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Holography for correlators in black hole microstates

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Abstract

In the AdS/CFT context black holes are dual to ensembles of 'heavy' CFT states whose conformal dimension scales as the central charge. The Strominger-Vafa [1] black hole, which admits an $\mathrm{AdS}_3 \times S_3$ decoupling limit and a dual description in terms of a two-dimensional CFT, provides an excellent model to study.

Among the dynamical quantities one can study, the four-point functions, with two heavy states and two light probes, provide a good observable to extract detailed informations from the black hole.
In the spirit of holographic description, since black hole regime is dual to a CFT at strong coupling, we need to use the dual gravitational description to extract the correlators. Since also, in supergravity approximation, the heavy states are not described by single particle modes, the Witten diagram technology is not so straightforward and the necessity of a new method emerges.
In this thesis we develop these techniques and then we generalize to more complex background geometries. A parallel analysis in the free CFT has also been provided with conformal blocks technology, which gives also a consistency check for the results. The consequences of these non trivial computations are extremely important in order to understand how the unitarity in the scattering processes is recovered. As we pointed out in those works, it turned out to be restored even in the supergravity approximation thanks to a mechanism that appears clearly in the CFT picture and that consists in an infinite sum of exchanged operators. This duality in the description is again fundamental and the exploration of the gravity picture of this mechanism, is a very interesting task.
Thus, besides the implications on black hole physics, four-point functions, thanks to their dynamical nature, give a powerful tool for going deep in the mechanism of the correspondence.

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Chapter 1

Introduction

Black holes are classical solutions that appear naturally in General Relativity (GR). The first black hole was written down for the first time a century ago by Schwarzschild, and it is a solution to the Einstein equations determined by one parameter, the mass. We can picture such a black hole as a region of spacetime with a singularity screened by a boundary called event horizon, in which things can fall, but nothing comes out. Over the years other black holes have been found generally characterized by solutions determined by a set of parameters like mass, charge and angular momentum, and governed by a set of uniqueness theorems. In the late 1960's and early 1970's, laws of classical black hole mechanics were discovered, which bear a striking resemblance to the laws of thermodynamics. By these laws, a black hole also has an entropy. It was first conjectured by Bekenstein [2] and later proven by Hawking [3] that this entropy is proportional to the area of the black hole horizon:

$$S_{\mathrm{BH}} = \frac{A_H c^3}{4 G_N \hbar}$$

A little bit of quantum mechanics entered in the game leading to the discovering of what we call Hawking process. The region of spacetime around the horizon of a black hole has curvature and hence a certain energy density. From the idea that in Quantum Field Theory (QFT) energy can decay into particle-antiparticle pair has led Hawking to perform a semiclassical analysis of QFT in black hole background. In this process, pairs will be created and once in a while one of the two falls into the black hole horizon, while the other escapes off to spatial infinity. The net result is that the black hole mass is lowered and energy, under the form of thermal radiation, escapes to infinity. In summary, the black hole behaves as a black body, with temperature proportional to the strength of the gravitational field at the horizon and whose conserved charges play the role of thermodynamic quantities.

Identifying the Bekenstein-Hawking entropy as the physical entropy of the black hole, gives rise to an immediate puzzle, namely the nature of the microscopic quantum mechanical degrees of freedom giving rise to that thermodynamic entropy. Another puzzle, the famous information problem or paradox, arose from Hawking computations, where he showed that the thermal radiation depends only on the conserved numbers of the black hole.

Within GR, there is no way out of these problems, indeed, traditional attempts to find microstates responsible for the entropy did not succeed, but rather it appeared that the black hole geometry was uniquely determined by conserved charges with no chance to have *hair*. One may think that the differences between the number of microstates $e^{S_{\mathrm{BH}}}$ of the hole are to be found by looking at planck sized neighborhood of the singularity, and these differences are not visible in the classical description. After all, the matter that made up the hole disappeared into the singularity. But this picture of the hole leads to the information paradox. Indeed, if the information about the microstate resides at the singularity then the outgoing radiation is insensitive to the details of the microstate, and when the hole has evaporated away we cannot recover the information contained in the matter which went to make the black hole. This is a violation of the unitarity of quantum mechanics and a severe contradiction with the way we understand physics of black holes. The information paradox has resisted attempts to the resolution for many years. The robustness of the paradox stems from the fact the it uses very few assumptions about the physics relevant to the black hole. One assumes that quantum gravity effects are confined to a small length scale like the planck size, and then notes that the curvature scales at the horizon are much larger than this length for large black holes. Thus it would appear that the precise theory of quantum gravity is irrelevant to the process of Hawking radiation and thus for resolution of the paradox.

In the context of string theory, which is a unified quantum theory of all interactions, including gravity, information should not be lost. As a consequence, the information paradox must be an artifact of the semiclassical approximation used to derive it. The information problem is therefore shifted to the problem of showing precisely how semi-classical arguments break down. This turns out to be a very difficult problem, and solving it is one of the foremost challenges in this area of string theory and several scenarios and attempts to resolution have been proposed during years.

Among the remarkable things about string theory, we have that it strongly constrained so we must make the black hole from objects predicted by the theory. Besides the fields coming from quantization of the fundamental string (F1) the theory contains a collection of extended objects of different dimensionality called D-branes. Generally, one makes black holes by taking branes in the theory and wrapping them on compact directions, so that from the non-compact space point of view this places a given mass and charge at a point in space, and with suitable choice of wrapped objects we can create a black hole. The quantities characterizing the hole will be a function of the details of the compact space and the parameters of string theory g_s and $\alpha' = \ell_s^2$, controlling respectively the strength of the quantum correction and finite sized effect of the string.

One of the first steps in string theory in the direction of shedding light on black hole puzzles, was done by Susskind, and later by others. He proposed [4] that there is a one-to-one correspondence between Schwarzschild black holes and fundamental string states. This is based on the fact that as one increases the string coupling g_s, the size of a highly excited string state becomes less than its Schwarzschild radius, so it must become a black hole. Conversely, as one decreases the coupling, the size of a black hole eventually becomes less than the string scale. At this point, the metric is no longer well defined near the horizon, so it can no longer be interpreted as a black hole. Susskind suggested that the configuration should be described in terms of some string state. At large values of the mass, the typical state consists of a small number of highly excited strings, so the black hole should reduce to such a state at weak coupling. So, basically, we can think about black holes and elementary string states as different ways of representing the same state. But to further substantiate this claim, one must show that the black holes have the same properties as elementary string states besides carrying the same quantum numbers. One of the features which seems to be common between black holes and elementary string states is that for both the degeneracy of states with given mass increases very rapidly with mass. However, Susskind found a discrepancy between the entropy of a free fundamental string S_{micro} and the Bekenstein-Hawking entropy S_{BH}, growing with different power of the mass. This discrepancy must be due to the counting of energy levels performed at different value of the string coupling, which makes incorrect the comparison. One way to avoid this problem is to study a configuration in string theory protected by supersymmetry, namely a BPS or extremal configuration, where states with a given mass and at a given value of the couplings move together as we change the couplings and are directly comparable at different values of g_s. A couple of years later, Sen [5] showed that the density of elementary string states of a given mass, correctly reproduces the entropy of some extremal black holes, with the same quantum numbers. He found the correct dependence on the charges of the holes, without fixing the exact relative coefficient. Few years later, Strominger e Vafa [1] constructed a class of five-dimensional extremal black holes and derived a relation between the black hole entropy and the microscopic counting of states on the branes. The configuration consists in a black hole constructed in type IIB string theory, by considering n_1 D1-brane wrapped on S^1, n_5 D5-brane on $T^4 \times S^1$ allowing n_p units of momentum along S^1. We refer to this system to the D1D5 black hole or D1D5 system. When the gravitational coupling is weak, the D-branes can be described via a worldvolume gauge theory. On the other hand, when the coupling is strong, the gauge theory is strongly coupled, but there is a gravity description as a black hole. One can count the degrees of freedom at weak coupling and compare to the Bekenstein-Hawking entropy at strong coupling. In general, one would not expect a weak and strong coupling answer to agree; however, thanks to supersymmetric nature of the black hole, these counts are protected from changes in coupling. When one performs this calculation, one finds exact agreement

