Sede Amministrativa: Università degli Studi di Padova

SCUOLA DI DOTTORATO DI RICERCA IN: Scienze Matematiche

INDIRIZZO: Matematica

CICLO: XXIV

Operators in Sobolev Morrey spaces

Direttore della Scuola: Ch.mo Prof. Paolo Dai Pra

Coordinatore d'indirizzo: Ch.mo Prof. Franco Cardin

Supervisore: Ch.mo Prof. Victor I. Burenkov
Ch.mo Prof. Massimo Lanza de Cristoforis

Dottoranda: Nurgul Kydyrmina

To my Father and Mother

Abstract

Morrey spaces were introduced by Charles Morrey in 1938. They are a useful tool in the regularity theory of partial differential equations, in real analysis and in mathematical physics.

In the nineties of the XX century an active study of general Morrey-type spaces characterized by a functional parameter has started to develop. A number of results on boundedness of classical operators in general Morrey-type spaces were obtained.

At the beginning of the XXI century there were new active developments in this area. In the last decade many mathematicians do research on smoothness spaces related to Morrey spaces. Among these spaces the Sobolev-type spaces play an important role.

In the thesis Sobolev spaces built on Morrey spaces are studied, which are also referred to as Sobolev Morrey spaces. These are spaces of functions which have derivatives up to certain order in Morrey spaces.

We analyze some basic properties of Morrey spaces and of Sobolev Morrey spaces. Then we consider the embedding and multiplication operators in Sobolev Morrey spaces. Finally, the dissertation provides a study of the composition operator in Sobolev Morrey spaces.

The results presented in the thesis have been obtained under supervision of Professors V.I. Burenkov and M. Lanza de Cristoforis.

Sunto

Gli spazi di Morrey sono stati introdotti da Charles Morrey nel 1938. Essi sono uno strumento utile nella teoria della regolarità per equazioni differenziali alle derivate parziali, in analisi reale ed in fisica matematica.

Negli anni novanta del XX secolo ha iniziato a svilupparsi un attivo studio degli spazi di Morrey di tipo generalizzato che sono caratterizzati da un parametro funzionale. È stato ottenuto un cero numero di risultati sulla limitatezza degli operatori classici negli spazi di Morrey di tipo generalizzato.

All'inizio del XXI secolo ci sono stati nuovi e attivi sviluppi in questa area. Nell'ultima decade molti matematici hanno svolto ricerche su spazi funzionali relativi agli spazi di Morrey. Tra questi spazi gli spazi di tipo Sobolev giocano un ruolo importante.

Nella tesi si studiano Spazi di Sobolev costruiti su spazi di Morrey, anche detti spazi di Sobolev Morrey. Questi sono spazi di funzioni che hanno derivate fino ad un certo ordine negli spazi di Morrey.

Si analizzano alcune proprietà di base degli spazi di Morrey e degli spazi di Sobolev-Morrey. Poi si considerano operatori di immersione e di moltiplicazione negli spazi di Sobolev Morrey. La terza parte della tesi presenta uno studio degli operatori di composizione negli spazi di Sobolev Morrey.

I risultati presentati nella tesi sono stati ottenuti sotto la supervisione dei Professori V.I. Burenkov and M. Lanza de Cristoforis.

Acknowledgements

First and foremost, I would like to express my sincere gratitude to both of my supervisors Prof. V.I. Burenkov and Prof. M. Lanza de Cristoforis for the continuous support of my Ph.D study and research, for their patience, motivation, and enthusiasm. Their guidance helped me in all the time of research and writing of this thesis. It has been an honor for me to be their Ph.D. student.

A special gratitude I give to Prof. E.S. Smailov who have always been a constant source of encouragement and inspiration during my graduate study. His unconditional support has been essential all these years.

I acknowledge the University of Padova for giving me the opportunity to study in Italy and for providing financial assistance.

Last but not the least, I would like to thank my family for their unflagging love and encouragement throughout my life. I thank my parents, Alim and Bagila, for their faith in me, for their prayers and supporting me whenever I needed it. I love you all dearly.

Contents

Introduction

This dissertation is devoted to Sobolev spaces built on Morrey spaces, also referred to as Sobolev Morrey spaces, *i.e.*, to the spaces of functions which have derivatives up to a certain order in Morrey spaces.

In the first part of the dissertation we analyze some basic properties of Morrey spaces and of Sobolev Morrey spaces. In particular,

(i) We characterize the functions in a Morrey space which can be approximated by smooth functions, as the functions which belong to a specific subspace of the Morrey space, which we call the 'little' Morrey space.

(ii) Contrary to the classical Sobolev spaces built on the L_p spaces with $p < \infty$, the Sobolev spaces built on Morrey spaces are not separable spaces even if $p < \infty$ and we cannot expect that the set of C^∞ functions of a Sobolev Morrey space be dense in a Sobolev Morrey space. However, we show that the functions in a Sobolev space built on little Morrey spaces can be approximated by C^∞ functions.

In the second part of the dissertation we consider the embedding and multiplication operators in Sobolev Morrey spaces. Namely,

(i) We prove a Sobolev Embedding Theorem for Sobolev Morrey spaces. The proof is based on the Sobolev Integral Representation Theorem and on a recent results on Riesz potentials in generalized Morrey spaces of Burenkov, Gogatishvili, Guliyev, Mustafaev [14] and on estimates on the Riesz potentials contained in the dissertation. We mention that a Sobolev Embedding Theorem for Sobolev Morrey spaces had been proved by Campanato [19, Thm. II.2, p. 75], for a subspace of our Sobolev Morrey space which corresponds to the closure of the set of smooth functions in our Sobolev Morrey space. The methods of the present dissertation are considerably different from those of Campanato.

(ii) We prove a multiplication Theorem for Sobolev Morrey spaces which extends to Sobolev Morrey spaces a known result of Zolesio [61] for classical Sobolev space (see also Valent [58], Runst and Sickel [51]).

Both the Sobolev Imbedding Theorem of (i) and the multiplication Theorem of (ii) have been proved for bounded domains with the cone property. We believe that one could prove the same type of results for unbounded domains with the cone property and in particular for the entire space.

Then in the third part of the dissertation, we consider the composition operator in Sobolev Morrey spaces, and as a first step we do so for Sobolev Morrey spaces of the first order. Let Ω be a bounded open subset of $\mathbb{R}^n$ with the cone property. Let $W_p^{1,\lambda}(\Omega)$ be the Sobolev space of functions with derivatives up to order 1 in the Morrey space $M_p^{\lambda}(\Omega)$ with exponents $\lambda \in [0, n/p]$, $p \in [1, +\infty]$.

Let Ω_1 be a bounded open subset of $\mathbb{R}$. Let $W_p^{1,\lambda}(\Omega, \Omega_1)$ denote the set of functions of $W_p^{1,\lambda}(\Omega)$ which map Ω to Ω_1.

Let $C^{0,1}(\bar{\Omega}_1)$ denote the space of Lipschitz continuous functions from $\bar{\Omega}_1$ to $\mathbb{R}$. Let r be a natural number. Let $C^r(\bar{\Omega}_1)$ denote the space of r times continuously differentiable functions from $\bar{\Omega}_1$ to $\mathbb{R}$.

Then we prove the following results.

(j) We prove that if $f \in C^{0,1}(\bar{\Omega}_1)$ and if $g \in W_p^{1,\lambda}(\Omega)$ has values in Ω_1, then the composite function $f \circ g$ belongs to $W_p^{1,\lambda}(\Omega)$ and the norm of $f \circ g$ can be estimated in terms of the norms of f and of g. We note that in case $\lambda = 0$, which corresponds to a classical Sobolev space such a result is well known (see Marcus and Mizel [35]).

(jj) We exploit an abstract scheme of Lanza de Cristoforis [30] and prove that if $(1 + \lambda) > n/p$, then the composition map T from $C^{r+1}(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$ which takes a pair (f, g) to the composite function $f \circ g$ is r-times continuously Fréchet differentiable. We note that in case $\lambda = 0$ the result of the present dissertation improves a corresponding result of Valent [58] for case $r = 1$.

(jjj) We prove that if $f \in C^{1,1}_{\mathrm{loc}}(\mathbb{R})$ and if $(1+\lambda) > n/p$, then the map which takes g to $f \circ g$ is Lipschitz continuous on the bounded subsets of $W^{1,\lambda}_p(\Omega)$. For a related result in the Besov space setting, we refer to Bourdaud and Lanza de Cristoforis [9].

We believe that our sufficient conditions on f of (j), (jj), (jjj) are optimal, just as they have been shown to be optimal in the frame of Sobolev spaces, which corresponds to case $\lambda = 0$ (see Appell and Zabreiko [4, Ch. 9], Runst and Sickel [51, Ch. 5], Bourdaud and Lanza de Cristoforis [9].)

We believe that by proving the Sobolev Imbedding Theorem of (i) and the multiplication Theorem of (ii) above for unbounded domains with the cone property, one could prove also the results of (j)–(jjj) for unbounded domains with the cone property and in particular for $\Omega = \mathbb{R}^n$.

The composition operator has been considered by several authors. For extensive references, we refer to the monographs of Appell and Zabreiko [4, Ch. 9], of Runst and Sickel [51], of Dudley and Norvaisa [23], and to the recent survey paper Bourdaud and Sickel [11]. In particular, the continuity, the Lipschitz continuity and the higher order differentiability of $f \circ g$ has a function of both f and g has long been investigated.

In the Sobolev space setting, we mention in particular Marcus and Mizel [34]–[40], Adams [3], Szigeti [56], [57], Valent [58], [59], Gol'dshtein and Reshetnyak [26], Drábek and Runst [22], Musina [41], Bourdaud and Meyer [10], Bourdaud [6], [7], Bourdaud and Kateb [8], Sickel [53]. As far as considering the differentiability of the composition operator when both the functions f and g belong to a Sobolev space, we mention a paper of Brokate and Colonuis [12], and of Lanza de Cristoforis [32].

The results of this dissertation will appear as joint work with the supervisors V.I. Burenkov and M. Lanza de Cristoforis.

Notation

$\mathbb{N}$ denotes the set of all natural numbers including 0. Throughout the paper, n is an element of $\mathbb{N} \setminus \{0\}$.

As usual, $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space, and

$$\mathbb{N}^n = \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{n}$$

is the set of multi-indices.

$\mathcal{P}(\mathbb{R}^n)$ – the linear space of polynomials with real coefficients and n real variables.

$B(x, r)$ – the open ball of radius $r > 0$ centered at the point $x \in \mathbb{R}^n$.

v_n – the volume of the unit ball in $\mathbb{R}^n$.

$\operatorname{supp} f$ – the support of a function f.

$f^{\leftarrow}(D)$ – f-preimage of a set D.

$D^\alpha \equiv \frac{\partial^{\alpha_1+\cdots+\alpha_n} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ – the (ordinary) derivative of the function f of order α, and

$D_w^\alpha \equiv \left(\frac{\partial^{\alpha_1+\cdots+\alpha_n} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}\right)_w$ – the weak derivative of the function f of order α.

For an arbitrary nonempty set $\Omega \subset \mathbb{R}^n$ we shall denote by:

$\operatorname{diam} \Omega$ – the diameter of Ω,

$\bar{\Omega}$ or $\mathrm{cl}(\Omega)$ – the closure of Ω,

χ_Ω – the characteristic function of Ω, *i.e.* $\chi_\Omega(\xi) = 1$ if $\xi \in \Omega$ and $\chi_\Omega(\xi) = 0$ if $\xi \in \mathbb{R}^n \setminus \Omega$,

$C^0(\Omega)$ – the space of functions continuous on Ω,

$C_b^0(\Omega)$ – the Banach space of functions continuous and bounded on Ω with the sup norm in Ω,

$C_{ub}^0(\Omega)$ – the Banach space of uniformly continuous and bounded functions on Ω with the sup norm in Ω,

$C^m(\Omega)$ ($m \in \mathbb{N}$) – the Banach space of m-times continuously differentiable functions on Ω,

$C^\infty(\Omega) = \bigcap\limits_{m\in\mathbb{N}} C^m(\Omega)$ – the space of infinitely continuously differentiable functions on Ω,

$C_c^\infty(\Omega)$ – the space of functions in $C^\infty(\Omega)$ with compact support.

For a measurable nonempty set $\Omega \subset \mathbb{R}^n$ we shall denote by:

$L_p(\Omega)$ ($1 \le p < \infty$) – the Banach space of functions f measurable on Ω such that the norm

$$\|f\|_{L_p(\Omega)} = \left(\int\limits_\Omega |f(x)|^p dx\right)^{\frac{1}{p}} < \infty.$$

$L_\infty(\Omega)$ – the Banach space of functions f measurable on Ω such that the norm

$$\|f\|_{L_\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < \infty.$$

For an open nonempty set $\Omega \subset \mathbb{R}^n$ we shall denote by:

$L_p^{\mathrm{loc}}(\Omega)$ ($1 \le p \le \infty$) – the set of functions f defined on Ω such that for each compact $K \subset \Omega$ $f \in L_p(K)$,

$B(\Omega) \equiv \{f\colon \Omega \to \mathbb{R}\colon f \text{ is bounded}\}$,

$\mathcal{M}(\Omega) \equiv \{f\colon \Omega \to \mathbb{R}\colon f \text{ is measurable}\}$,

$M(\Omega)$ – the factor space $\mathcal{M}(\Omega)/\Theta(\Omega)$, where $\Theta(\Omega)$ is the set of all functions defined on Ω which are equal to 0 almost everywhere on Ω,

$C^m(\bar{\Omega})$ – the subspace of $C^m(\Omega)$ of functions f such that f and its derivatives $D^\alpha f$ of order $|\alpha| \le m$ can be extended with continuity to $\bar{\Omega}$,

$\tilde{C}^\infty(\bar{\Omega})$ – the set of functions f from $\bar{\Omega}$ to $\mathbb{R}$ such that there exist an open neighborhood U of $\bar{\Omega}$ and a function $F \in C^\infty(U)$ such that the restriction of F to $\bar{\Omega}$ coincides with f.

Definition 0.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$. We denote by $C^{m,\alpha}(\bar{\Omega})$ the subspace of $C^m(\bar{\Omega})$ whose functions have m*th *order derivatives that are Hölder continuous with exponent $\alpha \in (0, 1]$.*

Definition 0.2. *By definition, a function f belongs to $C^\infty(\bar{\Omega})$ if $f \in C^\infty(\Omega)$ and for all $x \in \partial\Omega$ and for all $\alpha \in \mathbb{N}^n$ there exists the limit*

$$\lim_{\substack{y \to x \\ y \in \Omega}} D^\alpha f(y).$$

(By definition $D^\alpha f(x) = \lim_{\substack{y \to x \\ y \in \Omega}} D^\alpha f(y)$.)

Definition 0.3. *Let $p \in [1, +\infty]$. Let $\lambda > 0$, and m, k be two natural numbers such that $m < \lambda < m + k$. Then $f \in H_p^\lambda$ (Nikol'skii space) if and only if $f \in L_p$ and*

$$\sup_{|\alpha| = m,\, |h| \neq 0} |h|^{m-\lambda} \|\triangle_h^k D^\alpha f\|_{L_p} < +\infty.$$

This space does not depend on the choice of the integers k, m satisfying the inequality $m < \lambda < m + k$. We recall that H_p^λ is also known as the Besov space $B_{p,\infty}^\lambda$.

Definition 0.4. *Let V, Ω be open subsets of $\mathbb{R}^n$. We write*

$$V \subset\subset \Omega$$

if $V \subset \bar{V} \subset \Omega$ and $\bar{V}$ is compact, and say that V is compactly embedded in Ω.

Definition 0.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed space. By $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ we denote the normed space of the continuous linear maps of $\mathcal{X}$ to $\mathcal{Y}$ equipped with the topology of uniform convergence on the unit sphere of $\mathcal{X}$.*

Chapter 1

Morrey and Sobolev Morrey spaces

1.1 General Morrey spaces

Definition 1.1. *Let* Ω *be a Lebesgue measurable subset of* $\mathbb{R}^n$*. Let* $0 < p \leq +\infty$ *and let* w *be a measurable function from* $]0, +\infty[$ *to* $]0, +\infty[$*. Denote by* $\mathcal{M}_p^{w(\cdot)}(\Omega)$ *the space of all real-valued measurable functions on* Ω *for which*

$$\|f\|_{\mathcal{M}_p^{w(\cdot)}(\Omega)} = \sup_{x\in\Omega} \|w(\rho)\|f\|_{L_p(B(x,\rho)\cap\Omega)}\|_{L_\infty(0,\infty)} < \infty.$$

Definition 1.2. *Let* $0 < p \leq +\infty$*. Denote by* $\Lambda_{p,\infty}$ *the set of all measurable functions* w *from* $]0, +\infty[$ *to* $]0, +\infty[$ *which are not equivalent to* 0 *such that*

$$\|w(\rho)\|_{L_\infty(1,\infty)} < \infty, \quad \|w(\rho)\rho^{\frac{n}{p}}\|_{L_\infty(0,1)} < \infty.$$

In [15], [18] it is proved that, if w is a non-negative measurable function from $]0, +\infty[$ to $]0, +\infty[$ which are not equivalent to 0, then the space $\mathcal{M}_p^{w(\cdot)}(\Omega)$ is non-trivial, *i.e.* consists not only of functions f equivalent to 0 on Ω if, and only if, $w \in \Lambda_{p,\infty}$.

Definition 1.3. *If* $w_\lambda(\rho) = \begin{cases} \rho^{-\lambda}, & \rho \in]0, 1], \\ 1, & \rho \geq 1, \end{cases}$ *, then we set*

$$M_p^\lambda(\Omega) \equiv \mathcal{M}_p^{w_\lambda}(\Omega)$$

and the condition $w_\lambda \in \Lambda_{p,\infty}$ *means that* $0 \leq \lambda \leq \frac{n}{p}$*.*

Note that

Lemma 1.4.

$$\|f\|_{M_p^\lambda(\Omega)} = \max\left\{\sup_{x\in\Omega}\ \sup_{0<\rho<1}\ \rho^{-\lambda}\|f\|_{L_p(B(x,\rho)\cap\Omega)},\ \|f\|_{L_p(\Omega)}\right\}. \tag{1.1}$$

We find convenient to set

$$|f|_{\rho,w,p,\Omega} \equiv \sup_{x\in\Omega}\left\|w(r)\|f\|_{L_p(B(x,r)\cap\Omega)}\right\|_{L_\infty(0,\rho)} \quad \forall\rho\in]0,+\infty[,$$

and

$$|f|_{\rho,\lambda,p,\Omega} \equiv |f|_{\rho,w_\lambda,p,\Omega}$$

for all measurable functions f from Ω to $]0,+\infty[$ and for all functions w from $]0,+\infty[$ to $]0,+\infty[$. Clearly, $|f|_{\rho,w,p,\Omega}\in[0,+\infty]$.

Definition 1.5. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p\in[1,+\infty]$.*

(i) Let w be a function from $]0,+\infty[$ to $]0,+\infty[$. Then we define as generalized little Morrey space with weight w and exponent p the subspace

$$\mathcal{M}_p^{w,0}(\Omega) \equiv \left\{f\in\mathcal{M}_p^w(\Omega)\ :\ \lim_{\rho\to 0}|f|_{\rho,w,p,\Omega}=0\right\}$$

of $\mathcal{M}_p^w(\Omega)$.

(ii) Let $\lambda\in[0,+\infty[$. Then, in particular, the little Morrey space with exponents λ, p is the subspace

$$M_p^{\lambda,0}(\Omega) \equiv \left\{f\in M_p^\lambda(\Omega)\ :\ \lim_{\rho\to 0}|f|_{\rho,\lambda,p,\Omega}=0\right\}$$

of $M_p^\lambda(\Omega)$.

Example 1.6. *Let $0<\lambda<n/p$. Then*

1) The function $|x|^\alpha\in M_p^\lambda(B(0,1))$ if and only if $\alpha\geq\lambda-\frac{n}{p}$.

2) The function $|x|^\alpha\in M_p^{\lambda,0}(B(0,1))$ if and only if $\alpha>\lambda-\frac{n}{p}$.

Lemma 1.7. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p\in[1,+\infty]$. Let $w\in\Lambda_{p,\infty}$. Then $\mathcal{M}_p^{w,0}(\Omega)$ is a closed proper subspace of $\mathcal{M}_p^w(\Omega)$.*

Proof. Let $f\in\mathcal{M}_p^w(\Omega)$. Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathcal{M}_p^{w,0}(\Omega)$ which converges to f in $\mathcal{M}_p^w(\Omega)$. We want to prove that $f\in\mathcal{M}_p^{w,0}(\Omega)$.

Let $\varepsilon > 0$. Since $f_j \to f$ as $j \to \infty$, there exists $N_1 \in \mathbb{N}$ such that

$$|f - f_k|_{+\infty,w,p,\Omega} < \frac{\varepsilon}{2} \quad \forall\, k \geq N_1.$$

By definition 1.5, there exists $\delta > 0$ such that if $\rho \in]0, \delta[$, then

$$|f_{N_1}|_{\rho,w,p,\Omega} < \frac{\varepsilon}{2}.$$

Since $f = f - f_{N_1} + f_{N_1}$, we have

$$|f|_{\rho,w,p,\Omega} \leq |f - f_{N_1}|_{+\infty,w,p,\Omega} + |f_{N_1}|_{\rho,w,p,\Omega} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for all $\rho \in]0, \delta[$. $\square$

Lemma 1.8. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $p \in [1, +\infty]$. Then $\mathcal{M}_p^{r^{-\lambda}}(\Omega) = M_p^{\lambda}(\Omega)$. Moreover, the quasi-norm*

$$\|f\|_{\mathcal{M}_p^{\rho^{-\lambda}}(\Omega)} = \sup_{\substack{x\in\Omega \\ \rho>0}} \rho^{-\lambda} \|f\|_{L_p(B(x,\rho)\cap\Omega)} \quad \forall f \in \mathcal{M}_p^{\rho^{-\lambda}}(\Omega)$$

is equivalent to the quasi-norm

$$\|f\|_{M_p^{\lambda}(\Omega)} = \sup_{x\in\Omega} \|w_\lambda(\rho)\|f\|_{L_p(B(x,\rho)\cap\Omega)}\|_{L_\infty(0,\infty)} =$$
$$= \sup_{\substack{x\in\Omega \\ \rho>0}} w_\lambda(\rho)\|f\|_{L_p(B(x,\rho)\cap\Omega)} \quad \forall f \in M_p^{\lambda}(\Omega).$$

Proof. A simple calculation shows that

$$\|f\|_{\mathcal{M}_p^{\rho^{-\lambda}}(\Omega)} \leq \|f\|_{M_p^{\lambda}(\Omega)}.$$

Indeed, $\rho^{-\lambda} \leq w_\lambda(\rho)$ for all $\rho \in]0, +\infty[$ and thus

$$\|f\|_{\mathcal{M}_p^{\rho^{-\lambda}}(\Omega)} = \sup_{(x,\rho)\in\Omega\times]0,+\infty[} \rho^{-\lambda}\|f\|_{L_p(B(x,\rho)\cap\Omega)} \leq$$
$$\leq \sup_{(x,\rho)\in\Omega\times]0,+\infty[} w_\lambda(\rho)\|f\|_{L_p(B(x,\rho)\cap\Omega)} = \|f\|_{M_p^{\lambda}(\Omega)}.$$

Conversely,

$$\|f\|_{M_p^{\lambda}(\Omega)} = \sup_{(x,\rho)\in\Omega\times]0,+\infty[} w_\lambda(\rho)\|f\|_{L_p(B(x,\rho)\cap\Omega)} \leq$$

$$\leq \sup\left\{ \sup_{(x,\rho)\in\Omega\times]0,1]} \rho^{-\lambda}\|f\|_{L_p(B(x,\rho)\cap\Omega)},\ \sup_{(x,\rho)\in\Omega\times]1,+\infty[} 1\cdot\|f\|_{L_p(B(x,\rho)\cap\Omega)} \right\} =$$

$$= \sup \left\{ \sup_{(x,\rho)\in\Omega\times]0,1]} \rho^{-\lambda}\|f\|_{L_p(B(x,\rho)\cap\Omega)}, \sup_{(x,\rho)\in\Omega\times]1,+\infty[} \|f\|_{L_p(B(x,\rho)\cap\Omega)} \right\}.$$

Now we estimate

$$\sup_{(x,\rho)\in\Omega\times]1,+\infty[} \|f\|_{L_p(B(x,\rho)\cap\Omega)}.$$

If $1 < \rho < \operatorname{diam}\Omega$, then

$$\|f\|_{L_p(B(x,\rho)\cap\Omega)} \leq (\operatorname{diam}\Omega)^{\lambda}(\operatorname{diam}\Omega)^{-\lambda}\|f\|_{L_p(B(x,\rho)\cap\Omega)} \leq$$

$$\leq (\operatorname{diam}\Omega)^{\lambda} \sup_{(x,\rho)\in\Omega\times]1,\, \operatorname{diam}\Omega[} \rho^{-\lambda}\|f\|_{L_p(B(x,\rho)\cap\Omega)} \leq$$

$$\leq (\operatorname{diam}\Omega)^{\lambda} \sup_{(x,\rho)\in\Omega\times]0,+\infty[} \rho^{-\lambda}\|f\|_{L_p(B(x,\rho)\cap\Omega)} = (\operatorname{diam}\Omega)^{\lambda}\|f\|_{\mathcal{M}_p^{\rho^{-\lambda}}(\Omega)}.$$

If $\rho \geq \sup\{1, \operatorname{diam}\Omega\}$, then

$$\|f\|_{L_p(B(x,\rho)\cap\Omega)} = \|f\|_{L_p(\Omega)} \leq$$

$$\leq (\operatorname{diam}\Omega)^{\lambda}(\operatorname{diam}\Omega)^{-\lambda}\|f\|_{L_p(B(x,\, \operatorname{diam}\Omega)\cap\Omega)} \leq$$

$$\leq (\operatorname{diam}\Omega)^{\lambda} \sup_{(x,\rho)\in\Omega\times]0,+\infty[} \rho^{-\lambda}\|f\|_{L_p(B(x,\rho)\cap\Omega)} = (\operatorname{diam}\Omega)^{\lambda}\|f\|_{\mathcal{M}_p^{\rho^{-\lambda}}(\Omega)}.$$

Therefore,

$$\|f\|_{M_p^{\lambda}(\Omega)} \leq \max\{1, (\operatorname{diam}\Omega)^{\lambda}\}\|f\|_{\mathcal{M}_p^{\rho^{-\lambda}}(\Omega)},$$

and proof is complete. □

Lemma 1.9. *Let* Ω *be an open subset of* $\mathbb{R}^n$. *Let* $1 \leq p \leq +\infty$, $0 < \lambda < n/p$. *Then* $M_p^{\lambda}(\Omega) \subseteq L_p(\Omega)$.

Proof. Let $f \in M_p^{\lambda}(\Omega)$. We note that

$$w_{\lambda}(\rho)\|f\|_{L_p(B(x,\rho)\cap\Omega)} \leq \|f\|_{M_p^{\lambda}(\Omega)} < \infty \quad \text{for all } (x,\rho) \in \Omega\times]0,+\infty[.$$

Since $w_{\lambda}(\rho) = 1$ for $\rho \geq 1$, we obtain

$$\|f\|_{L_p(B(x,\rho)\cap\Omega)} \leq \|f\|_{M_p^{\lambda}(\Omega)} < \infty \quad \text{for all } (x,\rho) \in \Omega \times [1,+\infty[.$$

By taking supremum in $\rho \geq 1$ we get

$$\|f\|_{L_p(\Omega)} \leq \|f\|_{M_p^{\lambda}(\Omega)}.$$

□

Lemma 1.10. *Let* Ω *be an open subset of* $\mathbb{R}^n$. *Let* $0 < p < \infty$, $0 < \lambda < \frac{n}{p}$. *Then*

$$M_p^\lambda(\Omega) \not\subseteq L_q^{\mathrm{loc}}(\Omega)$$

for any $q > p$.

Proof. Without loss of generality, we can assume that $0 \in \Omega$ and $B(0,1) \subset \Omega$.

Let

$$f(x) = \begin{cases} (2^k k^{n-1})^{\frac{1}{p}}, & x \in B\left(0, \frac{1}{k}\right) \setminus B\left(0, \frac{1}{k} - 2^{-k} k^{-p\lambda-1}\right),\ k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$\left| B\left(0, \frac{1}{k}\right) \setminus B\left(0, \frac{1}{k} - 2^{-k} k^{-p\lambda-1}\right) \right| = v_n \left(\left(\frac{1}{k}\right)^n - \left(\frac{1}{k} - 2^{-k} k^{-\lambda p-1}\right)^n \right) \leq$$

$$\leq n\, v_n \left(\frac{1}{k}\right)^{n-1} 2^{-k} k^{-\lambda p-1} = \sigma_n 2^{-k} k^{-n-\lambda p-2}, \tag{1.2}$$

where v_n, σ_n respectively, is the volume, the surface area respectively, of the unit ball in $\mathbb{R}^n$. Similarly, since $\frac{1}{k} - 2^{-k} k^{-\lambda p-1} > \frac{1}{2k}$,

$$\left| B\left(0, \frac{1}{k}\right) \setminus B\left(0, \frac{1}{k} - 2^{-k} k^{-\lambda p-1}\right) \right| \geq 2^{1-n} \sigma_n 2^{-k} k^{-n-\lambda p-2}. \tag{1.3}$$

By using inequality (1.2) and the inequality

$$\sum_{k \geq a} \frac{1}{k^{\alpha+1}} \leq \left(1 + \frac{1}{\alpha}\right) \frac{1}{a^\alpha}, \quad \text{where } \alpha > 0,\ a \geq 1,$$

we get that for any $0 < r \leq 1$ and $x \in \mathbb{R}^n$

$$\|f\|_{L_p(B(x,r))}^p \leq \|f\|_{L_p(B(0,r))}^p =$$

$$= \sum_{\frac{1}{k} - 2^{-k} k^{-\lambda p-1} \leq r} 2^k k^{n-1} \left| B\left(0, \frac{1}{k}\right) \setminus B\left(0, \frac{1}{k} - 2^{-k} k^{-\lambda p-1}\right) \right| \leq$$

$$\leq \sigma_n \sum_{k \geq \frac{1}{2r}} \frac{1}{k^{\lambda p+1}} \leq \sigma_n \left(1 + \frac{1}{\lambda p}\right) 2^{\lambda p} r^{\lambda p}.$$

If $r \geq 1$ and $x \in \mathbb{R}^n$, then

$$\|f\|_{L_p(B(x,r))}^p \leq \|f\|_{L_p(B(0,r))}^p = \|f\|_{L_p(B(0,1))}^p = \sigma_n \left(1 + \frac{1}{\lambda p}\right) 2^{\lambda p}.$$

Therefore, $f \in M_p^\lambda(\Omega)$.

