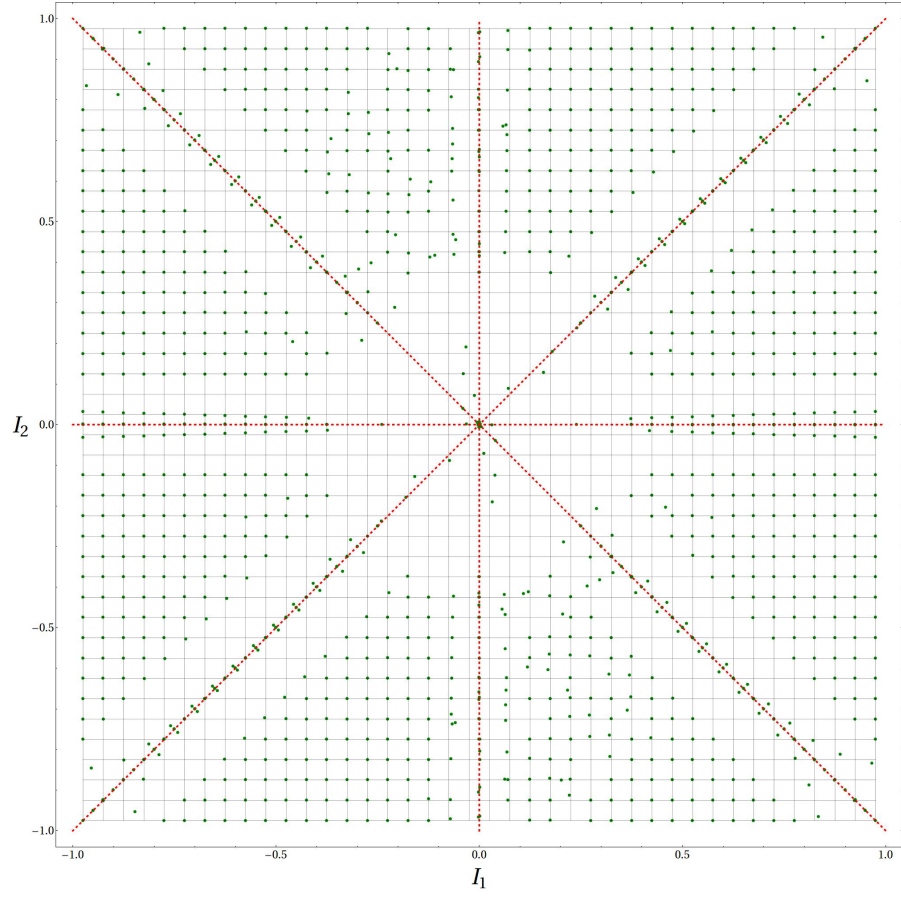


Figure 4.6: the same as figure 4.5, but with $\epsilon = 10^{-3}$.

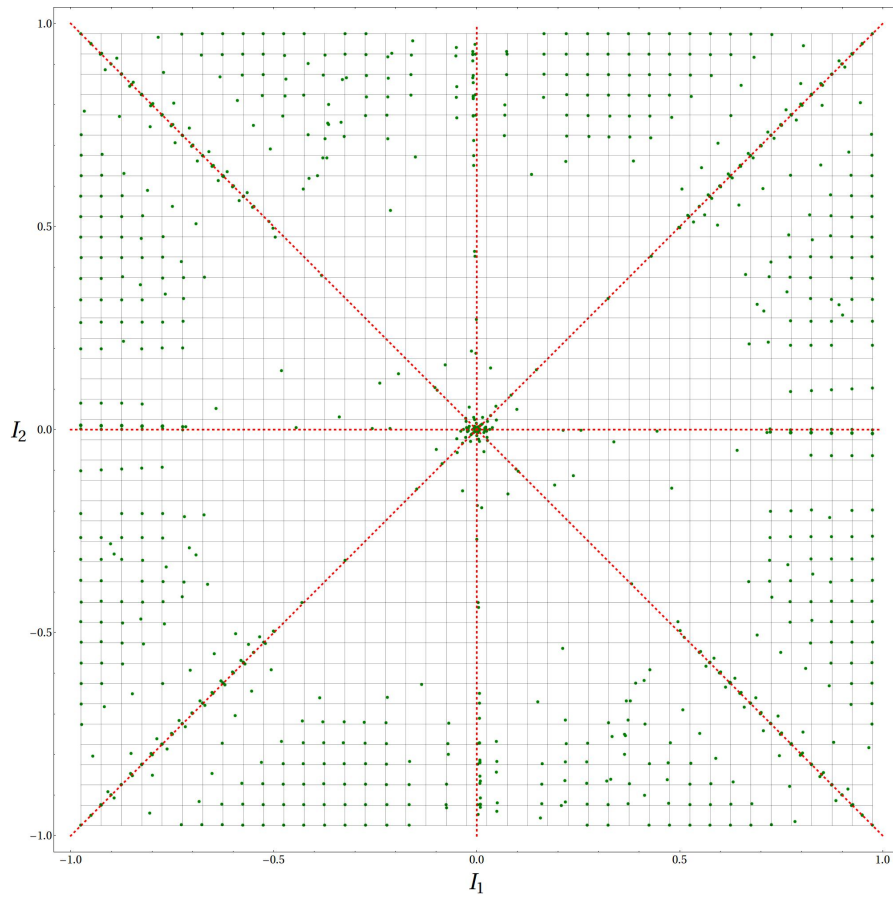


Figure 4.7: the same as figures 4.5 and 4.6, but with $\epsilon = 10^{-2}$.

4.2 Analysis of coherent states and their evolution through Husimi distributions

4.2.1 Introduction

We will now investigate the peculiarities of the perturbed dynamics related to the superintegrable character of the unperturbed Hamiltonian in contrast to the Liouville integrable case. This poses two questions:

1. Which states should be taken into consideration?
2. Which characteristics of the considered states should be analyzed?

We have already met a particular class of elements of $\mathcal{H}$, namely coherent states, that are in a one-to-one relation with the points of $\mathcal{M}$ and can be interpreted as the quantum states that are the most localized in the phase space around the corresponding point. Since a characteristic of perturbed superintegrable classical systems is the presence of chaotic motions confined to specific zones of the phase space, it is natural to compare coherent states relative to points that classically belong to these zones with those that are regular.

The evolution, under $\mathbf{h}_\epsilon$, of the coherent state relative to the point $p \in \mathcal{M}$, can be interpreted as the quantum counterpart of the motion of p under the flow of the classical Hamiltonian h_ϵ . To compare the quantum evolution with the classical analog we need some kind of representation of the states of $\mathcal{H}$ in the phase space. This can be achieved by considering their Husimi distribution, which is a real-valued function over $\mathcal{M}$.

4.2.2 Husimi distribution on $\mathcal{M}$

Given any (normalized) $|\psi\rangle \in \mathcal{H}$, we recall that its Husimi distribution $H_{|\psi\rangle}$ is the covariant symbol of the operator $|\psi\rangle\langle\psi|$. From Proposition 7, proved in the previous chapter, taking $\mathbf{f} = |\psi\rangle\langle\psi|$ and $\mathbf{g} = \mathbf{h}_\epsilon$, since the contravariant symbol $\check{h}_\epsilon$ of $\mathbf{h}_\epsilon$ is equal to $h_\epsilon + O(\hbar)$, we have that

$$\frac{dH_{|\psi(t)\rangle}}{dt} = \{H_{|\psi(t)\rangle}, h_\epsilon\} + O(\hbar)$$

where $H_{|\psi(t)\rangle}$ is the Husimi distribution of $|\psi(t)\rangle$ which is the evolution, under $\mathbf{h}_\epsilon$, at time t of $|\psi\rangle$. This shows that the Husimi distribution of a quantum state evolves (under the dynamics of the quantum Hamiltonian) approximately as a classical function under the flow of the classical Hamiltonian h_ϵ .

As previously done we identify $\mathbb{S}^2$ (with the north pole removed) by $\mathbb{C}$ and a point of coordinates $(x_1, x_2, x_3) \in \mathbb{S}^2$, $x_3 \neq 1$, with $z = (1 - x_3)^{-1}(x_1 + ix_2)$.

When this is not a source of confusion, we will indicate the coherent state relative to $z \in \mathbb{S}^2$ by $|z\rangle \in \mathcal{H}'$, with $\mathcal{H}'$ being the n -dimensional Hilbert space obtained quantizing the sphere. $|z\rangle$ is defined as:

$$|z\rangle = \sum_{k=0}^{n-1} \sqrt{\binom{n-1}{k}} \bar{z}^k |k\rangle$$

where $(|k\rangle)_{k=0, \dots, n-1}$ are the elements of the canonical base of $\mathcal{H}'$, i.e.

$$\mathbf{x}_3 |k\rangle = \lambda_k |k\rangle.$$

If $|\psi\rangle \in \mathcal{H}'$ is normalized, then the Husimi distribution of $|\psi\rangle$ is the function $H_{|\psi\rangle}$ defined on the sphere by:

$$H_{|\psi\rangle}(z) = \frac{|\langle z | \psi \rangle|^2}{\langle z | z \rangle}.$$

For any $\xi \in \mathbb{S}^2$, $H_{|\xi\rangle}$ attains its maximum in ξ and its level curves are circles around ξ which have smaller radius as $\hbar$ decreases. This is proved in the following

Proposition. *Let $\dim \mathcal{H}' = n$, there exists a function $H^{(n)} : [0, \pi] \rightarrow \mathbb{R}$ such that, for any pair of points $z, \xi \in \mathbb{S}^2$,*

$$H_{|\xi\rangle}(z) = H^{(n)}(\theta(\xi, z))$$

where $\theta(\xi, z)$ is the angle between ξ and z . An explicit formula of H^n is given by:

$$H^{(n)}(\theta) = \cos^{2(n-1)} \frac{\theta}{2}.$$

Proof. It is immediate to verify that if $g \in SO(3)$ then

$$|g\xi\rangle = U(g)|\xi\rangle$$

where $U(g)$ is an unitary matrix (independent from ξ). The map $g \mapsto U(g)$ defines a unitary representation of $SO(3)$ on $\mathcal{H}'$ and so

$$\langle z | g\xi \rangle = \langle z | U(g)|\xi\rangle = \langle z | U^\dagger(g^{-1})|\xi\rangle = \langle g^{-1}z | \xi \rangle$$

for every g, z , and ξ .

In particular $\theta(\xi, z) = \theta(\xi', z')$ for some $\xi, \xi', z, z' \in \mathbb{S}^2$ if and only if there exists $g \in SO(3)$ such that $g\xi = \xi'$ and $gz = z'$ and so

$$H_{|\xi'\rangle}(z') = H_{|g\xi\rangle}(gz) = H_{|\xi\rangle}(g^{-1}gz) = H_{|\xi\rangle}(z).$$

This implies that $H_{|\xi\rangle}(z) = H^{(n)}(\theta(\xi, z))$ for some function $H^{(n)}$. The explicit expression of $H^{(n)}$ is easily recovered considering the case $z = 0$. $\square$

$H^{(n)}$, for some values of n , is represented in figure (4.8).