$$S_{\mathrm{BH}} = S_{\mathrm{micro}} = 2\pi\sqrt{n_1 n_5 n_p}$$

Such agreements, besides to be one of the first and most remarkable result in microstate physics of black hole, contributed to Maldacena'a conjecture [6], where he stated that string theory in $d+1$-dimensional anti-de Sitter (AdS) space is dual to a d-dimensional Conformal Field Theory (CFT). This duality is the most explicit and powerful instance of gravitational holography and has proved a powerful tool in a large number of studies, comprising black hole. Indeed the D1D5 system provides a prototypical case because admits an AdS_3 decoupling limit and a dual description in terms of a two-dimensional CFT, often dubbed D1D5 CFT. In this dual description, microstates have a either a field theory picture, in terms of states/operators in the CFT, or a gravity picture, in terms of a gravity background solution. An interesting question that follows under this new light of dual description, is that the appearance of $e^{S_{\mathrm{BH}}}$ microstates leaves us with the question of how the microstates manifest themselves in gravity at strong coupling, where there is a black hole. For instance, we can take a single microstate of the weakly coupled gauge theory, and ask what happens to that state as we turn up the coupling. It is therefore fundamental to go beyond the counting problem and ask if the detailed understanding of the microstates of supersymmetric black hole can be used to shed any light to conceptual puzzles arising in black hole physics. This motivation underlines many developments, including the fuzzball proposal [7, 8] which is a conjecture and construction for black hole, coming from all the developments in this area briefly described so far.

The *fuzzball* proposal states that the black hole solution is an effective coarse-grained description that results from averaging over *horizon-free, nonsingular microstates* that have nontrivial structure differing from the *naive* black hole solution up to the horizon scale. This fact resolves the information paradox: the radiation leaves from the surface and carries information about the state of the system. For certain extremal black holes, like in the D1D5 system, one can explicitly construct solutions of supergravity that form a phase space of black hole microstates, which account for the entropy of the black hole. The solutions found are indeed nonsingular, horizon-free, and differ from each other up to the horizon scale. Asymptotically they approach the black hole solution. Furthermore, there is an explicit mapping between solutions and states of the D1D5 CFT. Let us emphasize, however, that the fuzzball proposal does not require the microstates to be well-described by supergravity. We only need the microstates to differ from the naive black hole solution and each other up to the would-be horizon. In fact, we expect generic states of the black hole not to have a well-defined geometric description; however, there may still be certain special states that have nice classical descriptions. AdS/CFT duality gives us a tool to construct microstates and to provide a dictionary between the field theory and the gravity side. Moreover one might wonder why we insist on using a gravitational description if we expect to have a unitary field theory description. Since all of the issues with black holes and gravity arise when considering the geometrical description, it is important to resolve those issues in the same language. For instance, if one is to resolve the Hawking information paradox, one must show what aspect of the argument fails. Otherwise, we are faced with the specter of giving up quantum mechanics, and therefore string theory and the AdS/CFT correspondence.

In order to introduce the work done in the dissertation, we briefly summarize what we have said so far. We have seen that different breakthroughs in the black hole physics area, led, firstly, to study black holes in string theory, and then to conjecture relations between string states and black holes, in order to provide a microscopic explanation for the thermodynamic entropy. A further step was to specialize to a very particular setup, the D1D5 system, in which the counting and matching could be done precisely and together with the AdS/CFT correspondence was possible to formulate a proposal and explicitly construct the microstates. Despite the fact that the research in this area takes many directions, among the most challenging steps to take we have the explicit construction of all the microstates for more realistic black holes and the generalization of the proposal to other systems. Besides this very difficult and demanding program, another important issue to focus on is the study of the already known microstates, and how they differ from the naive black hole. A useful way to address this question is to probe the microstates by switching on some perturbations around the given background solution. Thanks to AdS/CFT we can reduce the problem to the computation of correlation functions in the dual CFT, that has the advantage to be a unitary theory and questions about information loss in these background take a more clear form. The duality helps us with the technical difficulties arising when computing a correlator in a regime where the CFT is strongly coupled and where there is a black hole, that is the regime we are interested in. In this thesis, we focus in particular on four-point function often called HHLL correlators of the form

$$\mathcal{C}(z_i) = \langle O_H(z_1)\bar{O}_H(z_2)O_L(z_3)\bar{O}_L(z_4)\rangle$$

The O_H operators are *heavy* (H) operators, with conformal dimension scaling with the central charge c of the D1D5 CFT, and dual, in gravity, to a particular microstate background solution. The O_L operators are *light* (L) with weight of order one and corresponding in gravity to perturbation of the dual fields. One of our main motivations for performing these computations is to contrast the correlators computed in microstates with those computed in a black hole background. We mainly focus on two-charge solutions in D1D5, corresponding to a black hole that is not actually described by a regular black hole in classical supergravity, but by the singular geometry obtained by taking the zero temperature limit of the BTZ black hole. Nevertheless, this geometry shares some properties with macroscopic black holes: in particular, a correlator computed in this background vanishes at large Lorentzian time τ, and when three operators are put in fixed points it reads

$$\mathcal{C}_{\mathrm{BTZ}}(\tau,\sigma)=\frac{1}{2i\tau}\left[\frac{1}{1-e^{i(\sigma-\tau)}}+\frac{1}{1-e^{-i(\sigma+\tau)}}-1\right]$$

As first pointed out in [9], and more recently emphasised in [10] in the AdS_3 context, the late-time decay of correlators is one of the manifestations of the information loss problem. Indeed, Maldacena has emphasized that in a black hole background, correlators decay exponentially at late Lorentzian time. Intuitively, this means that information thrown into a black hole never returns. This behavior is forbidden in a field theory with a finite number of local degrees of freedom and so it provides a sharp signature of information loss.

Following the discussions coming from in [9, 11, 12, 13], the field theory dual to the black hole considered in those works is thermal and generically has a discrete spectrum and finite entropy. In such theories, time-like separated two-point correlation functions should be periodic functions of time (or quasi-periodic, see [11, 13] for definitions and details). In thermal field theories, finite entropy implies that the spectrum of the Hamiltonian is discrete. In such systems, there exist Poincaré recurrences given an initial configuration, because of the finite phase space volume the system generically evolves under unitary time evolution in such a way that it comes arbitrarily close to its initial state an infinite number of times. While often discussed in the context of classical physics, this phenomenon extends to the behavior of correlators in quantum theories. In particular, under some weak assumptions about the light operator O_L, one can prove that the correlator $\mathcal{C}(\tau)$, computed in a equilibrium state described by a density matrix ρ_β at temperature β, and given by

$$\mathcal{C}(\tau)=\mathrm{Tr}\left[\rho_\beta O_L(\tau)\bar{O}_L(0)\right] \tag{1.1}$$

is a periodic function of time. The details of the time dependence of the correlator depend very sensitively on the details of the spectrum, but generically the expected time T between order one recurrences is at least exponentially long in the entropy[1]. On the other hand, correlators computed in black hole spacetimes are damped in time (perturbed black holes ring with a quasi-normal frequency that has a non-zero imaginary part), and in particular are not periodic. They always take their maximum value at $\tau=0$, and due to the damping never come back to that value again. A more quantitative criterion to see information or unitarity loss can be used by defining the long time average that for unitary and finite entropy theories is bounded and given by [11, 13]

$$\frac{1}{T}\int^{T}|\mathcal{C}(\tau)|^2\,d\tau\sim\exp\left(-kS_\beta\right) \tag{1.2}$$

for some numerical constant k and S_β the entropy of the ensemble. It is clear that for large time decay functions the time-averaged value is zero. In summary, in unitary field theories, the short-time behavior is, in general, an exponential decay due to thermalization, but at long times the correlators will behave stochastically, returning its initial value an infinite number of times. The bulk dual to the short time exponential decay of the correlator is related to the fact that black holes swallow anything that is thrown into them. Restoring unitarity and resolving the information paradox, and matching with the CFT prediction, requires that the exact bulk correlator be periodic. Understading how this periodicity (and so unitarity) is restored in the bulk is clearly of great interest and subject of activity and in particular, a good proposal for describing black hole in context

[1]In [13] they estimate to be $T\sim\beta e^{S(\beta)}$. See the reference for details of derivation.

of AdS/CFT has to satisfy the constraints explained above either for pure or thermal states.