On the other hand, by (1.3) for any $q > p$

$$\|f\|^q_{L_q(B(0,r))} \geq \sum_{\frac{1}{k}\leq r} (2^k k^{n-1})^{\frac{q}{p}} \left| B\left(0, \frac{1}{k}\right) \setminus B\left(0, \frac{1}{k} - 2^{-k} k^{-\lambda p - 1}\right) \right| \geq$$

$$\geq 2^{1-n} \sigma_n \sum_{k \geq \frac{1}{r}} 2^{k\left(\frac{q}{p}-1\right)} k^{(n-1)\frac{q}{p} - n - \lambda p - 2} = \infty.$$

Hence, $f \notin L_q^{\mathrm{loc}}(\Omega)$. $\square$

Corollary 1.11. *Let Ω be an open subset of $\mathbb{R}^n$. Let $0 < p < \infty$, $0 < \lambda < \frac{n}{p}$. Then*

$$H_p^\lambda(\Omega) \subset M_p^\lambda(\Omega)$$

and this inclusion is strict.

Proof. The above inclusion was proved in [29] for $n = 1$ and in [49] for $n > 1$. The strictness of the inclusion follows since by the embedding theorem [48]

$$H_p^\lambda(\Omega) \subset L_q^{\mathrm{loc}}(\Omega)$$

with $q = \frac{np}{n - \lambda p} > p$. Hence, the function f constructed in the proof of the previous Lemma belongs to $M_p^\lambda(\Omega)$ but does not belong to $H_p^\lambda(\Omega)$. $\square$

Lemma 1.12. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$, $m_n(\Omega) < \infty$. Let $p \in [1, +\infty[$. Then the following statements hold.*

(i) If $\lambda \leq \frac{n}{p}$, then $L_\infty(\Omega) \subseteq M_p^\lambda(\Omega)$.

(ii) If $\lambda < \frac{n}{p}$, then $L_\infty(\Omega) \subseteq M_p^{\lambda,0}(\Omega)$.

Proof. (i) Let $f \in L^\infty(\Omega)$. Then we note that

$$\|f\|_{L_p(B(x,r)\cap\Omega)} \leq (m_n(B(x,r) \cap \Omega))^{\frac{1}{p}} \|f\|_{L_\infty(\Omega)}$$

and

$$\|f\|_{M_p^\lambda(\Omega)} = \sup_{x\in\Omega} \sup_{\rho>0} w_\lambda(\rho) \|f\|_{L_p(B(x,\rho)\cap\Omega)} \leq$$

$$\leq \max\left\{ \sup_{x\in\Omega} \sup_{0<\rho\leq 1} \rho^{-\lambda} (v_n \rho^n)^{\frac{1}{p}} \|f\|_{L_\infty(\Omega)},\right.$$

$$\left. \sup_{x\in\Omega} \sup_{\rho>1} (m_n(\Omega))^{\frac{1}{p}} \|f\|_{L_\infty(\Omega)} \right\} =$$

$$= \max\left\{ v_n^{\frac{1}{p}}, (m_n(\Omega))^{\frac{1}{p}} \right\} \|f\|_{L_\infty(\Omega)}.$$

(ii) Let $f \in L^\infty(\Omega)$. Then for all $\rho \in]0,1]$ we consider the norm

$$
\begin{aligned}
|f|_{\rho,w_\lambda,p,\Omega} = & \sup_{(x,r)\in\Omega\times]0,\rho[} w_\lambda(r)\|f\|_{L_p(B(x,r)\cap\Omega)} \leq \\
& \leq \sup_{(x,r)\in\Omega\times]0,\rho[} w_\lambda(r)(m_n(B(x,r)\cap\Omega))^{\frac{1}{p}}\|f\|_{L^\infty(\Omega)} \leq \\
& \leq \sup_{(x,r)\in\Omega\times]0,\rho[} r^{-\lambda}(m_n(B(x,r)))^{\frac{1}{p}}\|f\|_{L^\infty(\Omega)} \leq \\
& \leq \sup_{(x,r)\in\Omega\times]0,\rho[} r^{\frac{n}{p}-\lambda}v_n^{\frac{1}{p}}\|f\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad \rho \to 0.
\end{aligned}
$$

Hence, we obtain that $f \in M_p^{\lambda,0}(\Omega)$.

□

Corollary 1.13. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$. Let $p \in [1,+\infty[$, $\lambda \in [0,n/p]$. If $f \in L^\infty(\Omega)$ and $\operatorname{supp} f$ is compact, then $f \in M_p^\lambda(\Omega)$.*

Proof. Let $f \in L^\infty(\Omega)$. Then

$$
\begin{aligned}
\|f\|_{M_p^\lambda(\Omega)} = & \sup_{x\in\Omega}\sup_{\rho>0} w_\lambda(\rho)\|f\|_{L_p(B(x,\rho)\cap\Omega)} \leq \\
& \leq \sup_{x\in\Omega}\sup_{\rho>0} w_\lambda(\rho)(m_n(B(x,r)\cap\operatorname{supp} f))^{\frac{1}{p}}\|f\|_{L_\infty(\Omega)} \leq \\
& \leq \max\Bigg\{\sup_{x\in\Omega}\sup_{0<\rho\leq 1}\rho^{-\lambda}(v_n\rho^n)^{\frac{1}{p}}\|f\|_{L_\infty(\Omega)}, \\
& \sup_{x\in\Omega}\sup_{\rho>1}(m_n(\operatorname{supp} f))^{\frac{1}{p}}\|f\|_{L_\infty(\Omega)}\Bigg\} = \\
& = \max\left\{v_n^{\frac{1}{p}}, (m_n(\operatorname{supp} f))^{\frac{1}{p}}\right\}\|f\|_{L_\infty(\Omega)}.
\end{aligned}
$$

□

Corollary 1.14. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty[$, $\lambda \in [0,n/p]$. Then $C_c^\infty(\Omega) \subset M_p^\lambda(\Omega)$.*

Next we state a known result for Morrey spaces. For the sake of completeness we also give proofs.

Theorem 1.15. *Let Ω be a bounded open subset of $\mathbb{R}^n$. If $1 \leq p < +\infty$ then the following statements hold.*

(i) $M_p^0(\Omega) = L_p(\Omega)$;

(ii) If $\lambda = \frac{n}{p}$, then $M_p^\lambda(\Omega) = L_\infty(\Omega)$ both algebraically and topologically;

(iii) If $\frac{n}{p} < \lambda \le +\infty$, $1 \le p < +\infty$, *then* $M_p^\lambda(\Omega) = \{0\}$.

(iv) Let $0 < p \le q \le +\infty$ *and* $0 \le \lambda \le \frac{n}{p}$, $0 \le \nu \le \frac{n}{q}$. *If* $\frac{n}{q} - \nu \le \frac{n}{p} - \lambda$, *then* $M_q^\nu(\Omega) \hookrightarrow M_p^\lambda(\Omega)$.

Proof. (i) Let us take $f \in M_p^0(\Omega)$ and consider its norm in this space

$$\|f\|_{M_p^0(\Omega)} = \sup_{\substack{x\in\Omega\\ \rho>0}} w_0(\rho)\|f\|_{L_p(B(x,\rho)\cap\Omega)} = \sup_{\substack{x\in\Omega\\ \rho>0}} \|f\|_{L_p(B(x,\rho)\cap\Omega)} = \|f\|_{L_p(\Omega)}.$$

(ii) If $\lambda = \frac{n}{p}$, then by the Lebesgue Theorem

$$\|f\|_{\mathcal{M}_p^{\frac{n}{p}}(\Omega)} = \sup_{\substack{x\in\Omega\\ \rho>0}} \rho^{-\frac{n}{p}}\|f\|_{L_p(B(x,\rho)\cap\Omega)} = \upsilon_n^{\frac{1}{p}} \sup_{\substack{x\in\Omega\\ \rho>0}} \left(\frac{\int\limits_{B(x,\rho)\cap\Omega} |f(y)|^p dy}{\upsilon_n \rho^n}\right)^{\frac{1}{p}} \ge$$

$$\ge \upsilon_n^{\frac{1}{p}} \operatorname{ess\,sup} |f(x)| = \upsilon_n^{\frac{1}{p}} \|f\|_{L_\infty(\Omega)},$$

where υ_n is the volume of the unit ball in the space $\mathbb{R}^n$.

Suppose now that $f \in M_p^{\frac{n}{p}}(\Omega)$ and $f \notin L_\infty(\Omega)$. Since $f \notin L_\infty(\Omega)$, *i.e.* $\|f\|_{L_\infty(\Omega)} = \infty$, then for every $K > 0$ the set

$$S(f,K) = \{x \in \Omega\colon\ |f(x)| > K\}$$

has positive measure.

Denote by $\mathcal{A}$ the set of all points $x \in \Omega$ for which

$$|f(x)|^p = \lim_{\rho\to 0+} \frac{1}{m_n(B(x,\rho))} \int\limits_{B(x,\rho)} |f(y)|^p dy.$$

Thus, $\mathcal{A} \cap S(f,K)$ has a positive measure.

If $x \in \mathcal{A} \cap S(f,K)$, then

$$\lim_{\rho\to 0+} \frac{1}{m_n(B(x,\rho))} \int\limits_{B(x,\rho)\cap\Omega} |f(y)|^p dy = |f(x)|^p > K^p.$$

From this fact it easy to see that for every $K > 0$ there exists ρ such that

$$\frac{1}{m_n(B(x,\rho))} \int\limits_{B(x,\rho)\cap\Omega} |f(y)|^p dy > K,$$

and, thus,

$$K^{\frac{1}{p}} < \upsilon_n^{-\frac{1}{p}} \rho^{-\frac{n}{p}} \left(\int\limits_{B(x,\rho)\cap\Omega} |f(y)|^p dy \right)^{\frac{1}{p}} \leq \upsilon_n^{-\frac{1}{p}} \sup_{\substack{x\in\Omega \\ \rho>0}} \rho^{-\frac{n}{p}} \|f\|_{L_p(B(x,\rho)\cap\Omega)}.$$

So we have $\|f\|_{M_p^{\frac{n}{p}}(\Omega)} = \infty$. This contradicts the assumptions that $f \in M_p^{\frac{n}{p}}(\Omega)$.

Now let $f \in L_\infty(\Omega)$. For every ordered pair $(x,\rho) \in \Omega\times]0,+\infty[$ we have

$$\rho^{-\frac{n}{p}} \left(\int\limits_{B(x,\rho)\cap\Omega} |f(y)|^p dy \right)^{\frac{1}{p}} \leq \rho^{-\frac{n}{p}} \left(\int\limits_{B(x,\rho)\cap\Omega} dy \right)^{\frac{1}{p}} \|f\|_{L_\infty(\Omega)} \leq$$

$$\leq \rho^{-\frac{n}{p}} \upsilon_n^{\frac{1}{p}} \rho^{\frac{n}{p}} \|f\|_{L_\infty(\Omega)} = \upsilon_n^{\frac{1}{p}} \|f\|_{L_\infty(\Omega)} \quad \forall \rho \in]0,+\infty[.$$

Thus,

$$\|f\|_{M_p^{\frac{n}{p}}(\Omega)} \leq \upsilon_n^{\frac{1}{p}} \|f\|_{L_\infty(\Omega)}.$$

From this inequality follows the continuity of the identity operator $I:\, L_\infty(\Omega) \to M_p^{\frac{n}{p}}(\Omega)$.

(iii) Let $f \in M_p^\lambda(\Omega)$, then exploiting the Lebesgue Theorem we obtain

$$\|f\|_{M_p^\lambda(\Omega)} = \sup_{\substack{x\in\Omega \\ \rho>0}} \rho^{-\lambda} \|f\|_{L_p(B(x,\rho)\cap\Omega)} =$$

$$= \sup_{\substack{x\in\Omega \\ \rho>0}} \rho^{-\lambda+\frac{n}{p}} \left(\frac{\int\limits_{B(x,\rho)\cap\Omega} |f|^p dx}{\rho^n} \right)^{\frac{1}{p}} < +\infty \quad \Leftrightarrow \quad f(x) = 0 \text{ a.e.}$$

(iv) Let $f \in M_q^{\nu}(\Omega)$, then

$$
\begin{aligned}
\|f\|_{\mathcal{M}_p^{\rho^{-\lambda}}(\Omega)} &= \sup_{\substack{x\in\Omega\\ \rho>0}} \rho^{-\lambda}\|f\|_{L_p(B(x,\rho)\cap\Omega)} \le \\
&\le \sup_{\substack{x\in\Omega\\ \rho>0}} \rho^{-\lambda}[m_n(B(x,\rho)\cap\Omega)]^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L_q(B(x,\rho)\cap\Omega)} \le \\
&\le \max\left\{ \sup_{(x,\rho)\in\Omega\times]0,1]} \rho^{-\lambda}[m_n(B(x,\rho))]^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L_q(B(x,\rho)\cap\Omega)}, \right. \\
&\left. \sup_{(x,\rho)\in\Omega\times[1,+\infty[} \rho^{-\lambda}[m_n(\Omega)]^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L_q(B(x,\rho)\cap\Omega)} \right\} \le \\
&\le \max\left\{ \sup_{(x,\rho)\in\Omega\times]0,1]} \upsilon_n^{\frac{1}{p}-\frac{1}{q}}\rho^{-\lambda+\frac{n}{p}-\frac{n}{q}}\|f\|_{L_q(B(x,\rho)\cap\Omega)}, \right. \\
&\left. [m_n(\Omega)]^{\frac{1}{p}-\frac{1}{q}} \sup_{(x,\rho)\in\Omega\times[1,+\infty[} w_\nu(\rho)\|f\|_{L_q(B(x,\rho)\cap\Omega)} \right\} \le \\
&\le \max\left\{ \upsilon_n^{\frac{1}{p}-\frac{1}{q}}, [m_n(\Omega)]^{\frac{1}{p}-\frac{1}{q}} \right\} \|f\|_{M_q^{\nu}(\Omega)},
\end{aligned}
$$

where υ_n is the volume of the unit ball in the space $\mathbb{R}^n$.

□

Theorem 1.16. *Let Ω be an open subset of $\mathbb{R}^n$. Let p_1, $p_2 \in [1,+\infty]$ be such that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$. Let $\lambda_1,\lambda_2 \in [0,+\infty[$, $\lambda=\lambda_1+\lambda_2$. Then the pointwise multiplication is bilinear and continuous from $M_{p_1}^{\lambda_1}(\Omega)\times M_{p_2}^{\lambda_2}(\Omega)$ to $M_p^{\lambda}(\Omega)$ and maps $M_{p_1}^{\lambda_1,0}(\Omega)\times M_{p_2}^{\lambda_2}(\Omega)$ to $M_p^{\lambda,0}(\Omega)$ and $M_{p_1}^{\lambda_1}(\Omega)\times M_{p_2}^{\lambda_2,0}(\Omega)$ to $M_p^{\lambda,0}(\Omega)$*

Remark 1.17. *This statement proves the Hölder inequality for Morrey space $M_p^{\lambda}(\Omega)$:*

$$
\|fg\|_{M_p^{\lambda}(\Omega)} \le \|f\|_{M_{p_1}^{\lambda_1}(\Omega)}\|g\|_{M_{p_2}^{\lambda_2}(\Omega)} \quad \forall\,(f,g)\in M_{p_1}^{\lambda_1}(\Omega)\times M_{p_2}^{\lambda_2}(\Omega).
$$

Proof. Note that

$$
w_\lambda(\rho)=\begin{cases}\rho^{-\lambda}, & \rho\in]0,1],\\ 1, & \rho\ge 1,\end{cases} = \begin{cases}\rho^{-\lambda_1-\lambda_2}, & \rho\in]0,1],\\ 1, & \rho\ge 1,\end{cases} = w_{\lambda_1}(\rho)w_{\lambda_2}(\rho).
$$

Then, by Hölder inequality, we have

$$
\begin{aligned}
|fg|_{\rho,\lambda,p,\Omega} &= \sup_{(x,r)\in\Omega\times]0,\rho[} w_\lambda(r)\|fg\|_{L_p(B(x,r)\cap\Omega)} \le \\
&\le \sup_{(x,r)\in\Omega\times]0,\rho[} w_\lambda(r)\|f\|_{L_{p_1}(B(x,r)\cap\Omega)}\|g\|_{L_{p_2}(B(x,r)\cap\Omega)} \le
\end{aligned}
$$

$$\leq \sup_{(x,r)\in\Omega\times]0,\rho[} w_{\lambda_1}(r)\|f\|_{L_{p_1}(B(x,r)\cap\Omega)} \sup_{(x,r)\in\Omega\times]0,\rho[} w_{\lambda_2}(r)\|g\|_{L_{p_2}(B(x,r)\cap\Omega)} =$$

$$= |f|_{\rho,\lambda_1,p_1,\Omega}|g|_{\rho,\lambda_2,p_2,\Omega} \quad \text{for all} \quad \rho \in]0,+\infty].$$

Therefore, by taking $\rho = +\infty$, we deduce that $fg \in M_p^\lambda(\Omega)$ when $(f,g) \in M_{p_1}^{\lambda_1}(\Omega) \times M_{p_2}^{\lambda_2}(\Omega)$.

By letting $\rho \to 0$, we deduce that $fg \in M_p^{\lambda,0}(\Omega)$ when $(f,g) \in M_{p_1}^{\lambda_1,0}(\Omega) \times M_{p_2}^{\lambda_2}(\Omega)$.

The case when $f \in M_{p_1}^{\lambda_1}(\Omega)$ and $g \in M_{p_2}^{\lambda_2,0}(\Omega)$ can be analyzed in the same way. □

Theorem 1.18. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$. Let $\lambda \in [0,n/p]$. Then the pointwise multiplication is bilinear and continuous from $M_p^\lambda(\Omega) \times L_\infty(\Omega)$ to $M_p^\lambda(\Omega)$ and maps $M_p^{\lambda,0}(\Omega) \times L_\infty(\Omega)$ to $M_p^{\lambda,0}(\Omega)$.*

Proof. We show that if $f \in M_p^\lambda(\Omega)$, $g \in L_\infty(\Omega)$, then

$$|fg|_{\rho,\lambda,p,\Omega} = \sup_{(x,r)\in\Omega\times]0,\rho[} w_\lambda(r)\|fg\|_{L_p(B(x,r)\cap\Omega)} \leq$$

$$\leq \|g\|_{L_\infty(\Omega)} \sup_{(x,r)\in\Omega\times]0,\rho[} w_\lambda(\rho)\|f\|_{L_p(B(x,r)\cap\Omega)} =$$

$$= \|g\|_{L_\infty(\Omega)}|f|_{\rho,\lambda,p,\Omega} \quad \text{for all} \quad \rho \in]0,+\infty].$$

Hence, by taking $\rho = +\infty$, we deduce that $fg \in M_p^\lambda(\Omega)$ when $(f,g) \in M_p^\lambda(\Omega) \times L_\infty(\Omega)$.

By letting $\rho \to 0$, we deduce that $fg \in M_p^{\lambda,0}(\Omega)$ when $(f,g) \in M_p^{\lambda,0}(\Omega) \times L_\infty(\Omega)$. □

Lemma 1.19. *Let $A \subset \mathbb{R}^m$ be a measurable set. Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$. Let $\lambda \in [0,n/p]$. Suppose that f is a measurable from $A \times \Omega$ to $\mathbb{R}$. Let $f(\cdot,y) \in M_p^\lambda(\Omega)$ for almost all $y \in A$ and $\int_A \|f(\cdot,y)\|_{M_p^\lambda(\Omega)}\,dy < +\infty$. Then for almost all $x \in \Omega$ the integral $\int_A f(x,y)dy$ makes sense and Minkowski's inequality for Morrey spaces*

$$\left\|\int_A f(\cdot,y)dy\right\|_{M_p^\lambda(\Omega)} \leq \int_A \|f(\cdot,y)\|_{M_p^\lambda(\Omega)}\,dy$$

holds.

Proof. By Minkowski's inequality for the Lebesgue spaces and by the imbedding of $M_p^\lambda(\Omega)$ into $L_p(\Omega)$ we know that for almost all $x \in \Omega$ the integral $\int_A f(x,y)dy$ makes sense and defines almost everywhere a function of $L_p(\Omega)$.

Then by applying the Minkowski's inequality for the Lebesgue spaces in $(B(x,\rho)\cap\Omega)\times A$ for all $\rho \in]0,+\infty[$, we obtain the following inequality

$$\left\| \int_A f(\cdot,y)dy \right\|_{M_p^\lambda(\Omega)} = \sup_{(x,\rho)\in\Omega\times]0,+\infty[} w_\lambda(\rho) \left\| \int_A f(\cdot,y)dy \right\|_{L_p(B(x,\rho)\cap\Omega)} \le$$

$$\le \sup_{(x,\rho)\in\Omega\times]0,+\infty[} w_\lambda(\rho) \int_A \|f(\cdot,y)\|_{L_p(B(x,\rho)\cap\Omega)}\, dy =$$

$$= \int_A \sup_{(x,\rho)\in\Omega\times]0,+\infty[} w_\lambda(\rho)\, \|f(\cdot,y)\|_{L_p(B(x,\rho)\cap\Omega)}\, dy = \int_A \|f(\cdot,y)\|_{M_p^\lambda(\Omega)}\, dy.$$

□

1.2 Approximation by C^∞ functions in Morrey spaces

Definition 1.20. *If $\phi \in L^1(\mathbb{R}^n)$ and $t \in]0,+\infty[$, we denote by $\phi_t(\cdot)$ the function from $\mathbb{R}^n$ to $\mathbb{R}$ defined by*

$$\phi_t(x) \equiv t^{-n}\phi(x/t) \quad \forall\, x \in \mathbb{R}^n.$$

By the formula of change of variables in integrals, we conclude that

$$\int_{\mathbb{R}^n} \phi_t(x)dx = \int_{\mathbb{R}^n} \phi(x)dx \quad \forall\, t \in]0,+\infty[,$$

whenever $\phi \in L^1(\mathbb{R}^n)$.

Lemma 1.21. *Let $p \in [1,+\infty[$, $0 \le \lambda < \frac{n}{p}$ and $f \in M_p^{\lambda,0}(\mathbb{R}^n)$. Then*

$$\lim_{k\to\infty} f\chi_{B(0,k)} = f \quad \textit{in } M_p^\lambda(\mathbb{R}^n).$$

Proof. Consider for $0 < \rho \le 1$ the norm

$$\|f\chi_{B(0,k)} - f\|_{M_p^\lambda(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \|w_\lambda(r)\|f\chi_{B(0,k)} - f\|_{L_p(B(x,r))}\|_{L_\infty(0,\infty)} =$$

$$= \sup_{x\in\mathbb{R}^n} \|w_\lambda(r)\|f\|_{L_p(B(x,r)\setminus B(0,k))}\|_{L_\infty(0,\infty)} \le$$

$$\le \sup_{x\in\mathbb{R}^n} \|w_\lambda(r)\|f\|_{L_p(B(x,r))}\|_{L_\infty(0,\rho)} + \max\{\rho^{-\lambda},1\}\|f\|_{L_p(\mathbb{R}^n\setminus B(0,k))} =$$

$$= |f|_{\rho,\lambda,p,\mathbb{R}^n} + \rho^{-\lambda}\|f\|_{L_p(\mathbb{R}^n\setminus B(0,k))}.$$

Let $k\to\infty$, then since $f\in L_p(\mathbb{R}^n)$ (by Lemma 1.9) and $p<\infty$ we have

$$\lim_{k\to\infty}\|f\|_{L_p(\mathbb{R}^n\setminus B(0,k))} = 0.$$

Therefore, for all $0<\rho\le 1$

$$\overline{\lim_{k\to\infty}}\|f\chi_{B(0,k)} - f\|_{M_p^\lambda(\mathbb{R}^n)} \le |f|_{\rho,\lambda,p,\mathbb{R}^n}.$$

By passing to the limit as $\rho\to 0$, since $f\in M_p^{\lambda,0}(\mathbb{R}^n)$, we have

$$\overline{\lim_{k\to\infty}}\|f\chi_{B(0,k)} - f\|_{M_p^\lambda(\mathbb{R}^n)} = 0,$$

or equivalently

$$\lim_{k\to\infty}\|f\chi_{B(0,k)} - f\|_{M_p^\lambda(\mathbb{R}^n)} = 0.$$

□

Then we have the following result of approximation by convolution.

Theorem 1.22. *Let $\phi\in C_c^\infty(\mathbb{R}^n)$, $\int\limits_{\mathbb{R}^n}\phi(x)dx = 1$. Then the following statements hold.*

(i) Let $p\in[1,+\infty]$, $\lambda\in\left[0,\frac{n}{p}\right]$. If $f\in M_p^\lambda(\mathbb{R}^n)$ and $\varepsilon>0$, then the function $f*\phi_\varepsilon$ from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$f*\phi_\varepsilon \equiv \int\limits_{\mathbb{R}^n} f(x-y)\phi_\varepsilon(y)dy \quad \forall\, x\in\mathbb{R}^n$$

belongs to $M_p^\lambda(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)$ and

$$\|f*\phi_\varepsilon\|_{M_p^\lambda(\mathbb{R}^n)} \le \|\phi\|_{L_1(\mathbb{R}^n)}\|f\|_{M_p^\lambda(\mathbb{R}^n)} \quad \forall\, f\in M_p^\lambda(\mathbb{R}^n).$$

(ii) Let $p\in[1,+\infty]$, $\lambda\in\left[0,\frac{n}{p}\right]$. If $f\in M_p^{\lambda,0}(\mathbb{R}^n)$ and $\varepsilon>0$, then $f*\phi_\varepsilon$ belongs to $M_p^{\lambda,0}(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)$.

(iii) Let $p\in[1,+\infty[$. If $f\in M_p^{\lambda,0}(\mathbb{R}^n)$, then $f*\phi_\varepsilon$ belongs to $M_p^{\lambda,0}(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)\cap C^0_{ub}(\mathbb{R}^n)$ for all $\varepsilon\in]0,+\infty[$ and

$$\lim_{\varepsilon\to 0} f*\phi_\varepsilon = f \quad \text{in } M_p^\lambda(\mathbb{R}^n). \tag{1.4}$$

(iv) Let $p \in [1,+\infty[$. *Then*

$$\mathrm{cl}_{M_p^\lambda(\mathbb{R}^n)} C_c^\infty(\mathbb{R}^n) = M_p^{\lambda,0}(\mathbb{R}^n). \tag{1.5}$$

Proof. (i) Let $f \in M_p^\lambda(\mathbb{R}^n)$ and $\varepsilon > 0$, then

$$w_\lambda(\rho)\|f * \phi_\varepsilon\|_{L_p(B(x,\rho))} \le w_\lambda(\rho) \left\| \int_{\mathbb{R}^n} f(\xi - y)\phi_\varepsilon(y)dy \right\|_{L_{p,\xi}(B(x,\rho))} \le$$

$$\le w_\lambda(\rho) \int_{B(0,\varepsilon)} \|f(\xi - y)\|_{L_{p,\xi}(B(x,\rho))}|\phi_\varepsilon(y)|dy =$$

$$= w_\lambda(\rho) \left(\int_{\mathbb{R}^n} |\phi_\varepsilon(y)|dy \right) \sup_{y \in B(0,\varepsilon)} \|f(\xi - y)\|_{L_{p,\xi}(B(x,\rho))} =$$

$$= w_\lambda(\rho) \left(\int_{\mathbb{R}^n} |\phi(y)|dy \right) \sup_{y \in B(0,\varepsilon)} \|f(z)\|_{L_p(B(x-y,\rho))} =$$

$$= w_\lambda(\rho) \left(\int_{\mathbb{R}^n} |\phi(y)|dy \right) \sup_{z \in B(x,\varepsilon)} \|f\|_{L_p(B(z,\rho))} \le$$

$$\le w_\lambda(\rho) \left(\int_{\mathbb{R}^n} |\phi(y)|dy \right) \sup_{z \in \mathbb{R}^n} \|f\|_{L_p(B(z,\rho))} \quad \text{for all } \rho \in]0,+\infty[.$$

Thus, we have

$$\|f * \phi_\varepsilon\|_{M_p^\lambda(\mathbb{R}^n)} \le \left(\int_{\mathbb{R}^n} |\phi|dx \right) \|f\|_{M_p^\lambda(\mathbb{R}^n)} \quad \forall\, f \in M_p^\lambda(\mathbb{R}^n).$$

(ii) Let $f \in M_p^{\lambda,0}(\mathbb{R}^n)$ and $\varepsilon > 0$. In the proof of statement (i) we have proved that

$$w_\lambda(r)\|f * \phi_\varepsilon\|_{L_p(B(x,r))} \le$$
$$w_\lambda(r) \left(\int_{\mathbb{R}^n} |\phi(x)|dx \right) \sup_{x \in \mathbb{R}^n} \|f\|_{L_p(B(x,r))} \quad \text{for all } r \in]0,+\infty[.$$

Moreover,

$$w_\lambda(r)\|f\|_{L_p(B(x,r))} \le |f|_{\rho,\lambda,p,\mathbb{R}^n} \quad \forall\, x \in \mathbb{R}^n,\ r \in]0,\rho[,$$

and thus, by taking the supremum on $x \in \mathbb{R}^n$, we have

$$w_\lambda(r) \sup_{x \in \mathbb{R}^n} \|f\|_{L_p(B(x,r))} \le |f|_{\rho,\lambda,p,\mathbb{R}^n} \quad \forall\, r \in]0,\rho[.$$

Hence,

$$\sup_{r\in]0,\rho[} w_\lambda(r)\|f*\phi_\varepsilon\|_{L_p(B(x,r))} \le \left(\int_{\mathbb{R}^n} |\phi(x)|dx\right) |f|_{\rho,w,p,\mathbb{R}^n},$$

and

$$\lim_{\rho\to 0} |f*\phi_\varepsilon|_{\rho,\lambda,p,\mathbb{R}^n} = 0 \quad \forall\, f \in M_p^{\lambda,0}(\mathbb{R}^n).$$

(iii) Let $\eta > 0$. Since $f \in M_p^{\lambda,0}$, there exists $\rho_\eta > 0$ such that

$$|f|_{\rho,\lambda,p,\mathbb{R}^n} \le \eta \left(1 + \int_{\mathbb{R}^n} |\phi(y)|dy\right)^{-1} \qquad \forall\, \rho \in]0,\rho_\eta].$$

Then we have

$$\sup_{(x,r)\in\mathbb{R}^n\times]0,\rho_\eta[} w_\lambda(r)\|f - f*\phi_\varepsilon\|_{L_p(B(x,r))} \le$$

$$\le \sup_{(x,r)\in\mathbb{R}^n\times]0,\rho_\eta[} w_\lambda(r)\|f\|_{L_p(B(x,r))} + \sup_{(x,r)\in\mathbb{R}^n\times]0,\rho_\eta[} w_\lambda(r)\|f*\phi_\varepsilon\|_{L_p(B(x,r))} \le$$

$$\le |f|_{\rho_\eta,\lambda,p,\mathbb{R}^n} + \left(\int_{\mathbb{R}^n} |\phi(y)|dy\right) |f|_{\rho_\eta,\lambda,p,\mathbb{R}^n} =$$

$$= |f|_{\rho_\eta,\lambda,p,\mathbb{R}^n} \left(1 + \int_{\mathbb{R}^n} |\phi(y)|dy\right) \le \eta, \tag{1.6}$$

for all $\varepsilon \in]0,+\infty[$.

