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DISORDERED MODELS:
SPIN GLASSES AND DIRECTED POLYMERS

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Abstract

Disordered systems are among the most difficult and most fascinating problems in statistical mechanics. One speaks of disordered (or complex) system when the dynamics, or the structures that appears within the system, exhibits a rich variety of behaviours, while the microscopic entities the system is made of, and the interactions among these entities, are a priori simple. In this thesis we consider two famous examples of these systems: spin glasses and directed polymers in random environments.

In the first part of the thesis we study a variant of the Sherrington-Kirkpatrick (SK) model, *the SK model with ferromagnetic interactions*. More precisely the Hamiltonian that describes the model is given by

$$-H_N(\sigma) = \frac{\beta_1}{2N}\left(\sum_{i\le N}\sigma_i\right)^2 + \frac{\beta_2}{\sqrt{N}}\left(\sum_{i<j\le N} g_{i,j}\sigma_i\sigma_j\right) + h\sum_{i\le N}\sigma_i,$$

so it is a combination of a SK and Curie-Weiss Hamiltonian. Our aim is to extend the well known results obtained in the SK model, trying to describe this new model in the high temperature region. The main result is that the two key parameters, the *magnetization* $m = \frac{1}{N}\sum_{i=1}^N \sigma_i$ and the overlap $R_{1,2} = \frac{1}{N}\sum_{i\le N}\sigma_i^1\sigma_i^2$, are asymptotically close to constants μ and q, in the sense that $\nu(m_1-\mu)^2 \le K/N$ and $\nu(R_{1,2}-q)^2 \le K/N$ where ν denotes the average Gibbs measure. The two constants q and μ are the unique solutions of the so-called replica symmetric equations of this model. We then use this result to study the thermodynamical limit of the free energy and the behaviour of the Gibbs measure.

In the second part of the thesis we consider two models of directed polymers in random environment. First of all we consider a Brownian polymer b in a Gaussian environment $W(t,x)$ fully determined by its covariance function $Q(x)$. Then the Hamiltonian of the system is given by $H_t(\beta) = \int_0^t W(ds, b(s))$. In this case it is known that the thermodynamical limit of the free energy exists and it is expected that the polymer is in the strong disorder regime for low temperatures. We give a better estimate of the limit of the free energy in order to quantify how far we are from the weak disorder

regime. Then we modify the hypothesis on the covariance of the environment, to determine if one ever leaves the strong disorder regime. After this we consider a continuous time random walk on $\mathbb{Z}^d$ in a white noise potential, making a link between the last result and this new model.

Riassunto

Si parla di sistemi disordinati (o complessi) quando sono presenti eterogeneità a livello microscopico e per questo manifestano una ricca varietà di comportamenti a livello macroscopico. In questa tesi considereremo due dei più famosi esempi di questo tipo di sistemi: i vetri di spin e i polimeri diretti.
Nella prima parte della tesi consideriamo una variante del famoso modello di Sherrington-Kirkpatrick (SK), *il modello SK con interazioni ferromagnetiche*. In questo nuovo modello, oltre alle difficoltà dovute al disordine del modello SK, è presente una interazione ferromagnetica. Più precisamente l'Hamiltoniana che lo descrive è data da

$$-H_N(\sigma) = \frac{\beta_1}{2N}\left(\sum_{i\leq N}\sigma_i\right)^2 + \frac{\beta_2}{\sqrt{N}}\left(\sum_{i<j\leq N} g_{i,j}\sigma_i\sigma_j\right) + h\sum_{i\leq N}\sigma_i,$$

quindi è una combinazione dell'Hamiltoniana del modello SK e di quella del modello di Curie-Weiss. Il nostro scopo è stato estendere i risultati già noti nel modello SK in regime di alta temperatura. Il risultato principale riguarda i due parametri d'ordine del modello, la *magnetizzazione* $m = \frac{1}{N}\sum_{i=1}^N \sigma_i$ e l'overlap $R_{1,2} = \frac{1}{N}\sum_{i\leq N}\sigma_i^1\sigma_i^2$. Abbiamo dimostrato che essi convergono a due costanti, q e μ, nel senso che $\nu(m_1-\mu)^2 \leq K/N$ e $\nu(R_{1,2}-q)^2 \leq K/N$, dove ν è la media fatta rispetto alla misura di Gibbs. Le costanti q e μ sono le uniche soluzioni delle cosiddette equazioni di simmetria di replica in questo modello. Questo risultato è usato poi per trovare il limite termodinamico della energia libera e il comportamento della misura di Gibbs.

Nella seconda parte della tesi consideriamo invece due modelli di polimeri diretti in ambiente aleatorio. Prima di tutto consideriamo un polimero modellizzato come un moto Browniano b in un ambiente Gaussiano $W(t,x)$, dove il processo W è considerato a incrementi indipendenti ed è totalmente descritto da una funzione di covarianza $Q(x)$. L'Hamiltoniana del sistema è data da $H_t(\beta) = \int_0^t W(ds, b(s))$. In questo caso è noto che il limite termodinamico dell'energia libera esiste e ci si aspetta che, a basse temperature, il polimero stia in una regione di disordine forte. Abbiamo quindi cercato (e ottenuto) una stima migliore per il limite dell'energia libera per determinare

quanto lontano sia il polimero dalla regione di disordine debole. In seguito, per capire se il polimero lascia effettivamente la regione di disordine forte o no, abbiamo modificato le ipotesi sulla covarianza, e grazie a queste nuove ipotesi, ci siamo spinti a studiare la regione di regolarità logaritmica di W. Abbiamo cioè studiato una camminata aleatoria continua in $\mathbb{Z}^d$ in un ambiente disordinato dato da una successione di moti Browniani dando anche in questo caso dei risultati più precisi per il limite dell'energia libera (sempre a basse temperature).

Contents

Chapter 1

Introduction

Disordered systems are among the most difficult and most fascinating problems in statistical mechanics. One speaks of disordered (or complex) system when the dynamics, or the structures that appears within the system, exhibits a rich variety of behaviours, while the microscopic entities the system is made of, and the interactions among these entities, are a priori simple.

In the last years a lot of examples of systems characterised by disordered molecular aggregation have been found. Two typical examples are **polymers**, *i.e.* chemical compounds consisting of repeating units called monomers (like DNA or RNA), and **spin glasses**, *i.e.* metals containing random magnetic impurities in which the interaction between pairs of spins can be ferromagnetic or antiferromagnetic.

The chaotic behaviour of these systems is very peculiar: a small modification in the laws that rules the movement can completely change their macroscopic behaviour and this effect is more relevant as soon as the number of elements that compose the system increases. Therefore it is very difficult to make an estimate of the macroscopic behaviour: the results depend on a huge number of microscopic details.

To solve this problem, we give up the attempt of describing the macroscopic properties of a particular complex system. Namely, we want to calculate the probability that a system has a certain numbers of different states and the relations among them. Thus, we start from the behaviour of the components, assuming that the interactions among them are irrelevant. We also suppose that the statistical properties of the collective behaviour do not change if we make small modifications of the microscopical laws.

1.1 Statistical mechanics of disordered systems

Here is a brief overview, mainly taken from [3], of disordered mean-field models.

Statistical mechanics is a theory that attempts to explain the behaviour of systems that are composed of many individual components, like gases, liquids or crystalline solids. In order to do this, it treats the dynamical degrees of freedom of these large systems like random variables, distributed according to a particular probability measure. This is possible because, after a sufficiently large time, the dynamics relaxes to equilibrium and it forgets the initial conditions. Of course this kind of hypothesis can not always be satisfied: a typical example in which this fails are solid alloys. Suppose that we are studying a material made by a mixture of gold and iron (the classical spin glass), then some lattice sites will be occupied by the iron atoms and the others by the gold ones. If, for example, we heat the system, the atoms will become mobile and will change places, but at low temperatures this motion of the atoms is suppressed, and if we wait, even for a large time, the macroscopic realization of the material will not change. In this case we say that the positions of the atoms are 'frozen' and the system will not be in thermal equilibrium.

However, if we are interested in the *magnetic properties* of the system, we don't have to look at the position of the atoms, but at their magnetic moments, to be more precise, we have to look at the orientation of these moments, because the orientation is not 'frozen' and so it can be described by a Gibbs' measure. On the other hand we have to remember that to make a good description of the system we can not forget the position of the iron atoms because the interactions among them depend on their distance. So, if we suppose that we knew the positions x_i of the iron atoms we could write an Hamiltonian for the spin degrees of freedom

$$H(\sigma, x) = -\sum_{i,j} \sigma_i \sigma_j \phi(x_i, x_j)$$

and a Gibbs' measure

$$G(\sigma, x) = \frac{\exp(-\beta H(\sigma, x))}{Z(x)}.$$

We call such a system *disordered*. The problem is that, as we said, it is very difficult to study all possible systems for all possible placements of the x_i. Hence we have to suppose that only certain properties are important and that the microscopic details of these arrangements are negligible, *i.e.* we want to model the disordered system as a random model, by introducing a probability measure on the space of the possible realizations of the positions x_i of the iron atoms. This new type of randomness is called *quenched*.

As we said, what we would like to happen is that certain properties of our materials do not depend (or depend little) on the microscopic realizations, and are the same for almost all realizations of the disorder. Thus we will consider *averages* with respect to the disorder. We can do this computation basically in two ways,

Quenched average: we fix the disorder variables and we compute the thermodynamic quantities and then we perform the average over the disorder. For example, the *quenched free energy* is given as

$$p(\beta) = \frac{1}{\beta}\mathbf{E}\left[\log Z(x)\right]$$

Annealed average: we first compute the average of the partition function $Z(x)$. Thus the *annealed free energy* is

$$p(\beta) = \frac{1}{\beta}\log \mathbf{E}Z(x).$$

In the second way of averaging we treats the disorder variables as another degree of freedom, and thus we neglect the fact that they do not equilibrate on the same time-line. Therefore this operation is inappropriate in the situations we want to describe.

1.2 Outline of the thesis

This thesis is organized as follow.

In Chapter 2 we introduce the readers to the theory of spin glasses. After a short introduction in which we explain what is a spin glass and why these models are interesting, we describe the mean field approximation and the Sherrington-Kirkpatrick model. Then we introduce the model that we analyzed, the Sherrington-Kirkpatrick model with ferromagnetic interactions. With the help of the cavity method, we first study the behaviour of the order parameters of our model, the overlap and the magnetization. In particular we prove their convergence in L^2 to the unique solution of the replica-symmetric equations of this system. We then study the free energy, establishing its thermodynamical limit, and we finish studying the regularity of the system, in particular the behaviour of the Gibbs' measure.

These results are based on

A. Cadel, C. Rovira: *The Sherrington Kirkpatrick model with ferromagnetic interaction.*

In Chapter 3 we will study two models of directed polymer in random environment. First of all we consider a Brownian polymer in a Gaussian environment fully determined by its covariance function. In this case it is known that the thermodynamical limit of the free energy exists and it is expected that the polymer is in the strong disorder regime for low temperatures. We give a better estimate of the limit of the free energy in order to quantify how far we are from the weak disorder regime. Then we modify the hypothesis on the covariance of the environment, to determine if one ever leaves the strong disorder regime. After this we consider a continuous time random walk on $\mathbb{Z}^d$ in a white noise potential, making a link between the last result and this new model.
These results are based on

A. Cadel, S. Tindel, F. Viens: *Sharp asymptotics for the partition function of some continuous-time directed polymers.*

Chapter 2

Spin Glasses

2.1 Introduction

The movement of the electrons around the nucleus generates a microscopic magnetic moment called *spin*. This spin can be seen has a vector in a 3-dimensional space and for it only two directions are allowed: *up* or *down*. The theory that explains the macroscopic behaviour of matter using its microscopic properties is called *statistical mechanics*. According to this theory a system in equilibrium is described by means of an Hamiltonian that represents the energy of the system, so we have a link between a macroscopic energy and each configuration of the system (where, with 'configuration' we mean all the possible values taken from the spins, in this case $+1$ for up and -1 for down).

In the case of *spin glasses*, some pair of neighbouring spin want to be aligned, but some others prefer to be anti-aligned. In the first case we'll say that the interaction is ferromagnetic, while for the latter we talk about anti-ferromagnetic interaction. For any pair of spins the type of interaction is chosen randomly with the same probability. As a consequence, in the Hamiltonian we can't simply consider the interaction among pairs of spins (like we do for example for the pure ferromagnets) but we have to consider a random variable to show the type of the interaction (ferromagnetic or anti-ferromagnetic) among them.

Besides the presence of the randomness in the Hamiltonian, in order to have a spin glass we need another feature: there must be *frustration*. We say that a system is frustrated when the Hamiltonian cannot be written as the sum of many terms, all of which can be minimized by a single ground state configuration.

To see this let us make an example: consider a group of N people and suppose that each person knows each other. We also assume that any couple of individuals can be either friends or enemies and that the friendship-enmity relations are assigned randomly and independently for each couple. Now, one wants to divide the N individuals into two parties, so as to minimize social discomfort, *i.e.* putting as much as possible friends together and enemies apart. The system is obviously frustrated because if A is a friend of B and B a friend of C, we can say nothing about the relationship between A and C.

So we define the model in the following way: let $\Lambda = \{1, 2, \cdots, N\} = \{i\}_{i=1,\cdots,N}$ be a lattice. At each site i we assign a variable σ_i (the spin) such that $\sigma_i \in \{-1, 1\}$, so the set of all possible configurations is $\Sigma_N = \{-1, 1\}^N$. The Hamiltonian that describes the model is

$$-H_N(\sigma) = \sum_{\substack{i,j\in\Lambda \\ i\sim j}} J_{i,j}\sigma_i\sigma_j \tag{2.1}$$

where $i \sim j$ means that i and j are neighbouring sites and $J_{i,j}$ takes values $+1$ or -1 (respectively for ferromagnetic or anti-ferromagnetic interaction) with probability $1/2$ for any i, j. To make a connection with the previous example, for any of the 2^N possible ways the N people can be divided, assign at each person the variable

$$\begin{cases} \sigma_i = 1 \quad \text{if } i \text{ is in the first group} \\ \\ \sigma_i = -1 \quad \text{if } i \text{ is in the second group} \end{cases} \tag{2.2}$$

Furthermore, given a pair of people i and j set $J_{i,j} = 1$ if they are friends or $J_{i,j} = -1$ if they are enemies. Thus the problem to find the optimal division of the group is equivalent to find the minimum of the 'cost function'

$$-H_N(\sigma) = \sum_{1\leq i<j\leq N} J_{i,j}\sigma_i\sigma_j,$$

that is almost our Hamiltonian, in this case we consider the sum over *all* the 2^N configurations. Notice that in this example the role of disorder is played by the random choice of the relation, friendship or hostility.

At a given temperature T the state of the system is described by the *Gibbs' measure* associated with the Hamiltonian

$$G_N(\sigma) = \frac{1}{Z_N} \exp\left(-\frac{H_N(\sigma)}{kT}\right), \tag{2.3}$$

where k is the Boltzmann constant and Z_N is the normalizing factor that makes G_N a probability measure: it is the probability to find the system of N spins with energy levels H_N at the given configuration σ. To simplify things from now on we will introduce $\beta = (kT)^{-1}$, so

$$G_N(\sigma) = \frac{1}{Z_N} \exp\left(-\beta H_N(\sigma)\right).$$

The Hamiltonian (2.1) was introduced by Edwards and Anderson [16]. The Edwards-Anderson model is one of the most difficult models to analyze, from both the analytical and numerical point of view. This is due to frustration: it is non-trivial to say something about its *ground states*. The reason is that the couplings take both signs, favouring alignment or non-alignment of the spins and it is clearly impossible to satisfy the demands of all couplings. Thus it is not possible to guess the ground state configuration just from symmetry consideration (like in the study of ferromagnets!). To see this, consider a simple case in which the system is composed of four spins, denoted by 1, 2, 3, 4 and where $J_{1,2} = J_{2,3} = J_{3,4} = -J_{1,4} = +1$. In this case there are eight ground states configurations, not all connected by global spin-flip.

The general prescription of statistical mechanics is to calculate the thermal average of a physical quantity using the probability distribution defined in (2.3) for a given Hamiltonian. In principle it is possible to calculate the expectation value of any physical quantity using this distribution, but this is usually very difficult in practice, since we have to consider 2^N terms in the partition function Z_N. It seems quite natural, then, to use approximations. *Mean fields* theory is widely used in such situations. In other words, we start from the study of a simplified model maintaining the two fundamental features of disorder and frustration: the geometry of the lattice is broken so that every magnetic moment interacts with *all* others (and not only with the neighbouring ones).

2.1.1 The Sherrington-Kirkpatrick model

The Edwards-Anderson model was modified using the mean field approximation by Sherrington and Kirkpatrick [29] to find a solvable model for spin glasses. The Hamiltonian of the SK model, that for sake of simplicity we consider dependent also on the inverse of the temperature, is

$$-H_N(\sigma) = \frac{\beta}{\sqrt{N}} \sum_{1 \le i < j \le N} g_{i,j} \sigma_i \sigma_j, \tag{2.4}$$

and if we introduce an external magnetic field h, that we suppose positive, uniform in all directions and independent of our system, the Hamiltonian becomes

$$-H_N(\sigma) = \frac{\beta}{\sqrt{N}} \sum_{1\le i<j\le N} g_{i,j}\sigma_i\sigma_j + h\sum_{i\le N}\sigma_i.$$

The positive external field favours the $+$ spins over the - ones.
The $g_{i,j}$ are standard Gaussian random variables. This allow us to regard the Hamiltonian like a Gaussian process, thus we are in a simpler situation. It may appear too restrictive, but it turns out that the nature of the process that we consider is not really important, a large class of models have the same asymptotics as the corresponding Gaussian ones, at least at the level of the free energy.
The factor $1/\sqrt{N}$ is necessary in this case in order to have a good thermodynamic limit for the free energy per spin, *i.e.* in the infinite volume limit ($N \to \infty$) the free energy per site is finite and non trivial.
Notice that this Hamiltonian clearly defines a mean field model since any spin interacts with all others. Moreover, the two feature of spin glasses, disorder and frustration are present, since the couplings are not only random but they take a random sign also. Therefore $H_N(\sigma)$ is a centered Gaussian process totally characterized by its covariance function

$$\begin{aligned}\mathbf{Cov}\left(H_N(\sigma^1)H_N(\sigma^2)\right) &= \frac{1}{2N}\sum_{1\le i,j,l,k\le N}\mathbf{E}\left[g_{i,j}g_{l,k}\sigma_i^1\sigma_j^1\sigma_l^2\sigma_k^2\right]\\ &= \frac{1}{N}\sum_{1\le i,j\le N}\sigma_i^1\sigma_i^2\sigma_j^1\sigma_j^2\\ &= N\left(R_{1,2}\right)^2,\end{aligned}$$

where we introduced the order parameter of the SK model, the *overlap*

$$R_{1,2} = R_{1,2}(\sigma^1,\sigma^2) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^1\sigma_i^2. \tag{2.5}$$

Notice that the closer the overlap is to one, the closer the two configuration are to each other. It is also useful to recall that the overlap is related to the *Hamming distance* of σ^1, σ^2 that is the proportion of coordinates where σ^1, σ^2 differ

$$d(\sigma^1,\sigma^2) = \sharp\left\{i\le N : \sigma_i^1 \ne \sigma_i^2\right\},$$

in fact we can write this distance as

$$d(\sigma^1,\sigma^2) = \frac{N - \sigma^1\cdot\sigma^2}{2},$$

where with '$\cdot$' we mean the vector product in $\mathbb{R}^N$. Thus we have

$$R_{1,2}(\sigma^1, \sigma^2) = 1 - 2d(\sigma^1, \sigma^2).$$

We remark that we are interested in the study of the *quenched free energy*

$$p_N(\beta) = \frac{1}{N}\mathbf{E} \log Z_N$$

from which all thermodynamics quantities can be deduced. Let us remark also that we want to compute the *quenched* free energy and not the *annealed* free energy $N^{-1} \log \mathbf{E}Z_N$. The quenched average is the proper way to average: one computes, for fixed disordered variables, thermodynamics quantities and then performs an average over the disorder. This corresponds to consider the $(g_{i,j})_{i,j}$ variables as a frozen disorder, which does not take part to thermal equilibrium.

2.2 Ferromagnets and spin glasses

In ferromagnets each spin has a tendency to align with the one in its proximity. At high temperature the motion of the spins is so erratic that almost half of them are pointing up and the other half down, thus the net macroscopic magnetization is zero (the microscopic magnetic field generated by each spin cancel each other out). At low temperature, however, the spins become more sensible to their mutual interaction because their erratic movement is reduced. The fundamental feature of ferromagnets is that there exists a critical temperature T_c below which the spins exhibit a collective behaviour in that a majority of them point in the same direction (up or down). This is called *spontaneous magnetization*. As a consequence of this property, the individual magnetic moment of each spin sum up, creating a macroscopic magnetic field. In terms of the Gibbs' measure, this phenomenon can be explained in the following way: when the temperature is lowered, the measure tends to concentrate more and more around the configurations of minimal energy (the ground states of the system) and for the ferromagnets these states are those in which *all* the spins point in the same directions. In fact, the Hamiltonian of a ferromagnets is

$$-H_\Lambda(\sigma) = \sum_{\substack{i,j \in \Lambda \\ i \sim j}} \sigma_i \sigma_j$$

and so its minimum it is reached only if $\sigma_i = 1$ or $\sigma_i = -1$ for all i.

In the case of spin glasses, instead, there is no reason why the majority of the spins should be aligned. Indeed, one can think that due to the equal competition of ferromagnetic and anti-ferromagnetic interactions the total magnetization is zero and so the model is of no interest at all. Despite that, it was found experimentally that there is still a critical temperature below which the spins tend to behave similarly in some way.

Approximately, what happens is that above this critical temperature the spins are essentialy independent, *i.e.* their orientation is not influenced by the others in their proximity. But below T_c the spins show a cooperative behaviour that can be found in *more* than one typical configuration. Recall that in ferromagnets we have two typical configurations, one with all the spins pointing up and the other with all the spins pointing down. Instead in this case there are many pure states in which the spins behave following a sort of 'magnetic order'.

The article by Sherrington and Kirkpatrick, in which they used the 'replica symmetry' theory, turns out to be incorrect. Only some years later Giorgio Parisi proposed a new solution, known as the 'replica symmetry breaking' scheme. Parisi's theory stated that the Hamiltonian of the SK model has many ground states, which are highly disordered and which are not connected by simple transformations, but at the same time the set of all the ground states has a geometrical structure called *ultrametricity*, that is not modified when the disorder is changed.

In ferromagnets the difference between the regions above and below T_c is the value of magnetization. Here, as we already pointed out, we have to look at the *overlap* $R_{1,2}$. As a matter of fact, in spin glasses, above T_c the overlap is zero for typical configurations, while below T_c it can assume a range of non zero random values. This can be explained as follows. At low temperature the Gibbs' measure is peaked around the ground states. So the configurations of the two replicas will be closed to one of the ground states, but *not* necessarily the same one, which causes the non zero overlap.

Therefore the study of the overlap turned out to be one of the main step to understand the limiting behaviour of the system. It is natural then to investigate further its behaviour in the thermodynamical limit, and to obtain some extra informations on its exponential moments.

2.3 The SK model with ferromagnetic interactions

The high temperature regime of the SK model with external field has been widely studied (see e.g. [29], [30]), but the results on models with ferromagnetic interactions are scarce. The SK model with ferromagnetic interaction is a system with the difficulties due to the ferromagnetic interaction but with a familiar disorder, it appears as a first step in the study of models with this kind of interaction.

Our aim is to extend the well known results obtained in the SK model (see e.g. [30]) to this model, trying to describe the behaviour of the model in the high temperature region.

2.3.1 The model

The configuration space is again $\Sigma_N = \{-1,1\}^N$ and the energy of each configuration $\sigma \in \Sigma_N$ is represented by the Hamiltonian

$$-H_N(\sigma) = \frac{\beta_1}{2N}\left(\sum_{i\le N}\sigma_i\right)^2 + \frac{\beta_2}{\sqrt{N}}\left(\sum_{i<j\le N} g_{i,j}\sigma_i\sigma_j\right) + h\sum_{i\le N}\sigma_i.$$

The two parameters β_1 and β_2 play the role of two inverse temperatures. If $\beta_1 = 0$ the Hamiltonian is equivalent to the one of the Sherrington Kirkpatrick model. On the other hand, if $\beta_2 = 0$ the model reduces to Curie Weiss model, that is the canonical model for mean field (deterministic) ferromagnetic interaction. For this type of interaction, in which spins tend to align with the ones in their vicinity, we need a term proportional to $\sigma_i\sigma_j$ in the Hamiltonian, or, equivalently, we can consider the square $\left(\sum_{i\le N}\sigma_i\right)^2$ in order to write the Hamiltonian as a function of the *magnetization*

$$m_l = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^l. \tag{2.6}$$

Quite natural the Gibbs' measure of this model is

$$G_N(\sigma) = \frac{1}{Z_N}\exp(-\beta H_N(\sigma))$$

where Z_N is the disorder partition function, *i.e.* the sum over all configurations of the Boltzman factor $\exp(-\beta H_N(\sigma))$

$$Z_N = \sum_{\sigma\in\Sigma_N}\exp(-\beta H_N(\sigma)).$$

We will denote by $\langle f \rangle$ the average with respect to the Gibbs' measure of a function $f : \Sigma_N \to \mathbb{R}$ as well as for a function $f : \Sigma_N^n \to \mathbb{R}$. So

$$\langle f \rangle = \frac{1}{Z_N^n} \sum_{\sigma \in \Sigma_N^n} f(\sigma^1, \cdots, \sigma^n) \exp\left(- \sum_{l \leq n} H_N(\sigma^l) \right).$$

We write $\nu(f) = \mathbf{E} \langle f \rangle$ where $\mathbf{E}$ denotes the expectation with respect to the randomness in the Hamiltonian.

In this model we have to consider two order parameters and not only one as in the SK model. One of them is the same one considered in the SK model, that is the *overlap*

$$R_{l,l'} = \frac{1}{N} \sum_{i \leq N} \sigma_i^l \sigma_i^{l'}$$

where $\sigma^l, \sigma^{l'}$ are understood as two independent configurations under G_N, and the other is the *magnetization*, defined in (2.6).