The definition of coherent states and Husimi distribution of a state can be extended to $\mathcal{M} = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ and $\mathcal{H} = \mathcal{H}' \otimes \mathcal{H}' \otimes \mathcal{H}'$ in a straightforward way by tensor product.

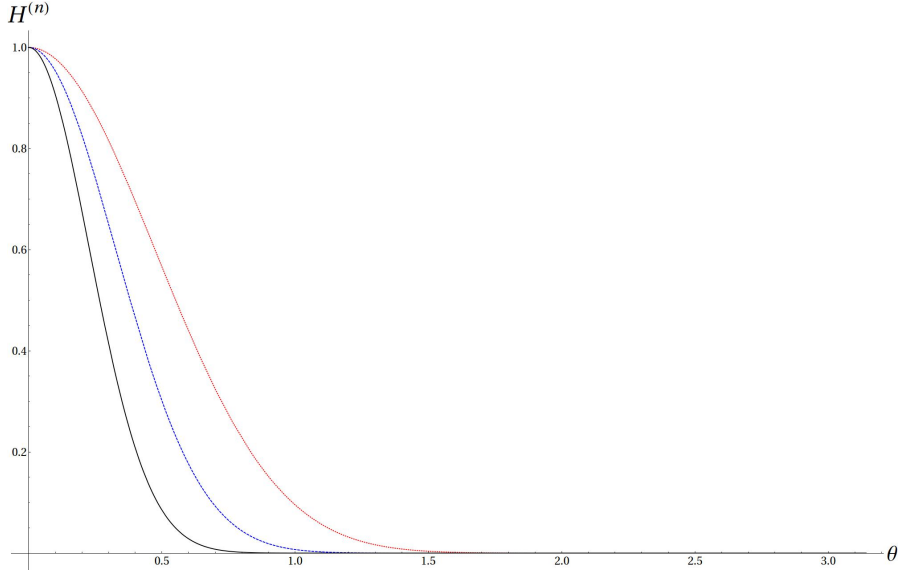


Figure 4.8: dependence of the Husimi distribution of a coherent state ψ from the angle taken from the point on which ψ is based. The values of n for the figures are 10, 20 and 40 (respectively represented by the colour red, blue and black).

4.2.3 Time evolution of Husimi distributions

Any state $|\psi\rangle \in \mathcal{H}$ can be expressed uniquely as

$$|\psi\rangle = \sum_{i=1}^{n^3} \psi(E_i) |E_i\rangle$$

where $(|E_i\rangle)_{i=1,\dots,n^3}$ is an orthonormal basis made of stationary states for $\mathbf{h}_\epsilon$, with $|E_i\rangle$ relative to the eigenvalue $E_i \in \mathbb{R}$ (as remarked before we consider only the case $E_i \neq E_j$ for $i \neq j$). Schrödinger equation determines the evolution, under the dynamics of $\mathbf{h}_\epsilon$, of $|\psi\rangle$ which is thus given by

$$|\psi(t)\rangle = \sum_{i=1}^{n^3} \exp\left(\frac{E_i t}{i\hbar}\right) \psi(E_i) |E_i\rangle$$

and the Husimi distribution of $|\psi(t)\rangle$ is thus defined by

$$H_{|\psi(t)\rangle}(z) = \sum_{i,j=1}^{n^3} \exp\left(\frac{(E_i - E_j)t}{i\hbar}\right) \psi(E_i) \bar{\psi}(E_j) \frac{\langle z | E_i \rangle \langle E_j | z \rangle}{\langle z | z \rangle}. \quad (4.1)$$

Now we face with the non trivial problem of extracting, and visualizing in a meaningful way, information from $H_{|\psi(t)\rangle}(z)$. In the first place we need a way to represent the history of the evolution of the Husimi distribution for a state $|\psi\rangle$. This

means that we need to define a time independent quantity which encodes information on the values that $H_{|\psi(t)\rangle}$ attains on the points of $\mathcal{M}$. In some sense this is the quantum equivalent of the classical orbit of a point $p \in \mathcal{M}$. However, while the orbit of a point is a subset of $\mathcal{M}$, the “quantum orbit” defined by the evolution of a state would rather be a function defined on $\mathcal{M}$.

In the second place we need to visualize this function and in particular we want to highlight its behaviour on the third sphere. In the classical case this is done simply by projecting the orbit on the third sphere. The quantum case is more complicate, since there is no canonical way to “project” a function defined on $\mathcal{M}$ so as to obtain a function defined on $\mathbb{S}^2$.

4.2.4 Time independent representation of the evolution

There exist several strategy that might allow to overcome the aforementioned difficulties. For instance it is possible to consider, provided it exists, the time average $\bar{H}_{|\psi\rangle}$ of $H_{|\psi(t)\rangle}$, which is the function defined on $\mathcal{M}$ by

$$\bar{H}_{|\psi\rangle}(z) := \lim_{t \rightarrow \infty} \frac{1}{2t} \int_{-t}^t H_{|\psi(s)\rangle}(z) ds$$

or to consider the function $\hat{H}_{|\psi\rangle}$ defined on $\mathcal{M}$ by

$$\hat{H}_{|\psi\rangle}(z) = \sup_{t \in \mathbb{R}} H_{|\psi(t)\rangle}(z).$$

Since $H_{|\psi(t)\rangle}(z)$ is the probability of finding, after a measurement, the state $|\psi(t)\rangle$ in the coherent state localized around z and so it measures, in some sense, the “localization” of $|\psi(t)\rangle$ in z , $\bar{H}_{|\psi\rangle}(z)$ can be interpreted as the average localization of $|\psi\rangle$ in z during its evolution, in this way $\bar{H}_{|\psi\rangle}$ attains large values on the points on which $|\psi\rangle$ is more likely to be found in the considered time interval.

On the other hand we might want to consider, for a fixed $0 < c < 1$ and for every $t \in \mathbb{R}$, the set $S_c(t) := (H_{|\psi(t)\rangle})^{-1}([c, 1])$. This is motivated by the fact that, if $|\psi\rangle$ is a coherent state, as $\hbar$ tends to zero, the corresponding $S_c(0)$ shrinks to the point on which $|\psi\rangle$ is based. In this way the union of the $S_c(t)$, as t varies over $\mathbb{R}$, can be interpreted as the “orbit” of $|\psi\rangle$. It is easily seen that this union is exactly the set $(\hat{H}_{|\psi\rangle})^{-1}([c, 1])$.

Both methods have the drawback that they do not take into account the fact that the integral of $H_{|\psi\rangle}$ over $\mathcal{M}$ is finite and independent of $|\psi\rangle$. In this way if $H_{|\psi(t)\rangle}$, for some t , is localized around some point, then its total variation on $\mathcal{M}$ will be large, while if it is uniformly spread over an extended region, as we expect for the chaotic case, it is characterized by a small total variation. Therefore both $\bar{H}_{|\psi\rangle}$ and $\hat{H}_{|\psi\rangle}$ might highlight these points on which $H_{|\psi(t)\rangle}$ has been localized for

some time t . A way to overcome this issue might be to consider the rescaling $H'_{|\psi(t)\rangle}$ of $H_{|\psi(t)\rangle}$ whose range is the entire interval $[0, 1]$. Using the fact that $H_{|\psi(t)\rangle}$ is defined (modulo a rescaling factor) as the squared absolute value of a polynomial, we have that $H_{|\psi(t)\rangle} \geq 0$ and there exists $z \in \mathcal{M}$ such that $H_{|\psi(t)\rangle}(z) = 0$. In this way:

$$H'_{|\psi(t)\rangle}(z) = \frac{H_{|\psi(t)\rangle}(z)}{\sup_{\mathcal{M}} H_{|\psi(t)\rangle}}.$$

From this it is possible to compute $\hat{H}'_{|\psi(t)\rangle}$ and $\bar{H}'_{|\psi(t)\rangle}$ as before.

Before investigating these functions we need some preliminary definitions.

Definition. The spectrum $(E_i)_{i=1, \dots, n^3}$ of $\mathbf{h}_\epsilon$ is non-resonant if for every collection of n^3 integer numbers $c_1, \dots, c_{n^3}$, non identically vanishing for all $i = 1, \dots, n^3$, we have that:

$$\sum_{i=1}^{n^3} c_i E_i \neq 0$$

Definition. If $|\psi\rangle \in \mathcal{H}$, $|\psi\rangle = \sum_{i=1}^{n^3} \psi(E_i) |E_i\rangle$ and $\alpha \in \mathbb{T}^{n^3}$, where we identify $\mathbb{T}^{n^3}$ with the subset of $\mathbb{C}^{n^3}$ defined by

$$\mathbb{T}^{n^3} = \{\alpha = (\alpha_1, \dots, \alpha_{n^3}) \in \mathbb{C}^{n^3} \mid |\alpha_i| = 1 \ \forall i = 1, \dots, n^3\},$$

we define the state $|\psi_\alpha\rangle \in \mathcal{H}$ as

$$|\psi_\alpha\rangle := \sum_{i=1}^{n^3} \alpha_i \psi(E_i) |E_i\rangle.$$

Remark. Since $|\psi_\alpha\rangle$ is a state, it is possible to consider its Husimi distribution $H_{|\psi_\alpha\rangle}$. In particular, fixed $z \in \mathcal{M}$, we can view $\alpha \mapsto H_{|\psi_\alpha\rangle}(z)$ as a real valued map on the n^3 -dimensional torus (and the same obviously goes for $H_{|\psi_\alpha\rangle}(z')$). It should moreover be observed that if we define $\alpha(t) = (\alpha_1(t), \dots, \alpha_{n^3}(t))$, for $t \in \mathbb{R}$, as

$$\alpha_i(t) = \exp\left(\frac{E_i t}{i\hbar}\right)$$

we have that $|\psi(t)\rangle = |\psi_{\alpha(t)}\rangle$. If we identify $\mathbb{T}^{n^3}$ with $\mathbb{R}^{n^3}/2\pi\mathbb{Z}^{n^3}$ with the map (using the fact that $\exp(ix) = \exp(ix')$ iff $x - x' \in 2\pi\mathbb{Z}$):

$$\mathbb{R}^{n^3} \ni (x_1, \dots, x_{n^3}) \mapsto \left(\exp\left(\frac{x_1}{i\hbar}\right), \dots, \exp\left(\frac{x_{n^3}}{i\hbar}\right)\right) \in \mathbb{T}^{n^3}$$

then we have that $t \mapsto \alpha(t)$ is a linear motion on the torus. △

Proposition 9. For any $|\psi\rangle \in \mathcal{H}$, $|\psi\rangle = \sum_{i=1}^{n^3} \psi(E_i) |E_i\rangle$, the functions $\bar{H}_{|\psi\rangle}$, $\bar{H}'_{|\psi\rangle}$, $\hat{H}_{|\psi\rangle}$ and $\hat{H}'_{|\psi\rangle}$ are well defined on $\mathcal{M}$ and, provided that the spectrum of $\mathbf{h}_\epsilon$ is non-degenerate,

$$\bar{H}_{|\psi\rangle} = \sum_{i=1}^{n^3} |\psi(E_i)|^2 H_{|E_i\rangle}. \quad (4.2)$$

