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ON MATRICES WITH THE EDMONDS-JOHNSON PROPERTY

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To my parents

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Abstract

The *strong Chvátal rank* of a rational matrix A is the smallest number t such that the polyhedron defined by the system $b \leq Ax \leq c$, $l \leq x \leq u$ has Chvátal rank at most t for all integral vectors b, c, l, u . Matrices with strong Chvátal rank at most 1 are said to have the *Edmonds-Johnson property*, since it was shown by Edmonds and Johnson [16] that any integral matrix $A = (\alpha_{ij})$ such that $\sum_i |\alpha_{ij}| \leq 2$ for each column index j has Chvátal rank at most 1.

While the class of integral matrices with strong Chvátal rank 0 is well understood, since it is the class of totally unimodular matrices, no general characterization is known for integral matrices with the Edmonds-Johnson property.

Another class of matrices known to have the Edmonds-Johnson property is the class of the edge-node incidence matrices of bidirected graphs with no odd- K_4 minors (Gerards and Schrijver [19]).

The matrices in such class, as well as the ones considered by Edmonds and Johnson in [16], are *totally half-modular*, that is, they are integral matrices such that for any nonsingular square submatrix B , $2B^{-1}$ is integral. Gerards and Schrijver posed the question of which totally half-modular matrices have the Edmonds-Johnson property, and proposed a characterization of this class in terms of excluded minors [18]. As far as we know, this question is wide open.

In Chapter 1 we provide some definitions and results that will be needed later.

In Chapter 2 we introduce the Edmonds-Johnson property, and survey some related results.

Our contributions are presented in the remaining two chapters.

In Chapter 3, we study systems of the form

$$\begin{aligned} b &\leq Mx \leq c \\ l &\leq x \leq u, \end{aligned} \tag{1}$$

for integral vectors b, c, l, u , where M is obtained from a totally unimodular matrix with two nonzero elements per row by multiplying by 2 some of its columns.

The case where M is obtained from the transpose of the incidence matrix of a bipartite graph by multiplying by 2 some of the columns, has been studied by Conforti et al. in [7]. In this case, they derived an explicit characterization of the inequalities defining the integer hull, and showed that the problem of maximizing a linear function cx , with c integral, over the integer hull of (1) can be solved in strongly polynomial time.

We give an explicit description of a totally dual integral system that describes the integer hull of the polyhedron P defined by (1). Since the inequalities of such totally dual integral system are Chvátal inequalities for P , this implies that the matrix M has the Edmonds-Johnson property. We also derive a strongly polynomial time algorithm to find an integral optimal dual solution for the problem of maximizing a linear function with integer coefficients over the totally dual integral system describing the integer hull of (1). The results in Chapter 3 are joint work with G. Zambelli [11].

In Chapter 4 we study totally half-modular matrices obtained from matrices $0, \pm 1$ with at most two nonzero entries per column by multiplying by 2 some of the columns. We show that the matrix

$$M_4 = \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

is the only minor minimal matrix in such class (up to multiplying rows and columns by -1) that does not have the Edmonds-Johnson property. We will give a formal definition of minor in Chapter 2. The theorem of Edmonds and Johnson discussed above is the special case where the sum of the absolute values of the entries in each column is at most 2. We will also show that, for each matrix M in this class that does not contain M_4 as a minor, one can minimize in polynomial time any linear function over the integer hull of $b \leq Mx \leq c$, $l \leq x \leq u$, for all integral vectors b, c, l, u . The results in Chapter 4 are joint work with A. Musitelli and G. Zambelli [10]. A partial result was shown by Del Pia and Zambelli [12].

Sommario

Il *rango forte di Chvátal* di una matrice razionale A è il più piccolo numero t tale che il poliedro definito dal sistema $b \leq Ax \leq c, l \leq x \leq u$ ha rango di Chvátal al più t per tutti i vettori interi b, c, l, u . Matrici con rango forte di Chvátal al più 1 si dicono avere la *proprietà di Edmonds-Johnson*, poiché fu mostrato da Edmonds e Johnson [16] che ogni matrice intera $A = (\alpha_{ij})$ tale che $\sum_i |\alpha_{ij}| \leq 2$ per ogni indice di colonna j ha rango di Chvátal al più 1.

Mentre la classe di matrici intere con rango forte di Chvátal 0 è ben caratterizzata, poiché è la classe delle matrici totalmente unimodulari, non è nota una caratterizzazione generale per matrici intere con la proprietà di Edmonds-Johnson.

Un'altra classe nota di matrici con la proprietà di Edmonds-Johnson è la classe delle matrici di incidenza arco-nodo di grafi biorientati senza minori odd- K_4 (Gerards e Schrijver [19]).

Le matrici in questa classe, come quelle considerate da Edmonds e Johnson in [16], sono *totalmente $\frac{1}{2}$ -modulari*, ovvero sono matrici intere tali che per ogni sottomatrice quadrata non singolare B , $2B^{-1}$ è intera. Gerards e Schrijver posero la domanda di quali matrici totalmente $\frac{1}{2}$ -modulari hanno la proprietà di Edmonds-Johnson, e proposero una caratterizzazione di questa classe in termini di minori esclusi [18]. Per quanto ne sappiamo, questa domanda è ancora aperta.

Nel Capitolo 1 forniamo alcune definizioni e risultati che saranno necessarie più avanti.

Nel Capitolo 2 introduciamo la proprietà di Edmonds-Johnson, ed esaminiamo qualche risultato correlato.

I nostri contributi sono presentati nei rimanenti due capitoli.

Nel Capitolo 3, studiamo sistemi nella forma

$$\begin{aligned} b &\leq Mx \leq c \\ l &\leq x \leq u, \end{aligned} \tag{2}$$

per vettori interi b, c, l, u , dove M è ottenuta da una matrice totalmente unimodulare con due elementi diversi da zero per riga moltiplicando per 2 alcune colonne.

Il caso in cui M è ottenuta dalla trasposta della matrice di incidenza di un grafo bipartito moltiplicando per 2 alcune colonne, è stato studiato da Conforti et al. in [7]. In questo caso, loro hanno derivato una caratterizzazione esplicita delle disuguaglianze che definiscono l'involuppo convesso dei punti interi, e hanno mostrato che il problema di massimizzare una funzione lineare cx , con c intero, nell'involuppo convesso dei punti interi di (2) può essere risolto in tempo fortemente polinomiale.

Noi diamo una descrizione esplicita di un sistema totally dual integral che descrive l'involuppo convesso dei punti interi del poliedro P definito da (2). Dato che le disuguaglianze di tale sistema totally dual integral sono disuguaglianze di Chvátal per P , questo implica che la matrice M ha la proprietà di Edmonds-Johnson. Inoltre deriviamo un algoritmo fortemente polinomiale per trovare una soluzione duale ottima intera per il problema di massimizzare una funzione lineare con coefficienti interi nel sistema totally dual integral che descrive l'involuppo convesso dei punti interi di (2). I risultati nel Capitolo 3 sono ottenuti in collaborazione con G. Zambelli [11].

Nel Capitolo 4 studiamo matrici totalmente $\frac{1}{2}$ -modulari ottenute da matrici $0, \pm 1$ con al più due elementi non zero per colonna moltiplicando per 2 alcune colonne. Mostriamo che la matrice

$$M_4 = \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

è l'unica matrice minimale rispetto alla relazione minore in tale classe (almeno di moltiplicare righe e colonne per -1) che non ha la proprietà di Edmonds-Johnson. Daremo una definizione formale di minore nel Capitolo 2. Il teorema di Edmonds e Johnson discusso sopra è il caso particolare in cui la somma dei valori assoluti degli elementi in ogni colonna è al più 2. Mostriamo anche che, per ogni matrice M in questa classe che non contiene M_4 come minore, si può minimizzare in tempo polinomiale ogni funzione lineare nell'involuppo convesso dei punti interi di $b \leq Mx \leq c$, $l \leq x \leq u$, per tutti i vettori interi b, c, l, u . I risultati nel Capitolo 4 sono ottenuti in collaborazione con A. Musitelli e G. Zambelli [10]. Un risultato parziale è stato mostrato da Del Pia e Zambelli [12].

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Chapter 1

Introduction

1.1 Graph theory

In this thesis we will work with graphs. Therefore, in this section we give a brief review of some elementary concepts in graph theory. We refer the reader to West [34] for graph theory notations and for proofs of the mentioned results.

1.1.1 Undirected graphs

An (*undirected*) *graph* is a pair $G = (V, E)$, where V is a finite set, and E is a multiset of unordered pairs of elements of V . The elements of V are called the *nodes* of G , and the elements of E are called the *edges* of G . We denote with vw an edge $\{v, w\}$ in E . The nodes v, w are called the *endnodes* of vw . If $v = w$ then the edge $vw = vv$ is called a *loop*. Given a graph G , we denote with $V(G)$ the set of nodes of G , with $E(G)$ the set of edges of G , and with $L(G)$ the set of loops in $E(G)$.

The term ‘multiset’ in the definition of graph means that a pair of nodes may occur several times in E . A pair occurring more than once in E is called a *parallel edge*. So distinct edges may be represented in E by the same pair. Nevertheless, we shall often speak of ‘an edge vw ’ or even of ‘the edge vw ’, where ‘an edge with endnodes v, w ’ would be more correct. A *loopless* graph is a graph having no loops, a *simple graph* is a graph having no loops or parallel edges.

We shall say that an edge vw *connects* the nodes v and w . The nodes v and w are *adjacent* if there is an edge connecting v and w . The edge vw is said to be *incident* with the node v and with the node w , and conversely. Two edges e and f are *adjacent* if they have an endnode in common. We say that a node is *isolated* if it is not incident with any edge.

If $U \subseteq V$, then $\delta_G(U)$ (or $\delta(U)$ when there is no ambiguity) denotes the set of edges in E that have exactly one endnode in U . Notice that $e \in \delta_G(U)$ for every loop e with endnode in U . When v is a node of G , we will write $\delta(v)$ instead of $\delta(\{v\})$. The number $|\delta_G(v)|$ is the *degree* of v in G .

A graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. If E' is the multiset of all edges of G with all endnodes in V' , then G' is said to be *induced* by V' , and we denote G' by $G[V']$, or by $G \setminus V''$, where $V'' = V \setminus V'$. When v is a node of G , we will write $G \setminus v$ instead of $G \setminus \{v\}$. If $V = V'$, then we denote G' by $G \setminus E''$, where $E'' = E \setminus E'$. When e is an edge of G , we will write $G \setminus e$ instead of $G \setminus \{e\}$.

A *walk* in the graph $G = (V, E)$ from v_0 to v_t , is a sequence of the form

$$W = v_0, e_1, v_1, e_2, \dots, v_{t-1}, e_t, v_t \quad (1.1)$$

where v_0, \dots, v_t are nodes and e_1, \dots, e_t are edges, such that $e_i = v_{i-1}v_i$ for $i = 1, \dots, t$. If there is no ambiguity, we often write walk W just as a sequence of nodes, i.e. $W = v_0, v_1, \dots, v_t$. The nodes v_0 and v_t are the *endnodes* of the walk, and we say that the walk *starts* in v_0 and *ends* in v_t . We call walk W a $v_0 - v_t$ -*walk*, and it is said to *connect* v_0 and v_t . We define $V(W) = \{v_0, \dots, v_t\}$, and $E(W) = \{e_1, \dots, e_t\}$.

Walk W is a *trail* if all edges in (1.1) are different. A trail T' is a *subtrail* of a trail T if T' is a subsequence of T , say $T' = v_i, e_{i+1}, v_{i+1}, e_{i+2}, \dots, v_{j-1}, e_j, v_j$, $i < j$, such that $V(T') = \{v_h : i \leq h \leq j\}$, $E(T') = \{e_h : i + 1 \leq h \leq j\}$.

Walk W is a *path* if all nodes and edges in (1.1) are different. If P is a path, a subtrail of P is called a *subpath* of P .

The *length* of walk W is t . The *distance* between two nodes r and s in a graph is the minimum length of a path connecting r and s .

If $v_0 = v_t$, walk W is called *closed*. A closed walk of length at least two and without repeated edges or nodes (except for the endnodes) is called a *cycle*. So, for the purpose of this thesis, loops are not cycles. An edge connecting two nodes of a cycle which are not connected by an edge of the cycle is called a *chord* of the cycle. An *Eulerian circuit* in a graph $G = (V, E)$ is a closed trail C such that $E(C) = E$. It is well known that a loopless graph G has an Eulerian circuit if and only if $|\delta(v)|$ is even for every $v \in V$.

A graph is *connected* if each two nodes of the graph are connected by a path. The (*connected*) *components* of a graph are its maximally connected subgraphs. A (*node*-)*cutset* of a graph G is a set $V' \subseteq V$ such that $G \setminus V'$ has strictly more connected components than G . If $\{v\}$ is a node-cutset of G , we call v a *cutnode* of G . An *edge-cutset* of a graph G is a set $E' \subseteq E$ such that $G \setminus E'$ has strictly more connected components than G . If $\{e\}$ is an edge-cutset of G , we call e a *cutedge* of G . A *block* of a graph G is a maximal subgraph of G that does not have a cutnode.

A graph having no cycles is said to be *acyclic*, and is called a *forest*. A *tree* is a connected forest. It is not difficult to see that the following are equivalent for a given simple graph $G = (V, E)$:

- (i) G is a tree;
- (ii) G contains no cycles and $|E| = |V| - 1$;
- (iii) G is connected and $|E| = |V| - 1$;
- (iv) any two nodes of G are connected by exactly one simple path.

If we add one new edge connecting two nodes of the tree, we obtain a graph with a unique cycle. Each tree with at least two nodes has at least two nodes of degree one.

A subgraph $G' = (V', E')$ of $G = (V, E)$ is a *spanning (sub)tree* of G if $V' = V$ and G' is a tree. Then G has a spanning tree if and only if G is connected.

In a graph $G = (V, E)$, we say that a set $E' \subseteq E$ is a *star centered at node v* if all the edges in E' are incident with $v \in V$. A *matching* is a set of pairwise disjoint edges. A matching covering all nodes is called a *perfect matching*.

A graph $G = (V, E)$ is called *bipartite* if V can be partitioned into two classes V_1 and V_2 such that each edge of G contains a node in V_1 and a node in V_2 . The sets V_1 and V_2 are called *sides*. It is easy to see that a graph G is bipartite if and only if G contains no cycles of odd length.

The *incidence matrix* of G is the matrix with rows and columns indexed by V and E , respectively, where the entry in position (v, e) is 1 if edge e is not a loop and it is incident with node v , is 2 if edge e is a loop incident with node v , and is 0 otherwise. Notice that the incidence matrix of a loopless graph is a $\{0, 1\}$ -matrix. The *edge-node incidence matrix* of G is the transpose of the incidence matrix of G .

1.1.2 Directed graphs

A *directed graph*, or *digraph*, is a pair $D = (V, A)$, where V is a finite set, and A is a finite multiset of ordered pairs of distinct elements of V . The elements of V are called the *nodes*, and the elements of A are called the *arcs* of D . The nodes v and w are called the *tail* and the *head* of the arc (v, w) , respectively.

So the difference with undirected graphs is that orientations are given to the pairs. Each directed graph gives rise to an *underlying* undirected graph,

in which we forget the orientation of the arcs. When there is no ambiguity, we use ‘undirected’ terminology for directed graphs.

We say that the arc (v, w) *enters* w and *leaves* v . If U is a set of nodes such that $v \notin U$ and $w \in U$, then (v, w) is said to *enter* U and to *leave* $V \setminus U$. If $U \subseteq V$, then $\delta_D^-(U)$ (or $\delta^-(U)$ when there is no ambiguity) denotes the set of arcs of D entering U , and $\delta_D^+(U)$ (or $\delta^+(U)$) denotes the set of arcs of D leaving U . $\delta^-(v)$ and $\delta^+(v)$ stand for $\delta^-(\{v\})$ and $\delta^+(\{v\})$. The *in-degree* of v is $|\delta^-(v)|$, while the *out-degree* of v is $|\delta^+(v)|$.

A *directed walk*, from v_0 to v_t , or a $v_0 - v_t$ -walk, in a digraph $D = (V, A)$ is a sequence of the form

$$W = v_0, a_1, v_1, a_2, \dots, v_{t-1}, a_t, v_t \quad (1.2)$$

where v_0, \dots, v_t are nodes and a_1, \dots, a_t are arcs, such that $a_i = (v_{i-1}, v_i)$ for $i = 1, \dots, t$. If there is no ambiguity, we often write directed walk W just as a sequence of nodes, i.e. $W = v_0, v_1, \dots, v_t$. Walk W is said to *start* in v_0 and to *end* in v_t . The nodes v_0 and v_t are the *endnodes* of the walk. The number t is the *length* of the walk W . Directed walk W is a *directed trail* if all arcs in (1.2) are different. Directed walk W is a *directed path* if all nodes and arcs in (1.2) are different.

A $v_0 - v_t$ -walk is *closed* if $v_0 = v_t$. A *directed cycle* is a closed directed walk of length at least two, without repeated nodes or arcs (except for its endnodes). We call *acyclic* a digraph with no directed cycles.

An (*undirected*) *walk* (resp. *trail*, *path*, *cycle*) is a walk (resp. trail, path, cycle) in the underlying undirected graph. In a natural way, an undirected walk or cycle in a directed graph has *forward* arcs and *backward* arcs.

A digraph D is *connected* if its underlying undirected graph is connected.

The *incidence matrix* of a digraph $D = (V, A)$ is the matrix with rows and columns indexed by V and A , respectively, where the entry in position (v, a) is $+1$ if v is the head of a , -1 if v is the tail of a , or 0 otherwise. The *edge-node incidence matrix* of D is the transpose of the incidence matrix of D .

1.1.3 Bidirected graphs

A *bidirected graph* is a triple $G = (V, E, \sigma)$, where (V, E) is an undirected graph (possibly with loops and parallel edges) and σ is a *signing* of (V, E) , i.e. a map that assigns to each $e \in E$ and $v \in e$ a *sign* $\sigma_{v,e} \in \{+1, -1\}$. We say that a nonloop edge $e = vw$ of G is a *++ edge* if $\sigma_{v,e} = \sigma_{w,e} = +1$, it is a *-- edge* if $\sigma_{v,e} = \sigma_{w,e} = -1$, and it is a *+− edge* otherwise. For convenience, we define $\sigma_{v,e} := 0$ if $v \notin e$. The *odd edges* of G are the *++ edges* and *--*

edges. Each bidirected graph gives rise to an *underlying* undirected graph, in which we forget σ . Sometimes, when no misunderstanding is possible, we use ‘undirected’ terminology for bidirected graphs.

A cycle C in G , is *even* if the number of odd edges in it is even. This is equivalent to saying that the sum of the signs on the edges in C is divisible by 4 ($\sum_{vw \in C} (\sigma_{v,vw} + \sigma_{w,vw}) \equiv_4 0$). Otherwise we say that C is *odd*. We call a bidirected graph *bipartite* if it does not contain any odd cycle.

G is *connected* if its underlying undirected graph is connected.

The *sign matrix* of a bidirected graph G is the $V \times E$ matrix $\Sigma(G) = (\sigma_{v,e})$. The *incidence matrix* of G is obtained from the sign matrix of G by multiplying by 2 the columns corresponding to the loops of G . The *edge-node incidence matrix* of G is the transpose of the incidence matrix of G .

1.2 Polyhedra and linear inequalities

In this section we introduce the fundamental notion of polyhedra. We refer the reader to Schrijver [29] for a more detailed introduction and for the proofs of the following theorems.

1.2.1 Polyhedra and polytopes

A set P of vectors in \mathbb{R}^n is called a (*convex*) *polyhedron* if

$$P = \{x : Ax \leq b\} \quad (1.3)$$

for some matrix A and vector b . Thus P is a polyhedron if it is the intersection of finitely many affine half-spaces, where an *affine half-space* is a set of the form $\{x : ax \leq \beta\}$ for some nonzero row vector a and some number β . If (1.3) holds, we say that $Ax \leq b$ *defines* or *determines* P .

The *convex hull* of a set X of vectors is the inclusionwise minimal convex set containing X , and is denoted by $\text{conv}(X)$. A set of vectors is a (*convex*) *polytope* if it is the convex hull of finitely many vectors. The following theorem gives a fundamental link between polytopes and polyhedrons.

Theorem 1.1. (*Minkowski-Weyl theorem for polytopes.*) *A set P is a polytope if and only if P is a bounded polyhedron.*

1.2.2 Faces

If c is a nonzero vector, P is a polyhedron, and $\delta = \max\{cx : x \in P\}$, the affine hyperplane $\{x : cx = \delta\}$ is called a *supporting hyperplane* of P . A

subset F of P is called a *face* of P if $F = P$ or if F is the intersection of P with a supporting hyperplane of P .

Theorem 1.2. *F is a face of P if and only if F is nonempty and $F = \{x \in P : A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.*

It is known that the dimension of F is $n - \text{rk}(A')$, where $\text{rk}(A')$ denotes the rank of A' .

It follows that:

- (i) P has only finitely many faces;
- (ii) each face is a nonempty polyhedron;
- (iii) if F is a face of P and $F' \subseteq F$, then: F' is a face of P if and only if F' is a face of F .

1.2.3 Minimal faces and vertices

A *minimal face* of P is a face not containing any other face. It is well known that a face F of P is a minimal face if and only if F is an affine subspace. In fact, there is the following result of Hoffman and Kruskal [25].

Theorem 1.3. *A set F is a minimal face of P , if and only if $\emptyset \neq F \subseteq P$ and $F = \{x : A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.*

In particular, all minimal faces of P have the same dimension, namely n minus the rank of A . If each minimal face of P consists of just one point, then P is called *pointed*. These points (or these minimal faces) are called the *vertices* of P . So each vertex is determined by n linearly independent equations from the systems $Ax = b$.

Remark 1.4. *A vector $\bar{x} \in P$ is a vertex of P if and only if it satisfies n linearly independent equations of the system $Ax = b$.*

Moreover, a vector $x \in P$ is a vertex of P if and only if x cannot be expressed as a convex combination of vectors in P .

1.2.4 A polynomial result in linear programming

An interesting result was obtained by Tardos [32, 33], who showed that any linear program $\max\{cx : Ax \leq b\}$ can be solved in at most $p(\text{size}(A))$ elementary arithmetic operations on numbers of size polynomially bounded by $\text{size}(A, b, c)$, where p is a polynomial. Thus the sizes of b and c do not contribute in the number of arithmetic steps. In particular the class of LP-problems where A is a $\{0, \pm 1\}$ -matrix has a strongly polynomial algorithm.

Theorem 1.5. *There exists an algorithm which solves a given rational LP-problem $\max\{cx : Ax \leq b\}$ in at most $P(\text{size}(A))$ elementary arithmetic operations on numbers of size polynomially bounded by $\text{size}(A, b, c)$, for some polynomial P .*

Corollary 1.6. *There exists a strongly polynomial algorithm for rational LP-problems with $\{0, \pm 1\}$ -constraint matrix.*

1.3 Integral polyhedra

A vector or matrix is called *integral* if its entries are all integers. Define, for any polyhedron P , the *integer hull* P_I of P by $P_I := \text{conv}(P \cap \mathbb{Z}^n)$, i.e. the convex hull of the integral vectors in P . So, given a rational matrix A , and rational vectors b and c , determining $\max\{cx : Ax \leq b, x \text{ integral}\}$ is equivalent to determining $\max\{cx : x \in P_I\}$, for $P := \{x : Ax \leq b\}$. We say that a rational polyhedron is *integral* if each face contains integral vectors. This is equivalent to saying that the polyhedron is the convex hull of the integral vectors contained in it.

We state the following result of Meyer [28], which is trivial if P is bounded.

Theorem 1.7. *For any rational polyhedron P , the set P_I is also a polyhedron.*

1.3.1 Totally unimodular matrices

An integral matrix is *unimodular* if it has determinant ± 1 . In this section we consider *totally unimodular matrices*, which are matrices with all subdeterminants equal to $+1$, -1 , or 0 (in particular, each entry is $+1$, -1 , or 0). Notice that A is totally unimodular if and only if the transpose of A is totally unimodular. Totally unimodular matrices yield a prime class of linear programming problems with integral optimum solutions.

In the present section we introduce the basic theory and examples of totally unimodular matrices. We state Hoffman and Kruskal's characterization of totally unimodular matrices, providing the link between total unimodularity and integer linear programming. Then we give some further characterizations, and we discuss some classes of examples of totally unimodular matrices.

Theorem 1.8. (Hoffman and Kruskal's theorem). *Let A be an integral matrix. Then A is totally unimodular if and only if for each integral vector b the polyhedron $\{x : x \geq 0, Ax \leq b\}$ is integral.*

Since for any totally unimodular matrix A also the matrix

$$\begin{pmatrix} I \\ -I \\ A \\ -A \end{pmatrix}$$

is totally unimodular, it follows that an integral matrix A is totally unimodular if and only if for all integral vectors b, c, l, u the vertices of the polytope $\{x : l \leq x \leq u, b \leq Ax \leq c\}$ are integral. Similarly, one may derive from Theorem 1.8 that an integral matrix A is totally unimodular if and only if one of the following polyhedra is integral, for each integral vector b and for some integral vector c :

$$\begin{aligned} &\{x : x \leq c, Ax \leq b\}, \{x : x \leq c, Ax \geq b\}, \\ &\{x : x \geq c, Ax \leq b\}, \{x : x \geq c, Ax \geq b\}. \end{aligned}$$

Formulated in terms of linear programming, Theorem 1.8 says the following.

Corollary 1.9. *An integral matrix A is totally unimodular if and only if for all integral vectors b and c both sides of the linear programming duality equation*

$$\max\{cx : x \geq 0, Ax \leq b\} = \min\{yb : y \geq 0, yA \geq c\}$$

are achieved by integral vectors x and y (if they are finite).

There are other characterizations of totally unimodular matrices. We collect some of them, together with the above characterization, in the following theorem. Characterizations (ii) and (iii) are due to Hoffman and Kruskal [25], characterization (iv) to Ghouila-Houri [20], characterization (v) to Camion [1, 2, 3].

Theorem 1.10. *Let A be a matrix with entries 0, +1, or -1. Then the following are equivalent:*

- (i) *A is totally unimodular, i.e. each square submatrix of A has determinant 0, +1, or -1;*
- (ii) *for each integral vector b the polyhedron $\{x : x \geq 0, Ax \leq b\}$ has only integral vertices;*
- (iii) *for all integral vectors b, c, l, u the polyhedron $\{x : l \leq x \leq u, b \leq Ax \leq c\}$ has only integral vertices;*

- (iv) *each collection of columns of A can be partitioned into two sets so that the sum of the columns in one set minus the sum of the columns in the other set is a vector with entries 0, +1, and -1;*
- (v) *the sum of the entries in any square submatrix with even row and column sums is divisible by four.*

We give three well known examples of totally unimodular matrices due to Hoffman and Kruskal [25], and Heller and Tompkins [24].

Example 1. Bipartite graphs

Let $G = (V, E)$ be an undirected graph, and let M be the $V \times E$ -incidence matrix of G . Then:

Theorem 1.11. *M is totally unimodular if and only if G is bipartite.*

So M is totally unimodular if and only if the rows of M can be split into two classes so that each column contains a 1 in each of these classes. Theorem 1.11 easily follows from Ghouila-Houri's characterization of totally unimodular matrices ((iv) in Theorem 1.10).

Example 2. Directed graphs

Let $D = (V, A)$ be a directed graph, and let M be the $V \times A$ -incidence matrix of D . Then M is totally unimodular. In other words:

Theorem 1.12. *A $\{0, \pm 1\}$ -matrix with in each column exactly one +1 and exactly one -1 is totally unimodular.*

Theorem 1.12 follows directly from Ghouila-Houri's characterization ((iv) in Theorem 1.10).

Example 3. A combination of Examples 1 and 2

Theorem 1.13. *Let M be a $\{0, \pm 1\}$ -matrix with exactly two nonzeros in each column. Let G be the bidirected graph whose incidence matrix is M . Then the following are equivalent:*

- (i) *M is totally unimodular;*
- (ii) *the rows of M can be split into two classes such that for each column if the two nonzeros in the column have the same sign then they are in different classes, and if they have opposite sign then they are both in one and the same class;*

(iii) G is bipartite.

This follows easily from Example 2 (multiply all rows in one of the classes by -1 , and we obtain a $\{0, \pm 1\}$ -matrix with in each column exactly one $+1$ and exactly one -1) and from (iv) and (v) of Theorem 1.10. Conversely, it includes Examples 1 and 2.

1.3.2 Total dual integrality

In Section 1.3.1 we introduced totally unimodular matrices, which are exactly those integral matrices A with the property that the polyhedron $P := \{x \geq 0 : Ax \leq b\}$ is integral for each integral vector b .

In this section we fix A and b , and we study integral polyhedra and the related notion of total dual integrality. We refer the reader to [29] for an extensive treatment on total dual integrality.

The basis for this section is the following result of Edmonds and Giles [14].

Theorem 1.14. *A rational polyhedron P is integral, if and only if each rational supporting hyperplane of P contains integral vectors.*

The following is an equivalent formulation.

Corollary 1.15. *Let $Ax \leq b$ be a system of rational linear inequalities. Then $\max\{cx : Ax \leq b\}$ is achieved by an integral vector x for each vector c for which the maximum is finite, if and only if $\max\{cx : Ax \leq b\}$ is an integer for each integral vector c for which the maximum is finite.*

We call a rational system $Ax \leq b$ of linear inequalities *totally dual integral*, abbreviated *TDI*, if the minimum in the LP-duality equation

$$\max\{cx : Ax \leq b\} = \min\{yb : y \geq 0, yA = c\}$$

has an integral optimum solution y for each integral vector c for which the minimum is finite. So by Corollary 1.9, if A is totally unimodular, then $Ax \leq b$ is TDI for each rational vector b .

Edmonds and Giles [14] showed that total dual integrality of $Ax \leq b$ implies that also the maximum has an integral optimum solution, if b is integral.

Corollary 1.16. *If $Ax \leq b$ is a TDI-system and b is integral, the polyhedron $\{x : Ax \leq b\}$ is integral.*

Note however that total dual integrality is not a property of just polyhedra. The systems

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

define the same polyhedron, but the first system is TDI and the latter not. Generally a TDI-system contains more constraints than necessary for just defining the polyhedron.

The following Theorem basically states that each face of a TDI-system is TDI.

Theorem 1.17. *Let $Ax \leq b, ax \leq \beta$ be a TDI-system. Then the system $Ax \leq b, ax = \beta$ is also TDI.*

We give a proof of this result from Schrijver [29].

Proof. Let c be an integral vector, with

$$\begin{aligned} & \max\{cx : Ax \leq b, ax = \beta\} \\ &= \min\{yb + (\lambda - \mu)\beta : y \geq 0, \lambda, \mu \geq 0, yA + (\lambda - \mu)a = c\} \end{aligned} \quad (1.4)$$

finite. Let $x^*, y^*, \lambda^*, \mu^*$ attain these optima (possibly being fractional). Let $c' := c + Na$ where N is an integer satisfying $N \geq \mu^* - \lambda^*$ and Na integer. Then the optima

$$\begin{aligned} & \max\{c'x : Ax \leq b, ax \leq \beta\} \\ &= \min\{yb + \lambda\beta : y \geq 0, \lambda \geq 0, yA + \lambda a = c'\} \end{aligned} \quad (1.5)$$

are finite, since $x := x^*$ is a feasible solution for the maximum, and $y := y^*, \lambda := \lambda^* + N - \mu^*$ is a feasible solution for the minimum.

Since $Ax \leq b, ax \leq \beta$ is TDI, the minimum (1.5) has an integer optimum solution, say, $\tilde{y}, \tilde{\lambda}$. Then $y := \tilde{y}, \lambda := \tilde{\lambda}, \mu := N$ is an integer optimum solution for the minimum in (1.4): it is feasible in (1.4), and it is optimum as:

$$\begin{aligned} \tilde{y}b + (\tilde{\lambda} - N)\beta &= \tilde{y}b + \tilde{\lambda}\beta - N\beta \\ &\leq y^*b + (\lambda^* + N - \mu^*)\beta - N\beta = y^*b + (\lambda^* - \mu^*)\beta. \end{aligned}$$

(Here \leq follows from the fact that $y^*, \lambda^* + N - \mu^*$ is a feasible solution for the minimum in (1.5), and $\tilde{y}, \tilde{\lambda}$ is an optimum solution for this minimum.) So the minimum in (1.4) has an integral optimum solution. \square

The following is an important result of Giles and Pulleyblank [21].

Theorem 1.18. *For each rational polyhedron P there exists a TDI-system $Ax \leq b$ with A integral and $P = \{x : Ax \leq b\}$. Here b can be chosen to be integral if and only if P is integral.*

Chandrasekaran [4] showed the following result.

Theorem 1.19. *Let $Ax \leq b$ be a TDI-system with A integral, and let c be an integral vector. Then an integral optimum solution for $\min\{yb : y \geq 0, yA = c\}$ can be found in polynomial time.*

1.3.3 Chvátal-Gomory cuts

Let H be a rational affine half-space $\{x : cx \leq \delta\}$, where c is a nonzero vector whose components are relatively prime integers (each rational affine half-space can be represented in this way), then clearly $H_I = \{x : cx \leq \lfloor \delta \rfloor\}$. Geometrically, H_I arises from H by shifting the bounding hyperplane of H until it contains integral vectors. Now define for any polyhedron P :

$$P' := \bigcap_{H \supseteq P} H_I$$

where the intersection ranges over all rational affine half-spaces H with $H \supseteq P$. (Clearly, we may restrict the intersection to half-spaces whose bounding hyperplane is a supporting hyperplane of P .) The polyhedron P' is called the *first (Chvátal) closure* of P . As $P \subseteq H$ implies $P_I \subseteq H_I$, it follows that $P_I \subseteq P'$. So $P \supseteq P' \supseteq P'' \supseteq \dots \supseteq P_I$.

The half-spaces H_I are called *Chvátal-Gomory cuts*, while their defining inequalities are called *Chvátal(-Gomory) inequalities*.

A constraint is *valid* for a set S if each element in S satisfies this constraint. Algebraically, if $P = \{x : Ax \leq b\}$, then any rational valid inequality for P is of the form $(\lambda A)x \leq \lambda b$ with $\lambda \geq 0$, where we can assume that λA is integral. Therefore any Chvátal-Gomory inequality can be written in the form $(\lambda A)x \leq \lfloor \lambda b \rfloor$, where $\lambda \geq 0$ is such that λA is integral. The vector λ is called a *Chvátal-Gomory multiplier*. The *first (Chvátal) closure* of a system $Ax \leq b$, is the first closure of the polyhedron defined by such system.

We say that a Chvátal inequality for a system of linear inequalities is *nontrivial* if it is not implied by such system. Two inequalities $\alpha x \leq \beta$ and $\alpha' x \leq \beta'$ valid for the first Chvátal closure of a system of linear inequalities are *equivalent* if they define the same face of the first Chvátal closure.

Lemma 1.20. *If A and b are integral, any nontrivial Chvátal inequality for $Ax \leq b$ is equivalent to an inequality of the form $(\lambda A)x \leq \lfloor \lambda b \rfloor$ such that $0 \leq \lambda < 1$, λA is integral, and λb is not integral.*

Proof. By definition, any Chvátal-Gomory inequality for P can be written in the form $(\lambda A)x \leq \lfloor \lambda b \rfloor$, where $\lambda \geq 0$ is such that λA is integral. Since $(\lambda - \lfloor \lambda \rfloor)A$ is integral, then $(\lambda - \lfloor \lambda \rfloor)Ax \leq \lfloor (\lambda - \lfloor \lambda \rfloor)b \rfloor$ is valid for the first Chvátal closure of $Ax \leq b$. Furthermore $(\lambda A)x \leq \lfloor \lambda b \rfloor$ is the sum of $(\lambda - \lfloor \lambda \rfloor)Ax \leq \lfloor (\lambda - \lfloor \lambda \rfloor)b \rfloor$ and $\lfloor \lambda \rfloor Ax \leq \lfloor \lambda \rfloor b$, where the last inequality is implied by $Ax \leq b$. Clearly, if λb is integral, then the Chvátal inequality $(\lambda A)x \leq \lfloor \lambda b \rfloor$ is implied by the system $Ax \leq b$. \square

Theorem 1.21. *For any rational polyhedron P , P' is a polyhedron again.*

In general, it was shown by Eisenbrand [17] that optimizing over the first Chvátal closure of a system of linear inequalities is \mathcal{NP} -complete.

Theorem 1.22. *For each rational polyhedron P there exists a number t such that $P^{(t)} = P_I$.*

(Here: $P^{(0)} := P$, $P^{(t+1)} := P^{(t)'}.$)

A direct consequence of Theorem 1.22 is Theorem 1.14: if each rational supporting hyperplane of a rational polyhedron P contains integral vectors, then $P = P'$ and hence $P = P_I$. This theorem also implies a result of Chvátal [5] for not-necessarily rational polytopes:

Corollary 1.23. *For each polytope P there exists a number t such that $P^{(t)} = P_I$.*

The *Chvátal rank* of a polyhedron P is the smallest number t such that $P^{(t)} = P_I$. The *Chvátal rank* of a system of linear inequalities is the Chvátal rank of the polyhedron defined by such system.

The following Lemma will be useful in the next Chapters.

Lemma 1.24. *Consider the systems*

$$\begin{aligned} ax &\leq \beta \\ Ax &\leq b \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} ax + s &= \beta \\ Ax &\leq b \\ s &\geq 0. \end{aligned} \tag{1.7}$$

Where A is an integral matrix, b and a are integral vectors, and β is integer. Then:

- (i) \bar{x} is in the first closure of (1.6) if and only if $(\beta - a\bar{x}, \bar{x})$ is in the first closure of (1.7).

(ii) (1.6) has Chvátal rank at most 1 if and only if (1.7) has Chvátal rank at most 1.

Proof. (i): Let \bar{x} be in the first closure of (1.6). By contradiction assume that $(\beta - a\bar{x}, \bar{x})$ is not in the first closure of (1.7). Since $(\beta - a\bar{x}, \bar{x})$ satisfies (1.7), there exists Chvátal inequality for (1.7) not satisfied by $(\beta - a\bar{x}, \bar{x})$. Such inequality can be written in the form $(\lambda a + \mu A)x + (\lambda - \gamma)s \leq \lfloor \lambda\beta + \mu b \rfloor$, where $\lambda a + \mu A$ and $\lambda - \gamma$ are integral. By substituting $s = \beta - ax$, and since $\lambda - \gamma$ is integer, it follows that \bar{x} does not satisfy $(\mu A + \gamma a)x \leq \lfloor \mu b + \gamma\beta \rfloor$, which is a Chvátal inequality for (1.6), a contradiction.