2.3.2 The cavity method

With this method we reduce a system with N spins into one with $N-1$ spins, creating a cavity, so we can think the last spin σ_N independent from the others. The main idea of the cavity method is to reorder in the Hamiltonian all the terms that depend on the last spin.

$$\begin{aligned}
-H_N(\sigma) &= \frac{\beta_1}{2N}\left(\sum_{i \leq N-1} \sigma_i \right)^2 + \frac{\beta_2}{\sqrt{N}} \left(\sum_{i<j \leq N-1} g_{i,j}\sigma_i\sigma_j \right) + h \sum_{i \leq N-1} \sigma_i \\
&+ \frac{\beta_1}{2N}\left({\sigma_N}^2 + 2\sigma_N \left(\sum_{i \leq N-1} \sigma_i \right) \right) \\
&+ \left(\frac{\beta_2}{\sqrt{N}} \sum_{i \leq N-1} g_{i,N}\sigma_i + h \right) \sigma_N \\
&= \frac{\beta_1}{2N}\left(\sum_{i \leq N-1} \sigma_i \right)^2 + \frac{\beta_2}{\sqrt{N}} \left(\sum_{i<j \leq N-1} g_{i,j}\sigma_i\sigma_j \right) + h \sum_{i \leq N-1} \sigma_i \\
&+ \frac{\beta_1}{2N} + \sigma_N \left[\frac{\beta_1}{N} \sum_{i \leq N-1} \sigma_i + \frac{\beta_2}{\sqrt{N}} \sum_{i \leq N-1} g_{i,N}\sigma_i + h \right]
\end{aligned}$$

The term that does not depend on the lat spin looks like an Hamiltonian of a system of $N-1$ spins. Indeed let $\rho = (\sigma_1, \cdots, \sigma_{N-1}) \in \Sigma_{N-1}$ and let

$$\beta_1^- = \frac{N-1}{N}\beta_1, \qquad \beta_2^- = \sqrt{\frac{N-1}{N}}\beta_2 \tag{2.7}$$

(they will play the role of β_1 and β_2 in our reduced system). The Hamiltonian becomes

$$-H_N(\sigma) = -H_{N-1,\beta_1^-,\beta_2^-}(\rho) + \frac{\beta_1}{2N} + \sigma_N\left(g(\rho) + h\right),$$

where $-H_{N-1,\beta_1^-,\beta_2^-}(\rho)$ is the Hamiltonian of the reduced system with $N-1$ spins and $g(\rho)$ is defined as

$$g(\rho) = \frac{\beta_1}{N}\sum_{i\leq N-1}\sigma_i + \frac{\beta_2}{\sqrt{N}}\sum_{i\leq N-1} g_{i,N}\sigma_i.$$

We will denote by $\langle\cdot\rangle_-$ the average with respect to the Gibbs' measure in Σ_{N-1} with reference to the Hamiltonian $-H_{N-1}(\rho)$.
For a function $f : \Sigma_N \to \mathbb{R}$ the following equality holds

$$\langle f\rangle = \frac{\langle Av f \exp \sigma_N(g(\rho)+h)\rangle_-}{Z},$$

where Av means average on the values $\sigma_N = \pm 1$ and

$$Z = \langle Av \exp \sigma_N(g(\rho)+h)\rangle_- = \langle \cosh(g(\rho)+h)\rangle_- .$$

Similarly, for functions in Σ_N^n, we have

$$\langle f\rangle = \frac{\left\langle Av f \exp \sum_{l\leq n}\sigma_N^l\left(g(\rho^l)+h\right)\right\rangle_-}{\left(\langle\cosh(g(\rho^l)+h)\rangle_-\right)^n}.$$

To simplify notation we will write $\epsilon_l = \sigma_N^l$.

In order to construct a continuous path between the original configuration and a configuration where the last spin is independent of the others, let us define for a function $f : \Sigma_N^n \to \mathbb{R}$

$$\langle f\rangle_t = \frac{\left\langle Av f \exp \sum_{l\leq n}\epsilon_l(g_t(\rho^l)+h)\right\rangle_-}{Z_t^n}, \tag{2.8}$$

where

$$\begin{aligned} g_t(\rho^l) &= \frac{\sqrt{t}\beta_2}{\sqrt{N}}\sum_{i\leq N-1} g_{i,N}\sigma_i^l + \sqrt{1-t}(\beta_2 z\sqrt{q}) \\ &+ \frac{t\beta_1}{N}\sum_{i\leq N-1}\sigma_i^l + (1-t)\beta_1\mu \end{aligned} \tag{2.9}$$

and

$$Z_t = \left\langle Av \exp \epsilon(g_t(\rho^l) + h) \right\rangle_- = \left\langle \cosh(g_t(\rho^l) + h) \right\rangle_- .$$

Moreover, let us write

$$\xi_{n,t} = \exp \sum_{l \le n} \epsilon_l (g_t(\rho^l) + h)$$

and

$$\nu_t(f) = \mathbf{E} \langle f \rangle_t .$$

Then, it will be simpler to compute $\nu_0(f)$ than $\nu_1(f) = \nu(f)$ and these two quantities are obviously related by

$$\nu(f) - \nu_0(f) = \int_0^1 \nu'_t(f) dt.$$

In the following lemma we show how to compute $\nu_0(f)$. The proof is an obvious extension of Lemma 2.4.4 in [30].

Lemma 2.3.1 *Let Y be the random variable defined as*

$$Y = \beta_2 z \sqrt{q} + \beta_1 \mu + h, \tag{2.10}$$

where z is a standard Gaussian random variable.
For any function $f^- : \Sigma_{N-1}^n \to \mathbb{R}$ and any subset I of $\{1, ..., n\}$ we have

$$\nu_0 \left(f^- \prod_{i \in I} \epsilon_i \right) = \boldsymbol{E}(\tanh Y)^{cardI} \nu_0(f^-) = \nu_0(\prod_{i \in I} \epsilon_i) \nu_0(f^-).$$

We now compute the derivative of $\nu_t(f)$ with respect to t.

Proposition 2.3.2

$$\begin{aligned} \nu_t'(f) &= \beta_2^2 \left(\sum_{1 \le l < l' \le n} \nu_t \left(f \epsilon_l \epsilon_{l'} (R_{l,l'} - q) \right) \right) \\ &- n\beta_2^2 \sum_{l \le n} \nu_t \left(f \epsilon_l \epsilon_{n+1} (R_{l,n+1} - q) \right) \\ &+ \beta_2^2 \frac{n(n+1)}{2} \nu_t \left(f \epsilon_{n+1} \epsilon_{n+2} (R_{n+1,n+2} - q) \right) \\ &+ \beta_1 \left(\sum_{l \le n} \nu_t (f \epsilon_l (m_l - \mu)) - n \nu_t (f \epsilon_{n+1} (m_{n+1} - \mu)) \right). \end{aligned} \tag{2.11}$$

Remark 2.3.3 *Define*

$$R^-_{l,l'} = \frac{1}{N}\sum_{i=1}^{N-1}\sigma_i^l\sigma_i^{l'} \qquad m_l^- = \frac{1}{N}\sum_{i=1}^{N-1}\sigma_i^l,$$

thus the following relations hold

$$\epsilon_l\epsilon_{l'}R_{l,l'} = \frac{1}{N} + \epsilon_l\epsilon_{l'}R^-_{l,l'} \qquad \epsilon_l m_l = \frac{1}{N} + \epsilon_l m_l^-. \tag{2.12}$$

Proof of Proposition 2.3.2. It suffices to prove

$$\begin{aligned}
\nu_t'(f) &= \beta_2^2\left(\sum_{1\le l<l'\le n}\nu_t\left(f\epsilon_l\epsilon_{l'}(R^-_{l,l'}-q)\right)\right)\\
&- n\beta_2^2\sum_{l\le n}\nu_t\left(f\epsilon_l\epsilon_{n+1}(R^-_{l,n+1}-q)\right)\\
&+ \beta_2^2\frac{n(n+1)}{2}\nu_t\left(f\epsilon_{n+1}\epsilon_{n+2}(R^-_{n+1,n+2}-q)\right)\\
&+ \beta_1\left(\sum_{l\le n}\nu_t(f\epsilon_l(m_l^- - \mu)) - n\nu_t(f\epsilon_{n+1}(m^-_{n+1}-\mu))\right)
\end{aligned} \tag{2.13}$$

and then use relations (2.12). This proof is an extension of the proof of Proposition 2.4.5 in [30]. After a similar computation, we get

$$\nu_t'(f) = I + II$$

where

$$\begin{aligned}
I &= \beta_2^2\left(\sum_{1\le l<l'\le n}\nu_t\left(f\epsilon_l\epsilon_{l'}(R^-_{l,l'}-q)\right)\right)\\
&- n\beta_2^2\sum_{l\le n}\nu_t\left(f\epsilon_l\epsilon_{n+1}(R^-_{l,n+1}-q)\right)\\
&+ \beta_2^2\frac{n(n+1)}{2}\nu_t\left(f\epsilon_{n+1}\epsilon_{n+2}(R^-_{n+1,n+2}-q)\right)
\end{aligned}$$

and

$$\begin{aligned}
II &= \mathbf{E}\left(\frac{\left\langle Avf\sum_{l\le n}\epsilon_l(f_1(\rho^l)-\beta_1\mu)\xi_{n,t}\right\rangle_-}{Z_t^n}\right)\\
&- n\mathbf{E}\left(\frac{\langle Avf\xi_{n,t}\rangle_-\,\langle Av\epsilon(f_1(\rho)-\beta_1\mu)\xi_{1,t}\rangle_-}{Z_t^{n+1}}\right)\\
&= \mathbf{E}\left(\frac{\left\langle Avf\sum_{l\le n}\epsilon_l f_1(\rho^l)\xi_{n,t}\right\rangle_-}{Z_t^n}\right)\\
&- n\mathbf{E}\left(\frac{\langle Avf(\epsilon_{n+1}f_1(\rho^{n+1}))\xi_{n+1,t}\rangle_-}{Z_t^{n+1}}\right)\\
&- \beta_1\mu\mathbf{E}\left(\frac{\left\langle Avf\sum_{l\le n}\epsilon_l\xi_{n,t}\right\rangle_-}{Z_t^n}-n\frac{\langle Avf\epsilon_{n+1}\xi_{n+1,t}\rangle_-}{Z_t^{n+1}}\right).
\end{aligned}$$

Then, using the definition of $\langle f\rangle_t$ given in (2.8)

$$II=\beta_1\left(\nu_t\left(f\sum_{l\le n}\epsilon_l(m_l^- -\mu)\right)-n\nu_t(f\epsilon_{n+1}(m_{n+1}^- -\mu))\right).$$

□

As a consequence of Proposition 2.3.2 we can bound $\nu_t(f)$ by $\nu(f)$

Proposition 2.3.4 *If f is a non-negative function on Σ_N^n we have*

$$\nu_t(f)\le\nu(f)\exp\left(4n^2\beta_2^2+4n\beta_1\right)$$

Proof. Since we can assume that $|q|\le 1$ and $|\mu|\le 1$, we have $|R_{1,2}-q|\le 2$ and $|m_1-\mu|\le 2$. So, from (2.11) we have

$$\nu_t'(f)\ge -2(2n^2\beta_2^2+2n\beta_1)\nu_t(f),$$

so

$$\frac{d}{dt}\log\nu_t(f)\ge -4n^2\beta_2^2-4n\beta_1.$$

To conclude, we only have to integrate from t to 1.

□

Proposition 2.3.5 *Consider a function $f:\Sigma_N^n\to\mathbb{R}$ and numbers α_1, α_2, τ_1, $\tau_2>1$ such that $\frac{1}{\alpha_1}+\frac{1}{\alpha_2}=1$ and $\frac{1}{\tau_1}+\frac{1}{\tau_2}=1$. Then*

$$\begin{aligned}
\nu(f) &\le \nu_0(f)+2n^2\beta_2^2\exp(4n^2\beta_2^2+4n\beta_1)\left(\nu|f|^{\tau_1}\right)^{\frac{1}{\tau_1}}\left(\nu|R_{1,2}-q|^{\tau_2}\right)^{\frac{1}{\tau_2}}\\
&+ 2n\beta_1\exp(4n^2\beta_2^2+4n\beta_1)\left(\nu|f|^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\left(\nu|m_1-\mu|^{\alpha_2}\right)^{\frac{1}{\alpha_2}}.
\end{aligned}$$

Proof. Notice that

$$\nu(f) = \nu_0(f) + \int_0^1 \nu_t'(f)dt \leq \nu_0(f) + \sup_{0\leq t\leq 1} |\nu_t'(f)|.$$

Using Hölder's inequality we have

$$|\nu_t(f\epsilon_l\epsilon_{l'}(R_{l,l'} - q))| \leq (\nu_t|f|^{\tau_1})^{\frac{1}{\tau_1}} (\nu_t|R_{l,l'} - q|^{\tau_2})^{\frac{1}{\tau_2}}$$

and

$$|\nu_t(f\epsilon_l(m_l - \mu))| \leq (\nu|f|^{\alpha_1})^{\frac{1}{\alpha_1}} (\nu|m_l - \mu|^{\alpha_2})^{\frac{1}{\alpha_2}}.$$

Then, thanks to (2.11) we get that

$$\begin{aligned} |\nu_t'(f)| &\leq 2n^2\beta_2^2 (\nu_t|f|^{\tau_1})^{\frac{1}{\tau_1}} (\nu_t|R_{1,2} - q|^{\tau_2})^{\frac{1}{\tau_2}} \\ &+ 2n\beta_1 (\nu_t|f|^{\alpha_1})^{\frac{1}{\alpha_1}} (\nu_t|m_l - \mu|^{\alpha_2})^{\frac{1}{\alpha_2}}. \end{aligned}$$

We then use Proposition 2.3.4.

□

2.3.3 L^2 convergence to the parameters

The replica-symmetric equations of this model are

$$\begin{cases} q = \mathbf{E}\tanh^2(\beta_2 z\sqrt{q} + \beta_1\mu + h) \\ \mu = \mathbf{E}\tanh(\beta_2 z\sqrt{q} + \beta_1\mu + h) \end{cases} \tag{2.14}$$

where z is as usual a standard Gaussian random variable. First of all, let us prove that q and μ are the unique solutions of this system. We will denote by Y the random variable defined in (2.10).

Lemma 2.3.6 *For β_1 and β_2 sufficiently small, the system (2.14) admits a unique solution.*

Proof. Thanks to Banach fixed point theorem it is sufficient to show that

$$\begin{aligned} T : [-1,1]\times[0,1] &\to [-1,1]\times[0,1] \\ (\mu, q) &\to (\varphi(\mu,q), \psi(\mu,q)) \end{aligned}$$

is a contraction, where

$$\begin{aligned} \varphi(\mu,q) &= \mathbf{E}\tanh(\beta_2 z\sqrt{q} + \beta_1\mu + h) \\ \psi(\mu,q) &= \mathbf{E}\tanh^2(\beta_2 z\sqrt{q} + \beta_1\mu + h). \end{aligned}$$

Thus we have to prove that

$$d(T(\mu,q),T(\mu',q')) \le \alpha d((\mu,q),(\mu',q'))$$

where $0 \le \alpha \le 1$ and so that

$$d^2(T(\mu,q),T(\mu',q')) \le \alpha^2 d^2((\mu,q),(\mu',q')).$$

Now

$$d^2(T(\mu,q),T(\mu',q')) = |\varphi(\mu,q)-\varphi(\mu',q')|^2 + |\psi(\mu,q)-\psi(\mu',q')|^2,$$

$$\begin{aligned}|\varphi(\mu,q)-\varphi(\mu',q')| &= |\varphi(\mu,q)-\varphi(\mu,q')+\varphi(\mu,q')-\varphi(\mu',q')|\\ &\le \sup_{\mu,q}\left|\frac{\partial\varphi}{\partial q}\right| |q-q'| + \sup_{\mu,q}\left|\frac{\partial\varphi}{\partial\mu}\right| |\mu-\mu'|\end{aligned}$$

and similarly for $\psi(\mu,q)$

$$\begin{aligned}|\psi(\mu,q)-\psi(\mu',q')| &= |\psi(\mu,q)-\psi(\mu,q')+\psi(\mu,q')-\psi(\mu',q')|\\ &\le \sup_{\mu,q}\left|\frac{\partial\psi}{\partial q}\right| |q-q'| + \sup_{\mu,q}\left|\frac{\partial\psi}{\partial\mu}\right| |\mu-\mu'|.\end{aligned}$$

The partial derivatives are

$$\begin{aligned}\frac{\partial\varphi}{\partial\mu} &= \frac{\partial}{\partial\mu}\mathbf{E}(\tanh Y)\\ &= \mathbf{E}(\beta_1(1-\tanh^2 Y))\\ \frac{\partial\varphi}{\partial q} &= \mathbf{E}\left(\frac{\beta_2 z}{2\sqrt{q}}(1-\tanh^2 Y)\right)\end{aligned}$$

so, using the integration by part formula

$$\mathbf{E}(z(1-\tanh^2 Y)) = \mathbf{E}(-2\beta_2\sqrt{q}\tanh Y(1-\tanh^2 Y)),$$

we have

$$\frac{\partial\varphi}{\partial q} = -\beta_2^2\mathbf{E}(\tanh Y(1-\tanh^2 Y)).$$

Therefore for $\psi(\mu,q)$ we obtain

$$\begin{aligned}\frac{\partial\psi}{\partial\mu} &= \frac{\partial}{\partial\mu}\mathbf{E}(\tanh^2 Y)\\ &= 2\beta_1\mathbf{E}(\tanh Y(1-\tanh^2 Y))\\ \frac{\partial\psi}{\partial q} &= \mathbf{E}\left(\frac{\beta_2 z}{\sqrt{q}}\tanh Y(1-\tanh^2 Y)\right)\\ &= \beta_2^2\mathbf{E}((1-\tanh^2 Y)(1-3\tanh^2 Y))\end{aligned}$$

where we used again the integration by part formula. Thus, knowing that $-1 \leq \tanh Y \leq 1$,

$$|\varphi(\mu, q) - \varphi(\mu', q')| \leq \beta_2^2|q - q'| + \beta_1|\mu - \mu'|$$

and using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we have

$$|\varphi(\mu, q) - \varphi(\mu', q')|^2 \leq 2\beta_2^4|q - q'|^2 + 2\beta_1^2|\mu - \mu'|^2.$$

Similarly for $\psi(\mu, q)$

$$|\psi(\mu, q) - \psi(\mu', q')|^2 \leq 8\beta_2^4|q - q'|^2 + 8\beta_1^2|\mu - \mu'|^2.$$

Thus, for β_1 e β_2 sufficiently small, it exists an α, $0 \leq \alpha \leq 1$, such that

$$|\varphi(\mu, q) - \varphi(\mu', q')|^2 + |\psi(\mu, q) - \psi(\mu', q')|^2 \leq \alpha(|q - q'|^2 + |\mu - \mu'|^2).$$

$\square$

Now we can focus our attention on the main theorem of this section. Let us assume that from now on our high temperature region will be determined by the following relations

$$\begin{cases} 16\beta_2^2 \exp(16\beta_2^2 + 8\beta_1) \leq \frac{1}{4}, \\ \\ 8\beta_1 \exp(16\beta_2^2 + 8\beta_1) \leq \frac{1}{4}. \end{cases} \tag{2.15}$$

Theorem 2.3.7 *For β_1 and β_2 satisfying (2.15) and for q and μ solutions of (2.14), the following inequalities hold*

$$Q_N := \nu((R_{1,2} - q)^2) \leq \frac{K}{N}, \tag{2.16}$$

$$M_N := \nu((m_1 - \mu)^2) \leq \frac{K}{N}. \tag{2.17}$$

Proof. Let $f = (m_1 - \mu)^2$. By symmetry we have

$$\nu(f) = \nu((\epsilon_1 - \mu)(m_1 - \mu))$$

and using Lemma 2.3.1 with $f^- = m_1^- - \frac{N-1}{N}\mu$ we have

$$\begin{aligned} \nu_0((\epsilon_1 - \mu)(m_1 - \mu)) &= \frac{1}{N}\nu_0((\epsilon_1 - \mu)^2) + \nu_0\left((\epsilon_1 - \mu)\, f^-\right) \\ &= \frac{1 - \mu^2}{N}. \end{aligned}$$

Applying Proposition 2.3.5 with $\alpha_1 = \alpha_2 = \tau_1 = \tau_2 = n = 2$ we get that

$$\begin{aligned} \nu(f) &\leq \nu_0((\epsilon_1 - \mu)(m_1 - \mu)) \\ &+ 16\beta_2^2 \exp(16\beta_2^2 + 8\beta_1)(\nu(R_{1,2} - q)^2)^{1/2}(\nu(m_1 - \mu)^2)^{1/2} \\ &+ 8\beta_1 \exp(16\beta_2^2 + 8\beta_1)\nu(m_1 - \mu)^2, \end{aligned}$$

and so

$$\begin{aligned} M_N &\leq \frac{1-\mu^2}{N} + 16\beta_2^2 \exp(16\beta_2^2 + 8\beta_1) M_N^{1/2} Q_N^{1/2} \\ &+ 8\beta_1 \exp(16\beta_2^2 + 8\beta_1) M_N. \end{aligned} \tag{2.18}$$

Using the same arguments for Q_N we obtain

$$\begin{aligned} Q_N &\leq \frac{1-q^2}{N} + 16\beta_2^2 \exp(16\beta_2^2 + 8\beta_1) Q_N \\ &+ 8\beta_1 \exp(16\beta_2^2 + 8\beta_1) Q_N^{1/2} M_N^{1/2}. \end{aligned} \tag{2.19}$$

Then the problem reduces to study the system (2.18)-(2.19). Observe that hypothesis (2.15) and relation $(Q_N M_N)^{\frac{1}{2}} \leq \frac{1}{2}(Q_N + M_N)$ yield that the system (2.18)-(2.19) implies that

$$\begin{cases} M_N \leq \frac{K}{N} + \frac{1}{8}(Q_N + M_N) + \frac{1}{4}M_N \\ \\ Q_N \leq \frac{K}{N} + \frac{1}{8}(Q_N + M_N) + \frac{1}{4}Q_N \end{cases}$$

and the result follows easily.

$\square$

2.3.4 Study of the free energy

Set

$$p_N(\beta_1, \beta_2) = \frac{1}{N}\mathbf{E} \log Z_N = \frac{1}{N}\mathbf{E}\left[\log\left(\sum_{\sigma\in\Sigma_N} \exp(-H_N(\sigma))\right)\right].$$

This quantity is closely related to the *free energy* considered by physicists, up to a scaling factor, and we call it the free energy of our system. In this section we will prove that the limit of $p_N(\beta_1, \beta_2)$, when $N \to \infty$, is the function

$$F(\beta_1, \beta_2) = \frac{\beta_2^2}{4}(1-q)^2 + \log 2 + \mathbf{E}\left(\log\cosh(\beta_2 z\sqrt{q} + \beta_1\mu + h)\right) - \frac{\beta_1\mu^2}{2}.$$

Theorem 2.3.8 *If β_1 and β_2 satisfy hypothesis (2.15) we have*

$$\lim_{N\to\infty} p_N(\beta_1,\beta_2) = F(\beta_1,\beta_2).$$

Proof. If we fix β_2 we have

$$\begin{aligned}|F_N(\beta_1,\beta_2) - p_N(\beta_1,\beta_2)| &\leq |F_N(0,\beta_2) - p_N(0,\beta_2)| \\ &+ \int_0^{\beta_1} \left| \frac{\partial F(x,\beta_2)}{\partial x} - \frac{\partial p_N(x,\beta_2)}{\partial x} \right| dt\end{aligned}$$

Thanks to Theorem 2.4.18 proved by Talagrand in [30], we know that

$$|F_N(0,\beta_2) - p_N(0,\beta_2)| \leq \frac{K}{N}.$$

Indeed, if $\beta_1 = 0$ we are in the case of SK model with external magnetic field.
Setting by $B(\sigma)$ the Boltzman factor, $B(\sigma) = \exp(-H_N(\sigma))$, and taking derivative in β_1 we have

$$\frac{\partial F}{\partial \beta_1} = -\frac{\mu^2}{2} + \mu \mathbf{E}(\tanh Y) = \frac{\mu^2}{2}$$

and

$$\begin{aligned}\frac{\partial p_N}{\partial \beta_1} &= \frac{1}{N}\mathbf{E}\left(\frac{1}{Z_N}\sum_\sigma \frac{\partial B(\sigma)}{\partial \beta_1}\right) \\ &= \frac{1}{2N^2}\mathbf{E}\left(\frac{1}{Z_N}\sum_\sigma \left(\sum_{i=1}^N \sigma_i\right)^2 B(\sigma)\right) \\ &= \frac{1}{2N^2}\mathbf{E}\left\langle \left(\sum_{i=1}^N \sigma_i\right)^2 \right\rangle.\end{aligned}$$

So we have to prove that

$$\left| \frac{1}{N^2}\mathbf{E}\left\langle \left(\sum_{i\leq N} \sigma_i\right)^2 \right\rangle - \mu^2 \right| \leq \frac{K}{\sqrt{N}}$$

and this is an easy computation applying Theorem 2.3.7.