If moreover the spectrum of $\mathbf{h}_\epsilon$ is non resonant, then

$$\hat{H}_{|\psi\rangle} = \left(\sum_{i=1}^{n^3} \sqrt{|\psi(E)|^2 H_{|E_i\rangle}} \right)^2. \quad (4.3)$$

Under the non resonance condition on the energies it is possible to express, for every $z \in \mathcal{M}$, $\bar{H}'_{|\psi\rangle}(z)$ as

$$\bar{H}'_{|\psi\rangle}(z) = \frac{1}{(2\pi\iota)^{n^3}} \int_{\mathbb{T}^{n^3}} H'_{|\psi_\alpha\rangle}(z) d\alpha_1 \wedge \dots \wedge d\alpha_{n^3} \quad (4.4)$$

and $\hat{H}'_{|\psi\rangle}(z)$ as

$$\hat{H}'_{|\psi\rangle}(z) = \sup_{\alpha \in \mathbb{T}^{n^3}} H'_{|\psi_\alpha\rangle}(z) \quad (4.5)$$

Proof. Fixed $z \in \mathcal{M}$, both $t \mapsto H_{|\psi(t)\rangle}(z)$ and $t \mapsto H'_{|\psi(t)\rangle}(z)$ are uniformly bounded on $\mathbb{R}$, ensuring the existence of the supremum and of the average. From equation (4.1) it is clear that the average over time removes the terms in the sum for which $E_i \neq E_j$, leaving us with the desired expression for $\bar{H}_\psi$.

For $t \in \mathbb{R}$, consider the function $t \mapsto \alpha(t) \in \mathbb{T}^{n^3}$ defined in the remark above. Since the non resonance condition holds, the points on the curve defined by $\alpha(t)$ form a dense set in $\mathbb{T}^{n^3}$. Therefore the sup taken in t over $\mathbb{R}$ of $H'_{|\psi_{\alpha(t)}\rangle}(z)$ is equal to the sup taken in α over $\mathbb{T}^{n^3}$ of $H'_{|\psi_\alpha\rangle}(z)$ (since the dependence from α of $H'_{|\psi_\alpha\rangle}(z)$ is continuous), allowing us to recover the formula for $\hat{H}'_{|\psi\rangle}$. The expression for $\bar{H}'_{|\psi\rangle}$ follows from the well known fact that the time average of a function over a non resonant linear orbit on the torus is equal to the space average of the same function over the torus. To obtain the explicit formula for $\hat{H}_{|\psi\rangle}$, fixed $z \in \mathcal{M}$, we define $\theta(z) \in \mathbb{T}^{n^3}$ by

$$\psi(E_i) \langle z | E_i \rangle = \theta_i(z) |\psi(E_i) \langle z | E_i \rangle|,$$

we have that

$$H_{|\psi(t)\rangle}(z) = \left| \sum_{i=1}^{n^3} \theta_i(z) \alpha_i(t) \sqrt{|\psi(E_i)|^2 H_{|E_i\rangle}(z)} \right|^2.$$

Using the dense character of the curve $\alpha(t)$ in $\mathbb{T}^{n^3}$ it is possible to find $t \in \mathbb{R}$ such that each $\alpha_i(t)$ is arbitrarily close to $\theta_i(z)$ and so

$$\hat{H}_{|\psi\rangle}(z) \geq \left(\sum_{i=1}^{n^3} \sqrt{|\psi(E_i)|^2 H_{|E\rangle}(z)} \right)^2$$

but it is obvious that this bound is optimal, giving the desired result. $\square$

As well as the non-degeneracy condition of the spectrum of $\mathbf{h}_\epsilon$, the stronger non-resonance condition should be checked case by case in the numerical experiments¹ (it should be stressed, however, that the expression of $\bar{H}_\psi$ does not require the non-resonance condition).

4.2.5 Visualizing the time evolution on the third sphere

We want now to analyze the behaviour of $H_{|\psi(t)\rangle}$ on the third sphere. As mentioned above there is no canonical way to “project” a function defined on $\mathcal{M}$ to recover a function on the third sphere. There are, however, several possibility for doing so.

Each sphere is endowed with a natural measure μ , in this way it is possible to obtain, from any continue function f on $\mathcal{M}$ taking values on $\mathbb{R}$, a function $\pi_3(f)$ defined on $\mathbb{S}^2$, by integration on the first two sphere. It is possible, for example, to take $\bar{H}_{|\psi\rangle}$, for some state $|\psi\rangle$, as the function to integrate. Under the non degeneracy condition, using the explicit formula (4.2), $\pi_3(\bar{H}_{|\psi\rangle})$ can be computed, by linearity, once the $\pi_3(H_{|E_i\rangle})$, for $i = 1, \dots, n^3$, are known.

Definition. Let $|\psi\rangle \in \mathcal{H}$, $|\phi_i\rangle_{i=1,\dots,n^2}$ be an orthonormal basis for the Hilbert space obtained by geometric quantization of the product of the first and the second sphere and $|\phi'_j\rangle_{j=1,\dots,n}$ be an orthonormal basis for the Hilbert space obtained by quantization of the third sphere (in this way $|\phi_i\rangle \otimes |\phi'_j\rangle$, for $i = 1, \dots, n^2$ and $j = 1, \dots, n$ defines a orthonormal basis for $\mathcal{H}$). If $|\psi\rangle \in \mathcal{H}$ is expressed as

$$|\psi\rangle = \sum_{i,j} \psi_{i,j} |\phi_i\rangle \otimes |\phi'_j\rangle$$

we define the quantum observable $A(|\psi\rangle)$ on $\mathcal{H}$ as

$$A(|\psi\rangle) := \sum_{j_1, j_2=1}^n \left(\sum_{i=1}^{n^2} \psi_{i,j_2} \bar{\psi}_{i,j_1} \right) |\phi'_{j_1}\rangle \langle \phi'_{j_2}|$$

¹Since the precision in numerical computations is finite, it is not possible to tell if a computed spectrum is non resonant. However if we consider the spectrum of $\mathbf{h}_\epsilon$, $(E_i)_{i=1,\dots,n^3}$, as an element of $\mathbb{R}^{n^3}$ we have that the subset where the non degeneracy condition does not hold, while dense, has measure 0. Moreover if the non-resonance condition is verified up to a high numerical precision there exist theorems ensuring that the difference between the time average and the space average on the torus is negligible.

Remark. Clearly the operator $A(|\psi\rangle)$ is independent of the choice of $|\phi_i\rangle$ and $|\phi'_j\rangle$. $\triangle$

Proposition. For any state $|\psi\rangle \in \mathcal{H}$ and for every $z_3 \in \mathbb{S}^2$,

$$\pi_3(H_{|\psi\rangle})(z_3) = (\langle z_3 | z_3 \rangle)^{-1} \langle z_3 | A(|\psi\rangle) | z_3 \rangle$$

Proof. We observe that $H_{|\psi\rangle}(z)$ can be viewed as the expectation value of the projection operator $\frac{|z\rangle\langle z|}{\langle z | z \rangle}$ on the state $|\psi\rangle$. Moreover, if $z = (z_1, z_2, z_3)$, with $z_i \in \mathbb{S}^2$, we have that

$$\frac{|z\rangle\langle z|}{\langle z | z \rangle} = \bigotimes_{i=1}^3 \frac{|z_i\rangle\langle z_i|}{\langle z_i | z_i \rangle}$$

where $|z_i\rangle \in \mathcal{H}_i$ represents the coherent state centred in z_i on the i^{th} -sphere (which is quantized using the Hilbert space $\mathcal{H}_i$ and we have that $\mathcal{H} = \bigotimes_{i=1}^3 \mathcal{H}_i$).

If we integrate this expression over the first two spheres, recalling the properties of coherent states, we obtain the operator

$$\mathbb{I}_{\mathcal{H}_1} \otimes \mathbb{I}_{\mathcal{H}_2} \otimes \frac{|z_3\rangle\langle z_3|}{\langle z_3 | z_3 \rangle}.$$

The expectation value of this observable in the state $|\psi\rangle$ is exactly the expectation value of $A(|\psi\rangle)$ in the state $|z_3\rangle$, allowing us to conclude. $\square$

In this way, from expression (4.2) we have, using the linearity of π_3 , that

$$\pi_3(\bar{H}_{|\psi\rangle}) = \sum_{i=1}^{n^3} |\psi(E_i)|^2 \pi_3(H_{|E_i\rangle})(z_3) = \sum_{i=1}^{n^3} |\psi(E_i)|^2 \frac{\langle z_3 | A(|E_i\rangle) | z_3 \rangle}{\langle z_3 | z_3 \rangle}.$$

Remark. In the same way we may as well compute $\pi_3(\hat{H}_{|\psi\rangle})$, $\pi_3(\bar{H}'_{|\psi\rangle})$ and $\pi_3(\hat{H}'_{|\psi\rangle})$. However no explicit simple formula is known for these cases. On the other hand we may consider the time dependent function $\pi_3(H_{|\psi(t)\rangle})$ on $\mathbb{S}^2$ and its time average

$$\overline{\pi_3(H_{|\psi\rangle})}(z) := \lim_{t \rightarrow \infty} \frac{1}{2t} \int_{-t}^t \pi_3(H_{|\psi(s)\rangle})(z) ds$$

for every $z \in \mathbb{S}^2$. It is easy to see that $\overline{\pi_3(H_{|\psi\rangle})} = \pi_3(\bar{H}_{|\psi\rangle})$. Alternatively we can define $\widehat{\pi_3(H_{|\psi\rangle})}(z) = \sup_t \pi_3(H_{|\psi(t)\rangle})(z)$. When the non-resonance condition is met we have:

$$\widehat{\pi_3(H_{|\psi\rangle})}(z) = \left(\sum_{i=1}^{n^3} \sqrt{|\psi(E_i)|^2 \pi_3(H_{|E_i\rangle})(z)} \right)^2.$$

In a completely analogous way we can define the functions $\widehat{\pi_3(H_{|\psi\rangle})}'$ and $\overline{\pi_3(H_{|\psi\rangle})}'$ using a rescaled version on $\mathbb{S}^2$ of $\pi_3(H_{|\psi\rangle})$, $\pi_3(H_{|\psi\rangle})'$. We have, under the non resonance condition, that

$$\widehat{\pi_3(H_{|\psi\rangle})}' = \frac{1}{(2\pi\iota)^{n^3}} \int_{\mathbb{T}^{n^3}} \pi_3(H_{|\psi_\alpha\rangle})'(z) d\alpha_1 \wedge \dots \wedge d\alpha_{n^3}$$

and

$$\overline{\pi_3(H_{|\psi\rangle})}' = \frac{1}{(2\pi\iota)^{n^3}} \int_{\mathbb{T}^{n^3}} \pi_3(H_{|\psi_\alpha\rangle})'(z) d\alpha_1 \wedge \dots \wedge d\alpha_{n^3}.$$

Those formulae are proved in the same way as the one given in proposition (9). $\triangle$

Another way to recover a function on $\mathbb{S}^2$, from a function f defined on $\mathcal{M}$, consists in fixing a continuous function $\phi : \mathbb{S}^2 \rightarrow \mathcal{M}$ such that $\phi_3 = id_{\mathbb{S}^2}$, where ϕ_3 is the projection of ϕ on the third sphere. Then, for every $z \in \mathbb{S}^2$, consider the function $\tau_\phi(f)$ defined on $\mathbb{S}^2$ by

$$\tau_\phi(f)(z) := f(\phi(z))$$

A choice for ϕ could be, for example, $\phi(z) = (z_1, z_2, z)$ for $z_1, z_2 \in \mathbb{S}^2$ fixed.