Focusing on pure states, the study of the correlators has the potential to shed light on the mechanism by which the information of the heavy state is encoded in the correlator. Indeed we will see that, in the correlators computed in pure state, the information is restored by a mechanism that has a dual interpretation.
On the gravity side, the result emerges thanks to taking into account the entire ten-dimensional horizonless microstates and the reduction down to three dimensions gives the additional pieces responsible for the difference from the naive computation. From the CFT point of view the result of the four-point function can be analyzed in terms of contributions from exchanged operators in the OPE channels, and the entire result could be recovered and explained as a sum of Virasoro blocks encoding the intermediate states. Despite the naive expectation in the black hole background, where the four-point function is just given by the identity block, whose behavior would lead to a large time decay of the correlator, in the microstate we considered, the result and the pathological features of the identity block are resolved by an infinite tower of exchanged operators giving a and consistent result [14, 15].

One of the fundamental result we will find is indeed, the pure state correlator [16]

$$\mathcal{C}_{\text{micro}}(\tau,\sigma) = \frac{1}{\eta}\frac{1}{1-e^{-2i\frac{\tau}{\eta}}}\left[\frac{1}{1-e^{i(\sigma-\tau)}}+\frac{1}{1-e^{-i(\sigma+\tau)}}-1\right]$$

with η being a parameter of the particular microstate considered and controlling the shift from the black hole. It can be seen that we no longer have a late time decay as in the BTZ case. The general behavior of the correlators studied is a time decay similar to the black hole case for $\tau < \eta$ and then an oscillating pattern that continue for large time. It can be shown that in correlators we considered here the long time average is non null as expected for unitary theory and the transition between the two regimes reflects the initial damping followed by the oscillating trend. This pattern confirms the general expectation that correlation functions in pure state don't decay in time and information is not lost.

Another interesting issue is more technical and relies in the possibility to extract a four-point correlator in AdS_3/CFT_2, a largely studied object in this context. In our case however, it is not straightforward to use the technology of the Witten diagrams to calculate the correlators above, since the heavy states correspond to multi-particle operators with a large conformal dimension and are not dual to a single supergravity mode. We bypass this issue by exploiting the known smooth geometries dual to the heavy states; then we use the standard AdS/CFT dictionary to calculate the HHLL correlators by studying the fluctuations of the supergravity field dual to the light operators in the asymptotycally AdS geometry describing the heavy operators.

In summary, this dissertation focuses on the gauge/gravity description of the D1D5 system to understand black holes and the fuzzball proposal, through the correlation functions. Some of the calculations adn techniques developed, however, have more general applications in purely AdS/CFT context. Since we are dealing with black holes in D1D5, we start in chapter 2 by explicitly constructing the solutions with an algebraic method available in string theory, the solution generating technique, that avoids the need to pass through the difficulty of solving very non-linear equations of motion. We firstly construct the D1D5P three-charge black hole, which has macroscopic area, and we then compute the entropy both from area law and from microscopic counting and we show the matching. We also look at the two-charge limit of this black hole, whose microstates are the mostly studied in this work, and finally we also give the recipe to generalize to rotating black holes.

In chapter 3 we present the duality in the D1D5 system. We define and discuss in details the D1D5 CFT and the gravity side of the system giving in the end the ingredients to define the mapping between the two sides. We provide a detailed dictionary for two-charge microstates that will be useful for our purposes.

In chapter 4 we start the computation of the four-point functions we anticipated above. In this chapter we analyse the 1/4 and 1/8-BPS microstates in the D1D5 CFT and their dual asymptotically $\mathrm{AdS}_3 \times S^3 \times \mathcal{M}$ geometries by studying the holographic correlators of two light operators in a heavy state. Computations are performed either at the free point of the CFT or at the gravity point and a matching of the two results, not expected a priori, is explained by studying the correlators in terms of conformal blocks. In chapter 5 we continue the study of four-point functions changing the heavy states but keeping the same light operators. This turns out into a very different scenario from the previous one: indeed the free and the gravity result doesn't match anymore and the correlator shares a very non-trivial dynamics whose mechanisms and discussed in the course of chapters.

In chapter 6 we present a class of correlators slightly different from the ones studied in the two previous chapters but having their own interest. We consider the change of the light operators as well the heavy operators. Heavy states are described, in the gravity picture, by a different background solution while the change of the light operators implies the study of a fluctuation dual to different fields from the ones dual to the light operators used so far. Since the light operators used in this chapter are constructed with bosonic fundamental fields of the CFT, that are superdescendants of fermions we seek for a relation between the correlators with those two different kind of light operators. This relation turns out to be a Ward Identity (WI) for correlators and it will be a powerful tool to test all the result obtained.

Finally in chapter 7 we provide the results of the same type of correlators computed in previous chapters but in the black hole background, dual to the thermal ensemble. Therefore the analysis of differences between the correlators computed in pure or thermal state is carried out focusing mostly on the late time behavior as proposed in this introduction as a powerful way to test information loss signals.

Chapter 2

Black holes in five dimensions

Black holes in string theory with macroscopically large entropy can all be constructed out of various brane constituents. Supersymmetric black hole in dimensions $d \geq 4$ may be constructed from BPS D-brane building blocks. Typically, however, they have zero horizon area and therefore non-macroscopic entropy[1]. The essential reason behind this slightly annoying fact comes from the supergravity field equations [17]. The sizes and shapes of internal manifolds, as well as the dilaton, turn out to be controlled by scalar fields, and the horizon area is related to these scalars. But in any given dimension, there are only a few independent charges on a black hole, and mostly these give rise to too few independent ratios to give all the scalar fields well-behaved vevs everywhere in spacetime. For stringy black holes made by compactifying on tori, the only asymptotically flat BPS black holes with macroscopic finite area occur with three charges in $d = 5$ and four charges in $d = 4$ [18]. We focus here on supersymmetric black holes in five dimensions constructed by a configuration of D1 and D5 branes.

2.1 Black holes in D1D5 system

We introduce the D1D5 system analyzing the asymptotically flat black hole solutions obtained by considering a brane configuration with n_1 D1-branes wrapped on S^1 and n_5 D5-branes wrapped on $S^1 \times T^4$ and allowing n_p unit of momentum along S^1. We are going to show how to find the solution for the three-charge non-rotating extremal black hole in $d = 5$ preserving $1/8$ of the total supersymmetries of type IIB, usually called D1D5P black hole. We start giving the ingredients to find the solution then we compute the conserved charges and the Bekenstein-Hawking entropy, and we finally match it with the microscopic counting [1]. In the end we will have a discussion about other black hole solutions in this system.

2.1.1 Solution generating technique

In general, finding new solutions of supergravity actions can be quite difficult because the equations of motion are very nonlinear. There is a solution-generating method available in string theory which is purely algebraic and relies in the fact that a d-dimensional vacuum solution in GR can be lifted to a ten-dimensional one by adding flat directions. Then, one can perform Lorentz transformations, along these directions still having a solution, that, after reducing down again to d-dimensions, solves the equations of motion in presence of non zero gauge fields whose fluxes give the charges of the black hole, related, at the end, with the parameters of the Lorentz transformations.
In the context of string theory these gauge fields gives, throughout a chain of dualities,

[1] It's still possible to compute the non zero area of the horizon by considering stringy corrections to supergravity action.

non zero value to the NSNS and RR fields which couple with objects like strings and D-branes. We use now this technique to obtain the black hole under consideration, explicitly showing the first steps and giving all the ingredients to obtain the final solution that we will use to compute the conserved charges and entropy.