Further, we have

$$\sup_{(x,r)\in\mathbb{R}^n\times[\rho_\eta,+\infty[} w_\lambda(r)\|f - f*\phi_\varepsilon\|_{L_p(B(x,r))} \le$$

$$\le \left(\sup_{r\in[\rho_\eta,+\infty[} w_\lambda(r)\right) \|f - f*\phi_\varepsilon\|_{L_p(\mathbb{R}^n)} \quad \text{for all } \varepsilon > 0.$$

Since $f \in L_p(\mathbb{R}^n)$ (by Lemma 1.9) and $p \in [1,+\infty[$ and $\int_{\mathbb{R}^n} |\phi(y)|dy = 1$, standard properties of approximate identities of convolution imply that

$$\lim_{\varepsilon\to 0} \|f - f*\phi_\varepsilon\|_{L_p(\mathbb{R}^n)} = 0.$$

Thus, there is exists $\varepsilon_\eta > 0$ such that

$$\left(\sup_{r\in[\rho_\eta,+\infty[} w_\lambda(r)\right) \|f - f*\phi_\varepsilon\|_{L_p(\mathbb{R}^n)} \le \eta \quad \forall\, \varepsilon \in]0,\varepsilon_\eta].$$

So, we obtain

$$\sup_{(x,r)\in\mathbb{R}^n\times[\rho_\eta,+\infty[} w_\lambda(r)\|f-f*\phi_\varepsilon\|_{L_p(B(x,r))}\le\eta \quad \forall\,\varepsilon\in]0,\varepsilon_\eta]. \tag{1.7}$$

By combining inequalities (1.6) and (1.7), we have

$$\sup_{(x,r)\in\mathbb{R}^n\times[0,+\infty[} w_\lambda(r)\|f-f*\phi_\varepsilon\|_{L_p(B(x,r))}\le\eta \quad \forall\,\varepsilon\in]0,\varepsilon_\eta].$$

Hence,

$$\lim_{\varepsilon\to0} f*\phi_\varepsilon=f \quad \text{in} \quad M_p^\lambda(\mathbb{R}^n).$$

(iv) Clearly, $C_c^\infty(\mathbb{R}^n)$ is contained in $M_p^{\lambda,0}(\mathbb{R}^n)$. Since $M_p^{\lambda,0}(\mathbb{R}^n)$ is closed in $M_p^\lambda(\mathbb{R}^n)$ (by Lemma 1.7), we obtain

$$\mathrm{cl}_{M_p^\lambda(\mathbb{R}^n)}C_c^\infty(\mathbb{R}^n)\subset\mathrm{cl}_{M_p^\lambda(\mathbb{R}^n)}M_p^{\lambda,0}(\mathbb{R}^n)=M_p^{\lambda,0}(\mathbb{R}^n).$$

Now let $f\in M_p^{\lambda,0}(\mathbb{R}^n)$. Then for all $k\in\mathbb{N}$

$$f\chi_{B(0,k)}*\phi_{\frac{1}{k}}\in C_c^\infty(\mathbb{R}^n).$$

Next, using (i) of Theorem 1.22, we obtain

$$\begin{aligned}
&\|f\chi_{B(0,k)}*\phi_{\frac1k}-f\|_{M_p^\lambda(\mathbb{R}^n)}\le\\
&\qquad\le\|(f\chi_{B(0,k)}-f)*\phi_{\frac1k}\|_{M_p^\lambda(\mathbb{R}^n)}+\|f*\phi_{\frac1k}-f\|_{M_p^\lambda(\mathbb{R}^n)}\le\\
&\qquad\qquad\le\|f\chi_{B(0,k)}-f\|_{M_p^\lambda(\mathbb{R}^n)}\|\phi\|_{L_1(\mathbb{R}^n)}+\|f*\phi_{\frac1k}-f\|_{M_p^\lambda(\mathbb{R}^n)}.
\end{aligned}$$

By Lemma 1.21

$$\lim_{k\to\infty}\|f\chi_{B(0,k)}-f\|_{M_p^\lambda(\mathbb{R}^n)}=0,$$

and by (iii) of Theorem 1.22

$$\lim_{k\to\infty}\|f*\phi_{\frac1k}-f\|_{M_p^\lambda(\mathbb{R}^n)}=0.$$

So, we obtain that

$$\lim_{k\to\infty}\|f\chi_{B(0,k)}*\phi_{\frac1k}-f\|_{M_p^\lambda(\mathbb{R}^n)}=0.$$

Thus, $f\in\mathrm{cl}_{M_p^\lambda(\mathbb{R}^n)}C_c^\infty(\mathbb{R}^n)$ and equality (1.5) holds.

□

Remark 1.23. *The limiting relation (1.4) does not hold for all* $f \in M_p^\lambda(\mathbb{R}^n)$. *For example,*

$$|x|^{\lambda-\frac{n}{p}}\chi_{B(0,1)} * \phi_\varepsilon \nrightarrow |x|^{\lambda-\frac{n}{p}}\chi_{B(0,1)} \quad \text{in } M_p^\lambda(\mathbb{R}^n) \tag{1.8}$$

as $\varepsilon \to 0$.

Indeed, function $|x|^{\lambda-\frac{n}{p}}\chi_{B(0,1)} * \phi_\varepsilon$ *belongs to* $C_c^\infty(\mathbb{R}^n)$. *If*

$$|x|^{\lambda-\frac{n}{p}}\chi_{B(0,1)} * \phi_\varepsilon \to |x|^{\lambda-\frac{n}{p}}\chi_{B(0,1)} \quad \text{in } M_p^\lambda(\mathbb{R}^n)$$

then by (iv) of Theorem 1.22 $|x|^{\lambda-\frac{n}{p}}\chi_{B(0,1)} \in \mathrm{cl}_{M_p^\lambda(\mathbb{R}^n)}C_c^\infty(\mathbb{R}^n) = M_p^{\lambda,0}(\mathbb{R}^n)$, *which is not true (see Example 1.6).*

Lemma 1.24. *Let* Ω *be an open subset of* $\mathbb{R}^n$. *Let* E_Ω *be the extension operator from* $M(\Omega)$ *to* $M(\mathbb{R}^n)$ *defined by*

$$E_\Omega f \equiv \begin{cases} f, & \text{in } \Omega, \\ 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

Let $p \in [1, +\infty]$, $\lambda \in [0, n/p]$. *Then for all* $f \in M_p^\lambda(\Omega)$

$$|E_\Omega f|_{\rho, w_\lambda, p, \mathbb{R}^n} \le 2^\lambda |f|_{2\rho, w_\lambda, p, \Omega} \tag{1.9}$$

for all $\rho \in]0, +\infty]$ *and*

$$\|E_\Omega f\|_{M_p^\lambda(\mathbb{R}^n)} \le 2^\lambda \|f\|_{M_p^\lambda(\Omega)}. \tag{1.10}$$

In particular, E_Ω *maps* $M_p^\lambda(\Omega)$ *to* $M_p^\lambda(\mathbb{R}^n)$ *and* $M_p^{\lambda,0}(\Omega)$ *to* $M_p^{\lambda,0}(\mathbb{R}^n)$.

Proof. Let $\rho \in]0, +\infty]$. Then we have

$$|E_\Omega f|_{\rho, w_\lambda, p, \mathbb{R}^n} = \sup_{(x,r)\in\mathbb{R}^n\times]0,\rho[} w_\lambda(r)\|E_\Omega f\|_{L_p(B(x,r))} =$$

$$= \sup_{0<r<\rho} \sup_{x\in\mathbb{R}^n} w_\lambda(r)\|f\|_{L_p(B(x,r)\cap\Omega)} =$$

$$= \sup_{0<r<\rho} \sup_{\substack{x\in\mathbb{R}^n \\ B(x,r)\cap\Omega\neq\varnothing}} w_\lambda(r)\|f\|_{L_p(B(x,r)\cap\Omega)}.$$

If $(x,r) \in \mathbb{R}^n\times]0,\rho[$ and $B(x,r)\cap\Omega \neq \varnothing$, then there exists $\xi(x) \in B(x,r)\cap\Omega$. By the triangle inequality, we have

$$B(x,r)\cap\Omega \subset B(\xi(x), 2r)\cap\Omega,$$

hence,

$$|E_\Omega f|_{\rho,w_\lambda,p,\mathbb{R}^n} \le \sup_{0<r<\rho} \sup_{\substack{x\in\mathbb{R}^n \\ B(x,r)\cap\Omega\neq\varnothing}} w_\lambda(r)\|f\|_{L_p(B(\xi(x),2r)\cap\Omega)} \le$$

$$\le \sup_{0<r<\rho} \sup_{\eta\in\Omega} w_\lambda(r)\|f\|_{L_p(B(\eta,2r)\cap\Omega)}.$$

We also note that

$$w_\lambda(\rho) \le 2^\lambda w_\lambda(2\rho) \quad \forall\,\rho\in[0,+\infty[.$$

Therefore,

$$|E_\Omega f|_{\rho,w_\lambda,p,\mathbb{R}^n} \le 2^\lambda \sup_{\eta\in\Omega} \sup_{0<r<\rho} w_\lambda(2r)\|f\|_{L_p(B(\eta,2r)\cap\Omega)} = 2^\lambda |f|_{2\rho,w_\lambda,p,\Omega}.$$

Inequality (1.10) follows by inequality (1.9). □

Theorem 1.25. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p\in[1,+\infty[$. Then the following statements hold.*

(i) $\mathrm{cl}_{M_p^\lambda(\Omega)}\left(M_p^{\lambda,0}(\Omega)\cap\tilde{C}^\infty(\bar\Omega)\cap C^0_{ub}(\bar\Omega)\right) =$

$$= \mathrm{cl}_{M_p^\lambda(\Omega)}\left(M_p^{\lambda,0}(\Omega)\cap C^\infty(\bar\Omega)\cap C^0_{ub}(\bar\Omega)\right) = M_p^{\lambda,0}(\Omega).$$

(ii) If $m_n(\Omega)<\infty$ and if $\lambda<\frac{n}{p}$, then $C^0_{ub}(\bar\Omega)\subseteq L^\infty(\Omega)\subseteq M_p^{\lambda,0}(\Omega)$ and

$$\mathrm{cl}_{M_p^\lambda(\Omega)}\left(\tilde{C}^\infty(\bar\Omega)\cap C^0_{ub}(\bar\Omega)\right) = \mathrm{cl}_{M_p^\lambda(\Omega)}\left(C^\infty(\bar\Omega)\cap C^0_{ub}(\bar\Omega)\right) = M_p^{\lambda,0}(\Omega).$$

(iii) If Ω is bounded and if $\lambda<\frac{n}{p}$, then

$$\tilde{C}^\infty(\bar\Omega)\subseteq C^\infty(\bar\Omega)\subseteq C^0_{ub}(\bar\Omega)\subseteq L^\infty(\Omega)\subseteq M_p^{\lambda,0}(\Omega)$$

and

$$\mathrm{cl}_{M_p^\lambda(\Omega)}\tilde{C}^\infty(\bar\Omega) = \mathrm{cl}_{M_p^\lambda(\Omega)}C^\infty(\bar\Omega) = M_p^{\lambda,0}(\Omega).$$

Proof. (i) First we observe that the sets $M_p^{\lambda,0}(\Omega)\cap\tilde{C}^\infty(\bar\Omega)\cap C^0_{ub}(\bar\Omega)$ and $M_p^{\lambda,0}(\Omega)\cap C^\infty(\bar\Omega)\cap C^0_{ub}(\bar\Omega)$ are contained in $M_p^{\lambda,0}(\Omega)$.

By Lemma 1.7 $M_p^{\lambda,0}(\Omega)$ is a closed subspace of $M_p^\lambda(\Omega)$.

Then the closure of the set $M_p^{\lambda,0}(\Omega)\cap\tilde{C}^\infty(\bar\Omega)\cap C^0_{ub}(\bar\Omega)$ is contained in $M_p^{\lambda,0}(\Omega)$.

Similarly, the closure of the set $M_p^{\lambda,0}(\Omega) \cap C^\infty(\bar{\Omega}) \cap C^0_{ub}(\bar{\Omega})$ is contained in $M_p^{\lambda,0}(\Omega)$.

Let now $f \in M_p^{\lambda,0}(\Omega)$. Then, by Lemma 1.27, there exists a bounded linear extension operator $E_\Omega : M_p^{\lambda,0}(\Omega) \to M_p^{\lambda,0}(\mathbb{R}^n)$ such that $E_\Omega f\big|_\Omega = f$ for all $f \in M_p^{\lambda,0}(\Omega)$.

Thus, we can approximate $E_\Omega f$ by smooth functions, which are, by definition, functions from $\tilde{C}^\infty(\bar{\Omega})$.

Therefore, $E_\Omega f$, as a limit, belongs to

$$\mathrm{cl}_{M_p^\lambda(\Omega)} \left(M_p^{\lambda,0}(\Omega) \cap \tilde{C}^\infty(\bar{\Omega}) \cap C^0_{ub}(\bar{\Omega}) \right) = \\ = \mathrm{cl}_{M_p^\lambda(\Omega)} \left(M_p^{\lambda,0}(\Omega) \cap C^\infty(\bar{\Omega}) \cap C^0_{ub}(\bar{\Omega}) \right).$$

(ii) Let $f \in C^0_{ub}(\bar{\Omega})$, then f is bounded and, thus, it is essentially bounded, *i.e.* $f \in L^\infty(\Omega)$ and hence

$$C^0_{ub}(\bar{\Omega}) \subseteq L^\infty(\Omega).$$

Then by Lemma 1.12 (ii) we have $L^\infty(\Omega) \subset M_p^{\lambda,0}(\Omega)$. Hence,

$$M_p^{\lambda,0}(\Omega) \cap \tilde{C}^\infty(\bar{\Omega}) \cap C^0_{ub}(\bar{\Omega}) = \tilde{C}^\infty(\bar{\Omega}) \cap C^0_{ub}(\bar{\Omega})$$

and

$$M_p^{\lambda,0}(\Omega) \cap C^\infty(\bar{\Omega}) \cap C^0_{ub}(\bar{\Omega}) = C^\infty(\bar{\Omega}) \cap C^0_{ub}(\bar{\Omega}).$$

Therefore, from (i) we obtain

$$\mathrm{cl}_{M_p^\lambda(\Omega)} \left(\tilde{C}^\infty(\bar{\Omega}) \cap C^0_{ub}(\bar{\Omega}) \right) = \mathrm{cl}_{M_p^\lambda(\Omega)} \left(C^\infty(\bar{\Omega}) \cap C^0_{ub}(\bar{\Omega}) \right) = M_p^{\lambda,0}(\Omega).$$

(iii) Let $f \in \tilde{C}^\infty(\bar{\Omega})$. By definition there exist an open neighborhood U of $\bar{\Omega}$ and a function $F \in C^\infty(U)$ such that $F|_{\bar{\Omega}} = f$. Then $f \in C^\infty(\bar{\Omega})$.

Now let $f \in C^\infty(\bar{\Omega})$. Then f is continuous.

Since Ω is bounded and $\bar{\Omega}$ is closed, then, by Cantor's Theorem, f is uniformly continuous. In this case f is also bounded. Thus, $f \in C^0_{ub}(\bar{\Omega})$.

Hence, we have

$$\tilde{C}^\infty(\bar{\Omega}) \cap C^0_{ub}(\bar{\Omega}) = \tilde{C}^\infty(\bar{\Omega}) \quad \text{and} \quad C^\infty(\bar{\Omega}) \cap C^0_{ub}(\bar{\Omega}) = C^\infty(\bar{\Omega}).$$

From (ii) we get

$$\mathrm{cl}_{M_p^\lambda(\Omega)}\tilde{C}^\infty(\bar{\Omega}) = \mathrm{cl}_{M_p^\lambda(\Omega)}C^\infty(\bar{\Omega}) = M_p^{\lambda,0}(\Omega).$$

□

1.3 Preliminaries on integral operators

Lemma 1.26. *Let* $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$*. If there exists* $R \in]0,+\infty[$ *such that* $f\big|_{\mathbb{R}^n\setminus B_n(0,R)}$ *is essentially bounded, then for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\int\limits_E |f|\, dm_n \le \varepsilon \quad \forall\, E \in \mathcal{L}_n, \ \ m_n(E) \le \delta.$$

Proof. Since $f\big|_{\mathbb{R}^n\setminus B_n(0,R)}$ is essentially bounded, we have $f \in L_\infty(\mathbb{R}^n\setminus B_n(0,R))$ and $f \in L_1(B_n(0,R))$.

We set

$$E_1 = E \cap B_n(0,R), \quad E_2 = E \setminus B_n(0,R), \quad \text{for all } \ E \in \mathcal{L}_n.$$

Now let $\varepsilon > 0$. Then, by absolute continuity of the Lebesgue integral, there exists $\delta_1(\varepsilon) > 0$ such that

$$\int\limits_{E_1} |f|\, dm_n \le \frac{\varepsilon}{2}$$

whenever $m_n(E_1) \le \delta_1(\varepsilon)$.

Next we suppose that

$$\delta_2(\varepsilon) = \frac{\varepsilon}{2\|f\|_{L_\infty(\mathbb{R}^n\setminus B_n(0,R))} + 1},$$

then for E_2 satisfying $m_n(E_2) \le \delta_2(\varepsilon)$ we obtain

$$\int\limits_{E_2} |f|\, dm_n \le \|f\|_{L_\infty(\mathbb{R}^n\setminus B_n(0,R))} m_n(E_2) < \|f\|_{L_\infty(\mathbb{R}^n\setminus B_n(0,R))}\delta_2(\varepsilon) =$$

$$= \|f\|_{L_\infty(\mathbb{R}^n\setminus B_n(0,R))} \frac{\varepsilon}{2\|f\|_{L_\infty(\mathbb{R}^n\setminus B_n(0,R))}} = \frac{\varepsilon}{2}.$$

Therefore,

$$\int\limits_E |f|\, dm_n = \int\limits_{E_1} |f|\, dm_n + \int\limits_{E_2} |f|\, dm_n \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Lemma 1.27. *Let* $n \in \mathbb{N} \setminus \{0\}$*. Let* $\lambda \in]0, n[$*. For each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\int_E \frac{1}{|\xi - \eta|^\lambda} \, d\eta \leq \varepsilon \quad \forall \, E \in \mathcal{L}_n, \;\; m_n(E) \leq \delta.$$

for all $\xi \in \mathbb{R}^n$*.*

Proof. Setting $\xi - \eta = z$, we have

$$\int_E \frac{1}{|\xi - \eta|^\lambda} \, d\eta = \int_{\xi - E} \frac{dz}{|z|^\lambda}.$$

Function $\frac{1}{|x|^\lambda} \in L_1(B(0,1))$ and $\frac{1}{|x|^\lambda} \in L_\infty(\mathbb{R}^n \setminus B_n(0,1))$.

Then, by applying the previous lemma, we complete the proof. □

Theorem 1.28. *Let* $n \in \mathbb{N} \setminus \{0\}$*. Let* $\alpha \in]0, n[$*.Let* Ω *be an open subset of* $\mathbb{R}^n$ *of finite measure. Let* Ω_1 *be an open subset of* $\mathbb{R}^n$*. Let*

$$D_{\Omega_1 \times \Omega} \equiv \{(x, y) \in \Omega_1 \times \Omega : x = y\}.$$

Let k *be a function from* $(\Omega_1 \times \Omega) \setminus D_{\Omega_1 \times \Omega}$ *to* $\mathbb{R}$ *such that* $k(x, \cdot)$ *is measurable in* $\Omega \setminus \{x\}$ *for all* $x \in \Omega_1$*. Assume that there exists* $c \in]0, +\infty[$ *such that*

$$|k(x, y)| \leq \frac{c}{|x - y|^{n-\alpha}} \quad \forall \, (x, y) \in (\Omega_1 \times \Omega) \setminus D_{\Omega_1 \times \Omega}.$$

Assume that if $\tilde{x} \in \Omega_1$*, there exists a subset* $N_{\tilde{x}}$ *of measure 0 of* Ω *such that* $\tilde{x} \notin \Omega \setminus N_{\tilde{x}}$ *and such that the function* $k(\cdot, y)$ *from* $\Omega_1 \setminus \{y\}$ *to* $\mathbb{R}$ *is continuous at* $\tilde{x}$ *for all* $y \in \Omega \setminus N_{\tilde{x}}$*.*

If $f \in L^\infty(\Omega)$*, then the function* $H[f]$ *from* Ω_1 *to* $\mathbb{R}$ *defined by*

$$H[f](x) \equiv \int_\Omega k(x, y) f(y) dy \quad \forall \, x \in \Omega_1$$

is continuous.

Proof. Let $x \in \Omega_1$. Clearly,

$$|k(x, y) f(y)| \leq \|f\|_{L^\infty(\Omega)} \frac{c}{|x - y|^{n-\alpha}} \tag{1.11}$$

for almost all $y \in \Omega$. Since $\frac{c}{|x-y|^{n-\alpha}}$ is integrable in $y \in \Omega$, we deduce that $k(x, y) f(y)$ is integrable in $y \in \Omega$.

Now let $\tilde{x} \in \Omega_1$. In order to prove the continuity of $H[f]$ at $\tilde{x}$, we want to apply the Vitali Convergence Theorem

By inequality (1.11), we have

$$\int_E |k(x,y)f(y)|dy \leq c\|f\|_{L^\infty(\Omega)} \int_E \frac{dy}{|x-y|^{n-\alpha}} \leq$$

$$\leq c\|f\|_{L^\infty(\Omega)} \sup_{x\in\mathbb{R}^n} \int_E \frac{dy}{|x-y|^{n-\alpha}},$$

for all $x \in \Omega_1$. Now let $\varepsilon > 0$. Lemma 1.27 implies that there exists $\delta > 0$ such that

$$\int_E \frac{dy}{|x-y|^{n-\alpha}} \leq \frac{\varepsilon}{1+c\|f\|_{L^\infty(\Omega)}} \quad \text{if } E \in \mathcal{L}_n,\ \ m_n(E) \leq \delta,\ x \in \mathbb{R}^n.$$

Then we have

$$\int_E |k(x,y)f(y)|dy \leq \varepsilon \quad \text{if } E \in \mathcal{L}_n,\ \ m_n(E) \leq \delta,\ x \in \Omega_1. \tag{1.12}$$

Now let $\{x_j\}_{j\in\mathbb{N}}$ be a sequence in $\Omega_1 \setminus \{\tilde{x}\}$ which converges to $\tilde{x}$ in Ω_1. Let $N_{\tilde{x}}$ be a subset of measure 0 of Ω as in assumptions. Then we have

$$\lim_{j\to\infty} k(x_j,y) = k(\tilde{x},y) \quad \forall\, y \in \Omega \setminus N_{\tilde{x}}, \tag{1.13}$$

and, in particular, for almost all $y \in \Omega$. Then (1.12) and (1.13) and the Vitali Convergence Theorem imply that

$$\lim_{j\to\infty} H[f](x_j) = H[f](\tilde{x}).$$

Hence, $H[f]$ is continuous at $\tilde{x}$. □

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. Consider the Riesz potential

$$(I_\alpha f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy, \quad 0 < \alpha < n.$$

In [14], in particular, the following statement is proved generalizing the results of [1], [17], [42], [47], [27] and [16].

Theorem 1.29. *Let condition*

$$1 < p \leq \infty,\ \ 0 < q \leq \infty \ \text{ and } \ n\left(\frac{1}{p}-\frac{1}{q}\right)_+ < \alpha < n, \tag{1.14}$$

or

$$p = 1, \ \ 0 < q < \infty \ \ \textit{and} \ \ n\left(1 - \frac{1}{q}\right)_+ < \alpha < n, \tag{1.15}$$

or

$$1 < p < q < +\infty \ \ \textit{and} \ \ \alpha = n\left(\frac{1}{p} - \frac{1}{q}\right) \tag{1.16}$$

be satisfied. Let also $u \in \Lambda_{p,\infty}$, $v \in \Lambda_{q,\infty}$ *and*

$$I(u,v) = \left\| v(t)t^{\frac{n}{q}} \int\limits_t^\infty \frac{s^{\alpha - \frac{n}{p} - 1}}{\|u\|_{L_\infty(s,\infty)}} ds \right\|_{L_\infty(0,\infty)} < \infty. \tag{1.17}$$

Then the operator I_α *is bounded from* $\mathcal{M}_p^{u(\cdot)}(\mathbb{R}^n)$ *to* $\mathcal{M}_q^{v(\cdot)}(\mathbb{R}^n)$.

Moreover, if condition (1.15) is satisfied, then condition (1.17) is necessary and sufficient for the boundedness of I_α *from* $\mathcal{M}_p^{u(\cdot)}(\mathbb{R}^n)$ *to* $\mathcal{M}_q^{v(\cdot)}(\mathbb{R}^n)$.

Theorem 1.30. *Let* $n \in \mathbb{N} \setminus \{0\}$. *Let* $1 \leq p \leq q < +\infty$. *Let* $0 \leq \lambda \leq \nu < \frac{n}{q}$. *Let*

$$\alpha \equiv \left(\nu - \frac{n}{q}\right) - \left(\lambda - \frac{n}{p}\right). \tag{1.18}$$

Then the following statements hold.

(i) If $\lambda < \nu$, *then the operator* I_α *is bounded from* $\mathcal{M}_p^{r^{-\lambda}}(\mathbb{R}^n)$ *to* $\mathcal{M}_q^{r^{-\nu}}(\mathbb{R}^n)$.

(ii) If $\lambda = \nu$ *and if* $1 < p < q$, *then* I_α *is bounded from* $\mathcal{M}_p^{r^{-\lambda}}(\mathbb{R}^n)$ *to* $\mathcal{M}_q^{r^{-\nu}}(\mathbb{R}^n)$.

(iii) If $\lambda = \nu$ *and if* $1 < p < q$, *then* I_α *is bounded from* $M_p^\lambda(\mathbb{R}^n)$ *to* $M_q^\nu(\mathbb{R}^n)$.

Remark 1.31. *If* $\nu = \lambda = 0$, *then* $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$, *and this is the classical Hardy-Littlewood-Sobolev theorem.*

Proof. (i) We recall that

$$t_+ = \begin{cases} t, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Since $p \leq q$, we have $n\left(\frac{1}{p} - \frac{1}{q}\right)_+ = n\left(\frac{1}{p} - \frac{1}{q}\right)$.

Assumptions $\lambda < \nu$ and (1.18) imply

$$\begin{cases} \alpha = \left(\nu - \frac{n}{q}\right) - \left(\lambda - \frac{n}{p}\right), \\ \lambda < \nu, \end{cases} \Rightarrow \begin{cases} \alpha = \nu - \lambda + \frac{n}{p} - \frac{n}{q}, \\ \nu - \lambda > 0, \end{cases} \Rightarrow n\left(\frac{1}{p} - \frac{1}{q}\right) < \alpha,$$

$$\begin{cases} \alpha = \left(\nu - \frac{n}{q}\right) - \left(\lambda - \frac{n}{p}\right), \\ 0 \le \lambda \le \nu < \frac{n}{q}, \end{cases} \Rightarrow \begin{cases} \alpha = \nu - \lambda + \frac{n}{p} - \frac{n}{q}, \\ \nu - \lambda < \frac{n}{q}, \end{cases} \Rightarrow \alpha < \frac{n}{p} \le n.$$

Therefore,

$$n\left(\frac{1}{p} - \frac{1}{q}\right) < \alpha < n,$$

and thus either condition (1.14) or (1.15) satisfied.

Now we want to prove that

$$I(r^{-\lambda}, r^{-\nu}) = \sup_{t>0} r^{-\nu}(t) t^{\frac{n}{q}} \int_t^{\infty} \frac{s^{\alpha - \frac{n}{p} - 1}}{r^{-\lambda}(s)} ds < \infty.$$

Indeed,

$$t^{-\nu + \frac{n}{q}} \int_t^{\infty} s^{\alpha + \lambda - \frac{n}{p} - 1} ds = t^{-\nu + \frac{n}{q}} \int_t^{\infty} s^{\nu - \frac{n}{q} - 1} ds =$$

$$= \frac{t^{-\nu + \frac{n}{q} + \nu - \frac{n}{q}}}{-\nu + \frac{n}{q}} = \left(-\nu + \frac{n}{q}\right)^{-1} < \infty.$$

By Theorem 1.29, operator I_α is linear and continuous from $\mathcal{M}_p^{r^{-\lambda}}(\mathbb{R}^n)$ to $\mathcal{M}_q^{r^{-\nu}}(\mathbb{R}^n)$.