□

2.3.5 Exponential moments

The central result of this section is to control the moments of order $2k$, $k > 0$ of $R_{1,2} - q$ and $m_1 - \mu$. Notice that these bounds will permit us to prove, at the end of this section, the following bounds

$$|\nu(R_{1,2} - q)| \le \frac{K}{N}, \qquad |\nu(m_1 - \mu)| \le \frac{K}{N}.$$

Theorem 2.3.9 *For all $k \ge 0$ and for β_1 and β_2 satisfying (2.15) the following inequalities hold*

$$\nu\left((R_{1,2} - q)^{2k}\right) \le \left(\frac{Lk}{N}\right)^k,$$

$$\nu\left((m_1 - \mu)^{2k}\right) \le \left(\frac{Lk}{N}\right)^k.$$

We speak about *exponential moments* because Theorem 2.3.9 imply

$$\begin{aligned} \nu\left(\exp\frac{N}{L}(R_{1,2} - q)^2\right) &\le L, \\ \nu\left(\exp\frac{N}{L}(m_1 - \mu)^2\right) &\le L. \end{aligned}$$

Indeed, using the relation $\exp x^2 = \sum_{k\ge1} x^{2k}/k!$ and the fact that $(k/L)^k \le k! \le k^k$, we have

$$\begin{aligned} \nu\left(\exp\frac{N}{L}(R_{1,2} - q)^2\right) &= \nu\left(\sum_{k\ge1}\left(\frac{N}{L}\right)^k \frac{(R_{1,2} - q)^{2k}}{k!}\right) \\ &= \sum_{k\ge1}\left(\frac{N}{L}\right)^k \frac{1}{k!}\nu(R_{1,2} - q)^{2k} \\ &\le L. \end{aligned}$$

We will prove Theorem 2.3.9 by induction, considering that we have already proved the induction step $k = 1$ in Theorem 2.3.7. The induction hypothesis is for all $l \le k$

$$\begin{aligned} \nu(R_{1,2} - q)^{2l} &\le \left(\frac{Ll}{N}\right)^l, \\ \nu(m_1 - \mu)^{2l} &\le \left(\frac{Ll}{N}\right)^l. \end{aligned} \tag{2.20}$$

To prove this theorem, anyway, we will need the following lemma, whose proof is an obvious extension of Lemma 2.5.2 in [30].

Lemma 2.3.10 *We assume (2.20) and* $k \leq N-1$. *Then, if* $L_0 \geq 4$ *we have*

$$\forall j \leq 2k \quad \nu(|R^-_{1,2} - q|^j) \leq \left(\frac{L_0(j+1)}{N}\right)^{j/2},$$

$$\nu(|R^-_{1,2} - q|^{2k}) \leq 3\left(\frac{L_0(k+1)}{N}\right)^{k},$$

$$\forall j \leq 2k \quad \nu(|m^-_1 - \mu|^j) \leq \left(\frac{L_0(j+1)}{N}\right)^{j/2},$$

$$\nu(|m^-_1 - \mu|^{2k}) \leq 3\left(\frac{L_0(k+1)}{N}\right)^{k}.$$

We can now prove Theorem 2.3.9

Proof of Theorem 2.3.9. Let $f = (m_1 - \mu)^{2k+2}$. Symmetry implies $\nu(f) = \nu(\hat{f})$ with

$$\hat{f} = (\epsilon_1 - \mu)(m_1 - \mu)^{2k+1}.$$

Using Proposition 2.3.5 with $n = 2$, $\tau_1 = \alpha_1 = \frac{2k+2}{2k+1}$, $\tau_2 = \alpha_2 = 2k+2$ and hypothesis (2.15) we have

$$\begin{aligned}
& \nu(f) \\
\leq\ & \nu_0(\hat{f}) + 8\beta_2^2 \exp(16\beta_2^2 + 8\beta_1)\Big(\nu\left((\hat{f})^{\frac{2k+2}{2k+1}}\right)\Big)^{\frac{2k+1}{2k+2}} \left(\nu\left((R_{1,2} - q)^{2k+2}\right)\right)^{\frac{1}{2k+2}} \\
+\ & 4\beta_1 \exp(16\beta_2^2 + 8\beta_1)\left(\nu\left((\hat{f})^{\frac{2k+2}{2k+1}}\right)\right)^{\frac{2k+1}{2k+2}} \left(\nu((m_1 - \mu)^{2k+2})\right)^{\frac{1}{2k+2}} \\
\leq\ & \nu_0(\hat{f}) + \frac{1}{4}\left[\nu\left((m_1 - \mu)^{2k+2}\right)\right]^{\frac{2k+1}{2k+2}} \left[\nu\left((R_{1,2} - q)^{2k+2}\right)\right]^{\frac{1}{2k+2}} \\
+\ & \frac{1}{4}\nu\left((m_1 - \mu)^{2k+2}\right). \qquad (2.21)
\end{aligned}$$

So, using that for any numbers a, $b < 1$ such that $a + b = 1$ and x, $y > 0$, we have

$$x^a y^b \leq x + y;$$

(2.21) becomes

$$\nu(f) \leq 2\nu_0(\hat{f}) + \frac{1}{2}\nu\left((R_{1,2} - q)^{2k+2}\right).$$

Letting

$$g = (R_{1,2} - q)^{2k+2}, \quad \hat{g} = (\epsilon_1\epsilon_2 - q)(R_{1,2} - q)^{2k+1},$$

and by similar arguments we have that

$$\nu(g) \leq 2\nu_0(\hat{g}) + \frac{1}{2}\nu(f).$$

So we have to study the inequalities

$$\begin{cases} \nu(f) \leq 2\nu_0\left((\epsilon_1-\mu)(m_1-\mu)^{2k+1}\right) + \frac{1}{2}\nu(g) \\ \nu(g) \leq 2\nu_0\left((\epsilon_1\epsilon_2-q)(R_{1,2}-q)^{2k+1}\right) + \frac{1}{2}\nu(f). \end{cases}$$

If we prove that

$$\begin{cases} \nu_0\left((\epsilon_1-\mu)(m_1-\mu)^{2k+1}\right) \leq 32L_0^k\left(\frac{k+1}{N}\right)^{k+1} \\ \nu_0\left((\epsilon_1\epsilon_2-q)(R_{1,2}-q)^{2k+1}\right) \leq 32L_0^k\left(\frac{k+1}{N}\right)^{k+1} \end{cases} \tag{2.22}$$

the system becomes

$$\begin{cases} \nu(f) \leq 64L_0^k\left(\frac{k+1}{N}\right)^{k+1} + \frac{1}{2}\nu(g) \\ \nu(g) \leq 64L_0^k\left(\frac{k+1}{N}\right)^{k+1} + \frac{1}{2}\nu(f) \end{cases}$$

and we conclude easily choosing $L_0 = 128$.
To prove (2.22) we use Lemma 2.3.1: it implies that $\nu_0((\epsilon_1-\mu)(m_1^- - \mu)^{2k+1}) = 0$ and hence

$$\nu_0\left((\epsilon_1-\mu)(m_1-\mu)^{2k+1}\right) = \nu_0\left((\epsilon_1-\mu)((m_1-\mu)^{2k+1} - (m_1^- - \mu)^{2k+1})\right).$$

Using the inequality $|x^{2k+1} - y^{2k+1}| \leq (2k+1)|x-y|(x^{2k}+y^{2k})$, Proposition 2.3.4 and relations (2.12) we have

$$\begin{aligned} \nu_0\left((\epsilon_1-\mu)(m_1-\mu)^{2k+1}\right) &\leq 4\frac{k+1}{N}\exp(16\beta_2^2+8\beta_1) \\ &\quad \times \left[\nu(m_1-\mu)^{2k} + \nu(m_1^- - \mu)^{2k}\right]. \end{aligned}$$

Since (2.15) holds, we can assume $\exp(16\beta_2^2 + 8\beta_1) \leq 2$. So using Lemma 2.3.10 and the induction hypothesis we have

$$\begin{aligned} \nu_0\left((\epsilon_1-\mu)(m_1-\mu)^{2k+1}\right) &\leq 8\frac{k+1}{N}\left(\left(\frac{L_0k}{N}\right)^k + 3\left(\frac{L_0(k+1)}{N}\right)^k\right) \\ &\leq 32\frac{k+1}{N}\left(\frac{L_0(k+1)}{N}\right)^k. \end{aligned}$$

Similarly we have

$$\nu_0\left((\epsilon_1\epsilon_2-q)(R_{1,2}-q)^{2k+1}\right) \leq 32\frac{k+1}{N}\left(\frac{L_0(k+1)}{N}\right)^k.$$

□

Theorem 2.3.9 allows us to control $\nu(R_{1,2}-q)$ and $\nu(m_1-\mu)$. Our next goal is to prove the following theorem

Theorem 2.3.11

$$|\nu(R_{1,2}-q)| \leq \frac{K}{N},$$

$$|\nu(m_1-\mu)| \leq \frac{K}{N}.$$

First, we need the following lemma:

Lemma 2.3.12 *For a function $f:\Sigma_N^n \to \mathbb{R}$ we have*

$$|\nu(f)-\nu_0(f)| \leq \frac{K(n)}{\sqrt{N}}\left(\nu(f^2)\right)^{1/2} \tag{2.23}$$

$$|\nu(f)-\nu_0(f)-\nu_0'(f)| \leq \frac{K(n)}{N}\left(\nu(f^2)\right)^{1/2}. \tag{2.24}$$

Proof. From Proposition 2.3.2, Proposition 2.3.4 and Theorem 2.3.9 we have

$$\begin{aligned}
|\nu(f)-\nu_0(f)| &\leq \int_0^1 |\nu_t'(f)|dt \\
&\leq \int_0^1 (2n^2\beta_2^2\nu_t(|f(R_{1,2}-q)|)+2n\beta_1\nu_t(|f(m_1-\mu)|))dt \\
&\leq \int_0^1 [2n^2\beta_2^2(\nu_t(f^2))^{1/2}(\nu_t((R_{1,2}-q)^2)^{1/2} \\
&\quad +2n\beta_1(\nu_t(f^2))^{1/2}(\nu_t(m_1-\mu)^2)^{1/2}]dt \\
&\leq \int_0^1 2n^2\beta_2^2\exp(4n^2\beta_2^2+4n\beta_1)(\nu(f^2))^{1/2}(\nu((R_{1,2}-q)^2)^{1/2} \\
&\quad +2n\beta_1\exp(4n^2\beta_2^2+4n\beta_1)(\nu(f^2))^{1/2}(\nu((m_1-\mu)^2)^{1/2}dt \\
&\leq \frac{K(n)}{\sqrt{N}}(\nu(f^2))^{1/2}
\end{aligned}$$

that implies (2.23). To prove (2.24) we use Theorem 2.3.9 with $k = 2$ and we have

$$\begin{aligned}
&|\nu(f) - \nu_0(f) - \nu_0'(f)| \\
&\quad \leq \sup_{0\leq t\leq 1} |\nu_t''(f)| \\
&\quad \leq L(n)(\nu(f^2))^{1/2}[(\nu(R_{1,2}-q)^4)^{1/2} + (\nu(m_1-\mu)^4)^{1/2} \\
&\qquad +(\nu((R_{1,2}-q)^2(m_1-\mu)^2))^{1/2}] \\
&\quad \leq L(n)(\nu(f^2))^{1/2}[(\nu(R_{1,2}-q)^4)^{1/2} + (\nu(m_1-\mu)^4)^{1/2} \\
&\qquad +\frac{1}{2}(\nu(R_{1,2}-q)^4 + \nu(m_1-\mu)^4)^{1/2}] \\
&\quad \leq \frac{K(n)}{N}(\nu(f^2))^{1/2}.
\end{aligned}$$

□

From now on set

$$\hat{q} = \mathbf{E}\tanh^4(\beta_2 z\sqrt{q} + \beta_1\mu + h)$$

$$\hat{\mu} = \mathbf{E}\tanh^3(\beta_2 z\sqrt{q} + \beta_1\mu + h).$$

Proof of Theorem 2.3.11. Let $f = m_1 - \mu$. By symmetry we have $\nu(f) = \nu(\epsilon_1 - \mu)$. Thanks to Lemma 2.3.12 we have

$$|\nu(f) - \nu_0(\epsilon_1-\mu) - \nu_0'(\epsilon_1-\mu)| \leq \frac{K(n)}{N}, \tag{2.25}$$

where $\nu_0(\epsilon_1 - \mu) = 0$ because of Lemma 2.3.1. To compute $\nu_0'(\epsilon_1 - \mu)$ we use (2.13) with $n = 1$ and Lemma 2.3.1. We have

$$\nu_0'(\epsilon_1-\mu) = \beta_2^2(\hat{\mu}-\mu)\nu_0(R_{1,2}^- - q) + \beta_1(1-q)\nu_0(m_1^- - \mu).$$

Since (2.23) imply that

$$\begin{cases} |\nu_0(R_{1,2}^- - q) - \nu(R_{1,2} - q)| \leq \frac{K}{N} \\ |\nu_0(m_1^- - \mu) - \nu(m_1-\mu)| \leq \frac{K}{N}, \end{cases} \tag{2.26}$$

(2.25) becomes

$$|\nu(f) - \beta_2^2(\hat{\mu}-\mu)\nu(R_{1,2}-q) - \beta_1(1-q)\nu(m_1-\mu)| \leq \frac{K(n)}{N}.$$

Reasoning analogously with $g = R_{1,2} - q$ we have

$$|\nu(R_{1,2} - q) - \beta_2^2(1 - 4q + 3\hat{q})\nu(R_{1,2} - q) - 2\beta_1(\mu - \hat{\mu})\nu(m_1 - \mu)| \leq \frac{K}{N}.$$

So we have to compute the system

$$\begin{cases} |(1 - \beta_2^2(1 - 4q + 3\hat{q}))\nu(R_{1,2} - q) - 2\beta_1(\mu - \hat{\mu})\nu(m_1 - \mu)| \leq \frac{K}{N} \\ |(1 - \beta_1(1 - q))\nu(m_1 - \mu) - \beta_2^2(\hat{\mu} - \mu)\nu(R_{1,2} - q)| \leq \frac{K}{N}. \end{cases}$$

There exist two constants L and L', such that $|L| + |L'| \leq \frac{K}{N}$, such that

$$\begin{cases} (1 - \beta_2^2(1 - 4q + 3\hat{q}))\nu(R_{1,2} - q) = 2\beta_1(\mu - \hat{\mu})\nu(m_1 - \mu) + L \\ (1 - \beta_1(1 - q))\nu(m_1 - \mu) = \beta_2^2(\hat{\mu} - \mu)\nu(R_{1,2} - q) + L' \end{cases}$$

and thus

$$\left[1 - \beta_1(1 - q) + \frac{2\beta_1\beta_2^2(\hat{\mu} - \mu)^2}{1 - \beta_2^2(1 - 4q + 3\hat{q})}\right]\nu(m_1 - \mu) = L' + \frac{L}{1 - \beta_2^2(1 - 4q + 3\hat{q})}$$

where

$$\frac{1}{2} < 1 - \beta_2(1 - 4q + 3\hat{q}) < 1,$$

because $\hat{q} \leq q$. Moreover we have

$$1 - \beta_1(1 - q) + \frac{2\beta_1\beta_2^2(\hat{\mu} - \mu)^2}{1 - \beta_2^2(1 - 4q + 3\hat{q})} \geq 1 - \beta_1(1 - q) > \frac{1}{2},$$

and so

$$\nu(m_1 - \mu) \leq 4L + L'.$$

□

2.3.6 Regularity of the system

One way of looking at the regularity of the system when $N \to \infty$ is to investigate the limit of the laws of the random variables $(\langle\sigma_1\rangle, \cdots, \langle\sigma_n\rangle)$. In fact, one way to study the self averaging phenomenon for the model is to show that those quantities converge to some independent and identically distributed centered random variables that can be clearly identified, by analogy with the fact that the magnetization vanishes for the Ising model at high temperature.

It turns out that the above sequence is formed by asymptotically i.i.d. random variables and the limit law of each one of them is the law of the random variable $Y = \tanh(\beta_2 z\sqrt{q}+\beta_1\mu+h)$, where z is as usual a standard Gaussian random variable.
The central theorem of this section reads as follows.

Theorem 2.3.13 *If β_1 and β_2 satisfy (2.15) we can find independent standard Gaussian random variables $\{z_i\}_{i\leq n}$ such that*

$$\boldsymbol{E}\left[\sum_{i\leq n}\left(\langle\sigma_i\rangle - \tanh(\beta_2 z_i\sqrt{q}+\beta_1\mu+h)\right)^2\right] \leq \frac{K}{N}$$

To prove it we need some preliminary results.

Lemma 2.3.14 *Denote by q_- and μ_- the solutions of (2.14) when β_1 and β_2 are replaced by β_1^- and β_2^- defined in (2.7). Then for β_1 and β_2 satisfying (2.15) we have*

$$\begin{aligned} |q-q_-| &\leq \frac{K}{N}, \\ |\mu-\mu_-| &\leq \frac{K}{N}. \end{aligned}$$

Proof. Clearly

$$|q(\beta_1,\beta_2)-q(\beta_1^-,\beta_2^-)| \leq \sup_{\beta_1,\beta_2}\left|\frac{\partial q}{\partial\beta_1}\right||\beta_1-\beta_1^-| + \sup_{\beta_1,\beta_2}\left|\frac{\partial q}{\partial\beta_2}\right||\beta_2-\beta_2^-|$$

Since system (2.14) can be seen as

$$\begin{cases} q = F_1(\beta_1,\beta_2,q(\beta_1,\beta_2),\mu(\beta_1,\beta_2)) \\ \\ \mu = F_2(\beta_1,\beta_2,q(\beta_1,\beta_2),\mu(\beta_1,\beta_2)), \end{cases}$$

for $i=1,2$ we have

$$\frac{\partial q}{\partial\beta_i} = \frac{\frac{\partial F_1}{\partial\beta_i}+\frac{\partial F_1}{\partial\mu}\frac{\partial\mu}{\partial\beta_i}}{1-\frac{\partial F_1}{\partial q}}$$

and

$$\frac{\partial \mu}{\partial\beta_i} = \frac{\frac{\partial F_2}{\partial\beta_i}+\frac{\partial F_2}{\partial q}\frac{\partial q}{\partial\beta_i}}{1-\frac{\partial F_2}{\partial \mu}}.$$

Computing these derivatives we can conclude that

$$\begin{aligned}|q(\beta_1,\beta_2)-q(\beta_1^-,\beta_2^-)| &= K(\beta_1,\beta_2)|\beta_2-\beta_2^-| \\ &\leq \frac{K(\beta_1,\beta_2)}{N},\end{aligned}$$

and similarly that

$$|\mu(\beta_1,\beta_2)-\mu(\beta_1^-,\beta_2^-)| \leq \frac{K(\beta_1,\beta_2)}{N}.$$

□

Set now

$$\begin{aligned} g(c) &= \frac{\beta_2}{\sqrt{N}}\sum_{i\leq N-1} g_{i,N}\left\langle\sigma_i\right\rangle_- + \frac{\beta_1}{N}\sum_{i\leq N-1}\left\langle\sigma_i\right\rangle_-, \qquad (2.27)\\ \|c\|^2 &= \sum_{i\leq N-1}\left\langle\sigma_i\right\rangle_-^2, \\ e &= \sum_{i\leq N}\left\langle\sigma_i\right\rangle. \end{aligned}$$

Lemma 2.3.15 *We can find a standard Gaussian random variable z, that depends only on $(g_{i,j})_{i<j\leq N}$ but it is probabilistically independent of the $(g_{i,j})_{i<j\leq N-1}$, such that*

$$\boldsymbol{E}\left((\langle\sigma_N\rangle-\tanh(\beta_2 z\sqrt{q}+\beta_1\mu+h))^2\right)\leq\frac{K}{N}$$

Proof. Let

$$z=\frac{1}{\|c\|}\sum_{i\leq N-1} g_{i,N}\left\langle\sigma_i\right\rangle_-$$

and let Y be the random variable defined in (2.10). Using the inequalities

$$|\tanh x-\tanh y|\leq|x-y| \quad \text{and} \quad (x+y)^2\leq 2x^2+2y^2 \qquad (2.28)$$

we have

$$\begin{aligned}\mathbf{E}\left((\langle\sigma_N\rangle-\tanh(Y))^2\right) &\leq 2\mathbf{E}\left((\langle\sigma_N\rangle-\tanh(g(c)+h))^2\right)\\ &+ 4\mathbf{E}\left(\left(\frac{\beta_2}{\sqrt{N}}\sum_{i\leq N-1} g_{i,N}\left\langle\sigma_i\right\rangle_- -\beta_2 z\sqrt{q}\right)^2\right)\\ &+ 4\mathbf{E}\left(\left(\frac{\beta_1}{N}\sum_{i\leq N-1}\left\langle\sigma_i\right\rangle_- -\beta_1\mu\right)^2\right). \qquad (2.29)\end{aligned}$$

We will prove in Lemma 2.3.16 that

$$\mathbf{E}\left((\langle\sigma_N\rangle - \tanh(g(c)+h))^2\right) \le \frac{K}{N}. \tag{2.30}$$

Using the definition of z and taking expectation in $(g_{i,N})_{i\le N-1}$, we have

$$\mathbf{E}\left(\left(\frac{\beta_2}{\sqrt{N}}\sum_{i\le N-1} g_{i,N}\langle\sigma_i\rangle_- - \beta_2 z\sqrt{q}\right)^2\right) = \beta_2^2\mathbf{E}\left(\frac{\|c\|}{\sqrt{N}} - \sqrt{q}\right)^2,$$

where

$$\left(\frac{\|c\|}{\sqrt{N}} - \sqrt{q}\right)^2 = \frac{\left(\frac{\|c\|^2}{N} - q\right)^2}{\left(\frac{\|c\|}{\sqrt{N}} + \sqrt{q}\right)^2} \le \frac{1}{q}\left(\frac{\|c\|^2}{N} - q\right)^2.$$

Moreover, using Lemma 2.3.14 we have

$$\begin{aligned}
\beta_2^2\mathbf{E}\left(\left(\frac{\|c\|}{\sqrt{N}} - \sqrt{q}\right)^2\right) &\le \frac{\beta_2^2}{q}\mathbf{E}\left(\left(\frac{1}{N}\sum_{i\le N-1}\langle\sigma_i\rangle_-^2 - q\right)^2\right)\\
&\le \frac{2\beta_2^2}{q}\mathbf{E}\left(\left(\langle R_{1,2}^-\rangle_- - q_-\right)^2\right) + \frac{K}{N},
\end{aligned}$$

and thanks to Jensen's inequality

$$\mathbf{E}\left(\langle R_{1,2}^-\rangle_- - q_-\right)^2 \le \mathbf{E}\left\langle\left(R_{1,2}^- - q_-\right)^2\right\rangle_- \le \frac{K}{N}.$$

So

$$\mathbf{E}\left(\left(\frac{\beta_2}{\sqrt{N}}\sum_{i\le N-1} g_{i,N}\langle\sigma_i\rangle_- - \beta_2 z\sqrt{q}\right)^2\right) \le \frac{K}{N}. \tag{2.31}$$

With a similar argument we have

$$\begin{aligned}
\mathbf{E}\left(\left(\frac{\beta_1}{N}\sum_{i\le N-1}\langle\sigma_i\rangle_- - \beta_1\mu\right)^2\right) &\le \beta_1^2\mathbf{E}\left(\left(\frac{1}{N}\sum_{i\le N-1}\langle\sigma_i\rangle_- - \mu_-\right)^2\right) + \frac{K}{N}\\
&\le \beta_1^2\mathbf{E}\left\langle\left(m_1^- - \mu_-\right)^2\right\rangle_- + \frac{K}{N}\\
&\le \frac{K}{N}.
\end{aligned} \tag{2.32}$$

The proof finishes putting together (2.29), (2.30), (2.31) and (2.32).

$\square$

Lemma 2.3.16 *For β_1 and β_2 satisfying (2.15) and $g(c)$ defined in (2.27) we have*

$$\boldsymbol{E}\left((\langle\sigma_N\rangle - \tanh(g(c)+h))^2\right) \leq \frac{K}{N}$$

.

Remark 2.3.17 *In the proof of Lemma 2.3.16 we will use Gronwall's inequality in the following way: let $g(t)$ be a function such that*

$$\begin{aligned} g(0) &\leq \frac{L}{N} \\ g'(t) &\leq Lg(t) \end{aligned} \tag{2.33}$$

Then $g(t) \leq \frac{L}{N}$.
Indeed, it follows that

$$g(t) = g(0) + \int_0^t g'(s)ds \leq \frac{L}{N} + L\int_0^t g(s)ds$$

and then we can use Gronwall's inequality.

Proof. Consider $g_t(\rho)$ defined in (2.9) and set $g_t(c)$ in a similar way

$$g_t(c) = \frac{\sqrt{t}\beta_2}{\sqrt{N}} \sum_{i\leq N-1} \langle\sigma_i\rangle_- + \sqrt{1-t}(\beta_2 z\sqrt{q}) + \frac{t\beta_1}{N}\sum_{i\leq N-1}\langle\sigma_i\rangle_- + (1-t)\beta_1\mu.$$

We consider the function

$$\varphi(t) = \mathbf{E}\left((U(t)-V(t))^2\right)$$

where

$$\begin{aligned} U(t) &= \langle\sigma_N\rangle_t\,, \\ V(t) &= \tanh(g_t(c)+h) = \frac{\langle Av\epsilon\exp(\epsilon(g_t(c)+h))\rangle_-}{\langle\cosh(g_t(c)+h)\rangle_-}, \end{aligned}$$

where $U(t)$ and $V(t)$ are defined similarly, putting $g_t(c)$ instead of $g_t(\rho)$. Obviously, our aim is to check that

$$\varphi(1) \leq \frac{K}{N}.$$

Since $g_0(\rho) = g_0(c)$, we have $\varphi(0) = \mathbf{E}((U(0) - V(0))^2) = 0$ and $\varphi(1) = \varphi(1) - \varphi(0)$.
Set $\varphi(t) = \varphi_1(t) - 2\varphi_2(t) + \varphi_3(t)$ where

$$\varphi_1(t) = \mathbf{E}(U^2(t)), \quad \varphi_2(t) = \mathbf{E}(U(t)V(t)), \quad \varphi_3(t) = \mathbf{E}(V^2(t)). \tag{2.34}$$

Then, it is enough to prove that

$$\begin{aligned} |\varphi(1) - \varphi(0)| &\leq |\varphi_1(1) - \varphi_1(0)| + 2|\varphi_2(1) - \varphi_2(0)| + |\varphi_3(1) - \varphi_3(0)| \\ &\leq \frac{K}{N}. \end{aligned}$$

We begin with the study of $\varphi_1(t)$. The same kind of computations will be useful to study $\varphi_2(t)$ and $\varphi_3(t)$, provided that we will prove that relation (2.24) also holds for $\varphi_2(t)$ and $\varphi_3(t)$.