The opposite implication is trivial, as each Chvátal inequality for (1.6) is also a Chvátal inequality for (1.7).

(ii): Assume that (1.6) has Chvátal rank at most 1, and let (\bar{s}, \bar{x}) be a vector in the first closure of (1.7). Notice that $\bar{s} = \beta - a\bar{x}$. By (i), \bar{x} is in the first closure of (1.6), thus $\bar{x} = \sum_{i \in I} y_i$, where y_i is integral and satisfies (1.6) for every $i \in I$. Notice that $(\beta - ay_i, y_i)$ is integral and satisfies (1.7) for every $i \in I$. Since $(\bar{s}, \bar{x}) = \sum_{i \in I} (\beta - ay_i, y_i)$, then (1.7) has Chvátal rank at most 1.

Conversely, assume that (1.7) has Chvátal rank at most 1, and let \bar{x} be a vector in the first closure of (1.6). By (i), $(\beta - a\bar{x}, \bar{x})$ is in the first closure of (1.7), thus $(\beta - a\bar{x}, \bar{x}) = \sum_{i \in I} (z_i, y_i)$, where z_i and y_i are integral and satisfy (1.7) for every $i \in I$. Hence $\bar{x} = \sum_{i \in I} y_i$, where y_i is integral and satisfies (1.6) for every $i \in I$. Hence (1.6) has Chvátal rank at most 1. \square

We state a result of Cook, Gerards, Schrijver, and Tardos [8].

Theorem 1.25. *For each rational matrix A there exists a number t such that for each column vector b one has: $\{x : Ax \leq b\}^{(t)} = \{x : Ax \leq b\}_I$.*

Chapter 2

The Edmonds-Johnson property

Theorem 1.25 states that for each rational matrix A there exists a number t such that for each column vector b one has: $\{x : Ax \leq b\}^{(t)} = \{x : Ax \leq b\}_I$.

This motivates the following definition. The *strong Chvátal rank* (or *cut-rank*) of a rational matrix A is the smallest number t such that the polyhedron defined by the system $b \leq Ax \leq c, l \leq x \leq u$ has Chvátal rank at most t for all integral vectors b, c, l, u . So Theorem 1.25 states that the (strong) Chvátal rank is a well-defined integer.

It follows from Theorem 1.8 that an integral matrix A has strong Chvátal rank 0 if and only if A is totally unimodular.

Similar characterizations for higher Chvátal ranks are not known. Matrices with strong Chvátal rank at most 1 are said to have the *Edmonds-Johnson property*.

In the following theorem we give some operations that preserve the Edmonds-Johnson property.

Theorem 2.1. *The class of matrices with the Edmonds-Johnson property is closed under the following operations:*

- (i) *permuting rows and columns;*
- (ii) *multiplying rows and columns by -1 ;*
- (iii) *deleting rows and columns;*
- (iv) *dividing by $k \in \mathbb{N}$, $k \geq 2$ a row where all entries are multiple of k ;*

(v) pivoting on a 1 entry, i.e. replacing matrix $\begin{pmatrix} 1 & g \\ f & D \end{pmatrix}$ by the matrix $\begin{pmatrix} -1 & g \\ f & D - fg \end{pmatrix}$, where f is a column vector and g a row vector.

Proof. Let A be a matrix with the Edmonds-Johnson property. By definition, the system $b \leq Ax \leq c, l \leq x \leq u$ has Chvátal rank at most 1 for all integral vectors b, c, l, u .

(i),(ii): If A' is obtained from A by permuting rows or columns, or multiplying them by -1 , then trivially A' has the Edmonds-Johnson property.

(iii): If A' is obtained from A by deleting a column, say corresponding to variable x_j , by taking $l_j = u_j = 0$, it follows trivially that A' has the Edmonds-Johnson property. If A' is obtained from A by deleting a row, say the i -th row, by taking $b_i = -\infty, c_i = +\infty$, it follows trivially that A' has the Edmonds-Johnson property.

(iv): Assume that A' is obtained from A by dividing by k a row, say the i -th, where all entries are multiple of k . The system $b' \leq A'x \leq c', l' \leq x \leq u'$ has Chvátal rank at most 1 for all integral b', c', l', u' because the system $b \leq Ax \leq c, l' \leq x \leq u'$ has Chvátal rank at most 1, where b and c are obtained from b' and c' by multiplying the i -th component by k .

(v): Assume that the matrix $\begin{pmatrix} 1 & g \\ f & D \end{pmatrix}$ has the Edmonds-Johnson property. We want to show that the system

$$\begin{pmatrix} b_0 \\ b \end{pmatrix} \leq \begin{pmatrix} -1 & g \\ f & D - fg \end{pmatrix} \begin{pmatrix} x_0 \\ x \end{pmatrix} \leq \begin{pmatrix} c_0 \\ c \end{pmatrix} \quad (2.1)$$

$$\begin{pmatrix} l_0 \\ l \end{pmatrix} \leq \begin{pmatrix} x_0 \\ x \end{pmatrix} \leq \begin{pmatrix} u_0 \\ u \end{pmatrix}$$

has Chvátal rank at most 1 for all integral $b_0, b, c_0, c, l_0, l, u_0, u$. Let $x'_0 = -x_0 + bx$. By substituting $x_0 = -x'_0 + bx$ in (2.1) we get

$$\begin{pmatrix} l_0 \\ b \end{pmatrix} \leq \begin{pmatrix} 1 & g \\ f & D \end{pmatrix} \begin{pmatrix} x'_0 \\ x \end{pmatrix} \leq \begin{pmatrix} u_0 \\ c \end{pmatrix} \quad (2.2)$$

$$\begin{pmatrix} b_0 \\ l \end{pmatrix} \leq \begin{pmatrix} x'_0 \\ x \end{pmatrix} \leq \begin{pmatrix} c_0 \\ u \end{pmatrix}.$$

Since \bar{x}'_0 is integer if and only if x_0 and x are integral, and \bar{x}_0 is integer if and only if x'_0 and x are integral, than one can easily show that the system (2.1) has Chvátal rank at most 1 if and only if the system (2.2) has Chvátal rank at most 1. Since (2.2) has Chvátal rank at most 1, also (2.1) has Chvátal rank at most 1. \square

2.1 Classes of matrices with the Edmonds-Johnson property

There are few known classes of matrices known to have the Edmonds-Johnson property. In this section we briefly present the two main examples, while in Chapter 3 and 4 we introduce two new classes.

2.1.1 The Edmonds and Johnson's class

Edmonds and Johnson [15, 16] derived the following theorem from Edmonds' characterization of the matching polytope [13].

Theorem 2.2. (*Edmonds and Johnson [16].*) *If $A = (\alpha_{ij})$ is an integral matrix such that $\sum_i |\alpha_{ij}| \leq 2$ for each column index j , then A has the Edmonds-Johnson property.*

We give a short proof of this result, essentially due to M. Singh [31].

Given a bidirected graph G and a subset F of its loops, we denote by $A(G, F)$ the matrix obtained from the sign matrix of G , $\Sigma(G)$, by multiplying by 2 the columns corresponding to the loops in F . In what follows let \mathcal{A} be the family of pairs (G, F) , where G is a bidirected graph and F is a subset of its loops. Thus Theorem 2.2, states that for each pair (G, F) in \mathcal{A} , $A(G, F)$ has the Edmonds-Johnson property.

In the remaining of the section, we denote with $A(G)$ the incidence matrix of a bidirected graph G . Notice that $A(G) = A(G, L(G))$.

Next we show that, in proving Theorem 2.2, we can reduce ourselves to study systems of the form

$$\begin{aligned} A(G)x &= c \\ x &\geq 0, \end{aligned} \tag{2.3}$$

for every loopless bidirected graph G , and $c \in \mathbb{Z}^{V(G)}$.

Notice that, if G is a loopless undirected graph, and $c \in \mathbb{Z}^V(G)$, then the integer hull of (2.3) is the c -matching polytope.

Lemma 2.3. *If (2.3) has Chvátal rank at most 1 for every loopless bidirected graph G and every integral c , then $A(G, F)$ has the Edmonds-Johnson property for every (G, F) in \mathcal{A} .*

Proof. At first we show that if (2.3) has Chvátal rank at most 1 for every loopless bidirected graph G and every integral c , then the system

$$\begin{aligned} A(G)x &= c \\ 0 &\leq x \leq u \end{aligned} \tag{2.4}$$

has Chvátal rank at most 1 for every loopless bidirected graph G and every integral c, u .

Let G be a loopless bidirected graph, let c, u be integral vectors and let \bar{x} be a vector in the first closure of (2.4). Now let G' , c' and \bar{x}' be obtained from G , c and \bar{x} in the following way. For each edge $e = v_1v_2$ in $E(G)$, add a new node v_e and replace e with the path $v_1, v_1v_e, v_e, v_e v_2, v_2$, such that v_1v_e and $v_e v_2$ have a +1 sign in the vertex v_e , the edge v_1v_e has in v_1 the same sign that e had in v_1 , while the edge $v_e v_2$ has in v_2 the opposite sign that e had in v_2 , decrease c'_{v_2} by $\sigma_{v_2, e}u_e$, set $c'_{v_e} = u_e$, and set $\bar{x}'_{v_1v_e} = \bar{x}_e$ and $\bar{x}'_{v_e v_2} = u_e - \bar{x}_e$. Notice that also G' is loopless.

By Lemma 1.24 (i), \bar{x}' is in the first closure of $A(G')x' = c'$, $x' \geq 0$. If the latter system has Chátal rank at most 1, then \bar{x}' is a convex combination of integral solutions. Hence \bar{x} is a convex combination of integral vectors satisfying (2.4).

Now we show that, if (2.4) has Chvátal rank at most 1 for every loopless bidirected graph G and every integral c, u , then (2.4) has Chvátal rank at most 1 for every bidirected graph G and every integral c, u .

Let G be a bidirected graph, let c, u be integral vectors and let \bar{x} be a vector in the first closure of (2.4). Now let G' be obtained from G by replacing every loop l incident with v_l with two parallel edges l_1, l_2 with endnodes v_l, w_l , where $w_l \notin V(G)$ and $w_l \neq w_p$, for every pair l, p of loops of G . Notice that G' is loopless. The sign of l_1 and l_2 in v_l is the sign that l had in v_l , l_1 has in w_l a +1 sign, and l_2 has in w_l a -1 sign. Let c' be obtained from c by setting $c_{w_l} = 0$ for every loop l of G . Let \bar{x}' and u' be obtained from \bar{x} and u by setting $\bar{x}'_{l_1} = \bar{x}'_{l_2} = \bar{x}_l$ and $u'_{l_1} = u'_{l_2} = u_l$ for every loop l of G . Clearly \bar{x}' is in the first closure of $A'x' = c'$, $0 \leq x' \leq u'$. If the latter system has Chátal rank at most 1, then \bar{x}' is a convex combination of integral solutions. Hence \bar{x} is a convex combination of integral vectors satisfying (2.4).

Now we show that if (2.4) has Chvátal rank at most 1 for every bidirected graph G and every integral c, u , then the system

$$\begin{aligned} b &\leq A(G)x \leq c \\ 0 &\leq x \leq u \end{aligned} \tag{2.5}$$

has Chvátal rank at most 1 for every bidirected graph G and every integral b, c, u .

Let G be a bidirected graph, let b, c, u be integral vectors and let \bar{x} be a vector in the first closure of (2.5). Now let G' be obtained from G by adding a new node v , a new edge $e_w = vw$ for every node $w \in V(G)$, and a loop l incident with v . Let c' be obtained from c by setting $c_v = c(V)$.

Let $u' \in \mathbb{Z}^{E(G')}$ be obtained from u by setting $u'_l = +\infty$, and by setting $u'_{e_w} = c_w - b_w$ for every new edge $e_w \in E(G') \setminus (E(G) \cup l)$. Let \bar{x}' be obtained from \bar{x} by setting $\bar{x}_{vw} = (c - A(G)\bar{x})_w$ for every $w \in V(G)$ and by setting $\bar{x}'_l = b(V)$. By Lemma 1.24 (i), \bar{x}' is in the first closure of $A'x' = c$, $0 \leq x' \leq u'$. If the latter system has Chátal rank at most 1, then \bar{x}' is a convex combination of integral solutions. Hence \bar{x} is a convex combination of integral vectors satisfying (2.5).

Now we show that if (2.5) has Chvátal rank at most 1 for every bidirected graph G and every integral b, c, u , then $A(G)$ has the Edmonds-Johnson property for every bidirected graph G .

Let b, c, l, u be integral vectors and let \bar{x} be a vector in the first closure of (2.3). Let $b' = b - Al$, $c' = c - Al$, $u' = u - l$, $\bar{x}' = \bar{x} - l$. It is easy to see that \bar{x}' is in the first closure of $b' \leq A(G)x' \leq c'$, $0 \leq x' \leq u'$. If the latter system has Chátal rank at most 1, then \bar{x}' is a convex combination of integral solutions, i.e. $\bar{x}' = \sum_{i \in I} \lambda_i \tilde{y}^i$, $0 \leq \lambda_i \leq 1$, $\sum_{i \in I} \lambda_i = 1$. Hence $\bar{x} = \sum_{i \in I} \lambda_i y^i$, where $y^i = \tilde{y}^i + l$ is a convex combination of integral vectors satisfying (2.3).

Finally we show that, if for every bidirected graph G , $A(G)$ has the Edmonds-Johnson property, then for every pair (G, F) in \mathcal{A} , $A(G, F)$ has the Edmonds-Johnson property.

Let (G, F) be a pair in \mathcal{A} , and let G' be obtained from G by adding a new node v adjacent to each node of G incident with a loop in $E(G) \setminus F$, and by removing all the loops in $E(G) \setminus F$. Clearly, $A(G, F)$ is obtained from $A(G')$ by removing the row corresponding to the node v . If $A(G')$ has the Edmonds-Johnson property, then by Theorem 2.1 (iii), also $A(G, F)$ has the Edmonds-Johnson property. \square

In the remaining of this thesis, whenever Z is a set, $Y \subseteq Z$, and z is a vector in \mathbb{R}^Z , we denote with $z(Y) = \sum_{i \in Y} z_i$.

Let G be a bidirected graph. It is well known that, for each $U \subseteq V(G)$ such that U is connected in G , and $c(U)$ is odd, the inequality

$$x(\delta(U)) \geq 1 \tag{2.6}$$

is a Chvátal inequality for the system (2.3). We call such constraint an *odd-cut inequality* for (2.3). In the remainder of this section we will show that the polyhedron defined by the system

$$\begin{aligned} A(G)x &= c \\ x(\delta(U)) &\geq 1 & U \subseteq V(G), U \text{ connected}, c(U) \text{ odd} \\ x &\geq 0, \end{aligned} \tag{2.7}$$

is integral for every loopless bidirected graph G and every integral vector c . By Lemma 2.3, this proves Theorem 2.2.

Now we give a lemma that uses standard uncrossing arguments (see [9, 22, 26, 27, 31] for details). This lemma is used in the proof of Theorem 2.2 and will also be used in Chapter 4.

From now on, given a set W of vectors, we denote by $\text{span}W$ the space generated by the vectors in W . Whenever Z is a set, and $Y \subseteq Z$, we denote by $\chi_Z(Y)$, or by $\chi(Y)$ when there is no ambiguity, the vector in $\{0, 1\}^Z$ defined by $\chi_Z(Y)_z = 1$ if $z \in Y$, and $\chi_Z(Y)_z = 0$ otherwise.

Lemma 2.4. (*Uncrossing Lemma.*) *Let $G = (V, E)$ be a graph, let $c \in \mathbb{Z}^V$, $\bar{x} \in \mathbb{R}^E$ with $\bar{x} > 0$. Let \mathcal{F} be the family of the subset $U \subseteq V$ with $c(U)$ odd, such that $\bar{x}(\delta(U)) = 1$. Then there exists a laminar subfamily \mathcal{L} of \mathcal{F} such that $\text{span}\{\chi(\delta(U)) : U \in \mathcal{L}\} = \text{span}\{\chi(\delta(U)) : U \in \mathcal{F}\}$.*

Proof. Given a family \mathcal{F} of subsets of V , we define $\text{span}\mathcal{F} = \text{span}\{\chi(\delta(U)) : U \in \mathcal{F}\}$. Given two subsets T and S of V , we say that T and S *intersect* if the sets $T \cap S$, $T \setminus S$, $S \setminus T$ are all nonempty.

Let $\mathcal{F} = \{U \subseteq V : \bar{x}(\delta(U)) = 1, c(U) \text{ odd}\}$. Let \mathcal{L} be a maximal independent laminar subfamily of \mathcal{F} . We want to show that $\text{span}\mathcal{L} = \text{span}\mathcal{F}$.

Otherwise there exists $T \in \mathcal{F}$ such that $\chi(\delta(T)) \notin \text{span}\mathcal{L}$. But then, since \mathcal{L} is a maximal independent laminar subfamily of \mathcal{F} , T intersects at least a set $S \in \mathcal{L}$. Among all $T \in \mathcal{F}$ such that $\chi(\delta(T)) \notin \text{span}\mathcal{L}$, let T be one that intersects the minimum number of sets in \mathcal{L} . Notice that, since $c(T)$ and $c(S)$ are odd, then either $c(S \cap T)$ and $c(S \cup T)$ are odd, or $c(S \setminus T)$ and $c(T \setminus S)$ are odd.

In the first case, it can be checked that $2 = \bar{x}(\delta(S)) + \bar{x}(\delta(T)) \geq \bar{x}(\delta(S \cap T)) + \bar{x}(\delta(S \cup T)) \geq 2$, hence both $S \cap T$ and $S \cup T$ are in \mathcal{F} . As \mathcal{L} is laminar, $T \cap S$ and $T \cup S$ intersect fewer sets from \mathcal{L} than T . Hence by our choice of T , $\chi(\delta(T \cap S)), \chi(\delta(T \cup S)) \in \text{span}\mathcal{L}$. Since $\bar{x} > 0$, $\chi(\delta(T)) = \chi(\delta(S \cap T)) + \chi(\delta(S \cup T)) - \chi(\delta(S))$ is in $\text{span}\mathcal{L}$, a contradiction.

In the second case, it can be checked that $2 = \bar{x}(\delta(S)) + \bar{x}(\delta(T)) \geq \bar{x}(\delta(S \setminus T)) + \bar{x}(\delta(T \setminus S)) \geq 2$, hence both $S \setminus T$ and $T \setminus S$ are in \mathcal{F} . As \mathcal{L} is laminar, $S \setminus T$ and $T \setminus S$ intersect fewer sets from \mathcal{L} than T . Hence by our choice of T , $\chi(\delta(S \setminus T)), \chi(\delta(T \setminus S)) \in \text{span}\mathcal{L}$. Since $\bar{x} > 0$, $\chi(\delta(T)) = \chi(\delta(S \setminus T)) + \chi(\delta(T \setminus S)) - \chi(\delta(S))$ is in $\text{span}\mathcal{L}$, a contradiction. \square

We say that a scalar x is *half-integer* if $2x$ is integer. Similarly, a vector or matrix A is called *half-integral* if $2A$ is integral.

Proof of Theorem 2.2.

By Lemma 2.3, we only need to show that system (2.7) defines an integral polyhedron for every loopless bidirected graph G and every integral vector c .

By contradiction, suppose that there exists a loopless bidirected graph G and an integral vector c such that the system (2.7) has a fractional vertex \bar{x} . Among all such counterexamples, choose G such that the pair $(|V(G)|, |E(G)|)$ is lexicographically minimal. Let σ be the signing of G .

Claim 2.1. G is connected.

Proof of claim. If not, let G' be a component of G such that $\bar{x}_e \notin \mathbb{Z}$ for some $e \in E(G')$. Let \bar{x}', c' be the restrictions of \bar{x}, c , to $E(G')$. Notice that G' is loopless and that $|V(G')| < |V(G)|$, hence the system (2.7) with respect to G', c' has integral vertices. Then clearly \bar{x}' is a fractional vertex of the first closure of system (2.7) with respect to G', c' , a contradiction. \diamond

Notice that $c(V(G))$ is even, as otherwise the inequality $x(\delta(V(G))) \geq 1$ is not satisfied by \bar{x} .

Claim 2.2. $\bar{x}_e > 0$ for every $e \in E(G)$.

Proof of claim. If not, assume that there exists an edge e in $E(G)$ with $\bar{x}_e = 0$. Let G' be obtained from G by removing e . Let \bar{x}' be the vector obtained from \bar{x} by removing the component corresponding to e . Since G' is loopless, $|V(G')| = |V(G)|$, and $|E(G')| < |E(G)|$, the system (2.7) with respect to G', c has integral vertices. Since the vector c has not changed, the odd-cut inequalities for $A(G')x' = c$, $x' \geq 0$ are exactly the odd-cut inequalities for (2.7). Moreover notice that $\bar{x}'(\delta(U)) = \bar{x}(\delta(U))$ for every $U \subseteq V(G)$. Hence \bar{x}' is a fractional vertex of the system (2.7) with respect to G', c , a contradiction. \diamond

Let $\mathcal{F} = \{U \subseteq V \mid \bar{x}(\delta(U)) = 1\}$.

By Claim 2.2, \bar{x} is the unique solution of the system

$$\begin{aligned} A(G)x &= c \\ x(\delta(U)) &= 1 \quad U \in \mathcal{F}. \end{aligned}$$

By Lemma 2.4, we can choose a laminar subfamily \mathcal{L} of \mathcal{F} such that \bar{x} is the unique solution of the system

$$\begin{aligned} A(G)x &= c \\ x(\delta(U)) &= 1 \quad U \in \mathcal{L}, \end{aligned} \tag{2.8}$$

and the elements of $\{\chi(\delta(U)) : U \in \mathcal{L}\}$ are not linear combination of rows of $A(G)$.

In particular, $|E| \leq |V| + |\mathcal{L}|$.

Claim 2.3. $G[S]$ is connected for every $S \in \mathcal{L}$.

Proof of claim. Otherwise, $G[S]$ has a connected component T with $c(V(T))$ odd, thus $\delta(T) \subseteq \delta(S)$. So by Claim 2.2, $1 \leq \bar{x}(\delta(T)) \leq \bar{x}(\delta(S)) = 1$. Therefore $\delta(T) = \delta(S)$, hence $\delta(S \setminus T) = \emptyset$, contradicting the fact that G is connected. \diamond

Claim 2.4. $\mathcal{L} \neq \emptyset$.

Proof of claim. Otherwise $|E| \leq |V|$. Thus G is a tree plus at most one edge. By Theorem 1.13, G is not bipartite, as otherwise $A(G)$ is totally unimodular, and \bar{x} is integral, a contradiction. Thus G contains a cycle C , and C is odd. It follows that \bar{x}_e is half-integer for every edge $e \in E(C)$. Since c is integral, then \bar{x}_e is integer for every $e \in E(G) \setminus E(C)$, and, since \bar{x} is not integral, then \bar{x}_e is fractional for every $e \in E(C)$. Since C is odd, there is an odd number of odd edges in C , thus $c(V(G))$ is odd, a contradiction. \diamond

Let S be a minimal element of \mathcal{L} .

Claim 2.5. $G[S]$ is not bipartite.

Proof of claim. Otherwise, by Theorem 1.13, the nodes in $G[S]$ can be partitioned into two subsets R, B such that, for every $e = vw \in E(G)$, v and w are in the same class of the partition if and only if $\sigma_{v,e} \neq \sigma_{w,e}$. Since $c(S)$ is odd, we may assume $c(R) > c(B)$, thus $c(R) - c(B) \geq 1$.

Let R^+ and R^- (resp. B^+ and B^-) be the sets of edges $vw \in \delta(R) \cap \delta(S)$, $v \in R$ (resp. $vw \in \delta(B) \cap \delta(S)$, $v \in B$), with respectively $\sigma_{v,vw} = +1$ and $\sigma_{v,vw} = -1$.

Then the odd-cut inequality relative to S , that is $\sum_{e \in R^+ \cup R^- \cup B^+ \cup B^-} x_e \geq 1$, is satisfied tightly by \bar{x} and it is linearly independent from the equations in the system $A(G)x = c$. By summing together all the equalities in $A(G)x = c$ corresponding to nodes in R and all the equalities in $-A(G)x = -c$ corresponding to nodes in B we get

$$\sum_{e \in R^+ \cup B^-} x_e - \sum_{e \in R^- \cup B^+} x_e = c(R) - c(B).$$

Hence $1 = \sum_{e \in R^+ \cup R^- \cup B^+ \cup B^-} \bar{x}_e \geq \sum_{e \in R^+ \cup B^-} \bar{x}_e - \sum_{e \in R^- \cup B^+} \bar{x}_e \geq 1$, because $\bar{x} \geq 0$. Thus $\sum_{e \in R^+ \cup R^- \cup B^+ \cup B^-} \bar{x}_e = \sum_{e \in R^+ \cup B^-} \bar{x}_e - \sum_{e \in R^- \cup B^+} \bar{x}_e$, therefore $R^- \cup B^+ = \emptyset$, because $\bar{x} > 0$. But then the inequality defined by S is linearly dependent from $Ax = c$, a contradiction. \diamond

In particular, $G[S]$ is not a tree. Now let G' be the bidirected graph obtained from G by shrinking S into a new node s , where we identify each edge $e = vw \in E(G)$ with $v \in S$ and $w \notin S$ with an edge $e = sw$ in $E(G')$ (edges with both endnodes in S are removed), and where the signing σ' of G' is defined as follows. For every edge $e = sw \in E(G')$, let $\sigma'_{s,e} = +1$ and $\sigma'_{w,e} = \sigma_{w,e}$, while for every edge e in $E(G')$ not incident with s and every endnode v of e , let $\sigma'_{v,e} = \sigma_{v,e}$. Let c' be defined by $c'_v = c_v$ for $v \in V \setminus S$ and $c'_s = 1$. Then the vector \bar{x}' obtained by restricting \bar{x} to the edges of G' is a point that satisfies system (2.7) with respect to G', c' . Furthermore, since $G[S]$ is connected but not a tree, G' has at most $|E(G)| - |S|$ edges. Notice that \bar{x}' satisfies exactly all the equations in (2.8), except for the node equalities corresponding to the nodes in S . Thus, among this equalities, at least $|E(G)| - |S|$ are linearly independent. Thus \bar{x}' is a vertex of the system (2.7) with respect to G', c' , thus by induction it is integral. In particular $|\delta(S)| = 1$ and, if $f = st$ is the unique edge leaving S , $\bar{x}_f = 1$.

Let $G'' = G \setminus \{f\}$. Now let \bar{x}'' be the restriction of \bar{x} to the edges in $E \setminus f$. Let $c''_v = c_v$ for $v \in V \setminus \{s, t\}$, $c''_s = c_s - \sigma_{s,st}$, $c''_t = c_t - \sigma_{t,st}$. We show that \bar{x}'' satisfies the system (2.7) with respect to G'', c'' . Clearly $A(G'')\bar{x}'' = c''$. Let $U \subseteq V(G'')$, such that U is connected in G'' , and $c''(U)$ odd. Since G'' is not connected, then $U \subseteq V(C)$, for a connected component C of G'' . Since s, t are in different connected components of G'' , by symmetry we assume $s \in V(C)$. Thus, $c''(U)$ is odd if and only either $c(U)$ is odd and $s \notin U$, or if $c(U)$ is even and $s \in U$. In the former case, $\bar{x}''(\delta(U)) = \bar{x}(\delta(U)) \geq 1$. In the latter case, $c(C \setminus U)$ is odd, thus $\bar{x}''(\delta(U)) = \bar{x}(\delta(C \setminus U)) \geq 1$. So \bar{x}'' satisfies the system (2.7) with respect to G'', c'' , hence by induction it is a convex combination of integral vectors y^1, \dots, y^k satisfying such system. Thus \bar{x} is a convex combination of $\begin{pmatrix} \bar{x}_{st} \\ y^i \end{pmatrix}$, $i = 1, \dots, k$, which are integral vectors satisfying (2.7) with respect to G, c , a contradiction. \square

2.1.2 The Gerards and Schrijver's class

In this section we state a Theorem by Gerards and Schrijver [19], that characterizes the Edmonds-Johnson property for integral matrices $A = (\alpha_{ij})$ that satisfy $\sum_j |\alpha_{ij}| \leq 2$ for each row index i .

In Section 2.1.1 we showed that if $A = (\alpha_{ij})$ is an integral $m \times n$ -matrix such that

$$\sum_{i=1}^m |\alpha_{ij}| \leq 2 \quad j = 1, \dots, n, \quad (2.9)$$

then A has the Edmonds-Johnson property.

This property is not maintained when passing to transposes: (2.9) may not be replaced by

$$\sum_{j=1}^n |\alpha_{ij}| \leq 2 \quad i = 1, \dots, m, \quad (2.10)$$

as the matrix

$$M(K_4) := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

the edge-node incidence matrix of the undirected graph K_4 , does not have the Edmonds-Johnson property. Gerards and Schrijver proved that $M(K_4)$ is essentially the only counterexample among the matrices satisfying (2.10):

Theorem 2.5. *An integral matrix satisfying (2.10) has the Edmonds-Johnson property if and only if it cannot be transformed to $M(K_4)$ by a series of the following operations:*

- (a) deleting or permuting rows or columns, or multiplying them by -1 ;
- (b) replacing matrix $\begin{pmatrix} 1 & g \\ f & D \end{pmatrix}$ by the matrix $D - fg$, where f is a column vector and g a row vector.

Notice that operation (b) corresponds to pivoting on a $+1$ entry, and then deleting the row and the column corresponding to the pivoted element.

Notice that we can restrict ourselves to integral matrices A satisfying

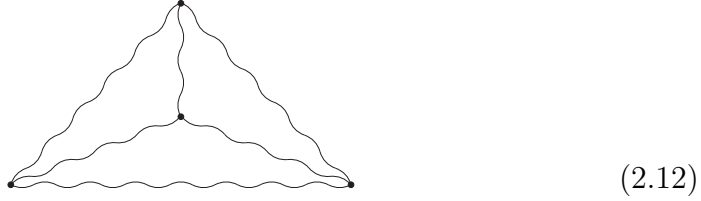
$$\sum_{j=1}^n |\alpha_{ij}| = 2 \quad (i = 1, \dots, m), \quad (2.11)$$

as rows with exactly one ± 1 simply correspond to bounds on one variable. For every integral matrix A satisfying (2.11), let $G(A)$ be the bidirected graph whose edge-node incidence matrix is A . Let σ be the signing of $G(A)$. Clearly, the columns of A correspond to the nodes of $G(A)$, and the rows to the edges.

Let A' be obtained by applying operation (b) in Theorem 2.5 to A . If the first row of A corresponds to an edge $vw \in E(G(A))$ with $\sigma_{v,vw} \neq \sigma_{w,vw}$, we get the ordinary graph contraction, thus $G(A')$ is obtained from $G(A)$ by

deleting the edge vw and identifying the nodes v and w . Otherwise, if the first row of A corresponds to an edge $vw \in E(G(A))$ with $\sigma_{v,vw} = \sigma_{w,vw}$, then $G(A')$ is obtained from $G(A)$ by deleting the edge, changing sign to $\sigma_{v,vr}$ for every node r incident with v , and identifying the nodes v and w . Thus we obtain the following equivalent form of Theorem 2.5.

Corollary 2.6. *A bidirected graph has the Edmonds-Johnson property if and only if it does not have a subgraph of the form*



where the wiggled lines stand for (pairwise openly disjoint) paths, such that each of the four cycles in (2.12) which have exactly three nodes of degree three, is odd.

A graph as in (2.12) is called an *odd- K_4* .

Gerards and Schrijver also described a smaller set of Chvátal-Gomory cuts which are sufficient to give the convex hull of the integral solutions.

Let A be matrix satisfying (2.11) with the Edmonds-Johnson property. Let $G = G(A)$, $V = V(G)$, and $E = E(G)$. If $x \in \mathbb{R}^V$, $b \in \mathbb{R}^E$, $e \in E$ and C is a subgraph of G , we denote with $x(e)$ the entry in position e of Ax (so $x(e) = \pm x_v \pm x_w$ if e connects v and w), and we define $x(C) := 1/2 \sum_{e \in E(C)} x(e)$, and $b(C) := \sum_{e \in E(C)} b_e$. So $Ax \leq b$ is the same as: $x(e) \leq b_e$ for $e \in E$. If C is an odd cycle, the corresponding *odd cycle inequality* is:

$$x(C) \leq \left\lfloor \frac{1}{2} b(C) \right\rfloor.$$

So it is a special type of Chvátal-Gomory cut. In fact, for bidirected graphs, the odd cycle inequalities imply all other Chvátal-Gomory cuts:

Proposition 2.7. *Let A be a matrix satisfying (2.11), let $G = G(A)$ with node set V and edge set E , and let $b \in \mathbb{Z}^E$. Then the first closure of the system $Ax \leq b$ is given by the system*

$$\begin{aligned} Ax &\leq b \\ x(C) &\leq \left\lfloor \frac{1}{2} b(C) \right\rfloor \quad C \text{ odd cycle of } G. \end{aligned}$$

2.2 A conjecture by Gerards and Schrijver

Characterizing integral matrices with the Edmonds-Johnson property seems complicated. However, Gerards and Schrijver [18] noticed that there are some openings if we restrict ourselves to *totally half-modular matrices*, i.e. integral matrices in which each nonsingular square submatrix B has a half-integral inverse. The reason seems that the ‘cuts’ H_I determine an affine space over $GF(2)$. Notice that the classes of matrices that we described in Sections 2.1.1 and 2.1.2 are totally half-modular, since it is well known that incidence matrices of bidirected graphs are totally half-modular.

We now show that the operations that preserve the Edmonds-Johnson property, introduced in Theorem 2.1, also preserve total half-modularity. At first we give an easy technical lemma that will be used in the next proof and later in Chapter 4.

Lemma 2.8. *An $m \times n$ matrix A is totally half-modular if and only if for every nonsingular $m \times m$ submatrix B of $(A \mathbb{I})$, B^{-1} is half-integral.*

Proof. Assume that for every nonsingular $m \times m$ submatrix B of $(A \mathbb{I})$, B^{-1} is half-integral, and let A' be a nonsingular square submatrix of A . Now, for an identity matrix \mathbb{I} of the appropriate size, the matrix $\begin{pmatrix} A' & 0 \\ 0 & \mathbb{I} \end{pmatrix}$ is a nonsingular $m \times m$ submatrix of $(A \mathbb{I})$, thus has half-integral inverse $\begin{pmatrix} A'^{-1} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$. Hence A'^{-1} is half-integral, and A is totally half-modular.

Conversely, assume that A is totally half-modular. Up to permuting rows and columns, we may assume that each nonsingular $m \times m$ submatrix A' of $(A \mathbb{I})$ is of the form $\begin{pmatrix} P & 0 \\ Q & \mathbb{I} \end{pmatrix}$, where P is a nonsingular square submatrix of A , thus has half-integral inverse by hypothesis, and where Q is a submatrix of A and thus is integral. As $A'^{-1} = \begin{pmatrix} P^{-1} & 0 \\ -QP^{-1} & \mathbb{I} \end{pmatrix}$, P^{-1} is half-integral, and Q is integral, then A'^{-1} is half-integral. \square

Theorem 2.9. *The class of totally half-modular matrices is closed under the following operations:*

- (i) *permuting rows and columns;*
- (ii) *multiplying rows and columns by -1 ;*

(iii) deleting rows and columns;

(iv) dividing by $k \in \mathbb{N}$, $k \geq 2$ a row where all entries are multiple of k ;

(v) pivoting on a 1 entry, i.e. replacing matrix $\begin{pmatrix} 1 & g \\ f & D \end{pmatrix}$ by the matrix

$$\begin{pmatrix} -1 & g \\ f & D - fg \end{pmatrix}, \text{ where } f \text{ is a column vector and } g \text{ a row vector.}$$

Proof. Let A be a totally half-modular matrix.

(i)-(iii): If A' is obtained from A by permuting rows or columns, or by multiplying rows or columns by -1 , or by deleting rows or columns, then trivially A' is totally half-modular.

(iv): Assume that A' is obtained from A by dividing by $k \in \mathbb{N}$, $k \geq 2$ a row where all entries are multiple of k . By (i) we can assume that it is the first row of A . Let B' be a nonsingular square submatrix of A' . Clearly we can assume that B' contains entries of first row of A , as otherwise B' is also a nonsingular square submatrix of A , and thus it has a half-integral inverse.

Let B be the nonsingular square submatrix of A corresponding to B' . Clearly B' is obtained from B by dividing by k the entries of the first row of B . Thus B'^{-1} is obtained from B^{-1} by multiplying by k the first column. Thus B'^{-1} is half-integral because B^{-1} is half-integral and $k \in \mathbb{N}$.

(v): Assume that the $m \times n$ matrix $A = \begin{pmatrix} 1 & g \\ f & D \end{pmatrix}$ is totally half-modular. By Lemma 2.8, for each $m \times m$ nonsingular submatrix B' of $B = \begin{pmatrix} 1 & g & 1 & 0 \\ f & D & 0 & \mathbb{I} \end{pmatrix}$, B'^{-1} is half-integral. Let $U = \begin{pmatrix} 1 & 0 \\ -f & \mathbb{I} \end{pmatrix}$, and notice that U is unimodular. Notice that $UB = \begin{pmatrix} 1 & g & 1 & 0 \\ 0 & D - fg & -f & \mathbb{I} \end{pmatrix} =: C$.

Now let C' be a $m \times m$ nonsingular submatrix of C . Clearly there exists an $m \times m$ nonsingular submatrix B' of B such that $UB' = C'$, thus $C'^{-1} = B'^{-1}U^{-1}$. Since U is unimodular, U^{-1} is integral, and since B'^{-1} is half-integral, it follows that C'^{-1} is half-integral. Thus, by Lemma 2.8, and by (i) and (ii), the matrix $\begin{pmatrix} -1 & g \\ f & D - fg \end{pmatrix}$ is totally half-modular. \square

Notice that operations (i)-(v) of Theorem 2.9 are exactly operations (i)-(v) of Theorem 2.1. Furthermore, since totally half-modular matrices have entries in $\{0, \pm 1, \pm 2\}$, then operation (iv) can be applied to such matrices only with $k = 2$.