As starting point we take the metric for the five-dimensional Schwarzchild black hole solution that solves Einstein equations in the vacuum

$$ds_5^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_3, \quad f(r) = 1 - \frac{m}{r^2} \tag{2.1.1}$$

where m is the mass of the Schwarchild black hole [2]and the metric of the three-sphere defined by

$$d\Omega_3 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2 \tag{2.1.2}$$

As a next step we add five flat directions y and z_i with $i = 6,7,8,9$ such that the new metric is lifted to a ten dimensional solution of Type II supergravity with zero RR and with NSNS field given by

$$ds_{10}^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_3 + dy^2 + d\hat{s}_4^2, \quad e^{2\Phi} = 1, \quad B_2 = 0 \tag{2.1.3}$$

with $d\hat{s}_4^2 \equiv dz^i dz^i$. Since the solution is translational invariant along the flat directions we firstly perform a boost along the y direction parametrized by δ_1

$$\begin{pmatrix} y \\ t \end{pmatrix} \to \begin{pmatrix} \cosh\delta_1 & \sinh\delta_1 \\ \sinh\delta_1 & \cosh\delta_1 \end{pmatrix} \begin{pmatrix} y \\ t \end{pmatrix} \tag{2.1.4}$$

and we get the solution (we omit the dependence on r in the functions)

$$ds_{10}^2 = Z_1\left(dy + \frac{K_1}{Z_1}dt\right)^2 - Z_1^{-1}dt^2 + d\hat{s}_4^2 \quad e^{2\Phi} = 1, \quad B_2 = 0 \tag{2.1.5}$$

with the following definitions

$$Z_1 = 1 + (1-f)\sinh^2\delta_1, \quad K_1 = (1-f)\sinh\delta_1\cosh\delta_1 \tag{2.1.6}$$

Let's consider the above solution as a solution of type IIA and let's perform a chain of duality transformations that map solutions in other solutions following a set of rules summarized in appendix B.

We first compactify the flat directions such that

$$y \sim y + 2\pi R, \quad z_i \sim z_i + 2\pi R_i \tag{2.1.7}$$

and then a T-duality along y direction gives the F1 solution in type IIB

$$\begin{aligned} ds_{10}^2 &= Z_1^{-1}[dy^2 - dt^2 + ds_4^2] + d\hat{s}_4^2, \quad ds_4^2 \equiv dr^2 + r^2 d\Omega_3 \\ B_2 &= \frac{K_1}{Z_1} dt \wedge dy \\ e^{2\Phi} &= Z_1^{-1} \end{aligned} \tag{2.1.8}$$

[2]Note that this is not the physical mass of the new solution we are going to construct.

Now we want to add the second charge, by boosting along y direction, with parameter δ_5. The solution, called F1P solution, reads

$$\begin{aligned} ds_{10}^2 &= \frac{1}{\sqrt{Z_1}}\left(Z_5 dy - \frac{K_1}{Z_1}dt\right)^2 - \frac{f\,dt^2}{Z_1 Z_5} + f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega_3 + d\hat{s}_4^2 \\ B_2 &= \frac{K_1}{Z_1} dt\wedge dy \\ e^{2\Phi} &= Z_1^{-1} \end{aligned} \tag{2.1.9}$$

S-duality brings the above solution in the D1P frame where the solution, still in type IIB, reads

$$\begin{aligned} ds_{10}^2 &= Z_1^{\frac{1}{2}}(ds_{10}^2)_{\text{F1P}} \\ C_2 &= \frac{K_1}{Z_1} dt\wedge dy \\ e^{2\Phi} &= Z_1 \end{aligned} \tag{2.1.10}$$

The complete chain of dualities to use on the solution from the D1P frame is

$$\begin{pmatrix} D1 \\ P \end{pmatrix}^{\text{IIB}} \xrightarrow[T_{6789}]{} \begin{pmatrix} D5 \\ P \end{pmatrix}^{\text{IIB}} \xrightarrow[\text{S}]{} \begin{pmatrix} NS5 \\ P \end{pmatrix}^{\text{IIB}} \xrightarrow[T_y]{} \begin{pmatrix} NS5 \\ F1 \end{pmatrix}^{\text{IIA}} \xrightarrow[T_1]{} \begin{pmatrix} NS5 \\ F1 \end{pmatrix}^{\text{IIB}} \xrightarrow[\text{S}]{} \begin{pmatrix} D5 \\ D1 \end{pmatrix}^{\text{IIB}} \tag{2.1.11}$$

where we have indicated the directions along which we have to T-dualize and the type of string theory where we end up with after the transformations. The last step to get a three-charge black hole is to perform a boost parametrize by δ_p along y and the solution reads[3]

$$\begin{aligned} ds_{10}^2 = &\frac{Z_p}{(Z_1 Z_5)^{\frac{1}{2}}}\left(dy + \frac{K_p}{Z_p}dt\right)^2 - \frac{f}{Z_p\,(Z_1 Z_5)^{\frac{1}{2}}}dt^2 \\ &+ (Z_1 Z_5)^{\frac{1}{2}}\,ds_4^2 + \left(\frac{Z_5}{Z_1}\right)d\hat{s}_4^2 \end{aligned} \tag{2.1.12}$$

$$e^{2\Phi} = \frac{Z_5}{Z_1}$$

We are now going to take the extremal limit on the solution defined by [4]

$$\begin{aligned} m\to 0, \quad &\delta_1\to\infty, \quad m\sinh^2\delta_1 \equiv Q_1 \\ m\to 0, \quad &\delta_5\to\infty, \quad m\sinh^2\delta_5 \equiv Q_5 \\ m\to 0, \quad &\delta_p\to\infty, \quad m\sinh^2\delta_p \equiv Q_p \end{aligned} \tag{2.1.13}$$

[3]For our purposes, we write down only the metric and the dilaton of the entire solution that contains also RR fields whose explicit solution can be easily found by following the chain of dualities and by using the respective rules.

[4]This definition will become clear in the next subsection where we will discuss the extremal limit and the BPS bound between mass and charge.

that leads to the extremal ten-dimensional solution of the D1D5P black hole solution

$$
\begin{aligned}
ds_{10}^2 &= (Z_1Z_5)^{-\frac{1}{2}}\left(-dt^2+dy^2+K_p(dt+dy)^2\right)+(Z_1Z_5)^{\frac{1}{2}}\,ds_4^2\\
&\quad+\left(\frac{Z_1}{Z_5}\right)^{\frac{1}{2}}d\hat{s}_4^2 \qquad (2.1.14)\\
e^{2\Phi} &= \frac{Z_1}{Z_5}
\end{aligned}
$$

with the function Z_i and K_i defined as the functions defined in (2.1.6) in the extremal limit (2.1.13).

To finally obtain the expression for the five dimensional metric in Einstein frame we have to compactify the solution (2.1.14) on $T^4 \times S^1$. Using the results in appendix B for compactification on T^4 and using the value of dilaton we can see that the six dimensional part metric ds_6^2 of the ten dimensional string frame is already in Einsten frame. To further reduce on S^1 we write it down in standard Kaluza Klein form (see [19] for details)

$$
ds_6^2 = e^{2\alpha\Phi}ds_{5(E)}^2 + e^{2\beta\Phi}(A_\mu dx^\mu + dy)(A_\mu dx^\mu + dy) \qquad (2.1.15)
$$

where $ds_{5(E)}^2$ is solution of Einstein equation in five dimensions coupled with gauge field A_μ. We fix the constants α and β by requiring that the dimensionally reduced action is of the Einstein-Hilbert form and that the dilaton field acquires a kinetic term with canonical normalization. Using the explicit form of $ds_{(6)}^2$ we can extract the form of $ds_{5(E)}^2$ that finally gives the five dimensional extremal three charges black hole

$$
\begin{aligned}
ds_{5(E)}^2 &= -\frac{dt^2}{\left(1+\frac{Q_1}{r^2}\right)^{\frac{2}{3}}\left(1+\frac{Q_5}{r^2}\right)^{\frac{2}{3}}\left(1+\frac{Q_p}{r^2}\right)^{\frac{2}{3}}}\\
&\quad+\left(1+\frac{Q_1}{r^2}\right)^{\frac{1}{3}}\left(1+\frac{Q_5}{r^2}\right)^{\frac{1}{3}}\left(1+\frac{Q_p}{r^2}\right)^{\frac{1}{3}}\left(dr^2+r^2d\Omega_3\right)
\end{aligned}
\qquad (2.1.16)
$$

This is the $d = 5$ black hole studied in [1] and it is a BPS supergravity solution preserving 1/8 of the supersymmetries of Type IIB string theory, with finite horizon in $r = 0$ and a nonzero Bekenstein-Hawking entropy.

2.1.2 Conserved charges and entropy

Being a BPS solution, the black hole constructed above corresponds to an extremal black hole, with a total mass given by

$$
M = M_1 + M_5 + M_p \qquad (2.1.17)
$$

can be expressed as a sum of three masses related by the three charges Q_i by means of an extremal bound.