Step 1. Using symmetry we have

$$\varphi_1(t) = \nu_t(\epsilon_1\epsilon_2)$$

and thanks to (2.24) we have that

$$|\varphi_1(1) - \varphi_1(0) - \varphi_1'(0)| \leq \frac{K}{N}.$$

So it is sufficient to prove that $\varphi_1'(0) \leq \frac{K}{N}$. Using Proposition 2.3.2 with $n = 2$ and $f = \epsilon_1\epsilon_2$ and Lemma 2.3.1 we have

$$\varphi_1'(0) = \beta_2^2(1 - 4q + 3\hat{q})\nu_0(R_{1,2}^- - q) + 2\beta_1(\mu - \hat{\mu})\nu_0(m_1^- - \mu)$$

and we can conclude using (2.26) and Theorem 2.3.11.

Step 2. Study of $\varphi_3(t)$. For a function $f : \Sigma_N^n \to \mathbb{R}$, set

$$\langle f \rangle_t' = \frac{\left\langle Av f \exp\left(\sum_{l\leq n} \epsilon_l(g_t(c) + h)\right)\right\rangle_-}{\langle \cosh^n(g_t(c) + h)\rangle_-}.$$

Using symmetry

$$\varphi_3(t) = \mathbf{E}\left\langle \epsilon_1\epsilon_2\right\rangle_t'.$$

The only difference between $\langle\cdot\rangle_t$ and $\langle\cdot\rangle_t'$ is that instead of $g_t(\rho^l)$ we will have $g_t(c)$. So, (2.13) remains valid, provided one replaces $\nu_t(\cdot)$ by $\mathbf{E}\left\langle\cdot\right\rangle_t'$, $R_{l,l'}^-$ by

$\frac{\|c\|^2}{N}$ and m_l^- by $e_- = \frac{1}{N}\sum_{i\leq N-1}\langle\sigma_i\rangle_-$. Thus

$$\begin{aligned}
\frac{d}{dt}\mathbf{E}(\langle f\rangle_t') &= \beta_2^2\sum_{1\leq l<l'\leq n}\mathbf{E}\left\langle f\epsilon_l\epsilon_{l'}\left(\frac{\|c\|^2}{N}-q\right)\right\rangle_t' \\
&- n\beta_2^2\sum_{l\leq n}\mathbf{E}\left\langle f\epsilon_l\epsilon_{n+1}\left(\frac{\|c\|^2}{N}-q\right)\right\rangle_t' \\
&+ \beta_2^2\frac{n(n+1)}{2}\mathbf{E}\left\langle f\epsilon_{n+1}\epsilon_{n+2}\left(\frac{\|c\|^2}{N}-q\right)\right\rangle_t' \\
&+ \beta_1\sum_{l\leq n}\mathbf{E}\left\langle f\epsilon_l(e_- -\mu)\right\rangle_t' \\
&- n\beta_1\mathbf{E}\left\langle f\epsilon_{n+1}(e_- -\mu)\right\rangle_t'. \qquad (2.35)
\end{aligned}$$

To prove that (2.24) holds, we have to verify first that (2.23) also holds. We will use Remark 2.3.17 to prove that $\mathbf{E}\left\langle\left(\frac{\|c\|^2}{N}-q\right)^2\right\rangle_t' \leq \frac{K}{N}$ and $\mathbf{E}\left\langle(e_- -\mu)^2\right\rangle_t' \leq \frac{K}{N}$. Some easy computations give that the functions

$$\psi(t) = \mathbf{E}\left\langle\left(\frac{\|c\|^2}{N}-q\right)^2\right\rangle_t' \quad \text{and} \quad \eta(t) = \mathbf{E}\left\langle(e_- -\mu)^2\right\rangle_t'$$

satisfy hypothesis (2.33). Indeed, for $t=0$, $\mathbf{E}\langle\cdot\rangle_0' = \mathbf{E}\langle\cdot\rangle_0 = \nu_0(\cdot)$. Thus, thanks to Proposition 2.3.4 and Theorem 2.3.7 we have

$$\begin{aligned}
\psi(0) &= \nu_0\left(\left(\frac{1}{N}\sum_{i\leq N-1}\langle\sigma_i\rangle_-^2 - q\right)^2\right) \\
&= \nu_0\left(\left\langle R_{1,2}^- - q\right\rangle_-^2\right) \\
&\leq \nu_0(R_{1,2}^- - q)^2 \\
&\leq \frac{K}{N}.
\end{aligned}$$

To show that $\psi'(t)\leq L\psi(t)$, it is enough to use (2.35).
We have to prove now that $\varphi_3'(0)\leq\frac{K}{N}$. Using (2.35) with $n=2$ and $f=\epsilon_1\epsilon_2$ we have

$$\varphi_3'(0) = \beta_2^2(1-4q+3\hat{q})\nu_0\left(\frac{\|c\|^2}{N}-q\right) + 2\beta_1(\mu-\hat{\mu})\nu_0(e_- -\mu),$$

and so $\varphi_3'(0) \leq \frac{K}{N}$.

Step 3. To study $\varphi_2(t)$ we will do the same things of Step 2.
Set

$$\langle f \rangle_t'' = \frac{\left\langle Avf \exp\left(\sum_{1\leq l\leq n} \epsilon_l (g_t(\rho^l)+h)\right) \exp(\sum_{n+1\leq l'\leq 2n} \epsilon_{l'}(g_t(c)+h))\right\rangle_-}{\langle \cosh^n(g_t(\rho)+h)\cosh^n(g_t(c)+h)\rangle_-}.$$

Then

$$\varphi_2(t) = \mathbf{E}\left\langle \epsilon_1\epsilon_2 \right\rangle_t''.$$

Now, in the adapted version of (2.13) some of the terms $R^-_{l,l'}$ are replaced by $N^{-1}\sum_{i\leq N-1} \sigma_i^l \langle \sigma_i \rangle_-$ and others by $\|c\|^2/N$, while the terms m_l^- are again replaced by e_-. Ideed deriving we have

$$
\begin{aligned}
\frac{d}{dt}\mathbf{E}(\langle f\rangle_t'') &= \beta_2^2 \sum_{1\le l<l'\le n} \mathbf{E}\left\langle f\epsilon_l\epsilon_{l'}(R^-_{l,l'}-q)\right\rangle_t'' - n\beta_2^2 \sum_{l\le n}\mathbf{E}\left\langle f\epsilon_l\epsilon_{2n+1}(R^-_{l,2n+1}-q)\right\rangle_t'' \\
&+ \beta_2^2\frac{n(n+1)}{2}\mathbf{E}\left\langle f\epsilon_{2n+1}\epsilon_{2n+2}(R^-_{2n+1,2n+2}-q)\right\rangle_t'' \\
&+ \beta_2^2 \sum_{n+1\le k<k'\le 2n}\mathbf{E}\left\langle f\epsilon_k\epsilon_{k'}\left(\frac{\|c\|^2}{N}-q\right)\right\rangle_t'' \\
&- n\beta_2^2 \sum_{n+1\le k\le 2n}\mathbf{E}\left\langle f\epsilon_k\epsilon_{2n+3}\left(\frac{\|c\|^2}{N}-q\right)\right\rangle_t'' \\
&+ \beta_2^2\frac{n(n+1)}{2}\mathbf{E}\left\langle f\epsilon_{2n+3}\epsilon_{2n+4}\left(\frac{\|c\|^2}{N}-q\right)\right\rangle_t'' \\
&+ \sum_{1\le l\le n}\sum_{n+1\le k\le 2n}\beta_2^2\mathbf{E}\left\langle f\epsilon_l\epsilon_k\left(\frac{1}{N}\sum_{i\le N-1}\sigma_i^l\left\langle\sigma_i\right\rangle_- - q\right)\right\rangle_t'' \\
&- n\beta_2^2\sum_{l\le n}\mathbf{E}\left\langle f\epsilon_l\epsilon_{2n+3}\left(\frac{1}{N}\sum_{i\le N-1}\sigma_i^l\left\langle\sigma_i\right\rangle_- - q\right)\right\rangle_t'' \\
&- n\beta_2^2\sum_{n+1\le k\le 2n}\mathbf{E}\left\langle f\epsilon_k\epsilon_{2n+1}\left(\frac{1}{N}\sum_{i\le N-1}\sigma_i^{2n+1}\left\langle\sigma_i\right\rangle_- - q\right)\right\rangle_t'' \\
&- n\beta_2^2\mathbf{E}\left\langle f\epsilon_{2n+1}\epsilon_{2n+3}\left(\frac{1}{N}\sum_{i\le N-1}\sigma_i^{2n+1}\left\langle\sigma_i\right\rangle_- - q\right)\right\rangle_t'' \\
&+ \frac{n(n+1)}{2}\beta_2^2\mathbf{E}\left\langle f\epsilon_{2n+1}\epsilon_{2n+4}\left(\frac{1}{N}\sum_{i\le N-1}\sigma_i^{2n+1}\left\langle\sigma_i\right\rangle_- - q\right)\right\rangle_t'' \\
&+ \frac{n(n+1)}{2}\beta_2^2\mathbf{E}\left\langle f\epsilon_{2n+2}\epsilon_{2n+3}\left(\frac{1}{N}\sum_{i\le N-1}\sigma_i^{2n+2}\left\langle\sigma_i\right\rangle_- - q\right)\right\rangle_t'' \\
&+ \beta_1\sum_{1\le l\le n}\mathbf{E}\left\langle f\epsilon_l(m_l^- -\mu)\right\rangle_t'' - n\beta_1\mathbf{E}\left\langle f\epsilon_{2n+1}(m^-_{2n+1}-\mu)\right\rangle_t'' \\
&+ \beta_1\sum_{n+1\le k\le 2n}\mathbf{E}\left\langle f\epsilon_k(e_- -\mu)\right\rangle_t'' - n\beta_1\mathbf{E}\left\langle f\epsilon_{2n+3}(e_- -\mu)\right\rangle_t''.
\end{aligned}
$$

The analysis of the problem proceeds similarly. So we have to prove that the functions $\psi(t)$, $\eta(t)$ and

$$\mathbf{E}\left\langle\left(\frac{1}{N}\sum_{i\leq N-1}\sigma_i\left\langle\sigma_i\right\rangle_- - q\right)^2\right\rangle_t''$$

verify hypothesis (2.33). We will verify this just for the last function, considering that for $\psi(t)$ and $\eta(t)$ the previous reasoning holds. In this case, we have that for $t=0$, $\mathbf{E}\left\langle\cdot\right\rangle_0'' = \mathbf{E}\left\langle\cdot\right\rangle_0 = \nu_0(\cdot)$, and we obtain

$$\begin{aligned}
\mathbf{E}\left\langle\left(\frac{1}{N}\sum_{i\leq N-1}\sigma_i\left\langle\sigma_i\right\rangle_- - q\right)^2\right\rangle_0'' &= \nu_0\left(\left(\frac{1}{N}\sum_{i\leq N-1}\sigma_i\left\langle\sigma_i\right\rangle_- - q\right)^2\right)\\
&= \frac{1}{N^2}\nu_0\left(\left(\sum_{i\leq N-1}\sigma_i\left\langle\sigma_i\right\rangle_-\right)^2\right)\\
&- 2q\nu_0\left(\frac{1}{N}\sum_{i\leq N-1}\sigma_i\left\langle\sigma_i\right\rangle_-\right) + q^2.
\end{aligned}$$

Notice that

$$\nu_0\left(\frac{1}{N}\sum_{i\leq N-1}\sigma_i\left\langle\sigma_i\right\rangle_-\right) = \nu_0(R_{1,2}^-) = \nu(R_{1,2}) + \frac{K}{N} = q + \frac{K}{N},$$

thanks to (2.26) and Theorem 2.3.11. Besides we have

$$\begin{aligned}
\frac{1}{N^2}\nu_0\left(\left(\sum_{i\leq N-1}\sigma_1\left\langle\sigma_i\right\rangle_-\right)^2\right) &= \nu_0(R_{1,2}^-R_{1,3}^-)\\
&\leq \left(\nu_0((R_{1,2}^-)^2)\right)^{1/2}\left(\nu_0((R_{1,3}^-)^2)\right)^{1/2}\\
&= q^2 + \frac{K}{N},
\end{aligned}$$

and so

$$\mathbf{E}\left\langle\left(\frac{1}{N}\sum_{i\leq N-1}\sigma_i\left\langle\sigma_i\right\rangle_- - q\right)^2\right\rangle_0'' \leq \frac{K}{N}.$$

On the other hand, it suffices to proceed as we did in Step 2 for $\psi(t)$ and $\eta(t)$, in order to obtain

$$\begin{aligned}
|\varphi_2'(0)| &\leq (1+4q+3\hat{q})\beta_2^2|\nu_0(R_{1,2}^- - q)| + 8\hat{q}\beta_2^2\left|\nu_0\left(\frac{\|c\|^2}{N} - q\right)\right| \\
&+ (8q+12\hat{q})\beta_2^2\left|\nu_0\left(\frac{1}{N}\sum_{i\leq N-1}\sigma_i\left\langle\sigma_i\right\rangle_- - q\right)\right| \\
&+ 2(\mu+\hat{\mu})\beta_1|\nu_0(m_1^- - \mu)| + 2(\mu+\hat{\mu})\beta_1|\nu_0(e_- - \mu)| \\
&\leq \frac{K}{N}
\end{aligned}$$

□

Corollary 2.3.18 *For β_1 and β_2 satisfying (2.15) we have*

$$\boldsymbol{E}\left(\langle\sigma_1\rangle - \langle\sigma_1\rangle_-\right)^2 \leq \frac{K}{N}.$$

Proof. We will proceed as in the previous proof. Set

$$\begin{aligned}
\hat{U}(t) &= \langle\sigma_1\rangle_t \\
\hat{V}(t) &= \langle\sigma_1\rangle_- = \frac{\langle Av\sigma_1\exp(\epsilon(g_t(c)+h))\rangle_-}{\langle\cosh(g_t(c)+h)\rangle_-}
\end{aligned}$$

and

$$\hat{\varphi}(t) = \mathbf{E}\left(\hat{U}(t) - \hat{V}(t)\right)^2.$$

Clearly $\hat{\varphi}(0) = 0$, and we only need to prove that

$$\begin{aligned}
|\hat{\varphi}(1) - \hat{\varphi}(0)| &\leq |\hat{\varphi}_1(1) - \hat{\varphi}_1(0)| + 2|\hat{\varphi}_2(1) - \hat{\varphi}_2(0)| + |\hat{\varphi}_3(1) - \hat{\varphi}_3(0)| \\
&\leq \frac{K}{N},
\end{aligned}$$

where we define $\hat{\varphi}_1(t)$, $\hat{\varphi}_2(t)$ e $\hat{\varphi}_3(t)$ as in (2.34). Using symmetry we have

$$\hat{\varphi}_1(t) = \nu_t(\sigma_1^1\sigma_1^2), \quad \hat{\varphi}_2(t) = \mathbf{E}\left\langle\sigma_1^1\sigma_1^2\right\rangle_t'', \quad \hat{\varphi}_3(t) = \mathbf{E}\left\langle\sigma_1^1\sigma_1^2\right\rangle_t'.$$

Notice that $\hat{\varphi}_3(t)$ does not depend on t. So $\hat{\varphi}_3{}'(0) = 0$ and we just have to prove that

$$|\hat{\varphi}_1{}'(0)| \leq \frac{K}{N}, \qquad |\hat{\varphi}_2{}'(0)| \leq \frac{K}{N}.$$

Using (2.13) and Lemma 2.3.1 we have

$$\begin{aligned}\hat{\varphi}_1{}'(0) &= \beta_2^2 q\nu_0(\sigma_1^1\sigma_1^2(R_{1,2}^- - q)) - 4\beta_2^2 q\nu_0(\sigma_1^1\sigma_1^2(R_{1,3}^- - q)) \\ &+ 3\beta_2^2 q\nu_0(\sigma_1^1\sigma_1^2(R_{3,4}^- - q)) + 2\beta_1\mu\nu_0(\sigma_1^1\sigma_1^2(m_1^- - \mu)) \\ &- 2\beta_1\mu\nu_0(\sigma_1^1\sigma_1^2(m_3^- - \mu)).\end{aligned}$$

Observe that from (2.23) we have

$$\begin{aligned}|\nu_0(\sigma_1^1\sigma_1^2(R_{l,l'}^- - q)) - \nu(\sigma_1^1\sigma_1^2(R_{l,l'} - q))| &\leq \frac{K}{N} \\ |\nu_0(\sigma_1^1\sigma_1^2(m_1^- - \mu)) - \nu(\sigma_1^1\sigma_1^2(m_1 - \mu))| &\leq \frac{K}{N},\end{aligned}$$

and using symmetry we can write

$$\begin{aligned}\nu(\sigma_1^1\sigma_1^2(R_{l,l'} - q)) &= \nu(R_{1,2}(R_{l,l'} - q)) \\ &= \nu((R_{1,2} - q)(R_{l,l'} - q)) + q\nu(R_{l,l'} - q). \quad (2.36)\end{aligned}$$

Thus

$$\begin{aligned}\hat{\varphi}_1{}'(0) &= \beta_2^2 q\nu(\sigma_1^1\sigma_1^2(R_{1,2} - q)) - 4\beta_2^2 q\nu(\sigma_1^1\sigma_1^2(R_{1,3} - q)) \\ &+ 3\beta_2^2 q\nu(\sigma_1^1\sigma_1^2(R_{3,4} - q)) + 2\beta_1\mu\nu(\sigma_1^1\sigma_1^2(m_1 - \mu)) \\ &- 2\beta_1\mu\nu(\sigma_1^1\sigma_1^2(m_3 - \mu)) + \frac{K}{N} \\ &= \beta_2^2 q\nu\left((R_{1,2} - q)^2\right) - 4\beta_2^2 q\nu\left((R_{1,2} - q)(R_{1,3} - q)\right) \\ &+ 3\beta_2^2 q\nu\left((R_{1,2} - q)(R_{3,4} - q)\right) + 2\beta_1\mu\nu\left((R_{1,2} - q)(m_1 - \mu)\right) \\ &- 2\beta_1\mu\nu\left((R_{1,2} - q)(m_3 - \mu)\right) + \frac{K}{N}.\end{aligned}$$

Then, using Cauchy-Schwarz's inequality and Theorem 2.3.7

$$|\hat{\varphi}_1{}'(0)| \leq \frac{K}{N}.$$

To prove that $\hat{\varphi}_2{}'(0) \leq \frac{K}{N}$ we proceed in a similar way. Deriving and using Lemma 2.3.1 we have

$$\begin{aligned}\hat{\varphi}_2{}'(0) &= \beta_2^2 q\nu_0(f(R_{1,2}^- - q)) - 4\beta_2^2 q\nu_0(f(R_{1,5}^- - q)) + 3\beta_2^2 q\nu_0(f(R_{5,6}^- - q)) \\ &- 3\beta_2^2 q\nu_0\left(f\left(\frac{1}{N}\sum_{i\leq N-1}\sigma_i^5\left\langle\sigma_i\right\rangle_- - q\right)\right) \\ &+ 3\beta_2^2 q\nu_0\left(f\left(\frac{1}{N}\sum_{i\leq N-1}\sigma_i^6\left\langle\sigma_i\right\rangle_- - q\right)\right) \\ &+ 2\beta_1\mu\nu_0(f(m_1^- - \mu)) - 2\beta_1\mu\nu_0(f(m_5^- - \mu)),\end{aligned}$$

so using symmetry we have $|\hat{\varphi}_2{}'(0)| \leq \frac{K}{N}$.

$\square$

We can now prove Theorem 2.3.13.

Proof of Theorem 2.3.13. We will proceed by induction over n considering that in Lemma 2.3.15 we proved the case $n = 1$. We suppose that it is true for n and we will prove it for $n + 1$.
Let $Y_i = \beta_2 z_i \sqrt{q} + \beta_1 \mu + h$ and $Y_i^- = \beta_2^- z_i \sqrt{q_-} + \beta_1^- \mu_- + h$. We have

$$\begin{aligned}
\sum_{i \leq n} \mathbf{E}\left((\langle\sigma_i\rangle - \tanh(Y_i))^2\right) &\leq K \sum_{i \leq n} \mathbf{E}\left(\langle\sigma_i\rangle - \langle\sigma_i\rangle_-\right)^2 \\
&+ K \sum_{i \leq n} \mathbf{E}\left(\langle\sigma_i\rangle_- - \tanh(Y_i^-)\right)^2 \\
&+ K \sum_{i \leq n} \mathbf{E}\left(\tanh(Y_i^-) - \tanh(Y_i)\right)^2.
\end{aligned}$$

From Corollary 2.3.18 we obtain

$$\sum_{i \leq n} \mathbf{E}\left(\langle\sigma_i\rangle - \langle\sigma_i\rangle_-\right)^2 \leq \frac{K(n)}{N}$$

and applying the induction hypothesis to the system with $N - 1$ spins and Hamiltonian

$$\begin{aligned}
-H_{N-1}(\rho) &= \frac{\beta_1^-}{2(N-1)} \left(\sum_{i \leq N-1} \sigma_i\right)^2 \\
&+ \frac{\beta_2^-}{\sqrt{N-1}} \left(\sum_{i<j \leq N-1} g_{i,j}\sigma_i\sigma_j\right) + h \sum_{i \leq N-1} \sigma_i
\end{aligned}$$

we have

$$\sum_{i \leq n} \mathbf{E}\left(\langle\sigma_i\rangle_- - \tanh(Y_i^-)\right)^2 \leq \frac{K}{N}.$$

Using inequalities (2.28) and Lemma 2.3.14 we can write

$$\begin{aligned}
\mathbf{E}\left(\tanh(Y_i^-) - \tanh(Y_i)\right)^2 &\leq 2\mathbf{E}\left(z_i(\beta_2^- \sqrt{q_-} - \beta_2\sqrt{q})\right)^2 \\
&+ 2\mathbf{E}(\beta_1^- \mu_- - \beta_1 \mu)^2 \\
&\leq \frac{K}{N},
\end{aligned}$$

and so

$$\sum_{i\le n}\mathbf{E}\left((\langle\sigma_i\rangle-\tanh(Y_i))^2\right)\le\frac{K}{N}.$$

To conclude we use Lemma 2.3.15 choosing $z_{n+1}=z$:

$$\sum_{i\le n}\mathbf{E}\left((\langle\sigma_i\rangle-\tanh(Y_i))^2\right)+\mathbf{E}\left((\langle\sigma_{n+1}\rangle-\tanh(\beta_2 z_{n+1}\sqrt{q}+\beta_1\mu+h))^2\right)\le\frac{K}{N}.$$

Let us remark that z is probabilistically independent from $(z_i)_{i\le n}$ because these are functions of the random variables $(g_{i,j})_{i<j\le N-1}$.

$\square$

2.3.7 Second order moments computations

A first step through central limit results is to give a more precise value to

$$\nu\left((R_{1,2}-q)^2\right),\quad \nu\left((m_1-\mu)^2\right),\quad \nu\left((R_{1,2}-q)(m_1-\mu)\right).$$

The estimates are established by our next Theorem.

Theorem 2.3.19

$$\begin{aligned}
\left|\nu\left((R_{1,2}-q)^2\right)-\frac{1}{N}(A_1+2B_1+E_1)\right| &\le \frac{K}{N^{3/2}},\\
\left|\nu\left((m_1-\mu)^2\right)-\frac{1}{N}(D_1+G_1)\right| &\le \frac{K}{N^{3/2}},\\
\left|\nu\left((R_{1,2}-q)(m_1-\mu)-\frac{1}{N}(C_1+F_1)\right)\right| &\le \frac{K}{N^{3/2}},
\end{aligned}$$

where A_1, B_1, C_1, D_1, E_1, F_1 are constants that we will define later.
We need to introduce some new notations and definitions.

Definition 2.3.20 *Set*

$$T_{l,l'}=\frac{(\sigma^l-b)\cdot(\sigma^{l'}-b)}{N};\quad T_l=\frac{(\sigma^l-b)\cdot b}{N};\quad T=\frac{b\cdot b}{N}-q;$$

$$U_l=m_l-\langle m_1\rangle;\qquad U=\langle m_1\rangle-\mu$$

where $b=\langle\sigma\rangle=(\langle\sigma_i\rangle)_{i\le N}$. Hence we have

$$R_{l,l'}-q=T_{l,l'}+T_l+T_{l'}+T$$

and

$$m_l-\mu=U_l+U.$$

Definition 2.3.21 *Set* $A = \beta_2^2(1-4q+3\hat{q}), B = \beta_1(\mu-\hat{\mu}), C = \beta_1(1-q), D = -2\beta_2^2(\mu-\hat{\mu}), E = \beta_2^2(1-2q+\hat{q}), F = \beta_1(\mu+\mu q-2\hat{\mu}), G = \beta_2^2(\hat{q}-q^2)$.