We can either consider the case $f = \bar{H}_{|\psi\rangle}, \bar{H}_{|\psi\rangle}$ (or one of their rescaled equivalent) for any state $|\psi\rangle \in \mathcal{H}$, or the time dependent function, defined on $\mathbb{S}^2$, $\tau_\phi(H_{|\psi(t)\rangle})$ and then define, in a completely straightforward way, $\tau_\phi(\widehat{H_{|\psi\rangle}})$ and $\tau_\phi(H_{|\psi\rangle})$ (and the same for the rescaled versions).

In this case τ_ϕ commutes with both of the operations of time average or sup when the non rescaled version are considered and the explicit formulae, when the required conditions of spectrum of $\mathbf{h}_\epsilon$ are met, are the same as (4.3) and (4.2) with $H_{|E_i\rangle}$ replaced by $\tau_\phi(H_{|E_i\rangle})$. For the rescaled case we can keep the formulae (4.5) and (4.4), with $H_{|\psi_\alpha\rangle}$ replaced by $\tau_\phi(H_{|\psi_\alpha\rangle})$ provided that τ_ϕ is applied *before* rescaling (and taking the subsequent average or sup over time).

If we assume that the quantum dynamics will closely follow the classical one, when considering large values of n it is probably sufficient to use the first method (i.e. integrating over the first two sphere) to obtain good results. When however n is small the domain on which the Husimi distribution of a coherent state is sensibly different from zero is *large* and contains points with different values of the fast action and of the resonant normal form which determine the shapes of the chaotic domains and of the adiabatic invariant level curves on the third sphere. In this case it is better to use the last method.

A resonant point $p_0 \in \mathcal{M}$ determines a value for the fast action $F(p_0)$ and of the resonant normal form $\epsilon E(p_0)$. Fixed now a point on the third sphere z it is possible to find $\phi(z) \in \mathcal{M}$ such that $F(\phi(z)) = F(p_0)$, $E(\phi(z)) = E(p_0)$ and $\phi(z)_3 = z$. ϕ can be used to define τ_ϕ .

It should be noted that ϕ is not uniquely defined. In our numerical computation we kept the fast angle equal to zero and the slow action as small (and positive) as possible (thus fixing the choice for the slow angle).

4.3 Results

Fixed the values for ϵ and n we proceed to compute $\tau_\phi(\bar{H}_{|\psi\rangle})$ for several coherent states $|\psi\rangle$.

In figures 4.9, 4.10, 4.11, 4.12, are represented the quantum analogues of the classical cases analyzed in the previous chapter. The result are somewhat promising but surely not definitive. In particular it can be seen that the Husimi distribution seems to spread along the level curves of the adiabatic invariant, however it is not clear if in the chaotic case it also spread over the chaotic region. This is mainly (in our opinion) for two reason:

1. it is not clear which threshold, under which the Husimi distribution should be considered equal to zero, is to be considered. If it is too large the Husimi distribution will look extremely localized, regardless to the initial condition while, on the other hand, it can spread all over the sphere when it is small;
2. as already mentioned, when n is small, the Husimi distribution of a coherent state is not localized. In this way even a regular evolution can cover a domain whose extension is comparable with the one of the chaotic zone.

Being able to perform the computation with very large values of n would probably be helpful in overcoming these difficulties, however we will see in the next section how this may be an unrealistic task, from the computational point of view.

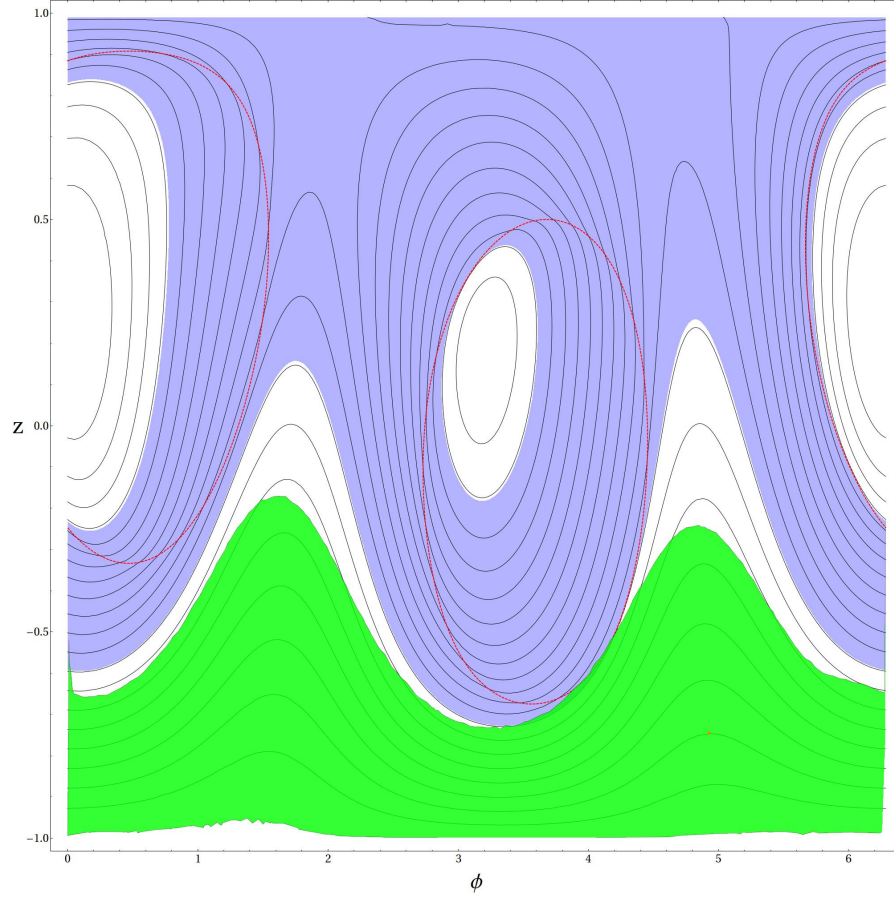


Figure 4.9: contour plots of $\tau_\phi(\bar{H}_{|\psi\rangle})$ on $\mathbb{S}^2$ (cylindrical coordinates). In this case $n = 40$ and $\epsilon = 10^{-3}$, the initial conditions are the same as the classical example in figure 2.5, which represents a regular resonant motion. The green area represents the points where $\tau_\phi(\bar{H}_{|\psi\rangle})$ is larger than a fixed threshold.

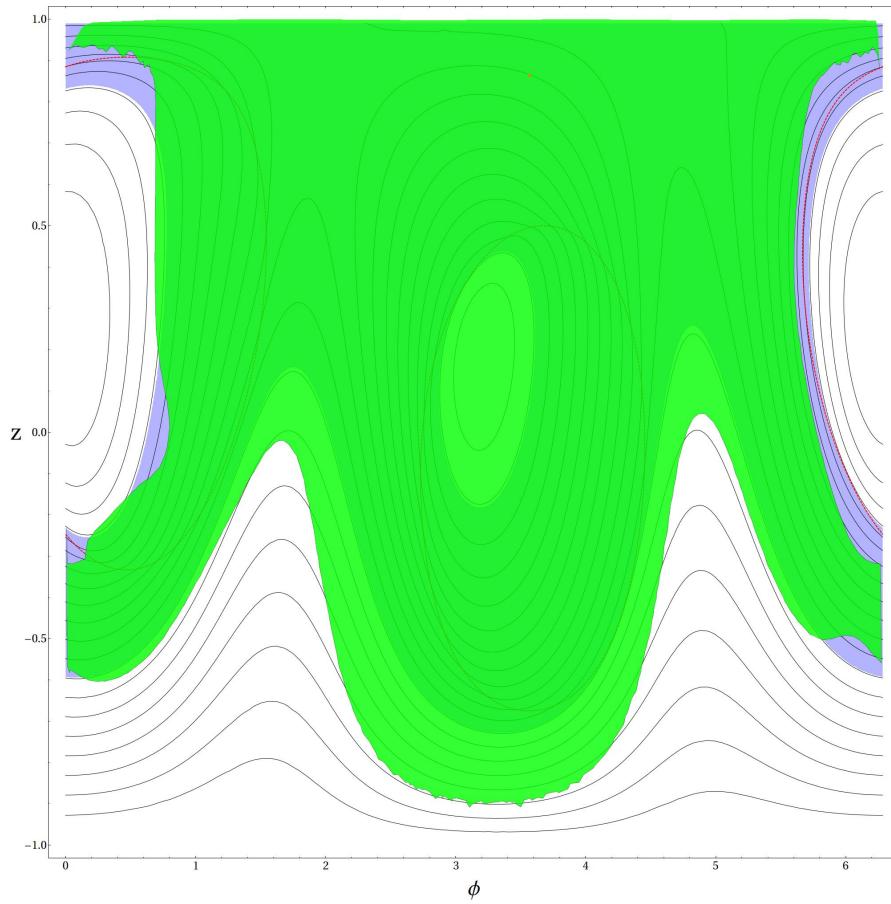


Figure 4.10: the same as 4.9, but with initial conditions are the same as in figure 2.6, which represents a chaotic resonant motion.