Since brane solutions follows superposition rules in order to have information about D1 charge we can just focus on the D1 solution that follows from (2.1.8) after S duality which further follows from (2.1.5) that, in the reduced theory corresponds to a Einstein solution coupled with gauge field. Thus, what we call charge of the D1 branes is intimately related to the flux of this gauge field in five dimensional theory and in the ten-dimensional version corresponds to the fluxes of the corresponding RR fields that couple with the branes. Concretely, we read the gauge field directly from (2.1.5)

$$A = \frac{K_1}{Z_1} dt \tag{2.1.18}$$

and its flux is given by the charge

$$Q \equiv \frac{1}{V_{S^3}} \int \star_3 dA \;=\; m \sinh\delta_1 \cosh\delta_1 \tag{2.1.19}$$

with V_{S^3} the volume of the three-sphere and the hodge operator defined on the three sphere.

To find the mass of the black hole we follow the same steps used to obtain the five dimensional black hole in Einstein frame and we extract the term

$$(g_{(5)})_{00} = \eta_{00} + h_{00} = -1 + \frac{m}{r^2}\left(1 + \frac{2}{3}\sinh^2\delta_1\right) \tag{2.1.20}$$

From Einstein equation in $d = 5$ we have the general result

$$R_{\mu\nu} = 8\pi G_5 \left(T_{\mu\nu} - \frac{g_{\mu\nu}}{3} T^\sigma_\sigma\right) \tag{2.1.21}$$

using the explicit form of the metric and the non relativistic assumption [19] $T_{00} >> T_{0i} >> T_{ij}$ we get, for the time components

$$\partial^2 h_{00} = -16\pi G_5 \frac{2}{3} T_{00} \tag{2.1.22}$$

The physical mass of the space time is now given by

$$M = \int d^4x\, T_{00} = -\frac{1}{16\pi G_5}\frac{2}{3}\int d^4x\, \partial^2 h_{00} \tag{2.1.23}$$

Using Stokes theorem and the explicit form of h_{00} in (2.1.20) we obtain

$$M_1 = V_{S^3}\left[\frac{3}{16\pi G_5} m \left(1 + \frac{2}{3}\sinh^2\delta_1\right)\right] \tag{2.1.24}$$

So we have explicit expressions for charge and mass of the five dimensional black hole that after dualities will become charges of the D1 branes. In the limit (2.1.13) we have the extremal bound[5]

$$16\pi G_5\, M = 2QV_{S^3} \tag{2.1.25}$$

Going in the direction of computing the entropy of the black hole and matching it with the microscopic counting become important to understand the charges Q_i in quantized units and to relate them with the number of branes that generates the black hole. We use the bound (2.1.38) to find the charge of the D1 brane in terms of its mass

$$Q_1 = \frac{16\pi G_5}{2(2\pi^2)} M_1 \tag{2.1.26}$$

The mass of a Dp-brane is given by $M_p = \tau_p V_p$ where τ_p is the tension of the brane and V_p is the volume of the manifold wrapped by the branes. For n_1 D1-branes on S^1 we

[5] The d-dimensional generalization of this expression reads $16\pi G_d\, M = (d-3)QV_{S^{d-2}}$

have

$$\tau_1 = \frac{1}{2\pi\alpha' g_s}, \quad V_{S^1} = 2\pi R, \quad M_1 = \frac{n_1 R}{\alpha' g_s} \tag{2.1.27}$$

Gathering the value of the mass, the bound (2.1.26) and the value of the Newton constant in five dimension we obtain [6]

$$Q_1 = \frac{(2\pi)^4}{V_4} g_s \alpha'^3 n_1 \tag{2.1.28}$$

For n_5 D5-branes and for n_p unit of momentum we have, following the same steps

$$Q_5 = n_5 g_s \alpha', \quad Q_p = n_p \frac{(2\pi)^4}{V_4} g_s^2 \frac{\alpha'^4}{R^2} \tag{2.1.29}$$

We have now all the ingredients to compute the Bekenstein Hawking entropy. It's possible to compute it directly from the five dimensional form in (2.1.16) or also in the ten dimensional Einstein frame solution coming from (2.1.14). In both case, using the expressions for the charges in (2.1.28), (2.1.29), we obtain the result

$$S_{BH} = \frac{A_5^E}{4G_5} = \frac{A_{10}^E}{4G_{10}} = 2\pi\sqrt{n_1 n_5 n_p} \tag{2.1.30}$$

Let's see now the entropy is reproduced by microscopic counting.

2.1.3 Microscopic Counting

In order to compute the entropy microscopically we have to find the degeneracies of states on D-branes. We firstly focus on D1D5 black hole whose counting of states can be performed [7, 20] in the F1P dual frame where we have a fundamental string winding the S^1 n_1 times and with an amount of momentum given by

$$P = \frac{n_p}{R} \tag{2.1.31}$$

Excitations on the strings come from the bosonic (B) and fermionic (F) quantum fields on the worldsheet X^μ, ψ^μ whose oscillator operators creates states with momentum and energy given by

$$e_k = |p_k| = \frac{2\pi k}{L} = \frac{k}{n_1 R} \tag{2.1.32}$$

where $L = 2\pi R n_1$ is the effective total length of the string. The respective partition functions read

$$Z_k^B = \sum_{n=0}^{\infty} e^{-\beta e_k n} = \frac{1}{1 - e^{-\beta e_k}}, \quad Z_k^F = \sum_{n=0}^{1} e^{-\beta e_k n} = 1 + e^{-\beta e_k} \tag{2.1.33}$$

and the total partition function is

$$Z = \left(\prod_{k=1}^{\infty} Z_k^B Z_k^F \right)^8 \tag{2.1.34}$$

[6]We have $G_5 = \frac{G_{10}}{V_4\, 2\pi R} = \frac{8\pi^6 g_s^2 \alpha'^4}{V_4\, 2\pi R}$

Since the entropy is given by the Legendre transformation of the free energy $F = \log Z$ we approximate the sum with an integral over k, in the large n_1 limit

$$\begin{aligned}
\sum_{k=1}^{\infty} \log Z_k^B &\simeq \int_0^{\infty} dk \, \log(1 - e^{-\frac{\beta}{n_1 R}k}) = \frac{\pi^2}{6}\frac{N_1 R}{\beta} \\
\sum_{k=1}^{\infty} \log Z_k^F &\simeq \int_0^{\infty} dk \, \log(1 + e^{-\frac{\beta}{n_1 R}k}) = \frac{\pi^2}{12}\frac{N_1 R}{\beta} \\
\log Z = 8\sum_{k=1}^{\infty} &\left(\log Z_k^B + \log Z_k^F\right) = 8\left[\left(1 + \frac{1}{2}\right)\frac{\pi^2}{6}\frac{N_1 R}{\beta}\right] \equiv c\frac{\pi^2}{6}\frac{n_1 R}{\beta}
\end{aligned} \tag{2.1.35}$$

where c is the central charge each space time direction of the fields on the worldsheet. We can find the internal energy of the string, given by its total momentum, from the free energy such that we obtain the relation

$$E = \frac{n_p}{R} = -\partial_\beta \log Z = \frac{c\pi^2}{6}\frac{n_1 R}{\beta^2} \tag{2.1.36}$$

thanks to which we can find $\beta = R\left(\frac{c\pi^2 n_1}{6 n_p}\right)^{\frac{1}{2}}$. Using the previous results, we get the entropy of the F1P system

$$S = \log Z + \beta E = 2\pi \left(\frac{c}{6}\right)^{\frac{1}{2}} (n_1 n_p)^{\frac{1}{2}} \tag{2.1.37}$$

The counting in the D1D5 system comes from the chain of dualities that sends n_1 into n_5 and n_p into n_1 and setting $c = 12$.

To obtain the entropy of the D1D5P system we have to consider the D1 branes in the D1D5P system, as an effective D1 brane with winding $n_1 n_5$, we can use the result of D1D5 entropy to obtain the entropy for the D1D5P black hole

$$S_{micro} = 2\pi\sqrt{n_1 n_5 n_p} \tag{2.1.38}$$

that reproduces, in the large n_1, n_5, n_p limit, the Bekenstein Hawking entropy (2.1.30).