We say that a matrix B is a *minor* of A if it arises from A by a series of operations (i)-(v). Let

$$A_4 = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix}.$$

Gerards and Schrijver [18] formulated the following conjecture:

Conjecture 2.1. *A totally half-modular matrix A has the Edmonds-Johnson property, if and only if it has no minor equal to A_4 or A_3 .*

Next we observe that the matrices A_4 and A_3 do not have the Edmonds-Johnson property. To this aim, we need the following technical lemma, that will also be needed in Chapter 4.

Lemma 2.10. *If A is a totally half-modular matrix and b is an integral vector, any nontrivial Chvátal inequality for $Ax \geq b$ is equivalent to an inequality of the form $(\lambda A)x \geq \lceil \lambda b \rceil$ such that λ has only $0, \frac{1}{2}$ entries, λA is integral, λb is not integral, and the positive components of λ correspond to linearly independent rows of A .*

Proof. By Lemma 1.20, the first Chvátal closure of $Ax \geq b$ is obtained by adding the inequalities $(\lambda A)x \geq \lceil \lambda b \rceil$ for all vectors $0 \leq \lambda < 1$ such that λA is integral and λb is not integral. By Caratheodory's theorem, we may assume that the positive components of λ correspond to linearly independent rows of A . As each nonsingular square submatrix of A has half-integral inverse, it follows that λ is half-integral. \square

Observation 2.11. *The matrices A_4 and A_3 do not have the Edmonds-Johnson property.*

Proof. Consider the system

$$A_4 x \geq 1, x \geq 0, \tag{2.13}$$

and let $\bar{x}_i = 1/3$ for $i = 1, \dots, 4$.

By Lemma 2.10, the nontrivial Chvátal inequalities for (2.13) are

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 1 & 1 \end{pmatrix} x \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}. \tag{2.14}$$

Notice that \bar{x} satisfies system (2.13), and satisfies tightly all the four Chvátal inequalities in (2.14), that are linearly independent. Thus \bar{x} is a fractional vertex of the first closure of (2.13).

Consider the system

$$A_3 x \geq \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, x \geq 0, \quad (2.15)$$

and let $\bar{x}_i = 1/2$ for $i = 1, 2, 3$.

By Lemma 2.10, the nontrivial Chvátal inequalities for (2.15) are

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} x \geq \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}. \quad (2.16)$$

Notice that \bar{x} satisfies system (2.15), and satisfies tightly all the Chvátal inequalities in (2.16). Since the constraint matrix of (2.16) has rank 3, then \bar{x} is a fractional vertex of the first closure of (2.15). \square

Notice that, in the class of matrices studied by Edmonds and Johnson (see Section 2.1.1), the matrices A_4 and A_3 do not appear.

In the class of matrices studied by Gerards and Schrijver (see Section 2.1.2) only A_4 appears. In fact, notice that A_4 can be obtained from $M(K_4)$ by pivoting the 1 entries in positions $(1, 2)$, $(2, 3)$, and $(3, 4)$, by multiplying by -1 the first column of the obtained matrix, and then by removing the first three rows.

In the remaining two chapters we present our contributions.

In Chapter 3, we study systems of the form

$$\begin{aligned} b &\leq Mx \leq c \\ l &\leq x \leq u, \end{aligned} \quad (2.17)$$

for integral vectors b, c, l, u , where M is obtained from a totally unimodular matrix with two nonzero elements per row by multiplying by 2 some of its columns.

The case where M is obtained from the transpose of the incidence matrix of a bipartite graph by multiplying by 2 some of the columns, has been studied by Conforti et al. in [7]. In this case, they derived an explicit characterization of the inequalities defining the integer hull.

We give an explicit description of a totally dual integral system that describes the integer hull of the polyhedron P defined by (2.17). Since the inequalities of such totally dual integral system are Chvátal inequalities for P , this implies that the matrix M has the Edmonds-Johnson property. Thus also in this class, the matrices A_4 and A_3 do not appear. The results in Chapter 3 are joint work with G. Zambelli [11].

In Chapter 4 we study totally half-modular matrices with entries in $\{0, \pm 1, \pm 2\}$ obtained from sign matrices of bidirected graphs by multiplying by 2 some of the columns. We show that the matrix

$$M_4 = \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \quad (2.18)$$

is the only minor minimal matrix in such class (up to multiplying rows and columns by -1) that does not have the Edmonds-Johnson property. Notice that this result extends Theorem 2.2 of Edmonds and Johnson. The results in Chapter 4 are joint work with A. Musitelli and G. Zambelli [10]. A partial result was shown by Del Pia and Zambelli [12].

Chapter 3

Bipartite vertex covers with parity conditions

In this chapter we study systems of the form $b \leq Mx \leq d$, $l \leq x \leq u$, where M is obtained from a totally unimodular matrix with two nonzero elements per row by multiplying by 2 some of its columns, and b, d, l, u are integral vectors. We give an explicit description of a totally dual integral system that describes the integer hull of the polyhedron P defined by the above inequalities. Since the inequalities of such totally dual integral system are Chvátal inequalities for P , our result implies that the matrix M has the Edmonds-Johnson property. We also derive a strongly polynomial time algorithm to find an integral optimal solution for the dual of the problem of maximizing a linear function with integer coefficients over the aforementioned totally dual integral system.

3.1 Introduction

Let M be a matrix obtained from a totally unimodular matrix with exactly two nonzero elements in every row by multiplying by 2 some of its columns, and let b, c, d, l, u be integral vectors of appropriate dimension. We consider problems of the form

$$\begin{aligned} \max \quad & c^\top x \\ b \leq \quad & Mx \leq d, \\ l \leq \quad & x \leq u, \\ & x \text{ integral.} \end{aligned} \tag{3.1}$$

The case where M is obtained from the transpose of the incidence matrix of a bipartite graph by multiplying by 2 some of the columns, and where

the variables are restricted to be nonnegative, has been studied by Conforti et al. in [7]. In this case, they showed that the above problem can be solved in strongly polynomial time. This was accomplished by expressing the integer hull of the system as the projection of some polyhedron in a higher dimensional space. From such extended formulation, they derived an explicit characterization of the inequalities defining the integer hull.

In this chapter we show, with a similar construction, how problem (3.1) can in fact be reduced to a weighted vertex covering problem on a certain extended graph. Using this construction, we describe a totally dual integral system defining the integer hull of the polyhedron defined by the constraint system of (3.1).

Note that, since $\begin{pmatrix} M \\ -M \end{pmatrix}$ is also obtained from a totally unimodular matrix with two nonzero elements per row by multiplying some of its columns by 2, it is sufficient to consider systems of the form

$$\begin{array}{ccc} Mx & \geq & b \\ l \leq & x & \leq u. \end{array} \quad (3.2)$$

The matrix M can be represented as a *bidirected graph* $G = (V, E, \sigma)$: the columns of M correspond to the nodes of G , and the rows to the edges. A row corresponds to an edge e connecting the two nodes where the nonzero elements occur, with $\sigma_{v,e} = +1$ if $M_{e,v} \in \{+1, +2\}$, with $\sigma_{v,e} = -1$ if $M_{e,v} \in \{-1, -2\}$, and with $\sigma_{v,e} = 0$ otherwise. Let I be the set of nodes of V corresponding to the columns of M with entries $0, \pm 2$. Let $L = V \setminus I$. Thus M is obtained from the edge-node incidence matrix of G by multiplying by 2 the columns corresponding to nodes in I .

For every bidirected edge $e \in E$ we call b_e the *requirement* of e . An *I-trail* in G is a trail $T = v_1, \dots, v_k$ in G such that $v_1 \in I$, $v_2, \dots, v_{k-1} \notin I$. An *I-path* in G is an *I-trail* in G which is a path in G . For any such *I-trail* in G we define

$$\begin{aligned} \gamma_1^T &= \sigma_{v_1, v_1 v_2}; \\ \gamma_i^T &= (\sigma_{v_i, v_{i-1} v_i} + \sigma_{v_i, v_i v_{i+1}})/2; \quad i = 2, \dots, k-1 \\ \gamma_k^T &= \sigma_{v_k, v_{k-1} v_k}. \end{aligned}$$

Notice that $\gamma_i^T \in \{0, \pm 1\}$ for every $i = 1, \dots, k$, and that it is possible that $v_s = v_t$ and $\gamma_s^T \neq \gamma_t^T$ for two indices $s \neq t$. Given an *I-trail* $P = v_1, \dots, v_k$ of G , the following inequalities are Chvátal-Gomory inequalities for (3.2), thus they are valid for its integer hull:

$$\begin{aligned}
\sum_{i=1}^k \gamma_i^P x_{v_i} &\geq \left\lceil \frac{\sum_{e \in P} b_e}{2} \right\rceil && \text{if } v_1, v_k \in I, \\
\sum_{i=1}^k \gamma_i^P x_{v_i} &\geq \left\lceil \frac{\sum_{e \in P} b_e + l_{v_k}}{2} \right\rceil && \text{if } v_k \notin I, \gamma_k^P = 1, \\
\sum_{i=1}^{k-1} \gamma_i^P x_{v_i} &\geq \left\lceil \frac{\sum_{e \in P} b_e - u_{v_k}}{2} \right\rceil && \text{if } v_k \notin I, \gamma_k^P = -1. \\
\sum_{i=1}^{k-1} \gamma_i^P x_{v_i} &\geq \left\lceil \frac{\sum_{e \in P} b_e + l_{v_k}}{2} \right\rceil && \\
\sum_{i=1}^k \gamma_i^P x_{v_i} &\geq \left\lceil \frac{\sum_{e \in P} b_e - u_{v_k}}{2} \right\rceil &&
\end{aligned} \tag{3.3}$$

In fact, all such inequalities are obtained by summing up the inequalities of $Mx \geq b$ corresponding to the edges of the I -trail, plus or minus the lower or upper bound on the variable corresponding to endnodes not in I , dividing the inequality thus obtained by 2, and rounding up the right-hand-side.

Theorem 3.1. *The system defined by (3.2) and (3.3), for every I -path P , is totally dual integral.*

Notice that, in Theorem 3.1 we only need inequalities of the form (3.3) when P is an I -path, rather than a general I -trail. We postpone the proof to Section 3.2.2. Our proof yields a strongly polynomial time algorithm that, given an integral cost vector c , finds an integral optimal solution for the dual of $\{\max c^\top x : x \text{ satisfies (3.2), (3.3) for every } I\text{-path } P\}$ whenever such problem has a finite optimum. Deriving a polynomial algorithm from the proof, however, is non-trivial, and it is accomplished in Section 3.4.

Edmonds and Giles [14] showed that if a system of linear inequalities with integer coefficients is totally dual integral, then the polyhedron defined by such a system is integral (cf. Corollary 1.16). Thus the above theorem implies the following.

Corollary 3.2. *The polyhedron defined by (3.2) and (3.3), for every I -path P , is integral.*

Since the inequalities in (3.3) are Chvátal inequalities for (3.2), Theorem 3.1 implies the following.

Corollary 3.3. *Let M be a matrix obtained from a totally unimodular matrix with two nonzero elements per row by multiplying some of its columns by 2. Then M has the Edmonds-Johnson property.*

3.2 Bipartite case

In this section we study a special case of problem (3.2), namely the case where M is obtained from a *nonnegative* totally unimodular matrix with two nonzero elements per row by multiplying by 2 some of its columns, and where all the variables are required to be nonnegative (i.e., $l = 0, u = +\infty$). In Section 3.3 we will see how the general case reduces to this simpler case.

In this special case the matrix M can be represented as an undirected graph $G = (V, E)$: the columns of M correspond to the nodes of G , and the rows to the edges. A row corresponds to an edge connecting the two nodes where the nonzero elements occur. By Theorem 1.11, G is bipartite. Let U, W be the sides of G . For every edge $e \in E$ we call b_e the *requirement* of e .

For every $i \in U \cup W$ we define $a_i = 2$ if $i \in I$, $a_i = 1$ if $i \in L$. Hence, given a cost vector $c \in \mathbb{Z}^{U \cup W}$, we consider the following problem:

$$\begin{aligned} \min \quad & \sum_{i \in U \cup W} c_i x_i \\ \text{s.t.} \quad & a_i x_i + a_j x_j \geq b_{ij} & ij \in E \\ & x_i \geq 0 & i \in U \cup W \\ & x_i \in \mathbb{Z} & i \in U \cup W. \end{aligned} \tag{3.4}$$

We will show the following.

Theorem 3.4. *The system of linear inequalities*

$$\begin{aligned} a_i x_i + a_j x_j &\geq b_{ij} & ij \in E \\ \sum_{i \in P} x_i &\geq \left\lceil \frac{b(P)}{2} \right\rceil & P \text{ } I\text{-path} \\ x_i &\geq 0 & i \in U \cup W \end{aligned} \tag{3.5}$$

is totally dual integral.

To show Theorem 3.4 we need to show that the problem

$$\begin{aligned} \max \quad & \sum_{e \in E} b_e y_e + \sum_P \left\lceil \frac{b(P)}{2} \right\rceil y_P \\ \text{s.t.} \quad & \sum_{e \ni i} a_i y_e + \sum_{P \ni i} y_P \leq c_i & i \in U \cup W \\ & y_e \geq 0 & e \in E \\ & y_P \geq 0 & P \text{ } I\text{-path}, \end{aligned} \tag{3.6}$$

has an integral optimal solution y for each vector $c \in \mathbb{Z}^{U \cup W}$ for which $\min\{c^\top x : x \text{ satisfies (3.5)}\}$ has a finite optimum. Since the latter problem is unbounded whenever c has a negative component, throughout this section we will assume that c is nonnegative.

We show how this problem can be reduced to a problem where the constraint matrix is the transpose of the incidence matrix of some extended bipartite graph.

3.2.1 The extended graph

Given a bipartite graph $G = (U \cup W, E)$, and a requirement vector $b \in \mathbb{Z}^E$, we define $\tilde{G}_{\emptyset, b} = (\tilde{U}_{\emptyset} \cup \tilde{W}_{\emptyset}, \tilde{E})$ as follows.

Let U', W' be copies of U, W , respectively, such that U, W, U', W' are pairwise disjoint. For every $i \in U \cup W$, we denote by i' the copy of i in $U' \cup W'$. Let $\tilde{U}_{\emptyset} = U \cup U', \tilde{W}_{\emptyset} = W \cup W'$. \tilde{E} contains the edges ij and $i'j'$ for every $ij \in E$ such that b_{ij} is odd, and the edges $i'j$ and ij' for every $ij \in E$ such that b_{ij} is even.

For $I \subseteq U \cup W$, the *extended graph* $\tilde{G}_{I, b} = (\tilde{U}_I \cup \tilde{W}_I, \tilde{E})$ is obtained from $\tilde{G}_{\emptyset, b}$ by identifying the two copies i, i' of every node $i \in I$, where \tilde{U}_I and \tilde{W}_I correspond to \tilde{U}_{\emptyset} and \tilde{W}_{\emptyset} . (Notice that we identify the set of edges of $\tilde{G}_{\emptyset, b}$ with that of $\tilde{G}_{I, b}$.)

For each node $i \in U \cup W$, the *images* of i in $\tilde{G}_{I, b}$ are the nodes i, i' (where $i = i'$ if $i \in I$). For each edge $ij \in E$, the *images* of ij in $\tilde{G}_{I, b}$ are the edges ij and $i'j'$ if b_{ij} is odd, the edges ij' and $i'j$ if b_{ij} is even. We say that a node $i \in \tilde{U}_I \cup \tilde{W}_I$ is the *symmetric* of another node $j \in \tilde{U}_I \cup \tilde{W}_I$, and we write $i = \text{sym}(j)$, when i, j are the images (possibly coincident) of the same element of $U \cup W$. We say that an edge $e_1 \in \tilde{E}$ is the *symmetric* of another edge $e_2 \in \tilde{E}$, and we write $e_1 = \text{sym}(e_2)$, when the two edges e_1 and e_2 are distinct images of the same edge $e \in E$.

To each edge e in \tilde{E} we assign a requirement \tilde{b}_e as follows. For each edge $ij \in E$ with b_{ij} odd we define $\tilde{b}_{ij} = \lfloor \frac{b_{ij}}{2} \rfloor$ and $\tilde{b}_{i'j'} = \lceil \frac{b_{ij}}{2} \rceil$, while for every edge $ij \in E$ such that b_{ij} is even we define $\tilde{b}_{i'j} = \tilde{b}_{ij'} = \frac{b_{ij}}{2}$.

To each node w of $\tilde{U}_I \cup \tilde{W}_I$ we assign a cost \tilde{c}_w equal to the cost of its corresponding node in $U \cup W$.

Now consider the following problem on $\tilde{G}_{I, b}$

$$\begin{aligned} \min \quad & \sum_{i \in \tilde{U}_I \cup \tilde{W}_I} \tilde{c}_i \tilde{x}_i \\ \text{s.t.} \quad & \tilde{x}_i + \tilde{x}_j \geq \tilde{b}_{ij} & ij \in \tilde{E} \\ & \tilde{x}_i \geq 0 & i \in \tilde{U}_I \cup \tilde{W}_I \end{aligned} \quad (3.7)$$

and its dual problem

$$\begin{aligned} \max \quad & \sum_{e \in \tilde{E}} \tilde{b}_e \tilde{y}_e \\ \text{s.t.} \quad & \sum_{e \ni i} \tilde{y}_e \leq \tilde{c}_i & i \in \tilde{U}_I \cup \tilde{W}_I \\ & \tilde{y}_e \geq 0 & e \in \tilde{E}. \end{aligned} \quad (3.8)$$

Note that the constraint matrix of (3.8) is the incidence matrix of a bipartite graph, and thus it is totally unimodular by Theorem 1.11. Thus, if problems (3.7) and (3.8) admit optimal solutions, then they admit optimal solutions

that are integral, provided that \tilde{b} and \tilde{c} are integral. The LP (3.7) is similar to an extended formulation introduced in [6] and used also in [7]. Note that, by construction of $\tilde{G}_{I,b}$, if x is a feasible solution for (3.4) then

$$\begin{aligned} \tilde{x}_i &= x_i & i \in I \\ \tilde{x}_i &= \left\lfloor \frac{x_i}{2} \right\rfloor & i \in (U \cup W) \setminus I \\ \tilde{x}_{i'} &= \left\lceil \frac{x_i}{2} \right\rceil & i' \in (U' \cup W') \setminus I \end{aligned}$$

is a feasible integral solution for (3.7) with the same objective value. If \tilde{x} is an integral feasible solution for (3.7) then

$$\begin{aligned} x_i &= \tilde{x}_i & i \in I \\ x_i &= \tilde{x}_i + \tilde{x}_{i'} & i \in L \end{aligned}$$

is a feasible solution for (3.4) with the same objective value. Hence (3.4), (3.7) and (3.8) have the same optimal value. Also, since any feasible solution of (3.4) is feasible for (3.5), then by weak duality the optimal value of (3.6) is at most the optimal value of (3.4), and therefore of (3.8).

3.2.2 Proof of Theorem 3.4

We prove Theorem 3.4 by showing how to derive an integral optimal solution for (3.6) from an integral optimal solution for (3.8). First, we need to prove a lemma.

We say that a digraph \tilde{D} is an *antisymmetric* orientation of $\tilde{G}_{I,b}$ if \tilde{D} is obtained from $\tilde{G}_{I,b}$ by orienting its edges so that, for any pair of symmetric edges e_1, e_2 of $\tilde{G}_{I,b}$, one of the two is oriented from \tilde{U}_I to \tilde{W}_I , and the other from \tilde{W}_I to \tilde{U}_I . For ease of notation, in the remainder, whenever we refer to an edge e of $\tilde{G}_{I,b}$, we also denote by e the arc of \tilde{D} obtained by orienting e .

We define the *cost* of each arc (u, v) from \tilde{U}_I to \tilde{W}_I as $\beta_{(u,v)} = \tilde{b}_{uv}$ and the *cost* of each arc (v, u) from \tilde{W}_I to \tilde{U}_I as $\beta_{(v,u)} = -\tilde{b}_{vu}$. Given a directed path or cycle S in \tilde{D} , we define the *cost* of S as $\beta(S)$.

Given a directed path (resp. a directed cycle) $S = v_1, v_2, \dots, v_n$ in \tilde{D} , $\text{sym}(S) = \text{sym}(v_n), \text{sym}(v_{n-1}), \dots, \text{sym}(v_2), \text{sym}(v_1)$ is a directed path (resp. a directed cycle), which we refer to as the *symmetric* of S in \tilde{D} . We say that a directed path or a directed cycle in \tilde{D} is *symmetric* if it coincides with its symmetric.

Observation 3.5. *A directed path S in \tilde{D} is symmetric if and only if S contains exactly one node $i \in I$ and $S = Q, i, \text{sym}(Q)$ for some directed path Q in \tilde{D} that ends in i .*

A directed cycle S is symmetric if and only if S contains exactly two distinct

nodes $i, j \in I$ and $S = i, Q, j, \text{sym}(Q), i$ for some directed path Q in \tilde{D} from i to j .

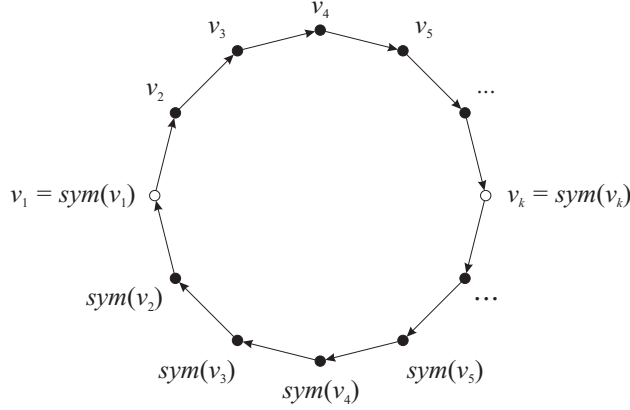


Figure 3.1: a symmetric directed cycle.

Proof. If $S = v_1, v_2, \dots, v_n$ is a symmetric directed path in \tilde{D} , then $S' = v_2, \dots, v_{n-1}$ is also symmetric, so by induction on the length of S we may assume that S' contains exactly one node $i \in I$ and $S' = Q', i, \text{sym}(Q')$ for some directed path Q' in \tilde{D} that ends in i . Since $(v_{n-1}, v_n) = \text{sym}(v_1, v_2)$, if we define $Q = v_1, v_2, Q', i$, then $S = Q, i, \text{sym}(Q)$.

Let S be a symmetric directed cycle. If S does not contain any node in L , then S consists of two distinct nodes $i, j \in I$ and of the two symmetric edges (i, j) and (j, i) . So we may assume that S contains a node $w \notin I$ and its symmetric. Since the two distinct paths in S with endnodes w and $\text{sym}(w)$ are both symmetric, the statement follows from the case of the symmetric directed path. \square

Lemma 3.6. *Let \tilde{D} be an antisymmetric orientation of $\tilde{G}_{\emptyset, b}$. Given a directed path S in \tilde{D} , then $-1 \leq \beta(S) + \beta(\text{sym}(S)) \leq 1$, where $\beta(S) + \beta(\text{sym}(S)) = 0$ if and only if the endnodes of S are either both in $U \cup W'$ or both in $U' \cup W$. Given a directed cycle S in \tilde{D} , then $\beta(S) + \beta(\text{sym}(S)) = 0$.*

Proof. Let S be a directed path or a directed cycle in \tilde{D} . One can readily verify that

$$\begin{aligned} \beta(S) + \beta(\text{sym}(S)) &= -|\{(u, v) \in S : u \in U, v \in W\}| \\ &\quad + |\{(v, u) \in S : v \in W, u \in U\}| \\ &\quad + |\{(u', v') \in S : u' \in U', v' \in W'\}| \\ &\quad - |\{(v', u') \in S : v' \in W', u' \in U'\}|. \end{aligned}$$

The arcs leaving $U \cup W'$ are either arcs from U to W or from W' to U' , while the arcs entering $U \cup W'$ are either arcs from W to U or from U' to W' . Therefore, by the above equation, $\beta(S) + \beta(\text{sym}(S))$ is the difference between the number of arcs in S entering $U \cup W'$ and the number of arcs in S leaving $U \cup W'$. Since S is a directed path or a directed cycle, the absolute value of this difference is at most 1, and it is 0 if and only if S is a directed path with endnodes either both in $U \cup W'$ or both in $U' \cup W$, or if S is a directed cycle. \square

Proof of Theorem 3.4.

Through this proof, we denote $\tilde{G}_{I,b} = (\tilde{U}_I \cup \tilde{W}_I, \tilde{E})$ simply by $\tilde{G} = (\tilde{U} \cup \tilde{W}, \tilde{E})$. Let \tilde{y} be an integral optimal solution of (3.8). We will show how to derive from \tilde{y} an integral solution y for (3.6) with value $\tilde{b}^\top \tilde{y}$, thus showing that y is optimal for (3.6). We say that an edge $e \in \tilde{E}$ is *loaded (for \tilde{y})* if $\tilde{y}_e > 0$, *unloaded (for \tilde{y})* if $\tilde{y}_e = 0$.

We prove the theorem by induction on $\sum_{i \in U \cup W} c_i$. If E contains an edge e such that both its images in \tilde{E} are unloaded, and if y' is an integral optimal solution for the instance of (3.6) on the graph $G' = (U \cup W, E \setminus \{e\})$ with value $\tilde{b}^\top \tilde{y}$, then the vector y , obtained by completing y' with $y_e = 0$ and with $y_P = 0$ for every I -path P in G that contains e , is an integral optimal solution for the problem (3.6) with value $\tilde{b}^\top \tilde{y}$. Thus from now on we will assume that for every $e \in E$, at least one of its two images in \tilde{E} is loaded. We will use two types of reductions, defined next.

Reduction (with respect to \tilde{y}) on the symmetric edges.

Suppose \tilde{G} contains a pair of symmetric edges e_1 and e_2 that are both loaded. For any edge $e \in E$ with images $e_1, e_2 \in \tilde{E}$, let $\gamma_e = \min\{\tilde{y}_{e_1}, \tilde{y}_{e_2}\}$, and define a new cost vector c' on the nodes of G by

$$c'_i = c_i - a_i \sum_{ij \in E} \gamma_{ij}, \quad i \in U \cup W.$$

We call *reduced problem (w.r.t. \tilde{y})* the instance of (3.6) on the graph G with costs on the nodes c' , and *extended reduced problem (w.r.t. \tilde{y})* the corresponding instance of (3.8). Notice that the vector \tilde{y}' defined by

$$\begin{aligned} \tilde{y}'_{e_1} &= \tilde{y}_{e_1} - \gamma_e \\ \tilde{y}'_{e_2} &= \tilde{y}_{e_2} - \gamma_e \end{aligned} \quad e \in E$$

is an integral optimal solution to the extended reduced problem, with value $\tilde{b}^\top \tilde{y} - \sum_{e \in E} b_e \gamma_e$.

Since $\sum_{i \in U \cup W} c'_i < \sum_{i \in U \cup W} c_i$, by induction the reduced problem has an integral optimal solution y' with value $\tilde{b}^\top y'$. Hence the vector y defined by

$$\begin{aligned} y_e &= y'_e + \gamma_e & e \in E \\ y_P &= y'_P & P \text{ } I\text{-path,} \end{aligned}$$

is a feasible integral solution for problem (3.6) with value $\tilde{b}^\top y$, thus y is optimal.

Thus we may assume that for every edge $e \in \tilde{E}$ exactly one among e and $\text{sym}(e)$ is loaded. Let \tilde{D} be the digraph obtained from \tilde{G} by orienting from \tilde{U} to \tilde{W} the unloaded edges, and from \tilde{W} to \tilde{U} the loaded edges. Note that \tilde{D} is an antisymmetric orientation of \tilde{G} . We denote by $\tilde{D}_{\emptyset, b}$ the digraph obtained from $\tilde{G}_{\emptyset, b}$ by orienting the unloaded edges of \tilde{E} from $U \cup U'$ to $W \cup W'$ and the loaded edges of \tilde{E} from $W \cup W'$ to $U \cup U'$. Notice that $\tilde{D}_{\emptyset, b}$ is an antisymmetric orientation of $\tilde{G}_{\emptyset, b}$, and that \tilde{D} can be obtained from $\tilde{D}_{\emptyset, b}$ by identifying the images of nodes in I .

Next we define the second type of reduction.

Reduction on a symmetric path or symmetric cycle of non-positive cost.

Let S be a symmetric directed cycle of \tilde{D} of non-positive cost, or a symmetric directed path of \tilde{D} of non-positive cost with $\tilde{c}_k - \sum_{e \ni k} \tilde{y}_e > 0$, where k is the only endnode of S incident with an unloaded arc of S . In the first case let $\gamma = \min\{\tilde{y}_e : e \in S, \tilde{y}_e > 0\}$, in the second one let γ be the minimum among $\tilde{c}_k - \sum_{e \ni k} \tilde{y}_e$, and $\min\{\tilde{y}_e : e \in S, \tilde{y}_e > 0\}$. Let c' be the cost vector on the nodes of G obtained from c by setting $c'_i = c_i$ for any node i whose images are not in S , and $c'_i = c_i - \gamma$ for any node i whose images are in S . We call *reduced problem* the instance of (3.6) on the graph G with costs c' , and *extended reduced problem* the corresponding instance of (3.8). Then

$$\begin{aligned} \tilde{y}'_e &= \tilde{y}_e - \gamma & e \in S, \tilde{y}_e > 0 \\ \tilde{y}'_e &= \tilde{y}_e & \text{otherwise} \end{aligned}$$

is an integral optimal solution for the extended reduced problem.

By Observation 3.5, S is the union of a path Q and its symmetric $\text{sym}(Q)$, and the endnodes in common among Q and $\text{sym}(Q)$ are the only nodes of S in I . Note that the set of edges of G whose images are in S define an I -path R in G . If S is a path, then the unique node of S in I is also the unique endnode of R in I , while if S is a cycle, then the two distinct nodes of S in I are the endnodes of R . Notice that

$$\sum_{e \in R} b_e = \sum_{e \in S} \tilde{b}_e = \sum_{\substack{e \in S \\ e \text{ loaded}}} \tilde{b}_e + \sum_{\substack{e \in S \\ e \text{ unloaded}}} \tilde{b}_e = \beta(S) + 2 \sum_{\substack{e \in S \\ e \text{ loaded}}} \tilde{b}_e. \quad (3.9)$$

The arcs of Q and of $\text{sym}(Q)$ induce directed paths Q' and $\text{sym}(Q')$ in $\tilde{D}_{\emptyset, b}$, respectively. Furthermore $\beta(S) = \beta(Q') + \beta(\text{sym}(Q'))$. By Lemma 3.6 $|\beta(Q') + \beta(\text{sym}(Q'))| \leq 1$. Since S has non-positive cost, $\beta(S) \in \{0, -1\}$. Thus, by (3.9),

$$\sum_{\substack{e \in S \\ e \text{ loaded}}} \tilde{b}_e = \left\lceil \frac{b(R)}{2} \right\rceil.$$

Hence the optimal value of the extended reduced problem is $\tilde{b}^\top \tilde{y} - \left\lceil \frac{b(R)}{2} \right\rceil \gamma$. Since $\sum_{i \in U \cup W} c'_i < \sum_{i \in U \cup W} c_i$, by induction there exists an integral solution y' for the reduced problem with value $\tilde{b}^\top \tilde{y}'$, thus the vector y defined by

$$\begin{aligned} y_e &= y'_e & e \in E \\ y_R &= y'_R + \gamma \\ y_P &= y'_P & P \text{ } I\text{-path, } P \neq R \end{aligned}$$

is an integral feasible solution for problem (3.6) with value $\tilde{b}^\top \tilde{y}$, hence y is optimal.

We define the *sources* of \tilde{D} as the elements of $\{u \in \tilde{U} : \sum_{e \ni u} \tilde{y}_e < \tilde{c}_u\} \cup \{v \in \tilde{W} : \sum_{e \ni v} \tilde{y}_e > 0\}$ and the *sinks* of \tilde{D} as the elements of $\{u \in \tilde{U} : \sum_{e \ni u} \tilde{y}_e > 0\} \cup \{v \in \tilde{W} : \sum_{e \ni v} \tilde{y}_e < \tilde{c}_v\}$. Let S be either a directed path in \tilde{D} from a source to a sink or a directed cycle, and ε be a positive number. We say that the solution \tilde{y}' is obtained by *augmenting \tilde{y} by ε on S* if $\tilde{y}'_e = \tilde{y}_e + \varepsilon$ for every unloaded edge $e \in S$, $\tilde{y}'_e = \tilde{y}_e - \varepsilon$ for every loaded edge $e \in S$, and $\tilde{y}'_e = \tilde{y}_e$ for every edge $e \in \tilde{E} \setminus S$. If ε is small enough, then \tilde{y}' is also a feasible solution for problem (3.8), with value $\tilde{b}^\top \tilde{y} + \varepsilon \beta(S)$ (notice that this is the standard notion of augmentation in flow theory, see for example [30]). Therefore, since \tilde{y} is an optimal solution, we have the following.

Observation 3.7. *If S is a directed path from a source to a sink or a directed cycle in \tilde{D} , then $\beta(S) \leq 0$. Furthermore, if $\beta(S) = 0$, then for $\varepsilon > 0$ small enough the solution obtained by augmenting \tilde{y} by ε on S is optimal for (3.8).*

Suppose now that \tilde{D} contains a directed cycle S . We show that in this case \tilde{D} contains a directed cycle C that either is symmetric or has at most one node in I . In fact, if S contains two or more nodes in I , let Q be a minimal directed path contained in S with endnodes in I and with no intermediate node in I . The directed graph induced by the arcs of $Q \cup \text{sym}(Q)$ is the union of arc-disjoint directed cycles, so it either contains a directed cycle with at most one node in I , or it is a symmetric directed cycle.

Case 1: C is a symmetric directed cycle in \tilde{D} .

By Observation 3.7, $\beta(C) \leq 0$, thus we can apply the reduction on the symmetric directed cycle C of non-positive cost, and we are done.

Case 2: C has at most one node in I .

The arcs in C form a directed cycle or a directed path in the digraph $\tilde{D}_{\emptyset, b}$, thus, by Lemma 3.6, $-1 \leq \beta(C) + \beta(\text{sym}(C)) \leq 1$, while by Observation 3.7 $\beta(C) \leq 0$ and $\beta(\text{sym}(C)) \leq 0$. Hence at least one among C and $\text{sym}(C)$ has cost zero, and we assume $\beta(C) = 0$. Note that C must cross an unloaded arc \bar{e} whose symmetric is not in C , otherwise all the unloaded arcs of C have their symmetric in C , thus C is symmetric. So we can augment \tilde{y} by $\min\{\tilde{y}_e : e \in C, \tilde{y}_e > 0\}$ on C thus getting another integral optimal solution \tilde{y}' where both \bar{e} and its symmetric have strictly positive value. Thus we can now apply the reduction w.r.t. \tilde{y}' on the symmetric edges, and we are done.

Hence we can assume that the digraph \tilde{D} is acyclic. Notice that every node not isolated in \tilde{D} with in-degree 0 is a source. In fact if j has in-degree 0 and strictly positive out-degree, then $\text{sym}(j)$ has out-degree 0 and strictly positive in-degree. So, if $j \in \tilde{U}$, then $\sum_{e \ni j} \tilde{y}_e = 0$ and $\sum_{e \ni \text{sym}(j)} \tilde{y}_e > 0$, if $j \in \tilde{W}$, then $\sum_{e \ni j} \tilde{y}_e > 0$. In the same way notice that every node not isolated in \tilde{D} with out-degree 0 is a sink.

Suppose that there exists a node i in I that is not isolated in \tilde{D} . Since \tilde{D} is acyclic, there exists a path Q from i to a node j of out-degree 0 in \tilde{D} and, since j has out-degree 0, $j \notin I$. Consider the directed walk $S = \text{sym}(j), \text{sym}(Q), i, Q, j$. Notice that, since \tilde{D} is acyclic, S must be a directed path. Since S is a directed path in \tilde{D} from a source to a sink, by Observation 3.7 $\beta(S) \leq 0$. Moreover, if k is the only endnode of S incident with an unloaded arc of S , then $\tilde{c}_k - \sum_{e \ni k} \tilde{y}_e > 0$, thus we may apply the reduction on the symmetric directed path of non-positive cost S , and we are done.

So we can assume that all the nodes in I are isolated in \tilde{D} .

Therefore there exists a directed path S in \tilde{D} from a node with in-degree 0 to a node with out-degree 0. Since both S and $\text{sym}(S)$ are directed paths in \tilde{D} from a source to a sink, by Observation 3.7 $\beta(S) \leq 0$ and $\beta(\text{sym}(S)) \leq 0$. By Lemma 3.6, $-1 \leq \beta(S) + \beta(\text{sym}(S)) \leq 1$. Hence at least one among S and $\text{sym}(S)$ has cost zero, and we assume it is $S = v_1, \dots, v_k$. Notice that S crosses an unloaded arc \bar{e} whose symmetric is not in S . In fact, if all the unloaded arcs of S have their symmetric in S , it must be $|e \in S : e \text{ loaded}| = |e \in S : e \text{ unloaded}| + 1$, since in S unloaded and loaded arcs alternate and since S is not symmetric. But then (v_1, v_2) and (v_{k-1}, v_k) are both loaded and at least one of them is the symmetric of an unloaded arc in S . By symmetry we may assume it is (v_1, v_2) , thus $\text{sym}(v_1)$ has out-degree 0, hence

$(\text{sym}(v_2), \text{sym}(v_1))$ is the last arc of S . A contradiction as it is unloaded. So we can augment \tilde{y} on S by the minimum among \tilde{c}_j for every endnode j of S incident with an unloaded arc of S , and $\min\{\tilde{y}_e : e \in S, \tilde{y}_e > 0\}$. Thus we get another integral optimal solution \tilde{y}' where both \bar{e} and its symmetric have strictly positive value. Hence we can now apply the reduction w.r.t. \tilde{y}' on the symmetric edges, and we are done. \square

We conclude the section with the following corollary, which will be used in the proof of Theorem 3.1.