(ii) If $\lambda = \nu$, then $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$ and, hence, condition (1.16) satisfied.

Now we prove that inequality (1.17) holds with $u(t) = v(t) = t^{-\lambda}$. Indeed,

$$t^{-\lambda + \frac{n}{q}} \int_t^{\infty} s^{\alpha + \lambda - \frac{n}{p} - 1} ds = t^{-\lambda + \frac{n}{q}} \int_t^{\infty} s^{\lambda - \frac{n}{q} - 1} ds =$$

$$= \frac{t^{-\lambda + \frac{n}{q} + \lambda - \frac{n}{q}}}{-\lambda + \frac{n}{q}} = \left(-\lambda + \frac{n}{q}\right)^{-1} < \infty.$$

Hence, Theorem 1.29 implies that I_α is continuous from $\mathcal{M}_p^{r^{-\lambda}}(\mathbb{R}^n)$ to $\mathcal{M}_q^{r^{-\nu}}(\mathbb{R}^n)$.

(iii) If $\lambda = \nu$, then $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$ and, therefore, condition (1.16) satisfied.

We want to prove that

$$I(w_\lambda, w_\lambda) = \sup_{t>0} w_\lambda(t) t^{\frac{n}{q}} \int_t^{\infty} \frac{s^{\alpha - \frac{n}{p} - 1}}{w_\lambda(s)} ds < \infty.$$

Let first $t \in]0,1]$. Then

$$t^{-\lambda+\frac{n}{q}} \int\limits_t^\infty s^{\alpha+\lambda-\frac{n}{p}-1} ds = t^{-\lambda+\frac{n}{q}} \int\limits_t^1 s^{\lambda-\frac{n}{q}-1} ds + t^{-\lambda+\frac{n}{q}} \int\limits_1^\infty s^{\lambda-\frac{n}{q}-1} ds =$$

$$= \frac{t^{-\lambda+\frac{n}{q}} - 1 - t^{-\lambda+\frac{n}{q}}}{\lambda - \frac{n}{q}} = \left(-\lambda + \frac{n}{q}\right)^{-1} < \infty.$$

Now let $t \geq 1$. Then $w_\lambda(t) = 1$ and

$$t^{\frac{n}{q}} \int\limits_t^\infty s^{\alpha-\frac{n}{p}-1} ds = t^{\frac{n}{q}} \int\limits_t^\infty s^{-\frac{n}{q}-1} ds = \frac{q}{n} < \infty.$$

Thus, (1.17) holds and Theorem 1.29 implies that I_α is continuous from $M_p^\lambda(\mathbb{R}^n)$ to $M_q^\nu(\mathbb{R}^n)$.

□

Lemma 1.32. *Let $p \in [1, +\infty[$, $\alpha \in]0, n[$, $\lambda \in [0, n/p]$. Let $q \in [1, p]$ be such that $(\alpha + \lambda) > \frac{n}{q}$. Let*

$$\mu_{w_\lambda, q} \equiv \max \left\{ 1, \frac{1}{(\alpha+\lambda) - \frac{n}{q}} \right\}. \tag{1.19}$$

Then we have

$$\int\limits_{E \cap \mathbb{B}_n(x,1)} \frac{|f(y)|dy}{|x-y|^{n-\alpha}} \leq$$
$$\leq m_n(E)^{\frac{1}{q}-\frac{1}{p}} \mu_{w_\lambda, q} (n + 2 - \alpha) v_n^{1-\frac{1}{q}} \|f\|_{M_p^\lambda(\mathbb{R}^n)} \quad \forall\, f \in M_p^\lambda(\mathbb{R}^n), \tag{1.20}$$

for all measurable subsets E of $\mathbb{R}^n$ of finite measure, and for all $x \in \mathbb{R}^n$.

Proof. The arguments of this proof are in part based on a development of the ideas of Campanato [19].

If $f \in M_p^\lambda(\mathbb{R}^n)$, then we know that $f|_{\mathbb{B}_n(x,r)} \in L_p(\mathbb{B}_n(x,r)) \subseteq L_1(\mathbb{B}_n(x,r))$ for all $x \in \mathbb{R}^n$ and $r \in]0, +\infty[$. In particular, $(\chi_E f)|_{\mathbb{B}_n(x,r)} \in L_1(\mathbb{B}_n(x,r))$ for all $x \in \mathbb{R}^n$ and $r \in]0, +\infty[$ and for all measurable subsets E of $\mathbb{R}^n$.

Now we fix $x \in \mathbb{R}^n$ and a measurable subset E of $\mathbb{R}^n$ of finite measure. The almost everywhere defined function from $]0, +\infty[$ to $[0, +\infty[$ which takes $s \in]0, +\infty[$ to $\int\limits_{\partial \mathbb{B}_n(x,s)} \chi_E |f| d\sigma$ is integrable in $]0, r[$ for all $r \in]0, +\infty[$. Then by the Fundamental Theorem of Calculus, the function $A_{E,x}$ from $[0,+\infty[$ to $[0,+\infty[$ defined by

$$A_{E,x}(\rho) \equiv \int_0^{\rho} \int_{\partial \mathbb{B}_n(x,s)} \chi_E |f| d\sigma \, ds \quad \forall \, \rho \in [0,+\infty[,$$

is locally absolutely continuous and

$$A'_{E,x}(\rho) \equiv \int_{\partial \mathbb{B}_n(x,\rho)} \chi_E |f| d\sigma,$$

for almost all $\rho \in [0,+\infty[$ (cf. e.g., Folland [25, 3.35]). By the Monotone Convergence Theorem, we have

$$\int_{E \cap \mathbb{B}_n(x,1)} \frac{|f(y)|dy}{|x-y|^{n-\alpha}} = \\ = \int_{\mathbb{B}_n(x,1)} \frac{\chi_E(y)|f(y)|dy}{|x-y|^{n-\alpha}} = \lim_{\varepsilon \to 0} \int_{\mathbb{B}_n(x,1) \setminus \mathbb{B}_n(x,\varepsilon)} \frac{\chi_E(y)|f(y)|dy}{|x-y|^{n-\alpha}}. \quad (1.21)$$

Now let $\varepsilon \in]0,1[$. Then we have

$$\int_{\mathbb{B}_n(x,1) \setminus \mathbb{B}_n(x,\varepsilon)} \frac{\chi_E(y)|f(y)|dy}{|x-y|^{n-\alpha}} = \\ = \int_{\varepsilon}^{1} s^{-n+\alpha} \int_{\partial \mathbb{B}_n(x,s)} \chi_E |f| d\sigma \, ds = \int_{\varepsilon}^{1} s^{-n+\alpha} A'_{E,x}(s) ds. \quad (1.22)$$

Then by integrating by parts, we obtain

$$\int_{\varepsilon}^{1} s^{-n+\alpha} A'_{E,x}(s) ds = \\ = \left[s^{-n+\alpha} A_{E,x}(s) \right]_{\varepsilon}^{1} - \int_{\varepsilon}^{1} (-n+\alpha) s^{-n+\alpha-1} A_{E,x}(s) ds, \quad (1.23)$$

(cf. e.g., Folland [25, ex.35 p.108]). Then the Hölder inequality and inequality

(1.19) imply that

$$\begin{aligned}
|A_{E,x}(\rho)| &\leq m_n(E\cap\mathbb{B}_n(x,\rho))^{1-\frac{1}{p}}\|f\|_{L_p(E\cap\mathbb{B}_n(x,\rho))} = \\
&= m_n(E\cap\mathbb{B}_n(x,\rho))^{\frac{1}{q}-\frac{1}{p}}\, m_n(E\cap\mathbb{B}_n(x,\rho))^{1-\frac{1}{q}}\|f\|_{L_p(E\cap\mathbb{B}_n(x,\rho))} \leq \\
&\leq m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}\rho^{n-\frac{n}{q}}\|f\|_{L_p(\mathbb{B}_n(x,\rho))} \leq \\
&\leq m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}\rho^{n-\alpha}\rho^{-n+\alpha}\rho^{n-\frac{n}{q}}w_\lambda^{-1}(\rho)w_\lambda(\rho)\|f\|_{L_p(\mathbb{B}_n(x,\rho))} \leq \\
&\leq \rho^{n-\alpha}m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}\rho^{\alpha-\frac{n}{q}}w_\lambda^{-1}(\rho)\|f\|_{M_p^\lambda(\mathbb{R}^n)} \leq \\
&\leq \rho^{n-\alpha}m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}\rho^{(\alpha+\lambda)-n/q}\|f\|_{M_p^\lambda(\mathbb{R}^n)} \qquad (1.24)
\end{aligned}$$

for all $\rho\in]0,1[$. Then by the second last line of inequality (1.24), we have

$$\begin{aligned}
&\left|\int_\varepsilon^1(-n+\alpha)s^{-n+\alpha-1}A_{E,x}(s)ds\right| \leq \\
&\leq m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}\|f\|_{M_p^\lambda(\mathbb{R}^n)}\left|\int_\varepsilon^1(-n+\alpha)s^{-n+\alpha-1}s^{n-\alpha}s^{(\alpha+\lambda)-\frac{n}{q}}ds\right| \leq \\
&\leq m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}\|f\|_{M_p^\lambda(\mathbb{R}^n)}(n-\alpha)\left|\int_\varepsilon^1 s^{(\alpha+\lambda)-\frac{n}{q}-1}ds\right| = \\
&= m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}\|f\|_{M_p^\lambda(\mathbb{R}^n)}(n-\alpha)\frac{1}{(\alpha+\lambda)-n/q}(1-\varepsilon) = \\
&= m_n(E)^{\frac{1}{q}-\frac{1}{p}}(n-\alpha)v_n^{1-\frac{1}{q}}\mu_{w_\lambda,q}\|f\|_{M_p^\lambda(\mathbb{R}^n)}(1-\varepsilon). \qquad (1.25)
\end{aligned}$$

Then by combining (1.22)–(1.25), we deduce that

$$\begin{aligned}
&\int_{\mathbb{B}_n(x,1)\setminus\mathbb{B}_n(x,\varepsilon)}\frac{\chi_E(y)|f(y)|dy}{|x-y|^{n-\alpha}} \leq \left|\int_\varepsilon^1 s^{-n+\alpha}A'_{E,x}(s)ds\right| \leq \\
&\leq |A_{E,x}(1)| + |\varepsilon^{-n+\alpha}A_{E,x}(\varepsilon)| + \left|\int_\varepsilon^1(-n+\alpha)s^{-n+\alpha-1}A_{E,x}(s)ds\right| \leq \\
&\leq 1^{n-\alpha}m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}1^{(\alpha+\lambda)-n/q}\|f\|_{M_p^\lambda(\mathbb{R}^n)}+ \\
&+\varepsilon^{-n+\alpha}\varepsilon^{n-\alpha}m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}\varepsilon^{(\alpha+\lambda)-n/q}\|f\|_{M_p^\lambda(\mathbb{R}^n)}+ \\
&+m_n(E)^{\frac{1}{q}-\frac{1}{p}}(n-\alpha)v_n^{1-\frac{1}{q}}\mu_{w_\lambda,q}\|f\|_{M_p^\lambda(\mathbb{R}^n)}(1-\varepsilon) \leq \\
&\leq m_n(E)^{\frac{1}{q}-\frac{1}{p}}\mu_{w_\lambda,q}v_n^{1-\frac{1}{q}}\|f\|_{M_p^\lambda(\mathbb{R}^n)}[1+1+(n-\alpha)(1-\varepsilon)].
\end{aligned}$$

Then the limiting relation (1.21) immediately implies the validity of inequality (1.20). □

Corollary 1.33. *Let* $p \in [1, +\infty[$*,* $\alpha \in]0, n[$*. Let* $\lambda \in [0, n/p]$*. Let* $\alpha + \lambda > \frac{n}{p}$*. Let* Ω *be an open subset of* $\mathbb{R}^n$*. Then the following statements hold.*

If $f \in M_p^\lambda(\mathbb{R}^n)$ *and if* $\int\limits_\Omega |f| dx < \infty$*, then the function from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *which takes* $x \in \mathbb{R}^n$ *to*

$$\int\limits_\Omega \frac{f(y)dy}{|x-y|^{n-\alpha}}$$

is bounded, and satisfies the following inequality

$$\sup_{x \in \mathbb{R}^n} \int\limits_\Omega \frac{|f(y)|dy}{|x-y|^{n-\alpha}} \leq$$

$$\leq \max\{1, ((\lambda + \alpha) - (n/p))^{-1}\}(n + 2 - \alpha) v_n^{1-\frac{1}{p}} \|f\|_{M_p^\lambda(\mathbb{R}^n)} + \int\limits_\Omega |f| dx. \quad (1.26)$$

If Ω *has finite measure, then the map* $I_{\alpha,\Omega}$ *from* $M_p^\lambda(\Omega)$ *to* $B(\mathbb{R}^n)$ *defined by*

$$I_{\alpha,\Omega} f(x) \equiv \int\limits_\Omega \frac{f(y)dy}{|x-y|^{n-\alpha}} \quad \forall\, x \in \mathbb{R}^n,$$

for all $f \in M_p^\lambda(\Omega)$ *is linear and continuous.*

Proof. By applying Lemma 1.32 with $E = \Omega$, we deduce that

$$\int\limits_\Omega \frac{|f(y)|dy}{|x-y|^{n-\alpha}} \leq \int\limits_{\Omega \cap \mathbb{B}_n(x,1)} \frac{|f(y)|dy}{|x-y|^{n-\alpha}} + \int\limits_{\Omega \setminus \mathbb{B}_n(x,1)} \frac{|f(y)|dy}{|x-y|^{n-\alpha}} \leq$$

$$\leq m_n\left(\mathbb{B}_n(x,1)\right)^{\frac{1}{q}-\frac{1}{p}} \mu_{w_\lambda,q}(n+2-\alpha) v_n^{1-\frac{1}{q}} \|f\|_{M_p^\lambda(\mathbb{R}^n)} + \int\limits_{\Omega \setminus \mathbb{B}_n(x,1)} \frac{|f|}{1^{n-\alpha}} dx \leq$$

$$\leq \mu_{w_\lambda,q}(n+2-\alpha) v_n^{1-\frac{1}{q}} \|f\|_{M_p^\lambda(\mathbb{R}^n)} + \int\limits_\Omega |f| dx,$$

for all $x \in \mathbb{R}^n$. Hence, inequality (1.26) follows. □

1.4 Sobolev spaces built on Morrey spaces

Definition 1.34. *A domain* $\Omega \subset \mathbb{R}^n$ *is called star-shaped with respect to the point* $y \in \Omega$ *if for all* $x \in \Omega$ *the segment* $[x, y] \subset \Omega$*. A domain* $\Omega \subset \mathbb{R}^n$ *is called star-shaped with respect to a point if for some* $y \in \Omega$ *it is star-shaped with respect to the point* y*.*

Definition 1.35. *A domain* $\Omega \subset \mathbb{R}^n$ *is called star-shaped with respect to the ball* $B \subset \Omega$ *if for all* $y \in B$ *and for all* $x \in \Omega$ *we have* $[x, y] \subset \Omega$*. A domain* $\Omega \subset \mathbb{R}^n$ *is called star-shaped with respect to a ball if for some ball* $B \subset \Omega$ *it is star-shaped respect to the ball* B*.*

Definition 1.36. *If* $0 < d \leq \operatorname{diam} B \leq \operatorname{diam} \Omega \leq D$*, we set that* Ω *is star-shaped with respect to a ball with the parameters* d, D*.*

Definition 1.37. *We call the set*

$$V_x \equiv V_{x,B} = \bigcup_{y \in B} (x, y)$$

a conic body with the vertex x *constructed on the ball* B*.*

A domain Ω star-shaped with respect to a ball B can be equivalently defined in the following way: for all $x \in \Omega$ the conic body $V_x \subset \Omega$.

Definition 1.38. *If* $r,\ h \in]0, +\infty[$*, then we define the cone* $K(r, h)$ *as follows*

$$K(r,h) \equiv \left\{ (x', x_n) \in \mathbb{R}^n : \ |x'| < \frac{r x_n}{h},\ x_n < h \right\},$$

if $n > 1$*, and*

$$K(r,h) \equiv]0, h[,$$

if $n = 1$*.*

We denote by $\mathcal{O}(n)$ the orthogonal group, *i.e.*, the set of $n \times n$ matrices R with real entries such that $RR^t = I = R^t R$.

Definition 1.39. *Let* Ω *be an open subset of* $\mathbb{R}^n$*.*

(i) Let $r,\ h \in]0, +\infty[$*. We say that* Ω *satisfies the cone property (or condition) with parameters* $r,\ h$ *provided that for each* $x \in \Omega$ *there exists* $R_x \in \mathcal{O}(n)$ *such that*

$$x + R_x(K(r,h)) \subseteq \Omega.$$

(ii) We say that Ω *satisfies the cone property (or condition), provided that there exists* $r,\ h \in]0, +\infty[$ *such that* Ω *satisfies the cone property (or condition) with parameters* $r,\ h$*.*

Lemma 1.40. *1. A bounded open set $\Omega \subset \mathbb{R}^n$ satisfies the cone condition if, and only if, there exist $s \in \mathbb{N}$ and bounded domains Ω_k, which are star-shaped with respect to the balls $B_k \subset \bar{B}_k \subset \Omega_k$, $k = 1, \dots, s$, such that $\Omega = \bigcup\limits_{k=1}^{s} \Omega_k$.*

2. An unbounded open set $\Omega \subset \mathbb{R}^n$ satisfies the cone condition if, and only if, there exist bounded domains Ω_k, $k \in \mathbb{N} \setminus \{0\}$, which are star-shaped with respect to the balls $B_k \subset \bar{B}_k \subset \Omega_k$, $k \in \mathbb{N} \setminus \{0\}$, and are such that

1) $\Omega = \bigcup\limits_{k=1}^{\infty} \Omega_k$,

2) $0 < \inf\limits_{k \in \mathbb{N} \setminus \{0\}} \operatorname{diam} B_k \le \sup\limits_{k \in \mathbb{N} \setminus \{0\}} \operatorname{diam} \Omega_k < \infty$,

3) the multiplicity of the covering $\aleph(\{\Omega_k\}_{k=1}^{\infty})$ is finite.

(cf. e.g., Burenkov[13, Ch. 3.2])

Lemma 1.41. *Let $m_0 \in \mathbb{N} \setminus \{0\}$, $1 \le p_1, \dots, p_{m_0}, q \le \infty$, $\lambda \in \left[0, \frac{n}{p_m}\right]$, for all $m = 1, \dots, m_0$, $0 \le \nu \le \frac{n}{q}$. Let $\Omega = \bigcup\limits_{k=1}^{s} \Omega_k$, where $s \in \mathbb{N} \setminus \{0\}$ and $\Omega_k \subset \mathbb{R}^n$ are bounded open sets. Furthermore, let f_m, $m = 1, \dots, m_0$, and g be measurable functions on Ω.*

Suppose that there exists $\sigma_m > 0$ such that

$$\|g\|_{\mathcal{M}_q^{\rho^{-\nu}}(\Omega_k)} \le \sum_{m=1}^{m_0} \sigma_m \|f_m\|_{\mathcal{M}_{p_m}^{\rho^{-\lambda}}(\Omega_k)}, \tag{1.27}$$

for all $m = 1, \dots, m_0$.

Then

$$\|g\|_{\mathcal{M}_q^{\rho^{-\nu}}(\Omega)} \le 2^{\nu} s^{\frac{1}{q}} \sum_{m=1}^{m_0} \sigma_m \|f_m\|_{\mathcal{M}_{p_m}^{\rho^{-\lambda}}(\Omega)}. \tag{1.28}$$

Proof. Let $q < \infty$.

$$\|g\|^q_{\mathcal{M}_q^{\rho^{-\nu}}(\Omega)} = \sup_{\substack{x\in\Omega \\ \rho>0}} \rho^{-\nu q} \|g\|^q_{L_q(B(x,\rho)\cap\Omega)} =$$

$$= \sup_{\substack{x\in\Omega \\ \rho>0}} \rho^{-\nu q} \int\limits_{B(x,\rho)\cap\Omega} |g(y)|^q dy \le \sup_{\substack{x\in\Omega \\ \rho>0}} \sum_{k=1}^{s} \rho^{-\nu q} \int\limits_{B(x,\rho)\cap\Omega_k} |g(y)|^q dy \le$$

$$\le \sum_{k=1}^{s} \sup_{\substack{x\in\Omega \\ \rho>0}} \rho^{-\nu q} \|g\|^q_{L_q(B(x,\rho)\cap\Omega_k)} \le \sum_{k=1}^{s} \sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \rho^{-\nu q} \|g\|^q_{L_q(B(x,\rho)\cap\Omega_k)}.$$

If $x \in \Omega_k$, then

$$\rho^{-\nu q} \|g\|^q_{L_q(B(x,\rho)\cap\Omega_k)} = \sup_{\substack{x\in\Omega_k \\ \rho>0}} \rho^{-\nu q} \|g\|^q_{L_q(B(x,\rho)\cap\Omega_k)} = \|g\|^q_{\mathcal{M}_q^{\rho^{-\nu}}(\Omega_k)}.$$

Let $x \in \mathbb{R}^n \backslash \Omega_k$. If $B(x,\rho) \cap \Omega_k = \varnothing$, then $\rho^{-\nu}\|g\|_{L_q(B(x,\rho)\cap\Omega_k)} = 0$. Thus, we can assume that $B(x,\rho) \cap \Omega_k \neq \varnothing$.

Let $\xi \in B(x,\rho) \cap \Omega_k$. By the triangle inequality, we have

$$B(x,\rho) \cap \Omega_k \subset B(\xi, 2\rho) \cap \Omega_k.$$

Hence,

$$\rho^{-\nu q}\|g\|^q_{L_q(B(x,\rho)\cap\Omega_k)} \leq 2^{\nu q} \sup_{\substack{\xi\in\Omega_k \\ \rho>0}} (2\rho)^{-\nu q}\|g\|^q_{L_q(B(\xi,2\rho)\cap\Omega_k)} = 2^{\nu q}\|g\|^q_{\mathcal{M}_q^{r-\nu}(\Omega_k)}.$$

By (1.27) and the Minkowski inequality it follows, that

$$\|g\|_{\mathcal{M}_q^{\rho-\nu}(\Omega)} \leq 2^{\nu}\left(\sum_{k=1}^{s}\|g\|^q_{\mathcal{M}_q^{\rho-\nu}(\Omega_k)}\right)^{\frac{1}{q}} \leq 2^{\nu}\left(\sum_{k=1}^{s}\left(\sum_{m=1}^{m_0}\sigma_m\|f_m\|_{\mathcal{M}_{p_m}^{\rho-\lambda}(\Omega_k)}\right)^q\right)^{\frac{1}{q}} \leq$$

$$\leq 2^{\nu}\sum_{m=1}^{m_0}\left(\sum_{k=1}^{s}\left(\sigma_m\|f_m\|_{\mathcal{M}_{p_m}^{\rho-\lambda}(\Omega_k)}\right)^q\right)^{\frac{1}{q}} = 2^{\nu}\sum_{m=1}^{m_0}\sigma_m\left(\sum_{k=1}^{s}\|f_m\|^q_{\mathcal{M}_{p_m}^{\rho-\lambda}(\Omega_k)}\right)^{\frac{1}{q}}.$$

Then

$$\left(\sum_{k=1}^{s}\|f_m\|^q_{\mathcal{M}_{p_m}^{\rho-\lambda}(\Omega_k)}\right)^{\frac{1}{q}} \leq \left(\sum_{k=1}^{s}\|f_m\|^q_{\mathcal{M}_{p_m}^{\rho-\lambda}(\Omega)}\right)^{\frac{1}{q}} = s^{\frac{1}{q}}\|f_m\|_{\mathcal{M}_{p_m}^{\rho-\lambda}(\Omega)}$$

and inequality (1.28) follows.

The case in which some $p_m = \infty$ is treated in a similar way with suprema replacing sums.

The case $q = \infty$ is trivial and the statement holds for $\Omega = \bigcup\limits_{i\in I}\Omega_i$, where I is an arbitrary set of indices:

$$\|g\|_{M_{\infty}^{\nu}(\Omega)} = \sup_{i\in I}\sup_{\substack{x\in\Omega \\ \rho>0}}\rho^{-\nu}\|g\|_{L_\infty(B(x,\rho)\cap\Omega_i)} \leq \sum_{m=1}^{m_0}\sigma_m\|f_m\|_{M_{p_m}^{\lambda}(\Omega)}.$$

□

Definition 1.42. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set. Let* $l \in \mathbb{N}$, $p \in [1,+\infty]$ *and* $\lambda \in \left[0, \frac{n}{p}\right]$. *Then we define the Sobolev space of order* l *built on the Morrey space* $M_p^{\lambda}(\Omega)$, *as the set*

$$W_p^{l,\lambda}(\Omega) \equiv \left\{f \in M_p^{\lambda}(\Omega) : D_w^{\alpha}f \in M_p^{\lambda}(\Omega)\ \forall\, \alpha \in \mathbb{N}^n, |\alpha| \leq l\right\},$$

where $D_w^{\alpha}f$ *is the weak derivative of* f.

Then we set

$$\|f\|_{W_p^{l,\lambda}(\Omega)} = \sum_{|\alpha|\leq l} \|D_w^\alpha f\|_{M_p^\lambda(\Omega)} \quad \forall\, f \in W_p^{l,\lambda}(\Omega).$$

In particular, $W_p^{0,\lambda}(\Omega) = M_p^\lambda(\Omega)$ *and* $W_p^{l,0}(\Omega) = W_p^l(\Omega)$*, where* $W_p^l(\Omega)$ *denotes the classical Sobolev space of exponents* l, p *in* Ω*. It is obvious that* $W_p^{l,\lambda}(\Omega) \subset W_p^l(\Omega)$*.*

Since $M_p^\lambda(\Omega)$ is a Banach space, one can exploit standard properties of the weak derivatives and prove the following.

Theorem 1.43. *Let* Ω *be an open subset of* $\mathbb{R}^n$*. Let* $p \in [1,+\infty]$ *and* $\lambda \in \left[0, \frac{n}{p}\right]$*. Then* $\left(W_p^{l,\lambda}(\Omega), \|\cdot\|_{W_p^{l,\lambda}(\Omega)}\right)$ *is a Banach space.*

Then we have the following remark, which provides a family of equivalent norms in $W_p^{l,\lambda}(\Omega)$.

Remark 1.44. *Let* Ω *be an open subset of* $\mathbb{R}^n$*. Let* $l \in \mathbb{N}$*,* $p \in [1,+\infty]$ *and* $\lambda \in \left[0, \frac{n}{p}\right]$*. Let* $c(l)$ *be the number of multi indexes* $\alpha \in \mathbb{N}^n$ *such that* $|\alpha| \leq l$*. Let* q *be a norm on* $\mathbb{R}^{c(l)}$*. Then the map* $q_{W_p^{l,\lambda}(\Omega)}$ *from* $W_p^{l,\lambda}(\Omega)$ *to* $[0,+\infty[$ *which takes* u *to* $q_{W_p^{l,\lambda}(\Omega)}(u) \equiv q\left((\|D_w^\alpha u\|_{M_p^\lambda(\Omega)})_{|\alpha|\leq l}\right)$ *is an equivalent norm in* $W_p^{l,\lambda}(\Omega)$*.*

In particular, $\left(\sum_{|\alpha|\leq l} \|D_w^\alpha u\|_{M_p^\lambda(\Omega)}^t\right)^{\frac{1}{t}}$ *for* $t \in [1,+\infty[$ *and* $\max_{|\alpha|\leq l} \|D_w^\alpha u\|_{M_p^\lambda(\Omega)}$ *are all equivalent norms on* $W_p^{l,\lambda}(\Omega)$*.*

Definition 1.45. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set. Let* $l \in \mathbb{N}$*,* $p \in [1,+\infty]$ *and* $\lambda \in \left[0, \frac{n}{p}\right]$*. Then we define the Sobolev space of order* l *built on the little Morrey space* $M_p^{\lambda,0}(\Omega)$*, as the set*

$$W_p^{l,\lambda,0}(\Omega) \equiv \left\{f \in M_p^{\lambda,0}(\Omega) : D_w^\alpha f \in M_p^{\lambda,0}(\Omega)\ \forall\, \alpha \in \mathbb{N}^n, |\alpha| \leq l\right\}.$$

Since $M_p^{\lambda,0}(\Omega)$ is a closed subspace of $M_p^\lambda(\Omega)$, we can easily deduce the validity of the following.

Theorem 1.46. *Let* Ω *be an open subset of* $\mathbb{R}^n$*. Let* $l \in \mathbb{N}$*,* $p \in [1,+\infty]$ *and* $\lambda \in \left[0, \frac{n}{p}\right]$*. Then* $W_p^{l,\lambda,0}(\Omega)$ *is a closed proper subspace of* $W_p^{l,\lambda}(\Omega)$*.*

Proof. Let $u \in W_p^{l,\lambda}(\Omega)$. Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence in $W_p^{l,\lambda,0}(\Omega)$ which converges to u in $W_p^{l,\lambda}(\Omega)$. We want to show that $u \in W_p^{l,\lambda,0}(\Omega)$.

Since $u_k \to u$ in $W_p^{l,\lambda}(\Omega)$ as $k \to \infty$, we have

$$D_w^\alpha u_k \to D_w^\alpha u \quad \forall\, |\alpha| \le l \quad \text{in} \quad M_p^\lambda(\Omega)$$

as $k \to \infty$.

We know that $M_p^{\lambda,0}(\Omega)$ is a closed subspace of $M_p^\lambda(\Omega)$. Therefore,

$$D_w^\alpha u \in M_p^{\lambda,0}(\Omega) \quad \forall\, |\alpha| \le l,$$

and, thus, $u \in W_p^{l,\lambda,0}(\Omega)$. □

Lemma 1.47. *Let $l \in \mathbb{N} \setminus \{0\}$. Let $m \in \mathbb{N}$, $m < l$. Let $1 \le p, q \le +\infty$, $0 \le \lambda \le \frac{n}{p}$, $0 \le \nu \le \frac{n}{q}$. Suppose that for each bounded domain $G \subset \mathbb{R}^n$ star-shaped with respect to a ball there exists $c_1 > 0$ such that for each $\beta \in \mathbb{N}^n$ satisfying $|\beta| \le m$ and for all $f \in W_p^{l,\lambda}(G)$*

$$\|D_w^\beta f\|_{M_q^\nu(G)} \le c_1 \|f\|_{W_p^{l,\lambda}(G)}.$$

Then for each open bounded set $\Omega \subset \mathbb{R}^n$ satisfying the cone condition there exists $c_2 > 0$ such that

$$\|D_w^\beta f\|_{M_q^\nu(\Omega)} \le c_2 \|f\|_{W_p^{l,\lambda}(\Omega)}$$

for each $\beta \in \mathbb{N}^n$ satisfying $|\beta| \le m$ and for all $f \in W_p^{l,\lambda}(\Omega)$.