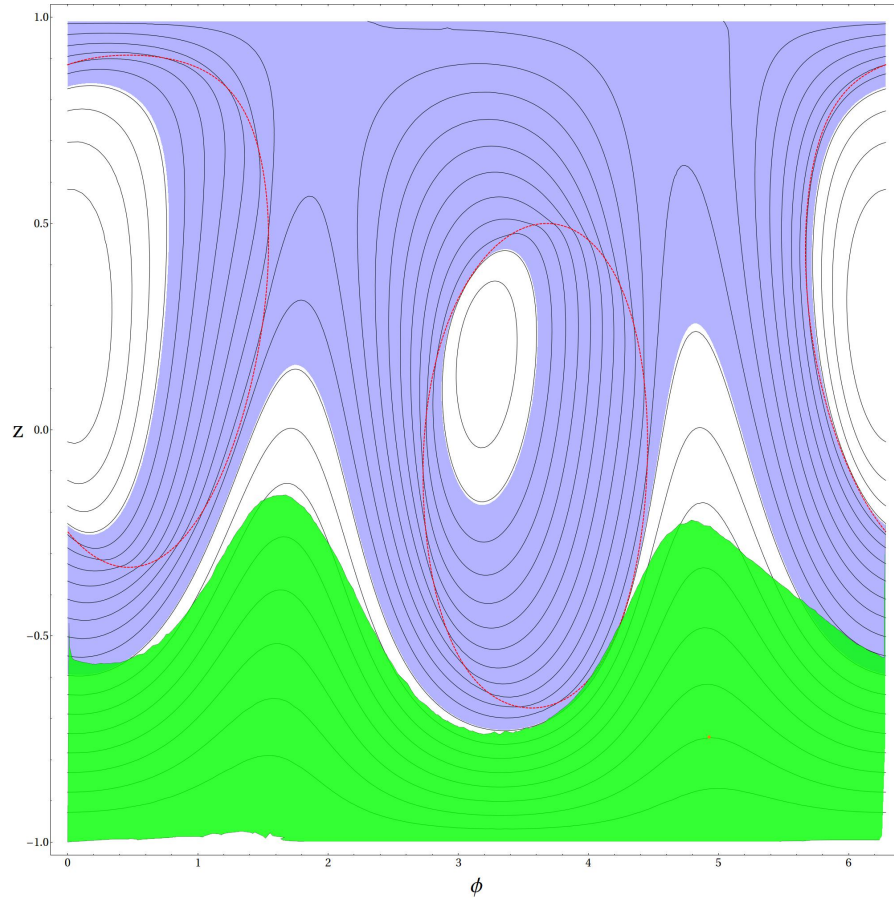


Figure 4.11: the same as 4.9, but with $\epsilon = 10^{-4}$.

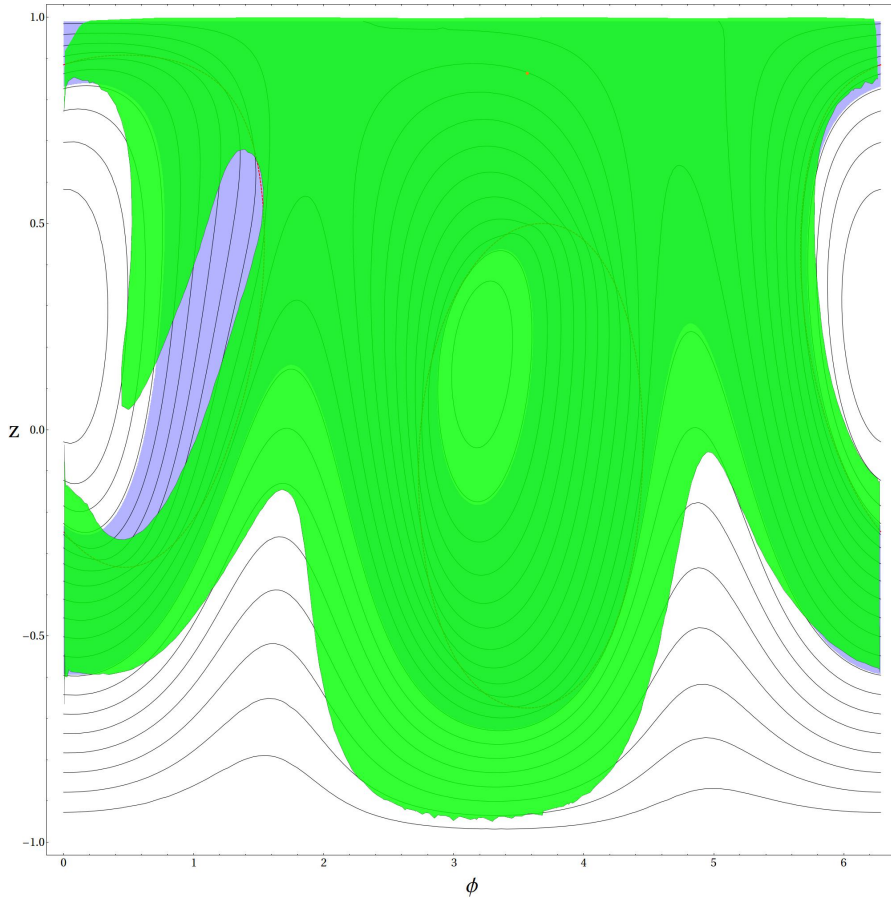


Figure 4.12: the same as 4.10, but with $\epsilon = 10^{-4}$.

4.4 Numerical considerations

Using the results of the previous chapter we are able to explicitly compute $\mathbf{h}_\epsilon$ for any value of ϵ and $\hbar$.

Fixed those parameters we can diagonalize the matrix $\mathbf{h}_\epsilon$. Since we are interested in the semi classical behaviour of the dynamics we need to work with $\hbar$ as small as possible. However the dimension of the quantum Hilbert space $\mathcal{H}$, on which $\mathbf{h}_\epsilon$ is defined, is $n^3 = 8\hbar^{-3}$. As $\hbar$ approaches 0, n tends to infinity. This poses serious difficulties in the numerical calculations (both regarding the time of execution of the numerical algorithms and the memory needed to store the data).

Until now we managed to work with n up to 40, which is equivalent to choosing $\hbar = \frac{1}{20}$, and results in $\dim \mathcal{H} = 64000$. The storage space needed for storing the eigenstates is almost $62Gb$ and the diagonalization process of $\mathbf{h}_\epsilon$ takes about 54 CPU hours to complete (the diagonalization procedure is the main bottleneck in the numerical work).

The code for the diagonalization has been written in the C language using the LaPACK library (See [14] for an analysis of the diagonalization algorithm which is implemented in the library). In principle it is possible to parallelize the algorithm that diagonalize $\mathbf{h}_\epsilon$ and experiments in this direction have been done with the library ScaLAPACK, with encouraging results. The advantages of parallelization is not only that it speeds up the calculations, but especially the fact that distributing the computation over different machines allows the use of a larger amount of RAM. The available RAM is in fact the main problem in the computations given that the required memory grows as n^6 . For instance already for the case $n = 43$, $283Gb$ of memory are needed, which are difficult to find on a single machine.

4.5 Perturbative approach

The numerical analysis outlined in the previous sections has the severe drawback that it depends on the diagonalization of a Hermitian matrix $\mathbf{h}_\epsilon$ of order n . As already mentioned the task of carrying on this procedure for high values of n , such as the ones needed for a meaningful approximation of the semiclassical limit, is, from a computational perspective, unrealistic. A possible solution, which relies on a tradeoff between the accuracy of the computed eigenvalues and the cost of their computation, might be found through a perturbative approach.

In quantum mechanics there are several procedures for solving the eigenvalue problem for a perturbed system. This section will be mainly devoted to the theory behind the so-called (degenerate and time independent) Rayleigh-Schrödinger method (a complete reference on the method can be found in [38]), while at the end

we will outline the basics principles of Van Vleck method, which, in this context, is less widely adopted, but it has the advantage of having a direct interpretation in term of classical perturbation theory and so it is promising when a comparison between quantum and classical behaviours is needed.

4.5.1 Rayleigh-Schrödinger method

The fundamental idea behind Rayleigh-Schrödinger method is that the eigenvalues, with their corresponding eigenstates, of the Hamiltonian $\mathbf{h}_\epsilon$ depend in a analytic way from the perturbative parameter ϵ . It is in this case possible to solve the equations that appear in the eigenvalue problem separately at each order in ϵ . Such a perturbative expansion, while being often divergent for a fixed value of $\epsilon \neq 0$ and thus not suitable for explicit computations, can give a relevant amount of information on the perturbed system even at its lowest order. In particular we will see how it allows to determine when the perturbed spectrum is non-degenerate, how it can explain the grid structure (numerically already found in the previous chapter) of the expectation values of the quantum actions in the non-resonant case and the “chaotic” regions present on the classical resonances and it allows us to approximately study, up to a correction of order ϵ , the motions on the third sphere caused by the perturbation.

The first proposition states a general result on the eigenvalues-eigenstates of a matrix which depends analytically from a parameter (the proof can be found in [29]):

Proposition 10 ([29]). *Let $\mathbf{a}_\epsilon$ be a $n \times n$ matrix depending on a parameter ϵ and let the dependence of the matrix elements of $\mathbf{a}_\epsilon$ on ϵ be analytic in some open set $\mathcal{D} \subseteq \mathbb{C}$. Under these assumptions there exists $P \subset \mathcal{D}$, with P finite on every compact set of $\mathbb{C}$, such that the eigenvalues of $\mathbf{a}_\epsilon$ and the projectors on the corresponding eigenspaces define analytic functions on $\hat{\mathcal{D}} = \mathcal{D} \setminus P$. If $\mathbf{a}_\epsilon$ is Hermitian for $\epsilon \in \mathbb{R}$ then the points of P on the real axis are removable singularities.*

In this context a function f is said to be analytic in $\hat{\mathcal{D}}$ if for every $\epsilon_0 \in \hat{\mathcal{D}}$ and $r > 0$ such that the open ball of radius r centred in ϵ_0 is entirely contained in $\hat{\mathcal{D}}$, then f admits a power series around ϵ_0 with radius of convergence $R \geq r$. The analyticity of the projectors is to be intended as the analyticity of their matrix elements on the considered domain. The analyticity of the projectors in $\hat{\mathcal{D}}$ implies that the multiplicity of each eigenvalue is constant on $\hat{\mathcal{D}}$. Conversely the points of P are exactly the ones where a crossing between different eigenvalues takes place. The last statement implies that for real ϵ crossing may still be present, but it is possible to find an appropriate permutation on the order of the eigenvalues (and

corresponding eigenstates) after every crossing such that the dependence from ϵ is analytic on the real axis.