Corollary 3.8. *Let $l \in \mathbb{Z}^{U \cup W}$. The system*

$$\begin{aligned} a_i x_i + a_j x_j &\geq b_{ij} & ij \in E \\ x_i &\geq l_i & i \in U \cup W \\ \sum_{i \in P} x_i &\geq \left\lceil \frac{b(P)}{2} \right\rceil & P = v_1, \dots, v_k \text{ } I\text{-path}, v_1, v_k \in I \\ \sum_{i \in P} x_i &\geq \left\lceil \frac{b(P) + l_{v_k}}{2} \right\rceil & P = v_1, \dots, v_k \text{ } I\text{-path}, v_k \notin I. \end{aligned} \quad (3.10)$$

is totally dual integral.

Proof. Let c be a vector in $\mathbb{Z}^{U \cup W}$. By Theorem 3.4 we know that the dual of the problem $\min\{c^\top x : x \text{ satisfies (3.5)}\}$ with integer requirements $b_{ij} - a_i l_i - a_j l_j$ has an integral optimal solution y^* . It is straightforward to check that the integral solution \bar{y} defined by $\bar{y}_e = y_e^*$, $e \in E$, $\bar{y}_P = y_P^*$, P I -path, $\bar{y}_i = c_i - \sum_{e \ni i} a_i y_e^* - \sum_{P \ni i} y_P^*$, $i \in U \cup W$, is optimal for $\min\{c^\top x : x \text{ satisfies (3.10)}\}$. \square

3.3 General case

In this section we prove Theorem 3.1 by reducing the general problem to the bipartite case studied in Section 3.2.

Proof of Theorem 3.1. First we show the following.

Claim. *The system defined by (3.2) and (3.3), for every I -trail P , is totally dual integral.*

Proof of claim. We show how to reduce this problem to the previous case. We define the undirected graph $G' = (V \cup V', E')$ as follows. Let V' be a copy of V such that $V \cap V' = \emptyset$. For every $i \in V$ we denote by i' the copy of i in V' and for every $X \subseteq V$ we denote by X' the subset of V' that contains only the copies of the nodes in X . E' contains the edge ii' for every $i \in V$, with

requirement 0, and the edge ij (resp. ij' , $i'j$, $i'j'$) for every edge $ij \in E$ with $\sigma_{i,ij} = \sigma_{j,ij} = +1$ (resp. $\sigma_{i,ij} = +1$ and $\sigma_{j,ij} = -1$, $\sigma_{i,ij} = -1$ and $\sigma_{j,ij} = +1$, $\sigma_{i,ij} = \sigma_{j,ij} = -1$), with the same requirement of the original bidirected edge ij . Let $b' \in \mathbb{Z}^{E'}$ be the vector of requirements on the edges in E' . Since the edge-node incidence matrix of G is totally unimodular and has two nonzero elements per row, it follows from Theorem 1.13 that V can be partitioned into two sets R, B such that every edge of G with the same sign in both its endnodes has one endnode in R and the other in B , while every edge with different signs in its endnodes is contained in R or B . Therefore every edge of G' has exactly one endnode in $R \cup B'$ and the other in $R' \cup B$, thus G' is bipartite.

If we define $a'_i = 2$ for $i \in I \cup I'$ and $a'_i = 1$ for $i \in L \cup L'$ then one can verify that a vector x satisfies $Mx \geq b$, $l \leq x \leq u$, if and only if the vector x' defined by $x'_i = -x'_{i'} = x_i$ for all $i \in V$ satisfies $a'_i x'_i + a'_j x'_j \geq b'_{ij}$ for all $ij \in E'$, $x'_i \geq l_i$ and $x'_{i'} \geq -u_i$ for all $i \in V$. Since the inequalities $x'_i \geq -x'_{i'}$, $i \in V$, are valid for the latter system, as they are the inequalities $a'_i x'_i + a'_{i'} x'_{i'} \geq b'_{ii'}$ for the edges ii' of G' , then the polyhedron defined by $Mx \geq b$, $l \leq x \leq u$ corresponds to the face of the polyhedron defined by $a'_i x'_i + a'_j x'_j \geq b'_{ij}$, $ij \in E'$, $x'_i \geq l_i$, $x'_{i'} \geq -u_i$, $i \in V$ given by $x'_i = -x'_{i'}$, $i \in V$.

Given an $I \cup I'$ -path P in G' , this determines an inequality as in (3.10) for the instance given by G' , b' and $I \cup I'$. Substituting x'_i for $-x'_{i'}$, for every $i \in V$, into such inequality, we obtain the inequality of (3.3) relative to the I -trail T obtained from P by identifying the pairs of nodes i, i' for every $i \in V$ such that i, i' are in P .

Since, by Corollary 3.8, the system obtained from $a'_i x'_i + a'_j x'_j \geq b'_{ij}$, $ij \in E'$, $x'_i \geq l_i$, $x'_{i'} \geq -u_i$, $i \in V$ by juxtaposing the inequalities of the form (3.10) relative to $I \cup I'$ -paths of G' is totally dual integral, and since, by Theorem 1.17, setting to equality some inequalities of a system preserves total dual integrality, then the system obtained from the above by setting $x'_i = -x'_{i'}$, $i \in V$, is totally dual integral, therefore also the system defined by (3.2) and (3.3) for every I -trail P is totally dual integral. This concludes the proof of the claim. \diamond

We conclude the proof of Theorem 3.1. Given a vector $c \in \mathbb{Z}^V$, we show how to get an integral optimal solution for the dual of $\min\{c^\top x : x \text{ satisfies (3.2), (3.3) for every } I\text{-path } P\}$ from an integral optimal solution y for the dual of $\min\{c^\top x : x \text{ satisfies (3.2), (3.3) for every } I\text{-trail } P\}$. In fact, if T is an I -trail that is not an I -path such that $y_T > 0$, then there exists a cycle C , a node j and two trails Q, R such that $T = Q, j, C, j, R$. Note that $S = Q, j, R$ is an I -trail with the same endnodes of T but with

less cycles than T . Since the edge-node incidence matrix of G is totally unimodular, by Theorem 1.10 (iv), the edges of C can be partitioned in two subsets C^1 and C^2 such that any two adjacent edges of C are contained in the same subset if and only if one of them has a -1 and the other has a $+1$ in their common endnode. Moreover, we may assume that C^1 has cost at least $\lceil b(C)/2 \rceil$. One can verify that, by our choice of the partition C^1, C^2 , the integral vector y' that is identical to y except for $y'_S = y_S + y_T, y'_e = y_e + y_T, \forall e \in C^1, y'_T = 0$, is feasible for the dual of $\min\{c^\top x : x \text{ satisfies (3.2), (3.3) for every } I\text{-trail } P\}$. Furthermore its objective value is at least that of y plus $y_T(\lceil b(S)/2 \rceil + \lceil b(C)/2 \rceil - \lceil (b(S) + b(C))/2 \rceil) \geq 0$, thus y' is also optimal. Since the total number of cycles contained in I -trails whose associated dual variables are positive strictly decreases, by repeating the argument we obtain an integral optimal solution for the dual of $\min\{c^\top x : x \text{ satisfies (3.2), (3.3) for every } I\text{-path } P\}$. \square

3.4 Polynomial time solvability

3.4.1 Bipartite case

The proof of Theorem 3.4 gives an algorithm (albeit not a polynomial time one) to derive an integral optimal solution y^* for (3.6) from an integral optimal solution \tilde{y} for (3.8), as follows. Initially we set $y^* := 0$. Each time we apply a reduction, we update the value of y^* and then apply our algorithm recursively on the reduced problem as long as the current vector c is not the all zero vector. Since each time we apply a reduction the value of some entry of c decreases, the total number of iterations is bounded by $\sum_{i \in U \cup W} c_i$, which is not a polynomial bound on the size of the problem.

More in detail: if \tilde{G} contains a pair of symmetric loaded edges, then for each $e \in E$ we update $y_e^* := y_e^* + \min\{\tilde{y}_{e_1}, \tilde{y}_{e_2}\}$, where e_1 and e_2 are the images of e in \tilde{G} , apply the reduction on the symmetric edges, and proceed recursively on the reduced problem.

If \tilde{D} has a directed cycle, we can find in polynomial time a directed cycle C that either is symmetric or has at most one node in I . If C is symmetric, then it has non-positive cost, thus we apply the reduction on C , update $y_R^* := y_R^* + \gamma$, where R is the I -path defined by the edges with images in C and γ is the minimum value of \tilde{y} on the loaded edges of C , and proceed recursively on the reduced problem.

Otherwise, we augment on the cycle among C and $\text{sym}(C)$ with cost zero by the smallest load on its edges, and apply the reduction on the symmetric edges.

If \tilde{D} is acyclic and there exists a non-isolated node in I , then we can find in polynomial time a symmetric directed path of non-positive cost S in \tilde{D} starting from some node of in-degree 0, we apply the reduction on S , update $y_R^* := y_R^* + \gamma$, where R is the I -path defined by the edges with images in S and γ is defined as in the proof, and proceed recursively on the reduced problem.

If all nodes of I are isolated, we can find in polynomial time a directed path S of cost zero from a node of in-degree zero to a node of out-degree zero. We augment on S by the minimum among \tilde{c}_j , for every endnode j of S incident with an unloaded arc of S , and $\min\{\tilde{y}_e : e \in S, \tilde{y}_e > 0\}$, and we apply the reduction on the symmetric edges.

Notice that each iteration can be performed in strongly polynomial time. While we cannot give a polynomial bound on the number of iterations of the algorithm described above, we can prove that the number of iterations in which we apply a reduction on a symmetric path or symmetric cycle of non-positive cost is bounded by the number of edges of G . In fact, each time we apply a reduction on a symmetric cycle, the number of loaded edges decreases by at least one. We apply the reduction on a symmetric path of non-positive cost S only when S starts in a node with in-degree 0 and ends in a node with out-degree 0. In this case, if k is the only endnode of S incident with an unloaded arc of S , we have

$$\tilde{c}_k - \sum_{e \ni k} \tilde{y}_e = \tilde{c}_k = \tilde{c}_{sym(k)} \geq \min\{\tilde{y}_e : e \in S, \tilde{y}_e > 0\}$$

since k is incident only with unloaded arcs, and $sym(k)$ is incident with a loaded arc of S . Thus, each time we apply a reduction on a symmetric path, the number of loaded edges decreases by at least one.

So, if on a given instance the algorithm described above does not perform any reduction on the symmetric edges, then it performs at most $|E|$ iterations. In particular, this happens if and only if the optimal solution y^* for (3.6) produced by the algorithm satisfies $y_e^* = 0$ for every $e \in E$. We will show next that we can reduce to this case, thus proving the following.

Theorem 3.9. *There is a strongly polynomial-time algorithm to compute an integral optimal solution for (3.6) for each integral vector c for which it has a finite optimum.*

Proof. Let x^* be an integral optimal solution for the problem $\min\{c^\top x : x \text{ satisfies (3.5)}\}$ and let \tilde{x} be the solution for (3.7) defined by $\tilde{x}_i = x_i^*$, $i \in I$, $\tilde{x}_i = \lfloor \frac{x_i^*}{2} \rfloor$, $\tilde{x}_{i'} = \lceil \frac{x_i^*}{2} \rceil$, $i \in L$. By Theorem 3.4, x^* is optimal if and only if \tilde{x} is optimal for (3.7), and $c^\top x^* = \tilde{c}^\top \tilde{x}$. Notice that this remains true even if c

is not an integral vector. Given $e = ij \in E$, let $\alpha^e \in \mathbb{R}^{U \cup W}$ be the coefficient vector of the constraint of (3.4) relative to e , that is $\alpha_i^e = a_i$, $\alpha_j^e = a_j$, $\alpha_k^e = 0$ for $k \in (U \cup W) \setminus \{i, j\}$.

Claim. *Given $\bar{e} \in E$ such that $\alpha^{\bar{e}} x^* = b_{\bar{e}}$, one can compute in strongly polynomial time the maximum γ such that x^* remains optimal for the problem $\min\{(c - \gamma \alpha^{\bar{e}})^\top x : x \text{ satisfies (3.5)}\}$.*

Proof of claim. Let $J = \{i \in \tilde{U}_I \cup \tilde{W}_I : \tilde{x}_i > 0\}$, $F = \{ij \in \tilde{E} : \tilde{x}_i + \tilde{x}_j > b_{ij}\}$. By complementary slackness, a vector $\tilde{y} \in \mathbb{R}^{\tilde{U}_I \cup \tilde{W}_I}$ is optimal for (3.8) if and only if \tilde{y} satisfies

$$\begin{aligned} \sum_{e \ni i} \tilde{y}_e &= \tilde{c}_i & i \in J \\ \sum_{e \ni i} \tilde{y}_e &\leq \tilde{c}_i & i \in (\tilde{U} \cup \tilde{W}) \setminus J \\ \tilde{y}_e &= 0 & e \in F \\ \tilde{y}_e &\geq 0 & e \in \tilde{E} \setminus F. \end{aligned} \tag{3.11}$$

Let $e_1, e_2 \in \tilde{E}$ be the images of \bar{e} in \tilde{E} . Let $\mu = \max\{s : s \leq \tilde{y}_{e_1}, s \leq \tilde{y}_{e_2}, \tilde{y} \text{ satisfies (3.11)}\}$. We show that $\gamma = \mu$.

We first show $\gamma \leq \mu$. Let $c' = c - \gamma \alpha^{\bar{e}}$, and $\tilde{c}' \in \mathbb{R}^{\tilde{U}_I \cup \tilde{W}_I}$ be the corresponding cost vector on the nodes of \tilde{G} . Since x^* is optimal for $\min\{(c - \gamma \alpha^{\bar{e}})^\top x : x \text{ satisfies (3.5)}\}$, then \tilde{x} is optimal for the problem (3.7) with respect to the cost vector \tilde{c}' . Hence there exists a vector \tilde{y}' that satisfies

$$\begin{aligned} \sum_{e \ni i} \tilde{y}'_e &= \tilde{c}'_i & i \in J \\ \sum_{e \ni i} \tilde{y}'_e &\leq \tilde{c}'_i & i \in (\tilde{U} \cup \tilde{W}) \setminus J \\ \tilde{y}'_e &= 0 & e \in F \\ \tilde{y}'_e &\geq 0 & e \in \tilde{E} \setminus F. \end{aligned}$$

Now the vector defined by $\tilde{y}_e = \tilde{y}'_e + \gamma$ if $e \in \{e_1, e_2\}$, $\tilde{y}_e = \tilde{y}'_e$ otherwise, satisfies (3.11) and $\gamma \leq \tilde{y}_{e_1}$, $\gamma \leq \tilde{y}_{e_2}$.

Now we show that $\gamma \geq \mu$. Let $c' = c - \mu \alpha^{\bar{e}}$, and $\tilde{c}' \in \mathbb{R}^{\tilde{U}_I \cup \tilde{W}_I}$ be the corresponding cost vector on the nodes of \tilde{G} . If \tilde{y} is the solution that satisfies (3.11) and maximizes s , then the vector defined by $\tilde{y}'_e = \tilde{y}_e - \mu$ if $e \in \{e_1, e_2\}$, $\tilde{y}'_e = \tilde{y}_e$ otherwise, satisfies (3.11) with respect to the cost vector \tilde{c}' . Hence \tilde{x} is optimal for the problem (3.7) with respect to the cost vector \tilde{c}' , and x^* is optimal for the problem $\min\{(c - \mu \alpha^{\bar{e}})^\top x : x \text{ satisfies (3.5)}\}$.

Finally, since the coefficients of the variables in (3.11) and in $s \leq \tilde{y}_{e_1}, s \leq \tilde{y}_{e_2}$, are $0, \pm 1$, γ can be computed in strongly polynomial time using an algorithm of Tardos [33] (see Corollary 1.6). \diamond

Let e^1, \dots, e^m be the edges in E such that $\alpha^e x^* = b_e, e \in \{e^1, \dots, e^m\}$. Set $c^0 = c$ and, for $k = 1, \dots, m$, let $c^k = c^{k-1} - \lfloor \gamma_k \rfloor \alpha^{e^k}$, where γ_k is the maximum γ such that x^* remains optimal for the problem $\max\{(c^{k-1} - \gamma \alpha^{e^k})^\top x : x \text{ satisfies (3.5)}\}$. By the previous claim, we can compute c^1, \dots, c^m in strongly polynomial time.

Given any integral optimal solution y^* for the dual of $\min\{c^m{}^\top x : x \text{ satisfies (3.5)}\}$, then the vector \bar{y} , defined by $\bar{y}_{e^k} = y_{e^k}^* + \lfloor \gamma_k \rfloor$ for every $k = 1, \dots, m$, $\bar{y}_P = y_P^*$ for every I -path P , is an integral optimal solution for (3.6). By definition of $\gamma_1, \dots, \gamma_m, c^1, \dots, c^m$, for every $e \in E$ we must have $y_e^* < 1$, thus $y_e^* = 0$. This concludes our proof, since we have shown above that in this case the algorithm given by the proof of Theorem 3.4 finds an integral optimal solution for $\min\{c^m{}^\top x : x \text{ satisfies (3.5)}\}$ in strongly polynomial time. This completes the proof of the claim. \square

3.4.2 General case

Theorem 3.10. *There is a strongly polynomial-time algorithm to compute an integral optimal solution for the dual of $\min\{c^\top x : x \text{ satisfies (3.2), (3.3) for every } I\text{-path } P\}$ whenever the problem has a finite optimum.*

Proof. We showed in Theorem 3.9 that an integral optimal solution for the dual of any problem of the form $\min\{c^\top x : x \text{ satisfies (3.5)}\}$ can be computed in strongly polynomial time for each integral vector c for which it has a finite optimum.

The proof of Corollary 3.8 shows how to obtain, in strongly polynomial time, an integral optimal solution for the dual of any problem of the form $\min\{c^\top x : x \text{ satisfies (3.10)}\}$ from an integral optimal solution for the dual of a problem of the form $\min\{c^\top x : x \text{ satisfies (3.5)}\}$ with integer requirements $b_{ij} - a_i l_i - a_j l_j$.

The proof of the Claim in the proof Theorem 3.1 shows how to reduce, in strongly polynomial time, any problem of the form $\min\{c^\top x : x \text{ satisfies (3.2), (3.3) for every } I\text{-trail } P\}$ to a problem of the form $\min\{\bar{c}^\top \bar{x} : \bar{x} \text{ satisfies (3.10)}\}$ in some auxiliary graph G' , but with a polynomial number of inequalities (of the form $x'_i + x'_{i'} \geq 0$) set to equality. The next claim shows that an integral optimal solution of the dual of any problem in the latter form can be computed in strongly polynomial time.

Finally, in the last part of the proof of Theorem 3.1, we showed how to get an integral optimal solution for the dual of $\min\{c^\top x : x \text{ satisfies (3.2), (3.3) for every } I\text{-path } P\}$ from an integral optimal solution for the dual of $\min\{c^\top x : x \text{ satisfies (3.2), (3.3) for every } I\text{-trail } P\}$. Notice that the procedure described terminates in strongly polynomial time.

Claim. Let $Ax \geq b, Cx \geq d$ be a totally dual integral system of linear inequalities, where $A \in \mathbb{Z}^{p \times n}$ and $C \in \mathbb{Z}^{q \times n}$. Let $\alpha = \max\{\|A\|_\infty, \|C\|_\infty\}$. Given $c \in \mathbb{Z}^n$, let γ be the q -dimensional vector with all entries equal to $n!\alpha^n\|c\|_\infty$ and $\bar{c} = c + C^\top \gamma$. If (y^*, u^*) is an integral optimal solution for the dual of

$$\min\{\bar{c}^\top x \mid Ax \geq b, Cx \geq d\}, \quad (3.12)$$

(where y and u are relative to the rows of A and C , respectively) then $(y^*, u^* - \gamma)$ is an integral optimal solution for the dual of

$$\min\{c^\top x \mid Ax \geq b, Cx = d\}, \quad (3.13)$$

provided that the latter has a finite optimum.

Proof of claim. Clearly $(y^*, u^* - \gamma)$ is integral and feasible for the dual of (3.13). We show it is optimal. Let \bar{x} be an optimal solution of (3.13), and (\bar{y}, \bar{u}) be an optimal basic solution for the dual of (3.13). Since (\bar{y}, \bar{u}) is basic, then the absolute values of its components are bounded above by $\|c\|_\infty$ times the maximum among the absolute values of the determinants of the square submatrices of (A^\top, C^\top) , which is at most $\alpha^n n!$. Therefore $\bar{u} \geq -\gamma$. Thus $(\bar{y}, \bar{u} + \gamma)$ is feasible for the dual of (3.12), \bar{x} is feasible for (3.12), and $\bar{c}\bar{x} = b^\top \bar{y} + d^\top \bar{u} + \gamma^\top C\bar{x} = b^\top \bar{y} + d^\top (\bar{u} + \gamma)$, thus \bar{x} and $(\bar{y}, \bar{u} + \gamma)$ are optimal for (3.12) and its dual, respectively. Thus $b^\top \bar{y} + \beta \bar{u} = b^\top y^* + \beta(u^* - \gamma)$, so $(y^*, u^* - \gamma)$ is an integral optimal solution for the dual of (3.13). This concludes the proof of the claim. \diamond

In particular, if the system $Ax \geq b, Cx \geq d$ is of the form (3.10), and the number of rows of C is bounded by some polynomial in n , then, for any $c \in \mathbb{Z}^n$, the problem of finding an integral dual solution of (3.13) can be reduced in strongly polynomial time to the problem of finding an integral dual solution of (3.12), which by Theorem 3.9 can be solved in strongly polynomial time. \square

Chapter 4

A class of matrices arising from bidirected graphs

In this chapter we study totally half-modular matrices obtained from $\{0, \pm 1\}$ -matrices with at most two nonzero entries per column by multiplying by 2 some of the columns. We show that the matrix

$$M_4 = \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

is the only minor minimal matrix in such class (up to multiplying rows and columns by -1) that does not have the Edmonds-Johnson property. Notice that this result extends Theorem 2.2 of Edmonds and Johnson. We will also show that, for each matrix M in this class that does not contain M_4 as a minor, one can minimize in polynomial time any linear function over the integer hull of $b \leq Mx \leq c$, $l \leq x \leq u$, for all integral vectors b, c, l, u . The results in this chapter are joint work with A. Musitelli and G. Zambelli [10].

4.1 Introduction

Given a bidirected graph G and a subset F of its edges, we denote by $A(G, F)$ the matrix obtained from $\Sigma(G)$ by multiplying by 2 the columns corresponding to the edges in F .

We will show in Lemma 4.5 that a matrix $A(G, F)$ is totally half-modular if and only if (G, F) satisfies the following.

Cycles condition: *no odd cycle of G contains edges in F .*

By Theorem 2.1 and Theorem 2.9, the class of matrices $A(G, F)$ with the Edmonds-Johnson property, and such that (G, F) satisfies the cycles condition is closed under taking minors.

In this chapter we characterize the pairs (G, F) that satisfy the cycles condition for which $A(G, F)$ has the Edmonds-Johnson property.

Our result implies that Conjecture 2.1 is true if A is a totally half-modular matrix with at most two nonzero entries per column and such that all the nonzero elements in a column have the same absolute value. In fact, the only minor that we need to exclude is A_3 , because A_4 never appears.

Let $G = (V, E, \sigma)$ be a bidirected graph and $F \subseteq E$. Given a node $v \in V$, the signing σ' obtained from σ by setting $\sigma'_{v,e} = -\sigma_{v,e}$ for all edges e incident with v is said to be obtained by *switching signs on the node v* .

Given $e = vw \in E$, the signing σ' obtained from σ by setting $\sigma'_{v,e} = -\sigma_{v,e}$, $\sigma'_{w,e} = -\sigma_{w,e}$, is said to be obtained by *switching signs on the edge e* .

Given a node $v \in V$, the pair (G', F') obtained from (G, F) by *deleting node v* is defined as follows. $V(G') = V \setminus \{v\}$, $E(G')$ contains all edges of E not incident with v and a loop on w for each edge $vw \in E$ with $v \neq w$. We will identify such loops in G' with the corresponding edges incident with v in G . The signing on the edges of G' coincides with σ on $G \setminus v$, while $F' = F \cap E(G')$.

Notice that our definition of node deletion is non-standard, since we do not remove all the edges incident with v , but we replace them with loops. (See Figure 4.1 for an example.)

Given a subset of nodes $U \subseteq V$, the pair (G', F') is obtained from (G, F) by *deleting the nodes in U* if (G', F') is obtained from (G, F) by deleting one by one the nodes in U .

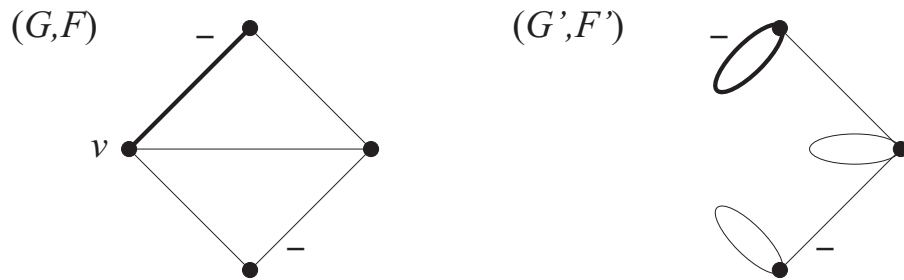


Figure 4.1: Node deletion: (G', F') is obtained from (G, F) by deleting the node v : the boldfaced edges represent the edges in F and in F' , and the signing on the edges is everywhere $+1$ except where a $-$ occurs.

Given an edge $e \in E$, (G', F') is obtained from (G, F) by *deleting edge*

e if $G' = (V, E \setminus \{e\}, \sigma')$ and $F' = F \setminus \{e\}$, where σ' coincides with σ on $E \setminus \{e\}$.

Let $e = vw \in E$ such that e is not a loop and $\sigma_{v,e} \neq \sigma_{w,e}$. We say that (G', F') is obtained from (G, F) by *contracting edge e* if G' is the bidirected graph obtained by replacing the nodes v, w with one new node $r \notin V$, by deleting all the edges vw with $\sigma_{v,vw} \neq \sigma_{w,vw}$, by replacing each edge vw with $\sigma_{v,vw} = \sigma_{w,vw}$ with a loop in r with sign $\sigma_{v,vw}$, which we identify with the original edge in E , by replacing each edge $tv, t \neq w$ or $tw, t \neq v$, with an edge tr in $E(G')$, which we identify with the original edge in E , and by letting the signing in G' coincide with σ on $E(G')$. Let F' be the union of F and the set of the loops in r corresponding to edges vw in G with the same sign in their endnodes.

Notice that, if $e = vw \in E \setminus F$, then $A(G', F')$ is obtained by pivoting the entry (v, e) in $A(G, F)$, and by removing the row corresponding to v and the column corresponding to e . Moreover, if $e = vw \in F$ and v is incident only with edges in F , then $A(G', F')$ is obtained by dividing by 2 the row corresponding to v , then by pivoting the entry (v, e) in $A(G, F)$, and by removing the row corresponding to v and the column corresponding to e .

Furthermore, if (G, F) contains an odd cycle C , then the pair (G', F') obtained from (G, F) by contracting one by one the edges in $E(C)$, contains a new loop $l \in F'$ in the node obtained from the contraction of C .

Given a pair (G, F) , we call a pair (G', F') a *minor* of (G, F) if it is obtained by the latter through some of the following operations:

- (O1) switching signs on a node or on an edge of G ;
- (O2) deleting a node or an edge in (G, F) ;
- (O3) contracting an edge in $E(G) \setminus (F \cup L(G))$;
- (O4) contracting an edge vw in $F \setminus L(G)$ such that $\delta(v) \subseteq F$.

By the above discussion, the class of pairs (G, F) that satisfy the cycles condition and such that $A(G, F)$ has the Edmonds-Johnson property is closed under taking minors.

One can verify that if the pair (G, F) , which we name G_4 , is as in figure 4.2, then $A(G, F)$ does not have the Edmonds-Johnson property.

The following is the main result of this chapter.

Theorem 4.1. *Given a pair (G, F) that satisfies the cycles condition, $A(G, F)$ has the Edmonds-Johnson property if and only if (G, F) does not contain G_4 as a minor.*

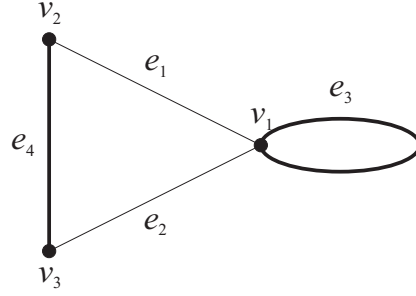


Figure 4.2: G_4 : the boldfaced edges represent the edges in F , and the signing σ of G is everywhere $+1$ except for $\sigma_{v_2, e_1} = -1$.

Notice that the following partial result was shown by Del Pia and Zambelli [12].

Theorem 4.2. *Given a pair (G, F) that satisfies the cycles condition and that does not contain G_3 as a minor, then $A(G, F)$ has the Edmonds-Johnson property.*

Where the pair (G, F) , which we name G_3 , is as in figure 4.3.

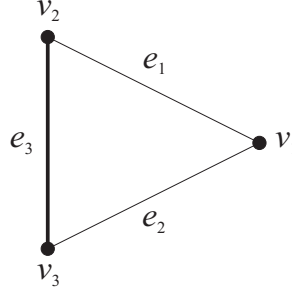


Figure 4.3: G_3 : the boldfaced edges represent the edges in F , and the signing σ of G is everywhere $+1$ except for $\sigma_{v_2, e_1} = -1$.

Notice that, in terms of bidirected graphs, Theorem 2.2 states that, given a bidirected graph G and a subset F of its loops, $A(G, F)$ has the Edmonds-Johnson property. Since for every bidirected graph G and for every subset F of its loops, (G, F) satisfies the cycles condition and does not contain G_4 as a minor, Theorem 4.1 reduces to Theorem 2.2 in this case.

Notice that Theorem 4.1 implies the following

Corollary 4.3. *Given a totally half-modular matrix A with at most two nonzero entries per column, such that all the nonzero elements in a column*

have the same absolute value, then A has the Edmonds-Johnson property if and only if it does not contain A_3 as a minor.

Proof. If A contains A_3 as a minor, then A does not have the Edmonds-Johnson property, because by Observation 2.11, A_3 does not have the Edmonds-Johnson property. Let (G, F) be a pair such that $A = A(G, F)$. If A does not contain A_3 as a minor, then (G, F) does not contain G_4 as a minor, since $A(G_4)$ contains A_3 as a minor (pivot on the +1 entry corresponding to the node v_1 and the edge e_1), and by Theorem 4.1, A has the Edmonds-Johnson property. \square

In what follows let \mathcal{C} be the family of pairs (G, F) , where G is a bidirected graph and F is a subset of its edges, such that (G, F) satisfies the cycles condition and does not contain G_4 as a minor.

Hence the matrices $A(G, F)$ with (G, F) in \mathcal{C} define another class of totally half-modular matrices with the Edmonds-Johnson property.

On the algorithmic side, we will show that, if $A(G, F)$ is a matrix in our class, one can minimize in polynomial time any linear function over the integer hull of $b \leq A(G, F)x \leq c$, $l \leq x \leq u$, for all integral vectors b, c, l, u . In contrast, Conforti et Al. [6] proved that deciding if a system of the form $A(G, F)x \geq b$ has an integral solution is \mathcal{NP} -complete even if $\Sigma(G)$ is totally unimodular. We recall that, by Theorem 1.13, $\Sigma(G)$ is totally unimodular if and only if G does not contain any odd cycle. Therefore, if $\Sigma(G)$ is totally unimodular, the cycles condition is always verified.

In the next section we show how we can reduce ourselves to study systems of the form $A(G, F)x = c$, $x \geq 0$; we describe the irredundant Chvátal inequalities for such systems and for the more general systems $A(G, F)x = c$, $0 \leq x \leq u$. Moreover we show how to separate such Chvátal inequalities in polynomial time. In Section 4.4 we finally prove Theorem 4.1.

4.2 First Chvátal closure

Let $G = (V, E, \sigma)$ be a bidirected graph and $F \subseteq E$. By definition $A(G, F)$ has the Edmonds-Johnson property if the system

$$\begin{aligned} b &\leq A(G, F)x \leq c \\ l &\leq x \leq u \end{aligned} \tag{4.1}$$

has Chvátal rank at most 1 for every $b, c \in \mathbb{Z}^V$, $l, u \in \mathbb{Z}^E$. Next we show that, in proving Theorem 4.1, we can reduce ourselves to study systems of the form

$$\begin{aligned} A(G, F)x &= c \\ x &\geq 0, \end{aligned} \tag{4.2}$$

where $c \in \mathbb{Z}^V$.

Lemma 4.4. *If (4.2) has Chvátal rank at most 1 for every (G, F) in \mathcal{C} and every integral c , then $A(G, F)$ has the Edmonds-Johnson property for every (G, F) in \mathcal{C} .*

Proof. At first we show that if (4.2) has Chvátal rank at most 1 for every (G, F) in \mathcal{C} and every integral c , then the system

$$\begin{aligned} A(G, F)x &= c \\ 0 \leq x \leq u \end{aligned} \tag{4.3}$$

has Chvátal rank at most 1 for every (G, F) in \mathcal{C} and every integral c, u .

Let (G, F) be a pair in \mathcal{C} , let c, u be integral vectors and let \bar{x} be a vector in the first closure of (4.3). Now let (G', F') , c' and \bar{x}' be obtained from (G, F) , c and \bar{x} in the following way. For each nonloop edge $e = v_1v_2$ in $E \setminus F$ (resp. in F), add a new node v_e and replace e with the path $v_1, v_1v_e, v_e, v_e v_2, v_2$ such that the edges v_1v_e and $v_e v_2$ are in $E(G') \setminus F'$ (resp. in F'), such that v_1v_e and $v_e v_2$ have a +1 sign in the vertex v_e , the edge v_1v_e has in v_1 the same sign that e had in v_1 , while the edge $v_e v_2$ has in v_2 the opposite sign that e had in v_2 , decrease c'_{v_2} by $\sigma_{v_2, e}u_e$ (resp. by $2\sigma_{v_2, e}u_e$), set $c'_{v_e} = u_e$ (resp. $c'_{v_e} = 2u_e$), and set $\bar{x}'_{v_1v_e} = \bar{x}_e$ and $\bar{x}'_{v_e v_2} = u_e - \bar{x}_e$. Similarly, for each loop e in $E \setminus F$ (resp. in F) incident with a node v , add a new node v_e and replace e with the edge vv_e and with a loop l_e in v_e such that the edges vv_e and l_e are in $E(G') \setminus F'$ (resp. in F'), such that vv_e and l_e have a +1 sign in the vertex v_e , the edge vv_e has in v the same sign that e had in v , set $c'_{v_e} = u_e$ (resp. $c'_{v_e} = 2u_e$), and set $\bar{x}'_{vv_e} = \bar{x}_e$ and $\bar{x}'_{l_e} = u_e - \bar{x}_e$. Notice that also (G', F') is in \mathcal{C} .

By Lemma 1.24 (i), \bar{x}' is in the first closure of $A(G', F')x' = c'$, $x' \geq 0$. If the latter system has Chvátal rank at most 1, then \bar{x}' is a convex combination of integral solutions. Hence \bar{x} can be expressed as a convex combination of integral vectors satisfying (4.3).

Now we show that the system

$$\begin{aligned} b \leq A(G, F)x &\leq c \\ 0 \leq x &\leq u \end{aligned} \tag{4.4}$$

has Chvátal rank at most 1 for every (G, F) in \mathcal{C} and every integral b, c, u .

Let (G, F) be a pair in \mathcal{C} , let b, c, u be integral vectors and let \bar{x} be a vector in the first closure of (4.4). Now let G' be obtained from G by adding a loop $e_v \notin E(G)$ on every node v of G . Notice that also (G', F) is in \mathcal{C} . Clearly $A(G', F) = (A \ \mathbb{I})$. Let $u' \in \mathbb{Z}^{E(G')}$ be obtained from u by setting $u'_{e_v} = c_v - b_v$ for every new loop e_v .

By Lemma 1.24 (i), $\bar{x}' = \begin{pmatrix} \bar{x} \\ c - A(G, F)\bar{x} \end{pmatrix}$ is in the first closure of the system $A(G', F)x' = c$, $0 \leq x' \leq u'$. If the latter system has Chátal rank at most 1, then \bar{x}' is a convex combination of integral solutions. Hence \bar{x} is a convex combination of integral vectors satisfying (4.4).

Finally we show that $A(G, F)$ has the Edmonds-Johnson property for every (G, F) in \mathcal{C} .

Let b, c, l, u be integral vectors and let \bar{x} be a vector in the first closure of (4.2). Let $b' = b - Al$, $c' = c - Al$, $u' = u - l$, $\bar{x}' = \bar{x} - l$. It is easy to see that \bar{x}' is in the first closure of $b' \leq A(G, F)x' \leq c'$, $0 \leq x' \leq u'$. If the latter system has Chátal rank at most 1, then \bar{x}' is a convex combination of integral solutions, i.e. $\bar{x}' = \sum_{i \in I} \lambda_i \tilde{y}^i$, $0 \leq \lambda_i \leq 1$, $\sum_{i \in I} \lambda_i = 1$. Hence $\bar{x} = \sum_{i \in I} \lambda_i y^i$, where $y^i = \tilde{y}^i + l$ is a convex combination of integral vectors satisfying (4.1). \square

To describe the Chvátal inequalities for our systems, we need total half-modularity. Thus at first we prove the following lemma.