Proof. Let Ω satisfy the cone condition with the parameters r, h. By Lemma 1.40

$$\Omega = \bigcup_{k=1}^{s} \Omega_k,$$

where $s \in \mathbb{N}$ and Ω_k are bounded domains star-shaped with respect to the balls $B_k \subset \bar{B}_k \subset \Omega_k$.

Then

$$\|D_w^\beta f\|_{M_q^\nu(\Omega_k)} \le c_1(\Omega_k) \|f\|_{W_p^{l,\lambda}(\Omega_k)}, \quad k = 1, \dots, s.$$

Hence, by Lemma 1.41

$$\|D_w^\beta f\|_{M_q^\nu(\Omega)} \le 2^\nu s^{\frac{1}{q}} \max_{k=1,\dots,s} c_1(\Omega_k) \|f\|_{W_p^{l,\lambda}(\Omega)}.$$

□

Chapter 2

The embedding and multiplication operators in Sobolev Morrey spaces

2.1 The Sobolev Embedding Theorem

First we introduce the following notation.

Definition 2.1. *Let* $p \in [1,+\infty]$, $l,n \in \mathbb{N}\backslash\{0\}$, $m \in \mathbb{N}$, $m \leq l$, $\lambda,\ \nu \in [0,+\infty[$. *Let* $l+\lambda-m-\nu \neq \frac{n}{p}$. *Then we set*

$$q^*(l,m,n,p,\lambda,\nu) \equiv \frac{n}{(n/p)-(l+\lambda-m-\nu)}.$$

If $\lambda = \nu = 0$, *then* $q^*(l,m,n,p,\lambda,\nu)$ *equals the classical Sobolev limiting exponent. If* $\lambda,\nu \in [0,+\infty[$, *then the exponent* $q^*(l,m,n,p,\lambda,\nu)$ *can be obtained from the classical one by replacing* l *by* $l+\lambda$ *and* m *by* $m+\nu$.

We note that if $l+\lambda-\nu \neq \frac{n}{p}$, then the equality which defines $q^*(l,0,n,p,\lambda,\nu)$ is equivalent to the equality

$$l = \left(\nu - \frac{n}{q^*(l,0,n,p,\lambda,\nu)}\right) - \left(\lambda - \frac{n}{p}\right).$$

We also note that

$$\frac{q^*(l,0,n,p,\lambda,\nu)}{p} > 1 \quad \text{whenever} \quad \begin{cases} l+\lambda > \nu, \\ l+\lambda-\nu < \frac{n}{p}, \end{cases}$$

and

$$q^*(l-m,0,n,p,\lambda,\nu)=q^*(l,m,n,p,\lambda,\nu).$$

Remark 2.2. *Before we prove an analogue of the Sobolev Embedding Theorem, we recall the following:*

(i) $M_p^\lambda(\mathbb{R}^n)$ *is continuously embedded into* $\mathcal{M}_p^{r^{-\lambda}}(\mathbb{R}^n)$.

(ii) If Ω *is a bounded domain, then we have* $\mathcal{M}_p^{r^{-\lambda}}(\Omega)=M_p^\lambda(\Omega)$ *with equivalent norms.*

We are now ready to prove the following Sobolev Embedding Theorem.

Theorem 2.3. *Let* $p\in[1,+\infty[$, $l,n\in\mathbb{N}\setminus\{0\}$, $m\in\mathbb{N}$, $m\le l$, $\lambda\in\left[0,\frac{n}{p}\right]$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *which satisfies the cone property. Then the following statements hold.*

(i) Let $l-m+\lambda<\frac{n}{p}$. *Let* $\nu\in]\lambda,(l-m)+\lambda]$. *Then* $W_p^{l,\lambda}(\Omega)$ *is continuously embedded into* $W^{m,\nu}_{q^*(l,m,n,p,\lambda,\nu)}(\Omega)$.

(ii) Let $l-m+\lambda<\frac{n}{p}$. *If* $p>1$, *then* $W_p^{l,\lambda}(\Omega)$ *is continuously embedded into* $W^{m,\lambda}_{q^*(l,m,n,p,\lambda,\lambda)}(\Omega)$.

(iii) Let $l-m+\lambda>\frac{n}{p}$. *Then* $W_p^{l,\lambda}(\Omega)$ *is continuously embedded into* $W_\infty^m(\Omega)$.

Proof. (i) First let $m=0$.

Let Ω be a bounded domain star-shaped with respect to the ball $B=B(x_0,r)$, $\bar{B}\subset\Omega$. Then by Sobolev's integral representation there exists $M_1>0$ such that

$$|f(x)|\le M_1\left(\int\limits_B|f|dy+\sum_{|\alpha|=l}\int\limits_{V_x}\frac{(D_w^\alpha f)(y)}{|x-y|^{n-l}}dy\right)$$

for almost all $x\in\Omega$ for each (cf. e.g., Burenkov [13, Ch.3 p.112]).

Hence,

$$\|f\|_{\mathcal{M}^{r^{-\nu}}_{q^*(l,0,n,p,\lambda,\nu)}(\Omega)}\le M_1\left(\int\limits_B|f|dy\cdot\|1\|_{\mathcal{M}^{r^{-\nu}}_{q^*(l,0,n,p,\lambda,\nu)}(\Omega)}+\right.$$

$$\left.+\sum_{|\alpha|=l}\left\|\int\limits_{\mathbb{R}^n}\frac{\Phi_\alpha(y)}{|x-y|^{n-l}}dy\right\|_{\mathcal{M}^{r^{-\nu}}_{q^*(l,0,n,p,\lambda,\nu)}(\Omega)}\right),$$

where $\Phi_\alpha(y) = \begin{cases} D_w^\alpha f(y), & \text{if} \quad y \in \Omega; \\ 0, & \text{if} \quad y \notin \Omega. \end{cases}$

Note that

$$\|1\|_{\mathcal{M}^{r-\nu}_{q^*(l,0,n,p,\lambda,\nu)}(\Omega)} = \sup_{x\in\Omega} \sup_{\rho>0} \rho^{-\nu} \|1\|_{L_{q^*(l,0,n,p,\lambda,\nu)}(B(x,\rho)\cap\Omega)} \leq$$

$$\leq \sup_{x\in\Omega} \max \left\{ \sup_{0<\rho\leq(\operatorname{diam}\Omega)} \upsilon_n^{\frac{1}{q^*(l,0,n,p,\lambda,\nu)}} \rho^{-\nu+\frac{n}{q^*(l,0,n,p,\lambda,\nu)}}, \sup_{\rho\geq(\operatorname{diam}\Omega)} \rho^{-\nu} m_n(\Omega)^{\frac{1}{q^*(l,0,n,p,\lambda,\nu)}} \right\} =$$

$$= \max \left\{ \upsilon_n^{\frac{1}{q^*(l,0,n,p,\lambda,\nu)}} m_n(\Omega)^{-\nu+\frac{n}{q^*(l,0,n,p,\lambda,\nu)}}, m_n(\Omega)^{-\nu+\frac{1}{q^*(l,0,n,p,\lambda,\nu)}} \right\} < \infty.$$

By Theorem 1.30 there exists $c > 0$ depending only on n, l, p, $q^*(l,0,n,p,\lambda,\nu)$ such that

$$\left\| \int\limits_{\mathbb{R}^n} \frac{\Phi_\alpha(y)}{|x-y|^{n-l}} dy \right\|_{\mathcal{M}^{r-\nu}_{q^*(l,0,n,p,\lambda,\nu)}(\mathbb{R}^n)} \leq c\|\Phi_\alpha\|_{\mathcal{M}^{r-\lambda}_p(\mathbb{R}^n)} \leq$$

$$\leq 2^\lambda c \|D_w^\alpha f\|_{\mathcal{M}^{r-\lambda}_p(\Omega)} \leq 2^\lambda c \|D_w^\alpha f\|_{M^\lambda_p(\Omega)}.$$

By Hölder inequality $\int\limits_B |f| dy \leq m_n(B)^{\frac{1}{p'}} \|f\|_{L_p(\Omega)}$.

Therefore, there exist $M_2 > 0$ and $M_3 > 0$ such that

$$\|f\|_{M^\nu_{q^*(l,0,n,p,\lambda,\nu)}(\Omega)} \leq \max\{1, (\operatorname{diam}\Omega)^\nu\} \|f\|_{\mathcal{M}^{r-\nu}_{q^*(l,0,n,p,\lambda,\nu)}(\Omega)} \leq$$

$$\leq \max\{1, (\operatorname{diam}\Omega)^\nu\} \|f\|_{\mathcal{M}^{r-\nu}_{q^*(l,0,n,p,\lambda,\nu)}(\mathbb{R}^n)} \leq$$

$$\leq M_2 \left(\|f\|_{L_p(\Omega)} + \sum_{|\alpha|=l} \left\| \int\limits_{\mathbb{R}^n} \frac{\Phi_\alpha(y)}{|x-y|^{n-l}} dy \right\|_{\mathcal{M}^{r-\nu}_{q^*}(\mathbb{R}^n)} \right) \leq$$

$$\leq M_3 \left(\|f\|_{M^\lambda_p(\Omega)} + \sum_{|\alpha|=l} \|D_w^\alpha f\|_{M^\lambda_p(\Omega)} \right) = M_3 \|f\|_{W^{l,\lambda}_p(\Omega)}, \quad \forall\, f \in W^{l,\lambda}_p(\Omega).$$

Hence, by Lemma 1.47, the statement of Theorem 2.3 follows.

Now let $\alpha : |\alpha| = m$. Then $D_w^\alpha f \in W_p^{l-|\alpha|,\lambda}(\Omega) = W_p^{l-m,\lambda}(\Omega)$. Hence, there exists a constant $c_1 > 0$ such that

$$\begin{aligned}
\|f\|_{W^{m,\nu}_{q^*(l,m,n,p,\lambda,\nu)}(\Omega)} &= \sum_{|\alpha|\leq m} \|D^\alpha_w f\|_{M^\nu_{q^*(l,m,n,p,\lambda,\nu)}(\Omega)} = \\
&= \sum_{|\alpha|\leq m} \|D^\alpha_w f\|_{M^\nu_{q^*(l-m,0,n,p,\lambda,\nu)}(\Omega)} \leq c_1 \sum_{|\alpha|\leq m} \|D^\alpha_w f\|_{W^{l-m,\lambda}_p(\Omega)} \leq \\
&\leq c_1 \sum_{|\alpha|\leq m} \sum_{|\gamma|\leq l-m} \|D^{\gamma+\alpha}_w f\|_{M^\lambda_p(\Omega)} \leq \\
&\leq c_1 \sum_{|\alpha|\leq l} \|D^\alpha_w f\|_{M^\lambda_p(\Omega)} = c_1 \|f\|_{W^{l,\lambda}_p(\Omega)}, \quad \forall\, f \in W^{l,\lambda}_p(\Omega).
\end{aligned}$$

(ii) This case can be analized as case (i) by replacing $q^*(l,m,n,p,\lambda,\nu)$ by $q^*(l,m,n,p,\lambda,\lambda)$.

(iii) Now $l - m + \lambda > \frac{n}{p}$. Let Ω be a bounded domain star-shaped with respect to the ball $B = \mathbb{B}_n(\xi, r_0)$, $\bar{B} \subset \Omega$. Then by Sobolev's integral representation there exists $c > 0$ such that

$$|f(x)| \leq c \left(\int_{\mathbb{B}_n(\xi,r_0)} |f| dx + \sum_{|\gamma|=l} \int_{V_x} \frac{|(D^\gamma_w f)(y)|}{|x-y|^{n-l}} dy \right), \tag{2.1}$$

for almost all $x \in \Omega$ and for all $f \in W^{l,\lambda}_p(\Omega)$, and where V_x denotes the conical body based on $\mathbb{B}_n(\xi, r_0)$ and with vertex x (cf. e.g., Burenkov [13, Ch.3 p.112]).

We first consider case $m = 0$. So we now assume that $l + \lambda > \frac{n}{p}$. We plan to estimate the supremum of $|f|$ by exploiting inequality (2.1). Since $\int_{\mathbb{B}_n(\xi,r_0)} |f| dx$ is a constant, it defines an element of $C^0_b(\Omega) \subseteq L_\infty(\Omega)$. Next we prove that the sum in the right hand side of (2.1) is bounded if $f \in W^{l,\lambda}_p(\Omega)$.

We plan to treat separately case $l < n$ and case $l \geq n$.

Let $l < n$. Since $l + \lambda > \frac{n}{p}$, we can invoke Corollary 1.33 and conclude that $I_{l,\Omega}$ is linear and continuous from $M^\lambda_p(\Omega)$ to $B(\mathbb{R}^n)$.

Since $V_x \subseteq \Omega$ for all $x \in \Omega$, we deduce that

$$\left| \int_{V_x} \frac{|h(y)|}{|x-y|^{n-l}} dy \right| \leq I_{l,\Omega}(|h|) \quad \forall\, x \in \Omega,$$

for all $h \in M^\lambda_p(\Omega)$. By the continuity of the restriction operator in Morrey spaces and by the above mentioned continuity of $I_{l,\Omega}$, we deduce that the map $J_{l,\Omega}$ from $M^\lambda_p(\Omega)$ to $B(\Omega)$ defined by

$$J_{l,\Omega} h(x) \equiv \int_{V_x} \frac{h(y)}{|x-y|^{n-l}} dy \quad \forall\, x \in \Omega,$$

for all $h \in M_p^\lambda(\mathbb{R}^n)$ satisfies the inequality

$$|J_{l,\Omega}h(x)| \leq |I_{l,\Omega}(|h|)(x)| \leq \|I_{l,\Omega}\|_{\mathcal{L}(M_p^\lambda(\Omega),B(\mathbb{R}^n))} \| \, |h| \, \|_{M_p^\lambda(\Omega)} \leq \\ \leq \|I_{l,\Omega}\|_{\mathcal{L}(M_p^\lambda(\Omega),B(\mathbb{R}^n))} \|h\|_{M_p^\lambda(\Omega)} \quad \forall \, x \in \Omega, \quad (2.2)$$

for all $h \in M_p^\lambda(\Omega)$. Then we deduce that

$$|f(x)| \leq c \left(\int\limits_{\mathbb{B}_n(\xi,r_0)} |f| dx + \sum_{|\gamma|=l} |J_{l,\Omega}[D_w^\gamma f](x)| \right) \leq$$

$$\leq c \left([m_n(\mathbb{B}_n(\xi,r_0))]^{1-\frac{1}{p}} \|f\|_{L_p(\Omega)} + \|I_{l,\Omega}\|_{\mathcal{L}(M_p^\lambda(\Omega),B(\mathbb{R}^n))} \sum_{|\gamma|=l} \|D_w^\gamma f\|_{M_p^\lambda(\Omega)} \right) \leq$$

$$\leq c \left([m_n(\mathbb{B}_n(\xi,r_0))]^{1-\frac{1}{p}} + \|I_{l,\Omega}\|_{\mathcal{L}(M_p^\lambda(\Omega),B(\mathbb{R}^n))} \right) \|f\|_{W_p^{l,\lambda}(\Omega)}, \qquad (2.3)$$

for almost all $x \in \Omega$ and for all $f \in W_p^{l,\lambda}(\Omega)$.

We now consider case $l \geq n$. The embedding of $M_p^\lambda(\Omega)$ into $L_p(\Omega)$ and inequality (2.1) and the Hölder inequality imply that

$$|f(x)| \leq c \left(\int\limits_{\mathbb{B}_n(\xi,r_0)} |f| dx + \sum_{|\gamma|=l} \int\limits_\Omega \frac{|(D_w^\gamma f)(y)|}{|x-y|^{n-l}} dy \right) \leq \qquad (2.4)$$

$$\leq c \left([m_n(\mathbb{B}_n(\xi,r_0))]^{1-\frac{1}{p}} \|f\|_{L_p(\Omega)} + \sum_{|\gamma|=l} \|D_w^\gamma f\|_{L_1(\Omega)} (\operatorname{diam} \Omega)^{l-n} \right) \leq$$

$$\leq c \left([m_n(\mathbb{B}_n(\xi,r_0))]^{1-\frac{1}{p}} \|f\|_{L_p(\Omega)} + (\operatorname{diam} \Omega)^{l-n} [m_n(\Omega)]^{1-\frac{1}{p}} \sum_{|\gamma|=l} \|D_w^\gamma f\|_{L_p(\Omega)} \right) \leq$$

$$\leq c \left([m_n(\mathbb{B}_n(\xi,r_0))]^{1-\frac{1}{p}} + (\operatorname{diam} \Omega)^{l-n} [m_n(\Omega)]^{1-\frac{1}{p}} \right) \|f\|_{W_p^{l,\lambda}(\Omega)},$$

for almost all $x \in \Omega$ and for all $f \in W_p^{l,\lambda}(\Omega)$. By Lemma 1.47, by the inequality (2.3) for case $l < n$ and by the inequality (2.4) for case $l \geq n$, we deduce the validity of statement (iii) in case $m = 0$.

Next we prove the statement (iii) in case $m > 0$. If $f \in W_p^{l,\lambda}(\Omega)$, then $D_w^\beta f \in W_p^{l-m,\lambda}(\Omega)$ for all $|\beta| \leq m$. Now by assumption, we have $(l-m)+\lambda > \frac{n}{p}$. Hence, case $m = 0$ with l replaced by $l - m$ implies that $W_p^{l-m,\lambda}(\Omega) \subseteq L_\infty(\Omega)$ and that there exists $c_1 > 0$ such that

$$\|g\|_{L_\infty(\Omega)} \leq c_1 \|g\|_{W_p^{l-m,\lambda}(\Omega)} \quad \forall \, g \in W_p^{l-m,\lambda}(\Omega).$$

Hence, $D_w^\beta f \in L_\infty(\Omega)$ for all $\beta \in \mathbb{N}^n$ such that $|\beta| \le m$, and

$$\|f\|_{W_\infty^m(\Omega)} \le \sum_{|\beta|\le m} \|D_w^\beta f\|_{L_\infty(\Omega)} \le$$
$$\le c_1 \sum_{|\beta|\le m} \|D_w^\beta f\|_{W_p^{l-m,\lambda}(\Omega)} \le c_1 \sum_{|\beta|\le m} \sum_{|\gamma|\le l-m} \|D_w^{\gamma+\beta} f\|_{M_p^\lambda(\Omega)} \le$$
$$\le c_1 \left(\sum_{|\beta|\le m} \sum_{|\gamma|\le l-m} 1^{|\gamma+\beta|} \right) \|f\|_{W_p^{l,\lambda}(\Omega)}$$

for all $f \in W_p^{l,\lambda}(\Omega)$. Hence, the proof of statement (iii) is complete. □

2.2 Approximation by C^∞ functions in Sobolev Morrey spaces

First we state a known Leibnitz formula for Sobolev spaces. For the proof one can see for example [24, 5.2.3]

Theorem 2.4. *Let $l \in \mathbb{N}$. Let $1 \le p < +\infty$. Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $u \in W_p^l(\Omega)$ and $|\alpha| \le l$. If $\zeta \in C_c^l(\Omega)$, then $\zeta u \in W_p^l(\Omega)$ and*

$$D_w^\alpha(\zeta u) = \sum_{\beta\le\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta \zeta D_w^{\alpha-\beta} u, \tag{2.5}$$

where $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $l \in \mathbb{N}$. Let $u, v \in L_1^{\mathrm{loc}}(\Omega)$. Moreover, assume that for any $\beta \in \mathbb{N}^n$ satisfying $|\beta| \le l$ there exists $1 \le p_\beta < \infty$ such that $D_w^\beta u \in L_{p_\beta}^{\mathrm{loc}}(\Omega)$ and $D_w^\gamma v \in L_{p'_\beta}^{\mathrm{loc}}(\Omega)$ for all $\gamma \in \mathbb{N}^n : |\gamma| \le l - |\beta|$. Then for any $\alpha \in \mathbb{N}^n$ satisfying $|\alpha| \le l$ the weak derivative $D_w^\alpha(uv)$ exists and the Leibnitz formula holds:*

$$D_w^\alpha(uv) = \sum_{0\le\beta\le\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D_w^\beta u D_w^{\alpha-\beta} v. \tag{2.6}$$

Proof. Let $u \in C^\infty(\Omega)$ and $v \in L_1^{\mathrm{loc}}(\Omega)$ be such that $D_w^\gamma v \in L_1^{\mathrm{loc}}(\Omega)$. Then

$$D_w^\alpha(uv) = \sum_{0\le\beta\le\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta u D_w^{\alpha-\beta} v.$$

Now let u be as in formulation, *i.e.* $D_w^\beta u \in L_{p_\beta}^{\mathrm{loc}}(\Omega)$. Let also

$$u_k(x) = u(x) * \varphi_{\frac{1}{k}}(x) \quad \forall\, k \in \mathbb{N},$$

where $\varphi_{\frac{1}{k}}(x)$ as in definition 1.20 with $t = \frac{1}{k}$.

Then, by Theorem 2.4, we have

$$D_w^\alpha(u_k v) = \sum_{0\le\beta\le\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta u_k D_w^{\alpha-\beta} v =$$

$$= \sum_{0\le\beta\le\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta[u * \varphi_{\frac{1}{k}}] D_w^{\alpha-\beta} v.$$

Properties of mollifiers imply that

$$u_k \to u \quad \text{in } L_{p_\beta}^{\mathrm{loc}}(\Omega),$$

$$D^\beta u_k \to D_w^\beta u \quad \text{in } L_{p_\beta}^{\mathrm{loc}}(\Omega),$$

as $k \to \infty$. Thus,

$$D_w^\alpha(uv) = \sum_{0\le\beta\le\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D_w^\beta u D_w^{\alpha-\beta} v.$$

□

Theorem 2.6. *Let $l \in \mathbb{N}$. Let $1 \le p < +\infty$, $0 \le \lambda \le \frac{n}{p}$. Let Ω be a bounded open subset of $\mathbb{R}^n$. Then for $u \in W_p^{l,\lambda,0}(\Omega)$ there exist functions $u_m \in C^\infty(\Omega) \cap W_p^{l,\lambda}(\Omega)$ such that*

$$u_m \to u \quad \textit{in } W_p^{l,\lambda}(\Omega).$$

Proof. We have $\Omega = \bigcup\limits_{i=1}^{\infty} \Omega_i$, where

$$\Omega_i := \left\{ x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > \frac{1}{i} \right\} \quad (i = 1, 2, \ldots).$$

Write $V_i := \Omega_{i+3} - \bar{\Omega}_{i+1}$.

Choose also any open set $V_0 \subset\subset \Omega$ so that $\Omega = \bigcup\limits_{i=0}^{\infty} V_i$. Now let $\{\zeta_i\}_{i=1}^\infty$ be a smooth partition of unity subordinate to the open sets $\{V_i\}_{i=0}^\infty$; that is, suppose

$$\begin{cases} 0 \le \zeta_i \le 1, \ \ \zeta_i \in C_c^\infty(V_i) \\ \sum\limits_{i=0}^{\infty} \zeta_i = 1 \qquad \text{on } \Omega. \end{cases}$$

Next, choose any function $u \in W_p^{l,\lambda,0}(\Omega)$. By Theorem 2.4 we know that

$$D_w^\alpha(u\zeta_i) = \sum_{0\le\beta\le\alpha} \left(\begin{array}{c} \alpha \\ \beta \end{array} \right) D_w^\beta u D_w^{\alpha-\beta}\zeta_i.$$

Since $D_w^\beta u \in M_p^{\lambda,0}(\Omega)$, $D_w^{\alpha-\beta}\zeta_i \in L_\infty(\Omega)$, by Theorem 1.18 $D_w^\beta u D_w^{\alpha-\beta}\zeta_i \in M_p^{\lambda,0}(\Omega)$. Therefore, $D_w^\alpha(u\zeta_i) \in M_p^{\lambda,0}(\Omega)$ for all $|\alpha| \le l$. Since $W_p^{l,\lambda,0}(\Omega)$ is a closed subspace of $W_p^{l,\lambda}(\Omega)$, we have $\zeta_i u \in W_p^{l,\lambda,0}(\mathbb{R}^n)$ and $\operatorname{supp}(\zeta_i u) \subset V_i$.

Fix $\delta > 0$. Choose then $\varepsilon_i > 0$ so small that $\phi_{\varepsilon_i} * (\zeta_i u)$ satisfies

$$\begin{cases} \|\phi_{\varepsilon_i} * (\zeta_i u) - \zeta_i u\|_{W_p^{l,\lambda}(\Omega)} \le \frac{\delta}{2^{i+1}}, & (i = 0, 1, \ldots) \\ \operatorname{supp}[\phi_{\varepsilon_i} * (\zeta_i u)] \subset W_i & (i = 1, \ldots), \end{cases}$$

for $W_i := \Omega_{i+4} - \bar{\Omega}_i \supset V_i$ $(i = 1, \ldots)$.

Write $v := \sum\limits_{i=0}^{\infty} \phi_{\varepsilon_i} * (\zeta_i u)$. This function belongs to $C^\infty(\Omega)$, since for each open set $V \subset\subset \Omega$ there are at most finitely many nonzero terms in the sum. Since $u = \sum\limits_{i=0}^{\infty} \zeta_i u$, we have for each $V \subset\subset \Omega$

$$\|v - u\|_{W_p^{l,\lambda}(V)} \le \sum_{i=0}^{\infty} \|\phi_{\varepsilon_i} * (\zeta_i u) - \zeta_i u\|_{W_p^{l,\lambda}(\Omega)} \le \delta \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = \delta.$$

Take the supremum over sets $V \subset\subset \Omega$, to conclude $\|v - u\|_{W_p^{l,\lambda}(\Omega)} \le \delta$. □

2.3 Multiplication Theorems for Sobolev Morrey spaces

Next we prove the following multiplication result which follows by the Hölder inequality.

Proposition 2.7. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$. *Let* $l \in \mathbb{N}$. *Let* $p, q, r \in [1, +\infty]$ *be such that* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. *Let also* $0 \le \lambda_1 \le \frac{n}{p}$, $0 \le \lambda_2 \le \frac{n}{q}$, $0 \le \lambda \le \frac{n}{r}$, $\lambda \equiv \lambda_1 + \lambda_2$. *Then if* $u \in W_p^{l,\lambda_1}(\Omega)$ *and* $v \in W_q^{l,\lambda_2}(\Omega)$, *we have* $uv \in W_r^{l,\lambda}(\Omega)$.

Moreover, there exists $c > 0$ *such that*

$$\|uv\|_{W_r^{l,\lambda}(\Omega)} \le c\|u\|_{W_p^{l,\lambda_1}(\Omega)}\|v\|_{W_q^{l,\lambda_2}(\Omega)} \quad \forall\, (u, v) \in W_p^{l,\lambda_1}(\Omega) \times W_q^{l,\lambda_2}(\Omega).$$

Proof. By Proposition 2.5, we know that the D_w^α weak derivative of u is delivered by the Leibnitz rule. Using Hölder's inequality for Morrey spaces, we obtain

$$\begin{aligned}\|uv\|_{W_r^{l,\lambda}(\Omega)} &= \sum_{|\alpha|\le l} \|D_w^\alpha(uv)\|_{M_r^\lambda(\Omega)} \le \\ &\le \sum_{|\alpha|\le l}\sum_{|\beta|\le\alpha}\left(\begin{array}{c}\alpha\\ \beta\end{array}\right)\left\|D_w^{\alpha-\beta}uD_w^\beta v\right\|_{M_r^\lambda(\Omega)} \le \\ &\le \sum_{|\alpha|\le l}\sum_{|\beta|\le\alpha}\left(\begin{array}{c}\alpha\\ \beta\end{array}\right)\left\|D_w^{\alpha-\beta}u\right\|_{M_p^{\lambda_1}(\Omega)}\left\|D_w^\beta v\right\|_{M_q^{\lambda_2}(\Omega)} \le \\ &\le c\|u\|_{W_p^{l,\lambda_1}(\Omega)}\|v\|_{W_q^{l,\lambda_2}(\Omega)}.\end{aligned}$$

□

We now extend a known multiplication result for Sobolev spaces of Zolesio [61] (see also Valent [58], Runst and Sickel [51]).

Theorem 2.8. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $l \in \mathbb{N}$. Let $p,\ q,\ r \in [1,+\infty]$, and let $0 \le \lambda_1 \le \frac{n}{p}$, $0 \le \lambda_2 \le \frac{n}{q}$, $0 \le \lambda \le \frac{n}{r}$. Assume that Ω has the cone property, and $p \ge r$, $q \ge r$ and*

$$\lambda_1 - \frac{n}{p} \ge \lambda - \frac{n}{r}, \quad \lambda_2 - \frac{n}{q} \ge \lambda - \frac{n}{r}, \tag{2.7}$$

$$\frac{l+\lambda_1+\lambda_2-\lambda}{n} > \frac{1}{p} + \frac{1}{q} - \frac{1}{r}, \tag{2.8}$$

$$\frac{l}{n} > \frac{1}{p} + \frac{1}{q} - \frac{1}{r}. \tag{2.9}$$

Then if $u \in W_p^{l,\lambda_1}(\Omega)$ and $v \in W_q^{l,\lambda_2}(\Omega)$, we have $uv \in W_r^{l,\lambda}(\Omega)$.