2.1.4 Rotating $d = 5$ black holes

One can also consider the rotating three-charge black hole obtained with the same procedure we showed but replacing the starting metric in (2.1.1) with a rotating solution instead of a Schwarchild solution. In [21] a seven parameter family of non extremal five-dimensional black hole solutions depending on mass, two angular momenta, three charges is constructed. Indeed in five dimensions the rotation is specified by two parameters a_1, a_2. Thus we start from the following metric

$$\begin{aligned}
ds_5^2 = &-\left(1 - \frac{m}{f(r)}\right) dt^2 + \left(\frac{r^2 f(r)}{(a_1^2 + r^2)(a_2^2 + r^2) - mr^2}\right) dr^2 \\
&+ f(r) d\theta^2 + \sin^2\theta \left(a_2^2 + r^2 + \frac{a_2^2 m \sin^2\theta}{f(r)}\right) d\phi^2 + \cos^2\theta\left(a_1^2 + r^2 + \frac{a_2^2 m \cos^2\theta}{f(r)}\right) d\psi^2 \\
&+ \frac{2a_1 a_2 m \cos^2\theta \sin^2\theta}{f(r)} d\phi d\psi + \frac{2a_2 m \sin^2\theta}{f(r)} d\phi dt + \frac{2a_1 m \cos^2\theta}{f(r)} d\psi dt
\end{aligned} \tag{2.1.39}$$

with $f(r)$ defined as

$$f(r) = r^2 + a_1^2 \sin^2\theta + a_2^2 \cos^2\theta \tag{2.1.40}$$

To obtain the three-charge rotating black hole in D1D5 frame we have to lift this solution to ten dimensions by adding five more flat direcitons and following the same steps as in the previous case. After the chain of boost and dualities we get the ten-dimensional non-extremal solution in string frame. We refer to [22] for complete expressions of the solution. To get the extremal solution it useful to define

$$\begin{aligned}\gamma_1 &\equiv -\frac{m}{\sqrt{Q_1 Q_5}}\left(a_1 \cosh\delta_1 \cosh\delta_5 \cosh\delta_p - a_2 \sinh\delta_1 \sinh\delta_5 \sinh\delta_p\right) = \frac{J_\psi}{\sqrt{Q_1 Q_5}}\\ \gamma_2 &\equiv -\frac{m}{\sqrt{Q_1 Q_5}}\left(a_2 \cosh\delta_1 \cosh\delta_5 \cosh\delta_p - a_1 \sinh\delta_1 \sinh\delta_5 \sinh\delta_p\right) = \frac{J_\phi}{\sqrt{Q_1 Q_5}}\end{aligned} \tag{2.1.41}$$

The extremal limit is defined as in (2.1.13) with additional suitable limits for a_1 and a_2 so that the angular momenta J_ϕ, J_ψ are held fixed. Inverting (2.1.41) and using the extremal limit (2.1.13) we find

$$\begin{aligned}a_1 &= -(\gamma_1+\gamma_2)\eta\sqrt{\frac{Q_p}{m}} - \frac{\gamma_1-\gamma_2}{4}\sqrt{\frac{m}{Q_p}} + O(m^{3/2})\\ a_2 &= -(\gamma_1+\gamma_2)\eta\sqrt{\frac{Q_p}{m}} + \frac{\gamma_1-\gamma_2}{4}\sqrt{\frac{m}{Q_p}} + O(m^{3/2})\end{aligned} \tag{2.1.42}$$

with

$$\eta \equiv \frac{Q_1 Q_5}{Q_1 Q_5 + Q_1 Q_p + Q_5 Q_p} \tag{2.1.43}$$

We thus see that for generic values of γ_1, γ_2 and Q_p the parameters a_1 and a_2 diverge when $m \to 0$. There are two exceptions: $Q_p = 0$ and $\gamma_1 + \gamma_2 = 0$. The latter case is the case studied in [21] and in this case the extremal limit on a_1, a_2 is defined so that they go to zero as $\sqrt{m}$ that gives

$$J_\phi = -J_\psi \equiv J \tag{2.1.44}$$

Computing the area of the horizon of this black hole is straightforward to get the result for the entropy

$$S = 2\pi\sqrt{n_1 n_5 n_p - J^2} \tag{2.1.45}$$

In the context of extremal solution we can take the limit of the three-charge black hole in the limit of zero momentum Q_p. This solution is the two-charge D1D5 black hole preserving 1/4 of the total supersymmetries. As we already mentioned, in the supergravity description this geometry is not actually a black hole, indeed the horizon in $r = 0$ coincides with the singularity and to get a finite horizon we have to go beyond the classical description and take into account string corrections. It is however of great interest because, all of the microstates solutions for this black hole have been found and explicitly written as smooth, horizonless supergravity solution with the same asymptotically charges of the naive solutions. We will mainly concentrate on this solution and on its microstates.

Chapter 3

Microscopy of D1D5 System

In this chapter we discuss the microscopic description of the D1D5 system coming from dynamic of the fundamental objects arising in string theory, in this system. Since we are dealing with black hole solutions in string theory, coming from a bound state of D-branes, it turns out to be important to understand the dynamics of these objects. D-branes are non perturbative objects present in string theory which couple with open and closed strings that can be viewed as elementary excitations of the branes. In particular the open string sector of massless excitations can be described by a quantum field theory on the brane while the closed string sector gives a theory of gravity in the bulk giving a dual description for the same physical system. The two sectors are in general coupled and governed by the moduli and couplings present in string theory, and depending on these couplings, one can consider either open string description of the brane configuration or a closed string description. The formulation of this duality have been made precise and treatable in the context of AdS/CFT correspondence that provides a dictionary and a mapping between the two sides.

We identify now the two sides and the couplings in our setup in order to estabilish the correspondence. Recalling the setup of chapter 2 we have type IIB string theory compactified on $T^4 \times S^1$ with a bound state of n_5 D5-branes wrapping the whole compact space and n_1 D1-branes wrapping the circle, S^1. The volume of T^4 will be denoted with V_4, while the radius of S^1 is R. Therefore the different regimes of the system will be controlled by the set of couplings

$$(n_1,\, n_5,\, V_4,\, R,\, g_s,\, \alpha') \tag{3.0.1}$$

where we also included the two fundamental parameters of string theory g_s, α'. We will work in the following region of parameter space:

$$V_4 \sim O(\alpha'^2), \quad R \gg \sqrt{\alpha'} \tag{3.0.2}$$

In closed string description, in order for classical to be a good description, we need the curvature to be small with respect to the string scale. We also need the string coupling to be small so that we do not need to consider quantum corrections. These requirements translate into a large charges Q_1 and Q_5 limit that leads to

$$n_1, n_5 \gg \frac{1}{g_s} \gg 1 \tag{3.0.3}$$

In order to make use of the AdS/CFT correspondence, we have to take a near-horizon limit that decouples the asymptotic flat physics from the AdS physics. This decoupling limit corresponds to going to the IR fixed point of the D-brane description, and is in effect a low-energy limit. As we will see in details in section 3.3, in this limit, the supergravity solutions becomes asymptotically $AdS_3 \times S^3 \times T^4$.

On the open string side, the effective field theory description of D-branes results from the open string zero modes. Working within perturbative string theory, the D-branes define boundary conditions for open strings. Let us recall the heuristic picture of how the bosonic degrees of freedom arise from open strings on a Dp-brane. The open strings with directions parallel to the branes have Neumann boundary conditions and give rise to a $U(n_p)$ gauge field. The open strings with directions perpendicular to the branes have Dirichlet boundary conditions and give adjoint scalars of the p+1-dimensional theory. The adjoint scalars describe the transverse oscillations of the D-branes. When all the D-branes are coincident, the gauge theory is said to be in the Higgs phase, and when some of the D-branes are separated, the gauge theory is said to be in the Coulomb phase. The gauge theory has 16 supercharges, since a stack of D-branes breaks half of the 32 supercharges of type IIB string theory. The coupling constant for the worldvolume gauge theory of a Dp-brane can be identified as

$$(g_{YM,p})^2 = g_s(2\pi)^{p-2}\alpha'^{\frac{p-3}{2}} \tag{3.0.4}$$

In the large-charge limit, then we want to keep the 't Hooft couplings small. Thus in our case the brane description should be weakly coupled when

$$1 \ll n_1, n_5 \ll \frac{1}{g_s} \tag{3.0.5}$$

There are three types of strings we may consider: 5-5 strings with both endpoints on D5 branes; 1-1 strings with both endpoints on D1 branes; and 1-5 and 5-1 strings with one endpoint on a D1 and one endpoint on a D5. It is common in literature to organize the fields coming from all these sectors in supersymmetry multiplet, ending up with a gauge theory with a set of fields or multiplet and a set of coupling constants given by (3.0.4). We refer to [23, 24] for a complete discussion on the gauge theory arising from open string sectors on branes and here we limit ourselves to mention that the Higgs branch of the D1D5 system flows in the infrared to two-dimensional $\mathcal{N} = (4, 4)$ SCFT with target space given by the orbifold

$$\mathcal{M} = \frac{(T^4)^N}{S_N}, \quad N = n_1 n_5 \tag{3.0.6}$$

AdS/CFT arises from open-closed string duality. The supergravity description arises from the low-energy behavior of the closed string modes. The CFT description that arises from the low-energy behavior of the open string modes. In fact, the behavior of the open string modes should first give rise to a nonconformal field theory. We then RG flow to the IR fixed point CFT. This corresponds to taking the near-horizon limit of the gravity description. Just as there is a twenty-dimensional near-horizon moduli space for the gravity description, there is a twenty-dimensional moduli space of CFTs as we will see in the next sections. This helps make the connection between RG flow and the near-horizon limit more precise.