Lemma 4.5. *$A(G, F)$ is totally half-modular if and only if (G, F) satisfies the cycles condition.*

Proof. Assume (G, F) satisfies the cycles condition, and let A' be a nonsingular square submatrix of $A(G, F)$. Let v_1, \dots, v_k be the nodes of G that correspond to the rows of A' and let e_1, \dots, e_k be the edges of G that correspond to the columns of A' . Let G' be the bidirected graph whose nodes are v_1, \dots, v_k and whose edges are e_1, \dots, e_k , where an edge e_i that has only one endnode among v_1, \dots, v_k is a loop of G' . Then A' is obtained from the sign matrix $\Sigma(G')$ of G' by multiplying by 2 some of its columns. If G' is not connected, then A' is a block diagonal matrix and so we can look at each block. Hence, by symmetry, we may assume that e_1, \dots, e_{k-1} induce a spanning tree of G' . If e_k is a loop of G' , then $\Sigma(G')$ is totally unimodular, thus A'^{-1} is half-integral because it is obtained from $\Sigma(G')^{-1}$ by dividing some rows by 2. Thus e_k has two distinct endnodes and it is contained in the unique cycle C of G' . One can readily verify that, if C is even, then $\Sigma(G')$

is singular, and so is A' . Therefore C is odd. Up to permuting rows and columns, we may assume that $\Sigma(G') = \begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}$, where P is the incidence matrix of the cycle C . It is easy to check that $\det(P) = \pm 2$, while R is totally unimodular, therefore P^{-1} is half-integral while R^{-1} is integral. One can verify that $\Sigma(G')^{-1} = \begin{pmatrix} P^{-1} & -P^{-1}QR^{-1} \\ 0 & R^{-1} \end{pmatrix}$. Notice that the first $|C|$ rows of $\Sigma(G')^{-1}$ are half-integral, while the other rows are integral. The matrix A'^{-1} is obtained from $\Sigma(G')^{-1}$ by dividing by 2 the rows corresponding to the edges in F . Since (G, F) satisfies the cycles condition, $C \cap F = \emptyset$, therefore A'^{-1} is obtained from $\Sigma(G')^{-1}$ by dividing by 2 some of the last $k - |C|$ rows. Hence A'^{-1} is half-integral.

Viceversa, if there exists an odd cycle C of G such that $C \cap F \neq \emptyset$, then $A(C, F \cap C)^{-1}$ is obtained by dividing by 2 some of the rows of $\Sigma(C)^{-1}$. Since C is odd, all the nonzero entries of $\Sigma^{-1}(C)$ have value $\pm 1/2$, hence $A(C, F \cap C)^{-1}$ is not half-integral. \square

At some point in our proof of Theorem 4.1 it will be convenient to introduce some upper bounds on the system (4.2). Hence in the following Lemma we describe the Chvátal inequalities for these more general systems.

Lemma 4.6. *Let (G, F) be a pair satisfying the cycles condition. Let $\alpha x \geq \beta$ be an irredundant nontrivial Chvátal inequality for*

$$\begin{aligned} A(G, F)x &= c \\ 0 &\leq x \leq u. \end{aligned} \tag{4.5}$$

Then there exists a connected set $U \subseteq V(G)$, and a partition E^0, E^u of the edges in $\delta(U) \setminus F$, such that $c(U) + u(E^u)$ is odd and $\alpha x \geq \beta$ is equivalent to

$$x(E^0) - x(E^u) \geq -u(E^u) + 1. \tag{4.6}$$

Furthermore, there is no nontrivial partition U_1, U_2 of U such that all the edges between U_1 and U_2 are in F .

Proof. Let $A = A(G, F)$. By Lemma 2.8 and Lemma 2.10, $\alpha x \geq \beta$ is equivalent to an inequality of the form $(\mu A + \gamma^0 - \gamma^u)x \geq \lceil \mu c - \gamma^u u \rceil$, where $\mu \in \{0, \pm \frac{1}{2}\}^{V(G)}$, $\gamma^0, \gamma^u \in \{0, \frac{1}{2}\}^{E(G)}$, $\mu A + \gamma^0 - \gamma^u$ is integral, and $\mu c - \gamma^u u$ is not integral. Let U be the set of nodes corresponding to non-zero entries of μ . Notice that all entries of μA are integer, except for the entries corresponding to edges in $\delta(U) \setminus F$, which have value $\pm \frac{1}{2}$. Hence there exists a partition E^u, E^0 of $\delta(U) \setminus F$ such that $\gamma_e^0 = \frac{1}{2}$ if $e \in E^0$, 0 otherwise, and $\gamma_e^u = \frac{1}{2}$ if $e \in E^u$, 0 otherwise. Since $c(U) + u(E^0)$ is odd, then

$\lceil \mu c - \gamma^u u \rceil = \mu c - \gamma^u u + \frac{1}{2}$. Since $\mu Ax = \mu c$ for every x that satisfies (4.5), $\alpha x \geq \beta$ is equivalent to $(\gamma^0 - \gamma^u)x \geq -\gamma^u u + \frac{1}{2}$. Multiplying the latter by 2, one obtains (4.6).

Finally, if there is a nontrivial partition U_1, U_2 of U such that all the edges between U_1 and U_2 are in F , let $E_h^0 = E^0 \cap \delta(U_h)$, $E_h^u = E^u \cap \delta(U_h)$, $h = 1, 2$. Notice that $\delta(U_h) \setminus F = E_h^0 \cup E_h^u$ and that $(\delta(U_1) \setminus F) \cap (\delta(U_2) \setminus F) = \emptyset$. Also, since $c(U) + u(E^u)$ is odd, by symmetry we may assume $c(U_1) + u(E_1^u)$ is odd and $c(U_2) + u(E_2^u)$ is even. Hence $x(E_1^0) - x(E_1^u) \geq -u(E_1^u) + 1$ is a Chvátal inequality, while $x(E_2^0) - x(E_2^u) \geq -u(E_2^u)$ is implied by (4.2). The sum of the two latter inequalities is precisely (4.6), contradicting the assumption that $\alpha x \geq \beta$ is irredundant. \square

We will refer to inequalities of the form (4.6) as *odd-cut inequalities (relative to U , E^0 , E^u)*. Notice that, when G is an undirected simple graph, $F = \emptyset$, c is the vector of all 1s, while u is the vector with all entries equal to $+\infty$, the odd-cut inequalities reduce to the well known ones for the perfect matching polytope.

Remark 4.7. An odd-cut inequality relative to U, E^0, E^u is satisfied by $\bar{x} \in \mathbb{R}^{E(G)}$ satisfying (4.5) if and only if $\sum_{e \in E^0} \bar{x}_e + \sum_{e \in E^u} (u_e - \bar{x}_e) \geq 1$.

Lemma 4.8. Let (G, F) be a pair satisfying the cycles condition, and $A = A(G, F)$. The first closure of $\{x : Ax = c, 0 \leq x \leq u\}$ is the intersection of the first closure of $\{x : Ax = c, 0 \leq x_e \leq u_e, \forall e \in E(G) \setminus F\}$ and the set $\{x : 0 \leq x_f \leq u_f, \forall f \in F\}$.

Proof. The statement follows from Lemma 4.6, and the fact that the odd-cut inequalities for $\{x : Ax = c, 0 \leq x \leq u\}$ are precisely the odd-cut inequalities for $\{x : Ax = c, 0 \leq x_e \leq u_e, \forall e \in E(G) \setminus F\}$. \square

Most of the times, in the proof of Theorem 4.1, we will be considering systems in the form (4.2). Hence in the following Lemma we describe the Chvátal inequalities for these simpler systems.

Lemma 4.9. Let (G, F) be a pair satisfying the cycles condition. Let $\alpha x \geq \beta$ be an irredundant nontrivial Chvátal inequality for (4.2). Then there exists a connected set $U \subseteq V(G)$, such that $c(U)$ is odd and $\alpha x \geq \beta$ is equivalent to

$$x(\delta(U) \setminus F) \geq 1. \quad (4.7)$$

Furthermore, there is no nontrivial partition U_1, U_2 of U such that all the edges between U_1 and U_2 are in F .

Proof. Directly from Lemma 4.6, by taking $u = +\infty$. \square

The following lemma will be useful in the proof of Theorem 4.1.

Lemma 4.10. *Let G be a bidirected graph, let $F \subseteq E(G)$, and let $I \subseteq F$. If the system $A(G, F)x = c, x \geq 0$ has Chvátal rank at most 1 for every integral c , then the system $A(G, F)x = c, x \geq 0, x_f \leq 1, f \in I$ has Chvátal rank at most 1 for every integral c .*

Proof. By contradiction assume that \bar{x} is a fractional vertex of the first closure of $A(G, F)x = c, x \geq 0, x_f \leq 1, f \in I$. Let $\bar{x}'_e = \bar{x}_e$ for every $e \notin I$, $\bar{x}'_f = \bar{x}_f - \lfloor \bar{x}_f \rfloor$ for every $f \in I$. Moreover let $c'_v = c_v - 2 \sum_{f \in \delta(v) \cap I} \sigma_{v,f} \lfloor \bar{x}_f \rfloor$. Clearly \bar{x}' satisfies $A(G, F)x = c', x \geq 0$. Notice that $c'_v \equiv_2 c_v$ for every $v \in V(G)$. Hence for every $U \subseteq V(G)$, $c'(U) \equiv_2 c(U)$. Thus the odd-cut inequalities for $A(G, F)x = c', x \geq 0$ and for $A(G, F)x = c, x \geq 0, x_f \leq 1, f \in I$ are the same, and since $\bar{x}'_e = \bar{x}_e$ for every $e \in E(G) \setminus F$, \bar{x}'_e is a vertex of the first closure of $A(G, F)x = c', x \geq 0$. A contradiction, as it is fractional. \square

4.2.1 Algorithmic aspects

We consider the separation problem for the odd-cut inequalities. Consider a pair (G, F) , where $G = (V, E, \sigma)$ is a bidirected graph and $F \subseteq E$. Let $c \in \mathbb{Z}^V$, and $\bar{x} \in R^E$ be a vector satisfying $A(G, F)x = c, x \geq 0$. Clearly the odd-cut inequalities relative to such system are valid for its first closure. We want to determine whether there exists an odd-cut inequality violated by \bar{x} .

Let $G' = (V, E', \sigma')$ be the bidirected graph obtained from G by deleting all edges in F , and by appending a loop f_v on $v \in V$ for each edge $f \in F$ incident with v , with sign $\sigma'_{v,f_v} = \sigma_{v,f}$. We let F' be the set of such loops, and $\sigma'_{v,e} = \sigma_{v,e}$ for each $e \in E' \setminus F'$.

Let $\bar{x}' \in R^{E'}$ be obtained from \bar{x} by removing the components relative to edges in F and by adding a component for each loop $f_v \in F'$ with value $\bar{x}'_{f_v} = \bar{x}_f$. Notice that \bar{x}' satisfies

$$\begin{aligned} A(G', F')x' &= c \\ x' &\geq 0, \end{aligned} \tag{4.8}$$

and the odd-cut inequalities for (4.8) are precisely the odd-cut inequalities for (4.2) since, by definition, the odd-cut inequalities are independent on the edges in F . Thus \bar{x}' violates an odd-cut inequality for (4.8) if and only if \bar{x} violates the same odd-cut inequality for (4.2). Thus we only need to determine if there exists an odd-cut inequality for (4.8) violated by \bar{x}' . Notice

that, by construction, the sum of the absolute values of the entries of each column of $A(G', F')$ is at most 2, hence by Theorem 2.2, $A(G', F')$ has the Edmonds-Johnson property, even when $A(G, F)$ does not. Therefore the odd-cut inequalities are sufficient to define the convex hull of integral solutions of (4.8). Edmonds and Johnson [16] show that the problem of finding an integral solution of (4.8) minimizing a given linear function can be solved in polynomial time, therefore also the separation problem over the convex hull of integral solutions of (4.8) can be solved in polynomial time with the ellipsoid method [23]. This shows the following.

Theorem 4.11. *There is a polynomial time algorithm that, given $\bar{x} \in \mathbb{R}^E$ satisfying $A(G, F)x = c$, $x \geq 0$, returns either an odd-cut inequality violated by \bar{x} , or determines that none exists.*

Corollary 4.12. *Let (G, F) satisfy the cycles condition. There is a polynomial time algorithm that, for any $\alpha \in \mathbb{R}^E$, finds a vector x^* in the first Chvátal closure of $A(G, F)x = c$, $x \geq 0$ minimizing αx .*

Proof. By Lemma 4.9, the only nontrivial inequalities valid for the first Chvátal closure of $A(G, F)x = c$, $x \geq 0$ are the odd-cut inequalities. Since these can be separated in polynomial time, with the ellipsoid method [23] we can solve the minimization problem in polynomial time. \square

Corollary 4.13. *Let (G, F) be in \mathcal{C} . There is a polynomial time algorithm that, for any $\alpha \in \mathbb{R}^E$, finds a integral vector x^* satisfying $A(G, F)x = c$, $x \geq 0$ minimizing αx .*

Proof. Follows immediately from Theorem 4.1 and Corollary 4.12. \square

4.3 Balanced bipartitions

Let G be a bidirected graph and $F \subseteq E(G)$. Given two disjoint sets R, B of the edges in $E(G)$, we say that R, B is a *balanced bipartition* of the edges in $R \cup B$, if for every $v \in V(G)$,

$$\frac{1}{2} \sum_{\substack{vw \in E(G) \setminus F \\ vw \in R}} \sigma_{v,vw} + \sum_{\substack{vw \in F \\ vw \in R}} \sigma_{v,vw} = \frac{1}{2} \sum_{\substack{vw \in E(G) \setminus F \\ vw \in B}} \sigma_{v,vw} + \sum_{\substack{vw \in F \\ vw \in B}} \sigma_{v,vw}. \quad (4.9)$$

In the proof of Theorem 4.1, it will be useful at times to determine balanced bipartitions.

Remark 4.14. *If there exists a balanced bipartition of the edges of $E(G)$, then*

- a) $|\delta_G(v) \setminus F|$ is even for every $v \in V$;
- b) for every component \bar{G} of $G \setminus F$ such that $L(\bar{G}) = \emptyset$, $|\delta_G(V(\bar{G}))|$ is congruent modulo 2 to the number of odd edges in $E(\bar{G})$.

Proof. Condition a) follows immediately from the fact that, given a balanced bipartition R, B ,

$$\frac{1}{2} \left(\sum_{\substack{vw \in E(G) \setminus F \\ vw \in R}} \sigma_{v,vw} - \sum_{\substack{vw \in E(G) \setminus F \\ vw \in B}} \sigma_{v,vw} \right)$$

is an integer.

To prove condition b), let \bar{G} be a connected component of $G \setminus F$ such that $L(\bar{G}) = \emptyset$. Summing equations 4.9 for all $v \in V(\bar{G})$, we obtain

$$\begin{aligned} & \sum_{vw \in R \cap E(\bar{G})} \frac{(\sigma_{v,vw} + \sigma_{w,vw})}{2} - \sum_{vw \in B \cap E(\bar{G})} \frac{(\sigma_{v,vw} + \sigma_{w,vw})}{2} = \\ & - \sum_{\substack{vw \in F \cap R \setminus L(G) \\ v, w \in V(\bar{G})}} (\sigma_{v,vw} + \sigma_{w,vw}) + \sum_{\substack{vw \in F \cap B \setminus L(G) \\ v, w \in V(\bar{G})}} (\sigma_{v,vw} + \sigma_{w,vw}) + \\ & - \sum_{\substack{vw \in \delta_G(V(\bar{G})) \cap R \\ v \in V(\bar{G})}} \sigma_{v,vw} + \sum_{\substack{vw \in \delta_G(V(\bar{G})) \cap B \\ v \in V(\bar{G})}} \sigma_{v,vw}. \end{aligned}$$

Hence the number of odd edges in $E(\bar{G})$ is congruent modulo 2 to $|\delta_G(V(\bar{G}))|$, since $\sigma_{v,vw} + \sigma_{w,vw} \equiv_2 0$ for every $vw \in F \setminus L(G)$ with $v, w \in V(\bar{G})$. \square

Let G be a bidirected graph and $F \subseteq E(G)$ such that $(G, F) \in \mathcal{C}$ and (G, F) satisfies conditions a) and b) of Remark 4.14.

Suppose that G has two distinct blocks B_1, B_2 such that, for $i = 1, 2$, either B_i contains an odd cycle or $E(B_i) \cap F \neq \emptyset$. Let w be a cutnode of G separating B_1 and B_2 . Choose $V_1, V_2 \subset V$ such that $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 = \{w\}$, $V(B_1) \subseteq V_1$, $V(B_2) \subseteq V_2$, the graphs induced by V_1 and V_2 are connected, there is no edge between $V_1 \setminus \{w\}$ and $V_2 \setminus \{w\}$. Partition the edges of $E(G)$ into two sets E_1, E_2 so that, for every $e \in E_i$, all endnodes of e are in V_i , $i = 1, 2$ (notice that such a partition is not uniquely defined, since a loop on w can be put arbitrarily in E_1 or E_2). Let $G_i = (V_i, E_i)$ and $F_i = E_i \cap F$, $i = 1, 2$.

Notice that, by condition a) of Remark 4.14, then either $|\delta_{G_1}(w) \setminus F_1|$ and $|\delta_{G_2}(w) \setminus F_2|$ are both odd, or they are both even. We define $(\tilde{G}_1, \tilde{F}_1)$ and $(\tilde{G}_2, \tilde{F}_2)$ as follows:

1. If $|\delta_{G_1}(w) \setminus F_1|$ and $|\delta_{G_2}(w) \setminus F_2|$ are odd, we define \tilde{G}_i the graph obtained from G_i by adding a loop ℓ_i on w with sign $+1$, and $\tilde{F}_i = F_i$, $i = 1, 2$.
2. If $|\delta_{G_1}(w) \setminus F_1|$ and $|\delta_{G_2}(w) \setminus F_2|$ are both even, and one among (G_1, F_1) and (G_2, F_2) does not satisfy condition b) of Remark 4.14, we define \tilde{G}_i the graph obtained from G_i by adding a loop ℓ_i on w with sign $+1$, and $\tilde{F}_i = F_i \cup \{\ell_i\}$, $i = 1, 2$.
3. If both (G_1, F_1) and (G_2, F_2) satisfy conditions a) and b) of Remark 4.14, let $(\tilde{G}_i, \tilde{F}_i) = (G_i, F_i)$, $i = 1, 2$.

We say that $(\tilde{G}_1, \tilde{F}_1)$ and $(\tilde{G}_2, \tilde{F}_2)$ are obtained from (G, F) by breaking B_1 and B_2 at w .

Lemma 4.15. *Let G be a bidirected graph and $F \subseteq E(G)$ such that $(G, F) \in \mathcal{C}$ and (G, F) satisfies conditions a) and b) of Remark 4.14. If $(\tilde{G}_1, \tilde{F}_1)$ and $(\tilde{G}_2, \tilde{F}_2)$ are obtained from $(G, F) \in \mathcal{C}$ by breaking B_1 and B_2 at w , then $(\tilde{G}_1, \tilde{F}_1)$ and $(\tilde{G}_2, \tilde{F}_2)$ are in \mathcal{C} , and they satisfy conditions a) and b) of Remark 4.14.*

Furthermore, if both $(\tilde{G}_1, \tilde{F}_1)$ and $(\tilde{G}_2, \tilde{F}_2)$ have a balanced bipartition, then also (G, F) has a balanced bipartition.

Proof. We first show that $(\tilde{G}_1, \tilde{F}_1)$ and $(\tilde{G}_2, \tilde{F}_2)$ satisfy conditions a) and b) of Remark 4.14. It is immediate to see that $(\tilde{G}_1, \tilde{F}_1)$ and $(\tilde{G}_2, \tilde{F}_2)$ satisfy condition a), and they trivially satisfy condition b) if case 1) or 3) applies. Assume case 2) applies, and that (G_1, F_1) violates condition b). Let \bar{G}_1 be a connected component of $G_1 \setminus F$ such that $L(\bar{G}_1) = \emptyset$ and the number of odd edges of $E(\bar{G}_1)$ is not congruent modulo 2 to $|\delta_{G_1}(V(\bar{G}_1))|$. Notice that any connected component of $G_1 \setminus F_1$ or $G_2 \setminus F_2$ that does not contain node w is also a connected component of $G \setminus F$. Hence $w \in V(\bar{G}_1)$, so the number of odd edges of $E(\bar{G}_1)$ is congruent modulo 2 to $|\delta_{\tilde{G}_1}(V(\bar{G}_1))|$, therefore $(\tilde{G}_1, \tilde{F}_1)$ satisfies condition b). If $(\tilde{G}_2, \tilde{F}_2)$ violates condition b), then the connected component \bar{G}_2 of $G_2 \setminus F$ that contains w satisfies $L(\bar{G}_2) = \emptyset$ and the number of odd edges of $E(\bar{G}_2)$ is not congruent modulo 2 to $|\delta_{\tilde{G}_2}(V(\bar{G}_2))|$, therefore the number of odd edges of $E(\bar{G}_2)$ is congruent modulo 2 to $|\delta_{G_2}(V(\bar{G}_2))|$. Thus the connected component \bar{G} of $G \setminus F$ containing w violates condition b).

We show that $(\tilde{G}_i, \tilde{F}_i) \in \mathcal{C}$, $i = 1, 2$. This is obvious if case 1) or 3) above apply, since (G_i, F_i) is a minor of (G, F) , so $(G_i, F_i) \in \mathcal{C}$. Since, in case 1), $(\tilde{G}_i, \tilde{F}_i)$ is obtained from (G_i, F_i) by adding a loop ℓ_i not in F_i , then also $(\tilde{G}_i, \tilde{F}_i) \in \mathcal{C}$. If case 2) applied, by symmetry it is sufficient to show that $(\tilde{G}_1, \tilde{F}_1)$ is a minor of (G, F) . Indeed, consider the pair (G', F') obtained

from (G, F) by contracting all the edges in $E_2 \setminus F$. Notice that $F' \setminus E_1$ is nonempty, since either $E(B_2) \cap F \neq \emptyset$, or B_2 contains an odd cycle C , and when we contract all edges in C we obtain a new loop in F' . Thus there exists an edge $f \in F' \setminus E_1$ with one endnode in w , say $f = vw$. Deleting from (G', F') all edges in $F' \setminus E_1$ except f , and deleting all nodes in $V(G') \setminus V_1$, we obtain $(\tilde{G}_1, \tilde{F}_1)$.

We finally show that, if both $(\tilde{G}_1, \tilde{F}_1)$ and $(\tilde{G}_2, \tilde{F}_2)$ have a balanced bipartition, then also (G, F) has a balanced bipartition. Let B_i, R_i be a balanced bipartition of $(\tilde{G}_i, \tilde{F}_i)$, for $i = 1, 2$. If case 3) above applies, then $B_1 \cup B_2, R_1 \cup R_2$ is a balanced bipartition of (G, F) . If cases 1) or 2) apply, assuming that $l_i \in B_i$, $i = 1, 2$, then $(B_1 \cup R_2) \setminus \{l_1\}, (R_1 \cup B_2) \setminus \{l_2\}$ is a balanced bipartition of (G, F) . \square

Next we give a few technical lemmas that will be used in the proof of Theorem 4.1 to show that certain pairs $(G, F) \in \mathcal{C}$ have balanced bipartitions.

Lemma 4.16. *Let (G, F) be a pair in \mathcal{C} such that $G \setminus F$ is connected. Assume that there exists a family \mathcal{T} of trails such that:*

- (C1) *each edge in $E(G) \setminus (F \setminus L(G))$ is contained in exactly one trail in \mathcal{T} and no edge in $F \setminus L(G)$ is contained in any trail in \mathcal{T} ;*
- (C2) *if a trail T in \mathcal{T} contains edges in $L(G) \setminus F$, then it contains exactly two of them and they are its first and last edge. If a trail T in \mathcal{T} does not contain edges in $L(G) \setminus F$, then it is closed and $|E(T) \cap F| \equiv_2 |\{vw \in E(T) \setminus F : \sigma_{v,vw} = \sigma_{w,vw}\}|$;*
- (C3) *for every $f \in F \setminus L(G)$, there exists a trail in \mathcal{T} that contains both endnodes of f .*

Then there exists a balanced bipartition of the edges in $E(G)$.

Proof. We give a construction for the bipartition. At first we give a balanced bipartition of the edges in $E(G) \setminus (F \setminus L(G))$, and later we use it to obtain a balanced bipartition of the edges in $E(G)$. We give a bipartition of the edges of $E(G) \setminus (F \setminus L(G))$ in the following way: For every trail $T = v_1, e_1, \dots, e_{k-1}, v_k$ in \mathcal{T} , e_j and e_{j-1} are in the same subset if and only if $\sigma_{v_j, e_j} \neq \sigma_{v_j, e_{j-1}}$, for every $j = 2, \dots, k-1$. Notice that this is a balanced bipartition of the edges in $E(G) \setminus (F \setminus L(G))$.

For every edge $f \in F \setminus L(G)$, we define a subtrail $T(f)$ of a trail in \mathcal{T} recursively in the following way. Among all the edges $f \in F \setminus L(G)$ for which

we have not yet defined $T(f)$, let $g = vw$ be the one such that there exists a subtrail of a trail in \mathcal{T} , from v to w , of minimal length. Let $T(g)$ be such minimal length subtrail. Notice that, by construction, for every edge $f \in F \setminus L(G)$, $T(f)$ is a subtrail of a trail in \mathcal{T} , the endnodes of $T(f)$ are the two endnodes of f , and all the interior nodes of $T(f)$ are different from the endnodes of f .

Now we show that, for every pair f, g of edges in $F \setminus L(G)$, either $T(f)$ is a subtrail of $T(g)$, or $T(g)$ is a subtrail of $T(f)$, or $E(T(f)) \cap E(T(g)) = \emptyset$.

By contradiction, let $f, g \in F \setminus L(G)$, $f \neq g$ be such that $E(T(f)) \cap E(T(g)) \neq \emptyset$, $T(f)$ is not a subtrail of $T(g)$, and $T(g)$ is not a subtrail of $T(f)$. Therefore, an internal node of $T(f)$ is an endnode of g , and an internal node of $T(g)$ is an endnode of f . By symmetry we can assume that $T(f)$ was defined before $T(g)$, thus the length of $T(f)$ is less than or equal to the length of $T(g)$. It follows that it is not possible that both the endnodes of g appear in $T(f)$, otherwise we would have defined $T(g)$ before $T(f)$. Thus, by considering the closed trail $T(f), f$, and by deleting the endnode of g that does not appear in $T(f)$, we get G_4 as a minor, a contradiction.

Now notice that, for every $f = vw \in F \setminus L(G)$, all the cycles contained in the closed trail $f, T(f)$ are even. Otherwise suppose that at least one cycle C contained in the closed trail $f, T(f)$ is not even. Notice that C cannot contain f , since (G, F) satisfies the cycles condition. Hence by contracting all the edges in C we get a new loop l in F . By using a cycle in $f, T(f)$ containing f , and a path in $T(f)$ from such cycle to l we get G_4 as a minor.

For the same reason, for every $f = vw$ in $F \setminus L(G)$, $T(f)$ does not contain loops in $F \cap L(G)$. Therefore, by construction, the edge e_v in $T(f)$ incident with v and the edge e_w in $T(f)$ incident with w are in the same subset if and only if $\sum_{\substack{e \in E(T(f)) \\ r \in e, r \neq v, w}} \sigma_{r,e} \equiv_4 0$. Since all the cycles contained in $f, T(f)$ are even, and since in $T(f)$ there are no loops in F , e_v and e_w are in the same subset if and only if $\sigma_{v,e_v} + \sigma_{v,f} + \sigma_{w,f} + \sigma_{w,e_w} \equiv_4 0$.

We say that an edge f in $F \setminus L(G)$ is *maximal* in $F \setminus L(G)$ if for every other edge g in $F \setminus L(G)$, either $T(g)$ is a subtrail of $T(f)$, or $E(T(f)) \cap E(T(g)) = \emptyset$.

Now we extend our bipartition of $E(G) \setminus (F \setminus L(G))$ to a bipartition of $E(G)$ by doing the following recursively for every f maximal among the edges in $F \setminus L(G)$ not in our bipartition. Let v be an endnode of f and let e be the only edge in $T(f)$ incident with v . Put f in the same side of the bipartition of e if and only if $\sigma_{v,f} = \sigma_{v,e}$, and switch the current bipartition of every edge in $T(f)$. By construction, the bipartition defined above is balanced. \square

Remark 4.17. Let $(G, F) \in \mathcal{C}$ such that $G \setminus F$ is connected, and let $\Delta \subseteq F \cap L(G)$. Assume that there exists a family \mathcal{T} of trails that satisfy conditions

(C1)-(C3) and such that $\Delta \subseteq \bar{T}$ for some $\bar{T} \in \mathcal{T}$. Let

$$T = l_1, T_1, l_2, \dots, T_{k-1}, l_k$$

be the subtrail of \bar{T} such that $\Delta = \{l_1, \dots, l_k\}$. Furthermore, assume that $\Delta = E(T) \cap F$, and that $(V(T), E(T))$ is bipartite.

Then the balanced bipartition given in the proof of Lemma 4.16 has the property that, for every $i = 1, \dots, k-1$, l_i and l_{i+1} are in the same side of the partition if and only if $\sum_{e \in \{l_i, E(T_i), l_{i+1}\}} \sum_{v \in e} \sigma_{v,e} \equiv_4 0$.

Lemma 4.18. *Let G be a bipartite bidirected graph and let $F \subseteq E(G)$ such that $G \setminus F$ is connected. If (G, F) satisfies the following*

- (i) F is a star;
- (ii) $|\delta_G(v) \setminus F|$ is even for every $v \in V(G)$;
- (iii) if $L(G) \subseteq F$, then $|L(G)|$ is even;

then there exists a balanced bipartition of $E(G)$.

Proof. Notice that, since F is a star and G is bipartite, then (G, F) does not contain the G_4 minor. Thus $(G, F) \in \mathcal{C}$.

If $L(G) \subseteq F$, then let T be an Eulerian circuit on the edges of $E(G) \setminus (F \setminus L(G))$. We want to prove that the family $\mathcal{T} = \{T\}$ satisfies conditions (C1)-(C3) of Lemma 4.16. It is immediate to see that (C1) and (C3) hold. We show that (C2) holds. By hypothesis (iii) in the statement, $|L(G)|$ is even, thus we need to show that $|\{vw \in E(G) \setminus F : \sigma_{v,vw} = \sigma_{w,vw}\}|$ is even. Since $|\delta_G(v) \setminus F|$ is even for every $v \in V(G)$, then $E(G) \setminus F$ can be partitioned into cycles, and each cycle in G is even, because G is bipartite, thus $|\{vw \in E(G) \setminus F : \sigma_{v,vw} = \sigma_{w,vw}\}|$ is even. Hence, by Lemma 4.16 there exists a balanced bipartition R, B of the edges in $E(G)$.

Otherwise $|L(G) \setminus F| \geq 1$, and since $|\delta_G(v) \setminus F|$ is even for every $v \in V(G)$, $|L(G) \setminus F|$ is even, hence $|L(G) \setminus F| \geq 2$. Let l_1 and l_2 be two loops in $L(G) \setminus F$. By Theorem 1.13, since G is bipartite, $V(G)$ can be partitioned into two sets V^1 and V^2 so that, for every edge $e = vw \in E(G) \setminus L(G)$, v, w are in distinct sets if and only if $\sigma_{v,e} = \sigma_{w,e}$. We define a new graph \tilde{G} obtained from G by adding a new vertex r and by replacing every loop l_v in $L(G) \setminus F$ incident with a node v and different from l_1 and l_2 , with an edge vr with $\sigma_{v,vr} = \sigma_{v,l_v}$, and with sign $\sigma_{r,vr} = -\sigma_{v,l_v}$ if $v \in V^1$, $\sigma_{r,vr} = \sigma_{v,l_v}$ if $v \in V^2$. Notice now that, by construction, for every edge $e = vw \in E(\tilde{G}) \setminus L(\tilde{G})$, v, w are not both in $V^1 \cup \{r\}$ or not both in V^2 if and only if $\sigma_{v,e} = \sigma_{w,e}$. By

Theorem 1.13, \tilde{G} is bipartite. As before, since \tilde{G} is bipartite and F is a star, $(\tilde{G}, F) \in \mathcal{C}$.

Notice that, if we identify every edge of \tilde{G} with its corresponding edge in G , a balanced bipartition of the edges of \tilde{G} defines a balanced bipartition of the edges in G , hence we only need to show that (\tilde{G}, F) has a balanced bipartition. Now let T be an Eulerian circuit on the edges of $E(\tilde{G}) \setminus (F \setminus L(\tilde{G}))$ such that its first edge is l_1 and its last edge is l_2 . Notice that $\mathcal{T} = \{T\}$ satisfies conditions (C1)-(C3) of Lemma 4.16 with respect to $\tilde{G}, F, L(\tilde{G})$. Hence, by Lemma 4.16 there exists a balanced bipartition of the edges in $E(\tilde{G})$. \square

Lemma 4.19. *Let $(G, F) \in \mathcal{C}$ such that G is bipartite and $G \setminus F$ is connected. Assume (G, F) satisfies conditions a) and b) of Remark 4.14 and that there exists a path P in $G \setminus F$ that passes through all the nodes of G incident with some edge in F . Then there exists a partition of the edges in $E(G) \setminus F$ in one closed trail, if $L(G) \subseteq F$, or, if $L(G) \setminus F \neq \emptyset$, in $|L(G) \setminus F|/2$ trails such that their first and last edge are in $L(G) \setminus F$, such that one of the trails passes through all the nodes incident with edges in F .*

Proof. Let P be a path in $E(G) \setminus F$ that passes through all the nodes incident with edges in F , and let v and w be its endnodes. Let G' be obtained from G by removing the edges in $E(P)$ and by adding one loop l_v in v , and one loop l_w in w . Notice that $G' \setminus F$ may be not connected. Now partition the edges in $E(G') \setminus F$ into $|L(G') \setminus F|/2$ trails starting and ending in loops in $E(G')$ and in closed trails. If the loops l_v and l_w are contained in different trails, say T_1 and T_2 , then we get from T_1, P, T_2 one new trail starting and ending in loops in $E(G) \setminus F$ different from the artificial l_v, l_w . Otherwise the loops l_v and l_w are contained in the same trail. Hence there exists a partition of the edges in $E(G) \setminus F$ in $|L(G) \setminus F|/2$ trails starting and ending in loops in $E(G) \setminus F$ and in closed trails, where one of these closed trail passes through all the nodes incident with edges in F . In both cases, since $G \setminus F$ is connected, we can always get the partition of the edges in $E(G) \setminus F$ required by recursively combining together the closed trails with another trail of the partition. \square

4.4 Proof of Theorem 4.1

Claim 4.1. *$A(G_4)$ does not have the Edmonds-Johnson property.*

Proof of claim. Notice that the matrix A_3 can be obtained from $A(G_4)$ by pivoting on the +1 entry corresponding to the node v_1 and the edge e_1 , and

then by removing the column corresponding to the edge e_1 . Since A_3 does not have the Edmonds-Johnson property, then by Theorem 2.1, $A(G_4)$ does not have the Edmonds-Johnson property. \diamond

In the remainder of the section we will prove that for any (G, F) in \mathcal{C} , the system (4.2) has Chátal rank at most 1 for any integral vector c . By Lemma 4.4, this will imply Theorem 4.1.

By contradiction, suppose that there exists a pair (G, F) in \mathcal{C} and an integral vector c such that the first closure of system (4.2) has a fractional vertex \bar{x} . Among all such counterexamples, choose $(G, F), c, \bar{x}$ such that the quadruple $(|V(G)|, |E(G) \setminus L(G)|, |E(G)|, \lfloor \sum_{e \in E(G)} \bar{x}_e \rfloor)$ is lexicographically minimal. Let $A = A(G, F)$.

Given a node v , if G' is obtained from G by switching sign on node v and $c' \in \mathbb{R}^{V(G)}$ is defined by $c'_u = c_u$, $u \in V(G) \setminus \{v\}$, $c'_v = -c_v$, then it is immediate to verify that \bar{x} is a vertex of the first closure of $A(G', F)x = c'$, $x \geq 0$, because for every $U \subseteq V(G)$, $c(U)$ is odd if and only if $c'(U)$ is odd. So, if $(G, F), c$, and \bar{x} are a minimal counterexample, then also $(G', F), c'$, and \bar{x} is a minimal counterexample. Hence, throughout the proof we will perform such switching whenever convenient.

Notice that (G, F) has at least an edge in $F \setminus L(G)$, otherwise, by Theorem 2.2, A has the Edmonds-Johnson property, contradicting our choice of (G, F) .

Furthermore, G is connected. If not, let G' be a component of G such that $\bar{x}_e \notin \mathbb{Z}$ for some $e \in E(G')$, let $F' = F \cap E(G')$, and let $A' = A(G', F')$. Let \bar{x}' and c' be the restrictions of \bar{x} and c , respectively, to $E(G')$. Notice that (G', F') is in \mathcal{C} and that $|V(G')| < |V(G)|$, hence the system $A'x' = c'$, $x' \geq 0$ has Chvátal rank at most 1. Then clearly \bar{x}' is a fractional vertex of the first closure of $A'x' = c'$, $x' \geq 0$, a contradiction.

We say that an inequality $\alpha x \leq \beta$ is *tight at \bar{x}* if $\alpha \bar{x} = \beta$. Notice that, since \bar{x} is a vertex of the first closure of (4.2), it satisfies at equality $|E(G)|$ linearly independent inequalities among the ones in (4.2) and the odd-cut inequalities.

Claim 4.2. $\bar{x}_e > 0$ for every $e \in E(G)$.

Proof of claim. If not, assume that there exists an edge e in $E(G)$ with $\bar{x}_e = 0$. Let (G', F') be obtained from (G, F) by deleting e , and let $A' = A(G', F')$. Let \bar{x}' be the vector obtained from \bar{x} by removing the component corresponding to e . Since (G', F') is in \mathcal{C} , $|V(G')| = |V(G)|$, $|E(G') \setminus L(G')| \leq |E(G) \setminus L(G)|$, and $|E(G')| < |E(G)|$, the system $A'x' = c$, $x' \geq 0$ has Chvátal rank at most 1. Since the vector c has not changed, the odd-cut inequalities

for $A'x' = c$, $x' \geq 0$ are exactly the odd-cut inequalities for (4.2). Moreover notice that $\bar{x}'(\delta(U) \setminus F') = \bar{x}(\delta(U) \setminus F)$ for every $U \subseteq V(G)$. Hence \bar{x}' is a fractional vertex of the first closure of $A'x' = c$, $x' \geq 0$, a contradiction. \diamond

Let $\mathcal{F} = \{U \subseteq V \mid \bar{x}(\delta(U) \setminus F) = 1\}$.

By Claim 4.2, \bar{x} is the unique solution of the system

$$\begin{aligned} A(G, F)x &= c \\ x(\delta(U) \setminus F) &= 1 \quad U \in \mathcal{F}. \end{aligned}$$

By Lemma 2.4, we can choose a laminar subfamily \mathcal{L} of \mathcal{F} such that \bar{x} is the unique solution of the system

$$\begin{aligned} A(G, F)x &= c \\ x(\delta(U) \setminus F) &= 1 \quad U \in \mathcal{L}, \end{aligned}$$

and the elements of $\{\chi(\delta(U) \setminus F) : U \in \mathcal{L}\}$ are not linear combination of rows of $A(G, F)$.