The following remark is fundamental for the next propositions

Remark 2.3.22 *Using symmetry we can prove that*

$$\begin{array}{l}
\forall l, l', k, k'; (l, l') \neq (k, k'), \qquad |\nu(T_{l,l'}T_{k,k'})| = 0, \\
\forall l, l', k, \qquad |\nu(T_{l,l'}T_k)| = 0, \\
\forall l, l', k, \qquad |\nu(T_{l,l'}U_k)| = 0, \\
\forall l, l', \qquad |\nu(T_{l,l'}T)| = 0, \\
\forall l, l', \qquad |\nu(T_{l,l'}U)| = 0, \\
\forall l, l'; l \neq l', \qquad |\nu(T_l T_{l'})| = 0, \\
\forall l, k; l \neq k, \qquad |\nu(T_l U_k)| = 0, \\
\forall l, \qquad |\nu(T_l T)| = 0, \\
\forall l, \qquad |\nu(T_l U)| = 0, \\
\forall k, k'; k \neq k', \qquad |\nu(U_k U_{k'})| = 0, \\
\forall k, \qquad |\nu(U_k T)| = 0, \\
\forall k, \qquad |\nu(U_k U)| = 0.
\end{array}$$

Proposition 2.3.23 *Let* $f^- : \Sigma_{N-1}^n \to \mathbb{R}$. *Then*

$$\nu_0'\left((\epsilon_1-\epsilon_2)(\epsilon_3-\epsilon_4)f^-\right) = E\nu_0\left(f^-(R_{1,3}^- - R_{1,4}^- - R_{2,3}^- + R_{2,4}^-)\right)$$

and

$$\begin{aligned}
&\left|\nu\left((\epsilon_1-\epsilon_2)(\epsilon_3-\epsilon_4)f^-\right) - E\nu\left(f^-(R_{1,3} - R_{1,4} - R_{2,3} + R_{2,4})\right)\right| \\
&\leq \frac{K(n)}{N}\left(\nu\left((f^-)^4\right)\right)^{1/4}.
\end{aligned} \tag{2.37}$$

Proof. Set

$$f = (\epsilon_1-\epsilon_2)(\epsilon_3-\epsilon_4)f^-.$$

Using Proposition 2.3.2 and considering $n \geq 4$, we have

$$\begin{aligned}
\nu_0'\left((\epsilon_1-\epsilon_2)(\epsilon_3-\epsilon_4)f^-\right) &= \beta_2^2 \sum_{1\leq l<l'\leq n} \nu_0\left(f\epsilon_l\epsilon_{l'}(R_{l,l'}^- - q)\right) \\
&- n\beta_2^2 \sum_{l\leq n} \nu_0\left(f\epsilon_l\epsilon_{n+1}(R_{l,n+1}^- - q)\right) \\
&+ \beta_2^2 \frac{n(n+1)}{2} \nu_0\left(f\epsilon_{n+1}\epsilon_{n+2}(R_{n+1,n+2}^- - q)\right) \\
&+ \beta_1 \sum_{l\leq n} \nu_0\left(f\epsilon_l(m_l^- - \mu)\right) \\
&- n\nu_0(f\epsilon_{n+1}(m_{n+1}^- - \mu)).
\end{aligned}$$

Lemma 2.3.1 implies that

$$\nu_0\left(f\epsilon_l\epsilon_{l'}(R^-_{l,l'}-q)\right)=\nu_0\left((\epsilon_1-\epsilon_2)(\epsilon_3-\epsilon_4)\epsilon_l\epsilon_{l'}\right)\nu_0\left(f^-(R^-_{l,l'}-q)\right).$$

Notice that

$$a(l,l')=\nu_0\left((\epsilon_1-\epsilon_2)(\epsilon_3-\epsilon_4)\epsilon_l\epsilon_{l'}\right)=0$$

if $l<l'$ except the cases $l\in\{1,2\}\,,l'\in\{3,4\}$ when we have

$$a(1,3)=-a(1,4)=-a(2,3)=a(2,4)=1-2q+\hat{q}.$$

Lemma 2.3.1 also implies that, $\forall l\geq 1$,

$$\nu_0\left(f\epsilon_l(m^-_l-\mu)\right)=\nu_0\left(\epsilon_l(\epsilon_1-\epsilon_2)(\epsilon_3-\epsilon_4)\right)\nu_0\left(f^-(m^-_l-\mu)\right)=0.$$

Then

$$\begin{aligned}\nu_0'(f) &= \beta_2^2(1-2q+\hat{q})\nu_0\left(f^-(R^-_{1,3}-R^-_{2,3}-R^-_{1,4}+R^-_{2,4})\right)\\ &= E\nu_0\left(f^-(R^-_{1,3}-R^-_{2,3}-R^-_{1,4}+R^-_{2,4})\right).\end{aligned}$$

Using (2.24) we have

$$\begin{aligned}&|\nu_0\left((\epsilon_1-\epsilon_2)(\epsilon_3-\epsilon_4)f^-\right)-E\nu_0\left(f^-(R^-_{1,3}-R^-_{2,3}-R^-_{1,4}+R^-_{2,4})\right)|\\ &\leq\frac{K}{N}\left(\nu(f^2)\right)^{1/2}\end{aligned}$$

and to conclude the proof we only need to use the following inequalities

$$\begin{aligned}&|\nu_0\left(f^-(R^-_{1,3}-R^-_{2,3}-R^-_{1,4}+R^-_{2,4})\right)-\nu_0\left(f^-(R_{1,3}-R_{2,3}-R_{1,4}+R_{2,4})\right)|\\ &\leq\frac{K}{N}\nu(|f^-|),\end{aligned}$$

$$\begin{aligned}&|\nu_0\left(f^-(R_{1,3}-R_{2,3}-R_{1,4}+R_{2,4})\right)-\nu\left(f^-(R_{1,3}-R_{2,3}-R_{1,4}+R_{2,4})\right)|\\ &\leq\frac{K}{N}\left(\nu(|f^-|^4)\right)^{1/4}.\end{aligned}$$

□

With similar arguments we can also prove the next two propositions

Proposition 2.3.24 *Let $f^-:\Sigma^n_{N-1}\to\mathbb{R}$. Then*

$$\begin{aligned}\nu_0'\left((\epsilon_1-\epsilon_2)\epsilon_3 f^-\right) &= \beta_2^2(1-q)\nu_0\left(f^-(R^-_{1,3}-R^-_{2,3})\right)\\ &+ \beta_2^2(q-\hat{q})\sum_{4\leq l\leq n}\nu_0\left(f^-(R^-_{1,l}-R^-_{2,l})\right)\\ &- n\beta_2^2(q-\hat{q})\nu_0\left(f^-(R^-_{1,n+1}-R^-_{2,n+1})\right)\\ &+ \beta_1(\mu-\hat{\mu})\nu_0\left(f^-(m^-_1-m^-_2)\right). \qquad (2.38)\end{aligned}$$

Besides, if f^- does not depend on ρ^3, we have

$$\begin{aligned}&\Big|\nu\left((\epsilon_1-\epsilon_2)\epsilon_3 f^-\right)-\beta_2^2(1-4q+3\hat{q})\nu(f^-(R_{1,3}-R_{2,3}))\\&\quad-\beta_2^2(q-\hat{q})\sum_{4\le l\le n}\nu(f^-(R_{1,l}-R_{2,l}-R_{1,n+1}+R_{2,n+1}))\\&\quad-\beta_1(\mu-\hat{\mu})\nu(f^-(m_1-m_2))\Big|\le\frac{K}{N}\left(\nu\left((f^-)^4\right)\right)^{1/4}.\end{aligned}\tag{2.39}$$

Proposition 2.3.25 *Let $f^-:\Sigma_{N-1}^n\to\mathbb{R}$. Then*

$$\begin{aligned}\nu_0'\left((\epsilon_1-\epsilon_2)f^-\right) &= \beta_2^2(\mu-\hat{\mu})\sum_{3\le l\le n}\nu_0\left(f^-(R^-_{1,l}-R^-_{2,l})\right)\\&- n\beta_2^2(\mu-\hat{\mu})\nu_0\left(f^-(R^-_{1,n+1}-R^-_{2,n+1})\right)\\&+ \beta_1(1-q)\nu_0\left(f^-(m_1^--m_2^-)\right).\end{aligned}\tag{2.40}$$

Besides, if f^- does not depend on ρ^3, we have

$$\begin{aligned}&\Big|\nu\left((\epsilon_1-\epsilon_2)f^-\right)+2\beta_2^2(\mu-\hat{\mu})\nu(f^-(R_{1,3}-R_{2,3}))\\&\quad-\beta_2^2(\mu-\hat{\mu})\sum_{4\le l\le n}\nu(f^-(R_{1,l}-R_{2,l}-R_{1,n+1}+R_{2,n+1}))\\&\quad-\beta_1(1-q)\nu(f^-(m_1-m_2))\Big|\le\frac{K}{N}\left(\nu\left((f^-)^4\right)\right)^{1/4}.\end{aligned}\tag{2.41}$$

In fact, if f^- does not depend on ρ^3, using symmetry we can write

$$\nu_0(f^-(R^-_{1,n+1}-R^-_{2,n+1}))=\nu_0(f^-(R^-_{1,3}-R^-_{2,3})).$$

Proposition 2.3.26 *If β_1 and β_2 satisfy hypothesis (2.15) we have*

$$\left|\nu(T_{1,2}^2)-\frac{A_1}{N}\right|\le\frac{K}{N^{3/2}}$$

where

$$A_1=\frac{E}{\beta_2^2(1-E)}.$$

Proof. Using again symmetry we can write

$$\begin{aligned}\nu(T_{1,2}^2) &= \nu\left(\frac{(\sigma^1-\sigma^2)\cdot(\sigma^3-\sigma^4)}{N}\frac{(\sigma^1-\sigma^5)\cdot(\sigma^3-\sigma^6)}{N}\right)\\&= \frac{1}{N}\nu\left((\epsilon_1-\epsilon_2)(\epsilon_3-\epsilon_4)(\epsilon_1-\epsilon_5)(\epsilon_3-\epsilon_6)\right)\\&+ \nu\left((\epsilon_1-\epsilon_2)(\epsilon_3-\epsilon_4)f^-\right)\end{aligned}\tag{2.42}$$

where

$$f^- = R^-_{1,3} - R^-_{5,3} - R^-_{1,6} + R^-_{5,6}.$$

Moreover we have

$$\nu_0\left((\epsilon_1-\epsilon_2)(\epsilon_3-\epsilon_4)(\epsilon_1-\epsilon_5)(\epsilon_3-\epsilon_6)\right) = 1-2q+\hat{q}$$

and

$$\left(\nu\left((f^-)^4\right)\right)^{1/4} \le \frac{K}{\sqrt{N}}.$$

Using (2.37) and (2.42)

$$\begin{aligned}
&\nu\left(f^-(R_{1,3}-R_{5,3}-R_{1,6}+R_{5,6})\right)\\
&\quad = \frac{1}{E}\nu\left((\epsilon_1-\epsilon_5)(\epsilon_3-\epsilon_6)f^-\right) + \frac{K}{N^{3/2}}\\
&\quad = -\frac{1}{NE}\nu\left((\epsilon_1-\epsilon_2)(\epsilon_3-\epsilon_4)(\epsilon_1-\epsilon_5)(\epsilon_3-\epsilon_6)\right)\\
&\qquad +\frac{1}{E}\nu(T^2_{1,2}) + \frac{K}{N^{3/2}}\\
&\quad = -\frac{1-2q+\hat{q}}{NE} + \frac{1}{E}\nu(T^2_{1,2}) + \frac{K}{N^{3/2}}.
\end{aligned}$$

We conclude noting that

$$\left|\nu\left(f^-(R_{1,3}-R_{5,3}-R_{1,6}+R_{5,6})\right) - \nu\left((R_{1,3}-R_{5,3}-R_{1,6}+R_{5,6})^2\right)\right| \le \frac{K}{N^{3/2}}$$

implies

$$\left|\frac{1-E}{E}\nu(T^2_{1,2}) - \frac{1-2q+\hat{q}}{NE}\right| \le \frac{K}{N^{3/2}}$$

□

In a similar way we can prove all the next propositions that yield to the proof of Theorem 2.3.19

Proposition 2.3.27 *If β_1 and β_2 satisfy hypothesis (2.15) we have*

$$\left|\nu(T^2_1) - \frac{B_1}{N}\right| \le \frac{K}{N^{3/2}},$$

where

$$B_1 = \frac{(1-C)(q-\hat{q}) + BE(\mu-\hat{\mu})}{(1-E)\left[(1-A)(1-C)-BD\right]}.$$

Proposition 2.3.28 *If β_1 and β_2 satisfy hypothesis (2.15) we have*

$$\left|\nu(U_1T_1) - \frac{C_1}{N}\right| \leq \frac{K}{N^{3/2}},$$

where

$$\begin{aligned} C_1 &= \frac{(\mu-\hat{\mu})(1-A)+D(q-\hat{q})}{(1-E)\left[(1-A)(1-C)-BD\right]} \\ &- \frac{BD(\mu-\hat{\mu})}{(1-C)\left[(1-A)(1-C)-BD\right]}. \end{aligned}$$

Proposition 2.3.29 *If β_1 and β_2 satisfy hypothesis (2.15) we have*

$$\left|\nu(U_1^2) - \frac{D_1}{N}\right| \leq \frac{K}{N^{3/2}},$$

where

$$\begin{aligned} D_1 &= \frac{(1-q)(1-A)+D(\mu-\hat{\mu})}{\left[(1-A)(1-C)-BD\right]} \\ &- \frac{BD^2(\mu-\hat{\mu})}{(1-C)^2\left[(1-A)(1-C)-BD\right]}. \end{aligned}$$

Proposition 2.3.30 *If β_1 and β_2 satisfy hypothesis (2.15) we have*

$$\left|\nu(T^2) - \frac{E_1}{N}\right| \leq \frac{K}{N^{3/2}}$$

and

$$\left|\nu(UT) - \frac{F_1}{N}\right| \leq \frac{K}{N^{3/2}},$$

where E_1 is such that

$$\begin{aligned} \left[(1-A)(1-C)-FD\right]E_1 &= (\hat{q}-q^2)(1-C) \\ &+ \beta_2^2\left((1-C)(\hat{q}-q^2)+2F(\hat{\mu}-\mu q)\right)A_1 \\ &+ 2\beta_2^2\left((1-C)(q^2+2q-3\hat{q}+\frac{2F^2}{\beta_1})\right)B_1 \\ &+ 2\beta_1\left((1-C)(\hat{\mu}-\mu q)+2F(q-\mu^2)\right)C_1 \end{aligned}$$

and F_1 such that

$$\begin{aligned} (1-C)F_1 &= \hat{\mu}-\mu q+\beta_2^2(\hat{\mu}-\mu q)A_1+2\beta_2^2(\mu+\mu q-2\hat{\mu})B_1 \\ &+ 2\beta_1(q-\mu^2)C_1+\beta_2^2(\hat{\mu}-\mu)E_1. \end{aligned}$$

Proposition 2.3.31 *If β_1 and β_2 satisfy hypothesis (2.15) we have*

$$\left|\nu(U^2) - \frac{G_1}{N}\right| \leq \frac{K}{N^{3/2}}$$

where G_1 is such that

$$\begin{aligned}(1-C)G_1 &= q - \mu^2 + \beta_2^2(\mu + \mu q - 2\hat{\mu})C_1 \\ &+ \beta_1(q-\mu^2)D_1 + \beta_2^2(\hat{\mu} - \mu)F_1.\end{aligned}$$

Chapter 3

Polymers

3.1 Introduction

3.1.1 Background and motivation

Models for directed polymers in a random environment have been introduced in the physical literature [15, 20, 21, 26] for two main reasons. First, they provide a reasonably realistic model of a particle under the influence of a random medium, for which a number of natural questions can be posed, in terms of the asymptotic behaviour for the path of the particle. The second point is that, in spite of the fact that polymers seem to be some more complicated objects than other disordered systems such as spin glasses, a lot more can be said about their behaviour in the low temperature regime, as pointed out in [17, 20]. At a mathematical level, after two decades of efforts, a substantial amount of information about different models of polymer is now available, either in discrete or continuous space settings (see [13, 25, 27] and [5, 23] respectively).

Directed polymers in random environment can be thought of as paths of stochastic processes interacting with a quenched disorder, depending on both time and space. Each path is weighted not only according to its length, but also according to the random impurities (disorder) that it meets along its route.

In order to simplify, we want to suppress entanglement and U-turns of the polymer, so we would like to deal with *self-avoiding walks*. In the lattice case, by this we mean a random walk that cannot visit again the sites it has already visited. The problem is that it is very difficult to deal with this kind of walks. To avoid this, we impose a simpler constraint: we work with *directed walks*, *i.e.* we force one of the coordinates to be strictly increasing.

Thus the polymer is supposed to live in $(1+d)$-dimensional lattice and to stretch in the direction of the first coordinate.

The most important question in these models is:

how does the disorder affect the global shape of the polymer?

The answer for the random walk-type models can be find for example in [12] and it is

- If $d \geq 3$ and β small enough, the impurities do not affect the global shape of the polymer (*weak disorder phase*).
- If either
  (i) $d \leq 2$ and $\beta \neq 0$ or
  (ii) $d \geq 3$ and β large enough
  then the impurities change drastically the global shape of the polymer (*strong disorder phase*).

It is also known [12, 13, 6] that strong disorder implies *localization*: the paths are pinned down to clouds of favourable impurities, while weak disorder implies *delocalization*. That is, the polymer localizes in the strong disorder regime in a few specific corridors, but spreads out in a diffusive way in the weak disorder regime. Similarly, the free energy essentially depends on the features of the relevant clouds, exhibits large fluctuation, as well as other thermodynamic quantities.

3.1.2 The models

Our work can be seen as a part of the global project interested in describing the polymer's asymptotic behaviour, beyond the spin glass case. Except for some toy models such as the REM or GREM [3, 30], little is known about the low temperature behaviour of the free energy for spin glasses systems, at least at a completely rigorous level. We shall see in this paper that polymer models are amenable to computations in this direction: we work to obtain some sharp estimates on the free energy of two different kind of polymers in continuous time, for which some scaling arguments seem to bring more information than in the discrete time setting. Here, in a strict polymer sense, time can also be interpreted as the length parameter of a directed polymer.

A word about random media appellations: we believe the term "random environment" normally implies that the underlying randomness is allowed to change over time; the appellation "random scenery" or "random landscape" is more specifically used for an environment that does not change over time;

the models we consider herein fall under the time-varying "environment" umbrella. We now give some brief specifics about these models.

(1) We first consider a Brownian polymer in a Gaussian environment: the polymer itself is modeled by a Brownian motion $b = \{b_t; t \geq 0\}$, defined on a complete filtered probability space $(\mathcal{C}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (P_b^x)_{x\in\mathbb{R}^d})$, where P_b^x stands for the Wiener measure starting from the initial condition x. The corresponding expected value is denoted by E_b^x, or simply by E_b when $x = 0$.

The random environment is represented by a centered Gaussian random field W indexed by $\mathbb{R}_+ \times \mathbb{R}^d$, defined on another independent complete probability space $(\Omega, \mathcal{G}, \mathbf{P})$. Denoting by $\mathbf{E}$ the expected value with respect to $\mathbf{P}$, the covariance structure of W is given by

$$\mathbf{E}\left[W(t,x)W(s,y)\right] = (t \wedge s) \cdot Q(x-y), \tag{3.1}$$

for a given homogeneous covariance function $Q : \mathbb{R}^d \to \mathbb{R}$ satisfying some regularity conditions that will be specified later on. In particular, the function $t \mapsto [Q(0)]^{-1/2} W(t,x)$ will be a standard Brownian motion for any fixed $x \in \mathbb{R}^d$; for every fixed $t \in \mathbb{R}_+$, the process $x \mapsto t^{-1/2}W(t,x)$ is a homogeneous Gaussian field on $\mathbb{R}^d$ with covariance function Q. Notice that the homogeneity assumption is made here for sake of readability, but could be weakened for almost all the results we will show. The interested reader can consult [18] for the types of tools needed for such generalizations.

Once b and W are defined, the polymer measure itself can be described as follows: for any $t > 0$, the energy of a given path (or configuration) b on $[0,t]$ is given by the *Hamiltonian*

$$-H_t(b) = \int_0^t W(ds, b_s). \tag{3.2}$$

A completely rigorous meaning for this integral will be given later, but for the moment, observe that for any fixed path b, $H_t(b)$ is a centered Gaussian random variable with variance $tQ(0)$. Based on this Hamiltonian, for any $x \in \mathbb{R}^d$, and a given constant β (interpreted as the inverse of the temperature of the system), we define our (random) polymer measure G_t^x (with $G_t := G_t^0$) as follows:

$$dG_t^x(b) = \frac{e^{-\beta H_t(b)}}{Z_t^x} dP_b^x(b), \quad \text{with} \quad Z_t^x = E_b^x\left[e^{-\beta H_t(b)}\right]. \tag{3.3}$$

(2) The second model we consider is the continuous time random walk on $\mathbb{Z}^d$ in a white noise potential, which can be defined similarly to the Brownian polymer above: the polymer is modeled by a continuous time random walk $\hat{b} = \{\hat{b}_t; t \geq 0\}$ on $\mathbb{Z}^d$, defined on a complete filtered probability space $(\hat{\mathcal{C}}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t\geq 0}, (\hat{P}^x_{\hat{b}})_{x\in\mathbb{Z}^d})$. The corresponding expected value will be denoted by $\hat{E}^x_{\hat{b}}$, or simply by $\hat{E}_{\hat{b}}$ when $x = 0$. Notice that $\hat{b}$ can be represented in terms of its jump times $\{\tau_i; i \geq 0\}$ and its positions $\{x_i; i \geq 0\}$ between the jumps, as $\hat{b}_t = \sum_{i\geq 0} x_i \mathbf{1}_{[\tau_i,\tau_{i+1})}(t)$. Then, under $\hat{P}_{\hat{b}}$, $\tau_0 = x_0 = 0$, the sequence $\{\tau_{i+1} - \tau_i; i \geq 0\}$ is i.i.d with common exponential law $\mathcal{E}(2d)$, and the sequence $\{x_i; i \geq 0\}$ is a nearest neighbour symmetric random walk on $\mathbb{Z}^d$.

In this context, the random environment $\hat{W}$ will be defined as a sequence $\{\hat{W}(., z); z \in \mathbb{Z}^d\}$ of Brownian motions, defined on another independent complete probability space $(\hat{\Omega}, \hat{\mathcal{G}}, \hat{\mathbf{P}})$. Just like in the Brownian case described above, the covariance structure we assume on $\hat{W}$ is of the following type:

$$\hat{\mathbf{E}}\left[\hat{W}(t,x)\hat{W}(s,y)\right] = [t \wedge s]\,\hat{Q}(x-y), \tag{3.4}$$

for a covariance function $\hat{Q}$ defined on $\mathbb{Z}^d$. Note that the case where $\hat{Q}\,(z) = 0$ for all z except $\hat{Q}\,(0) > 0$, is the case where Brownian motions in the family $\{\hat{W}\,(\cdot, z)\,; z \in \mathbb{Z}^d\}$ are independent, *i.e.* the case of space-time white noise. The Hamiltonian of our system can be defined formally similarly to the continuous case, as

$$-\hat{H}_t(\hat{b}) = \int_0^t \hat{W}(ds, \hat{b}_s).$$

Notice however that, since b is a piecewise constant function, the Hamiltonian $\hat{H}_t(\hat{b})$ can also be written as

$$-\hat{H}_t(\hat{b}) = \sum_{i=0}^{N_t} \hat{W}(\tau_{i+1}, x_i) - \hat{W}(\tau_i, x_i), \tag{3.5}$$

where N_t designates the number of jumps of $\hat{b}$ before time t, and $\tau_{N_t+1} = t$ by convention. Once the Hamiltonian $\hat{H}_t$ is defined, a Gibbs-type measure $\hat{G}_t$ can be introduced similarly to (3.3) in the Brownian case.

As mentioned before, our aim is to give some sharp estimates on the free energies $p(\beta)$ and $\hat{p}(\beta)$ of the two systems described above, for large β. The quantities of interest are defined asymptotically as

$$p(\beta) = \lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\log(Z_t)\right], \quad \text{and} \quad \hat{p}(\beta) = \lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\log(\hat{Z}_t)\right].$$

3.1.3 Related models

The models which we have discussed so far and that we are going to study along this chapter, have a number of related models in the literature. We now mention some of them.

- **Simple random walk.** Polymers are chemical compounds consisting of repeating units, called monomers. Following this definition we see that an important (and natural) example of an abstract polymer is given by a d-dimensional *random walk* where the increments are thought of monomers.
  To be more specific, we can consider the random walk $(\omega_n)_{n\geq 1}$ and we can suppose that the i.i.d. random variables $\left\{\eta(n,x); n,x \in \mathbb{N}\times\mathbb{Z}^d\right\}$ are the random impurities. Thus the energy of the model is given by
  $$-H_n(\omega) = \beta \sum_{j=1}^{n} \eta(j,\omega_j).$$
  This model has already been studied by many authors, see for example [5, 21, 26].

- **Gaussian random walk.** The Hamiltonian of this model takes the same form of the previous one, however here the random walk $(\omega_n)_{n\geq 1}$ is the summation of independent standard Gaussian random variables and the random environment has a certain mild correlation in x variables. The important benefit of this model is that with the Gaussian variables a Girsanov-type path transformation can be used.

- **Crossing Brownian motion in a soft Poissonian potential.** This model is studied in [36, 37, 38] and it is described in terms of Brownian motion and Poisson point. However the Brownian motion here is 'undirected', *i.e.* it travels through the Poisson points distributed in $\mathbb{R}^d$, not in space-time as ours.

3.1.4 The weak and strong disorder phase

We recall that in the *strong disorder phase* the presence of the random environment makes qualitative difference in the large time behaviour of the polymer, while in the *weak disorder phase* the random environment is irrelevant.

It is well-known (see *e.g.* [27] for the Brownian case) that the free energy $p(\beta)$, that we already defined as

$$p(\beta) = \lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\log Z_t\right],$$

is bounded from above by $Q(0)\beta^2/2$. It is then possible to separate a region of *weak disorder* from a region of *strong disorder* according to the value of $p(\beta)$: we will say that the polymer is in the weak disorder regime if $p(\beta) = Q(0)\beta^2/2$, while the strong disorder regime is defined by the strict inequality $p(\beta) < Q(0)\beta^2/2$. These two notions have some nice interpretations in terms of the behaviour of the particle under the Gibbs' measure (see *e.g.* [6, 14]), and it is expected, for any model of polymer in a random environment, that the strong disorder regime is attained whenever β is large enough. It is then natural to ask if one can obtain a sharper information than $p(\beta) < Q(0)\beta^2/2$ in the low temperature phase. Indeed, on the one hand, this may quantify in a sense how far we are from the weak disorder regime, and how much localization there is on our Gibbs' measure G_t.

In [27] Rovira and Tindel linked these definitions of weak and strong disorder regime with another relevant quantity for the study of disordered models: *the overlap*. For this model we can define the overlap as

$$\frac{1}{t}\int_0^1 \left\langle Q(b_s^1 - b_s^2)\right\rangle_s ds.$$

Observe that, since $Q(x)$ is usually a decreasing function of $|x|$, the last quantity really measures how close b_s^1 and b_s^2 are.

If we define the following process

$$V_t = Z_t \exp\left(-\frac{\beta^2 Q(0) t}{2}\right),$$

in terms of the partition function Z_t, then it is easily seen that V is a positive martingale, that converges almost surely. Set

$$V_\infty = \lim_{t\to\infty} V_t,$$

by Kolmogorov's 0-1 law we have

$$\mathbf{P}(V_\infty = 0) \in \{0, 1\}.$$

It is easy to prove that $V_\infty > 0$ implies $p(\beta) = \beta^2 Q(0)/2$ and hence a weak disorder type behaviour of the polymer. Thus we can adopt the following definition: we will say that the polymer is in the weak disorder regime if $V_\infty > 0$ almost surely, while it is in the strong disorder regime if $V_\infty = 0$ almost surely.