The chapter is structured as follow: in section 3.1 we present and discuss the D1D5 CFT at the free orbifold point. We will set the definitions and notations used in the entire thesis and we provide all the ingredients necessary. We discuss the symmetries of the theory, the spectrum, and the microstates constructed in literature and used in this work. In section 3.2 we give the gravity picture of the D1D5 system by exploiting the general solution in this system and the equation it has to satisfy. Moreover we also specialize in the microstates solution dual to the CFT state. In the last section we discuss more in detail the AdS/CFT correspondence and we provide the dictionary between the microstates presented before.

3.1 D1D5 CFT at the orbifold point

The field theory describing open strings on the D1D5 brane worldvolume, flows in the infrared to a two dimensional superconformal field theory with base space given by the cylinder (t, y) of radius R and as a target space the orbifold $(T^4)^N/S_N$ with $N = n_1 n_5$. We find more convenient to Wick rotate to Euclidean time and define the coordinates

$$t_e \equiv i\tau \equiv \frac{it}{R}, \quad \sigma \equiv \frac{y}{R} \tag{3.1.1}$$

and map the cylinder to a dimensionless complex plane

$$z = e^{\tau+i\sigma}, \quad \bar{z} = e^{\tau-i\sigma} \tag{3.1.2}$$

breaking the theory into left and right-movers[1].

We define the theory in terms of field content, symmetries, and the spectrum of states/operators, giving a free field realizations of them, providing, in the last subsection we the chiral spectrum, relevant for the supergravity analysis.

3.1.1 Field content and symmetries

The symmetries of the theory are generated by the $\mathcal{N} = (4,4)$ superconformal algebra, spanned, at every point in the complex plane[2], by the following local operators: the stress energy tensor $T(z)$, four supersymmetry currents $G^{\alpha A}(z)$, and a $SO(4)_R = SU(2)_L \times SU(2)_R$ R-symmetry current $J^a(z)$. We also have a global symmetry $SO(4)_I = SU(2)_1 \times SU(2)_2$ implementing the symmetry of the torus, and existing only at the free point. We use the following conventions for the fundamental representation indices of the various symmetry groups

$$\begin{array}{llll} \alpha,\beta & SU(2)_L, & \dot{\alpha},\dot{\beta} & SU(2)_R \\ A,B & SU(2)_1, & \dot{A},\dot{B} & SU(2)_2 \end{array} \tag{3.1.3}$$

while for the vectorial representation of $SO(4)_I$ we use the indices i, j and the usual Pauli matrices $(\sigma^i)^{\dot{A}A}$ for changing basis. The vectorial indices of $SU(2)_{L/R}$ are denoted with a, b, c.

In the free field description the theory has a formulation in terms of bosons and fermions

$$\left(X^{A\dot{A}}(z,\bar{z})_{(r)}, \quad \psi^{\alpha\dot{A}}_{(r)}(z), \, \bar{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z})\right) \tag{3.1.4}$$

Since the target space is an orbifold, coming from a discrete identification of a symmetric product of a manifold, it is natural to introduce the concept of a copy or *strand* with an associate index (r) with $r = 1, \cdots, N$, giving the number of the strand. Still because of the form of the target space, all the fields, operators or states can be written as symmetryzed object with respect the strand index. Moreover, the central charge of a single copy is given by $c = 6$ while the central charge of the orbifold theory is $c = 6N$. In the following, we use the parenthetical subscript index (r) when we are explicitly referring to a single strand and we omit the index when we are dealing with object in the orbifold

[1] We refer to the left movers sector (L) as the holomorphic sector, depending only on z, and the right sector (R) as the antiholomorphic, depending on $\bar{z}$.

[2] We refer only to the left sector with straightforward generalization to the right sector.

theory.

The algebra of the fields on a single strand is given by the OPEs[3]

$$\begin{aligned} X^{A\dot{A}}(z)_{(r)} X^{B\dot{B}}(w)_{(s)} &\sim \frac{\epsilon^{\dot{A}\dot{B}}\epsilon^{AB}}{(z-w)^2}\delta_{rs} \\ \psi^{\alpha\dot{A}}(z)_{(r)} \psi^{\beta\dot{B}}(w)_{(s)} &\sim \frac{\epsilon^{\alpha\beta}\epsilon^{AB}}{(z-w)^2}\delta_{rs} \end{aligned} \tag{3.1.6}$$

In terms of free fields we can express the generators of the theory as

$$\begin{aligned} T(z) &= \frac{1}{2}\sum_{r=1}^{N} \epsilon_{\dot{A}\dot{B}}\epsilon_{AB}\partial X^{A\dot{A}}_{(r)}\partial X^{B\dot{B}}_{(r)} + \frac{1}{2}\sum_{r=1}^{N}\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}}\psi^{\alpha\dot{A}}_{(r)}\partial\psi^{\beta\dot{B}}_{(r)} \\ G^{\alpha A}(z) &= \sum_{r=1}^{N}\psi^{\alpha\dot{A}}_{(r)}\partial X^{\dot{B}A}_{(r)}\epsilon_{\dot{A}\dot{B}} \\ J^a(z) &= \frac{1}{4}\sum_{r=1}^{N}\epsilon_{\dot{A}\dot{B}}\psi^{\alpha\dot{A}}_{(r)}\epsilon_{\alpha\beta}(\sigma^{*a})^{\beta}_{\gamma}\psi^{\gamma\dot{B}}_{(r)} \end{aligned} \tag{3.1.7}$$

which gives the superconformal current algebra OPEs

$$\begin{aligned} T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \\ G^{\alpha A}(z)G^{\beta B}(w) &\sim -\frac{c}{3}\frac{\epsilon^{\alpha\beta}\epsilon^{AB}}{(z-w)^3} + \epsilon^{AB}\epsilon^{\beta\gamma}(\sigma^{*a})^{\alpha}_{\gamma}\left[\frac{2J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)}\right] - \epsilon^{\alpha\beta}\epsilon^{AB}\frac{T(w)}{(z-w)} \\ J^a(z)J^b(w) &\sim \frac{c}{12}\frac{\delta^{ab}}{(z-w)^2} + i\epsilon^{ab}_{c}\frac{J^c(w)}{(z-w)} \\ T(z)J^a(w) &\sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)} \\ T(z)G^{\alpha A}(w) &\sim \frac{3}{2}\frac{G^{\alpha A}(w)}{(z-w)^2} + \frac{\partial G^{\alpha A}(w)}{(z-w)} \\ J^a(z)G^{\alpha A}(w) &\sim \frac{1}{2}(\sigma^{*a})^{\alpha}_{\beta}\frac{G^{\beta A}(w)}{(z-w)} \end{aligned} \tag{3.1.8}$$

[3] We use the basis for $SU(2)_{L/R}$ with $\alpha,\ \beta = +,-,3$ and with the following convention for the relevanti constant tensors

$$\begin{aligned} &\delta^{\pm\pm} = \delta_{\pm\pm} = 0, \quad \delta^{\mp\pm} = \delta_{\mp\pm} = 2 \\ &\epsilon^{\pm\pm} = \epsilon_{\pm\pm} = \frac{1}{2}, \quad \epsilon_{\pm\mp} = \epsilon^{\pm\mp} = 0 \end{aligned} \tag{3.1.5}$$