In particular, $|E| \leq |V| + |\mathcal{L}|$.

Claim 4.3. *For every $e \in E(G)$, $0 < \bar{x}_e < 1$. Furthermore, for every $e \in E(G) \setminus F$ there exists $U \in \mathcal{L}$ such that $e \in \delta(U)$.*

Proof of claim. By Claim 4.2, $\bar{x}_e > 0$ for every e in $E(G)$.

We show that $\bar{x}_f < 1$ for any f in F . Let f be an edge in F , and suppose $\bar{x}_f \geq 1$. Notice that, possibly by switching the signs on the endnode/s of f , we can assume that f has a sign +1 on its endnode/s. Let \bar{x}' be obtained from \bar{x} by decreasing by 1 the component corresponding to f and let c' be obtained from c by decreasing by 2 the component/s corresponding to the endnode/s of f . Since $\lfloor \sum_{e \in E(G)} \bar{x}'_e \rfloor < \lfloor \sum_{e \in E(G)} \bar{x}_e \rfloor$, by minimality of $(G, F), c, \bar{x}$ the system $Ax = c', x \geq 0$ has Chvátal rank at most 1. Notice that each entry of c' has the same parity as the corresponding entry of c , therefore the odd-cut inequalities for $Ax = c', x \geq 0$ are exactly the odd-cut inequalities for (4.2). Since such inequalities do not depend on the values on the edges in F , then \bar{x}' is a fractional vertex of the first closure of the system $Ax = c', x \geq 0$, a contradiction.

We show next that for every e in $E(G) \setminus F$ there exists $U \in \mathcal{L}$ such that $e \in \delta(U)$.

By contradiction, suppose there exists $e \in E(G) \setminus F$ such that for every $U \in \mathcal{L}$, $e \notin \delta(U)$. If $e = vw$ is not a loop, possibly by switching signs on

v we may assume that $\sigma_{v,e} \neq \sigma_{w,e}$. Let (G', F') be obtained from (G, F) by contracting e , let r be the node obtained from the contraction of vw , and let $A' = A(G', F)$. Let \bar{x}' be the vector obtained from \bar{x} by removing the components corresponding to the removed edges, and let c' be obtained from c by removing the components corresponding to v and w and adding a component relative to r with value $c'_r = c_v + c_w$. Since (G', F') is in \mathcal{C} and $|V(G')| < |V(G)|$, the system $A'x' = c', x' \geq 0$ has Chvátal rank at most 1. Notice that \bar{x}' satisfies the system $A'x' = c', x' \geq 0$, and that the equation $(A'x')_r = c'_r$ is the sum of $(Ax)_v = c_v$ and $(Ax)_w = c_w$. Furthermore, the odd-cut inequalities for $A'x' = c', x' \geq 0$ are exactly the odd-cut inequalities for (4.2) relative to sets $U \subseteq V(G)$ that either contain both v and w or none of them. Notice that all the odd-cut inequalities corresponding to sets in \mathcal{L} are of this form, since $e \notin \delta(U)$ for every $U \in \mathcal{L}$, hence \bar{x}' is in the first closure of $A'x' = c', x' \geq 0$ and, since $\bar{x}_e > 0$, it satisfies at equality $|E(G)| - 1 \geq |E(G')|$ linearly independent inequalities among $A'x' = c', x' \geq 0$ and the odd-cut inequalities corresponding to sets in \mathcal{L} . Therefore \bar{x}' is a vertex of the first closure of $A'x' = c', x' \geq 0$, so it is integral. Hence \bar{x}_e must be the only fractional entry in \bar{x} , which is impossible since $(A\bar{x})_v = c_v$ which is integer.

If e is a loop on node v , notice that, in this case, the column relative to e in the constraint matrix of the system $Ax = c, x(\delta(U) \setminus F) \geq 1, U \in \mathcal{L}$, is the vector of all zeroes except in row A_v . Since the columns of said matrix are linearly independent, then e is the only loop of G on v . Let (G', F') be obtained from (G, F) by deleting node v and let $A' = A(G', F')$. Clearly (G', F') is in \mathcal{C} . Let $\bar{x}' \in \mathbb{Z}^{E(G')}$ be the vector obtained from \bar{x} by removing the components corresponding to the deleted loops on v , and let $c' \in \mathbb{Z}^{V(G')}$ be obtained from c by removing the component corresponding to v . Since $|V(G')| < |V(G)|$, $A'x' = c', x' \geq 0$ has Chvátal rank at most 1. Notice that A' is obtained from A by removing the row corresponding to v and the column relative to e . Thus \bar{x}' is in the first closure of $A'x' = c', x' \geq 0$, since the odd-cut inequalities for the latter system are the odd-cut inequalities for (4.2) relative to sets $U \subseteq V(G) \setminus \{v\}$. Since $U \subseteq V(G) \setminus \{v\}$ for every $U \in \mathcal{L}$, all odd-cut inequalities for (4.2) corresponding to sets in \mathcal{L} are valid for the first closure of $A'x' = c', x' \geq 0$. Since $\bar{x}_e > 0$, \bar{x}' satisfies at equality $|E(G)| - 1 = |E(G')|$ linearly independent inequalities among $A'x' = c'$ and the odd-cut inequalities relative to sets $U \in \mathcal{L}$. Hence \bar{x}' is a vertex of the first closure of $A'x' = c', x' \geq 0$, therefore \bar{x}' is integral. Thus \bar{x}_e is the only fractional entry of \bar{x} , which is impossible since $(A\bar{x})_v = c_v$ which is integer.

Therefore, given e in $E(G) \setminus F$, $\bar{x}_e \leq 1$ since there exists $U \in \mathcal{L}$ such that $e \in \delta(U)$, and since $\bar{x}(\delta(U)) = 1$. We show that, given e in $E(G) \setminus F$, $\bar{x}_e < 1$. By contradiction, suppose that there exists an edge e in $E(G) \setminus F$

such that $\bar{x}_e = 1$. Possibly by switching signs on the endnode/s of e , we may assume that e has sign $+1$ on its endnode/s. Let $U \subseteq V(G)$ define an odd-cut inequality in \mathcal{L} such that $e \in \delta(U)$. Hence e is the only edge in $\delta(U) \setminus F$. Let (G', F') be obtained from (G, F) by deleting e , and let $A' = A(G', F')$. Let c' be obtained from c by subtracting 1 to the entries relative to the endnode/s of e , and let \bar{x}' be the vector obtained from \bar{x} by removing the component corresponding to e . Since (G', F') is in \mathcal{C} , $|V(G')| = |V(G)|$, $|E(G') \setminus L(G')| \leq |E(G) \setminus L(G)|$, and $|E(G')| < |E(G)|$, the system $A'x' = c'$, $x' \geq 0$ has Chvátal rank at most 1.

We show that \bar{x}' is in the first closure of $A'x' = c'$, $x' \geq 0$. Clearly \bar{x}' satisfies the system, so we need to show that it satisfies the odd-cut inequalities. Let $S \subseteq V(G')$ such that $c'(S)$ is odd. By Lemma 4.9 we may assume that S cannot be partitioned into two nonempty sets S_1, S_2 such that the only edges in G' between S_1 and S_2 are in F . Therefore, since $\delta_{G'}(U) \subseteq F$, either $S \subseteq U$ or $S \subseteq V(G) \setminus U$. If no endnode of e is in S , then clearly $\bar{x}'(\delta_{G'}(S) \setminus F) \geq 1$. Hence we assume $e \in \delta_G(S)$, hence $c(S)$ is even. If $S \subseteq U$, then $c(U \setminus S)$ is odd, hence $\bar{x}'(\delta_{G'}(S) \setminus F) = \bar{x}(\delta_G(U \setminus S) \setminus F) \geq 1$. If $S \subseteq V(G) \setminus U$, then $c(U \cup S)$ is odd, hence $\bar{x}'(\delta_{G'}(S) \setminus F) = \bar{x}(\delta_G(U \cup S) \setminus F) \geq 1$.

Therefore \bar{x}' is in the first closure of $A'x' = c'$, $x' \geq 0$, hence it is a convex combination of integral vectors y^1, \dots, y^k satisfying such system. Thus $\bar{x} = \begin{pmatrix} \bar{x}_e \\ \bar{x}' \end{pmatrix}$ is a convex combination of $\begin{pmatrix} \bar{x}_e \\ y^i \end{pmatrix}$, $i = 1, \dots, k$, which are integral vectors satisfying (4.2), a contradiction. \diamond

Claim 4.4. G does not contain a cycle in F .

Proof of claim. By contradiction, suppose C is a cycle in F . Since $(G, F) \in \mathcal{C}$, the cycle C is even and so there is an even number of nodes of C incident with two edges of C with the same sign on that node. Hence the edges of C can be partitioned in two subsets R and B such that any two adjacent edges of C are contained in the same subset if and only if they have distinct signs on their common endnode. Now let $y = \bar{x} + \epsilon\chi(R) - \epsilon\chi(B)$ and let $z = \bar{x} - \epsilon\chi(R) + \epsilon\chi(B)$. By Claim 4.3, there exists $\epsilon > 0$ such that both y and z satisfy all the inequalities of (4.2). Moreover, both y and z satisfy all the odd-cut inequalities for (4.2), since they only involve variables relative to edges in $E(G) \setminus F$. Since $\bar{x} = \frac{1}{2}(y + z)$, \bar{x} is not a vertex of the first closure of (4.2), a contradiction. \diamond

Claim 4.5. Each node in $V(G)$ is incident with at least one edge in $E(G) \setminus F$.

Proof of claim. By contradiction let v be a node in $V(G)$ incident only with edges in F . Notice that we can assume that G contains at least two nodes,

otherwise it is clear that $A(G, F)$ has the Edmonds-Johnson property. Thus, since G is connected, there exists an edge in $F \setminus L(G)$ incident with v . Thus let $f = vw$ be an edge in $F \setminus L(G)$.

Possibly by switching sign on v , we may assume that $\sigma_{v,f} \neq \sigma_{w,f}$. Notice that c_v is even, as otherwise the odd-cut corresponding to the set $\{v\}$ is not satisfied.

Let (G', F') be obtained from (G, F) by contracting f , let r be the node obtained from the contraction of v and w , and let $A' = A(G', F')$. Clearly, (G', F') is in \mathcal{C} . Let \bar{x}' be the vector obtained from \bar{x} by removing the component corresponding to f , and let c' be obtained from c by removing the components corresponding to v and w and adding a new component relative to r with value $c'_r = c_v + c_w$. Since $|V(G')| < |V(G)|$, the system $A'x' = c'$, $x' \geq 0$ has Chvátal rank at most 1. Since c_v is even, c'_r has the same parity as c_w . Thus, given $U \subseteq V(G) \setminus \{v\}$, then $c(U)$ is odd if and only if $c'(U')$ is odd, where $U' = U$ if $w \notin U$, $U' = U \setminus \{w\} \cup \{r\}$ if $w \in U$. Moreover, notice that $\delta(U) \setminus F = \delta_{G'}(U') \setminus F$, therefore the odd-cut inequalities for $A'x' = c'$, $x' \geq 0$ are exactly the odd-cut inequalities for (4.2) relative to U with $U \subseteq V(G) \setminus \{v\}$. Notice that these inequalities are precisely the odd-cut inequalities for (4.2). Clearly \bar{x}' satisfies also the system $A'x' = c'$, $x' \geq 0$, so it is in its first closure.

Since the inequality $(A'x')_r = c'_r$ is the sum of $(Ax)_w = c_w$ and $(Ax)_v = c_v$, and since all the odd-cut inequalities for (4.2) are also valid for $A'x' = c'$, $x' \geq 0$, there are $|E(G)| - 1 = |E(G')|$ linearly independent inequalities valid for the first closure of $A'x' = c'$, $x' \geq 0$ tight at \bar{x}' , hence \bar{x}' is a vertex of such first closure, and it is therefore integral. So $\bar{x}_e = \bar{x}'_e$ is integer for every e in $E(G')$, contradicting Claim 4.3. \diamond

4.4.1 Half-integrality for some special cases

Claim 4.6. *If $G \setminus F$ is connected and $V(G) \notin \mathcal{L}$, then \bar{x} is half-integral.*

Proof of claim. Let U be a maximal set in the laminar family \mathcal{L} . Notice that, since \mathcal{L} is laminar, for every $S \in \mathcal{L}$ either $S \subseteq U$ or $S \subseteq V(G) \setminus U$. Since $V(G) \notin \mathcal{L}$, then $U \subset V(G)$. Since $G \setminus F$ is connected, there exists at least one edge $e \in E(G) \setminus (F \cup L(G))$ such that $e \in \delta(U)$.

Let $e = vw$ and let v be the endnode of e in U . Now let (G', F) be obtained from (G, F) by deleting e and adding loops e_v, e_w on v and w with signs $\sigma_{v,e}$ and $\sigma_{w,e}$, respectively. Let $A' = A(G', F)$. One can readily verify that (G', F) is in the class \mathcal{C} , $|V(G')| = |V(G)|$, and $|E(G') \setminus L(G')| < |E(G) \setminus L(G)|$, thus $A'x' = c$, $x' \geq 0$ has Chvátal rank at most 1. Now let \bar{x}' be obtained from \bar{x} by replacing the component corresponding to the edge

e with two components equal to the replaced one and that correspond to the new loops e_v and e_w . Clearly \bar{x}' satisfies the system $A'x' = c, x' \geq 0$. Each odd-cut inequality of the latter system is satisfied by \bar{x}' since, for every $S \subseteq V$, $\bar{x}'(\delta_{G'}(S) \setminus F) \geq \bar{x}(\delta_G(S) \setminus F)$, where equality holds if and only if $|S \cap \{v, w\}| \leq 1$. Thus \bar{x}' is in the first closure of $A'x' = c, x' \geq 0$. Since U is a maximal set in \mathcal{L} , $|S \cap \{v, w\}| \leq 1$ for every $S \in \mathcal{L}$, hence

$$\bar{x}'(\delta_{G'}(S) \setminus F) = 1 \quad \text{for every } S \in \mathcal{L}. \quad (4.10)$$

Therefore, since \bar{x} satisfies tightly $|E(G)|$ inequalities of the first closure of (4.2), \bar{x}' satisfies tightly $|E(G)| = |E(G')| - 1$ linearly independent inequalities of the first closure of $A'x' = c, x' \geq 0$. Hence \bar{x}' lies on a face of dimension 1 of the first closure of $A'x' = c, x' \geq 0$. This implies that \bar{x}' is a convex combination of two vertices y and z of the first closure of such system, i.e. $\bar{x}' = \lambda y + (1 - \lambda)z$, $0 \leq \lambda \leq 1$. Since $A'x' = c, x' \geq 0$ has Chvátal rank at most 1, then y and z are integral, so $0 < \lambda < 1$. By (4.10), $y(\delta_{G'}(S) \setminus F) = z(\delta_{G'}(S) \setminus F) = 1$ for every $S \in \mathcal{L}$. By Claim 4.3, each edge $h \in E(G) \setminus F$ is in $\delta(S)$ for some set $S \in \mathcal{L}$, thus each edge $h \in E(G') \setminus (F \cup \{e_w\})$ is in $\delta(S)$ for some set $S \in \mathcal{L}$. Therefore $y_h, z_h \in \{0, 1\}$ for every $h \in E(G') \setminus (F \cup \{e_w\})$.

Thus, since $\bar{x}'_{e_v} = \bar{x}'_{e_w} = \bar{x}_e < 1$, we can assume that $y_{e_v} = 1$ and $z_{e_v} = 0$ and that precisely one among y_{e_w} and z_{e_w} is 0. Hence $\bar{x}_e = \lambda$. If $z_{e_w} = 0$, then $y_{e_w} = 1$ since $\bar{x}'_{e_w} = \lambda y_{e_w}$, thus if we define two points $\bar{y}, \bar{z} \in \mathbb{R}^{E(G)}$ by $\bar{y}_h = y_h$, $h \in E(G) \setminus \{e\}$, $\bar{y}_e = 1$, and $\bar{z}_h = z_h$, $h \in E(G) \setminus \{e\}$, $\bar{z}_e = 0$, then \bar{y} and \bar{z} are integral points satisfying (4.2) and $\bar{x} = \lambda \bar{y} + (1 - \lambda)\bar{z}$, contradicting the fact that \bar{x} is a vertex of the first closure of (4.2). Therefore $y_{e_w} = 0$ and $z_{e_w} = k$ for some positive integer k . Since $\bar{x}_e = \lambda y_{e_v} + (1 - \lambda)z_{e_v} = \lambda y_{e_v} + (1 - \lambda)z_{e_w}$, then $\lambda = k/(k+1)$. If $k = 1$, then all components of \bar{x} are equal to $1/2$ and we are done. Thus we may assume that $k \geq 2$. This implies $\bar{x}_e = k/(k+1) > 1/2$.

Moreover, $w \notin S$ for every $S \in \mathcal{L}$, otherwise $e_w \in \delta_{G'}(S)$ and $z(\delta_{G'}(S) \setminus F) = 1$ would imply $z_{e_w} = 1$. Notice also that since $z(\delta_{G'}(U) \setminus F) = 1$ there exists an edge $g \in \delta_{G'}(U) \setminus F$ such that $z_g = 1$. Hence $\delta_G(U) \setminus F = \{e, g\}$ and $\bar{x}_g = 1 - \lambda = 1/(k+1) < 1/2$. If g is not a loop, then we may apply to g the same argument as above, thus obtaining that $\bar{x}_g > 1/2$, a contradiction. Thus g is a loop.

Next we show that \bar{x}' is in the first closure of the system $A'x' = c, x' \geq 0, x_{e_w} \leq 1$. By Lemma 4.6, the irredundant non trivial inequalities for the first closure of $A'x' = c, x' \geq 0, x_{e_w} \leq 1$ are valid for the first closure of $A'x' = c, x' \geq 0$, or they are of the form $x'(\delta_{G'}(S) \setminus (F \cup \{e_w\})) - x'_{e_w} \geq 0$ for $S \subseteq V(G)$ such that $c(S)$ is even, $w \in S$, and S cannot be partitioned into two nonempty sets S_1 and S_2 such that all the edges between S_1 and S_2 are in F . Since all

edges of $E(G')$ between U and $V(G) \setminus U$ are in F , then the latter inequalities are defined by sets S contained in $V(G) \setminus U$ such that $w \in S$ and $c(S)$ is even. We only need to show that \bar{x}' satisfies such inequalities. Given $S \subseteq V(G) \setminus U$ with $w \in S$ and $c(S)$ even, $c(U \cup S)$ is odd, hence $\bar{x}(\delta_G(U \cup S) \setminus F) \geq 1$. Notice that $\delta_{G'}(S) \setminus (F \cup \{e_w\}) = \delta_G(U \cup S) \setminus (F \cup \{g\})$, hence

$$\begin{aligned} \bar{x}'(\delta_{G'}(S) \setminus (F \cup \{e_w\})) - \bar{x}'_{e_w} &= \bar{x}(\delta_G(U \cup S) \setminus F) - \bar{x}_e - \bar{x}_g \\ &\geq 1 - \frac{k}{k+1} - \frac{1}{k+1} = 0. \end{aligned}$$

Hence \bar{x}' lies on the face of dimension 1 of the first closure of $A'\bar{x}' = c$, $x' \geq 0$, $x'_{e_w} \leq 1$ defined by $A'x' = c$ and $x'(\delta_{G'}(S) \setminus F) = 1$, $S \in \mathcal{L}$. Notice that y is a vertex of such face. Let z' be the other vertex. Notice that z' is strictly inside the segment between y and z , i.e. $z' = \mu y + (1 - \mu)z$ for some $0 < \mu < 1$, so $z'_{e_w} > 0$ and z' is not integral. Therefore z' is not a vertex of the first closure of $A'x' = c$, $x' \geq 0$. Thus there exists a set $W \subset V(G) \setminus U$ such that $w \in W$, $c(W)$ is even, $\bar{x}'(\delta_{G'}(W) \setminus (F \cup \{e_w\})) - \bar{x}'_{e_w} = 0$, and the $|E(G')|$ inequalities

$$\begin{aligned} A'x' &= c \\ x'(\delta_{G'}(S) \setminus F) &\geq 1, \quad S \in \mathcal{L} \\ x'(\delta_{G'}(W) \setminus (F \cup \{e_w\})) - x'_{e_w} &\geq 0 \end{aligned}$$

are linearly independent. Notice that, since $w \notin S$ for every $S \in \mathcal{L}$, then all of the above inequalities are valid also for the first closure of $A'x' = 0$, $x'_h \geq 0$ $h \in E(G') \setminus \{e_w\}$, $x'_{e_w} \leq 1$.

$$\begin{aligned} A'x' &= 0, \\ x'_h &\geq 0, \quad h \in E(G') \setminus \{e_w\}, \\ x'_{e_w} &\leq 1 \end{aligned}$$

Let G'' be the bidirected graph $G'' = (V(G'), E(G'), \sigma')$, where $\sigma'_h = \sigma_h$ for every $h \in E(G') \setminus \{e_w\}$, and $\sigma'_{e_w} = -\sigma_{e_w}$, and let $A'' = A(G'', F)$. Notice that A'' is obtained from A' by multiplying by -1 the column of A' relative to e_w . Let $c'' \in \mathbb{R}^{V(G)}$ be defined by $c''_u = c_u$, $u \in V(G) \setminus \{v\}$, and $c''_w = c_w - 1$. Clearly $(G'', F) \in \mathcal{C}$, thus $A''x'' \geq c''$, $x'' \geq 0$ has Chvátal rank at most 1. It is not difficult to see that the point \bar{z}'' defined by $\bar{z}''_h = \bar{z}'_h$, $h \in E(G') \setminus \{e_w\}$, and $\bar{z}''_{e_w} = 1 - \bar{z}'_{e_w}$ is in the first closure of $A''x'' \geq c''$, $x'' \geq 0$. Furthermore, $c''(S) = c(S)$ is odd for every $S \in \mathcal{L}$ and $z''(\delta_{G''}(S) \setminus F) = 1$, $c''(W) = c(W) - 1$ is odd and $z''(\delta_{G''}(W) \setminus F) = 1$. Thus z'' satisfies at equality $|E(G'')|$ linearly independent inequalities, therefore it is a fractional vertex of the first closure of $A''x'' \geq c''$, $x'' \geq 0$, a contradiction. \diamond

Claim 4.7. *If G is bipartite, $G \setminus F$ is connected, and $L(G) \cap F = \emptyset$, then \bar{x} is half-integral.*

Proof of claim. Since all the cycles in the graph are even, by a theorem of Heller and Tompkins [24], the nodes in G can be partitioned into two subsets R, B such that, for every $e = vw \in E(G)$, v and w are in the same class of the partition if and only if $\sigma_{v,e} \neq \sigma_{w,e}$. Let L_R^+ and L_R^- (resp. L_B^+ and L_B^-) be the sets of loops of G with respectively a $+1$ and a -1 sign, incident with nodes in R (resp. in B). By symmetry, we may assume $c(R) \geq c(B)$.

If $V(G) \notin \mathcal{L}$, then by Claim 4.6 \bar{x} is half-integral. Thus we assume that $V(G) \in \mathcal{L}$. Then the odd-cut inequality relative to $V(G)$, that is $\sum_{l \in L_R^+ \cup L_R^- \cup L_B^+ \cup L_B^-} x_l \geq 1$, is satisfied tightly by \bar{x} and it is linearly independent from the equations in the system $A(G, F)x = c$. By summing together all the equalities in $A(G, F)x = c$ corresponding to nodes in R and all the equalities in $-A(G, F)x = -c$ corresponding to nodes in B we get

$$\sum_{l \in L_R^+ \cup L_B^-} x_l - \sum_{l \in L_R^- \cup L_B^+} x_l = c(R) - c(B).$$

Since $c(V(G))$ is odd, $c(R) - c(B)$ is odd, hence $c(R) - c(B) \geq 1$. Hence $1 = \sum_{l \in L_R^+ \cup L_R^- \cup L_B^+ \cup L_B^-} \bar{x}_l \geq \sum_{l \in L_R^+ \cup L_B^-} \bar{x}_l - \sum_{l \in L_R^- \cup L_B^+} \bar{x}_l \geq 1$, because $\bar{x} \geq 0$. Thus $\sum_{l \in L_R^+ \cup L_R^- \cup L_B^+ \cup L_B^-} \bar{x}_l = \sum_{l \in L_R^+ \cup L_B^-} \bar{x}_l - \sum_{l \in L_R^- \cup L_B^+} \bar{x}_l$, therefore $L_R^- \cup L_B^+ = \emptyset$, because $\bar{x} > 0$. But then the inequality defined by $V(G)$ is linearly dependent from $A(G, F)x = c$, a contradiction. \diamond

The following claim will be useful in the remaining of the proof.

Claim 4.8. *If $\bar{x}_e = \frac{1}{2}$ for every $e \in E(G)$, then (G, F) satisfies the conditions a) and b) of Remark 4.14.*

Proof of claim. Since $\bar{x}_e = \frac{1}{2}$ for every $e \in E(G)$ and \bar{x} satisfies $A(G, F)\bar{x} = c$, $\bar{x}(\delta_G(v) \setminus F)$ is an integer for every $v \in V$, hence $|\delta_G(v) \setminus F|$ is even and a) is satisfied.

Given a connected component \bar{G} of $G \setminus F$ such that $L(\bar{G}) = \emptyset$, then $\delta_G(V(\bar{G})) \setminus F = \emptyset$, hence $c(V(\bar{G}))$ is even, otherwise $V(\bar{G})$ defines an odd-cut inequality violated by \bar{x} . Notice that, since $A(G, F)\bar{x} = c$,

$$c(V(\bar{G})) = \frac{1}{2} \sum_{vw \in E(\bar{G})} (\sigma_{v,vw} + \sigma_{w,vw}) + \sum_{\substack{vw \in F \setminus L(G) \\ v, w \in V(\bar{G})}} (\sigma_{v,vw} + \sigma_{w,vw}) + \sum_{\substack{vw \in \delta_G(V(\bar{G})) \\ v \in V(\bar{G})}} \sigma_{v,vw}$$

Even edges of $E(\bar{G})$ contribute 0 to the right-hand-side of the latter expression, each odd edge of $E(\bar{G})$ contributes ± 1 , edges in $F \setminus L(G)$ with both endnodes in $V(\bar{G})$ contribute 0 or ± 2 , while edges in $\delta_G(V(\bar{G}))$ contribute ± 1 . Hence the number of odd edges in $E(\bar{G})$ is congruent modulo 2 to $|\delta_G(V(\bar{G}))|$. \diamond

4.4.2 Structure of (G, F)

Given a cycle C and a family $\{e_i, i \in I\}$ of chords of C , we say that $\{e_i, i \in I\}$ is a *family of non-crossing chords of C* if for every pair of chords $e_i, e_j, i, j \in I$, each path in C between the two endnodes of e_i , contains either both the endnodes of e_j or none of them.

Claim 4.9. $F \setminus L(G)$ does not contain two disjoint edges with all endnodes in the same block of $G \setminus F$.

Proof of claim. Otherwise let $f = vw$ and $f' = v'w'$ be such edges. We show that (G, F) contains a cycle C in $G \setminus F$ such that f and f' are non-crossing cords of C . Since v, w, v', w' are distinct and in the same block of $G \setminus F$, such block is 2-connected. Let P_1 be a path in $G \setminus F$ from v to v' that does not pass through w . Notice that, by switching v' with w' we can assume that P_1 does not pass through w' . Now let P_2 be a path in $G \setminus F$ from w' to w that does not pass through v . Notice that P_2 does not intersect P_1 , as otherwise we get G_4 as a minor. Now let P_3 be a path in $G \setminus F$ from w to v that does not pass through v' . Notice that P_3 does not intersects P_1 or P_2 , as otherwise we get G_4 as a minor. Now let P_4 be a path in $G \setminus F$ from v' to w' that does not pass through v . Notice that P_4 does not intersects P_1, P_2 , or P_3 as otherwise we get G_4 as a minor. Hence $C = v, P_1, v', P_4, w', P_2, w, P_3, v$ is a cycle in $G \setminus F$, and f and f' are non-crossing cords of C .

Now we show that the edges in $F \setminus L(G)$ form a family of non-crossing chords of C .

Let $f'' \notin \{f, f'\}$ be an edge in $F \setminus L(G)$. Notice that f'' is a chord of C , as otherwise by considering a shortest path from an endnode of f'' to a node in C , we get G_4 as a minor. Thus let C'' be a subpath of C from one endnode of f'' to the other. Clearly, if an internal node of C'' is adjacent to another edge $f''' \in F \setminus L(G)$, then C'' must contain also the other endnode of f''' , as otherwise by deleting one endnode of f''' we get G_4 as a minor. Thus the edges in $F \setminus L(G)$ form a family of non-crossing chords of C . This implies that $G \setminus F$ is connected.

Now we show that there are no loops in F . If not, let $l \in F \cap L(G)$. By considering a shortest path from the endnode of l to a node in $V(C)$, we get the minor G_4 .

We also show that G is bipartite. If not, let C' be an odd cycle, let (G', F') be obtained from (G, F) by contracting the edges in C' , and let l be the new loop in F' obtained from the contraction of C' . By considering a shortest path from the endnode of l to a node in $V(C)$, we get the minor G_4 .

Moreover, for every edge $f \in F$, the endnodes of f form a cutset for G . Otherwise, by symmetry assume that $\{v, w\}$ is not a cutset of G . Thus there exists a shortest path P from an internal node of P_3 to an endnode of f' , that does not pass through v and w . By considering the cycle v, w, P_3 , the path P , and the edge f' , we get G_4 as a minor.

Hence by Claim 4.7, \bar{x} is half-integral. Thus by Claim 4.8 (G, F) satisfies conditions a) and b) of Remark 4.14.

Let \mathcal{T} be a family of trails obtained by partitioning the edges in $E(G) \setminus (E(C) \cup F)$ into trails that start and end in loops in $E(G) \setminus F$ and/or into closed trails. Notice that $\mathcal{T}' = \mathcal{T} \cup \{C\}$ satisfies conditions (C1)-(C3) of Lemma 4.16.

Hence by Lemma 4.16 there exists a balanced bipartition R, B of $E(G)$. Now let $y = \bar{x} + 1/2\chi(R) - 1/2\chi(B)$ and let $z = \bar{x} - 1/2\chi(R) + 1/2\chi(B)$. y and z are integral, they satisfy (4.2) and $\bar{x} = 1/2(y + z)$, a contradiction. \diamond

Let $f = vw, f' = v'w'$ be two edges in $F \setminus L(G)$ such that v, w, v', w' are in the same connected component of $G \setminus F$. We say that f' is *nested in* f if every path in $G \setminus F$ from v to w contains the nodes v', w' . We say that f and f' are *nested* if f' is nested in f or f is nested in f' .

Claim 4.10. *Let $f = vw, f' = v'w'$ be two edges in $F \setminus L(G)$ such that v, w, v', w' are in the same connected component of $G \setminus F$. Then one of the following holds:*

- (i) *f and f' are adjacent, say $v = v'$, and for any two distinct nodes $s, t \in \{v, w, w'\}$ there exists a path in $G \setminus F$ between s and t that does not pass through $\{v, w, w'\} \setminus \{s, t\}$;*
- (ii) *f and f' are nested;*
- (iii) *one among v and w , say v is a cutnode of $G \setminus F$ separating w from $\{v', w'\} \setminus \{v\}$.*

Proof of claim. By contradiction assume that none of the cases above applies to f, f' . At first we show that f and f' are not adjacent. Otherwise assume $v = v'$. By Claim 4.4, $w \neq w'$. Since we are not in case (i), by symmetry either there is not a path in $G \setminus F$ from v to w that does not pass through w' , or there is not a path in $G \setminus F$ from w to w' that does not pass through

v . The first case cannot happen because otherwise f' is nested in f , thus case (ii) applies. The second case cannot happen because otherwise v is a cutnode of $G \setminus F$ separating w from $\{v', w'\} \setminus \{v\}$, thus case (iii) applies.

Thus all the nodes v, w, v', w' are pairwise disjoint. Since v, w are in the same connected component of $G \setminus F$, and since f and f' are not nested, there is a path P from v to w in $G \setminus F$ that does not contain both v' and w' . Clearly P does not contain any node among v' and w' , as otherwise we get G_4 as a minor by deleting the endnode of f' that does not appear in P . In the same way let P' be a path from v' to w' in $G \setminus F$ that does not contain any node among v and w .

Now let S be a path in $G \setminus F$ with an endnode in P and the other endnode in P' such that all its internal nodes are not in $V(P) \cup V(P')$. Clearly one endnode of S is an endnode of f , say v , and the other endnode of S is an endnode of f' , say v' . In fact otherwise by symmetry one endnode of S is an internal node of P . By contracting all the edges in S , some edges in P' , and by deleting one endnode of f' we get G_4 as a minor.

Now we show that any other path S' in $G \setminus F$ with an endnode in P , and the other endnode in P' such that all its internal nodes are not in $V(P) \cup V(P')$, has endnodes v, v' . We show this by contradiction. In fact, if its endnodes are w, v' or v, w' , then we get G_4 by deleting respectively w' or w , and by contracting some edges. Otherwise, if its endnodes are w, w' , then $V(S) \cap V(S') = \emptyset$, otherwise we get G_4 as a minor. Hence f and f' are contained in the same block of $G \setminus F$, but this contradicts Claim 4.9.

Thus case (iii) applies, a contradiction. \diamond

Claim 4.11. *Let $f = vw$ be an edge in $F \setminus L(G)$ with both endnodes in a connected component \bar{G} of $G \setminus F$, and let f' be an edge in F with exactly an endnode $v' \in V(\bar{G})$. Then one of the following holds:*

- (i) f and f' are adjacent;
- (ii) one among v and w , say v is a cutnode of $G \setminus F$ separating w from v' .

Proof of claim. By contradiction assume that none of the cases above applies to f, f' . Since f and f' are not adjacent, $v' \notin f$. Since case (ii) does not apply, there exists a path P_w in $G \setminus F$ from w to v' that does not contain v . Symmetrically, there exists a path P_v in $G \setminus F$ from v to v' that does not contain w . By considering the edge f , and the paths P_v and P_w , we get G_4 as a minor, a contradiction. \diamond

Claim 4.12. *Let $f = vw$ be an edge in $F \setminus L(G)$ with both endnodes in a connected component \bar{G} of $G \setminus F$, and let C be an odd cycle in \bar{G} . Then one among v and w , say v is a cutnode of $G \setminus F$ separating w from $V(C) \setminus \{v\}$.*

Proof of claim. By contradiction assume that the Claim does not hold. Since v is not a cutnode of $G \setminus F$ separating w from $V(C) \setminus \{v\}$, then there exists a path P_w in $G \setminus F$ from w to a node in $V(C) \setminus \{v\}$ that does not contain v . Symmetrically, there exists a path P_v in $G \setminus F$ from v to a node in $V(C) \setminus \{w\}$ that does not contain w .

Notice that C does not contain v or w , as otherwise we get an odd cycle containing f by using the paths P_v, P_w and the edge f .

Thus $V(C) \cap \{v, w\} = \emptyset$. Let (G', F') be the obtained from (G, F) by contracting all the edges of C . Let l be the new loop in F' obtained from the contraction of F , and let v' be the endnode of l .

By considering the edge f , the loop l , and the paths from the endnodes of f to v' in (G', F') corresponding to P_v and P_w , we get G_4 as a minor, a contradiction. \diamond

Given a subset Δ of F , let $G^{split(\Delta)}$ be obtained from G by deleting all the edges in Δ and by adding, for every node $v \in V(G)$ that was incident with at least one removed edge, a loop l_v in v , with sign $+1$ if and only if $\sum_{f \in \Delta} \sigma_{v,f} \bar{x}_f \geq 0$, with sign -1 otherwise. Let $\Delta^{split(\Delta)}$ be the set of these new loops in $G^{split(\Delta)}$ and let $F^{split(\Delta)} = F \cap E(G^{split(\Delta)}) \cup \Delta^{split(\Delta)}$. Let $A^{split(\Delta)} = A(G^{split(\Delta)}, F^{split(\Delta)})$. Let $\bar{x}^{split(\Delta)} \in \mathbb{R}^{E(G^{split(\Delta)})}$ be obtained from \bar{x} by removing the components corresponding to the edges in Δ , and by setting, for every loop l_v in $\Delta^{split(\Delta)}$, $\bar{x}_{l_v}^{split(\Delta)} = \sum_{f \in \Delta} \sigma_{v,f} \bar{x}_f$.

Claim 4.13. *Let $\Delta \subseteq F$ be such that $\Delta \setminus L(G) \neq \emptyset$, the graph induced by Δ is connected, and $(G^{split(\Delta)}, F^{split(\Delta)})$ does not contain G_4 as a minor. Then:*

- (i) $\Delta \cap L(G) = \emptyset$;
- (ii) $G \setminus \Delta$ is connected;
- (iii) *If Δ is a star centered at a node v , then $\bar{x}^{split(\Delta)} = \lambda y + (1 - \lambda)z$, where $0 < \lambda < 1$, where y, z are integral and satisfy $A^{split(\Delta)} x' = c, x' \geq 0, x'_e \leq 1, \forall e \in E(G^{split(\Delta)}) \setminus \{l_v\}$. Moreover for every set $U \in \mathcal{L}$, $|\delta(U) \setminus F| = 2$.*
- (iv) *If $\Delta = \{f\}$, with $f = vw \in F \setminus L(G)$, then \bar{x} is half-integral.*

Proof of claim. Notice that $(G^{split(\Delta)}, F^{split(\Delta)})$ is in \mathcal{C} . Since $|V(G^{split(\Delta)})| = |V(G)|$, and $|E(G^{split(\Delta)}) \setminus L(G^{split(\Delta)})| < |E(G) \setminus L(G)|$, it follows that the system $A^{split(\Delta)} x' = c, x' \geq 0$ has Chvátal rank at most 1. Notice that $A^{split(\Delta)}$ is obtained from A by removing the columns relative to the edges in Δ , and by adding columns, relative to the loops in $\Delta^{split(\Delta)}$, where the

column relative to a loop l_v incident with a node v is zero everywhere except the entry relative to v with value $2\sigma_{v,l_v}$. Notice that the space spanned by the columns of $A^{split(\Delta)}$ contains the space spanned by the columns of A .