Moreover, there exists a positive number c such that

$$\|uv\|_{W_r^{l,\lambda}(\Omega)} \le c\|u\|_{W_p^{l,\lambda_1}(\Omega)}\|v\|_{W_q^{l,\lambda_2}(\Omega)} \quad \forall\,(u,v) \in W_p^{l,\lambda_1}(\Omega) \times W_q^{l,\lambda_2}(\Omega). \tag{2.10}$$

Proof. Let $\alpha \in \mathbb{N}^n$, $|\alpha| \le l$. Then we set

$$Q_\alpha[u,v] = \sum_{\beta\le\alpha}\left(\begin{array}{c}\alpha\\ \beta\end{array}\right) D_w^{\alpha-\beta}uD_w^\beta v \quad \text{for all } (u,v) \in W_p^{l,\lambda_1}(\Omega) \times W_q^{l,\lambda_2}(\Omega).$$

We want to prove that Q_α defines a bilinear and continuous map from $W_p^{l,\lambda_1}(\Omega) \times W_q^{l,\lambda_2}(\Omega)$ to $M_r^\lambda(\Omega)$.

It is sufficies to show that if $\beta \le \alpha$, $|\alpha| \le l$, then

(A) the map from $W_p^{l-|\alpha-\beta|,\lambda_1}(\Omega)\times W_q^{l-|\beta|,\lambda_2}(\Omega)$ to $M_r^{\lambda}(\Omega)$ which takes (u,v) to uv is bilinear and continuous.

Before we perform the proof in several steps, we remark the following:

If either $W_p^{l-|\alpha-\beta|,\lambda_1}(\Omega)$ or $W_q^{l-|\beta|,\lambda_2}(\Omega)$ is continuously embedded into $L_\infty(\Omega)$ then (A) is true.

Indeed, if $W_p^{l-|\alpha-\beta|,\lambda_1}(\Omega)\hookrightarrow L_\infty(\Omega)$, we know that

$$W_q^{l-|\beta|,\lambda_2}(\Omega)\hookrightarrow M_q^{\lambda_2}(\Omega)\hookrightarrow M_r^{\lambda}(\Omega)$$

and the product

$$L_\infty(\Omega)\times M_r^{\lambda}(\Omega)\to M_r^{\lambda}(\Omega)$$

is continuous.

Now we split the proof into several steps:

(a) $\max\left\{l-|\alpha-\beta|+\lambda_1-\frac{n}{p}, l-|\beta|+\lambda_2-\frac{n}{q}\right\}>0;$

(b) $\max\left\{l-|\alpha-\beta|+\lambda_1-\frac{n}{p}, l-|\beta|+\lambda_2-\frac{n}{q}\right\}<0;$

(c) $\max\left\{l-|\alpha-\beta|+\lambda_1-\frac{n}{p}, l-|\beta|+\lambda_2-\frac{n}{q}\right\}=0.$

We begin by considering the case (a). Assume that

$$\max\left\{l-|\alpha-\beta|+\lambda_1-\frac{n}{p}, l-|\beta|+\lambda_2-\frac{n}{q}\right\}=l-|\alpha-\beta|+\lambda_1-\frac{n}{p}>0.$$

Then Theorem 2.3 (ii) implies that $W_p^{l-|\alpha-\beta|,\lambda_1}(\Omega)$ is continuously embedded into $L_\infty(\Omega)$. Thus, by remark above (A) follows.

If

$$\max\left\{l-|\alpha-\beta|+\lambda_1-\frac{n}{p}, l-|\beta|+\lambda_2-\frac{n}{q}\right\}=l-|\beta|+\lambda_2-\frac{n}{q}>0,$$

then by similar way we see that $W_q^{l-|\beta|,\lambda_2}(\Omega)$ is continuously embedded into $L_\infty(\Omega)$ and accordingly the truth of (A) follows.

Our next step will be proving of the case (b).

Here we distinguish the following cases

$$\begin{cases}(b1) & l-|\alpha-\beta|=0;\\ (b2) & l-|\alpha-\beta|>0.\end{cases}\qquad\begin{cases}(b3) & l-|\beta|=0;\\ (b4) & l-|\beta|>0.\end{cases}$$

Before we continue the proof we note that in case $(b1)$

$$l = |\alpha - \beta| \leq |\alpha| \leq l \Leftrightarrow |\alpha| = l, |\beta| = 0, \tag{2.11}$$

and in case $(b3)$

$$l = |\beta| \leq |\alpha| \leq l \Leftrightarrow \alpha = \beta.$$

So we can conclude that $(b1)$ and $(b3)$ cannot be both true.

We start from the case $(b1)$-$(b4)$. Thus, we have $W_p^{l-|\alpha-\beta|,\lambda_1}(\Omega) = W_p^{0,\lambda_1}(\Omega) = M_p^{\lambda_1}(\Omega)$ and by (2.11) $W_q^{l-|\beta|,\lambda_2}(\Omega) = W_q^{l,\lambda_2}(\Omega)$.

We now choose $\nu \in]\lambda_2, \lambda_2+l]$, then we set $q_1 = \frac{nq}{n-q(l+\lambda_2-\nu)}$. Since $\lambda_2+l < \frac{n}{q}$, Theorem 2.3(i) implies the following embedding

$$W_q^{l,\lambda_2}(\Omega) \hookrightarrow M_{q_1}^{\nu}(\Omega)$$

Next we define $r_1 \in [1, +\infty]$ such that $\frac{1}{r_1} = \frac{1}{p} + \frac{1}{q_1}$.

Using Hölder's inequality for Morrey spaces, we get that the pointwise product is bilinear and continuous from $M_p^{\lambda_1}(\Omega) \times M_{q_1}^{\nu}(\Omega)$ to $M_{r_1}^{\lambda_1+\nu}(\Omega)$.

Now we want to prove that $M_{r_1}^{\lambda_1+\nu}(\Omega) \hookrightarrow M_r^{\lambda}(\Omega)$. This embedding holds provided the following two conditions.

$$\frac{1}{r_1} = \frac{1}{p} + \frac{1}{q_1} = \frac{1}{p} + \frac{n - q(l+\lambda_2-\nu)}{nq} = \frac{1}{p} + \frac{\frac{n}{q} - (l+\lambda_2-\nu)}{n} \leq \frac{1}{r},$$

and

$$(\lambda_1 + \nu) - \frac{n}{r_1} \geq \lambda - \frac{n}{r}.$$

We can rewrite the second condition as follows

$$\frac{\lambda_1+\nu}{n} - \frac{1}{r_1} \geq \frac{\lambda}{n} - \frac{1}{r},$$

$$\frac{\lambda_1+\nu}{n} - \frac{1}{p} - \frac{\frac{n}{q} - (l+\lambda_2-\nu)}{n} \geq \frac{\lambda}{n} - \frac{1}{r},$$

$$\frac{l+\lambda_1+\lambda_2-\lambda}{n} \geq \frac{1}{p} + \frac{1}{q} - \frac{1}{r}.$$

So we have the two conditions

$$\begin{cases} \frac{l}{n} + \frac{\lambda_2-\nu}{n} \geq \frac{1}{p} + \frac{1}{q} - \frac{1}{r}, \\ \frac{l+\lambda_1+\lambda_2-\lambda}{n} \geq \frac{1}{p} + \frac{1}{q} - \frac{1}{r}. \end{cases}$$

By assumptions (2.8), (2.9) we can choose ν sufficiently close to λ_2 such that the above inequalities hold as strict inequalities.

Therefore, by Theorem 1.15(iii) $M_{r_1}^{\lambda_1+\nu}(\Omega)$ is continuously embedded into $M_r^{\lambda}(\Omega)$. Hence, $uv \in M_r^{\lambda}(\Omega)$. Moreover, there exists $c>0$ such that

$$\|uv\|_{M_r^{\lambda}(\Omega)} \leq c\|u\|_{M_p^{\lambda_1}(\Omega)}\|v\|_{W_q^{l,\lambda_2}(\Omega)} \quad \forall\,(u,v) \in M_p^{\lambda_1}(\Omega) \times W_q^{l,\lambda_2}(\Omega).$$

Case $(b2)$-$(b3)$ can be analyzed as case $(b1)$-$(b4)$ by switching roles of $W_p^{l-|\alpha-\beta|,\lambda_1}(\Omega)$ and of $W_q^{l-|\beta|,\lambda_2}(\Omega)$.

Finally, we consider the case $(b2)$-$(b4)$.

As above we choose $\gamma \in]\lambda_1, \lambda_1+l]$ and $\mu \in]\lambda_2, \lambda_2+l]$. Then we have the following embeddings

$$W_p^{l-|\alpha-\beta|,\lambda_1}(\Omega) \hookrightarrow M_{p^*}^{\gamma}(\Omega),$$

and

$$W_q^{l-|\beta|,\lambda_2}(\Omega) \hookrightarrow M_{q^*}^{\mu}(\Omega),$$

where $p^* = \frac{np}{n-p(l-|\alpha-\beta|+\lambda_1-\gamma)}$, $q^* = \frac{nq}{n-q(l-|\beta|+\lambda_2-\mu)}$.

Next we define r^* such as $\frac{1}{r^*} = \frac{1}{p^*} + \frac{1}{q^*}$.

Hölder's inequality for Morrey spaces implies that the pointwise product is bilinear and continuous from $M_{p^*}^{\gamma}(\Omega) \times M_{q^*}^{\mu}(\Omega)$ to $M_{r^*}^{\gamma+\mu}(\Omega)$.

Now if we recall Theorem 1.15(iii), we see that the embedding $M_{r^*}^{\gamma+\mu}(\Omega) \hookrightarrow M_r^{\lambda}(\Omega)$ holds provided that

$$r \leq r^*, \ \ (\gamma+\mu) - \frac{n}{r^*} \geq \lambda - \frac{n}{r}. \tag{2.12}$$

As before, we can rewrite this conditions as

$$\frac{1}{r^*} = \frac{1}{p^*} + \frac{1}{q^*} =$$

$$= \frac{n-p(l-|\alpha-\beta|+\lambda_1-\gamma)}{np} + \frac{n-q(l-|\beta|+\lambda_2-\mu)}{nq} =$$

$$= \frac{1}{p} + \frac{1}{q} - \frac{l}{n} - \frac{l-|\alpha|}{n} - \frac{\lambda_1-\gamma}{n} - \frac{\lambda_2-\mu}{n} \leq \frac{1}{r},$$

$$\frac{l}{n} + \frac{l-|\alpha|}{n} + \frac{\lambda_1-\gamma}{n} + \frac{\lambda_2-\mu}{n} \geq \frac{1}{p} + \frac{1}{q} - \frac{1}{r},$$

$$\frac{l}{n} + \frac{l-|\alpha|}{n} + \frac{\lambda_1-\gamma}{n} + \frac{\lambda_2-\mu}{n} \geq \frac{l}{n} + \frac{\lambda_1-\gamma}{n} + \frac{\lambda_2-\mu}{n} \geq \frac{1}{p} + \frac{1}{q} - \frac{1}{r}.$$

and

$$(\gamma+\mu)-\frac{n}{r^*}\geq\lambda-\frac{n}{r},$$

$$\frac{\gamma+\mu}{n}-\frac{1}{r^*}\geq\frac{\lambda}{n}-\frac{1}{r},$$

$$\frac{\gamma+\mu}{n}-\frac{1}{p}-\frac{1}{q}+\frac{l}{n}+\frac{l-|\alpha|}{n}+\frac{\lambda_1-\gamma}{n}+\frac{\lambda_2-\mu}{n}\geq\frac{\lambda}{n}-\frac{1}{r},$$

$$\frac{l+(l-|\alpha|)+\lambda_1+\lambda_2-\lambda}{n}\geq\frac{1}{p}+\frac{1}{q}-\frac{1}{r},$$

$$\frac{l+(l-|\alpha|)+\lambda_1+\lambda_2-\lambda}{n}\geq\frac{l+\lambda_1+\lambda_2-\lambda}{n}\geq\frac{1}{p}+\frac{1}{q}-\frac{1}{r}.$$

As before, by assumptions (2.8), (2.9) we can choose γ and μ sufficiently close to λ_1 and λ_2 respectively such that the above inequalities hold as strict inequalities. Thus, we finish the case (b).

Next we will see case (c). We consider the following three separate cases

(c1) $l-|\alpha-\beta|+\lambda_1-\frac{n}{p}<0,\ \ l-|\beta|+\lambda_2-\frac{n}{q}=0;$

(c2) $l-|\alpha-\beta|+\lambda_1-\frac{n}{p}=0,\ \ l-|\beta|+\lambda_2-\frac{n}{q}<0;$

(c3) $l-|\alpha-\beta|+\lambda_1-\frac{n}{p}=0,\ \ l-|\beta|+\lambda_2-\frac{n}{q}=0.$

We start with case (c1).

If $l-|\beta|=0$, then $\lambda_2=\frac{n}{q}$ and $W_q^{l-|\beta|,\lambda_2}(\Omega)=L_\infty(\Omega)$. Therefore, our remark implies (A).

If $l-|\beta|\geq 1$, we consider separately

(c11) $q>1$;

(c12) $\lambda_2>0$;

(c13) $\lambda_2=0$, $q=1$.

Let first $l-|\beta|\geq 1$, $q>1$. Then we choose $\tilde{q}\in]1,q[$ so close to q so that

$$\frac{l+\lambda_1+\lambda_2-\lambda}{n}>\frac{1}{p}+\frac{1}{\tilde{q}}-\frac{1}{r},\quad \frac{l}{n}>\frac{1}{p}+\frac{1}{\tilde{q}}-\frac{1}{r}.$$

Since $\tilde{q}<q$ we have

$$\begin{cases}\lambda_2+l<\frac{n}{\tilde{q}}, & \text{if } l-|\alpha-\beta|=0,\\ \lambda_2+l-|\beta|<\frac{n}{\tilde{q}}, & \text{if } l-|\alpha-\beta|\geq 1.\end{cases}$$

But we have already proved such case (see cases (b1)-(b4) and (b2)-(b4)). Thus (A) is true.

Now we see case (c12): $l-|\beta|\geq 1$, $\lambda_2>0$. Then we choose $\tilde{\nu}\in]0,\lambda_2[$ so close to λ_2 so that

$$\frac{l+\lambda_1+\tilde{\nu}-\lambda}{n}>\frac{1}{p}+\frac{1}{q}-\frac{1}{r}.$$

Since $\tilde{\nu}<\lambda_2$ we have

$$\begin{cases} \tilde{\nu}+l<\frac{n}{q}, & \text{if } l-|\alpha-\beta|=0,\\ \tilde{\nu}+l-|\beta|<\frac{n}{q}, & \text{if } l-|\alpha-\beta|\geq 1.\end{cases}$$

But we have already proved such case (see cases (b1)-(b4) and (b2)-(b4)). Thus (A) is true.

Finally we consider case (c13). Since $\lambda_2=0$, $q=1$, we have

$$W_q^{l-|\beta|,\lambda_2}(\Omega)=W_1^{l-|\beta|}(\Omega).$$

Condition $l-|\beta|=n$ implies that

$$W_1^{l-|\beta|}(\Omega)\hookrightarrow L_\infty(\Omega).$$

Hence, by our remark (A) follows.

Case (c2) can be analyzed as case (c1) by switching roles of $W_p^{l-|\alpha-\beta|,\lambda_1}(\Omega)$ and of $W_q^{l-|\beta|,\lambda_2}(\Omega)$.

Now we consider case (c3).

If $l-|\alpha-\beta|=0$ or $l-|\beta|=0$, then $\lambda_1=\frac{n}{p}$ or $\lambda_2=\frac{n}{q}$ respectively and $W_p^{l-|\alpha-\beta|,\lambda_1}(\Omega)=L_\infty(\Omega)$ or $W_q^{l-|\beta|,\lambda_2}(\Omega)=L_\infty(\Omega)$. Therefore, our remark implies (A).

Thus we can assume that

$$\min\{l-|\alpha-\beta|,l-|\beta|\}\geq 1.$$

In such a case, we can split the proof of case (c3) into the following five cases.

(c31) $p>1$, $q>1$;

(c32) $p>1$, $\lambda_2>0$;

(c33) $\lambda_1>0$, $q>1$;

(c34) $\lambda_1>0$, $\lambda_2>0$;

(c35) either $(p, \lambda_1) = (1, 0)$ or $(q, \lambda_2) = (1, 0)$.

In case (c31) we choose $\tilde{p} \in]1, p[$ so close to p and $\tilde{q} \in]1, q[$ so close to q so that

$$\frac{l + \lambda_1 + \lambda_2 - \lambda}{n} > \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} - \frac{1}{r}, \quad \frac{l}{n} > \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} - \frac{1}{r}.$$

Since $\tilde{p} < p$, $\tilde{q} < q$ we have

$$\begin{cases} \lambda_1 + l - |\alpha - \beta| < \frac{n}{\tilde{p}} \\ \lambda_2 + l - |\beta| < \frac{n}{\tilde{q}}. \end{cases}$$

But we have already proved such case (see case (b)). Thus (A) is true.

Now we see case (c32): $p > 1$, $\lambda_2 > 0$. Then we choose $\tilde{p} \in]1, p[$ so close to p and $\tilde{\nu} \in]0, \lambda_2[$ so close to λ_2 so that

$$\frac{l + \lambda_1 + \tilde{\nu} - \lambda}{n} > \frac{1}{\tilde{p}} + \frac{1}{q} - \frac{1}{r}, \quad \frac{l}{n} > \frac{1}{\tilde{p}} + \frac{1}{q} - \frac{1}{r}.$$

Since $\tilde{p} < p$, $\tilde{\nu} < \lambda_2$ we have

$$\begin{cases} \lambda_1 + l - |\alpha - \beta| < \frac{n}{\tilde{p}}, \\ \tilde{\nu} + l - |\beta| < \frac{n}{q}. \end{cases}$$

But we have already proved such case (see case (b)). Thus (A) is true.

Case (c33): $\lambda_1 > 0$, $q > 1$. Then we choose $\tilde{\gamma} \in]0, \lambda_1[$ so close to λ_1 and $\tilde{q} \in]1, q[$ so close to q so that

$$\frac{l + \tilde{\gamma} + \lambda_2 - \lambda}{n} > \frac{1}{p} + \frac{1}{\tilde{q}} - \frac{1}{r}, \quad \frac{l}{n} > \frac{1}{p} + \frac{1}{\tilde{q}} - \frac{1}{r}.$$

Since $\tilde{q} < q$, $\tilde{\gamma} < \lambda_1$ we have

$$\begin{cases} \tilde{\gamma} + l - |\alpha - \beta| < \frac{n}{p}, \\ \lambda_2 + l - |\beta| < \frac{n}{\tilde{q}}. \end{cases}$$

But we have already proved such case (see case (b)). Thus (A) is true.

Case (c34): $\lambda_1 > 0$, $\lambda_2 > 0$. We choose $\tilde{\gamma} \in]0, \lambda_1[$ so close to λ_1 and $\tilde{\nu} \in]0, \lambda_2[$ so close to λ_2 so that

$$\frac{l + \tilde{\gamma} + \tilde{\nu} - \lambda}{n} > \frac{1}{p} + \frac{1}{q} - \frac{1}{r}.$$

Since $\tilde{\gamma} < \lambda_1$, $\tilde{\nu} < \lambda_2$ we have

$$\begin{cases} \tilde{\gamma} + l - |\alpha - \beta| < \frac{n}{p}, \\ \tilde{\nu} + l - |\beta| < \frac{n}{q}. \end{cases}$$

But we have already proved such case (see case (b)). Thus (A) is true.

Case (c35): either $(p, \lambda_1) = (1, 0)$ or $(q, \lambda_2) = (1, 0)$. By Sobolev embedding theorem we have either

$$W_p^{l-|\alpha-\beta|,\lambda_1}(\Omega) = W_1^{l-|\alpha-\beta|}(\Omega) \hookrightarrow L_\infty(\Omega).$$

or

$$W_q^{l-|\beta|,\lambda_2}(\Omega) = W_1^{l-|\beta|}(\Omega) \hookrightarrow L_\infty(\Omega).$$

Hence, by our remark (A) follows.

We now show that if $\alpha \in \mathbb{N}^n$ and $|\alpha| \leq l$, then

$$D_w^\alpha(uv) = Q_\alpha[u, v] \quad \forall (u, v) \in W_p^{l,\lambda_1}(\Omega) \times W_q^{l,\lambda_2}(\Omega),$$

an equality which implies the validity of the Leibnitz rule. To do so, we take

$$p_1 \in [1, p], \;\; p_1 < \infty, \quad q_1 \in [1, q], \;\; q_1 < \infty,$$

such that

$$\begin{aligned} p_1 \geq 1, \quad 0 - \tfrac{n}{p_1} \geq 0 - \tfrac{n}{1}, \\ q_1 \geq 1, \quad 0 - \tfrac{n}{q_1} \geq 0 - \tfrac{n}{1}, \end{aligned}$$

and

$$\frac{l + 0 + 0 - 0}{n} > \frac{1}{p_1} + \frac{1}{q_1} - \frac{1}{1}.$$

Then we note that

$$W_p^{l,\lambda_1}(\Omega) \times W_q^{l,\lambda_2}(\Omega) \hookrightarrow W_p^l(\Omega) \times W_q^l(\Omega) \hookrightarrow W_{p_1}^l(\Omega) \times W_{q_1}^l(\Omega).$$

Let now u_k and v_k be sequences in $C^\infty(\Omega) \bigcap W_{p_1}^l(\Omega)$ and $C^\infty(\Omega) \bigcap W_{q_1}^l(\Omega)$ respectively such that u_k converges to u in $W_{p_1}^l(\Omega)$ and v_k converges to v in $W_{q_1}^l(\Omega)$.

Next we observe that the pointwise product from $W_{p_1}^l(\Omega) \times W_{q_1}^l(\Omega)$ to $L_1(\Omega)$ is continuous.

Indeed, we know that $uv = D_w^0(uv) = Q_0[u, v]$ is continuous from $W_{p_1}^l(\Omega) \times W_{q_1}^l(\Omega)$ to $L_1(\Omega)$. Since the pointwise product is bilinear and continuous from $L_1(\Omega) \times L_\infty(\Omega)$ to $L_1(\Omega)$ we conclude that

$$\int_\Omega u_k v_k D^\alpha \varphi dx \to \int_\Omega uv D^\alpha \varphi dx.$$

Then

$$\int_{\Omega} uvD^{\alpha}\varphi dx = \lim_{k\to\infty}\int_{\Omega} u_k v_k D_w^{\alpha}\varphi dx = \lim_{k\to\infty}\int_{\Omega} Q_{\alpha}[u_k, v_k]\varphi dx.$$

Since Q_α is continuous from $W^l_{p_1}(\Omega) \times W^l_{q_1}(\Omega)$ to $L_1(\Omega)$, we have

$$\lim_{k\to\infty}\int_{\Omega} Q_{\alpha}[u_k, v_k]\varphi dx = \int_{\Omega} Q_{\alpha}[u, v]\varphi dx,$$

and therefore

$$\int_{\Omega} uvD_w^{\alpha}\varphi dx = \int_{\Omega} Q_{\alpha}[u, v]\varphi dx.$$

□

Corollary 2.9. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *which satisfies the cone property. Let* $p, q \in [1, +\infty]$. *Let* $l \in \mathbb{N}\backslash\{0\}$. *Let* $\lambda_1 \in [0, n/p]$, $\lambda_2 \in [0, n/q]$,

$$p \geq q, \quad \lambda_1 - \frac{n}{p} \geq \lambda_2 - \frac{n}{q}, \tag{2.13}$$

and

$$l > \frac{n}{p}. \tag{2.14}$$

Then the pointwise product from $W_p^{l,\lambda_1}(\Omega) \times W_q^{l,\lambda_2}(\Omega)$ *to* $W_q^{l,\lambda_2}(\Omega)$ *is bilinear and continuous and the Leibnitz rule (2.6) holds.*

Proof. It suffices to choose $r = q$, $\lambda = \lambda_2$ in Theorem 2.8. We note that for such a specific choice of the exponents, condition (2.9) implies the validity of condition (2.8). □

Then we have the following immediate consequence of the previous corollary.

Corollary 2.10. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *which satisfies the cone property. Let* $p \in [1, +\infty]$. *Let* $l \in \mathbb{N}\backslash\{0\}$. *Let* $\lambda_1 \in [0, n/p]$,

$$l > \frac{n}{p}. \tag{2.15}$$

Then the pointwise product from $W_p^{l,\lambda_1}(\Omega) \times W_p^{l,\lambda_1}(\Omega)$ *to* $W_p^{l,\lambda_1}(\Omega)$ *is bilinear and continuous and the Leibnitz rule (2.6) holds.*

Proof. It suffices to choose $p = q$, $\lambda_2 = \lambda_1$ in the previous corollary. □

We now prove in case $l = 1$ the following stronger result.

Proposition 2.11. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *which satisfies the cone property. Let* $p \in [1, +\infty]$. *Let* $\lambda \in [0, n/p]$,

$$1 + \lambda > \frac{n}{p}. \tag{2.16}$$

Then the pointwise product from $W_p^{1,\lambda}(\Omega) \times W_p^{1,\lambda}(\Omega)$ *to* $W_p^{1,\lambda}(\Omega)$ *is bilinear and continuous and the Leibnitz rule (2.6) holds.*

Proof. We want to prove that if $u, v \in W_p^{1,\lambda}(\Omega)$, then $uv \in W_p^{1,\lambda}(\Omega)$.

To do so, we observe that $\forall\, j \in \{1, \dots, n\}$

$$(uv)_{x_j} = u_{x_j} v + u v_{x_j},$$

$$u_{x_j} \in M_p^\lambda(\Omega), \;\; v_{x_j} \in M_p^\lambda(\Omega).$$

Since $1 + \lambda > \frac{n}{p}$, Theorem 2.3 (iii) implies that $W_p^{1,\lambda}(\Omega)$ is continuously embedded into $L_\infty(\Omega)$. Then by Theorem 1.18 the pointwise product is bilinear and continuous from $M_p^\lambda(\Omega) \times L_\infty(\Omega)$ to $M_p^\lambda(\Omega)$ and from $L_\infty(\Omega) \times M_p^\lambda(\Omega)$ to $M_p^\lambda(\Omega)$. Thus,

$$(u_{x_j} v) \in M_p^\lambda(\Omega), \;\; (u v_{x_j}) \in M_p^\lambda(\Omega),$$

and, therefore, $(uv)_{x_j} \in M_p^\lambda(\Omega)$ for all $j \in \{1, \dots, n\}$. $\square$

Chapter 3

The composition operator in Sobolev Morrey spaces

3.1 Composition operator in Morrey spaces

Lemma 3.1. *Let Ω be an open subset of $\mathbb{R}^n$. Let Ω_1 be a subset of $\mathbb{R}$. Let $y \in \Omega_1$. Let f be a Lipschitz continuous map from Ω_1 to $\mathbb{R}$. Let $g \in \mathcal{M}(\Omega)$ be such that $g(x) \in \Omega_1$ for almost all $x \in \Omega$. Then*

$$|f(g(x))| \leq \mathrm{Lip}(f)|g(x)| + \mathrm{Lip}(f)|y| + |f(y)| \tag{3.1}$$

for almost all $x \in \Omega$.

Proof.

$$\begin{aligned}|f(g(x))| \leq |f(g(x)) - f(y)| + |f(y)| \leq \mathrm{Lip}(f)|g(x) - y| + |f(y)| \leq \\ \leq \mathrm{Lip}(f)|g(x)| + \mathrm{Lip}(f)|y| + |f(y)|\end{aligned}$$

for almost all $x \in \Omega$. □

Lemma 3.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $p \in [1, +\infty]$. Let $\lambda \in \left[0, \frac{n}{p}\right]$. Then $1 \in M_p^\lambda(\Omega)$.*

Proof. If $p = +\infty$, then we have $\lambda = 0$ and accordingly $M_p^\lambda(\Omega) = L_p(\Omega) = L_\infty(\Omega)$ and $1 \in L_\infty(\Omega) = M_p^\lambda(\Omega)$.

Now let $p \in [1, +\infty[$. Then we have

$$w_\lambda(r)\|1\|_{L_p(\mathbb{B}_n(x,r)\cap\Omega)} \leq w_\lambda(r)\left(m_n(\mathbb{B}_n(x,r)\cap\Omega)\right)^{\frac{1}{p}}$$

for all $x \in \Omega$ and $r \in]0, +\infty[$. Hence,

$$w_\lambda(r)\|1\|_{L_p(\mathbb{B}_n(x,r)\cap\Omega)} \leq v_n^{\frac{1}{p}}\, r^{\frac{n}{p}-\lambda} \leq v_n^{\frac{1}{p}}$$

for all $x \in \Omega$ and $r \in]0, 1]$ and

$$w_\lambda(r)\|1\|_{L_p(\mathbb{B}_n(x,r)\cap\Omega)} \leq (m_n(\Omega))^{\frac{1}{p}}$$

for all $x \in \Omega$ and $r \in]1, +\infty[$ and accordingly $1 \in M_p^\lambda(\Omega)$ and

$$\|1\|_{M_p^\lambda(\Omega)} \leq \sup\left\{ v_n^{\frac{1}{p}}, (m_n(\Omega))^{\frac{1}{p}} \right\}.$$

□

Now we consider the case of Morrey spaces and we introduce the following sufficient condition.