Rovira and Tindel were then able to relate the behaviour of V_t and of the overlap in the following way:

$$V_\infty > 0 \quad \text{a.s.} \Leftrightarrow \int_0^\infty \left\langle Q(b_s^1 - b_s^2) \right\rangle_s ds < \infty.$$

In other words, if the system is in the strong disorder phase, then localization occurs, namely

$$\int_0^\infty \left\langle Q(b_s^1 - b_s^2) \right\rangle_s ds = \infty,$$

coherently with what we said previously.

3.1.5 Summary of results

We now describe our main results. Our principal result in continuous space will be obtained in terms of the regularity of Q in a neighbourhood of 0. In particular, we shall assume some upper and lower bounds on Q of the form

$$c_0|x|^{2H} \le Q(0) - Q(x) \le c_1|x|^{2H}, \qquad \text{for all } x \text{ such that } |x| \in [0, r_0], \tag{3.6}$$

for a given exponent $H \in (0,1]$ and $r_0 > 0$. It should be noticed that condition (3.6) is equivalent to assuming that W has a specific almost-sure modulus of continuity in space, of order $|x|^H \log^{1/2}(1/|x|)$, *i.e.* barely failing to be H-Hölder continuous (see [31] for details). Then, under these conditions, we will get the following conclusions.

Theorem 3.1.1 *Assume that the function Q satisfies condition (3.6). Then the following hold true:*

1. *If $H \in [1/2, 1]$, we have for some constants $C_{0,d}$ and $C_{1,d}$ depending only on Q and d, for all $\beta \ge 1$,*

$$C_{0,d}\beta^{4/3} \le p(\beta) \le C_{1,d}\beta^{2-2H/(3H+1)}.$$

2. *If $H \in (0, 1/2]$, we have for some constants β_Q, $C'_{0,d}$, and $C'_{1,d}$ depending only on Q and d, for all $\beta \ge \beta_Q$,*

$$C'_{0,d}\beta^{2/(1+H)} \le p(\beta) \le C'_{1,d}\beta^{2-2H/(3H+1)}.$$

Corresponding almost sure results on $t^{-1}\mathbf{E}\left[\log(Z_t)\right]$ also hold, as seen in Corollary 3.1.3 and Proposition 3.2.1 below. Let us make a few elementary comments about the above theorem's bounds, which are also summarized in Figure 3.1.5. First of all, the exponent of β in those estimates is decreasing with H, which seems to indicate a stronger disorder when the Gaussian field W is smoother in space. Furthermore, in the case $H \in [1/2, 1]$, the gap between the two estimates decreases as H increases to 1; for $H = 1/2$, we get bounds with the powers of β equal to $4/3$ and $8/5$; and for $H = 1$, the bounds are $4/3$ and $3/2$. It should be noted that the case $H = 1/2$ is our least sharp result, while the case $H = 1$ yields the lowest power of β; one should not expect lower powers for any potential W even if W is so smooth that it is C^∞ in space: indeed, unless W is highly degenerate, the lower bound in (3.6) should hold with $H = 1$, while the upper bound will automatically be satisfied with $H = 1$. The case of small H is more interesting. Indeed, we can rewrite the lower and upper bounds above as

$$C'_{0,d}\beta^{2-2H+F(H)} \leq p(\beta) \leq C'_{1,d}\beta^{2-2H+G(H)}$$

where the functions F and G satisfy $F(x) = 2x^2 + O(x^3)$ and $G(x) = 6x^2 + O(x^3)$ for x near 0. We therefore see that the asymptotic β^{2-2H} is quite sharp for small H, but that the second order term in the expansion of the power of β for small H, while bounded, is always positive.

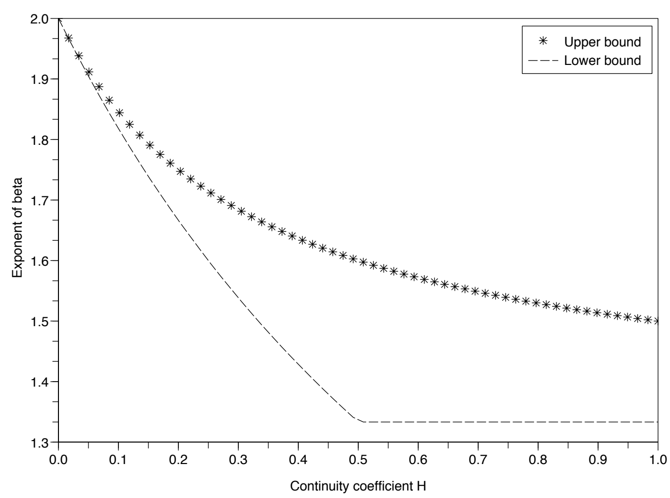

Figure 3.1: Exponent of β in function of H

Using ideas introduced in [18] to deal with spatially non-homogeneous media, it is possible to extend Theorem 3.1.1. The reader will check that the first of the following two corollaries is trivial to prove using the tools in this thesis. The second corollary requires techniques in [18], and can also be proved directly by using sub-Gaussian concentration results (see [35]). Neither corollary assumes that W is spatially homogeneous. One will note that no assertion on the existence of $p(\beta)$ is made in these corollaries, but that the first corollary already implies strong disorder for large β in the sense that $\limsup t^{-1}\mathbf{E}[\log(Z_t)] < \beta^2 Q(0)/2$. [18] can be consulted for conditions under which $p(\beta)$ exists even if W is not spatially homogeneous.

Corollary 3.1.2 *In the non homogeneous case, the following bounds are satisfied:*

- [Upper bound] *Assume that for some $r_0, c_1 > 0$, for all $x, y \in \mathbb{R}^d$ such that $|x-y| \leq r_1$, the spatial canonical metric of W is bounded above as*
$$\delta^2(x,y) := \mathbf{E}\left[(W(1,x) - W(1,y))^2\right] \leq c_1 |x-y|^{2H}.$$
Then, replacing $p(\beta)$ by $\limsup_{\beta\to\infty} t^{-1}\mathbf{E}[\log(Z_t)]$, the two upper bound results in Theorem 3.1.1 hold.

- [Lower bound] *Assume that for some $r_0, c_0 > 0$, for all $x, y \in \mathbb{R}^d$ such that $|x-y| \leq r_0$, we have*
$$\delta^2(x,y) := \mathbf{E}\left[(W(1,x) - W(1,y))^2\right] \geq c_0 |x-y|^{2H}.$$
Then, replacing $p(\beta)$ by $\liminf_{\beta\to\infty} t^{-1}\mathbf{E}[\log(Z_t)]$, the two lower bound results in Theorem 3.1.1 hold.

Corollary 3.1.3 *Under the hypotheses of Corollary 3.1.2, its conclusions also hold $\mathbf{P}$-almost surely with $\limsup_{\beta\to\infty} t^{-1}\mathbf{E}[\log(Z_t)]$ replaced by $\limsup_{\beta\to\infty} t^{-1}\log(Z_t)$, and similarly for the $\liminf$'s.*

Since our estimates become sharper as $H \to 0$, and also due to the fact that the behaviour of $p(\beta)$ is nearly quadratic in β for small H (*i.e.* approaching the weak disorder regime), we decided to explore further the region of logarithmic spatial regularity for W, in order to determine whether one ever leaves the strong disorder regime. Namely, we also examine the situation of a covariance function Q for which there exist positive constants c_0, c_1, and r_1 such that for all x with $|x| \leq r_1$,

$$c_0 \log^{-2\gamma}(1/|x|) \leq Q(0) - Q(x) \leq c_1 \log^{-2\gamma}(1/|x|), \qquad (3.7)$$

where γ is a given positive exponent. Assumption (3.7) implies that W is not spatially Hölder-continuous for any exponent $H \in (0,1]$. Moreover, the theory of Gaussian regularity implies that, if $\gamma > 1/2$, W is almost-surely continuous in space, with modulus of continuity proportional to $\log^{-\gamma+1/2}(1/|x|)$, while if $\gamma \leq 1/2$, W is almost-surely not uniformly continuous on any interval in space, and in fact is unbounded on any interval. We will then establish the following result, which is optimal, up to multiplicative constants.

Theorem 3.1.4 *Assume condition (3.7) where $\gamma > 0$. We have for a constant $D_{0,Q}$ that depends only on Q and $D_{1,d}$ depending on Q and d, for all β large enough,*

$$D_{0,Q}\beta^2 \log^{-2\gamma}(\beta) \leq p(\beta) \leq D_{1,d}\beta^2 \log^{-2\gamma}(\beta).$$

Besides giving a sharp result up to constants for the free energy $p(\beta)$, the last result will allow us to make a link between our Brownian model and the random walk polymer described by the Hamiltonian (3.5). Indeed, the following result will also be proved in the sequel.

Theorem 3.1.5 *Assume that $\hat{Q}(0) - \hat{Q}(2) > 0$, where $\hat{Q}$ has been defined at (3.4). Then the free energy $\hat{p}(\beta)$ of the random walk polymer $\hat{b}$ satisfies, for β large enough:*

$$D'_{0,d}\beta^2 \log^{-1}(\beta) \leq \hat{p}(\beta) \leq D'_{1,d}\beta^2 \log^{-1}(\beta), \tag{3.8}$$

for two constants $D'_{0,d}$ and $D'_{1,d}$ depending only on Q and d.

Relation (3.8) will be obtained here thanks to some simple arguments, which allow the extension to spatially inhomogeneous media.

In relation with the continuous space model considered at Theorem 3.1.4, we see that to obtain the same behaviour as with space-time white noise in discrete space, we need to use precisely the environment W in $\mathbb{R}^d$ with the logarithmic regularity corresponding to $\gamma = 1/2$ in (3.7). As mentioned before, this behaviour of W happens to be exactly at the threshold in which W becomes almost-surely discontinuous and unbounded on every interval. Nevertheless such a W is still function-valued. Hence, for the purpose of understanding the polymer partition function, there is no need to study the space-time white noise in continuous space, for which $W(t,\cdot)$ is not a bonafide function (only a distribution), and for which the meaning of Z_t itself is difficult to even define. Another way to interpret the coincidence of behaviours for "*space-time white noise in $\mathbb{R}_+ \times \mathbb{Z}^d$*" and for "$\gamma = 1/2$" is to say that both models for W are function-valued and exhibit spatial discontinuity: indeed, in discrete space, one extends $W(t,\cdot)$ to $\mathbb{R}^d$ by making it piecewise constant, in order to preserve independence. The fact that the limit in Theorem 3.1.4 depends on γ does prove, however, that the continuous-space polymer model under logarithmic regularity is richer than the discrete-space one.

Let us say a few words now about the methodology we have used in order to get our results. It is inspired by the literature on Lyapounov exponents for stochastic PDEs [7, 8, 9, 18, 33, 34]; our upper bound results rely heavily on the estimation of the supremum of some well-chosen Gaussian fields, using such results as Dudley's so-called entropy upper bound, and the Borell-Sudakov inequality (see [1] or [35]); our lower bound results are obtained more "by hand", by isolating very simple polymer configurations b or $\hat{b}$ which maximize the random medium's increments in the Hamiltonian $H_t(b)$ or $\hat{H}_t(\hat{b})$, and showing that these configurations contain enough weight to provide lower bounds. It turns out that these estimation procedures works better when the configuration b is simple enough, such as a piecewise constant or linear function. For the upper bound in the continuous case, a careful discretization of our Brownian path will thus have to be performed in order to get our main results; the resulting proof cannot exploit the discrete case itself because of the different nature of the discrete and continuous environments.

3.2 Preliminaries; the partition function

In this section, we will first recall some basic facts about the definition and the simplest properties of the partition functions Z_t and $\hat{Z}_t$ which have been already considered in the introduction. We will also give briefly some notions of Gaussian analysis which will be used later on.

We begin with basic information about the partition function of the Brownian polymer. Recall that W is a centered Gaussian field on $\mathbb{R}_+ \times \mathbb{R}^d$, defined by its covariance structure (3.1). The Hamiltonian $H_t(b)$ given by (3.2) can be defined more rigorously through a Fourier transform procedure: there exists (see *e.g.* [9] for further details) a centered Gaussian independently scattered $\mathbb{C}$-valued measure ν on $\mathbb{R}_+ \times \mathbb{R}^d$ such that

$$W(t,x) = \int_{\mathbb{R}_+\times\mathbb{R}^d} \mathbf{1}_{[0,t]}(s) e^{\imath ux} \nu(ds,du), \tag{3.9}$$

where the simple notation ux stands for the inner product $u\cdot x$ in $\mathbb{R}^d$. For every test function $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$, set now

$$\nu(f) \equiv \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(s,u)\nu(ds,du). \tag{3.10}$$

While the random variable $\nu(f)$ may be complex-valued, to ensure that it is real valued, it is sufficient to assume that f is of the form $f(s,u) = f_1(s)\, e^{\imath u f_2(s)}$ for real valued functions f_1 and f_2. Then the law of ν is defined by the following covariance structure: for any such test functions $f, g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$, we have

$$\mathbf{E}\left[\nu(f)\nu(g)\right] = \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(s,u)\overline{g(s,u)}\hat{Q}(du)ds, \tag{3.11}$$

where the finite positive measure $\hat{Q}$ is the Fourier transform of Q (see [32] for details).

From (3.9), we see that the Itô-stochastic differential of W in time can be understood as $W(ds,x) := \int_{u\in\mathbb{R}^d} e^{\imath u x}\nu(ds,du)$, or even, if the measure $\hat{Q}(du)$ has a density $f(u)$ with respect to the Lebesgue measure, which is typical, as

$$W(ds,x) := \int_{u\in\mathbb{R}^d} e^{\imath u x}\sqrt{f(u)}M(ds,du)$$

where M is a white-noise measure on $\mathbb{R}_+ \times \mathbb{R}^d$, *i.e.* a centered independently scattered Gaussian measure with covariance given by $\mathbf{E}\left[M(A)\, M(B)\right] = m_{Leb}(A \cap B)$ where m_{Leb} is Lebesgue's measure on $\mathbb{R}_+ \times \mathbb{R}^d$.

We can go back now to the definition of $H_t(b)$: invoking the representation (3.9), we can write

$$-H_t(b) := \int_0^t W(ds, b_s) = \int_0^t \int_{\mathbb{R}^d} e^{\imath u b_s}\nu(ds,du), \tag{3.12}$$

taking this expression as a definition of $H_t(b)$ for each fixed path b; it can be shown (see [9]) that the right hand side of the above relation is well defined for any Hölder continuous path b, by a L^2-limit procedure.

We use as the definition of the partition function Z_t^x, its expression in (3.3), and set its expectation under $\mathbf{P}$ as

$$p_t(\beta) := \frac{1}{t}\mathbf{E}\left[\log\left(Z_t^x\right)\right], \tag{3.13}$$

usually called the free energy of the system. It is easily seen that $p_t(\beta)$ is independent of the initial condition $x \in \mathbb{R}^d$, thanks to the spatial homogeneity of W. Thus, in the following chapters, x will be understood as 0 when not specified, and E_b, Z_t will stand for E_b^0, Z_t^0, etc... We summarize some basic results on $p_t(\beta)$ and Z_t established in [27].

Proposition 3.2.1 *For all $\beta > 0$ there exists a constant $p(\beta) > 0$ such that*

$$p(\beta) := \lim_{t\to\infty} p_t(\beta) = \sup_{t\geq 0} p_t(\beta). \tag{3.14}$$

Furthermore, the function p satisfies:

1. *The map $\beta \mapsto p(\beta)$ is a convex nondecreasing function on $\mathbb{R}_+$.*
2. *The following upper bound holds true:*
$$p(\beta) \leq \frac{\beta^2}{2} Q(0). \tag{3.15}$$
3. *$\mathbf{P}$-almost surely, we have*
$$\lim_{t\to\infty} \frac{1}{t} \log Z_t = p(\beta). \tag{3.16}$$

For the random walk polymer on $\mathbb{Z}^d$, the Hamiltonian $\hat{H}_t(\hat{b})$ is easier to define, and can be expressed in a simple way by (3.5). Recall then that $\hat{Z}_t, \hat{p}_t(\beta)$ are defined as:

$$\hat{Z}_t = \hat{E}_{\hat{b}} \left[e^{-\beta \hat{H}_t(\hat{b})} \right], \quad \text{and} \quad \hat{p}_t(\beta) = \hat{\mathbf{E}} \left[\log(\hat{Z}_t) \right].$$

Then, using the same kind of arguments as in [27] (see also [4]), we get the following:

Proposition 3.2.2 *The same conclusions as in Proposition 3.2.1 hold true for the random walk polymer $\hat{b}$.*

3.3 Estimates of the free energy: continuous space

In this section, we will proceed to the proof of Theorems 3.1.1 and, 3.1.4, by means of some estimates for some well-chosen Gaussian random fields.

The hypothesis we use guarantees that there is some $H \in (0, 1)$ such that W is no more than H-Hölder continuous in space. Accordingly, we define the homogeneous spatial *canonical metric* δ of W by

$$\delta^2(x-y) := \mathbf{E}\left[(W(1,x) - W(1,y))^2 \right] = 2\left(Q(0) - Q(x-y) \right), \tag{3.17}$$

for all $x, y \in \mathbb{R}^d$. Our hypotheses on δ translate immediately into statements about Q via this formula.

In our results below, we have also tried to specify the dependence of our constants on the dimension of the space variable. An interesting point in that respect is given in the lower bound of Subsection 3.3.2 below, which has to do with weak versus strong disorder in very high-dimensional cases.

3.3.1 Upper bound in the Brownian case

The upper bound in Theorem 3.1.1 follows immediately from the following proposition, which proves in particular that strong disorder holds for all $H \in (0, 1]$.

Proposition 3.3.1 *Assume that there exist a number $H \in (0, 1]$ and numbers c_1, r_1 such that for all $x, y \in \mathbb{R}^d$ with $|x - y| \le r_1$ we have*

$$\delta(x-y) < c_1 |x-y|^H . \tag{3.18}$$

Then there exists a constant C depending only on Q and a constant β_0 depending only on r_1 and d, such that for all $\beta \ge \beta_0$,

$$p(\beta) \le C d^{\frac{7H}{1+3H}} \beta^{\frac{2+4H}{1+3H}}.$$

Proof. Let us divide the proof in several steps:

Step 1: Strategy. From relation (3.14), we have

$$p(\beta) \le \limsup_{t\to\infty} p_t(\beta).$$

Our strategy is then to give an estimation of $p_t(\beta)$ for a *discretized* path $\tilde{b} \in \varepsilon\mathbb{Z}^d$ that stays close to b and proceeds only by jumps. Thanks to this substitution, and using Hölder's and Jensen's inequalities, we shall obtain

$$\begin{aligned}
\mathbf{E}[\log(Z_t)] &= \mathbf{E}\left[\log E_b\left[\exp\left(-\beta\left[H_t(b) - H_t(\tilde{b})\right]\right)\exp - \left(\beta H_t(\tilde{b})\right)\right]\right] \\
&\le \frac{1}{2}\mathbf{E}\left[\log E_b\left[\exp\left(-2\beta\left[H_t(b) - H_t(\tilde{b})\right]\right)\right]\right] \\
&\quad + \frac{1}{2}\mathbf{E}\left[\log E_b\left[\exp\left(-2\beta H_t(\tilde{b})\right)\right]\right] \\
&\le \frac{1}{2}\log E_b\left[\exp 2\beta^2 \int_0^t \left(\delta(b_s - \tilde{b_s})\right)^2 ds\right] \\
&\quad + \frac{1}{2}\mathbf{E}\left[\log E_b\left[\exp\left(-2\beta H_t(\tilde{b})\right)\right]\right].
\end{aligned} \tag{3.19}$$

Notice that the first term on the right-hand side represents the error made by considering the discretized path $\tilde{b}$ instead of b, but thanks to hypothesis (3.18) and the definition of $\tilde{b}$ we will easily control it.

Step 2: The discretized path. Let us describe now the discretized process we shall use in the sequel: we will approximate the Brownian path b with a path that stays in $\varepsilon\mathbb{Z}^d$, where ε is a small positive number. Let b^j be the j-th component of the d-dimensional path b. Let T_1^j be the first time that b^j exits the interval $(-\varepsilon, \varepsilon)$ and T_{i+1}^j be the first time after T_i^j that b^j exits $(b_{T_i^j} - \varepsilon, b_{T_i^j} + \varepsilon)$. So, for a fixed component j, the times $(T_{i+1}^j - T_i^j)_{i=0}^\infty$ are i.i.d. and the successive positions $x_m^j = b_{T_m^j}^j$, which are independent of the jump times, form a one-dimensional symmetric random walk on $\varepsilon\mathbb{Z}$ in discrete time.

Now let $(T_n)_{n=0}^\infty$ be the increasing sequence of all the $(T_m^j)_{j,m}$ and let $(x_n)_{n=0}^\infty$ be the nearest neighbor path in $\varepsilon\mathbb{Z}^d$ with $x_0 = 0$ whose j-th component takes the same step as x_m^j at time T_m^j. We define the *discretized path* $\tilde{b}$ as the path that jumps to site x_n at time T_n and it is constant between jumps.

Remark 3.3.2 *At any time s, each coordinate of $\tilde{b}_s$ is within ε of the corresponding one of b_s. So the distance separating the two paths is never more than $\varepsilon\sqrt{d}$. Thus we have, for all $s \geq 0$, $|b_s - \tilde{b}_s| \leq \varepsilon d^{1/2}$.*

Remark 3.3.3 *Thanks to Remark 3.3.2 we can now control the error term we have defined at relation (3.19). In fact, owing to Hypothesis (3.18), we have*

$$\frac{1}{2t}\log E_b\left[\exp 2\beta^2\int_0^t\left(\delta(b_s-\tilde{b_s})\right)^2 ds\right]$$
$$\leq \frac{1}{2t}\log E_b\left[\exp\left(2\beta^2C^2\int_0^t|b_s-\tilde{b_s}|^{2H}dt\right)\right] \leq C\beta^2\varepsilon^{2H}d^H,$$

where we recall that C is a constant depending on Q that can change from line to line.

Plugging this last inequality into (3.19), and defining

$$p_t^\varepsilon(\beta) = \frac{1}{t}\mathbf{E}\left[\log E_b\left[\exp\left(-2\beta H_t(\tilde{b})\right)\right]\right],$$

we have thus obtained the following estimate for $p_t(\beta)$:

$$p_t(\beta) \leq C\beta^2\varepsilon^{2H}d^H + \frac{1}{2}p_t^\varepsilon(\beta). \tag{3.20}$$

We shall try now to get some suitable bounds on $p_t^\varepsilon(\beta)$.

Step 3: Study of $p_t^\varepsilon(\beta)$. Let N_t^j be the number of jumps of the j-th component of $\tilde{b}$ up to time t. For a multi-index $k = (k_1, \cdots, k_d)$ let $|k| = k_1 + \cdots + k_d$, so the total number of jumps of $\tilde{b}$ up to time t is $|N_t| = N_t^1 + \cdots + N_t^d$. Denote by $\mathcal{S}(t,n)$ the simplex of all possible sequences of n jump times up to time t, namely

$$\mathcal{S}(t,n) = \{\mathbf{t} = (t_0, \cdots, t_n) : 0 = t_0 \leq \cdots \leq t_n \leq t\} . \tag{3.21}$$

The set of the first k_j jump times of the j-th component of $\tilde{b}$ is a point $(t_i^j)_{i=1}^{k_j}$ in $\mathcal{S}(t, k_j)$. Given the set of all jump times $\left\{t_i^j : j \in [1, \cdots, d] \,; i \in [1, \cdots, k_j]\right\}$, let $\left\{\tilde{t}_l : l \in [0, |k|+1]\right\}$ be the same set but ordered and with the convention $\tilde{t}_0 = 0$, $\tilde{t}_{|k|+1} = t$. And finally let $\tilde{x}_l$ be the value of $\tilde{b}$ between the two jump times $\tilde{t}_l$ and $\tilde{t}_{l+1}$. Denote by $\mathcal{P}_n$ the set of all such $\tilde{x}$, *i.e.* the set of all nearest-neighbor random walk paths of length k starting at the origin.

Then if we fix $|N_t| = |k|$, we can write

$$H_t(\tilde{b}) = X\left(|k|, (\tilde{t}_l)_{l=1}^{|k|}, (\tilde{x}_l)_{l=1}^{|k|}\right),$$

where

$$X\left(|k|, (\tilde{t}_l)_{l=1}^{|k|}, (\tilde{x}_l)_{l=1}^{|k|}\right) = \sum_{i=0}^{|k|} \left[W(\tilde{t}_{i+1}, \tilde{x}_i) - W(\tilde{t}_i, \tilde{x}_i)\right].$$

Thanks to these notations, we have

$$\begin{aligned} tp_t^\varepsilon(\beta) &= \mathbf{E}\left[\log E_b\left[\exp(-2\beta H_t(\tilde{b}))\right]\right] \\ &= \mathbf{E}\left[\log E_b\left[\exp\left(-2\beta X\left(|N_t|, (\tilde{t}_l)_{l=1}^{|N_t|}, (\tilde{x}_l)_{l=1}^{|N_t|}\right)\right)\right]\right]. \end{aligned}$$

So we can write the expectation with respect to b as:

$$\begin{aligned} E_b\left[\exp(-2\beta H_t(\tilde{b}))\right] = \sum_{n\geq 1} & E_b\left[\exp(-2\beta H_t(\tilde{b}))\Big| |N_t| \in [t\alpha(n-1), t\alpha n]\right] \\ & \times P_b\left[|N_t| \in [t\alpha(n-1), t\alpha n]\right]. \end{aligned}$$

The number of jumps of the discretized path $\tilde{b}$ in a given interval $[0,t]$ will play a crucial role in our optimization procedure. For a parameter $\alpha > 0$ which will be fixed later on, let us thus define

$$T_{n\alpha} = \left\{\left(k, \tilde{t}, \tilde{x}\right) : k \leq tn\alpha, \tilde{t} \in \mathcal{S}(t,k), \tilde{x} \in \mathcal{P}_k\right\}.$$

Then the following estimates will be essential for our future computations:

$$P_b\left[N_t^j > n\alpha t\right] \leq \exp\left(-\frac{t}{2}(\alpha n\varepsilon)^2 + t\alpha n\right) \tag{3.22}$$

$$\mathbf{E}\left[\sup_{T_{n\alpha}} X(k,\tilde{t},\tilde{x})\right] \leq Ktd\sqrt{n\alpha}, \tag{3.23}$$

where K is a constant that depends on the covariance of the environment Q. Inequality (3.22) can be found textually in [18]. Inequality (3.23) is established identically to equation (30) in [18], with the minor difference that the total number of paths in $\mathcal{P}_m$ is not 2^m but $(2d)^m$, which, in the inequality above (30) near the bottom of page 33 in [18], accounts for a factor $e^{1+\log(6d)} = 6ed$ instead of e^{c_1} therein, hence the factor d in (3.23).