We show that there are $|E(G)|$ linearly independent inequalities valid for the first closure of $A^{split(\Delta)}x' = c$, $x' \geq 0$ tight at $\bar{x}^{split(\Delta)}$. Let $\alpha^s x \geq \beta^s$, $s = 1, \dots, t$, be the odd-cuts inequalities for (4.2) tight at \bar{x} . Since \bar{x} is a vertex and $0 < \bar{x} < 1$ (by Claim 4.3), the system $Ax = c$, $\alpha^s x \geq \beta^s$, $s = 1, \dots, t$, has rank $|E(G)|$. Notice that $A^{split(\Delta)}\bar{x}^{split(\Delta)} = c$, $\alpha^s \bar{x}^{split(\Delta)} = \beta^s$, $s = 1, \dots, t$. Furthermore, since the space spanned by the columns of $A^{split(\Delta)}$ contains the space spanned by the columns of A , the system $A^{split(\Delta)}x' = c$, $\alpha^s x' \geq \beta^s$, $s = 1, \dots, t$ has rank at least $|E(G)|$.

Therefore $|E(G^{split(\Delta)})| \geq |E(G)|$, thus $|\Delta^{split(\Delta)}| \geq |\Delta|$. Also, since the graph induced by Δ is connected, by Claim 4.4, $\Delta \setminus L(G)$ forms a tree. Hence $|\Delta^{split(\Delta)}| = |\Delta \setminus L(G)| + 1$, so $|L(G) \cap \Delta| \leq 1$. Since every tree has at least two leafs, there exists a loop l in $\Delta^{split(\Delta)}$ such that $\bar{x}_l^{split(\Delta)}$ is fractional.

(i): If there is one loop in $\Delta \cap L(G)$, then $|E(G)| = |E(G^{split(\Delta)})|$. Hence $\bar{x}^{split(\Delta)}$ is a fractional vertex of the first closure of $A^{split(\Delta)}x' = c$, $x' \geq 0$, a contradiction by our minimality assumption. Therefore there are no edges in $\Delta \cap L(G)$.

(ii): If $G \setminus \Delta$ is not connected, let G' be a connected component of $G^{split(\Delta)}$ and let G'' be the union of all the other connected components of $G^{split(\Delta)}$. Let $F' = F^{split(\Delta)} \cap E(G')$, $F'' = F^{split(\Delta)} \cap E(G'')$, $\Delta' = \Delta^{split(\Delta)} \cap E(G')$, $\Delta'' = \Delta^{split(\Delta)} \cap E(G'')$. Let $A' = A(G', F')$ and $A'' = A(G'', F'')$. Let \bar{x}' (resp. \bar{x}'') be the restriction of $\bar{x}^{split(\Delta)}$ to the edges of G' (resp. G''). Let c' and c'' be the restriction of c to G' and G'' respectively. Notice that by Claim 4.5, $E(G') \setminus \Delta^{split(\Delta)} \neq \emptyset$, $E(G'') \setminus \Delta^{split(\Delta)} \neq \emptyset$, hence both \bar{x}' and \bar{x}'' have at least a fractional component. Consider the systems

$$A'x' = c' \quad , \quad x' \geq 0 \tag{4.11}$$

$$A''x'' = c'' \quad , \quad x'' \geq 0 \tag{4.12}$$

Clearly (G', F') and (G'', F'') are in \mathcal{C} . Since $|V(G')| < |V(G)|$ and $|V(G'')| < |V(G)|$, both systems (4.11) and (4.12) have Chvátal rank at most 1.

Since \bar{x} is a vertex of the first closure of (4.2), there are $|E(G)|$ linearly independent inequalities valid for the first closure tight at \bar{x} . By Lemma 4.9, and since $\bar{x} > 0$, such inequalities are either in the system $Ax = c$, or they are odd-cut inequalities for (4.2). By Lemma 4.9, given an irredundant odd-cut inequality, say relative to some set $U \subseteq V(G)$, U cannot be partitioned into two nonempty sets U_1 and U_2 such that every edge between U_1 and U_2 is in F .

Therefore either $U \subseteq V(G')$ or $U \subseteq V(G'')$, and the corresponding odd-cut inequality is valid either for the first closure of (4.11) or (4.12), respectively. Notice that each equation of $Ax = c$ corresponding to a node incident with no edge in Δ is an equation of either $A'x' = c'$ or $A''x'' = c''$. Moreover, to each equation of $Ax = c$ corresponding to a node v incident with some edges in Δ , corresponds an equation of either $A'x' = c'$ or $A''x'' = c''$ which is linearly independent from all the other equations of $Ax = c$ and all the odd-cut inequalities of either (4.11) or (4.12), since it is the only one that has a coefficient different from zero for the loop in Δ' or Δ'' incident with v . Therefore the $|E(G)|$ inequalities valid for the first closure of either (4.11) or (4.12), that correspond to the $|E(G)|$ linearly independent inequalities valid for the first closure of (4.2) tight at \bar{x} , are all linearly independent.

Since Δ is connected, $|E(G)| \geq |E(G')| + |E(G'')| - 1$. Therefore either \bar{x}' is a vertex of the first closure of (4.11), or \bar{x}'' is a vertex of the first closure of (4.12), then either \bar{x}' or \bar{x}'' is integral. In particular, $\bar{x}_e = \bar{x}'_e$ for an edge $e \in E(G') \setminus \Delta'$ or $\bar{x}_e = \bar{x}''_e$ for an edge $e \in E(G'') \setminus \Delta''$ is integer, contradicting Claim 4.3.

(iii): Now assume that Δ is a star centered at node $v \in V(G)$. Notice that $\bar{x}^{split(\Delta)}$ satisfies the system

$$A^{split(\Delta)}x' = c, x' \geq 0, x'_f \leq 1, \forall f \in F^{split(\Delta)} \setminus \{l_v\}, \quad (4.13)$$

that has Chvátal rank at most 1 by Lemma 4.10. By Lemma 4.8, the odd-cut inequalities for the latter system are the odd-cut inequalities for $A^{split(\Delta)}x' = c, x' \geq 0$. Moreover $|E(G)| = |E(G^{split(\Delta)})| - 1$. Hence $\bar{x}^{split(\Delta)}$ is contained in a face of dimension 1 of the first closure of $A^{split(\Delta)}x' = c, x' \geq 0, x'_f \leq 1, \forall f \in F^{split(\Delta)} \setminus \{l_v\}$, thus it is a convex combination of two integral vertices, y and z , satisfying the latter system, i.e. $\bar{x}^{split(\Delta)} = \lambda y + (1 - \lambda)z$ where $0 < \lambda < 1$. Moreover, by Claim 4.3, each edge in $E(G^{split(\Delta)}) \setminus F^{split(\Delta)}$ is in $\delta(U)$ for an odd-cut inequality of the latter system defined by a set U , tight at $\bar{x}^{split(\Delta)}$.

It follows that every edge in $E(G^{split(\Delta)}) \setminus F^{split(\Delta)} = E(G) \setminus F$ is in $\delta(U')$ for an odd-cut inequality of (4.13) defined by a set U' , tight at $\bar{x}^{split(\Delta)}$. It follows that $y_e, z_e \in \{0, 1\}$ for every e in $E(G^{split(\Delta)}) \setminus \{l_v\}$ and so y and z satisfy the system $A^{split(\Delta)}x' = c, x' \geq 0, x'_e \leq 1, \forall e \in E(G^{split(\Delta)}) \setminus \{l_v\}$. Since also y and z satisfy (4.13), and satisfy tightly all the odd-cut inequalities of (4.13) satisfied tightly by $\bar{x}^{split(\Delta)}$, it follows that for every set $U \in \mathcal{L}$, $|\delta(U) \setminus F| = 2$.

(iv): Now assume $\Delta = \{f\}$, with $f = vw \in F \setminus L(G)$, and let f_v and f_w be the loops in $F^{split(f)}$ incident with v and w respectively. In this case notice that $\bar{x}_{f_v}^{split(f)} = \bar{x}_{f_w}^{split(f)} = \bar{x}_f$. By applying the argument in (iii) to the system $A^{split(f)}x' = c, x' \geq 0, x'_f \leq 1, \forall f \in F^{split(f)}$, it follows that

$\bar{x}^{split(\Delta)} = \lambda y + (1 - \lambda)z$, where $0 < \lambda < 1$, where y, z are integral and satisfy $A^{split(\Delta)}x' = c, x' \geq 0, x' \leq 1$. By Claim 4.3, $\bar{x}_f \notin \mathbb{Z}$. Hence we can assume $y_{f_v} = 0$, and $z_{f_v} = 1$. Notice that $y_{f_w} = 1$ and $z_{f_w} = 0$. If not, let $\bar{y}, \bar{z} \in \mathbb{Z}^{E(G)}$ be obtained from y, z , respectively, by replacing the two components relative to f_v, f_w with one component relative to f with value $\bar{y}_f = y_{f_v} = y_{f_w}, \bar{z}_f = z_{f_v} = z_{f_w}$, respectively. Then $\bar{x} = \lambda \bar{y} + (1 - \lambda)\bar{z}$, a contradiction. Hence $\bar{x}_{f_v}^{split(f)} = 1 - \lambda$ and $\bar{x}_{f_w}^{split(f)} = \lambda$. Since $\bar{x}_{f_v}^{split(f)} = \bar{x}_{f_w}^{split(f)} = \bar{x}_f$, then $\lambda = 1/2$. Hence $\bar{x}^{split(f)}$ is half-integral and so \bar{x} is half-integral. \diamond

Claim 4.14. *Let $f = vw$ and $f' = vw'$ be distinct edges in $F \setminus L(G)$ such that, for any two distinct nodes $s, t \in \{v, w, w'\}$ there exists a path in $G \setminus F$ between s and t that does not pass through $\{v, w, w'\} \setminus \{s, t\}$. Then v is a cutnode of G .*

Furthermore, if \bar{G} is the subgraph of G induced by the node v and the connected component of $G \setminus v$ containing w and w' ,

- (i) $E(\bar{G}) \cap F$ is a star centered at v ;
- (ii) \bar{G} is bipartite;
- (iii) $\bar{G} \setminus F$ is connected.

Proof of claim. In this case, by applying Claim 4.10 to both f and f' , and by noticing that there are no cycles in \bar{F} by Claim 4.4, it follows that each edge in $F \setminus L(G)$ in \bar{G} different from f and f' , is either incident with v , or is nested in f and in f' . We show that there are no edges in $F \setminus L(G)$ nested in f and in f' and not incident with v . Let $f'' = v''w''$ be such an edge, f'' is not adjacent to f or f' . Let v'' be an endnode of f'' such that there exists a path in $E(G) \setminus F$ from v'' to w that does not pass through w'' . Now let (G', F') be obtained from (G, F) by removing all the edges incident with v and different from f and f' , and by contracting f . By considering a path in (G', F') from w to v'' that does not pass through w' and w'' and a path in (G', F') from w' to v'' that does not pass through w and w'' , we get G_4 as a minor. Hence all the edges in $F \setminus L(G)$ in \bar{G} are incident with v .

By Claim 4.11, each edge in F with exactly an endnode in $V(\bar{G})$ is incident with v . Moreover, by Claim 4.12, all the cycles in \bar{G} are even. Thus $E(\bar{G}) \cap F$ is a star centered at v , and \bar{G} is bipartite.

By applying Claim 4.13 (ii) to the set of edges in $E(\bar{G}) \cap F \setminus L(\bar{G})$ it follows that $\bar{G} \setminus F$ is connected.

Now we show that v is a cutnode of $G \setminus F$. By contradiction assume that v is not a cutnode of $G \setminus F$. In this case \bar{G} is the connected component of $G \setminus F$ that contains v, w, w' . In particular Δ is an edge-cutset of G , and

$(G^{split(F)}, F^{split(F)}) \in \mathcal{C}$. Let Δ be the family of edges in F with exactly an endnode in $V(\bar{G})$. Notice that the edges in Δ are all incident with v . By Claim 4.13 (ii), $\bar{G} = G$. Since $(G^{split(F)}, F^{split(F)})$ does not contain G_4 as a minor, and the graph induced by F is connected, Claim 4.13 (i) implies that there are no loops in F . By Claim 4.7, \bar{x} is half-integral. Thus by Claim 4.8, (G, F) satisfies conditions a) and b) of Remark 4.14. By Lemma 4.18, there exists a balanced bipartition R, B of the edges in $E(G)$. Now let $y = \bar{x} + 1/2\chi(R) - 1/2\chi(B)$ and let $z = \bar{x} - 1/2\chi(R) + 1/2\chi(B)$. y and z are integral, they satisfy (4.2) and $\bar{x} = 1/2(y + z)$, a contradiction.

Thus v is a cutnode of $G \setminus F$. Since each edge in F with exactly an endnode in $V(\bar{G})$ is incident with v , then v is also a cutnode of G . \diamond

Claim 4.15. *If $f \in F \setminus L(G)$ has both endnodes in the same connected component of $G \setminus F$, at least one endnode of f is a cutnode of $G \setminus F$.*

Proof of claim. Let v and w be the endnodes of f , and let \bar{G} be the connected component of $G \setminus F$ containing v, w . By contradiction assume that v and w are not cutnodes of $G \setminus F$. If there is another edge $f' \in F \setminus L(G)$ with both endnodes in \bar{G} , then by Claim 4.4, f' is not parallel to f . By Claim 4.10, f and f' are in case (i), (ii), or (iii) of such Claim. Case (iii) cannot happen because v, w are not cutnodes of $G \setminus F$. Case (i) cannot happen, as otherwise we contradict Claim 4.14 (i) because v, w are not cutnodes of $G \setminus F$. Thus case (ii) applies, and since v and w are not cutnodes of $G \setminus F$, f' is nested in f . Hence all the edges in $F \setminus L(G)$ with both endnodes in \bar{G} different from f are nested in f .

By Claim 4.11, all the edges in F with exactly one endnode in \bar{G} are adjacent to f . Notice that such edges in F with exactly one endnode in \bar{G} are of two types, the loops in \bar{G} , and the edges in $F \setminus L(G)$ with exactly one endnode in \bar{G} .

We now show that there are no edges in F with exactly one endnode in \bar{G} . Otherwise let $\Delta \subseteq F$ contain such edges and f . Notice that, since $\Delta \setminus \{f\}$ is a cut of G in F , and by the structure of the edges in F with endnodes in \bar{G} , then $(G^{split(\Delta)}, F^{split(\Delta)})$ does not contain G_4 as a minor. By Claim 4.13 (i), all the edges in F with exactly one endnode in \bar{G} are not in $L(G)$. Thus we contradict Claim 4.13 (ii), since $G \setminus \Delta$ is not connected. Thus $G \setminus F$ is connected, $\bar{G} = G$, and $L(G) \cap F = \emptyset$. Let P be a path in $G \setminus F$, from v to w . By the above discussion, P passes through all the nodes of G incident with some edge in F .

By applying Claim 4.13 (iv) to $\Delta = \{f\}$, \bar{x} is half-integral. Thus by Claim 4.8, (G, F) satisfies conditions a) and b) of Remark 4.14. Moreover, by Claim 4.12, all the cycles in $G \setminus F$ are even, thus G is bipartite.

By Lemma 4.19, there exists a partition of the edges in $E(G) \setminus F$ in one closed trail, if $L(G) \subseteq F$, or, if $L(G) \setminus F \neq \emptyset$, in $|L(G) \setminus F|/2$ trails such that their first and last edge are in $L(G) \setminus F$, such that one of the trails passes through all the nodes incident with edges in F .

Notice that such partition satisfies conditions (C1)-(C3) of Lemma 4.16.

Hence by Lemma 4.16 there exists a balanced bipartition R, B of $E(G)$. Now let $y = \bar{x} + 1/2\chi(R) - 1/2\chi(B)$ and let $z = \bar{x} - 1/2\chi(R) + 1/2\chi(B)$. y and z are integral, they satisfy (4.2) and $\bar{x} = 1/2(y + z)$, a contradiction. \diamond

For any block B of G containing a cut in F and any connected component Q of $B \setminus F$, we define the graph $H_{Q,B}$ as follows. Its node set denoted by $V_{Q,B}$ is the set of nodes in Q incident with edges in $\delta_B(Q) \setminus L(Q)$, and two nodes v and w are adjacent in $H_{Q,B}$ if and only if there exists a path from v to w in Q all of whose intermediate nodes are not in $V_{Q,B}$.

Claim 4.16. *Assume that $G \setminus F$ is not connected, let B be a block of G containing a cut in F , and let Q be a connected component of $B \setminus F$. Let (G', F') be the pair obtained from (G, F) by deleting all the nodes in $V(G) \setminus V(Q)$. Then we have:*

- (i) *the nodes in $V_{Q,B} = \{v_1, \dots, v_k\}$ can be ordered in such a way that v_i is a cut-node of G' that disconnects v_{i-1} and v_{i+1} for every $i = 2, \dots, k-1$;*
- (ii) *if vw is in $\delta_B(Q) \setminus L(Q)$ and $v \in \{v_2, \dots, v_{k-1}\}$, then $\{v, w\}$ is a node-cutset of B separating v_1 from v_k .*

Proof of claim. First, we prove that $H_{Q,B}$ is a tree. Suppose by contradiction that $H_{Q,B}$ contains a cycle C . Let v, v' and v'' be three nodes in C . By definition of $H_{Q,B}$, there exists a path, say $P_{v,v'}$, in Q from v to v' that does not contain v'' . Similarly, there exists a path, say $P_{v',v''}$, in Q from v' to v'' that does not contain v . In the same way, we define a path $P_{v,v''}$. Let us denote by V_C the set of nodes belonging to the paths $P_{v,v'}$, $P_{v',v''}$, and $P_{v,v''}$. Let w, w' and w'' be nodes in B such that $vw, v'w'$ and $v''w'' \in \delta_B(Q) \setminus L(Q)$. From the definition of Q , it follows that these edges belong to F .

Suppose that w, w' and w'' are distinct nodes. Since B is not a cutedge of G , B is 2-connected. So we consider a shortest path P in $B \setminus \{v\}$ from w to the node set $(V_C \setminus \{v\}) \cup \{w', w''\}$. W.l.o.g, we may assume that the nodes v' and w' are not in P (otherwise v'' and w'' are not in P). Let P' be the path $vw, P, w''v''$ if P ends in w' , and vw, P otherwise. By deleting w' and considering the subgraph consisting of a loop in F at v' and the paths $P_{v,v'}$, $P_{v',v''}$ and P' , we see that B contains the minor G_4 , a contradiction. Similarly, if exactly two of the nodes w, w' and w'' are identical, we can see the minor G_4 .

Now we may suppose that $w = w' = w''$. From Claim 4.13 (ii) it follows that there exists an edge $(\bar{v}, \bar{w}) \in \delta_B(Q) \setminus L(Q)$, where $\bar{v} \in Q$ and $\bar{w} \neq w$. Then consider a shortest path $P_{\bar{v}}$ in Q from \bar{v} to the set V_C . If $P_{\bar{v}}$ ends either in v , in v' , or in v'' , then we may assume that it ends in v' and by deleting \bar{w} and considering the subgraph consisting of a loop in F at \bar{v} and the paths $P_{\bar{v}}, P_{v,v'}, P_{v',v''}$ and the edges vw and $v''w$, we see that B contains the minor G_4 , a contradiction. Otherwise, we may assume that $P_{\bar{v}}$ ends in a node of $P_{v,v'}$ different from v and v' . Then, by deleting \bar{w} and considering the subgraph consisting of a loop in F at \bar{v} and the paths $P_{\bar{v}}, P_{v,v'}$ and the edges vw and $v'w$, we see the minor G_4 , a contradiction. This terminates the proof that $H_{Q,B}$ is a tree.

Now we show that the graph $H_{Q,B}$ is a path. Suppose by contradiction that $H_{Q,B}$ has a node v_0 of degree at least 3. Let v, v' and v'' be three neighbors of v_0 in $H_{Q,B}$. As above, we can define the paths $P_{v,v'}, P_{v',v''}$ and $P_{v,v''}$, and obtain a contradiction. Hence, the graph $H_{Q,B}$ is a path.

Let us prove statement (i). Let $2 \leq i \leq k-1$. Since $H_{Q,B}$ is a path, the sets of nodes $\{v_1, \dots, v_{i-1}\}$ and $\{v_{i+1}, \dots, v_k\}$ are contained in two different connected components of $Q \setminus \{v_i\}$, say K_1 and K_2 respectively. Suppose by contradiction that v_i does not disconnect v_{i-1} and v_{i+1} in G' . Then there exists a path in $G' \setminus \{v_i\}$ joining K_1 and K_2 with at least one edge in F . As $v_i \in V_{Q,B}$, there exists an edge $v_iw \in \delta_B(Q)$. By deleting w , we can see that B contains the minor G_4 , a contradiction.

Let us prove statement (ii). Let $e = vw \in \delta_B(Q) \setminus L(Q)$ for some $v \in \{v_2, \dots, v_{k-1}\}$. Suppose by contradiction that there exists a path P from v_1 to v_k in the graph $B \setminus \{v, w\}$. Let P' be a path in Q from v_1 to v_k . Statement (i) implies that there exists a cycle in P, P' containing v and an edge in F . By deleting w and considering the latter cycle, we see that B contains the minor G_4 , a contradiction. \diamond

Claim 4.17. *Assume that $G \setminus F$ is not connected, let B be a block such that $B \setminus F$ is not connected, and let Q be a connected component of $B \setminus F$. Let $V_{Q,B} = \{v_1, \dots, v_k\}$ be defined as in statement Claim 4.16 (i). Let V'' be the set of nodes $w \in V(Q)$ such that in Q there exist a path from w to v_1 that does not pass through v_k , and a path from w to v_k that does not pass through v_1 . Let (G'', F'') be the subgraph of G induced by the nodes in V'' . Then:*

- (i) *for any $1 \leq l, l' \leq k$, there exists a path in B from v_l to $v_{l'}$ that does not contain any node in $V(Q) \setminus \{v_l, v_{l'}\}$;*
- (ii) *each edge in $F'' \setminus L(G'')$ has endnodes in $\{v_1, v_k\} \cup \{v \in V'' : v \text{ is a cutnode of } G \setminus F \text{ separating } v_1 \text{ and } v_k\}$;*

- (iii) for each pair of edges $f = vw, f' = v'w'$ in $F'' \setminus L(G'')$, either f and f' are nested, or one among v and w , say v is a cutnode of $G'' \setminus F''$ separating w from $\{v', w'\} \setminus \{v\}$;
- (iv) each edge in $F \setminus W_B$ with exactly one endnode in V'' is incident with v_1 or v_k ;
- (v) all the cycles in G'' are even.

Proof of claim. Let us prove (i). For every edge $f = v_iw \in \delta_B(Q) \setminus L(Q)$, $2 \leq i \leq k-1$, we define G_f as the subgraph of G induced by the connected component of $G \setminus \{v_i, w\}$ containing v_1 , and by the nodes $\{v_i, w\}$. Since B is 2-connected, we observe that G_f is 2-connected. For any two distinct edges $f = v_iw$ and $f' = v_{i'}w'$ in $\delta_B(Q) \setminus L(Q)$, $2 \leq i \leq i' \leq k-1$, we have that $G_f \subset G_{f'}$ or $G_{f'} \subset G_f$, where $G_f \subset G_{f'}$ if $i < i'$. Indeed, by Claim 4.16 (ii) applied to f and f' , either v_1 and f are in $G_{f'}$ in which case $G_f \subset G_{f'}$, or v_1 and f' are in G_f in which case $G_{f'} \subset G_f$. Moreover, if $G_f \subset G_{f'}$, then $\{v_i, w\}$ is a node-cutset of $G_{f'}$ separating v_1 from any endnode of f' that is not an endnode of f .

Let $v_1w_0 \in \delta_B(Q) \setminus L(Q)$ for some node w_0 . If $k = 2$, then since B is 2-connected there exists a path from w_0 to v_k that does not pass through v_1 and so it contains no node in $V(Q) \setminus v_2$, which implies (i).

Now assume $k \geq 3$. Let f_1, \dots, f_s be the edges in $\delta_B(Q) \setminus L(Q)$ incident with one of the nodes v_2, \dots, v_{k-1} . We may assume that $G_{f_i} \subset G_{f_{i+1}}$ for $i = 1, \dots, s-1$. Let w_i be the endnode of f_i in $G \setminus Q$ for $i = 1, \dots, s$. Notice that $f_1 = v_2w_1$, $f_s = v_{k-1}w_s$ and $w_0 \in G_{f_1}$. Since G_{f_1} is 2-connected, there exists a path P_0 from w_0 to w_1 in $G_{f_1} \setminus v_1$. Since v_2w_1 is the unique edge in $\delta_B(Q) \setminus L(Q)$ and in G_{f_1} , P_0 does not contain any node in $V(Q) \cup \{v_2\}$. Let $v_kw_{s+1} \in \delta_B(Q) \setminus L(Q)$. As before, we can show that there exists a path P_s from w_s to w_{s+1} in G that does not contain any node in $V(Q) \cup V(G_{f_s}) \setminus \{w_s\}$.

Now we show that for $i = 1, \dots, s-1$, there exists a path P_i from w_i to w_{i+1} in $G_{f_{i+1}}$ not containing any node of Q . Let $1 \leq i \leq s-1$, $v_j, v_{j'}$ such that $f_i = v_jw_i$ and $f_{i+1} = v_{j'}w_{i+1}$. Since $\{v_j, w_i\}$ is a node-cutset of $G_{f_{i+1}}$ separating the node-set $\{v_1, w_0, \dots, w_{i-1}\} \setminus w_i$ from $\{v_{j'}, w_{i+1}\} \setminus \{v_j, w_i\}$, notice that the subgraph H of G induced by the nodes in $V(G_{f_{i+1}}) \setminus V(G_{f_i}) \cup \{v_j, w_i\}$ is 2-connected and f_i and f_{i+1} are the only edges in $E(H) \cap (\delta_B(Q) \setminus L(Q))$. This implies that there exists a path P_i from w_i to w_{i+1} in H not containing any node of Q .

Therefore, for any $1 \leq l, l' \leq k$, choose w_i and $w_{i'}$ ($0 \leq i, i' \leq s+1$) such that $v_lw_i, v_{l'}w_{i'} \in \delta_B(Q) \setminus L(Q)$. By considering the path $v_l, v_lw_i, P_i, \dots, P_{i'-1}, v_{i'}w_{i'}, v_{l'}$, this proves (i).

Now we show (ii). Let $f = vw \in F'' \setminus L(G'')$. Suppose first that v, w are both distinct from v_1 . Then, by Claim 4.11 applied to the edges f, v_1w_0 , we may assume that v is a cutnode of $G'' \setminus F''$ separating v_1 and w . If $w = v_k$, then we are done. So assume $w \neq v_k$. If w does not separate v and v_k , then by deleting w and considering a path from v_1 to v_k that contains no node in $V(Q) \setminus \{v_1, v_k\}$ (by (i)), and a path from v_1 to v_k passing through v in $Q \setminus w$, we get the minor G_4 , a contradiction. Thus w is a cut-node of $G'' \setminus F''$ separating v and v_k . It follows that v and w are cut-nodes of $G'' \setminus F''$ separating v_1 and v_k .

Now we may assume $v = v_1$ and $w \neq v_1, v_k$. By applying Claim 4.11 to the edges f, v_kw_{s+1} , we have that w is a cut-node of $G'' \setminus F''$ separating v_1 and v_k . Hence w is a cut-node of $G \setminus F$ separating v_1 and v_k . This concludes the proof of (ii).

Let us show (iii). We observe that case (i) of Claim 4.10 does not hold. In fact otherwise, we can assume by symmetry $v = v'$, then from (ii) it follows that one of the nodes v, w, w' separates the other two in $G'' \setminus F''$, a contradiction. Then (iii) directly follows from Claim 4.10.

We show that statement (iv) holds. Otherwise, let f be an edge in $F \setminus W_B$ with exactly one endnode, say v , in V'' , such that f is not incident with v_1 or v_k . Notice that by (i) there exists a path P_1 in B from v_1 to v_k that does not contain any node in $V(Q) \setminus \{v_1, v_k\}$. By considering P_1 , and a path P_2 in $G \setminus F$ from v_1 to v_k that passes through v , we get the minor G_4 .

Finally, we prove (v). Suppose by contradiction that there exists an odd cycle C in G'' . If $v_1, v_k \notin V(C)$, then by contracting all the edges of C , we get a new loop in F , and now we can get the minor G_4 as in the above proof of statement (iv). So we may assume by symmetry that $v_1 \in V(C)$. Let $v \in V(C) \setminus \{v_1\}$. By definition of G'' , there exists a path from v to v_k in $G'' \setminus F''$ that does not pass through v_1 . We may assume that such a path does not contain any node in $V(C) \setminus v$ (otherwise we consider some subpath). Then, using (i), we can find an odd cycle containing an edge in F , a contradiction. \diamond

4.4.3 \bar{x} is half-integral

In the following two claims we show that $\bar{x}_e = 1/2$ for every e in $E(G)$. In Claim 4.18 we show this in the case $G \setminus F$ connected. In Claim 4.19 we show this in the remaining case $G \setminus F$ not connected, and, for every maximal set \bar{V} of nodes connected in $G \setminus F$, we give the structure of the nodes in \bar{V} incident with edges in $\delta(V) \setminus L(G)$.

Claim 4.18. *If $G \setminus F$ is connected, $\bar{x}_e = 1/2$ for every e in $E(G)$.*

Proof of claim. Assume that for every two edges $f = vw$ and $f' = v'w'$ in $F \setminus L(G)$, either f and f' are nested, or one among v and w , say v is a cutnode of $G \setminus F$ separating w from $\{v', w'\} \setminus \{v\}$. Then then let f be an edge in $F \setminus L(G)$ that is not nested in any other edge in $F \setminus L(G)$. Notice that $(G^{split(f)}, F^{split(f)})$ does not contain G_4 as a minor, hence, by Claim 4.13 (iv), \bar{x} is half-integral.

Otherwise, by Claim 4.10, in G there are two edges $f = v_0v_1$ and $f' = v_0v_2$ in $F \setminus L(G)$ such that for any two distinct nodes $s, t \in \{v_0, v_1, v_2\}$ there exists a path in $G \setminus F$ between s and t that does not pass through $\{v_0, v_1, v_2\} \setminus \{s, t\}$. Let \bar{G} is the subgraph of G induced by the node v_0 and the connected component of $\bar{G} \setminus v_0$ containing v_1 and v_2 . By Claim 4.14, v_0 is a cutnode of G , $E(\bar{G}) \cap F$ is a star centered at v_0 , and \bar{G} is bipartite.

Let $\Delta = E(\bar{G}) \cap F$. Notice that $(G^{split(\Delta)}, F^{split(\Delta)})$ is in \mathcal{C} . Hence by Claim 4.13 (iii), $\bar{x}^{split(\Delta)} = \lambda y'' + (1 - \lambda)z''$, where $0 < \lambda < 1$, and where y'' and z'' are integral and satisfy

$$A^{split(\Delta)}x'' = c, x'' \geq 0, x''_e \leq 1, \forall e \in E(G^{split(\Delta)}) \setminus \{l_{v_0}\},$$

where l_{v_0} is the edge in $\Delta^{split(\Delta)}$ incident with v_0 . Moreover for every set $U \in \mathcal{L}$, $|\delta(U) \setminus F| = 2$.

It follows that $y''_e, z''_e \in \{0, 1\}$ for every e in $E(G^{split(\Delta)}) \setminus \{l_{v_0}\}$ and so $\bar{x}''_e \in \{\lambda, 1 - \lambda\}$ for every e in $E(G^{split(\Delta)}) \setminus \{l_{v_0}\}$. Hence $\bar{x}_e \in \{\lambda, 1 - \lambda\}$ for every e in $E(G)$, since for every edge in $E(G)$ there exists an edge in $E(G^{split(\Delta)}) \setminus \{l_{v_0}\}$ with its same value. If $\lambda = 1/2$ we are done, so assume $\lambda \neq 1/2$. Let y and z be the following vectors in $\{0, 1\}^{E(G)}$, $y_e = 1 - z_e = \begin{cases} 1 & \text{if } \bar{x}_e = \lambda \\ 0 & \text{otherwise} \end{cases}$. By the above discussion, $(Ay)_s = (Az)_s = c_s$ for every $s \neq v_0$.

If $V(G) \notin \mathcal{L}$, by Claim 4.6 \bar{x} is half-integral, hence we assume $V(G) \in \mathcal{L}$ and so $|L(G) \setminus F| = 2$, say $L(G) \setminus F = \{l_1, l_2\}$. We show that there exists an edge $\bar{e} = vw$ in $E(G) \setminus (F \cup L(G))$ such that there is only one set U in \mathcal{L} with $\bar{e} \in \delta(U)$. If not, by Claim 4.3, for every edge $e \in E(G) \setminus (F \cup L(G))$ there are at least two sets U_1^e, U_2^e in \mathcal{L} with $e \in \delta(U_1^e)$, $e \in \delta(U_2^e)$. Now consider the undirected graph Γ whose vertex set is $E(G) \setminus F$ and where two elements e_1, e_2 in $V(\Gamma)$, that correspond to two edges $e_1, e_2 \in E(G) \setminus F$, are adjacent in Γ if and only if there exists a set $U \in \mathcal{L}$ with $e_1, e_2 \in \delta(U) \setminus F$. If Γ contains a cycle C , then the odd-cuts in \mathcal{L} that correspond to the edges

of Γ in C can be written in the form

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

where k is the number of edges in C . Since all the odd-cut inequalities corresponding to sets in \mathcal{L} are linearly independent, k is odd, and the only solution of this system is $x_1 = \cdots = x_k = 1/2$, hence $\lambda = 1/2$ and $\bar{x}_e = 1/2$ for every $e \in E(G)$. Hence we assume that Γ does not contain any cycle. Notice that Γ has no loops and no parallel edges. Note that the edge-node incidence matrix of Γ is the constraint matrix of the system of odd-cut inequalities corresponding to elements in \mathcal{L} . Notice that by assumption all the nodes of Γ except the two corresponding to l_1, l_2 have degree at least two, and the nodes corresponding to l_1, l_2 have degree at least one by Claim 4.3, then Γ is a path from the node of Γ corresponding to l_1 to the node of Γ corresponding to l_2 . But, since $V(G) \in \mathcal{L}$, the nodes of Γ corresponding to l_1 and l_2 are adjacent, hence Γ contains only one edge. Hence $\mathcal{L} = \{V(G)\}$. Since $G \setminus F$ is connected, and since there exists $U \in \mathcal{L}$ such that $e \in \delta(U)$ for every $e \in E(G) \setminus F$ (by Claim 4.3), then G contains only one node. In this case it is clear that $A(G, F)$ has the Edmonds-Johnson property, a contradiction.

Now notice that, by switching signs on the endnodes of \bar{e} , we can assume that $\sigma_{v, \bar{e}} \neq \sigma_{w, \bar{e}}$. Now let (G', F') be obtained from (G, F) by contracting \bar{e} , and let r be the node obtained from the contraction of \bar{e} . Let $A' = A(G', F')$.

Let \bar{x}' be the vector obtained from \bar{x} by removing the component corresponding to the removed edges, and let c' be obtained from c by removing the components corresponding to v and w and adding a component relative to r with value $c'_r = c_v + c_w$. Since (G', F') is in \mathcal{C} , and $|V(G')| < |V(G)|$, then the system $A'x' = c', x' \geq 0$ has Chvátal rank at most 1. Notice that \bar{x}' satisfies the system $A'x' = c', x' \geq 0$, and that the equation $(A'x')_r = c'_r$ is the sum of $(Ax)_v = c_v$ and $(Ax)_w = c_w$. Furthermore, the odd-cut inequalities for $A'x' = c', x' \geq 0$ are exactly the odd-cut inequalities for (4.2) relative to sets $U \subseteq V(G)$ that either contain both v and w or none of them. Hence \bar{x}' is in the first closure of $A'x' = c', x' \geq 0, x'_f \leq 1$ for every $f \in F'$, which has Chvátal rank at most 1 by Lemma 4.10. Since all the odd-cut inequalities corresponding to sets in \mathcal{L} are of this form except for one, and since $\bar{x}_{\bar{e}} > 0$, then \bar{x}' satisfies at equality $|E(G)| - 2 = |E(G')| - 1$ linearly independent

inequalities among $A'x' = c'$, $x' \geq 0$, $x'_f \leq 1$ for every $f \in F'$ and the odd-cut inequalities corresponding to sets in \mathcal{L} .

Hence there exist two integral vertices of the first closure of the system $A'x' = c'$, $x' \geq 0$, $x'_f \leq 1$, $f \in F'$ such that $\bar{x}' = \lambda'y' + (1 - \lambda')z'$, where $0 < \lambda' < 1$. Notice that by Claim 4.3, $y'_e, z'_e \in \{0, 1\}$ for every e in $E(G)$. Hence by possibly switching y' with z' we can assume $\lambda' = \lambda$. This implies that for every $e \in E(G')$, $y'_e = y_e$, $z'_e = z_e$. Hence $(Ay)_s = (Az)_s = c_s$ for every $s \notin \{v, w\}$, $(A'y')_r = (Ay)_v + (Ay)_w = (Az)_v + (Az)_w = c_v + c_w$. Without loss of generality we can assume that $v \neq v_0$. But since we know that $(Ay)_s = (Az)_s = c_s$ for every $s \neq v_0$, hence $(Ay)_w = c_v + c_w - (Ay)_v = c_w$. Hence both y and z are integral, satisfy the system $Ax = c$, $x \geq 0$, and $\bar{x} = \lambda y + (1 - \lambda)z$, a contradiction. \diamond

Claim 4.19. *If $G \setminus F$ is not connected, then $\bar{x}_e = 1/2$ for every e in $E(G)$.*

Proof of claim. We denote by Q_1, \dots, Q_l the connected components of $G \setminus F$ that have nodes in B . By statement Claim 4.16 (i), let us denote by P_i a path in Q_i containing all nodes in $V_{Q_i, B}$ and having its end-nodes, say z_i^s and z_i^t , in $V_{Q_i, B}$.