Remark. It should be stressed that keeping the same ordering of the eigenvalues on the real axis (where they are real, thanks to the Hermiticity of $\mathbf{a}_\epsilon$) does not result, in general, in an analytic dependence. For instance consider:

$$\mathbf{a}_\epsilon = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}.$$

In this case keeping the ordering in the eigenvalues results in:

$$\begin{aligned} \lambda_1(\epsilon) &= -|\epsilon| \\ \lambda_2(\epsilon) &= |\epsilon| \end{aligned}$$

instead of the correct choice:

$$\begin{aligned} \lambda_1(\epsilon) &= -\epsilon \\ \lambda_2(\epsilon) &= \epsilon. \end{aligned}$$

△

We can apply this result to our Hamiltonian $\mathbf{h}_\epsilon$. We have the following:

Proposition. *Assuming that the eigenvalues of $\mathbf{h}_{\bar{\epsilon}}$ are simple for some $\bar{\epsilon}$, then there exists $\delta > 0$ such that the eigenvalues and the corresponding eigenspaces of $\mathbf{h}_\epsilon$ depend analytically on ϵ on the set*

$$\mathcal{S} = \{z \in \mathbb{C} \mid |\Im(z)| < \delta\}.$$

and in $\mathcal{S}$ the eigenvalues are simple with the exception of a finite number of points on the real axis.

Proof. Since the eigenvalues of $\mathbf{h}_{\bar{\epsilon}}$ are simple then, by continuity, there exists a neighbourhood $\mathcal{F}$ of $\bar{\epsilon}$ such that the eigenvalues of $\mathbf{h}_\epsilon$ are simple for $\epsilon \in \mathcal{F}$. Since P is discrete it means that on $\hat{\mathcal{D}}$ the eigenvalues are simple.

We have that the dependence of $\mathbf{h}_\epsilon$ on ϵ is affine, and so $\mathcal{D} = \mathbb{C}$. If we define the characteristic polynomial of $\mathbf{h}_\epsilon$ (with ϵ as a parameter) for $\lambda \in \mathbb{C}$:

$$C_\epsilon(\lambda) = \det(\mathbf{h}_\epsilon - \lambda)$$

then P is defined as the zero set of $\epsilon \mapsto \tilde{C}(\epsilon) := \Delta C_\epsilon$ where ΔC_ϵ is the discriminant of C_ϵ . It is immediate to check that $\tilde{C}(\epsilon)$ is polynomial in ϵ , and so P is finite. This in turn implies that (since the real points of P need not to be counted as singularities) the domain of analyticity we consider contains a strip around the real axis. □

Remark. It should be stressed that the power series defining the eigenvalues are, in general, divergent as ϵ become too large, for instance consider:

$$\mathbf{a}_\epsilon = \begin{pmatrix} 1 & \epsilon \\ \epsilon & -1 \end{pmatrix}$$

we have that the eigenvalues can be exactly computed: $\lambda_\pm(\epsilon) = \pm\sqrt{1+\epsilon^2}$. A perturbative approach will give their Taylor expansion:

$$\lambda_\pm(\epsilon) = \mp \sum_{k=0}^{\infty} \frac{(-1)^k (2k-3)!!}{2^k k!} \epsilon^{2k}$$

but obviously this series has no meaning when $|\epsilon| > 1$, since for $\epsilon = \pm i$ $\lambda_\pm$ has a singularity. $\triangle$

In general, as n grows, δ is typically closer to 0 (since the cardinality of P increases) and so it is not convenient to consider a power expansion for the eigenvalues or the projectors for a fixed value of ϵ (especially when considering the semi classical limit), nevertheless this expansion is useful when, as in our case, we want to investigate characteristics of the system typical of the asymptotic behaviour $\epsilon \rightarrow 0$. Moreover if we restrict our analysis to a sufficiently small neighbourhood of the origin it is immediate to see that it is possible to give an analytic expression to the eigenstates (rather than to the projectors on the eigenspaces) of $\mathbf{h}_\epsilon$, since it is possible to fix in a consistent way the phase of each eigenstate.

From now on for simplicity we will assume (for more general results see for instance [29]) that all the eigenvalues of $\mathbf{h}_\epsilon$ are simple on $\hat{\mathcal{D}}$. Since we are interested in the power expansions in ϵ for the eigenstates and eigenvalues we will restrict our analysis to a sufficiently small neighbourhood of $\epsilon = 0$.

Let $(|\psi_k(\epsilon)\rangle)_{k=1,\dots,n}$ be a basis of $\mathcal{H}$ made by eigenstates of $\mathbf{h}_\epsilon$ with $\lambda_k(\epsilon)$ being the eigenvalue relative to $|\psi_k(\epsilon)\rangle$. Since the eigenvalues are simple in $\hat{\mathcal{D}}$, then the $|\psi_k(\epsilon)\rangle$ are, up to a proportionality factor, well-defined. In particular we have that $(|\psi_k(0)\rangle)_{k=1,\dots,n}$ defines a basis of $\mathcal{H}$ of eigenstates of $\mathbf{h}_0$. If we write

$$\begin{aligned} \lambda_k(\epsilon) &= \sum_{\alpha \geq 0} \lambda_k^{(\alpha)} \epsilon^\alpha \\ |\psi_k(\epsilon)\rangle &= \sum_{\alpha \geq 0} |\psi_k^{(\alpha)}\rangle \epsilon^\alpha. \end{aligned}$$

By definition $|\psi_k(\epsilon)\rangle = |\psi_k^{(0)}\rangle$ and $\lambda_k(\epsilon) = \lambda_k^{(0)}$ and so we have that $|\psi_k^{(0)}\rangle$ is an eigenstate of $\mathbf{h}_0$ relative to the eigenvalue $\lambda_k^{(0)}$. We already know that the eigenvalues of $\mathbf{h}_0$ are not simple and so for every λ in the spectrum of $\mathbf{h}_0$ the

choice of an orthogonal basis of eigenstates relative to λ is not unique. This means that the perturbation $\mathbf{f}$ “chooses” from every eigenspace of $\mathbf{h}_0$ a preferred basis.

Let $\lambda^{(0)}$ be an element of the spectrum of $\mathbf{h}_0$, and $\mathbf{p}$ be the orthogonal projection operator on the correspondent d -dimensional eigenspace. Let $\lambda_{k_1}(\epsilon), \dots, \lambda_{k_d}(\epsilon)$ be the family of eigenvalues of $\mathbf{h}_\epsilon$ such that $\lambda_{k_i}(0) = \lambda^{(0)}$, and let $|\psi_{k_i}(\epsilon)\rangle, \dots, |\psi_{k_d}(\epsilon)\rangle$ correspondent eigenstates. We have that

Proposition. $|\psi_{k_d}(0)\rangle = |\psi_{k_d}^{(0)}\rangle$ is an eigenstate for the operator $\mathbf{pfp}$. If λ_d is the corresponding eigenvalue then $\lambda_{k_d}^{(1)} = \lambda_d$.

Proof. By definition we have that $\lambda_{k_d}(\epsilon)$ and $|\psi_{k_d}(\epsilon)\rangle$ satisfy:

$$(\mathbf{h}_0 + \epsilon \mathbf{f} - \lambda_{k_d}(\epsilon))|\psi_{k_d}(\epsilon)\rangle = 0. \quad (4.6)$$

If we define $\mathbf{q} = \mathbb{I} - \mathbf{p}$, applying $\mathbf{p}$ on the left of (4.6) (recalling that we have $\mathbf{p}\mathbf{h}_0 = \mathbf{h}_0\mathbf{p} = \lambda^{(0)}\mathbf{p}$) we obtain

$$(\lambda_0 + \epsilon \mathbf{pfp} - \lambda_{k_d}(\epsilon))\mathbf{p}|\psi_{k_d}(\epsilon)\rangle + \epsilon \mathbf{pfq}|\psi_{k_d}(\epsilon)\rangle = 0.$$

Using the fact that $|\psi_{k_d}(0)\rangle$ is an eigenstate of $\mathbf{h}_0$ relative to λ_0 , so that $\mathbf{p}|\psi_{k_d}^{(0)}\rangle = |\psi_{k_d}^{(0)}\rangle$ and $\mathbf{q}|\psi_{k_d}^{(0)}\rangle = 0$, and that $\lambda_{k_d}(0) = \lambda_0$, we expand the equation up to the first order in ϵ obtaining (the first order in the expansion vanishes):

$$\epsilon(\mathbf{pfp} - \lambda_{k_d}^{(1)})|\psi_{k_d}^{(0)}\rangle = 0$$

which concludes the proof. □

Remark. If all the eigenvalues of $\mathbf{pfp}$ are simple for each eigenspace of $\mathbf{h}_0$ then the previous result fixes $|\psi_k^{(0)}\rangle$ for all k . When an eigenvalue of $\mathbf{pfp}$ is not simple it is necessary to project equation (4.6) onto the corresponding eigenspace and expand it up to the second order. This procedure can be carried until the degeneracy is lift (i.e. $|\psi_k^{(0)}\rangle$ are all uniquely defined). While we will not discuss this problem further we just point out that this procedure eventually comes to an end after a finite number of steps (more details can be found in [29]). Conversely we observe that, even if we do not know a priori if the eigenvalues of $\mathbf{h}_\epsilon$ are simple on $\hat{\mathcal{D}}$, if the eigenvalues of $\mathbf{pfp}$ are simple for each eigenspace of $\mathbf{h}_0$ then there should exists an $\epsilon_0 \neq 0$ such that all the eigenvalues of $\mathbf{h}_\epsilon$ for $0 < \epsilon < \epsilon_0$ are simple (if $\epsilon \neq 0$ is small enough then the linear part in the expansion for the eigenvalues in power of ϵ is dominant and so all the eigenvalues originating from the same unperturbed value split, but do not cross with eigenvalues relatives to a different unperturbed eigenvalues). In turn this implies that on $\hat{\mathcal{D}}$ all the eigenvalues of $\mathbf{h}_\epsilon$ are simple.

Before considering the higher order corrections for the eigenvalues and eigenstates we observe that even the zeroth-order approximation $|\psi_k^{(0)}\rangle$ gives us important information on the system $\mathbf{h}_\epsilon$. For instance if $|\phi\rangle \in \mathcal{H}$ is a generic state we can define $\phi_k(\epsilon)$ (we suppose the eigenstates of $\mathbf{h}_\epsilon$ normalized) by $\phi_k(\epsilon) = \langle \psi_k(\epsilon) | \phi \rangle$. If ϵ is small enough we have that the eigenvalues of $\mathbf{h}_\epsilon$ are simple and so

$$\bar{H}_{|\phi\rangle} = \sum_{k=1}^n |\phi_k(\epsilon)|^2 H_{|\psi_k(\epsilon)\rangle}$$

It is easy to see however that it is possible to take the limit for ϵ going to zero uniformly on $\mathcal{M}$ obtaining

$$\lim_{\epsilon \rightarrow 0} \bar{H}_{|\phi\rangle} = \sum_{k=1}^n |\phi_k(0)|^2 H_{|\psi_k(0)\rangle}$$

Where $\phi_k(0) = \langle \psi_k^{(0)} | \phi \rangle$ can be easily computed once the lowest order of the power expansion for the eigenstates is known.