Proposition 3.3. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Let Ω_1 be a subset of $\mathbb{R}$. Let $y \in \Omega_1$. Let $p \in [1, +\infty]$. Let $\lambda \in \left[0, \frac{n}{p}\right]$. Let $g \in M_p^\lambda(\Omega)$ be such that $g(x) \in \Omega_1$ for almost all $x \in \Omega$. Let f be a measurable function from Ω_1 to $\mathbb{R}$. Assume that there exists $a, b > 0$ such that*

$$|f(y)| \leq a|y| + b, \quad y \in \Omega_1. \tag{3.2}$$

Then $f \circ g \in M_p^\lambda(\Omega)$ and for any $y \in \Omega_1$

$$\|f \circ g\|_{M_p^\lambda(\Omega)} \leq a\|g\|_{M_p^\lambda(\Omega)} + b\|1\|_{M_p^\lambda(\Omega)}. \tag{3.3}$$

Proof.

$$\|f \circ g\|_{M_p^\lambda(\Omega)} \leq \|a|g| + b\|_{M_p^\lambda(\Omega)} \leq a\|g\|_{M_p^\lambda(\Omega)} + b\|1\|_{M_p^\lambda(\Omega)}.$$

□

Remark 3.4. *If f is a Lipschitz continuous function on Ω_1, then Lemma 3.1 implies that condition (3.2) is satisfied with $a = \mathrm{Lip}(f)$, $b = \mathrm{Lip}(f)|y| + |f(y)|$. Hence, by Proposition 3.3 for any $y \in \Omega_1$*

$$\|f \circ g\|_{M_p^\lambda(\Omega)} \leq \mathrm{Lip}(f)\|g\|_{M_p^\lambda(\Omega)} + \|1\|_{M_p^\lambda(\Omega)}(\mathrm{Lip}(f)|y| + |f(y)|). \tag{3.4}$$

Corollary 3.5. *Let conditions of Proposition 3.3 are satisfied. If also* $0 \in \Omega_1$ *and* f *is a Lipschitz continuous function on* Ω_1, *then*

$$\|f \circ g\|_{M_p^\lambda(\Omega)} \leq \mathrm{Lip}(f)\|g\|_{M_p^\lambda(\Omega)} + |f(0)| \cdot \|1\|_{M_p^\lambda(\Omega)}.$$

Corollary 3.6. *Let conditions of Proposition 3.3 are satisfied. If also* $0 \in \Omega_1$, $f(0) = 0$ *and* f *is a Lipschitz continuous function on* Ω_1, *then*

$$\|f \circ g\|_{M_p^\lambda(\Omega)} \leq \mathrm{Lip}(f)\|g\|_{M_p^\lambda(\Omega)}.$$

Corollary 3.7. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$. *Let* $p \in [1, +\infty]$. *Let* $\lambda \in \left[0, \frac{n}{p}\right]$. *Let* f *be a locally Lipschitz continuous function from* $\mathbb{R}$ *to itself. Then*

$$T_f[M_p^\lambda(\Omega) \cap L_\infty(\Omega)] \subseteq M_p^\lambda(\Omega) \cap L_\infty(\Omega).$$

(Note that in general $M_p^\lambda(\Omega) \nsubseteq L_\infty(\Omega)$*).*

Proof. Let $g \in M_p^\lambda(\Omega) \cap L_\infty(\Omega)$. We set $\Omega_1 = \left[-\|g\|_{L_\infty(\Omega)}, \|g\|_{L_\infty(\Omega)}\right]$. Since Ω_1 is a finite segment, f is Lipschitz continuous on Ω_1. Hence, by Corollary 3.5

$$\|f \circ g\|_{M_p^\lambda(\Omega)} < +\infty.$$

We also have

$$\|f \circ g\|_{L_\infty(\Omega)} \leq \|f\|_{L_\infty(\Omega_1)} < +\infty.$$

So, $f \circ g \in M_p^\lambda(\Omega) \cap L_\infty(\Omega)$. ☐

3.2 Composition operator in Sobolev Morrey spaces

Next we try to understand whether the Lipschitz continuity of a function f of a real variable is enough to ensure that $T_f[W_p^{1,\lambda}(\Omega)] \subseteq W_p^{1,\lambda}(\Omega)$ under suitable conditions on the exponents. To do so, we face the problem of taking the distributional derivatives of the composite function $f \circ g$, and we expect to prove that

$$D_j(f \circ g) = (f' \circ g)D_j g \quad \forall j \in \{1, \dots, n\}.$$

However, it is not clear what $f' \circ g$ should mean. Indeed, f' is defined only up to a set of measure zero N_f and $g^{\leftarrow}(N_f)$ may have a positive measure, and even fill the whole of Ω, and $(f' \circ g)(x)$ makes no sense when $x \in g^{\leftarrow}(N_f)$. Classically, one circumvents such a difficulty by introducing a result of de la Vallée-Poussin which states that both $D_j(f \circ g)$ and $D_j g$ vanish on $g^{\leftarrow}(N_f)$. Accordingly, it suffices to define $(f' \circ g)(x)$ when $x \in \Omega \setminus g^{\leftarrow}(N_f)$, and to replace $(f' \circ g)(x)$ by 0 in $g^{\leftarrow}(N_f)$. We find convenient to introduce a symbol for the function which equals $(f' \circ g)(x)$ when $x \in \Omega \setminus g^{\leftarrow}(N_f)$ and 0 elsewhere. Then we introduce the following.

Definition 3.8. *Let Ω be an open subset of $\mathbb{R}^n$. Let Ω_1 be a measurable subset of $\mathbb{R}$. Let g be a measurable function from Ω to $\mathbb{R}$. Let the set $N_g \equiv \{x \in \Omega\colon\ g(x) \notin \Omega_1\}$ have measure zero.*

Let H be a Borel subset of Ω_1 of measure zero. Let h be a Borel measurable function from $\Omega \setminus H$ to $\mathbb{R}$. Let $h\tilde{\circ}g$ be the function from Ω to $\mathbb{R}$ defined by

$$h\tilde{\circ}g \equiv \begin{cases} 0, & \text{if } x \in g^{\leftarrow}(H) \cup N_g, \\ (h \circ g)(x), & \text{if } x \in \Omega \setminus (g^{\leftarrow}(H) \cup N_g). \end{cases} \tag{3.5}$$

By definition, the function $h\tilde{\circ}g$ is measurable. Next we note that the following holds.

Lemma 3.9. *Let Ω, Ω_1, h, H be as in Definition 3.8. Let g, g_1 be a measurable functions from Ω to $\mathbb{R}$ such that $g(x)$, $g_1(x) \in \Omega_1$ for almost all $x \in \Omega$. If $g(x) = g_1(x)$ for almost all $x \in \Omega$, then $(h\tilde{\circ}g)(x) = (h\tilde{\circ}g_1)(x)$ for almost all $x \in \Omega$.*

Proof. Let N be a measurable subset of measure zero of Ω such that $g(x) = g_1(x)$ and $g(x)$, $g_1(x) \in \Omega_1$ for all $x \in \Omega \setminus N$. Since N has measure zero, it suffices to show that $(h\tilde{\circ}g)(x) = (h\tilde{\circ}g_1)(x)$ for all $x \in \Omega \setminus N$.

If $x \in (\Omega \setminus N) \cap g^{\leftarrow}(H)$, then $g_1(x) = g(x) \in H$ and $x \in (\Omega \setminus N) \cap g^{\leftarrow}(H)$, and accordingly $(h\tilde{\circ}g_1)(x) = 0 = (h\tilde{\circ}g)(x)$. If instead $x \in (\Omega \setminus N) \cap (\Omega \setminus g^{\leftarrow}(H))$, then $g_1(x) = g(x) \notin H$ and accordingly $x \in (\Omega \setminus N) \cap (\Omega \setminus g^{\leftarrow}(H))$ and $(h\tilde{\circ}g_1)(x) = (h \circ g_1)(x) = (h \circ g)(x) = (h\tilde{\circ}g)(x)$. Hence, $(h\tilde{\circ}g_1)(x) = (h\tilde{\circ}g)(x)$ for all $x \in \Omega \setminus N$. □

By the previous Lemma, it makes sense to introduce the following.

Definition 3.10. *Let Ω, Ω_1, h, H be as in Definition 3.8. If G is an equivalence class of measurable functions g from Ω to $\mathbb{R}$ such that $g(x) \in \Omega_1$ for almost all $x \in \Omega$, then we define $h\tilde{\circ}G$ to be the equivalence class of measurable functions from Ω to $\mathbb{R}$ which are equal to $h\tilde{\circ}g$ almost everywhere for at least a $g \in G$.*

If Ω be an open subset of $\mathbb{R}^n$. We say that $g \in L_1^{\mathrm{loc}}(\Omega)$ is zero on a subset A of Ω provided that $\tilde{g}(x) = 0$ for almost all $x \in A$, for at least a representative $\tilde{g}$ of g (and thus for all representatives of g).

Remark 3.11. *Let g_1, $g_2 \in \mathcal{L}_1^{\mathrm{loc}}(\Omega)$. Let $g_1 = g_2$ almost everywhere in Ω. If A is a subset of $\mathbb{R}$, then the symmetric difference $g_1^{\leftarrow}(A) \triangle g_2^{\leftarrow}(A)$ has measure zero. Indeed, $g_1^{\leftarrow}(A) \triangle g_2^{\leftarrow}(A) \subseteq \{x \in \Omega\colon\ g_1(x) \neq g_2(x)\}$.*

Then we have the following n dimensional form of a result of de la Vallée-Poussin [21]. For a proof, we refer to Marcus and Mizel [34].

Theorem 3.12. *Let Ω be an open subset of $\mathbb{R}^n$. Let $g \in W_1^{1,\mathrm{loc}}(\Omega)$ and if N is a subset of $\mathbb{R}$ of measure zero, then $(D_1 g, \dots, D_n g) = 0$ on $\tilde{g}^{\leftarrow}(N)$ for any representative $\tilde{g}$ of g.*

Then we introduce the following form of the chain rule (see Marcus and Mizel [34]).

Proposition 3.13. *Let Ω be an open subset of $\mathbb{R}^n$. Let Ω_1 be an interval of $\mathbb{R}$. Let f be Lipschitz continuous function from Ω_1 to $\mathbb{R}$. Let*

$$W_1^{1,\mathrm{loc}}(\Omega, \Omega_1) \equiv \{g \in W_1^{1,\mathrm{loc}}(\Omega)\colon\ \tilde{g}(x) \in \Omega_1 \text{ for almost all } x \in \Omega \\ \text{for all representatives } \tilde{g} \text{ of } g\}.$$

Let N_f be the subset of Ω_1 such that $\Omega_1 \setminus N_f$ is the set of points of Ω_1 where f is differentiable. Let $g \in W_1^{1,\mathrm{loc}}(\Omega, \Omega_1)$. Let $f'\tilde{\circ}g$ be defined as in Definition 3.10 (with $h = f'$, $H = N_f$). Then the chain rule

$$D_j(f \circ g) = (f'\tilde{\circ}g) D_j g, \tag{3.6}$$

holds in the sense of distributions in Ω for all $j \in \{1, \dots, n\}$.

Then we introduce the following sufficient condition for Sobolev Morrey spaces of order one.

Proposition 3.14. *Let Ω be a bounded open subset of $\mathbb{R}^n$ which satisfies the cone property. Let Ω_1 be an interval of $\mathbb{R}$. Let $p \in [1, +\infty]$. Let $\lambda \in \left[0, \frac{n}{p}\right]$. Let f be Lipschitz continuous function from Ω_1 to $\mathbb{R}$. Let*

$$W_p^{1,\lambda}(\Omega, \Omega_1) \equiv \{g \in W_p^{1,\lambda}(\Omega) \colon\ \tilde{g}(x) \in \Omega_1 \text{ for almost all } x \in \Omega \\ \text{for all representatives } \tilde{g} \text{ of } g\}.$$

Then

$$T_f[W_p^{1,\lambda}(\Omega, \Omega_1)] \subseteq W_p^{1,\lambda}(\Omega).$$

Let N_f be the subset of Ω_1 such that $\Omega_1 \setminus N_f$ is the set of points of Ω_1 where f is differentiable. Let $g \in W_p^{1,\lambda}(\Omega, \Omega_1)$. Let $f'\tilde{\circ}g$ be defined as in Definition 3.10 (with $h = f'$, $H = N_f$). Then $f'\tilde{\circ}g \in L_\infty(\Omega)$ and the chain rule formula (3.6) holds in the sense of distributions in Ω for all $j \in \{1, \dots, n\}$. Moreover,

$$\|f \circ g\|_{W_p^{1,\lambda}(\Omega)} \leq \\ \leq \{(\mathrm{Lip}(f)|y| + |f(y)|) + \mathrm{Lip}(f)\}(\|g\|_{W_p^{1,\lambda}(\Omega)} + \|1\|_{M_p^\lambda(\Omega)}), \quad (3.7)$$

for all $g \in W_p^{1,\lambda}(\Omega, \Omega_1)$ and for all $y \in \Omega_1$.

Proof. By Remark 3.4, we know that inequality (3.4) holds for all $g \in W_p^{1,\lambda}(\Omega, \Omega_1) \subseteq M_p^\lambda(\Omega)$ and for all $y \in \Omega_1$.

Now let $g \in W_p^{1,\lambda}(\Omega, \Omega_1)$. Let $\tilde{g}$ be a representative of g. Let $N_{\tilde{g}}$ be a subset of measure 0 of Ω such that

$$\tilde{g} \in \Omega_1 \quad \forall x \in \Omega \setminus N_{\tilde{g}}.$$

If $x \in \Omega \setminus (N_{\tilde{g}} \cup \tilde{g}^{\leftarrow}(N_f))$, then

$$|f'(\tilde{g}(x))| = \left| \lim_{\eta \to \tilde{g}(x)} \frac{f(\tilde{g}(x)) - f(\eta)}{\tilde{g}(x) - \eta} \right| \leq \mathrm{Lip}(f).$$

Since $f'\tilde{\circ}\tilde{g} = 0$ for all $x \in \tilde{g}^{\leftarrow}(N_f)$, we conclude that

$$|(f'\tilde{\circ}\tilde{g})(x)| \leq \mathrm{Lip}(f) \quad \text{a.e. in } \Omega.$$

and accordingly that $f'\tilde{\circ}g \in L_\infty(\Omega)$ and that $\|f'\tilde{\circ}g\|_{L_\infty(\Omega)} \leq \mathrm{Lip}(f) < +\infty$. Then by the multiplication Theorem 1.18 and by the membership of D_jg in $M_p^\lambda(\Omega)$, we have $(f'\tilde{\circ}g)D_jg \in M_p^\lambda(\Omega)$ and

$$\|(f' \circ g)D_jg\|_{M_p^\lambda(\Omega)} \leq \mathrm{Lip}(f)\|D_jg\|_{M_p^\lambda(\Omega)} \tag{3.8}$$

for all $j \in \{1, \dots, n\}$. Thus the right hand side of equality (3.6) belongs to $M_p^\lambda(\Omega)$ for all $g \in W_p^{1,\lambda}(\Omega)$.

By the formula (3.6) for the chain rule, the inequalities (3.3), (3.8) imply that

$$\|f \circ g\|_{W_p^{1,\lambda}(\Omega)} = \|f \circ g\|_{M_p^\lambda(\Omega)} + \sum_{j=1}^{n} \|(f'\tilde{\circ}g)D_jg\|_{M_p^\lambda(\Omega)} \leq$$

$$\leq \mathrm{Lip}(f)\|g\|_{M_p^\lambda(\Omega)} + \|1\|_{M_p^\lambda(\Omega)}(\mathrm{Lip}(f)|y| + |f(y)|) + \sum_{j=1}^{n} \mathrm{Lip}(f)\|D_jg\|_{M_p^\lambda(\Omega)} \leq$$

$$\leq \{(\mathrm{Lip}(f)|y| + |f(y)|) + \mathrm{Lip}(f)\}(\|g\|_{W_p^{1,\lambda}(\Omega)} + \|1\|_{M_p^\lambda(\Omega)}), \tag{3.9}$$

for all $g \in W_p^{1,\lambda}(\Omega)$ and $y \in \Omega_1$, and thus inequality (3.7) holds true.

□

Corollary 3.15. *Let Ω be a bounded open subset of $\mathbb{R}^n$ which satisfies the cone property. Let $p \in [1, +\infty]$. Let $\lambda \in \left[0, \frac{n}{p}\right]$. Let f be a function from $\mathbb{R}$ to itself. Then the following statements hold.*

(i) If $(1+\lambda) > \frac{n}{p}$ and if f is locally Lipschitz continuous, then $T_f[W_p^{1,\lambda}(\Omega)] \subseteq W_p^{1,\lambda}(\Omega)$.

(ii) If $(1+\lambda) \leq \frac{n}{p}$ and if f is Lipschitz continuous, then $T_f[W_p^{1,\lambda}(\Omega)] \subseteq W_p^{1,\lambda}(\Omega)$.

Proof. We first consider statement (i). The Sobolev Embedding Theorem implies that $W_p^{1,\lambda}(\Omega)$ is continuously embedded into $L_\infty(\Omega)$. Thus if $g \in W_p^{1,\lambda}(\Omega)$, there exists a bounded subset Ω_1 of $\mathbb{R}$ such that $g(x) \in \Omega_1$ for almost all $x \in \Omega$. Since $f\big|_{\Omega_1}$ is Lipschitz continuous, Proposition 3.14 implies that $f \circ g \in W_p^{1,\lambda}(\Omega)$.

Statement (ii) is an immediate consequence of Proposition 3.14 with $\Omega_1 = \mathbb{R}$.

□

We summarize in the following statement some facts we need in the sequel in case $(1+\lambda) > \frac{n}{p}$ and which are immediate consequence of Proposition 2.11 and Proposition 3.14.

Corollary 3.16. *Let* $p \in [1, +\infty]$, $\lambda \in \left[0, \frac{n}{p}\right]$, $(1+\lambda) > \frac{n}{p}$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *which satisfies the cone property. Let* Ω_1 *be a bounded open interval of* $\mathbb{R}$. *Then the following statements hold.*

(i) $W_p^{1,\lambda}(\Omega)$ *is a Banach algebra.*

(ii) If $(f, g) \in C^{0,1}(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$, *then* $f \circ g \in W_p^{1,\lambda}(\Omega)$. *Moreover, there exists an increasing function* ψ *from* $[0, +\infty[$ *to itself such that*

$$\|f \circ g\|_{W_p^{1,\lambda}(\Omega)} \leq \|f\|_{C^{0,1}(\bar{\Omega}_1)} \psi(\|g\|_{W_p^{1,\lambda}(\Omega)}) \tag{3.10}$$

for all $(f, g) \in C^{0,1}(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$.

3.3 Continuity of the composition operator in Sobolev Morrey spaces

Corollary 3.16 shows that if $(1+\lambda) > \frac{n}{p}$ the composition T maps $C^{0,1}(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$ to $W_p^{1,\lambda}(\Omega)$. Now we want to understand for which f's the composition operator T_f is continuous in $W_p^{1,\lambda}(\Omega, \Omega_1)$. By following [31],[30], the idea is that if f is a polynomial, then T_f is continuous in $W_p^{1,\lambda}(\Omega)$. Indeed, for $(1+\lambda) > \frac{n}{p}$, the space $W_p^{1,\lambda}(\Omega)$ is a Banach algebra. Then we exploit inequality (3.10) to show that if f is a limit of polynomials, then T_f is continuous. Actually, such a scheme can be applied in a somewhat abstract general setting, which we now introduce. Let $\mathcal{X}$ be a Banach algebra with unity. Let $m \in \mathbb{N} \setminus \{0\}$. In the applications of the present notes we are interested in the specific case $m = 1$, but here we present a more general case, which can be applied to analyze vector valued functions of Sobolev Morrey.

We first note that if p belongs to the space $\mathcal{P}(\mathbb{R}^m)$ of polynomials in m real variables with real coefficients, then it makes perfectly sense to compose

p with some $x \equiv (x_1, \ldots, x_m) \in \mathcal{X}^m$. Namely, if p is defined by the equality

$$p(\xi_1, \ldots, \xi_m) \equiv \sum_{|\eta| \leq deg\, p,\ \eta \in \mathbb{N}^m} a_\eta \xi_1^{\eta_1} \ldots \xi_m^{\eta_m}, \quad \text{with } a_\eta \in \mathbb{R},\ (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m, \tag{3.11}$$

then we set

$$\tau_p[x] \equiv \sum_{|\eta| \leq deg\, p,\ \eta \in \mathbb{N}^m} a_\eta x_1^{\eta_1} \ldots x_m^{\eta_m}, \quad \forall\, x \equiv (x_1, \ldots, x_m) \in \mathcal{X}^m, \tag{3.12}$$

where the product between the x_j's is that of $\mathcal{X}$, and where we understand that x^0 is the unit element of $\mathcal{X}$, for all $x \in \mathcal{X}$. Next we state the following result of [30, Thm. 3.1].

Theorem 3.17. *Let $\| \cdot \|_{\mathcal{Y}}$ be a norm on $\mathcal{P}(\mathbb{R}^m)$. Let $\mathcal{Y}$ be the completion of $\mathcal{P}(\mathbb{R}^m)$ with respect to the norm $\| \cdot \|_{\mathcal{Y}}$. Let $\mathcal{X}$ be a real commutative Banach algebra with unity. Let $\tilde{\mathcal{X}}$ be a real Banach space. Assume that there exists a linear continuous and injective map $\mathcal{J}$ of $\mathcal{X}$ into $\tilde{\mathcal{X}}$. Let $\mathcal{A}$ be a subset of $\mathcal{X}^m$. Assume that there exists an increasing function ψ of $[0, +\infty)$ to itself such that*

$$\|\mathcal{J}[p(x_1, \ldots, x_m)]\|_{\tilde{\mathcal{X}}} \leq \|p\|_{\mathcal{Y}}\, \psi\left(\|(x_1, \ldots, x_m)\|_{\mathcal{X}^m}\right), \tag{3.13}$$

for all $(p, (x_1, \ldots, x_m)) \in \mathcal{P}(\mathbb{R}^m) \times \mathcal{A}$. Then there exists a unique map $\tilde{A}$ of $\mathcal{Y} \times \mathcal{A}$ to $\tilde{\mathcal{X}}$ such that the following two conditions hold

(i) $\tilde{A}[p, x] = \mathcal{J}[p(x)]$, *for all* $(p, x) \in \mathcal{P}(\mathbb{R}^m) \times \mathcal{A}$.

(ii) For all fixed $x \equiv (x_1, \ldots, x_m) \in \mathcal{A}$, the map $y \longmapsto \tilde{A}[y, x]$ is continuous from $\mathcal{Y}$ to $\tilde{\mathcal{X}}$.

Furthermore, the map $\tilde{A}[\cdot, x]$ of (ii) is linear, and $\tilde{A}$ is continuous from $\mathcal{Y} \times \mathcal{A}$ to $\tilde{\mathcal{X}}$, and if $y \in \mathcal{Y}$, $y = \lim_{j \to \infty} p_j$ in $\mathcal{Y}$, $p_j \in \mathcal{P}(\mathbb{R}^m)$, $x \equiv (x_1, \ldots, x_m) \in \mathcal{A}$, then

(iii) $\tilde{A}[y, x] = \lim_{j \to \infty} \mathcal{J}[p_j(x)]$ *in* $\tilde{\mathcal{X}}$;

(iv) $\|\tilde{A}[y, x]\|_{\tilde{\mathcal{X}}} \leq \|y\|_{\mathcal{Y}}\, \psi\left(\|x\|_{\mathcal{X}^m}\right)$.

We shall call $\tilde{A}[y, x]$ the 'abstract' composition of y and x.

We now turn to apply the above theorem to the case of Sobolev Morrey spaces. To do so, we need the following.

Proposition 3.18. *Let* Ω_1 *be a nonempty bounded open interval of* $\mathbb{R}$. *Then* $C^1(\bar{\Omega}_1)$ *is a completion of the space* $(\mathcal{P}(\mathbb{R}), \|\cdot\|_{C^{0,1}(\bar{\Omega}_1)})$.

Proof. We first note that

$$\sup_{\bar{\Omega}_1} |f'| = \mathrm{Lip}(f) \quad \forall\, f \in C^1(\bar{\Omega}_1).$$

Then we have

$$\|f\|_{C^1(\bar{\Omega}_1)} = \|f\|_{C^0(\bar{\Omega}_1)} + \|f'\|_{C^0(\bar{\Omega}_1)} = \|f\|_{C^0(\bar{\Omega}_1)} + \mathrm{Lip}(f) = \|f\|_{C^{0,1}(\bar{\Omega}_1)}$$

for all $f \in C^1(\bar{\Omega}_1)$. Hence,

$$\|f\|_{C^{0,1}(\bar{\Omega}_1)} = \|f\|_{C^1(\bar{\Omega}_1)} \quad \forall\, p \in \mathcal{P}(\mathbb{R}).$$

Since $C^1(\bar{\Omega}_1)$ is a Banach space and the restriction map from $\mathcal{P}(\mathbb{R})$ to $C^1(\bar{\Omega}_1)$ which takes $p \in \mathcal{P}(\mathbb{R})$ which takes p to $p\big|_{\bar{\Omega}_1}$ is a linear isometry from $(\mathcal{P}(\mathbb{R}), \|\cdot\|_{C^1(\bar{\Omega}_1)})$ to $\left(\{p\big|_{\bar{\Omega}_1} : \ p \in \mathcal{P}(\mathbb{R})\}, \|\cdot\|_{C^1(\bar{\Omega}_1)}\right)$ it suffices to show that $\{p\big|_{\bar{\Omega}_1} : \ p \in \mathcal{P}(\mathbb{R})\}$ is dense in $C^1(\bar{\Omega}_1)$. Let $f \in C^1(\bar{\Omega}_1)$. Let $x_0 \in \bar{\Omega}_1$. Then

$$f(x) = f(x_0) + \int\limits_{x_0}^{x} f'(t)dt \quad \forall\, x \in \bar{\Omega}_1.$$

Now by the Weierstrass approximation Theorem, there exists a sequence $\{q_j\}_{j \in \mathbb{N}}$ in $\mathcal{P}(\mathbb{R})$ such that

$$\lim_{j \to \mathbb{N}} q_j = f' \quad \text{uniformly in } \bar{\Omega}_1.$$

Then if we set

$$p_j(x) \equiv f(x_0) + \int\limits_{x_0}^{x} q_j(t)dt \quad \forall\, x \in \mathbb{R},$$

for all $j \in \mathbb{N}$, we have

$$\|f - p_j\|_{C^1(\bar{\Omega}_1)} \leq \sup_{x \in \bar{\Omega}_1} \left| \int\limits_{x_0}^{x} (f' - q_j)dt \right| + \|f' - q_j\|_{C^0(\bar{\Omega}_1)} \leq$$

$$\leq m_1(\Omega_1)\|f' - q_j\|_{C^0(\bar{\Omega}_1)} + \|f' - q_j\|_{C^0(\bar{\Omega}_1)} \leq$$

$$\leq (1 + m_1(\Omega_1))\|f' - q_j\|_{C^0(\bar{\Omega}_1)} \quad \forall\, j \in \mathbb{N},$$

and accordingly $\lim\limits_{j \to \infty} \|f - p_j\|_{C^1(\bar{\Omega}_1)} = 0$. □

Then by applying Theorem 3.17, we obtain the following.

Theorem 3.19. *Let* $p \in [1, +\infty]$, $\lambda \in \left[0, \frac{n}{p}\right]$, $(1+\lambda) > \frac{n}{p}$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *which satisfies the cone property. Let* Ω_1 *be a bounded open interval of* $\mathbb{R}$. *Then the composition operator* T *is continuous from* $C^1(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$ *to* $W_p^{1,\lambda}(\Omega)$.

Proof. We set $\|\cdot\|_{\mathcal{Y}} = \|\cdot\|_{C^{0,1}(\bar{\Omega}_1)}$, $\mathcal{X} = \tilde{\mathcal{X}} = W_p^{1,\lambda}(\Omega)$, $\mathcal{A} = W_p^{l,\lambda}(\Omega, \Omega_1)$, $\mathcal{J}$ equal to the identity map, $m = 1$. As we have shown, $C^1(\bar{\Omega}_1)$ is a completion of $(\mathcal{P}(\mathbb{R}), \|\cdot\|_{C^{0,1}(\bar{\Omega}_1)})$. By Corollary 3.16, $W_p^{1,\lambda}(\Omega)$ is a Banach algebra and there exists a function ψ as in (3.10). Then by Theorem 3.17, there exists a unique map $\tilde{A}$ from $C^1(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$ to $W_p^{1,\lambda}(\Omega)$ such that the following two conditions hold

(i) $\tilde{A}[p, g] = \tau_p[g]$ for all $(p, g) \in \mathcal{P}(\mathbb{R}) \times \mathcal{A}$.

(ii) For each fixed $g \in W_p^{1,\lambda}(\Omega, \Omega_1)$, the map from $C^1(\bar{\Omega}_1)$ to $W_p^{1,\lambda}(\Omega)$ which takes f to $f \circ g$ is continuous.

Moreover, $\tilde{A}$ is continuous. Clearly, T satisfies (i), and inequality (3.10) implies that T satisfies (ii). Hence, we must necessarily have

$$\tilde{A}[f, g] = T[f, g] \quad \forall (f, g) \in C^1(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1).$$

As a consequence, T is continuous on $C^1(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$. □

3.4 Lipschitz continuity of the composition operator in Sobolev Morrey spaces

Next we prove a Lipschitz continuity statement for the composition operator. For related results in Besov spaces, we refer to Bourdaud and Lanza de Cristoforis [9].

Theorem 3.20. *Let* $p \in [1, +\infty]$, $\lambda \in \left[0, \frac{n}{p}\right]$, $(1+\lambda) > \frac{n}{p}$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *which satisfies the cone property. If* $f \in C^{1,1}_{\mathrm{loc}}(\mathbb{R})$, *then* T_f *maps* $W_p^{1,\lambda}(\Omega)$ *to itself and Lipschitz continuous on the bounded subsets of* $W_p^{1,\lambda}(\Omega)$.