Defining $Y_{n\alpha} = \sup_{T_{n\alpha}} X(k,\tilde{t},\tilde{x})$, we can now bound $p_t^\varepsilon(\beta)$ as follows:

$$\begin{aligned} tp_t^\varepsilon(\beta) &\leq \mathbf{E}\left[\log(A+B)\right] \\ &\leq \mathbf{E}\left[(\log A)_+\right] + \mathbf{E}\left[(\log B)_+\right] + \log 2, \end{aligned}$$

where

$$A = P_b\left[|N_t| \leq \alpha t\right] \exp\left(2\beta Y_\alpha\right)$$

$$B = \sum_{n\geq 1} P_b\left[|N_t| \in [n\alpha t, (n+1)\alpha t]\right] \exp\left(2\beta Y_{\alpha(n+1)}\right).$$

We will now bound the terms A and B separately.

Step 4: The factor A. We can bound $P_b\left[|N_t| \leq \alpha t\right]$ by 1 and we easily get, invoking (3.23),

$$\mathbf{E}\left[(\log A)_+\right] \leq 2\beta\mathbf{E}\left[Y_\alpha\right] \leq 2\beta K dt\sqrt{\alpha}. \tag{3.24}$$

Step 5: The factor B. Let $\mu = \mathbf{E}\left[Y_{\alpha(n+1)}\right]$. Since X is a Gaussian field and since it is easy to show that

$$\sigma^2 := \sup_{(m,\tilde{t},\tilde{x})} \mathbf{Var}(X(k,\tilde{t},\tilde{x})) \leq tQ(0),$$

the so called Borell-Sudakov inequality (see [1] or [35]) implies that, for a constant $a > 0$,

$$\mathbf{E}\left[\exp\left(a\left|Y_{\alpha n} - \mu\right|\right)\right] \leq 2\exp\left(\frac{a^2\sigma^2}{2}\right) = 2\exp\left(\frac{a^2 tQ(0)}{2}\right). \tag{3.25}$$

Fix now a number $\gamma \in (1/2, 1)$ and let us denote $\log_+(B) = (\log B)_+$. We have

$$\frac{1}{t^\gamma}\mathbf{E}\left[\log_+ B\right] = \mathbf{E}\left[\log_+\left(\sum_{n\geq 1} P_b\left[|N_t| \in [nt\alpha, (n+1)t\alpha]\right]\exp\left(2\beta Y_{\alpha(n+1)}\right)\right)^{t^{-\gamma}}\right]$$

$$\leq \mathbf{E}\left[\log_+\left(\sum_{n\geq 1} P_b\left[|N_t| > nt\alpha\right]\exp\left(2\beta(Y_{\alpha(n+1)} - \mu)\right)\exp\left(2\beta Ktd\sqrt{\alpha(n+1)}\right)\right)^{t^{-\gamma}}\right],$$

where we used that (3.23) implies $\mu \leq Kdt\sqrt{(n+1)\alpha}$. We also know that for any sequence of non-negative reals $(x_n)_n$ the following holds: $\left(\sum_n x_n\right)^{t^{-\gamma}} \leq \sum_n x_n^{t^{-\gamma}}$. Thus we have

$$\frac{1}{t^\gamma}\mathbf{E}\left[\log_+ B\right]$$

$$\leq \mathbf{E}\left[\log_+\left(\sum_{n\geq 1}\left(P_b\left[|N_t| > nt\alpha\right]\right)^{t^{-\gamma}}\exp\left(\frac{2\beta}{t^\gamma}(Y_{\alpha(n+1)} - \mu)\right)\exp\left(2t^{1-\gamma}\beta Kd\sqrt{\alpha(n+1)}\right)\right)\right]$$

$$\leq \mathbf{E}\left[\log_+\left[d^{t^{-\gamma}}\sum_{n\geq 1}\exp\left(\frac{2\beta}{t^\gamma}(Y_{\alpha(n+1)} - \mu)\right)\exp\left(-\frac{t^{1-\gamma}}{2}y_n\right)\right]\right],$$

where we used estimate (3.22) in the following way:

$$\begin{aligned}
P_b\left[|N_t| > nt\alpha\right] &\leq \sum_{j=1}^{d} P_b\left[N_t^j > \frac{nt\alpha}{d}\right] \\
&= dP_b\left[N_t^1 > \frac{nt\alpha}{d}\right] \\
&\leq d\exp\left(-\frac{t}{2}\left(\frac{\alpha n\varepsilon}{d}\right)^2 + \frac{t\alpha n}{d}\right),
\end{aligned}$$

and where we have obtained:

$$y_n = \left(\frac{\varepsilon\alpha n}{d}\right)^2 - \frac{2\alpha n}{d} - 4\beta Kd\sqrt{\alpha(n+1)}.$$

Now, bounding $\log_+(x)$ from above by $\log(1+x)$, for $x \geq 1$, and using Jensen's inequality, we have:

$$\frac{1}{t^\gamma}\mathbf{E}\left[\log_+ B\right] \leq \log\left[1 + \sum_{n\geq 1}\mathbf{E}\left[\exp\left(\frac{2\beta}{t^\gamma}(Y_{\alpha(n+1)} - \mu)\right)\right]\exp\left(\frac{-t^{1-\gamma}}{2}y_n\right)\right],$$

so, using (3.25), it is readily checked that

$$\frac{1}{t^\gamma}\mathbf{E}\left[\log_+ B\right] \le \log\left[1 + 2\exp\left(\frac{2\beta^2 Q(0)}{t^{2\gamma-1}}\right)\sum_{n\ge 1}\exp\left(\frac{-t^{1-\gamma}}{2}y_n\right)\right].$$

In order for the series above to converge, we must choose α so as to compensate the negative terms in y_n. Specifically, we choose

$$\left(\frac{\alpha\varepsilon}{d}\right)^2 = 16\beta K d\sqrt{\alpha}, \qquad i.e. \qquad \alpha = (16\beta K d^3\varepsilon^{-2})^{2/3}. \tag{3.26}$$

With this choice, we end up with:

$$y_n = \left(\frac{\alpha\varepsilon}{d}\right)^2\left(n^2 - \frac{2dn}{\alpha\varepsilon^2} - \frac{1}{4}\sqrt{n+1}\right).$$

Now we remark that:

$$\text{If we choose } \varepsilon, \beta \text{ such that } \beta\varepsilon \ge d^{-3/2} \quad \Rightarrow \quad \frac{\alpha\varepsilon^2}{d} = (16K\beta\varepsilon)^{2/3}\, d \ge 4, \tag{3.27}$$

so that

$$y_n \ge \left(\frac{\alpha\varepsilon}{d}\right)^2\left(n^2 - \frac{n}{2} - \frac{1}{4}\sqrt{n+1}\right),$$

and since $n^2 - \frac{n}{2} - \frac{\sqrt{n+1}}{4} \ge \frac{n}{8}$, we get

$$\begin{aligned}
&\sum_{n\ge 1}\exp\left(-\frac{t^{1-\gamma}}{2}\left(\frac{\alpha\varepsilon}{d}\right)^2\left(n^2 - \frac{2dn}{\alpha\varepsilon^2} - \frac{1}{4}\sqrt{n+1}\right)\right)\\
&\le \sum_{n\ge 1}\exp\left(-\frac{t^{1-\gamma}}{2}\left(\frac{\alpha\varepsilon}{d}\right)^2\frac{n}{8}\right)\\
&= \frac{1}{1 - \exp\left(-\frac{1}{16}t^{1-\gamma}\left(\frac{\alpha\varepsilon}{d}\right)^2\right)} - 1.
\end{aligned}$$

Notice that this last term can be made smaller than 1 if t is large enough. Hence we can write a final estimate on $\mathbf{E}\left[\log_+ B\right]$ as follows: for large t we have

$$\begin{aligned}
\frac{1}{t^\gamma}\mathbf{E}\left[\log_+ B\right] &\le \log\left[1 + 2d^{t^{-\gamma}}\exp\left(\frac{2\beta^2 Q(0)}{t^{2\gamma-1}}\right)\right]\\
&\le \log(1 + 2d^{t^{-\gamma}}) + \frac{2\beta^2 Q(0)}{t^{2\gamma-1}}.
\end{aligned} \tag{3.28}$$

Final step. Using inequalities (3.24) and (3.28) and the value of α, we can estimate $p_t^\varepsilon(\beta)$ in the following way:

$$\begin{aligned} p_t^\varepsilon(\beta) &\leq 2\beta K d\sqrt{\alpha} + \frac{\log 2}{t} + \frac{\log(1+2d^{t^{-\gamma}})}{t^{1-\gamma}} + \frac{2\beta^2 Q(0)}{t^\gamma} \\ &\leq 2\beta K d\sqrt{\alpha} + o(1). \end{aligned}$$

So using the value of α given in (3.26) we have

$$p_t^\varepsilon(\beta) \leq C\frac{\beta^{4/3}d^2}{\varepsilon^{2/3}} + o(1), \tag{3.29}$$

where C is a constant that depends on Q and that can change from line to line. Putting this result in (3.20) and taking the limit for $t \to \infty$ we get

$$\limsup_{t\to\infty} p_t(\beta) \leq C\left(\beta^2 d^H \varepsilon^{2H} + d^2\beta^{4/3}\varepsilon^{-2/3}\right).$$

In order to make this upper bound as small as possible we can choose ε such that

$$\beta^2 d^H \varepsilon^{2H} = d^2\beta^{4/3}\varepsilon^{-2/3}, \qquad i.e. \qquad \varepsilon = d^{\frac{6-3H}{2+6H}}\beta^{-\frac{1}{1+3H}},$$

so that

$$\limsup_{t\to\infty} p_t(\beta) \leq C\beta^{\frac{2+4H}{1+3H}} d^{\frac{7H}{1+3H}},$$

which is the announced result. We then only need to check for what values of β we are allowed to make this choice of ε. Condition (3.18) states that we must use $\varepsilon \leq r_1$. This is equivalent to $\beta \geq \beta_0 =: (r_1)^{-1-3H} d^{3-3H/2}$. One can check in this case that the restriction on ε, β in (3.27) is trivially satisfied.

□

3.3.2 Lower bound in the Brownian case

In the following proposition, which implies the lower bound in Theorem 3.1.1, we shall also try to specify the dependence of the constants with respect to the dimension d. Let us state an interesting feature of this dependence. The proof of the proposition below shows that the results it states hold only for $\beta \geq \beta_0 = cd^{1-H/2}$. One may ask the question of what happens to the behaviour of the partition function when the dimension is linked to the inverse temperature via the relation $\beta = \beta_0$, and one allows the dimension to be very large. The lower bounds on the value $p(\beta)$ in the proposition below will then increase, and while they must still not exceed the global bound $\beta^2 Q(0)/2$, the behaviour for large β turns out to be quadratic in many cases. The reader will check that, when $H > 1/2$, this translates as $p(\beta) \geq c\beta^{2/(2-H)}$ which is quadratic when $H = 1$, and $p(\beta) \geq c\beta^2$ for all $H \leq 1/2$. This is an indication that for extremely high dimensions and inverse temperatures, for $H \leq 1/2$ or $H = 1$, strong disorder may not hold. Strong disorder for Brownian polymers may break down for complex, infinite-dimensional polymers.

Proposition 3.3.4 *Recall that δ has been defined at (3.17) and assume that there exist a number $H \in (0, 1]$ and some positive constants c_2, r_2 such that for all $x, y \in \mathbb{R}^d$ with $|x - y| \leq r_2$, we have*

$$\delta(x-y) > c_2 |x-y|^H . \tag{3.30}$$

Then if $H \leq 1/2$, there exists a constant C depending only on Q, and a constant β_0 depending only on Q and d, such that, for all $\beta > \beta_0$,

$$p(\beta) \geq C d^{\frac{2H-1}{H+1}} \beta^{\frac{2}{H+1}}.$$

On the other hand if $H > 1/2$, there exists a constant C' depending only on Q, and a constant β_0' depending only on Q and d, such that for all $\beta > \beta_0'$

$$p(\beta) \geq C' d^{\frac{2H-1}{3}} \beta^{\frac{4}{3}}.$$

Proof. Here again, we divide the proof in several steps.

Step 1: Strategy. From relation (3.14), we have

$$p(\beta) = \sup_{t \geq 0} p_t(\beta),$$

where $p_t(\beta)$ is defined by equation (3.13). So a lower bound for $p(\beta)$ will be obtained by evaluating $p_t(\beta)$ for any fixed value t. Additionally, by the positivity of the exponential factor in the definition of Z_t, one may include as a factor inside the expectation E_b the sum of the indicator functions of any disjoint family of events of Ω_b. In fact, we will need only two events, which will give the main contribution to Z_t at a logarithmic scale.

Step 2: Setup. Let $A_+(b)$ and $A_-(b)$ be two disjoints events defined on the probability space Ω_b under P_b, which will be specified later on. Set

$$X_b = -\beta H_{2t} = \beta \int_0^{2t} W(ds, b_s).$$

Conditioning by the two events $A_+(b)$ and $A_-(b)$ and using Jensen's inequality we have

$$\mathbf{E}(\log Z_t) \geq \log\left(\min\left\{P_b(A_+); P_b(A_-)\right\}\right) + \mathbf{E}\left[\max\left\{\tilde{Z}_+; \tilde{Z}_-\right\}\right], \tag{3.31}$$

where

$$\tilde{Z}_+ := E_b\left[X_b \mid A_+\right] \quad \text{and} \quad \tilde{Z}_- := E_b\left[X_b \mid A_-\right].$$

These two random variables form a pair of centered jointly Gaussian random variables: indeed they are both limits of linear combinations of values of a single centered Gaussian field. Thus this implies

$$\mathbf{E}\left[\max\left\{\tilde{Z}_+;\tilde{Z}_-\right\}\right] = \frac{1}{\sqrt{2\pi}}\left(\mathbf{E}\left[\left(\tilde{Z}_+ - \tilde{Z}_-\right)^2\right]\right)^{1/2}.$$

Therefore we only have to choose sets A_+ and A_- not too small, but still decorrelated enough so that condition (3.30) guarantees a certain amount of positivity in the variance of $\tilde{Z}_+ - \tilde{Z}_-$.

Step 3: Choice of A_+ and A_-. Let f be a positive increasing function. We take

$$A_+ = \left\{f(t) \le b_s^i \le 2f(t), \forall i = 1,\ldots,d, \quad \forall s \in [t,2t]\right\}$$

and

$$A_- = \left\{-2f(t) \le b_s^i \le -f(t), \forall i = 1,\ldots,d, \quad \forall s \in [t,2t]\right\}.$$

In other words, we force each component of our trajectory b to be, during the entire time interval $[t,2t]$, in one of two boxes of edge size $f(t)$ which are at a distance of $2f(t)$ from each other. Because these two boxes are symmetric about the starting point of b, the corresponding events have the same probability. While this probability can be calculated in an arguably explicit way, we give here a simple lower bound argument for it. Using time scaling, the Markov property of Brownian motion, the notation $a = f(t)/\sqrt{t}$, we have

$$\begin{aligned}
P_b(A_+) &= \prod_{i=1}^d P_b\left(\forall s \in [1,2] : b_s^i \in [a,2a]\right) \\
&= \prod_{i=1}^d \frac{1}{2\pi}\int_a^{2a} P_b\left(\forall s \in [0,1] : b_s^i + y \in [a,2a]\right) e^{-y^2/2}dy \\
&\ge \prod_{i=1}^d \frac{1}{2\pi}\int_{5a/4}^{7a/4} P_b\left(\forall s \in [0,1] : b_s^i + y \in \left[y - \frac{a}{4}, y + \frac{a}{4}\right]\right) e^{-y^2/2}dy \\
&= \left[P_b\left(b_1^1 \in [5a/4,7a/4]\right) P_b\left(\forall s \in [0,1] : |b_s^1| \le a/4\right)\right]^d. \qquad (3.32)
\end{aligned}$$

Step 4: Estimation of $\tilde{Z}_+$ and $\tilde{Z}_-$. It was established in [18] that in dimension $d = 1$

$$\mathbf{E}\left[\left(\tilde{Z}_+ - \tilde{Z}_-\right)^2\right] \ge \beta^2 \int_t^{2t} \mathbf{E}\left[\left(\delta(x_{s,+}^* - x_{s,-}^*)\right)^2\right] ds$$

where the quantities $x^*_{s,+}$ and $x^*_{s,-}$ are random variables such that for all $s \in [t, 2t]$: $x^*_{s,+} \in [f(t), 2f(t)]$ and $x^*_{s,-} \in [-2f(t), -f(t)]$. In dimension $d \geq 1$ the result still holds. In fact in this case we have $x^*_{s,+}, x^*_{s,-} \in \mathbb{R}^d$, so it is sufficient to take each component of the $x^*_{s,+}$ in the interval $[f(t), 2f(t)]$ and each component of $x^*_{s,-}$ in $[-2f(t), -f(t)]$, so their distance is greater than $d^{1/2} f(t)$. Thus, using condition (3.30), we have

$$\mathbf{E}\left[\left(\tilde{Z}_+ - \tilde{Z}_-\right)^2\right] \geq \beta^2 \int_t^{2t} C \left|x^*_{s,+} - x^*_{s,-}\right|^{2H} ds \geq Ct\beta^2 d^H \left(f(t)\right)^{2H}, \quad (3.33)$$

where as usual C is a constant that can change from line to line. Hence, we obtain:

$$\begin{aligned} \mathbf{E}\left[\max\left\{\tilde{Z}_+; \tilde{Z}_-\right\}\right] &= \frac{1}{\sqrt{2\pi}} \left(\mathbf{E}\left[\left(\tilde{Z}_+ - \tilde{Z}_-\right)^2\right]\right)^{1/2} \\ &\geq C\beta\sqrt{t}\left(f(t)\right)^H d^{H/2}, \end{aligned} \quad (3.34)$$

Observe that in order to use condition (3.30) we have to impose $f(t) \leq r_2$.

Step 5: The case $H \leq 1/2$. It is possible to prove that in this case the optimal choice for f is $f(t) = \sqrt{t}$, which corresponds to $a = 1$, so that $P_b(A_+)$ is a universal constant that does not depend on t. Thus we have, from (3.31), (3.32) and (3.34), for any $t > 0$,

$$p_{2t}(\beta) = \frac{\mathbf{E}\left[\log Z_{2t}\right]}{2t} \geq \frac{d \log C}{2t} + C\beta d^{H/2} t^{\frac{H-1}{2}}. \quad (3.35)$$

Now we may maximize the above function over all possible values of $t > 0$. To make things simple, we choose t so that the second term equals twice the first, yielding t of the form

$$t = Cd^{\frac{2-H}{H+1}} \beta^{-\frac{2}{H+1}}$$

and therefore

$$\sup_{t>0} p_{2t}(\beta) \geq Cd^{\frac{2H-1}{H+1}} \beta^{\frac{2}{H+1}}.$$

This result holds as long as the use of condition (3.30) can be justified, namely as long as $f(t) \leq r_2$. This is achieved as soon as $\beta > \beta_0$ where $\beta_0 = Cr_2^{-H-1} d^{1-H/2}$, and since $H \leq 1/2$, $\beta_0 \geq Cd^{3/4}$.

Step 6: The case $H > 1/2$. In this case we consider $f(t) = ct^{\alpha}$, for a given $\alpha \in [0, 1/2)$ and some constant c chosen below. Thus we have $a = ct^{\alpha-1/2}$. In this case, if a is larger than a universal constant K_u, the result (3.32) yields that, for some constant C, we have

$$P_b(A_+) \geq \prod_{i=1}^{d} \exp(-Ca^2) = \exp(-Cc^2 dt^{2\alpha-1}).$$

So, using again condition (3.30) and relation (3.34) we obtain

$$p_{2t}(\beta) \geq -Cdt^{2\alpha-2} + C\beta d^{H/2} t^{\alpha H-1/2},$$

where the constant C may also include the factor c^2. Again, choosing t so that the second term equals twice the first, we have

$$t = Cd^{\frac{1-H/2}{\alpha(H-2)+3/2}} \beta^{-\frac{1}{\alpha(H-2)+3/2}}, \tag{3.36}$$

and so

$$\sup_{t>0} p_{2t}(\beta) \geq Cd^{\frac{H-1/2}{\alpha(H-2)+3/2}} \beta^{-\frac{2\alpha-2}{\alpha(H-2)+3/2}}.$$

In order to maximize the power of β in the lower bound for $\sup_{t>0} p_t(\beta)$ we should find the maximum of the function

$$g(\alpha) = \frac{2-2\alpha}{\alpha(H-2)+3/2}$$

for $0 \leq \alpha < 1/2$. Since this function is monotone decreasing when $H > 1/2$, the maximum is reached for $\alpha = 0$, so $g(0) = 4/3$.

Recall once again that, in order to apply condition (3.30) in the computations above, we had to assume $f(t) \leq r_2$; since now $f(t)$ is the constant c, we only need to choose $c = r_2$. We also had to impose $a = r_2 t^{-1/2} > K_u$, which translates as $\beta > \beta_0' := (K_u/r_2)^{4/3} d^{1-H/2}$.

□

3.3.3 Logarithmic regularity scale

As mentioned in the introduction, the special shape of our Figure 3.1.5 induces us to explore the regions of low spatial regularity for W, in order to investigate some new possible scaling in the strong disorder regime. In other words, we shall work in this section under the assumptions that there exist positive constants c_0, c_1, and r_1, and $\beta \in (0,\infty)$, such that for all x, y with $|x-y| \leq r_1$,

$$c_0 \log^{-\gamma}(1/|x-y|) \leq \delta(x-y) \leq c_1 \log^{-\gamma}(1/|x-y|), \tag{3.37}$$

where $\gamma > 0$. Assumption (3.37) implies that W is not spatially Hölder-continuous for any exponent $H \in (0,1]$. Moreover, the theory of Gaussian regularity implies that, if $\gamma > 1/2$, W is almost-surely continuous in space, with modulus of continuity proportional to $\log^{-\gamma+1/2}(1/|x-y|)$, while if $\gamma \le 1/2$, W is almost-surely not uniformly continuous on any interval in space. The case $\gamma = 1/2$, which is the threshold between continuous and discontinuous W, is of special interest, since it can be related to the discrete space polymer which will be studied in the next section. The main result which will be proved here is the following:

Theorem 3.3.5 *Assume condition (3.37). We have for some constants C_0 and C_1 depending only on Q, for all β large enough,*

$$C_0 \frac{\beta^2}{d} \log^{-2\gamma}\left(\frac{\beta}{\sqrt{d}}\right) \le p(\beta) \le C_1 \beta^2 \log^{-2\gamma}\left(\frac{\beta}{\sqrt{d}}\right).$$

Proof. *Step 1: Setup.* Nearly all the calculations in the proof of Propositions 3.3.1 and 3.3.4 are still valid in our situation.

Step 2: Lower bound. For the lower bound, reworking the argument in Step 2 in the proof of Proposition 3.3.4, using the function $\log^{-\gamma}(x^{-1})$ instead of the function x^H, we obtain the following instead of (3.33):

$$\mathbf{E}\left[(Z_+ - Z_-)^2\right] \ge t\,(\beta c_0)^2 \left(\log\left(\frac{1}{\sqrt{d}f(t)}\right)\right)^{-2\gamma},$$

which implies, instead of (3.35) in Step 5 of that proof, the following:

$$p_{2t}(\beta) \ge \frac{d\log C}{2t} + C\beta t^{-1/2}\left(\log\left(\frac{1}{\sqrt{d}f(t)}\right)\right)^{-\gamma}.$$

In other words, now choosing $f(t) = t^{1/2}$ as we did in the case $H < 1/2$ (recall that we are in the case of small H, as stated in the introduction),

$$p_{2t}(\beta) \ge \frac{d\log C}{2t} + C\beta t^{-1/2}\left(\log\left(\frac{1}{\sqrt{dt}}\right)\right)^{-\gamma}.$$

Now choose t such that the second term in the right-hand side above equals twice the first, *i.e.*

$$t^{1/2}\log^{-\gamma}\left(\frac{1}{\sqrt{dt}}\right) = Cd\beta^{-1}.$$

For small t, the function on the left-hand side is increasing, so that the above t is uniquely defined when β is large. We see in particular that when β is large, t is small, and we have $t^{-1} \le \beta^2$. This fact is then used to imply

$$\frac{1}{t} = \left(\frac{C\beta}{d}\right)^2 \left(\log\left(\frac{1}{\sqrt{dt}}\right)\right)^{-2\gamma} \ge 2\left(C\beta\right)^2 \log^{-2\gamma}\left(\beta\right).$$

Therefore, for some constants β_2 and c depending only on Q, for the t chosen above with $\beta \ge \beta_2$,

$$p_{2t}(\beta) \ge \frac{C\beta^2}{d}\left(\log\left(\frac{\beta}{\sqrt{d}}\right)\right)^{-2\gamma}.$$

Step 3: Upper bound. Here, returning to the proof of Proposition 3.3.1, the upper bound (3.29) in the final step of that proof holds regardless of δ, and therefore, using the result of Remark 3.3.3 with $\delta\left(r\right) = \log^{-\gamma}\left(1/r\right)$, we immediately get that there exists c depending only on Q such that for all $\varepsilon < r_1$ and all $\beta > \beta_3$,

$$\limsup_{t\to\infty} p_t(\beta) \le C\beta^2\left(\log\left(\frac{1}{\varepsilon\sqrt{d}}\right)\right)^{-2\gamma} + Cd^2\beta^{4/3}\varepsilon^{-2/3},$$

as long as one is able to choose ε so that $\beta\varepsilon \ge 1$. By equating the two terms in the right-hand side of the last inequality above, we get

$$\varepsilon\left(\log\left(\frac{1}{\varepsilon\sqrt{d}}\right)\right)^{-3\gamma} = Cd^3\beta^{-1}.$$

Since the function $\varepsilon \mapsto \varepsilon\log^{-3\gamma}\left(1/(\varepsilon\sqrt{d})\right)$ is increasing for small ε, the above equation defines ε uniquely when β is large, and in that case ε is small. We also see that for any $\theta > 0$, for large β, $1/\varepsilon \ge \beta^{1-\theta}$. Therefore we can write, for $\beta \ge \beta_3$, almost surely,

$$\limsup_{t\to\infty} p_t(\beta) \le C\left(1-\theta\right)^{-2\gamma}\beta^2\left(\log\left(\frac{\beta}{\sqrt{d}}\right)\right)^{-2\gamma}.$$

This finishes the proof of the theorem.