Another information on the perturbed system we can recover from the lowest perturbative order is related to the distribution of the expectation values of the actions on the eigenstates.

Let $\mathcal{H}_{m_1, m_2} \subset \mathcal{H}$, $m_i = -\frac{n-1}{2}, \dots, \frac{n-1}{2}$ be the eigenspace of $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$ relative to the eigenvalues $\hbar m_1$ and $\hbar m_2$. Let $\mathcal{H}_e \subset \mathcal{H}$ be the eigenspace of $\mathbf{h}_0$ relative to the eigenvalue $\frac{1}{n(2+n)} + \frac{2(3+n)}{2(1+n)(2+n)} e^2$. It is immediate to see that

$$\mathcal{H}_e = \bigoplus_{m_1^2 + m_2^2 = e^2} \mathcal{H}_{m_1, m_2}$$

Since $\mathcal{H}_e$ is not, in general, an eigenspace for $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$ the expectation values, evaluated on the elements of $\mathcal{H}_e$, of these observables is not uniquely defined but ranges over the convex hull of the points with coordinates $(\hbar m_1, \hbar m_2)$, with $m_1^2 + m_2^2 = e^2$. When we perturb the system, however, the degeneracy is lifted and to every eigenspace of $\mathbf{h}_\epsilon$ is associated a single point. As seen above, the perturbed eigenstates that originate from the unperturbed eigenspace $\mathcal{H}_e$ are computed, as the lowest order, as the eigenstates of $\mathbf{pfp}$, where $\mathbf{p}$ is the orthogonal projection onto $\mathcal{H}_e$.

In our case, if $|\psi\rangle \in \mathcal{H}_{m_1, m_2}$, then the orthogonal projection of $\mathbf{f}|\psi\rangle$ onto $\mathcal{H}_{n_1, n_2}$ is non zero only if $(m_1 - n_1, m_2 - n_2)$ is one of the following: $(0, 0)$, $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$, $(1, -1)$, $(-1, 1)$. In particular if we consider $\mathbf{pfp}$ we have the following possibilities:

- $e = 0$, in this case $\mathcal{H}_e = \mathcal{H}_{0,0}$;

- $e = \frac{1}{4}$, in this case $\mathcal{H}_e = \bigoplus_{m_i=\pm\frac{1}{2}} \mathcal{H}_{m_1,m_2}$. The image under **pf** of any $\mathcal{H}_{m_1,m_2}$ is not contained in itself and it is not possible to find any proper subspace of $\mathcal{H}_e$ invariant under **pf** which is a direct sum of some $\mathcal{H}_{m_1,m_2}$;
- $e = \frac{5}{2}$, in this case $\mathcal{H}_e = \bigoplus \mathcal{H}_{m_1,m_2}$ where the sum is taken over the values of the indexes $m_1, m_2 \in \{\pm\frac{1}{2}, \pm\frac{3}{2}\}$, $m_1 \neq m_2$. Also in this case the image under **pf** of any $\mathcal{H}_{m_1,m_2}$ is not contained in itself and it is not possible to find any proper subspace of $\mathcal{H}_e$ invariant under **pf** which is a direct sum of some $\mathcal{H}_{m_1,m_2}$;
- $\mathcal{H}_e$ contains one of the following subspaces:

$$\begin{aligned}
- \mathcal{H}_m^{(1,0)} &:= \mathcal{H}_{-\frac{1}{2},m} \oplus \mathcal{H}_{\frac{1}{2},m}; \\
- \mathcal{H}_m^{(0,1)} &:= \mathcal{H}_{m,-\frac{1}{2}} \oplus \mathcal{H}_{m,\frac{1}{2}}; \\
- \mathcal{H}_m^{(1,-1)} &:= \mathcal{H}_{m,m+1} \oplus \mathcal{H}_{m+1,m}.
\end{aligned}$$

In this case we have that $\mathcal{H}_m^\alpha$, for any α , is invariant under **pf** and so its matrix has a block-diagonal structure where each block represents a subspace $\mathcal{H}_m^\alpha$. As a consequence we have that the lowest order of the expansion of the corresponding eigenstates is contained in $\mathcal{H}_m^\alpha$ and so the expectation value of the action on these states are concentrated along a segment which is orthogonal to the classical resonance represented by α ;

- when a direct summand $\mathcal{H}_{m_1,m_2}$ of $\mathcal{H}_e$ does not fall in one of the previous cases, then $\mathcal{H}_{m_1,m_2}$ is invariant under **pf**. This means that the corresponding eigenstates is contained, at its lowest perturbative order, in $\mathcal{H}_{m_1,m_2}$. In this way we have that the expectation value of the action on these states is concentrated around the point $(\hbar m_1, \hbar m_2)$.

Summarizing we can say that the eigenstates of the actions define an equispaced grid on $\mathbb{R}^2$, each eigenstate of $\mathbf{h}_\epsilon$, provided ϵ is small enough, falls in a different category which determines the expectation values of the actions:

- Fully resonant. In this case the expectation values fall in polygon with corners $(\frac{\hbar}{2}, \frac{3\hbar}{2})$, $(-\frac{\hbar}{2}, \frac{3\hbar}{2})$, $(-\frac{3\hbar}{2}, \frac{\hbar}{2})$, $(-\frac{3\hbar}{2}, -\frac{\hbar}{2})$, $(-\frac{\hbar}{2}, -\frac{3\hbar}{2})$, $(\frac{\hbar}{2}, -\frac{3\hbar}{2})$, $(\frac{3\hbar}{2}, -\frac{\hbar}{2})$, $(\frac{3\hbar}{2}, \frac{\hbar}{2})$. $12n$ eigenstates are contained there (and $4n$ of those are contained in the square centred in the origin and with the side of length $\hbar$).
- (1,0) resonant. In this case the expectation values are concentrated along a segment with length $\hbar$, middle point in $(0, \hbar m)$ for some $|m| > \frac{3}{2}$ and orthogonal to the y axis. $2n$ eigenstates are contained there.

- $(0, 1)$ resonant. In this case the expectation values are concentrated along a segment with length $\hbar$ and middle point in $(\hbar m, 0)$ for some $|m| > \frac{3}{2}$ and orthogonal to the x axis. $2n$ eigenstates are contained there.
- $(1, -1)$ resonant. In this case the expectation values are concentrated along a segment with length $\sqrt{2}\hbar$ and middle point in $(\hbar m + \frac{\hbar}{2}, \hbar m + \frac{\hbar}{2})$ for some $m > \frac{3}{2}$ or $m < -2$ and orthogonal to the bisector of the first quadrant. $2n$ eigenstates are contained there.
- non resonant. In this case the expectation values are concentrated around a point of the grid which is far from the resonances above. for each point there are n eigenstates.

This analysis is obviously coarse, nevertheless this explain with a very good accuracy the situation we have met numerically.

It is possible to push the analysis to higher orders. Let $|\psi(\epsilon)\rangle$ be an eigenstate of $\mathbf{h}_\epsilon$ with eigenvalue $\lambda(\epsilon)$. If $\lambda(\epsilon) = \sum_{k=0}^{\infty} \lambda^{(k)} \epsilon^k$, and $|\psi(\epsilon)\rangle = \sum_{k=0}^{\infty} |\psi^{(k)}\rangle \epsilon^k$, then $|\psi^{(0)}\rangle$ is a simultaneous eigenstate of $\mathbf{h}_0$, with eigenvalue $\lambda^{(0)}$, and of $\mathbf{pfp}$, where $\mathbf{p}$ is the orthogonal projection on the eigenspace of $\mathbf{h}_0$ relative to $\lambda^{(0)}$, with eigenvalues $\lambda^{(1)}$. $|\psi\rangle$ can be written as $|\psi\rangle = |\psi_{\mathbf{p}}\rangle + |\psi_{\mathbf{q}}\rangle$ where $|\psi_{\mathbf{p}}\rangle = \mathbf{p}|\psi\rangle$. In an analogous way $|\psi^{(k)}\rangle = |\psi_{\mathbf{p}}^{(k)}\rangle + |\psi_{\mathbf{q}}^{(k)}\rangle$. Obviously we have that $|\psi_{\mathbf{p}}^{(0)}\rangle = |\psi^{(0)}\rangle$ and $|\psi_{\mathbf{q}}^{(0)}\rangle = 0$. It should be observed that the definition of $|\psi\rangle$ is not unique, but determined up to a normalization factor. If we assume that $|\psi(0)\rangle$ is normalized then, for ϵ small enough, $\langle \psi(0) | \psi(\epsilon) \rangle \neq 0$. In this case it is not restrictive to assume $\langle \psi(0) | \psi(\epsilon) \rangle = 1$, which is equivalent to $\langle \psi^{(0)} | \psi^{(k)} \rangle = 0$ when $k \neq 0$. If we define the operator $\mathbf{q} = \mathbb{I}_{\mathcal{H}} - \mathbf{p}$ and $\pi = \mathbf{p} - |\psi^{(0)}\rangle \langle \psi^{(0)}|$ we have that, while neither $(\mathbf{h}_0 - \lambda^{(0)})$ nor $(\mathbf{f} - \lambda^{(1)})$ are invertible, there exist operators $\mathbf{d}_0$ and $\mathbf{d}_1$ such that:

$$\begin{aligned} \mathbf{d}_0 \mathbf{q} &= \mathbf{d}_0 = \mathbf{q} \mathbf{d}_0 \\ \mathbf{d}_0 \pi &= \mathbf{d}_1 = \pi \mathbf{d}_0 \\ \mathbf{d}_0 (\mathbf{h}_0 - \lambda^{(0)}) &= \mathbf{q} = (\mathbf{h}_0 - \lambda^{(0)}) \mathbf{d}_0 \\ \mathbf{d}_1 (\mathbf{f} - \lambda^{(1)}) &= \pi = (\mathbf{f} - \lambda^{(1)}) \mathbf{d}_1 \end{aligned}$$

Basically $\mathbf{d}_0$ and $\mathbf{d}_1$ represent the inverse of $(\mathbf{h}_0 - \lambda^{(0)})$ and $(\mathbf{f} - \lambda^{(1)})$ on the subspaces on which these inverse are defined. We have that

Proposition. *The coefficients in the perturbative expansion of the eigenvalues and*

eigstates of $\mathbf{h}_0$ satisfy (here $k \geq 1$):

$$\begin{aligned}\lambda^{(k+1)} &= \langle \psi^{(0)} | \mathbf{f} | \psi_q^{(k)} \rangle \\ |\psi_q^{(k)}\rangle &= -\mathbf{d}_0 \mathbf{f} |\psi^{(k-1)}\rangle + \sum_{i=1}^{k-1} \lambda^{(i)} \mathbf{d}_0 |\psi^{(k-i)}\rangle \\ |\psi_p^{(k)}\rangle &= -\mathbf{d}_1 \mathbf{f} |\psi_q^{(k)}\rangle + \sum_{i=1}^{k-1} \lambda^{(i+1)} \mathbf{d}_1 |\psi^{(k-i)}\rangle\end{aligned}$$