Proof. Let $\mathcal{B}$ be a bounded subset of $W^{1,\lambda}_p(\Omega)$. Since $W^{1,\lambda}_p(\Omega)$ is continuously embedded into $L_\infty(\Omega)$, the set $\mathcal{B}$ is a bounded subset of $L_\infty(\Omega)$ and there exists a closed interval B of $\mathbb{R}$ such that

$$\left[-\|g\|_{L_\infty(\Omega)}, \|g\|_{L_\infty(\Omega)}\right] \subseteq B \quad \forall\, g \in \mathcal{B}.$$

Now let g_1, $g_2 \in \mathcal{B}$. Since f is continuously differentiable, we can write

$$(f \circ g_2)(x) - (f \circ g_1)(x) = \\ = \int_0^1 f'[g_1(x) + t(g_2(x) - g_1(x))](g_2(x) - g_1(x))dt \quad \forall\, x \in \Omega.$$

Next we fix $x \in \Omega$, $r \in]0, +\infty[$. By the Minkowski inequality for integrals, we have

$$w_\lambda(r)\|f \circ g_2 - f \circ g_1\|_{L_p(\Omega\cap\mathbb{B}_n(x,r))} \leq$$

$$\leq \int_0^1 w_\lambda(r)\|f'[g_1(\cdot) + t(g_2(\cdot) - g_1(\cdot))](g_2(\cdot) - g_1(\cdot))\|_{L_p(\Omega\cap\mathbb{B}_n(x,r))} \leq$$

$$\leq \int_0^1 w_\lambda(r) \sup_B |f'|\, \|g_2 - g_1\|_{L_p(\Omega\cap\mathbb{B}_n(x,r))} \leq$$

$$\leq \sup_B |f'|\, \|g_2 - g_1\|_{M^\lambda_p(\Omega)}.$$

Hence,

$$\|f \circ g_2 - f \circ g_1\|_{M^\lambda_p(\Omega)} \leq \sup_B |f'|\, \|g_2 - g_1\|_{M^\lambda_p(\Omega)}. \tag{3.14}$$

Next we fix $j \in \{1, \dots, n\}$ and we try to estimate

$$\|(D_j)_w\, \{f \circ g_2 - f \circ g_1\}\|_{M^\lambda_p(\Omega)} = \tag{3.15}$$

$$= \|f'(g_2)(D_j)_w\, g_2 - f'(g_1)(D_j)_w\, g_1\}\|_{M^\lambda_p(\Omega)} \leq$$

$$\leq \|f' \circ g_2 - f' \circ g_1\|_{L_\infty(\Omega)}\, \|(D_j)_w\, g_2\|_{M^\lambda_p(\Omega)} +$$

$$+\|f' \circ g_1\|_{L_\infty(\Omega)}\, \|(D_j)_w\, g_2 - (D_j)_w\, g_1\|_{M^\lambda_p(\Omega)} \leq$$

$$\leq \mathrm{Lip}(f'|_B)\|g_2 - g_1\|_{L_\infty(\Omega)} \sup_{g\in\mathcal{B}} \|g\|_{W^{1,\lambda}_p(\Omega)} + \sup_B |f'|\, \|g_2 - g_1\|_{W^{1,\lambda}_p(\Omega)} \leq$$

$$\leq \left\{\mathrm{Lip}(f'|_B)\|I\|_{\mathcal{L}(W^{1,\lambda}_p(\Omega), L_\infty(\Omega))} \sup_{g\in\mathcal{B}} \|g\|_{W^{1,\lambda}_p(\Omega)} + \sup_B |f'|\right\} \|g_2 - g_1\|_{W^{1,\lambda}_p(\Omega)}.$$

By inequalities (3.14) and (3.15), we conclude that

$$\begin{aligned}\|f\circ g_2 - f\circ g_1\|_{W_p^{1,\lambda}(\Omega)} \leq \Big\{ &(1+n)\sup_B |f'| + \\ &+ n\mathrm{Lip}(f'|_B)\|I\|_{\mathcal{L}(W_p^{1,\lambda}(\Omega),L_\infty(\Omega))}\sup_{g\in\mathcal{B}}\|g\|_{W_p^{1,\lambda}(\Omega)}\Big\}\|g_2-g_1\|_{W_p^{1,\lambda}(\Omega)}.\end{aligned}$$

□

3.5 Differentiability properties of the composition operator in Sobolev Morrey spaces

Next we turn to the question of differentiability, and by following [30], we note that the following holds.

Lemma 3.21. *Let $\mathcal{X}$ be a commutative real Banach algebra with unity $1_{\mathcal{X}}$. Let $\mathcal{P}(\mathbb{R}^m)$ be the set of real polynomials in m real variables. Let $p \in \mathcal{P}(\mathbb{R}^m)$ be defined by*

$$p(\eta) \equiv \sum_{|\gamma|\leq deg\, p} a_\gamma x_1^{\gamma_1}\dots x_m^{\gamma_m}, \quad \forall\, \eta \equiv (\eta_1,\dots,\eta_m)\in\mathbb{R}^m.$$

The map τ_p of $\mathcal{X}^m$ to $\mathcal{X}$ defined by setting

$$\tau_p[x_1,\dots,x_m] \equiv \sum_{|\gamma|\leq deg\, p} a_\gamma x_1^{\eta_1}\dots x_m^{\eta_m}, \quad \forall\,(x_1,\dots,x_m)\in\mathcal{X}^m,$$

with the understanding that $x^0 \equiv 1_{\mathcal{X}}$, for all $x \in \mathcal{X}$, is of class $C^r(\mathcal{X}^m,\mathcal{X})$, for all $r\in\mathbb{N}\cup\{\infty\}$. Furthermore, the differential of $\tau_p[\cdot]$ at $x^\sharp \equiv (x_1^\sharp,\dots,x_m^\sharp)$ is delivered by the map

$$\mathcal{X}^m \ni (h_1,\dots,h_m) \mapsto \sum_{i=1}^m \tau_{\frac{\partial p}{\partial x_i}}[x^\sharp] * h_i \in \mathcal{X}.$$

Once more, we plan to proceed by approximation and show that T_f is of class C^r if f is a limit of polynomials with an appropriate norm. As we shall see, it turns out that a right choice for the norm is the following

$$\|p\|_{\mathcal{Y}_r} = \sum_{|\gamma|\leq r,\,\gamma\in\mathbb{N}^m}\|D^\gamma p\|_{\mathcal{Y}}, \;\; \forall p\in\mathcal{P}(\mathbb{R}^m). \tag{3.16}$$

Then we define $\mathcal{Y}_r$ to be the completion of the space $(\mathcal{P}(\mathbb{R}^m), \|\cdot\|_{\mathcal{Y}_r})$. As is well known, $\mathcal{Y}_r$ is unique up to a linear isometry, and we always choose $\mathcal{Y}_r \subseteq \mathcal{Y}$. Then we have the following obvious

Remark 3.22. *If* $r, s \in \mathbb{N}$, $s \leq r$, *then*

$$\mathcal{Y}_r \subseteq \mathcal{Y}_s, \quad \|y\|_{\mathcal{Y}_s} \leq \|y\|_{\mathcal{Y}_r}, \quad \forall\, y \in \mathcal{Y}_r.$$

Now we have the following (cf. Lanza de Cristoforis [30, Thm. 2.4]).

Theorem 3.23. *Let* $r, s \in \mathbb{N}$, $\gamma \in \mathbb{N}^m$, $r - |\gamma| = s$. *Let* $\| \cdot \|_{\mathcal{Y}}$ *be a norm on* $\mathcal{P}(\mathbb{R}^m)$, *and let* $\| \cdot \|_{\mathcal{Y}_r}$ *be the norm defined in (3.16), and let* $\mathcal{Y}_r$ *be the completion of* $(\mathcal{P}(\mathbb{R}^m), \| \cdot \|_{\mathcal{Y}_r})$. *Then there exists one and only one linear and continuous operator of* $\mathcal{Y}_r$ *to* $\mathcal{Y}_s$ *which coincides with the ordinary partial derivation of multi index* γ *on the elements of* $\mathcal{P}(\mathbb{R}^m)$. *By abuse of notation, we shall denote such operator by* D^γ, *just as the usual partial derivative of multi index* γ. *We have*

$$D^\gamma y = \lim_{j \to \infty} D^\gamma p_j \quad \text{in } \mathcal{Y}_s, \text{ whenever } \lim_{j \to \infty} p_j = y \text{ in } \mathcal{Y}_r, \tag{3.17}$$

and

$$\|y\|_{\mathcal{Y}_r} = \sum_{|\gamma| \leq r,\, \gamma \in \mathbb{N}^m} \|D^\gamma p\|_{\mathcal{Y}}, \ \ \forall\, y \in \mathcal{Y}_r.$$

With analogy with the usual derivations, Dy *denotes the matrix* $(D_1 y, \dots, D_m y)$.

Then we state the following result of Lanza de Cristoforis [30, Thm. 4.1].

Theorem 3.24. *Let* $r \in \mathbb{N} \backslash \{0\}$. *Let* $\| \cdot \|_{\mathcal{Y}}$ *be a norm on* $\mathcal{P}(\mathbb{R}^m)$. *Let* $\mathcal{Y}_r$ *be the completion of* $\mathcal{P}(\mathbb{R}^m)$ *with respect to the norm* $\| \cdot \|_{\mathcal{Y}_r}$ *defined in (3.16). Let* $\mathcal{X}$ *be a real commutative Banach algebra with unity. Let* $\tilde{\mathcal{X}}$ *be a real Banach space. Assume that there exists a linear continuous and injective map* $\mathcal{J}$ *of* $\mathcal{X}$ *into* $\tilde{\mathcal{X}}$. *Let* $(\cdot) * (\cdot)$ *be a continuous and bilinear map of* $\tilde{\mathcal{X}} \times \mathcal{X}$ *to* $\tilde{\mathcal{X}}$. *Let '*∗*' satisfy the following condition:*

$$\mathcal{J}[x_1] * x_2 = \mathcal{J}[x_1, x_2], \ \ \forall x_1, x_2 \in \mathcal{X}. \tag{3.18}$$

Let $\mathcal{A}$ *be an open subset of* $\mathcal{X}^m$. *Assume that there exists an increasing function* ψ *of* $[0, +\infty)$ *to itself satisfying condition (3.13), for all* $(p, x) \in \mathcal{P}(\mathbb{R}^m) \times \mathcal{A}$. *Then the restriction of the map* $\tilde{A}$ *of Theorem 3.17 to* $\mathcal{Y}_r \times \mathcal{A}$ *is of class* C^r *from* $\mathcal{Y}_r \times \mathcal{A}$ *to* $\tilde{\mathcal{X}}$. *(Note that* $\mathcal{Y}_r \subseteq \mathcal{Y}_0$, *and that* $\mathcal{Y}_0$ *equals the space* $\mathcal{Y}$ *defined* *in Theorem 3.17.) Furthermore, the differential of* $\tilde{A}$ *at each* $(y^{\#}, x^{\#}) \in \mathcal{Y}_r \times \mathcal{A}$ *is given by*

$$(u, v) \longmapsto \tilde{A}[v, x^{\#}] + \sum_{l=1}^{m} \tilde{A}[D_l y^{\#}, x^{\#}] * w_l,$$

for all $(u, v) \equiv (u, (w_1, \dots, w_m)) \in \mathcal{Y}_r \times \mathcal{X}^m$*. (For the definition of* $D_l y^{\#}$*, see Theorem 3.23)*

Now that we have introduced the above result on the r times differentiability of $\tilde{A}$, we introduce a formula for the differentials $d^s\tilde{A}$ of order $s = 1, \dots, r$ of $\tilde{A}$ of [30, p. 932]

Proposition 3.25. *Let all the assumptions of Theorem 3.24 hold. Let* $r, s \in \mathbb{N}$*,* $1 \leq s \leq r$*. The differential of order* s *of* $\tilde{A}$ *at* $(y^{\#}, x^{\#}) \in \mathcal{Y}_r \times \mathcal{A}$*, which can be identified with an element of* $\mathcal{L}^{(s)}(\mathcal{Y}_r \times \mathcal{X}^m, \tilde{\mathcal{X}})$*, is delivered by the formula*

$$\begin{aligned} & d^s\tilde{A}[y^{\#}, x^{\#}]((v_{[1]}, w_{[1]}), \dots (v_{[s]}, w_{[s]})) = \\ = & \sum_{j=1}^{s} \sum_{l_1, \dots, \hat{l}_j, \dots, l_s = 1}^{m} \left\{ \tilde{A}[D_{l_s} \cdots \widehat{D_{l_j}} \cdots D_{l_1} v_{[j]}, x^{\#}] \right\} * \left(w_{s,l_s} \cdots \widehat{w_{j,l_j}} \cdots w_{1,l_1} \right) + \\ & + \sum_{l_1, \dots, l_s = 1}^{m} \left\{ \tilde{A}[D_{l_s} \cdots D_{l_1} y^{\#}, x^{\#}] \right\} * (w_{s,l_s} \cdots w_{1,l_1}), \end{aligned} \tag{3.19}$$

for all $(v_{[j]}, w_{[j]} \equiv (w_{j,1}, \dots, w_{j,n})) \in \mathcal{Y}_r \times \mathcal{X}^m$*,* $j = 1, \dots, s$*. In (3.19), the symbols* $l_1, \dots, l_s$ *denote summation indexes ranging from 1 to m.*

Next we return to the applications to Sobolev Morrey spaces, and we prove the following.

Proposition 3.26. *Let* $r \in \mathbb{N} \backslash \{0\}$*. Let* Ω_1 *be a nonempty bounded interval of* $\mathbb{R}$*. Then* $C^{r+1}(\overline{\Omega}_1)$ *is a completion of the space* $(\mathcal{P}(\mathbb{R}, \| \cdot \|_{\mathcal{Y}_r})$*, where*

$$\|p\|_{\mathcal{Y}_r} \equiv \sum_{l=0}^{r} \left\| \frac{d^l}{dt^l} p \right\|_{C^{0,1}(\overline{\Omega}_1)} \qquad \forall\, p \in \mathcal{P}(\mathbb{R}).$$

If $f \in C^{r+1}(\overline{\Omega}_1)$ *and if* $\{p_j\}_{j \in \mathbb{N}}$ *is a sequence of* $\mathcal{P}(\mathbb{R})$ *which converges to* f *in the* $\| \cdot \|_{\mathcal{Y}_r}$*-norm and if* $l \in \{0, \dots, r\}$*, then*

$$\frac{d^l}{dt^l} f = \lim_{j \to \infty} \frac{d^l}{dt^l} p_j, \tag{3.20}$$

in $C^{r-l+1}(\overline{\Omega}_1)$*.*

Proof. As we have already pointed out

$$\|f\|_{C^{0,1}(\overline{\Omega}_1)} = \|f\|_{C^1(\overline{\Omega}_1)} \quad \forall\, f \in C^1(\overline{\Omega}_1).$$

Hence,

$$\|p\|_{\mathcal{Y}_r} = \sum_{l=0}^{r} \left(\left\| \frac{d^l}{dt^l} p \right\|_{C^0(\overline{\Omega}_1)} + \left\| \frac{d^{l+1}}{dt^{l+1}} p \right\|_{C^0(\overline{\Omega}_1)} \right) \quad \forall\, p \in \mathcal{P}(\mathbb{R}),$$

and

$$\|p\|_{C^{r+1}(\overline{\Omega}_1)} \leq \|p\|_{\mathcal{Y}_r} \leq 2\|p\|_{C^{r+1}(\overline{\Omega}_1)} \quad \forall\, p \in \mathcal{P}(\mathbb{R}).$$

Hence, the norm $\|\cdot\|_{\mathcal{Y}_r}$ is equivalent to the norm $\|\cdot\|_{C^{r+1}(\overline{\Omega}_1)}$ on $\mathcal{P}(\mathbb{R})$. Since $C^{r+1}(\overline{\Omega}_1)$ is a Banach space and the restriction map which takes p in $\mathcal{P}(\mathbb{R})$ to $p\big|_{\overline{\Omega}_1}$ in $C^{r+1}(\overline{\Omega}_1)$ is linear isometry of $(\mathcal{P}(\mathbb{R}), \|\cdot\|_{C^{r+1}(\overline{\Omega}_1)})$ onto $\left(\{p\big|_{\overline{\Omega}_1} : \ p \in \mathcal{P}(\mathbb{R})\}, \|\cdot\|_{C^{r+1}(\overline{\Omega}_1)}\right)$, it suffices to show that for each $f \in C^{r+1}(\overline{\Omega}_1)$, there exists a sequence of polynomials $\{p_j\}_{j\in\mathbb{N}}$ in $\mathcal{P}(\mathbb{R})$ such that

$$f = \lim_{j\to\infty} p_j\big|_{\overline{\Omega}_1} \quad \text{in } C^{r+1}(\overline{\Omega}_1),$$

i.e., $\{p\big|_{\overline{\Omega}_1} : \ p \in \mathcal{P}(\mathbb{R})\}$ is dense in $C^{r+1}(\overline{\Omega}_1)$. We already know that such a statement is true for $r = 0$. We now assume that the statement holds for r and we prove it for $r + 1$.

By inductive assumption, there exists a sequence of polynomials $\{q_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} q_j\big|_{\overline{\Omega}_1} = f' \quad \text{in } C^r(\overline{\Omega}_1).$$

Now let $x_0 \in \overline{\Omega}_1$. Then

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)dt \quad \forall\, x \in \overline{\Omega}_1.$$

Then we set

$$p_j \equiv f(x_0) + \int_{x_0}^{x} q_j(t)dt \quad \forall\, x \in \overline{\Omega}_1.$$

Clearly, $p_j \in \mathcal{P}(\mathbb{R})$ for all $j \in \mathbb{N}$. Since $\lim_{j\to\infty} q_j\big|_{\overline{\Omega}_1} = f'$ uniformly in $\overline{\Omega}_1$, the inequality

$$|f(x) - p_j(x)| \leq m_1(\overline{\Omega}_1)\|f' - q_j\|_{C^0(\overline{\Omega}_1)} \leq$$
$$\leq m_1(\overline{\Omega}_1)\|f' - q_j\|_{C^r(\overline{\Omega}_1)} \quad \forall\, x \in \overline{\Omega}_1,$$

shows that $\lim_{j\to\infty} \|f' - p_j\|_{C^0(\bar{\Omega}_1)} = 0$. Hence,

$$\lim_{j\to\infty} \|f' - p_j\|_{C^0(\bar{\Omega}_1)} + \sum_{l=0}^{r} \left\| \frac{d^l}{dt^l} f' - \frac{d^l}{dt^l} q_j \right\|_{C^0(\bar{\Omega}_1)} = 0$$

and accordingly

$$\lim_{j\to\infty} \|f' - p_j\|_{C^{r+1}(\bar{\Omega}_1)} = 0$$

Equality (3.20) is a well-known corollary of the theorem on passing to the limit under the differentiation sign. □

Remark 3.27. *Under the assumptions of the previous theorem, the operator D^γ defined by (3.17) coincides with the ordinary D^γ-differentiation in $C^{r+1}(\bar{\Omega}_1)$.*

Theorem 3.28. *Let $p \in [1, +\infty]$, $\lambda \in \left[0, \frac{n}{p}\right]$, $(1+\lambda) > \frac{n}{p}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ which satisfies the cone property. Let Ω_1 be a bounded open interval of $\mathbb{R}$. Then the composition operator T from $C^{r+1}(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$ to $W_p^{1,\lambda}(\Omega)$ defined by*

$$T[f, g] \equiv f \circ g \quad \forall\, (f, g) \in C^{r+1}(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$$

is of class C^r. If $(f_0, g_0) \in C^{r+1}(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$, then the first order differential of T at (f_0, g_0) is given by the formula

$$dT[f_0, g_0](v, w) = v \circ g_0 + f'(g_0)w$$

for all $(v, w) \in C^{r+1}(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$.

If $s \in \{1, \dots, r\}$, then the s-th order differential of T at (f_0, g_0) is given by the formula

$$d^sT[f_0, g_0][(v_{[1]}, w_{[1]}), \dots, (v_{[s]}, w_{[s]})] = \\ = \sum_{j=1}^{s} \frac{d^{s-1} v_{[j]}}{dt^{s-1}} \circ g_0 w_{[1]} \dots \widehat{w_{[j]}} \dots w_{[s]} + \frac{d^s f_0}{dt^s} \circ g_0 w_{[1]} \dots w_{[s]},$$

for all $(v_{[1]}, w_{[1]}), \dots, (v_{[s]}, w_{[s]}) \in C^{r+1}(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$.

Proof. We set $\|\cdot\|_{\mathcal{Y}_r} = \|\cdot\|_{C^{0,1}(\bar{\Omega}_1)}$, $\mathcal{X} = \tilde{\mathcal{X}} = W_p^{1,\lambda}(\Omega)$, $\mathcal{A} = W_p^{l,\lambda}(\Omega, \Omega_1)$, $\mathcal{J}$ equal to the identity map, $m = 1$. As we have shown, $C^{r+1}(\bar{\Omega}_1)$ is a completion of $(\mathcal{P}(\mathbb{R}), \|\cdot\|_{\mathcal{Y}_r})$. By Corollary 3.16, $W_p^{1,\lambda}(\Omega)$ is a Banach algebra and there exists a function ψ as in (3.10). As we have already proved in the proof of Theorem 3.19, the abstract composition $\tilde{A}$ of Theorem 3.17 coincides with T. Then we can invoke Theorem 3.24 and Proposition 3.25 conclude that T is of class C^r from $\mathcal{Y}_r \times \mathcal{A} = C^{r+1}(\bar{\Omega}_1) \times W_p^{1,\lambda}(\Omega, \Omega_1)$ to $W_p^{1,\lambda}(\Omega)$ and that the formulas for the differentials hold. $\square$

Bibliography

[1] D.R. Adams, *A note on Riesz potentials*, Duke Math. **42** (1975), 765–778.

[2] D.R. Adams, *On the Existence of Capacitary Strong Type Estimates in* $\mathbb{R}^n$, Ark. for Mat. **14** (1976), 125–140.

[3] R.A. Adams, *Sobolev spaces*, New York: Academic press, 1975.

[4] J. Appell and P.P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge Tracts in Mathematics 95, Cambridge University Press, 1990.

[5] R. Böhme and F. Tomi, *Zur Struktur der lösungsmenge des Plateauproblems*, Math. Z. **133** (1973), 1-29.

[6] G. Bourdaud, *Le calcul fonctionnel dans les espaces de Sobolev*, Invent. Math. **104** (1991), 435–446.

[7] G. Bourdaud, *The Functional Calculus in Sobolev Spaces*, Function Spaces, differential operators and nonlinear analysis. Teubner Texte in Mathematics **133** (1993), 127–142.

[8] G. Bourdaud and M.E.D. Kateb, *Calcul fonctionnel dans les espaces de Sobolev fractionnaires*, Math. Z. **210** (1992), 607–613.

[9] G. Bourdaud and M. Lanza de Cristoforis, *Regularity of the symbolic calculus in Besov algebras*, Studia Mathematica **184** (2008), 271–298.

[10] G. Bourdaud and Y. Meyer, *Fonctions qui opèrent sur les espaces de Sobolev*, Journal of Functional Analysis **97** (1991), 351–360.

[11] G. Bourdaud and W. Sickel, *Composition operators on function spaces with fractional order of smoothness*, RIMS Kokyuroku Bessatsu **B26** (2011), 93–132.

[12] M. Brokate and F. Colonuis, *Linearizing Equations with State-Dependent Delays*, Applied Mathematics and Optimization **21** (1990), 45–52.

[13] V.I. Burenkov, *Sobolev spaces on domains*, Stuttgart: B. G. Teubner Verlagsgesellschaft mbH, 1998.

[14] V.I. Burenkov, A. Gogatishvili, V.S. Guliyev, R.Ch. Mustafaev, *Boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal. **35(1)** (2011), 67–87.

[15] V.I. Burenkov and H.V. Guliyev, *Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces*, Studia Math. **163(2)** (2004), 157–176.

[16] V.I. Burenkov and V.S. Guliyev, *Necessary and sufficient conditions for boundedness of the Riesz operator in local Morrey-type spaces*, Potential Anal. **30(3)** (2009), 211–249.

[17] V.I. Burenkov, H.V. Guliyev, V.S. Guliyev, *Necessary and sufficient conditions for the boundedness of the Riesz operator in the local Morrey-type spaces*, Dokl. Ross. Akad. Nauk **412(5)** (2007), 585–589.

[18] V.I. Burenkov, P. Jain, T.V. Tararykova, *On boundedness of the Hardy operator in Morrey-type spaces*, Eurasian Mathematical Journal **2** (2011), no. 1, 52–80.

[19] S. Campanato, *Proprieta di inclusione per spazi di Morrey*, Ricerche Mat. **12** (1963), 67–86.

[20] F. Chiarenza and M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function*, Rend Mat. **7** (1987), 273–279.

[21] Ch.J. de la Vallée-Poussin, *Sur l'intégrale de Lebesgue*, Trans. Amer. Math. Soc. **16** (1915), 435–501.

[22] P. Drábek and T. Runst, *On the existence of Solutions of a Semi-linear Elliptic Boundary Value Problem with Superlinear Nonlinearities*, Zeitschrift für Analysis und ihre Anwendungen **21** (1990), 105–112.

[23] R.M. Dudley and R. Norvaisa, *Concrete functional calculus*, Springer, 2011.

[24] L.C. Evans, *Partial Differential Equations*, American Mathematical Society: Graduate Studies in Mathematics, V.19, 1999.

[25] G.B. Folland, *Real analysis: modern techniques and their applications*, United States of America: A Wiley-Interscience publication, 1999.

[26] V.M. Gol'dshtein and Y.G. Reshetnyak, *Quasiconformal Mappings and Sobolev spaces*, Kluwer Academic Publishers, 1990. Revised translation from the original in Russian: Vvedenie v teorii͡u funkt͡siĭ s obobshchennymi proizvodnymi i kvazikonformnye otobrazhenii͡a, Nauka, 1983.

[27] V.S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in R_n*, Doctor of Sciencies, Mat. Inst. Steklova, Moscow, 1994 (in Russian).

[28] A. Kufner, O. Jonh, and S. Fuĉik, *Function spaces*, Leyden: Noordhoff International Publishing, 1977.

[29] B. Kuttner, *Some theorems on fractional derivatives*, Proc. London Math. Soc. **3:2** (1953), 480–497.

[30] M. Lanza de Cristoforis, *Differentiability properties of an abstract autonomous composition operator*, J. London Math. Soc. (2) **61** (2000), no. 3, 923–936.

[31] M. Lanza de Cristoforis, *Differentiability properties of a composition operator*, Rendiconti del Circolo Matematico di Palermo, Serie II **56** (1998), 157–165.

[32] M. Lanza de Cristoforis, *Differentiability properties of the autonomous composition operator in Sobolev spaces*, Zeitschrift für Analysis und ihre Anwendungen **16** (1997), 631–651.

[33] M. Lanza de Cristoforis, *Higher order differentiability properties of the composition and of the inversion operator*, Indagationes Matematicae, N.S **5** (1994), 457–482.

[34] M. Marcus and J. Mizel, *Absolute continuity on tracks and mappings of Sobolev spaces*, Arch. Rat. Mech. Anal. **45** (1972), 294–320.

[35] M. Marcus and V.J. Mizel, *Complete characterization of functions which act, via superposition, on Sobolev spaces*, Trans. Amer. Math. Soc. **251** (1979), 187–218.

[36] M. Marcus and J. Mizel, *Continuity of certain Nemitsky operators on Sobolev Spaces and the chain rule*, J. Analyse Math. **28** (1975), 303–334.

[37] M. Marcus and J. Mizel, *Every Superpositionof Operator mapping one Sobolev Spaces into another is continuous*, J. Func. Anal. **33** (1979), 217–229.

[38] M. Marcus and J. Mizel, *Nemitsky Operators on Sobolev Spaces*, Archive for Rat. Mech.and Anal. **51** (1973), 347–370.

[39] M. Marcus and J. Mizel, *Superposition mappings which operate on Sobolev Spaces*, Nonlinear Anal. Theory, Methods Appl. **2** (1978), 257–258.

[40] M. Marcus and J. Mizel, *Transformation by Functions in Sobolev Spaces and Lower Semicontinuity for Parametric Variational Problems*, Bull. of the Amer. Math. Soc. **79** (1973), 790–795.

[41] R. Musina, *On the continuity of the Nemitsky operator induced by a Lipschitz continuous map*, Proceedingd of the American Mathematical Society **111** (1991), 1029–1041.

[42] T. Mizuhara, *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo (1991), 183–189.

[43] C.B. Morrey, *Multiple integral problem in the calculus of variations and related topics*, Un. of California, Publ. 1, 1943.

[44] C.B. Morrey, *Second order elliptic systems of differential equations*, Ann. of Math. Studies, n. 3, Princeton Univ. Press, 1954.

[45] C.B. Morrey, *Second order elliptic equations in several variables and Hölder continuity*, Math. Zeit. 72, 1959.

[46] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), no. 1, 126–166.

[47] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166** (1994), 95–103.

[48] S. M. Nikolskii, *Approximation of functions of several variables and embedding theorems*, Nauka, Moscow, 1969; English translation: Springer-Verlag, 1975.

[49] S. M. Nikolskii, *On a property of the* H_r^p *classes*, Ann. Univ. Sci. Budapest, Sect. Math. **34** (1960/1961), 205–216.

[50] Yu.G. Reshetnyak, *Space Mappings with Bounded Distortion*, Translation of Mathematical Monographs **73**, American Mathematical Society, 1989. Translation from the original in Russian: Prostranstvennye otobrazhenii͡a s ogranichennym iskazheniem. Nauka, 1982.

[51] T. Runst and W. Sickel, *Sobolev Spaces of Fractional order, Nemytskij Operators and Nonlinear Partial Differential Equations*, De Gruyter, Berlin, 1996.

[52] J. Serrin and D.E. Varberg, *Chain rule for the derivatives and change of variables formula for the Lebesgue integral*, The American Mathematical Monthly **76** (1969), 514–520.

[53] W. Sickel, *Superposition of Functions in Sobolev Spaces of Fractional order. A survey*, Banach Center Publications, Polish Acad. Science, Warszawa **27** (1992), 481–497.

[54] W. Sickel, *Smoothness spaces related to Morrey spaces – a survey. I*, Eurasian Mathematical Journal **3(3)** (2012), 110–149.

[55] W. Sickel, *Smoothness spaces related to Morrey spaces – a survey. II*, Eurasian Mathematical Journal **4(1)** (2013), 82–124.

[56] F. Szigeti, *Multivariable Composition of Sobolev Functions*, Acta Sci. Math. **48** (1985), 469–476.

[57] F. Szigeti, *On Nemytsky Operators on Sobolev Spaces*, Z. Angew. Math. Mech. **63**, T. 332 (1983), 5.

[58] T. Valent, *A property of multiplication in Sobolev spaces. Some applications*, Rend. Sem. Mat. Univ. Padova **74** (1985), 63–73.

[59] T. Valent, *Boundary value problems of finite elasticity – local theorems on existence, uniqueness and analytic dependence on data*, Springer Tracts Nat. Philos. **31**, Springer, 1988.

[60] W. Yuan, W. Sickel and D. Yang, *Morrey and Campanato Meet Besov, Lizorkin and Triebel*, Lecture Notes in Mathematics Vol. 2005, Springer, Berlin, 2010.

[61] J.L. Zolesio, *Multiplication dans les espaces de Besov*, Proceedings of the Royal Society of Edinburgh, Sect. A78 **1-2** (1977/78), 113–117.