We now show that there exists an edge in F whose end-nodes are equal to z_i^k and $z_{i'}^{k'}$ respectively, for some $1 \leq i, i' \leq l$, $i \neq i'$ and $k, k' \in \{s, t\}$.

Suppose that this is not true. For any edge $f = z_i^k w \in \delta_B(Q_i)$ for some $1 \leq i \leq l$ and $k \in \{s, t\}$, we define the subgraphs K_f^s and K_f^t of B as follows. We know that $w \in Q_j$ for some $j \neq i$, $w \neq z_j^s$ and $w \neq z_j^t$. Using Claim 4.16 (i), we denote by K_f^s and K_f^t the connected components of $B \setminus \{z_i^k, w\}$ containing z_j^s and z_j^t respectively. Let $f' = z_{i'}^{k'} w'$ such that $\min\{|V(K_{f'}^s)|, |V(K_{f'}^t)|\} = \min_{\substack{f \in \delta_B(Q_i) \\ 1 \leq i \leq l}} \{|V(K_f^s)|, |V(K_f^t)|\}$. W.l.o.g, we may assume that $|V(K_{f'}^s)| \leq |V(K_{f'}^t)|$ and $w \in Q_1$. Since $z_1^s \in K_{f'}^s$ is not adjacent to $z_{i'}^{k'}$ and $z_1^s \in V_{Q_1, B}$, there exists an edge $f'' = z_1^s w'' \in K_{f'}^s$ for some $w'' \notin Q_1$. Then, we may assume that $f' \notin K_{f''}^s$ (otherwise $f' \notin K_{f''}^t$). Since $\delta_B(K_{f'}^s) \subseteq \{z_{i'}^{k'}, w'\}$, it follows that $V(K_{f''}^s) \subsetneq V(K_{f'}^s)$, contradicting the definition of f' .

Let $f = z_i^k z_{i'}^{k'}$ for some $1 \leq i, i' \leq l$, $i \neq i'$ and $k, k' \in \{s, t\}$. We show that $(G^{split(f)}, F^{split(f)})$ does not contain G_4 as a minor, which implies that \bar{x} is half-integral by Claim 4.13.

Let f_i^k and $f_{i'}^{k'}$ be the new loops in $G^{split(f)}$ in z_i^k and $z_{i'}^{k'}$ respectively. By contradiction assume that $(G^{split(f)}, F^{split(f)})$ contains G_4 as a minor. Since (G, F) does not contain G_4 as a minor, by symmetry, we can assume that the loop of G_4 is f_i^k . Then there exists a cycle C that passes through $z_{i'}^{k'}$, a node $v \in Q_i$ incident with two edges in C not in $F^{split(f)}$, and a path P from

z_i^k to v that does not contain any node of C . Let $V_{Q_i,B} = \{v_1, \dots, v_r\}$. We may assume $z_i^k = v_1$. Since $z_{i'}^{k'} \notin Q_i$ and $v \in Q_i$, there exist two nodes v_l and $v_{l'}$ in $V_{Q_i,B} \cap C$, $1 < l < l' \leq r$, $v_l \neq v$ and $v_{l'} \neq v$, such that the paths in C from v to v_l and from v to $v_{l'}$ not containing $z_{i'}^{k'}$ are contained in the graph obtained from (G, F) by deleting all the nodes in $V(G) \setminus V(Q_i)$. Then v_l does not disconnect v_1 and $v_{l'}$ in such graph, contradicting statement Claim 4.16 (i). \diamond

4.4.4 Shrinkable pairs of edges

Assume that $G \setminus F$ is not connected, and let W be the set of edges in F with endnodes in different components of $G \setminus F$. For each block B of G , let $W_B = W \cap E(B)$. Two adjacent edges $uw, vw \in W_B$, are *consecutive* if there is no edge $rw \in W_B$ such that $\{r, w\}$ is a cutset of B separating u and v .

Let uw, vw be two edges in W incident with w . Notice that, by switching signs on u, v and w we can assume that $\sigma_{u,uw} = \sigma_{v,vw} = +1$ and that at least one among $\sigma_{w,uw}$ and $\sigma_{w,vw}$ is equal to $+1$. We say that (G', F') is obtained from (G, F) by *shrinking* uw and vw if $V(G') = V(G)$, $E(G') = E(G) \setminus \{uw, vw\} \cup \{uv\}$, $F' = F \setminus \{uw, vw\} \cup \{uv\}$, where the signing σ' on the edges of G' is defined by $\sigma'_{u,uv} = \sigma_{u,uw}$, $\sigma'_{z,e} = \sigma_{z,e}$ for every $e \in E(G') \setminus \{uv\}$, $z \in e$, and $\sigma'_{v,uv} = +1$ if $\sigma_{w,uw} \neq \sigma_{w,vw}$, $\sigma'_{v,uv} = -1$ if $\sigma_{w,uw} = \sigma_{w,vw}$. Notice that each cycle C' in G' that contains uv is even, since the corresponding cycles in G obtained from C' by removing the edge uv and by adding the two edges uw, vw are even, and since $\sigma_{u,uw} + \sigma_{w,uw} + \sigma_{v,vw} + \sigma_{w,vw} \equiv_4 \sigma'_{u,uv} + \sigma'_{v,uv}$. Thus (G', F') satisfies the cycles condition, and $A(G', F')$ is totally half-modular. By shrinking uw and vw , (G', F') may contain the minor G_4 , thus we say that two edges uw, vw in W are *shrinkable* if the graph obtained from (G, F) by shrinking uw and vw does not contain G_4 as a minor. Notice that, by shrinking uw and vw , for each block B' of G' , the nodes of B' are all contained in the same block of G .

Claim 4.20. *If there exists a node w in a block B of G such that w is incident with at least two edges in W_B , then there are two shrinkable edges in W_B incident with w .*

Proof of claim. Let $w \in B$ be a node incident with at least two edges in W_B . Let wu, vw be two consecutive edges in W_B , and let (G', F') be the pair obtained by shrinking wu, vw . Let B' be the block(s) corresponding to B in G' . If (G', F') does not contain G_4 as a minor we are done. Hence we assume that (G', F') contains G_4 as a minor.

Fact 1: In B' there exists a cycle C such that, up to switching the roles of u and v , then $v, w \in V(C)$, $u \notin V(C)$, v is incident with two edges in $E(C) \setminus F'$, and w is incident with at least one edge in $E(C) \cap F'$. Moreover, $\{v, w\}$ is a cutset of B .

Proof of fact 1: Since (G', F') contains G_4 as a minor, in G' there is a cycle C that contains at least one edge in F' , a node $c \in V(C)$ incident with two edges in $E(C) \setminus F'$, and a path P from c to a node d such that $V(P) \cap V(C) = \{c\}$, such that $E(P) \cap F' = \emptyset$, and d is incident with an edge $f = dt$ (possibly $t = d$) in F' incident with no other node in $V(C) \cup V(P)$. Since (G, F) does not contain G_4 as a minor, then $uv \in E(C) \cup \{f\}$.

We show that $uv = f$. Otherwise, suppose that $uv \in E(C)$. In this case the nodes in $V(C)$ are all in the block B of G . If $w \in V(C) \setminus \{c\}$, then the edges in $C \setminus \{uv\} \cup \{uw, vw\}$ form two cycles in G . Let C' be the one passing through c . Notice that $E(C') \cap F \neq \emptyset$, c is incident with two edges in $E(C') \setminus F$, and $V(C') \cap (V(P) \cup \{t\}) = \{c\}$. Thus (G, F) contains G_4 as a minor, a contradiction. Thus $w \in V(P) \cup \{t\}$. Let \bar{C} be the shortest subpath of C containing c as an internal node and with endnodes that are incident in G with edges in W_B . Notice that such path exists because the nodes u, v are incident in G with edges in W_B . Moreover, all the nodes in \bar{C} are in the same connected component of $G \setminus F$, since the two edges in C incident with c are not in F . Let \bar{u} (resp. \bar{v}) be the endnode of \bar{C} in the subpath of C from c to u (resp. v) that does not pass through v (resp. u). Let \bar{G} be the graph obtained from B by deleting all the nodes not in the connected component of $G \setminus F$ containing c .

Assume $t \notin V(B)$. By Claim 4.17 (i), there exists a path S in B from \bar{u} to \bar{v} that contains no node in $V(\bar{G}) \setminus \{\bar{u}, \bar{v}\}$. Since $t \notin V(B)$, then $t \notin V(S)$. Hence by using S , \bar{C} , and P , we see that (G, F) contains G_4 as a minor, a contradiction.

Hence $t \in V(B)$. Thus all the nodes in $V(P) \cup \{t\}$ are in B , and all the nodes in $V(P)$ are in \bar{G} , as $E(P) \subseteq E(G) \setminus F$.

Assume $f \in W_B$. Since d is in \bar{G} , then by Claim 4.16 (i) one among \bar{u}, \bar{v}, d is a cutnode of \bar{G} separating the other two. But the only possibility is that $d = c$ and d is a cutnode of \bar{G} separating \bar{u} and \bar{v} . So P has length zero. Since $w \in V(P) \cup \{t\}$, then $w \in \{d, t\}$. By Claim 4.16 (ii), $\{d, t\}$ is a cutset of B separating \bar{u} and \bar{v} , thus $\{d, t\}$ separates u and v , but this contradicts the choice of wu, wv to be consecutive.

Thus $f \notin W_B$, hence $dt \in E(\bar{G})$. Notice that, by Claim 4.16 (i), one among \bar{u}, \bar{v} and w is a cutnode of \bar{G} separating the remaining two. Since there is a path in \bar{G} between \bar{v} (resp. \bar{u}) and every node in $V(P) \cup \{t\}$ that

does not pass through \bar{u} (resp. \bar{v}), then w is a cutnode separating \bar{u} and \bar{v} . Therefore $w = c$. Since $\bar{u}, \bar{v} \in V(\bar{G})$, by Claim 4.17 (i) there exists a path T in B between \bar{u} and \bar{v} that does not contain any node in $V(\bar{G}) \setminus \{\bar{u}, \bar{v}\}$. Let C' be the cycle $\bar{u}, \bar{C}, \bar{v}, T, u$ in G . Then $t \notin V(C')$, $w \in V(C')$, w is incident with two edges in $E(C') \setminus F$, $E(C')$ contains at least an edge in W_B (because the inner nodes of T are in $V(G) \setminus V(\bar{G})$), $V(P) \cap V(C') = \{w\}$, therefore C and P form a G_4 minor.

Hence $uv = f$, and we can assume that $v \in V(P)$. Clearly $w \in V(C)$, as otherwise by considering C , P , and the edge vw we see that (G, F) contains G_4 as a minor. Notice that all the nodes in $V(C) \cup V(P) \cup \{u\}$ are in B . If not, then C is not in B , thus by considering C , P , and the edge among vw we see that (G, F) contains G_4 as a minor. Moreover w is incident with at least one edge in $E(C) \cap F$, as otherwise by considering C and the edge uw , (G, F) contains G_4 as a minor.

Let \bar{C} be the shortest subpath of C containing c as an internal node and with endnodes that are incident in G with edges in W_B . Notice that such path exists since $E(P) \subseteq E(G) \setminus F$, and since v and w are in different components of $B \setminus F$, as $vw \in W_B$. Let c', c'' be the endnodes of \bar{C} . All the nodes in \bar{C} are in the same connected component of $G \setminus F$, since the two edges in C incident with c are not in F . Let \bar{G} be the graph obtained from G by deleting all the nodes not in the connected component of $G \setminus F$ containing \bar{C} . By Claim 4.16 (i), one among c', c'', v is a cutnode of \bar{G} separating the other two. The only possibility is that $v = c$, and v is a cutnode of \bar{G} separating c' and c'' .

By Claim 4.16 (ii), this implies that $\{v, w\}$ is a cutset of B separating P' and P'' , where P' and P'' are the two disjoint paths in C from v to w . This concludes the proof of Fact 1.

Let $zw \in W_B$ such that $\{z, w\}$ is a cutset of B . Let V_1 be a maximal set of nodes connected in $B \setminus \{z, w\}$, and let B_1 be the subgraph of B induced by $V_1 \cup \{z, w\}$. Let B_2 be the subgraph of B induced by $V(B) \setminus V_1$.

Fact 2: If there exist edges $wr', wr'' \notin F$ in B_1 , and B_2 respectively, then given any two consecutive edges $uw, vw \in W_B$, uw, vw are shrinkable.

Proof of fact 2: Clearly $r', r'' \neq z$, as $zw \in W_B$. We first show that in this case, for every edge ws in W_B , $\{w, s\}$ is a cutset of B . We know that there exists at least another edge in W_B incident with w different from zw . Since B is 2-connected, there exists a path P' (resp. P''), from r' (resp. r'') to z , that does not pass through w . Clearly P' is contained in B_1 , and P'' is contained in B_2 .

Let \bar{C} be the shortest subpath of z, P', w, P'', z containing w as an internal node and with endnodes that are incident in G with edges in W_B . Notice that such path exists since z and w are in different components of $B \setminus F$, as $zw \in W_B$. Let c', c'' be the endnodes of \bar{C} . All the nodes in \bar{C} are in the same connected component of $G \setminus F$, since $wr', wr'' \notin F$. Let \bar{G} be the graph obtained from G by deleting all the nodes not in the connected component of $G \setminus F$ containing \bar{C} . By Claim 4.16 (i), w is a cutnode of \bar{G} separating the other two. By Claim 4.16 (ii), for every edge ws in W_B , $\{w, s\}$ is a cutset of B . Therefore s is in $V(P') \cup V(P'') \setminus \{w\}$.

Thus we can order the edges $f_1 = ww_1, \dots, f_u = ww_l$ in W_B incident with w in such a way that for every $i = 1, \dots, l-1$, the edges f_i and f_{i+1} are consecutive. We define the sets $U_1, \dots, U_{l+1} \subseteq V(B)$ such that U_i contains the nodes $v \in V(B)$ such that there is a path from v to w_i that does not contain nodes in $\{w, w_1, \dots, w_l\} \setminus \{w_i\}$, and a path from v to w_{i-1} that does not contain nodes in $\{w, w_1, \dots, w_l\} \setminus \{w_{i-1}\}$ for every $i = 1, \dots, l+1$, where $w_0 = w_{l+1} = w$. Notice that we can assume $r' \in U_1$ and $r'' \in U_{l+1}$, therefore $c' \in U_1$ and $c'' \in U_{l+1}$.

Next we show that, for every edge $wy \in E(B) \setminus W_B$, $y \in U_1 \cup U_{l+1}$. If not, suppose that $y \in U_i$, $i \in \{2, \dots, l\}$. Thus $y \in \bar{G}$. Let s be the first node incident with edges in W_B in a path from y to w_i in the subgraph of G induced by $U_i \cup \{w_i, w_{i-1}\}$. Then s is in \bar{G} , hence, by Claim 4.16 (i), one among s, c', c'' is a cutnode of \bar{G} separating the other two, a contradiction.

Now let (G', F') be obtained from (G, F) by shrinking two consecutive edges ww_i, ww_{i+1} , $1 \leq i \leq l-1$. We show that (G', F') does not contain the minor G_4 . Otherwise, by Fact 1 and by symmetry, there exists a cycle S in B' passing through w_i and w and not through w_{i+1} such that w_i is incident with two edges in $E(S) \setminus F$. Therefore all nodes in S are contained in $U_i \cup \{w_{i-1}, w_i\}$. Thus, $i = 1$. Let S' and S'' be the two distinct subpaths of S between w and w_1 . Then one among S' and S'' , say S' , does not pass through c' . Since $ww_1 \in W_B$, then w_1 is not in $V(\bar{G})$ hence there exists an edge of S' in W_B . Let c be the node in a S' incident with an edge in $W_B \cap E(S')$ closest to w . By the choice of f_1 , $c \neq w$. Notice that, between any two distinct nodes among c, c', c'' there is a path in \bar{G} that does not pass through the third one. But by Claim 4.16 (i), one among c, c', c'' should be a cutnode of \bar{G} separating the remaining two nodes, a contradiction. This concludes the proof of Fact 2.

By Fact 2, we assume that in B_1 there is no edge $wr \notin F$, $r \neq z$. Notice that we can choose the node z in such a way that in B_1 there is no other edge $wt \in W_B$ such that $\{w, t\}$ is a node cutset of B .

Clearly, for every edge $vw \in F$ with $v \in B_1$, then $vw \in W_B$, otherwise we have an edge $wr \notin F$ with $r \neq z$, and $r \in V(B_1)$.

Assume that there exist $u, v \in B_1$ such that $uw, vw \in W_B, u, v \neq z$. Then we may choose uw, vw consecutive. If $\{u, w\}$ or $\{v, w\}$ is a cutset, then we contradict the minimality of B_1 . So $\{u, w\}$ and $\{v, w\}$ are not cutsets, and the result follows from Fact 1.

Thus there is at most one edge $vw \in W_B$ with $v \in B_1$. Then there is one, otherwise w has no other neighbor in B_1 except for z , thus z is a cutnode of B , contradicting the fact that B is a block. Notice that, by Claim 4.13 (ii), there exists an edge e in $E \setminus F$ incident with w , therefore $e \in B_2$. Now let (G', F') obtained from (G, F) by shrinking zw, vw and assume that (G', F') contains G_4 as a minor.

Since in $G \setminus vw$ every path from v to w passes through z , then by Fact 1, there exists a cycle C passing through z and w and not through v such that the two edges in C incident with z are in $E \setminus F$. Clearly C must be contained in B_2 . Moreover, z and v are in the same connected component of $G \setminus F$, as otherwise (G, F) contains G_4 as a minor by considering the cycle C , and a path in B_1 from z to v in B_1 that contains edges in F . Since $zw \in W_B$, each of the two disjoint paths in C from z to w contains an edge in W_B .

Let \bar{C} be the shortest subpath of C containing z as an internal node and with endnodes that are incident in G with edges in W_B . Notice that such path exists since z and w are in different components of $B \setminus F$, as $zw \in W_B$. Let c', c'' be the endnodes of \bar{C} . All the nodes in \bar{C} are in the same connected component of $G \setminus F$, since $wr', wr'' \notin F$. Let \bar{G} be the graph obtained from G by deleting all the nodes not in the connected component of $G \setminus F$ containing \bar{C} . Notice that there are three disjoint paths in \bar{G} from z to, respectively, v, c' , and c'' , where $z \neq c', c'', v$. But this contradicts Claim 4.16 (i). \diamond

Claim 4.21. *In each block B of G , each node is incident with at most one edge in W_B .*

Proof of claim. Otherwise there exists a node w in a block B of G such that w is incident with at least two edges in W_B . By Claim 4.20, there are two shrinkable edges wu, wv in W_B incident with w . Let (G', F') be the pair obtained by shrinking wu, wv , and let $A' = A(G', F')$.

Notice that, by switching signs on u, v and w we can assume that $\sigma_{u, uw} = \sigma_{v, vw} = +1$ and that at least one among $\sigma_{w, uw}$ and $\sigma_{w, vw}$ is equal to $+1$. Now let c' be defined by $c'_z = c_z$ for every $z \in V(G') \setminus \{v, w\}$, $c'_v = c_v$ and $c'_w = c_w$ if $\sigma_{w, uw} \neq \sigma_{w, vw}$, $c'_v = c_v - 2$ and $c'_w = c_w - 2$ otherwise. Now let $\bar{x}'_e = 1/2$ for every $e \in E(G')$.

Since $|V(G')| = |V(G)|$ and $|E(G') \setminus L(G')| < |E(G) \setminus L(G)|$, then the system $A'x' = c', x' \geq 0$ has Chvátal rank at most 1. We show that \bar{x}' is

in the first closure of the system $A'x' = c', x' \geq 0$. Clearly \bar{x}' satisfies such system. Moreover, the odd-cut inequalities for $A'x' = c', x' \geq 0$ are satisfied by \bar{x}' , since the same inequalities are satisfied by \bar{x} , since for every $z \in V(G')$, c'_z has the same parity of c_z , and since the edges in $E(G) \setminus F$ are exactly the same edges in $E(G') \setminus F'$.

We show that \bar{x}' satisfies tightly $|E(G')| = |E(G)| - 1$ linearly independent inequalities valid for the first closure of the system $A'x' = c', x' \geq 0$.

Since \bar{x} is a vertex of the first closure of the system (4.2), it satisfies tightly $|E(G)|$ linearly independent inequalities, $\alpha^i x \geq \beta^i$, $i = 1, \dots, |E(G)|$, valid for the first closure of (4.2). Let M be the $|E(G)| \times |E(G)|$ matrix, such that its i -th row is equal to α^i . Now let M' be the $|E(G)| \times |E(G)| - 1$ matrix obtained from M by substituting the two columns corresponding to the edges $wu, uv \in E(G)$ with a new column corresponding to the new edge $uv \in E(G')$, which is the sum of the two removed columns if $\sigma_{w,wu} \neq \sigma_{w,uv}$ and is the difference between the column corresponding to the edge uw and the one corresponding to edge vw otherwise. Notice that each row of M' corresponds to an inequality of the first closure of $A'x' = c', x' \geq 0$ satisfied tightly by \bar{x}' , and that M' has rank $|E(G')| = |E(G)| - 1$.

Hence \bar{x}' is a fractional vertex of the first closure of $A'x' = c', x' \geq 0$, a contradiction. \diamond

4.4.5 The end: finding a balanced bipartition

In the remainder of the chapter we show that there exists a balanced bipartition R, B for (G, F) . Notice that this will conclude the proof of Theorem 4.1. Indeed, the vectors $y = \bar{x} + \frac{1}{2}\chi(R) - \frac{1}{2}\chi(B)$ and $z = \bar{x} + \frac{1}{2}\chi(B) - \frac{1}{2}\chi(R)$ are integral, since $\bar{x}_e = \frac{1}{2}$ for every $e \in E(G)$, y and z satisfy the system (4.2), since R, B is a balanced bipartition, and $\bar{x} = \frac{1}{2}(y + z)$, contradicting the fact that \bar{x} is a vertex of the first closure of (4.2).

Notice that, since $\bar{x}_e = \frac{1}{2}$ for every $e \in E(G)$, by Claim 4.8, (G, F) satisfies the conditions a) and b) of Remark 4.14.

We construct recursively a family \mathcal{G} of pairs (\tilde{G}, \tilde{F}) in \mathcal{C} satisfying conditions a) and b) of Remark 4.14 as follows. Initially $\mathcal{G} = \{(G, F)\}$. Until there exists $(\tilde{G}, \tilde{F}) \in \mathcal{G}$ such that \tilde{G} has two distinct blocks B_1, B_2 such that, for $i = 1, 2$, either B_i contains an odd cycle or $E(B_i) \cap \tilde{F} \neq \emptyset$, let w be a cutnode of \tilde{G} separating B_1 and B_2 , let $(\tilde{G}_1, \tilde{F}_1)$ and $(\tilde{G}_2, \tilde{F}_2)$ be obtained from (\tilde{G}, \tilde{F}) by breaking B_1 and B_2 at w , and let $\mathcal{G} := \mathcal{G} \setminus \{(\tilde{G}, \tilde{F})\} \cup \{(\tilde{G}_1, \tilde{F}_1), (\tilde{G}_2, \tilde{F}_2)\}$. The process ends with a family \mathcal{G} such that, for each element (\tilde{G}, \tilde{F}) of \mathcal{G} , (\tilde{G}, \tilde{F}) has at most one block B such that B contains an odd cycle or $E(B) \cap \tilde{F} \neq \emptyset$.

By Lemma 4.15, if every (\tilde{G}, \tilde{F}) in \mathcal{G} has a balanced bipartition, then (G, F) has a balanced bipartition.

Let (\tilde{G}, \tilde{F}) be an element of \mathcal{G} . We show that (\tilde{G}, \tilde{F}) has a balanced bipartition.

Claim 4.22. *If $\tilde{G} \setminus \tilde{F}$ is connected, then there exists a balanced bipartition of $E(\tilde{G})$.*

Proof of claim. Notice that each edge in $\tilde{F} \cap L(\tilde{G})$ is either an edge of G , or was introduced in the construction of \mathcal{G} .

Assume that for every two edges $f = vw$ and $f' = v'w'$ in $\tilde{F} \setminus L(\tilde{G})$, either f and f' are nested, or one among v and w , say v , is a cutnode of $\tilde{G} \setminus \tilde{F}$ separating w from $\{v', w'\} \setminus \{v\}$. Choose an edge $f = vw$ in $\tilde{F} \setminus L(\tilde{G})$ not nested in other edges in $\tilde{F} \setminus L(\tilde{G})$. Assume that there is an edge $f' = v'w' \in \tilde{F} \setminus L(\tilde{G})$, $f' \neq f$, not nested in f . Then one among v and w , say v , is a cutnode of $\tilde{G} \setminus \tilde{F}$ separating w from $\{v', w'\} \setminus \{v\}$. Since f is not nested in other edges in $\tilde{F} \setminus L(\tilde{G})$, w is a cutnode of \tilde{G} , contradicting the construction of \mathcal{G} . Hence in \tilde{G} , all the edges in $\tilde{F} \setminus L(\tilde{G})$ are nested in f .

We show that all the edges in $\tilde{F} \cap L(\tilde{G})$ are incident with v or w . Assume that there is an edge $f' = v'v' \in \tilde{F} \cap L(\tilde{G})$ not adjacent to f . Then by Claim 4.11 and 4.12, one among v and w , say v , is a cutnode of $\tilde{G} \setminus \tilde{F}$ separating w from v' . Since f is not nested in other edges in $\tilde{F} \setminus L(\tilde{G})$, v is a cutnode of \tilde{G} , contradicting the construction of \mathcal{G} . Thus all the edges in $\tilde{F} \cap L(\tilde{G})$ are incident with v or w .

Similarly, we show that \tilde{G} is bipartite. Assume that there is an odd cycle C in \tilde{G} . Then by Claim 4.12, one among v and w , say v , is a cutnode of $\tilde{G} \setminus \tilde{F}$ separating w from $V(C) \setminus \{v\}$. Since f is not nested in other edges in $\tilde{F} \setminus L(\tilde{G})$, then v is a cutnode of \tilde{G} , contradicting the construction of \mathcal{G} . Thus \tilde{G} is bipartite.

Let P be a path in $\tilde{G} \setminus \tilde{F}$ from v to w . Clearly P passes through all the nodes of \tilde{G} incident with some edge in \tilde{F} .

By Lemma 4.19, there exists a partition of the edges in $E(\tilde{G}) \setminus \tilde{F}$ in one closed trail, if $L(\tilde{G}) \subseteq \tilde{F}$, or, if $L(\tilde{G}) \setminus \tilde{F} \neq \emptyset$, in $|L(\tilde{G}) \setminus \tilde{F}|/2$ trails such that their first and last edge are in $L(\tilde{G}) \setminus \tilde{F}$, such that one of the trails, say T , passes through all the nodes incident with edges in \tilde{F} .

Let \mathcal{T} be the family of trails obtained from the above partition of the edges in $E(\tilde{G}) \setminus \tilde{F}$, by adding in T all the edges in $\tilde{F} \cap L(\tilde{G})$. Clearly \mathcal{T} satisfies (C1)-(C3) of Lemma 4.16, thus there exists a balanced bipartition of $E(\tilde{G})$.

Thus, by Claim 4.10, in \tilde{G} there are two edges $f = vw$ and $f' = vw'$ in $\tilde{F} \setminus L(\tilde{G})$ such that for any two distinct nodes $s, t \in \{v, w, w'\}$ there exists a path in $G \setminus F$ between s and t that does not pass through $\{v, w, w'\} \setminus \{s, t\}$.

We show that the edges in \tilde{F} , form a star centered at v . Notice that, by the construction of \mathcal{G} , \tilde{G} has at most one block B such that B contains an odd cycle or $E(B) \cap \tilde{F} \neq \emptyset$. Thus by Claim 4.14 and by the construction of \mathcal{G} , the edges in \tilde{F} that were edges of G , form a star centered at v . Any other edge $l \in \tilde{F}$ is obtained from the construction of \mathcal{G} , and $l \in L(\tilde{G})$. Thus assume that $l = v'v'$ is an edge in $\tilde{F} \cap L(\tilde{G})$ not adjacent to f introduced in the construction of \mathcal{G} . Then by applying Claim 4.11 and 4.12 to both f and f' , it follows that v is a cutnode of $\tilde{G} \setminus \tilde{F}$ separating w from v' . Since all the edges in $\tilde{F} \setminus L(\tilde{G})$ are incident with v , then v is a cutnode of \tilde{G} , contradicting the construction of \mathcal{G} . Thus all the edges in $\tilde{F} \cap L(\tilde{G})$ are incident with v or w .

Similarly, we show that \tilde{G} is bipartite. Assume that there is an odd cycle C in \tilde{G} . Then by applying Claim 4.12 to f and f' , it follows that v is a cutnode of $\tilde{G} \setminus \tilde{F}$ separating w from $V(C) \setminus \{w\}$. Since all the edges in $\tilde{F} \setminus L(\tilde{G})$ are incident with v , then v is a cutnode of \tilde{G} , contradicting the construction of \mathcal{G} . Thus \tilde{G} is bipartite.

Thus, by Lemma 4.18, there exists a balanced bipartition of $E(\tilde{G})$. \diamond

Thus $\tilde{G} \setminus \tilde{F}$ is not connected. Let \bar{V} be a maximal set of nodes connected in $\tilde{G} \setminus \tilde{F}$. Let (\bar{G}, \bar{F}) be obtained from (\tilde{G}, \tilde{F}) by deleting all the nodes in $V(\tilde{G}) \setminus \bar{V}$. Notice that (\bar{G}, \bar{F}) is in \mathcal{G} , and that it satisfies conditions a) and b) of Remark 4.14. Notice that the loops in $E(\bar{G}) \cap \bar{F}$ are of three types: loops of F incident with some node in $V(\bar{G})$; artificial loops created in the construction of the family \mathcal{G} ; and loops corresponding to edges of F with one endnode in $V(\bar{G})$ and one in $V(\tilde{G}) \setminus V(\bar{G})$ obtained by deleting the endnode in $V(\tilde{G}) \setminus V(\bar{G})$. Let $\Delta \subseteq \bar{F}$ be the set of loops in $E(\bar{G}) \cap \bar{F}$ of the last type, and let Θ be the set of the loops in $E(\bar{G}) \cap \bar{F}$ of the other two types. Notice that we identify the loops in Δ with the original edges in \tilde{G} . By Claim 4.21 each node of \bar{G} is incident with at most one edge in Δ , since all the edges in Δ correspond to edges in the same block of \tilde{G} .

Let v_1, \dots, v_k be an ordering of the nodes in $V(\bar{G})$ incident with edges in Δ as in Claim 4.16 (i), and let $P = v_1, P_1, v_2, \dots, v_{k-1}, P_{k-1}, v_k$ be a path in $\bar{G} \setminus \bar{F}$ from v_1 to v_k , where P_i is a path from v_i to v_{i+1} for every $i = 1, \dots, k-1$. Let l_i be the only loop in Δ incident with v_i for every $i = 1, \dots, k$.

Notice that, by the construction of \mathcal{G} , and by Claim 4.17, it follows that:

- \bar{G} is bipartite;

- each edge in $\bar{F} \setminus L(\bar{G})$ has endnodes in $\{v_1, v_k\} \cup \{v \in \bar{G} : v \text{ is a cutnode of } \bar{G} \setminus \bar{F} \text{ separating } v_1 \text{ and } v_k\}$;
- each edge in Θ is incident with v_1 or v_k .

Thus P passes through all the nodes of \bar{G} incident with some edge in \bar{F} .

Claim 4.23. *There exists a balanced bipartition for the edges in $E(\bar{G})$ such that two loops l_i and l_{i+1} in Δ , are in the same side of the partition if and only if $\sum_{e \in \{l_i, E(P_i), l_{i+1}\}} \sum_{v \in e} \sigma_{v,e} \equiv_4 0$, for every $i = 1, \dots, k-1$.*

Proof. By Lemma 4.19, there exists a partition of the edges in $E(\bar{G}) \setminus \bar{F}$ in one closed trail, if $L(\bar{G}) \subseteq \bar{F}$, or, if $L(\bar{G}) \setminus \bar{F} \neq \emptyset$, in $|L(\bar{G}) \setminus \bar{F}|/2$ trails such that their first and last edge are in $L(\bar{G}) \setminus \bar{F}$, such that one of the trails passes through all the nodes incident with edges in \bar{F} . Let \mathcal{T} be such family of trails, and let $T \in \mathcal{T}$ be the one that passes through all the nodes incident with edges in \bar{F} . Let S be a shortest subtrail of T containing all the nodes v_1, \dots, v_k . Since v_1, \dots, v_k are all cutnodes of \bar{G} and since $P = v_1, P_1, v_2, \dots, v_{k-1}, P_{k-1}, v_k$ is a path in $\bar{G} \setminus \bar{F}$, it follows that S can be written in the form $S = v_1, S_1, v_2, \dots, v_{k-1}, S_{k-1}, v_k$.

Let S' be obtained from S by adding to S all the edges in $\bar{F} \cap L(\bar{G})$ in such a way that there exists a subtrail of S' containing all the edges in Δ and no edge in Θ . Notice that this can be easily done since the endnodes of S are v_1 and v_k , since all the edges in Δ are incident with the nodes v_1, \dots, v_k , and since all the edges in Θ are incident with v_1, v_k . Let T' be obtained from T by replacing its subtrail S with S' . Let $\mathcal{T}' = \mathcal{T} \setminus \{T\} \cup \{T'\}$. Clearly \mathcal{T}' satisfies (C1)-(C3) of Lemma 4.16, thus there exists a balanced bipartition of $E(\bar{G})$. Moreover, by Remark 4.17, l_i and l_{i+1} in Δ , are in the same side of the partition if and only if $\sum_{e \in \{l_i, E(S_i), l_{i+1}\}} \sum_{v \in e} \sigma_{v,e} \equiv_4 0$, for every $i = 1, \dots, k-1$. For every $i = 1, \dots, k-1$, let E_i^1 be the edges in the symmetric difference between $E(P_i)$ and $E(S_i)$, and let $E_i^2 = E(P_i) \cup E(S_i) \setminus E_i^1$. The edges in E_i^1 form a union of cycles, that are balanced since \bar{G} is bipartite. Thus $\sum_{vw \in E_i^1} (\sigma_{v,vw} + \sigma_{w,vw}) \equiv_4 0$, and $2 \sum_{vw \in E_i^2} (\sigma_{v,vw} + \sigma_{w,vw}) \equiv_4 0$. Thus $\sum_{vw \in E(S_i)} (\sigma_{v,vw} + \sigma_{w,vw}) + \sum_{vw \in E(P_i)} (\sigma_{v,vw} + \sigma_{w,vw}) \equiv_4 0$. Since $\sum_{vw \in E(P_i)} (\sigma_{v,vw} + \sigma_{w,vw})$ is even, it follows that $\sum_{vw \in E(S_i)} (\sigma_{v,vw} + \sigma_{w,vw}) \equiv_4 \sum_{vw \in E(P_i)} (\sigma_{v,vw} + \sigma_{w,vw})$ for every $i = 1, \dots, k-1$.

Thus l_i and l_{i+1} in Δ , are in the same side of the partition if and only if $\sum_{e \in \{l_i, E(P_i), l_{i+1}\}} \sum_{v \in e} \sigma_{v,e} \equiv_4 0$, for every $i = 1, \dots, k-1$. \square

Claim 4.24. *There exists a balanced bipartition of the edges in $E(\tilde{G})$.*

Proof of claim. Let W be the set of edges in \tilde{F} with endnodes in different components of $\tilde{G} \setminus \tilde{F}$. Consider the pair $(\tilde{G}^{split(W)}, \tilde{F}^{split(W)})$. Notice that \tilde{G} is a connected component of $\tilde{G}^{split(W)}$, and that $\tilde{F} = E(\tilde{G}) \cap \tilde{F}^{split(W)}$.

Notice that, since by Claim 4.21 there are not two adjacent edges in W , then by Claim 4.23 there exists a balanced bipartition, as in Claim 4.23, for the edges in each connected component of $\tilde{G}^{split(W)}$. Thus we also have a balanced bipartition for the edges in $\tilde{G}^{split(W)}$. We now show that we can combine all these bipartitions to get a balanced bipartition of the edges in $E(\tilde{G})$.

Let \tilde{G}' be obtained from \tilde{G} by contracting all the edges in $E(\tilde{G}) \setminus \tilde{F}$ and by removing all the loops. Notice that there is a one-to-one relation between the edges of \tilde{G}' and the edges in W in \tilde{G} . Now let $T \subseteq E(\tilde{G}')$ be a spanning tree of \tilde{G}' .

Clearly, for every edge in T we can assume that the two corresponding loops in $\tilde{G}^{split(W)}$ are in the same side of the bipartition. If for every edge f in W , the two corresponding loops in $\tilde{F}^{split(W)}$ are in the same side of the bipartition, then we are done by assigning to f to that side of the bipartition. Let $W^+ \subseteq W$ contain the edges such that the two corresponding loops in $\tilde{G}^{split(W)}$ are in the same side of the bipartition, and let $W^- = W \setminus W^+$. Thus $E(T) \subseteq W^+$. Thus we only need to show that $W = W^+$.

By contradiction, among all the edges in W^- , let vw be one that minimizes the distance between v and w in the graph obtained from $\tilde{G}^{split(W)}$ by replacing each pair of loops corresponding to an edge in W^+ with the original edge in \tilde{G} . Notice that for every edge in W^- , such distance is always finite, since $T \subseteq W^+$. Moreover let P be the path from v to w that realizes this minimum length. Now let $C = v, vw, P, v$ be the cycle in \tilde{G} corresponding to the edges in P and the edge vw . By the minimality of vw it follows that the cycle C has no chords in \tilde{F} .

Notice that the cycle C corresponds in $\tilde{G}^{split(W)}$ to a family \mathcal{S} of paths whose first and last edge are loops in $\tilde{F}^{split(W)}$ corresponding to edges in W . Since the two loops corresponding to vw in $\tilde{G}^{split(W)}$ are not in the same side of the bipartition, it follows that in \mathcal{S} there is an odd number of paths whose first and last edge are in different sides of the bipartition. By Claim 4.23, this means that in \mathcal{S} there is an odd number of paths S such that $\sum_{vw \in E(S)} (\sigma_{v,e} + \sigma_{w,e}) \equiv_2 2$. But this implies that C is an odd cycle of \tilde{G} that contains edges in \tilde{F} , a contradiction. \diamond

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