Proof. If we expand the eigenvalue problem (4.6) in power of ϵ and collect the same orders we obtain:

$$\begin{aligned}(\mathbf{h}_0 - \lambda^{(0)})|\psi^{(0)}\rangle &= 0 \\ \mathbf{h}_0 |\psi^{(k)}\rangle + \mathbf{f} |\psi^{(k-1)}\rangle - \sum_{i=0}^k \lambda^{(i)} |\psi^{(k-i)}\rangle &= 0 \quad k \geq 1.\end{aligned}$$

The first equation reminds us that $|\psi^{(0)}\rangle$ is an eigenstate of $\mathbf{h}_0$ with eigenvalue $\lambda^{(0)}$. Applying $\mathbf{p}$ and $\mathbf{q}$ on the left of the second equation we obtain:

$$\begin{aligned}(\mathbf{pfp} - \lambda^{(1)})|\psi_p^{(0)}\rangle &= 0 \\ \mathbf{p}(\mathbf{f} - \lambda^{(1)})\mathbf{p}|\psi_p^{(k-1)}\rangle + \mathbf{p}\mathbf{f}\mathbf{q}|\psi_q^{(k-1)}\rangle - \sum_{i=2}^k \lambda^{(i)} |\psi_p^{(k-i)}\rangle &= 0 \quad k \geq 2 \\ \mathbf{q}(\mathbf{h}_0 - \lambda^{(0)})\mathbf{q}|\psi_q^{(k)}\rangle + \mathbf{q}\mathbf{f}|\psi^{(k-1)}\rangle - \sum_{i=1}^k \lambda^{(i)} |\psi_q^{(k-i)}\rangle &= 0 \quad k \geq 1\end{aligned}$$

From the first equation we know that $|\psi_p^{(0)}\rangle$ is an eigenstate of $\mathbf{pfp}$ with eigenvalue $\lambda^{(1)}$. The formula for the eigenvalues is obtained by multiplying the second equation on the left by $\langle \psi^{(0)} |$, since

$$\langle \psi^{(0)} | \mathbf{p}(\mathbf{f} - \lambda^{(1)})\mathbf{p} = \langle \psi_p^{(0)} | (\mathbf{pfp} - \lambda^{(1)}) = 0$$

and

$$\langle \psi^{(0)} | \psi_p^{(k-i)} \rangle = \langle \psi^{(0)} | \psi^{(k-i)} \rangle = \delta_{ki}.$$

The recursive formulae for $|\psi_p^{(k)}\rangle$ and $|\psi_q^{(k)}\rangle$ are instead obtained by multiplication on the left of the second and the third equation by $\mathbf{d}_1$ and $\mathbf{d}_0$ respectively (and by redefining the index k in the equation for $|\psi_p^{(k)}\rangle$ and performing trivial simplifications). $\square$

Remark. While the expressions for high orders become soon very complicated and suitable mostly for automated computations only, the expression for the first perturbative order is still fairly simple:

$$\begin{aligned}
|\psi_{\mathbf{q}}^{(1)}\rangle &= -\mathbf{d}_0 \mathbf{f} |\psi^{(0)}\rangle \\
\lambda^{(2)} &= \langle \psi^{(0)} | \mathbf{f} | \psi_{\mathbf{q}}^{(1)} \rangle \\
&= - \langle \psi^{(0)} | \mathbf{f} \mathbf{d}_0 \mathbf{f} | \psi^{(0)} \rangle \\
|\psi_{\mathbf{p}}^{(1)}\rangle &= -\mathbf{d}_1 \mathbf{f} |\psi_{\mathbf{q}}^{(1)}\rangle \\
&= \mathbf{d}_1 \mathbf{f} \mathbf{d}_0 \mathbf{f} |\psi^{(0)}\rangle .
\end{aligned}$$

It is interesting to notice that if we consider the expectation value $a_\alpha(\epsilon)$ of $\mathbf{x}_{\alpha 3}$, for $\alpha = 1, 2$, in the state $|\psi(\epsilon)\rangle$, we have that $a_\alpha(\epsilon)$ is an analytic function of ϵ , $a_\alpha(\epsilon) = \sum_{k=0}^{\infty} a_\alpha^{(k)} \epsilon^k$. $a_\alpha^{(0)}$ is given by $\langle \psi^{(0)} | \mathbf{x}_{\alpha 3} | \psi^{(0)} \rangle$, while the first order approximation is given by:

$$a_\alpha^{(1)} = 2\Re(\langle \psi^{(0)} | \mathbf{x}_{\alpha 3} | \psi^{(1)} \rangle) \quad (4.7)$$

Since $\mathbf{x}_{\alpha 3}$ commutes with $\mathbf{h}_0$ we have that $\langle \psi^{(0)} | \mathbf{x}_{\alpha 3} | \psi_{\mathbf{q}}^{(1)} \rangle = 0$ and so, using the previous formulae:

$$a_\alpha^{(1)} = 2\Re(\langle \psi^{(0)} | \mathbf{x}_{\alpha 3} \mathbf{d}_1 \mathbf{f} \mathbf{d}_0 \mathbf{f} | \psi^{(0)} \rangle)$$

When $|\psi^{(0)}\rangle$ is a “non resonant” eigenstate, for what said before, we have that $\mathbf{x}_{\alpha 3} |\psi^{(0)}\rangle = k_\alpha |\psi^{(0)}\rangle$ for some (k_1, k_2) . In this way, since $\langle \psi^{(0)} | \mathbf{d}_1 = 0$, $a_\alpha^{(1)} = 0$. In the $(0, 1)$ resonant case we have that $\mathbf{x}_{13} |\psi^{(0)}\rangle = k_1 |\psi^{(0)}\rangle$ for some k_1 and so $a_1^{(1)} = 0$. Similarly in the $(1, 0)$ resonant case we have that $\mathbf{x}_{23} |\psi^{(0)}\rangle = k_2 |\psi^{(0)}\rangle$ for some k_2 and so $a_2^{(1)} = 0$. On the other hand it is easy to see that in the $(1, -1)$ case $(\mathbf{x}_{13} + \mathbf{x}_{23}) |\psi^{(0)}\rangle = k_3 |\psi^{(0)}\rangle$ for some k_3 , and so $a_1^{(1)} + a_2^{(1)} = 0$. This shows that the first order variation (in ϵ) of a_α is absent in the non-resonant case, while in the resonant case it take place along the fast drift line orthogonal to the resonance, and equation (4.7) allows to compute the correction. $\triangle$

4.5.2 Van Vleck method

Even if the Rayleigh-Schrödinger method, outlined above, explains successfully important phenomena. Its formalism is somewhat unsatisfying when a connection with classical perturbation theory is wanted. In this case Van Vleck method is more appropriate.

The idea is to find a unitary transformation $\mathbf{u}$ of $\mathcal{H}$ “close” to the identity such that

$$\mathbf{u}^{-1} \mathbf{h}_\epsilon \mathbf{u} = \mathbf{h}_0 + \epsilon \mathbf{g} + \epsilon^2 \mathbf{f}' \quad (4.8)$$

where $[\mathbf{h}_0, \mathbf{g}] = 0$. To define $\mathbf{u}$ we consider the self-adjoint operator $\mathbf{k}$ such that

$$\mathbf{u} = \exp\left(\frac{\epsilon \mathbf{k}}{i\hbar}\right)$$

(we see that $\mathbf{u}$ is unitary for every ϵ and, as ϵ tends to zero $\mathbf{u}$ tends to the identity on $\mathcal{H}$). Using the expression for $\mathbf{u}$ in (4.8) and expanding in power of ϵ we have that, neglecting terms of order 2 or higher in ϵ ,

$$\frac{1}{i\hbar}[\mathbf{k}, \mathbf{h}_0] = \mathbf{f} - \mathbf{g} \quad (4.9)$$

where the quantum version of the fundamental equation of Hamiltonian perturbation theory can be recognized. Let $0 < i, j < n$ such that $(\mathbf{h}_0)_{ii} = \lambda_i^{(0)}$ and $(\mathbf{h}_0)_{jj} = \lambda_j^{(0)}$. We have that (4.9) is solved by:

$$(\lambda_i^{(0)} - \lambda_j^{(0)})\mathbf{k}_{ij} = i\hbar(\mathbf{f}_{ij} - \mathbf{g}_{ij}). \quad (4.10)$$

Formally (4.10) can be solved by choosing

$$\mathbf{g}_{ij} = \begin{cases} \mathbf{f}_{ij} & \lambda_i^{(0)} - \lambda_j^{(0)} = 0 \\ 0 & \lambda_i^{(0)} - \lambda_j^{(0)} \neq 0 \end{cases}$$

and

$$\mathbf{k}_{ij} = \begin{cases} 0 & \lambda_i^{(0)} - \lambda_j^{(0)} = 0 \\ \frac{i\hbar}{\lambda_i^{(0)} - \lambda_j^{(0)}} \mathbf{f}_{ij} & \lambda_i^{(0)} - \lambda_j^{(0)} \neq 0 \end{cases}$$

However it is easy to see that the first order of $\mathbf{f}'$ in (4.8) is (again, the result corresponds to an analogous one in classical mechanics):

$$\mathbf{f}' = \frac{1}{i\hbar} \left[\frac{\mathbf{f} + \mathbf{g}}{2}, \mathbf{k} \right] + O(\epsilon)$$

An immediate consequence of this is that if, for some $i \neq j$, we have $\lambda_i^{(0)} - \lambda_j^{(0)} \sim \epsilon$ then $\mathbf{f}'_{ij}$ is of order ϵ and not ϵ^2 . This is exactly the same problem which is encountered in the classical Nekhoroshev theory where is addressed by introducing resonant normal forms. In the quantum case we need to redefine $\mathbf{k}$ and $\mathbf{g}$ by replacing $\lambda_i^{(0)} - \lambda_j^{(0)} = 0$ with $|\lambda_i^{(0)} - \lambda_j^{(0)}| \sim \sqrt{\epsilon}$. In this way $\mathbf{k}$ results bounded, however $\mathbf{g}$ does not commute with $\mathbf{h}_0$ anymore.

Clearly this procedure can be carried on for multiple steps, trying to push the perturbation order to ϵ^k for some $k > 1$.

While a precise analysis has not been carried on yet for our case it appears reasonable that the truncated quantum normal form (which at the first order is $\mathbf{h}_0 + \epsilon \mathbf{g}$) will display features similar to the classical ones, as the presence of diagonal blocks relative to the non-resonant eigenspaces along with resonances in the form of interacting blocks relatives to different actions.

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