□

3.4 Estimates of the free energy: discrete space

Recall that, up to now, we have obtained our bounds on the free energy in the following manner: the upper bound has been computed by evaluation of the supremum of a well-chosen random Gaussian field, while the lower bound has been obtained by introducing two different events, depending on the Brownian configuration, which capture most of the logarithmic weight of our polymer distribution. This strategy also works in the case of the random walk polymer whose Hamiltonian is described by (3.5), without many additional efforts, but a separate proof is still necessary. This section shows how this procedure works, resulting in the proof of Theorem 3.1.5.

Quantities referring to the random walk polymer have been denoted by $\hat{b}, \hat{W}, \hat{E}_{\hat{b}}, \hat{\mathbf{E}}$, etc... In this section, for notational sake, we will omit the hats in the expressions above, and write instead $b, W, E_b, \mathbf{E}$ like in the Brownian case. We expect that this change in the notation does not introduce any confusion.

Recall that we have assumed the following simple non-degeneracy condition on Q:

$$c_Q := \sup_{1\leq i\leq d} \left(Q(0) - Q(2e_i)\right)^{1/2} > 0, \tag{3.38}$$

where e_i, $i = 1, \cdots, d$ are the unit vectors in $\mathbb{Z}^d$. Condition (3.38), which is used only in the lower bound result, is extremely weak. It essentially covers all possible homogeneous covariance functions, except the trivial one $Q(x) \equiv Q(0)$ for all x, which is the case where W does not depend on x at all, in which case the Hamiltonian has no effect. Indeed, assume that there exists an $x_0 \in \mathbb{Z}^d$ such that $W(t, 0)$ and $W(t, x_0)$ are not (a.s.) equal. Then $Q(x_0) < Q(0)$.

3.4.1 Lower bound for the random walk polymer

The lower bound announced in Theorem 3.1.5 is contained in the following:

Proposition 3.4.1 *Assume condition (3.38) holds true. Then there exists a constant $\beta_0 > 0$, which depends on d and on c_Q and a constant $C > 0$, which depend only on c_Q, such that if $\beta > \beta_0$ then almost surely*

$$\lim_{t\to\infty} \frac{1}{t} \log Z_t \geq C \frac{\beta^2}{\log \beta}.$$

Proof. Invoking Proposition 3.2.2, we know that $\lim_{t\to\infty} p_t(\beta)$ exists and equals $\sup_{t\geq 0} p_t(\beta)$. Therefore, to find a lower bound on the latter, it is sufficient to choose a fixed value t and evaluate $p_t(\beta)$ at that value.

We express Z_t by using the fact that each component of b is constant between its jump times, which are uniformly distributed on the simplex, given N_t the total number of jumps before time t, which is a Poisson r.v. with parameter $2dt$. We also know that the visited sites $(x_k)_{k=1}^{N_t}$ are uniformly distributed on the set of all nearest-neighbor paths of length N_t, given N_t. Note that $x_0 = 0$ is the position of b before τ_1, the first jump time of the random walk b. We then obtain a lower bound on $p_t(\beta)$ by throwing out, in the expectation defining Z_t, all the paths b that do not jump exactly once before time t. We also throw out all jump positions that are not $\pm e_i$, where e_i stands for a unit vector in $\mathbb{Z}^d$ such that $c_Q = (Q(0)-Q(2e_i))^{1/2}$. Therefore, we have

$$Z_t \geq P_b\left[N_t = 1\right] \frac{1}{2d} \int_0^t \frac{ds}{t} \left(e^{\beta W(s,0)+\beta W([s,t],e_i)} + e^{\beta W(s,0)+\beta W([s,t],-e_i)}\right),$$

where $W\left([s,t],x\right) := W\left(t,x\right) - W\left(s,x\right)$. Here, given $N_t = 1$, $1/(2d)$ is the weight of the path that jumps to $\pm e_i$. Also, given $N_t = 1$, $\mathbf{1}_{[0,t]}\left(s\right) ds/t$ is the law of the single jump time τ_1.

So we obtain

$$Z_t \geq dte^{-2td} \int_0^t \frac{ds}{t} \left(e^{\beta W(s,0)+\beta W([s,t],e_i)} + e^{\beta W(s,0)+\beta W([s,t],-e_i)}\right),$$

and hence, according to Jensen's inequality,

$$\begin{aligned}\frac{1}{t}\mathbf{E}(\log Z_t) &\geq \frac{\log t}{t} - 2d \\ &\quad + \beta \int_0^t \frac{ds}{t^2}\mathbf{E}\left[\max\left(W([s,t],e_i); W([s,t],-e_i)\right)\right]\end{aligned}$$

Now we evaluate the expected maximum above. The vector $\left(W\left([s,t],e_i\right), W\left([s,t],-e_i\right)\right)$ is jointly Gaussian with common variances $\sqrt{t-s}Q(0)$ and covariance $\sqrt{t-s}Q(2)$. Therefore

$$\begin{aligned}\mathbf{E}\left[\max\left(W\left([s,t],e_i\right), W\left([s,t],-e_i\right)\right)\right] &= \frac{1}{2}\mathbf{E}\left[\left|W\left([s,t],e_i\right) - W\left([s,t],-e_i\right)\right|\right] \\ &= \frac{1}{\sqrt{2\pi}}\left(\mathbf{Var}\left[W\left([s,t],e_i\right) - W\left([s,t],-e_i\right)\right]\right)^{1/2} \\ &= \frac{1}{\sqrt{\pi}}\sqrt{t-s}\sqrt{Q(0)-Q(2e_i)}. \quad (3.39)\end{aligned}$$

Thus, recalling condition (3.38), we obtain

$$\frac{1}{t}\mathbf{E}(\log Z_t) \geq \frac{\log t}{t} - 2d + \frac{2\beta}{3\sqrt{\pi t}}c_Q$$

Furthermore, we have the freedom to choose t any way we want. If we choose $t = C\log^2\beta/\beta^2$, we end up with

$$\frac{1}{t}\mathbf{E}(\log Z_t) \geq \frac{\beta^2}{\log\beta}\left(-\frac{2}{C}+\frac{2c_Q}{3\sqrt{c\pi}}\right) + \frac{\beta^2}{C\log^2\beta}\left(\log C + 2\log\log\beta\right) - 2d. \tag{3.40}$$

Then, in order to complete the proof, we only need to choose C such that

$$-\frac{2}{C}+\frac{2c_Q}{3\sqrt{C\pi}} > 0, \quad \textit{i.e.} \quad C > \frac{9\pi}{Q(0)-Q(2)},$$

and β large enough so that the second term in (3.40), namely

$$\frac{\beta^2}{C\log^2\beta}\left(\log C + 2\log\log\beta\right) - 2d,$$

is nonnegative.

□

3.4.2 Upper bound for the random walk polymer

The upper bound result in Theorem 3.1.5 can be summarized in the following proposition.

Proposition 3.4.2 *Under the assumption that $Q(0) < \infty$, there exists a constant $\beta_0' > 0$, which depends on Q and on d, and a constant $C > 0$, which depend only on Q, such that if $\beta > \beta_0'$ then almost surely*

$$\lim_{t\to\infty}\frac{1}{t}\log Z_t \leq Cd^3\frac{\beta^2}{\log\beta}.$$

Proof. Define $\mathcal{S}(t,n)$, k_j, t_i^j, $\tilde{t}_l$, N_t, $\mathcal{P}_n$ and $\tilde{x}_l$ like in Step 3 of the proof of Proposition 3.3.1. Then if we fix $N_t = m$, we can define

$$X\left(m,\tilde{t},\tilde{x}\right) := \sum_{i=0}^{m}\left\{W\left(\tilde{t}_{i+1},\tilde{x}_i\right) - W\left(\tilde{t}_i,\tilde{x}_i\right)\right\}.$$

Let α be a fixed positive number which will be chosen later. Let $I_\alpha = \cup_{m\leq\alpha t}J_m$, where $J_m := \{m\}\times\mathcal{S}_{m,t}\times\mathcal{P}_m$, and set also $Y_\alpha = \sup_{I_\alpha}X$. As in the Brownian case, we can bound $\mathbf{E}[\log Z_t]$ above as follows:

$$\mathbf{E}[\log Z_t] \leq \mathbf{E}[\log(A+B)] \leq \mathbf{E}\left[\log_+ A\right] + \mathbf{E}\left[\log_+ B\right] + \log 2, \tag{3.41}$$

where

$$A := P_b\left[N_t \le \alpha t\right] \exp\left(\beta Y_\alpha\right) \tag{3.42}$$
$$B := \sum_{m > \alpha t} P_b\left[N_t = m\right]\, E_b\left[\exp\left(\beta X\left(m, \tilde{t}, \tilde{x}\right)\right) \Big| N_t = m\right],$$

and where we used the notation $\log_+ A = \left(\log A\right)_+ = \max(\log A, 0)$.

Step 1: The term A. As in the continuous case, we have that

$$\mathbf{E}\left[\sup_{T_{n\alpha}} X(k, \tilde{t}, \tilde{x})\right] \le K t d \sqrt{n\alpha}, \tag{3.43}$$

where K is a constant that depends on the covariance of the environment Q. So, bounding $P_b\left[N_t \le \alpha t\right]$ by one we immediately have

$$\mathbf{E}\left[\log_+ A\right] \le \beta \mathbf{E}\left[Y_\alpha\right] \le \beta K d t \sqrt{\alpha}. \tag{3.44}$$

Step 2: The term B. The term B defined in (3.42) can be bounded as follows:

$$\begin{aligned}
&\mathbf{E}\left[\log B_+\right] \\
&= \mathbf{E}\left[\log_+ \sum_{m > \alpha t} P_b\left[N_t = m\right] \sum_{\tilde{x} \in \mathcal{P}_m} \frac{1}{(2d)^m} \int_{\mathcal{S}_{m,t}} \exp\left(\beta X\left(m, \tilde{t}, \tilde{x}\right)\right) d\tilde{t}\right] \\
&= \mathbf{E}\left[\log_+ \sum_{n \ge 1} \sum_{m \in [\alpha n t, \alpha(n+1)t]} P_b\left[N_t = m\right] \sum_{\tilde{x} \in \mathcal{P}_m} \frac{1}{(2d)^m} \int_{\mathcal{S}_{m,t}} \exp\left(\beta X\left(m, \tilde{t}, \tilde{x}\right)\right) d\tilde{t}\right] \\
&\le \mathbf{E}\left[\log_+ \sum_{n \ge 1} P_b\left[N_t > \alpha n t\right] \exp\left(\beta Y_{(n+1)\alpha}\right)\right].
\end{aligned}$$

So, using the fact that for $t > 1$, the power t^{-1} of a sum is less than the sum of the terms raised to the power t^{-1}, followed by Jensen's inequality, we have, similarly to what we did in the proof of Proposition 3.3.1,

$$\frac{1}{t}\mathbf{E}\left[\log_+ B\right] \le \log\left(1 + \sum_{n \ge 1} \left(P_b\left[N_t > \alpha n t\right]\right)^{t^{-1}} \mathbf{E}\left[\exp\left(\frac{\beta Y_{(n+1)\alpha}}{t}\right)\right]\right).$$

Using once again well-known results from Gaussian supremum analysis (see for instance [1] or [35] on the Borell-Sudakov inequality), it holds that for some universal constant K_u, and for any $\alpha, x > 0$,

$$\begin{aligned}
\mathbf{E}\left[\exp\left(x Y_\alpha\right)\right] &\le \exp\left(x \mathbf{E}\left[Y_\alpha\right]\right) \exp\left(x^2 K_u \max_{(m, \tilde{t}, \tilde{x}) \in I_\alpha} \mathbf{Var}\left[X\left(m, \tilde{t}, \tilde{x}\right)\right]\right) \\
&\le \exp\left(x d K t \sqrt{\alpha}\right) \exp\left(x^2 K_u t Q(0)\right).
\end{aligned}$$

where in the last line we used (3.43) and the fact that $\mathbf{E}[X\left(m,\tilde{t},\tilde{x}\right)^2]$ is always equal to $Q\left(0\right)t$. So we have

$$\mathbf{E}\left[\exp\left(\frac{\beta Y_{(n+1)\alpha}}{t}\right)\right] \leq \exp\left(\beta dK\sqrt{\alpha(n+1)} + \frac{\beta^2 K_u Q(0)}{t}\right).$$

If we choose t such that $t > (2\beta K_u Q(0))/(dK\alpha^{1/2})$, the estimate on B becomes

$$\frac{1}{t}\mathbf{E}\left[\log_+ B\right] \leq \log\left\{1 + \sum_{n\geq 1}\left(P_b\left[N_t > \alpha n t\right]\right)^{t^{-1}} \exp\left(\beta dK_u\sqrt{\alpha}\left(\sqrt{n+1}+\frac{1}{2}\right)\right)\right\}. \tag{3.45}$$

Step 3: The tail of N_t. It is possible to derive by hand a very simple estimate of the tail of N_t by using the Stirling-type bound $m! \geq m^m 3^{-m}$, but the following concentration-type estimate is presumably well-known, and can be found in many places including [22, pages 16-19]: for all $\alpha \geq 1$,

$$P_b\left[N_t > \alpha t\right] \leq \exp\left(-\alpha t\log\left(\frac{\alpha}{2d}\right) - t(\alpha - 2d)\right),$$

so if we set $\alpha' = \alpha/2d$ and we assume $\alpha' \geq \exp\left(1 - 1/2d\right)$ we have

$$P_b\left[N_t > \alpha t\right] \leq \exp\left(-t\alpha' \log\alpha'\right). \tag{3.46}$$

Step 4: Grouping our estimates and choosing α. From (3.45) and (3.46) we have

$$\frac{1}{t}\mathbf{E}\left[\log_+ B\right] \leq \log\left\{1 + \sum_{n\geq 1}\exp\left(-\alpha' n\log\alpha' n + d\beta K_u\sqrt{\alpha}\left(\sqrt{n+1}+\frac{1}{2}\right)\right)\right\}.$$

Here we see that in order to exploit the negativity of the first term in the above exponential, it is sufficient to require

$$\alpha' \log\alpha' = 4dK_u\beta\sqrt{\alpha}. \tag{3.47}$$

Indeed, since $n \geq 1$, we then have that the term inside the exponential is

$$\begin{aligned}
&\alpha' n \log \alpha' n - \beta d K_u \sqrt{\alpha}\left(\sqrt{n+1}+\frac{1}{2}\right) \\
&= \alpha' n \log \alpha' n - \frac{1}{4}\alpha' \log \alpha' \left(\sqrt{n+1}+\frac{1}{2}\right) \\
&\geq \alpha' n \log \alpha' - \frac{1}{4}\alpha' \log \alpha' \left(\sqrt{n+1}+\frac{1}{2}\right) \\
&= \frac{1}{2}\alpha' n \log \alpha' n + \left(\frac{1}{2}n - \frac{1}{4}\left(\sqrt{n+1}+\frac{1}{2}\right)\right)\alpha' \log \alpha' \\
&\geq \frac{1}{2}\alpha' n \log \alpha',
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{1}{t}\mathbf{E}\left[\log_+ B\right] &\leq \log\left\{1+\sum_{n\geq 1}\exp\left(-\frac{1}{2}\alpha' n \log \alpha'\right)\right\} \\
&= \log\left\{1+\sum_{n\geq 1}\left[\exp\left(-\frac{1}{2}\alpha' \log \alpha'\right)\right]^n\right\} \\
&= \log\left\{1+\frac{1}{\exp\left(\frac{1}{2}\alpha' \log \alpha'\right)-1}\right\} := c_d.
\end{aligned}$$

The restriction $\alpha' \geq \exp(1-1/2d)$ implies that c_d is a constant that depends on the dimension d only. Combining this with (3.41) and (3.44), we get

$$\frac{1}{t}\mathbf{E}\left[\log Z_t\right] \leq \frac{\log 2}{t} + c_d + dK_u\beta\sqrt{\alpha}. \tag{3.48}$$

Step 5: Conclusion. Let us now reformulate the definition of α' in (3.47): it is easy enough to see that, with

$$x := \left(4d\sqrt{2d}\beta K_u\right)^2, \tag{3.49}$$

the equation $\alpha' = x/\log^2 \alpha'$ has a unique solution α' when x exceeds e, and that α' also exceeds e in that case: indeed $\alpha' = e$ when $x = e$ and $d\alpha'/dx = \left(\log^2 \alpha' + 2\log \alpha'\right)^{-1} > 0$ for all $\alpha' \geq e$. Therefore, since $\log^2 \alpha' > 1$, we can write $\alpha' \leq x$, and thus we also have:

$$\alpha' = \frac{x}{\log^2 \alpha'} \geq \frac{x}{\log^2 x}. \tag{3.50}$$

This lower bound on α' allows us immediately to derive the following upper bound on α':

$$\alpha' = \frac{x}{\log^2 \alpha'} \le \frac{x}{\log^2\left(x/\log^2 x\right)} = \frac{x}{\left(\log x - 2\log(\log x)\right)^2}. \tag{3.51}$$

Since there exists x_0 such that, for any $x > x_0$, we have

$$\log x > 4\log(\log x), \tag{3.52}$$

and we can recast expression (3.51) into:

$$\alpha' \le \frac{4x}{\log^2 x} = \frac{(4d\sqrt{2d}\beta K_u)^2}{(\log\beta + \log 4d\sqrt{2d}K_u)^2} \le (4d\sqrt{2d}K_u)^2 \frac{\beta^2}{\log^2\beta},$$

which yields the following inequality on α (recall that $\alpha = 2d\alpha'$):

$$\alpha \le (8d^2K_u)^2 \frac{\beta^2}{\log^2\beta}.$$

Thus we get the following bound, valid for t large enough:

$$\frac{\mathbf{E}[\log(Z_t)]}{t} \le \frac{\log 2}{t} + c_d + 8d^3K_u^2\frac{\beta^2}{\log\beta}.$$

Taking limits as t tends to ∞ and choosing β so that

$$\beta^2 > \frac{c_d \log\beta}{8d^3K_u^2}, \tag{3.53}$$

the theorem is proved with $C = 16K_u^2$.

Finally, let us observe that the theorem holds for β large enough. In fact, analyzing the conditions we used above, we only have to take $\beta \ge \beta_0'$, where

$$\beta_0' = \max(K_d, \beta^*, \beta_*),$$

and the constants K_d, β^*, β_* will be specified below. This is due to the fact that we assumed $\alpha' \ge \exp(1 - 1/2d)$ and this implies, via (3.50), that

$$x \ge 2d\exp(1-1/2d)\left(\log 2d + 1 - \frac{1}{2d}\right),$$

and therefore, from (3.49), we have to take $\beta \ge K_d$, where K_d is a constant that depends only on the dimension d. In addition, according to (3.52) and (3.53), β_* and β^* are the solutions to the following equations:

$$\beta^2 = \frac{c_d\log\beta}{8d^3K_Q^2} \quad \text{and} \quad \log(4d\sqrt{2d}K\beta) = 4\log(\log(4d\sqrt{2d}K\beta)).$$

$\square$

Bibliography

[1] R. Adler, *An introduction to continuity, extrema and related topics for general Gaussian processes*, Institute of Math. statistics **12** Hayward (1990).

[2] X. Bardina, D. Márquez-Carreras, C. Rovira, S. Tindel, *The p-spin interaction model with external field*, Potential Analysis **21** (2004), 311-362.

[3] A. Bovier, *Statistical Mechanics of Disordered Systems*, Cambridge Series in Statistical and Probabilistic Mathematics (2006).

[4] A. Cadel, D. Márquez-Carreras, C. Rovira, S. Tindel, *A model of continuous time polymer on the lattice*, In preparation (2007).

[5] P. Carmona, Y. Hu, *On the partition function of a directed polymer in a Gaussian random environment*, Probab. Theory Relat. Fields, **124** (2002), 431-457.

[6] P. Carmona, Y. Hu, *Strong disorder implies strong localization for directed polymers in a random environment*, ALEA Lat. Am. J. Probab. Math. Stat **2**, (2006), 217–229.

[7] R. Carmona, L. Koralov, S.A. Molchanov, *Asymptotics for the almost sure Lyapunov exponent for the solution of the parabolic Anderson problem*, Random Oper. Stochastic Equations **9** no. 1, (2001), 77–86.

[8] R. Carmona, S.A. Molchanov, *Parabolic Anderson problem and intermittency*, Mem. Amer. Math. Soc. **108**, (1994).

[9] R. Carmona, F. Viens, *Almost-sure exponential behavior of a stochastic Anderson model with continuous space parameter*, Stochastics and Stochastic Reports **62**, (1998), 251-273.

[10] F. Comets, G. Giacomin, J. L. Lebowitz, *The Sherrington-Kirkpatrick model with short range ferromagnetic interactions*, C. R. Acad. Sci. Paris Sér. I Math. **328** no. 1 (1999), 57-62.

[11] F. Comets, F. Guerra, F. L. Toninelli, *The Ising-Sherrington-Kirkpatrick model in a magnetic field at high temperature*, J. Stat. Phys. **120** no. 1-2 (2005), 147–165.

[12] F. Comets, T. Shiga, N. Yoshida, *Probabilistic analysis of directed polymers in a random environment: a review*, Adv. Stud. Pure Math. (2004) **39**, 115-142 (2003)

[13] F. Comets, N. Yoshida, *Brownian directed polymers in random environment*, Comm. Math. Phys **254** no. 2, (2005), 257–287.

[14] F. Comets, N. Yoshida, *Directed polymers in random environment are diffusive at weak disorder*, Ann. Probab. **34** no. 5, (2006), 1746–1770.

[15] B. Derrida, H. Spohn, *Polymers on disordered trees, spin glasses, and traveling waves. New directions in statistical mechanics (Santa Barbara, CA, 1987)*, J. Statist. Phys. **51**, no. 5-6, (1988), 817–840.

[16] S. F. Edwards, P. W. Anderson, *Theory of spin glasses*, J. Phys. F. no **5**, 1965 (1975).

[17] D. Fisher, D. Huse, *Directed paths in random potential*, Phys. Rev. B. **43** no 13, (1991), 10728-10742.

[18] I. Florescu, F. Viens, *Sharp estimation for the almost-sure Lyapunov exponent of the Anderson model in continuous space*, in press in Probab. Theory and Related Fields, (2005).

[19] G. Giacomin, *Random Polymer models*, Imperial College Press, World Scientific (2007).

[20] D. Huse, C. Henley, (1985). *Phys. Rev. Lett.* **54**, 2708

[21] J. Imbrie, T. Spencer, *Diffusion of directed polymers in a random environment*, J. Statist. Phys. **52** no. 3-4, (1988), 609–626.

[22] P. Massart, *Concentration Inequalities and Model Selection.* Ecole d'Eté de Probabilités de Saint-Flour XXXIII - 2003; Lecture Notes in Mathematics **1896**, (2007), Springer.

[23] O. Méjane, *Upper bound of a volume exponent for directed polymers in a random environment*, Ann. Inst. H. Poincaré Probab. Statist. **40** no. 3, (2004), 299–308.

[24] H. Nishimori, *Statistical Physics of Spin Glasses and Information Processing*, Oxford University Press, (2001).

[25] M. Petermann, *Superdiffusivity of polymers in random environment.* Ph.D. Thesis, Univ. Zürich, (2000).

[26] M.S.T. Piza, *Directed polymers in a random environment: some results on fluctuations*, J. Statist. Phys. **89** no. 3-4, (1997), 581–603.

[27] C. Rovira, S. Tindel, *On the Brownian-directed polymer in a Gaussian random environment*, J. Funct. Anal. **222** no. 1, (2005), 178–201.

[28] B. Roynette, M. Yor, *Penalising Brownian Paths: Rigorous Results and Meta-Theorems*, Preprint, (2007).

[29] D. Sherrington, S. Kirkpatrick, *A solvable model of a spin glass*, Phys. Rev. Lett. 35 (1975), 1792-1796.

[30] M. Talagrand, *Spin Glasses: A Challenge for Mathematicians*, Springer-Verlag, (2003).

[31] S. Tindel, C. A. Tudor, F. Viens, S*harp Gaussian regularity on the circle, and applications to the fractional stochastic heat equation*, J. Funct. Anal. **217** no. 2, (2004), 280–313.

[32] S. Tindel, F. Viens, *On space-time regularity for the stochastic heat equations on Lie groups*, J. Funct. Analysis **169** no. 2, (1999), 559-603.

[33] S. Tindel, F. Viens, *Almost sure exponential behaviour for a parabolic SPDE on a manifold*, Stochastic Process. Appl. **100**, (2002), 53–74.

[34] S. Tindel, F. Viens, *Relating the almost-sure Lyapunov exponent of a parabolic SPDE and its coefficients' spatial regularity*, Potential Anal. **22** no. 2, (2005), 101–125.

[35] F. Viens, A. Vizcarra, *Supremum Concentration Inequality and Modulus of Continuity for Sub-nth Chaos Processes*, to appear in J. Functional Analysis, (2007) 27 pages.

[36] M. V. Wüthrich, *Scaling identity for crossing Brownian motion in a Poissonian potential*, Probab. Theory Related Fields **112** (1998), no. 3, 299-319.

[37] M. V. Wüthrich, *Superdiffusive behavior of two-dimensional Brownian motion in a Poissonian potential*, Ann. Probab. **26** (1998), no. 3, 1000-1015.

[38] M. V. Wüthrich, *Fluctuation results for Brownian motion in a Poissonian potential*, Ann. Inst. H. Poincaré Probab. Statist. **34** (1998), no. 3, 279-308.

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