Using the mode decomposition of an operator on the plane, [4] the set of OPEs in (3.1.6) generates the infinite dimensional affine algebra

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \\
\left[J^a_m, J^b_n\right] &= \frac{c}{12}m\,\delta^{ab}\delta_{m+n,0} + i\epsilon^{abc}J^c_{m+n} \\
\left\{G^{\alpha A}_m, G^{\beta B}_m\right\} &= -\frac{c}{6}\left(m^2-\frac{1}{4}\right)\epsilon^{\alpha\beta}\epsilon^{AB}\delta_{m+n,0} + (m-n)\epsilon^{AB}\epsilon^{\beta\gamma}(\sigma^{*a})^\alpha_\gamma J^a_{m+n} - \epsilon^{AB}\epsilon^{\alpha\beta}L_{m+n} \\
\left[J^a_m, G^{\alpha A}_n\right] &= \frac{1}{2}(\sigma^{*a})^\alpha_\beta G^{\beta A}_{m+n} \\
[L_m, J^a_n] &= -nJ^a_{m+n} \\
\left[L_m, G^{\alpha A}_n\right] &= -\left(\frac{m}{2}-n\right)G^{\alpha A}_{m+n}
\end{aligned} \tag{3.1.10}$$

The subalgebra restricted to the modes $\left\{L_0, L_{\pm 1}, J^a_0, G^{\alpha A}_{\pm\frac{1}{2}}\right\}$ is anomaly free and generates the group $SU(1,1|2)$.

3.1.2 Spectrum

The Hilbert space of a theory is, in general, a sum of irreducible representations of the symmetry algebra. Irreducible representations of an affine algebra, that is the algebra we have in our theory, can be constructed starting from an highest weight state $|\phi\rangle$, called *primary* and annihilated by all the positive modes of the generators, and acting upon it with all the negative modes, generating all the *descendants*. The set of all primaries is given by the theory. There is however different definitions of primary, depending on the basis of generators we want to work with. In particular we define

$$\begin{aligned}
&\textit{Virasoro primary:} \quad L_n|\phi\rangle = 0 \quad n>0 \\
&\textit{Affine primary:} \quad J^a_n|\phi\rangle = 0 \quad n>0
\end{aligned} \tag{3.1.11}$$

In general the states are labelled by the eigenvalue of the Cartan subalgebra spanned by $\{L_0,\, J^3_0\}$, which means we may label states by their conformal dimension h and spin m. Once we chose the basis, we have to identify the primaries and then we can generate all the other descendants and fill the spectrum.

In D1D5 CFT there are different Hilbert space sectors, given by different boundary conditions that in turn, give integer or half integer mode expansion for the operators. In particular we have two differents boundary conditions on the fermions that splits the Hilbert space into two Ramond (R) and Neveu-Schwarz (NS) sectors

$$\begin{aligned}
\text{R:} \quad \psi^{\alpha\dot{A}}_{(r)}(e^{2\pi i}z) &= -\psi^{\alpha\dot{A}}_{(r)}(z) \\
\text{NS:} \quad \psi^{\alpha\dot{A}}_{(r)}(e^{2\pi i}z) &= \psi^{\alpha\dot{A}}_{(r)}(z)
\end{aligned} \tag{3.1.12}$$

The above conditions are reversed if we came back to the cylinder coordinates and R sector is given by periodic boundary conditions while the NS by antiperiodic ones.

[4] For operators of conformal dimension Δ we have

$$\mathcal{O}_n = \oint \frac{dz}{2\pi i}\mathcal{O}(z)z^{\Delta+m-1}, \quad \mathcal{O}(z) = \sum_n \mathcal{O}_n z^{-\Delta-n} \tag{3.1.9}$$

Because of the orbifold nature of the theory we can also have mixed boundary, that for a generic operator $\mathcal{O}(z)$ read

$$\begin{aligned} \text{Untwisted:} \quad \mathcal{O}_{(r)}(e^{2\pi i}z) = (-1)^{\epsilon}\mathcal{O}_{(r)}(z) \\ \text{Twisted:} \quad \mathcal{O}_{(r)}(e^{2\pi i}z) = (-1)^{\epsilon}\mathcal{O}_{(r+1)}(z) \end{aligned} \tag{3.1.13}$$

where $\epsilon = 0, 1$ needs to take into account the correct periodicity of the R or NS sector in case we are dealing with fermionic operators.

We will treat the untwisted and twisted case seperately and inside each of these sector, we discuss the NS and R sectors working in the complex plane, in the left sector, with straightforward generalization in the right sector.

Untwisted sector

In the untwisted sector we have the theory that is a collection of N strands of *length* or *twist* $k = 1$.

In the Neveu-Schwarz sector, the boundary conditions for bosons and fermions read

$$\begin{aligned} \partial X^{A\dot{A}}_{(r)}(e^{2\pi i}z) = \partial X^{A\dot{A}}_{(r)}(z), \quad \bar{\partial} X^{A\dot{A}}_{(r)}(e^{-2\pi i}\bar{z}) = \bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z}) \\ \psi^{\alpha\dot{A}}_{(r)}(e^{2\pi i}z) = \psi^{\alpha\dot{A}}_{(r)}(z), \quad \bar{\psi}^{\alpha\dot{A}}_{(r)}(e^{-2\pi i}\bar{z}) = \bar{\psi}^{\alpha\dot{A}}_{(r)}(\bar{z}) \end{aligned} \tag{3.1.14}$$

that gives the mode expansion

$$\begin{aligned} \partial X^{A\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \alpha^{A\dot{A}}_{(r)n} z^{-n-1}, \quad \bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{\alpha}^{A\dot{A}}_{(r)n} \bar{z}^{-n-1} \\ \psi^{\alpha\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi^{\alpha\dot{A}}_{(r)n} z^{-n-\frac{1}{2}}, \quad \bar{\psi}^{\alpha\dot{A}}_{(r)}(\bar{z}) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \bar{\psi}^{\alpha\dot{A}}_{(r)n} \bar{z}^{-n-\frac{1}{2}} \end{aligned} \tag{3.1.15}$$

Similar expansion and conditions are valid for other operators, like the ones generating the currents algebra (3.1.6). As the above, the bosonic generators have integer mode expansion, while fermionic generators are be expanded in terms of half integers modes.

The first primary we have is the vacuum state, associated with the identity operator, and it corresponds to a primary with $(h, m) = (0, 0)$. It can be defined in each copy the tensor product of a vacuum state for the bosons and for the fermions, and each of them is in turn, a product of a vacuum state for the holomorphic sector and one for the antiholomorphic sector.

The bosonic vacuum state on one copy is defined as[5]

$$\alpha^{A\dot{A}}_{(r)n}|0\rangle^{X}_{(r)} = 0, \quad \bar{\alpha}^{A\dot{A}}_{(r)n}|0\rangle^{X}_{(r)} = 0, \quad \forall n \geq 0,\ A, \dot{A} = 1, 2 \tag{3.1.16}$$

whereas the vacuum state in the NS fermionic sector is defined as

$$\psi^{\alpha\dot{A}}_{(r)n}|0\rangle^{\text{NS}}_{(r)} = 0, \quad \bar{\psi}^{\dot{\alpha}\dot{A}}_{(r)n}|0\rangle^{\text{NS}}_{(r)} = 0, \quad \forall n \geq 0,\ A = 1, 2,\ \alpha, \dot{\alpha} = \pm \tag{3.1.17}$$

[5]It is not strictly necessary that the boson zero modes annihilate the vacuum, but relaxing this condition allows momentum along one of the directions of the T4, which is a charge that we don't want.

In the Ramond sector we have

$$
\begin{aligned}
\partial X^{A\dot{A}}_{(r)}(e^{2\pi i}z) &= \partial X^{A\dot{A}}_{(r)}(z), \quad \bar{\partial} X^{A\dot{A}}_{(r)}(e^{-2\pi i}\bar{z}) = \bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z}) \\
\psi^{\alpha\dot{A}}_{(r)}(e^{2\pi i}z) &= -\psi^{\alpha\dot{A}}_{(r)}(z), \quad \bar{\psi}^{\alpha\dot{A}}_{(r)}(e^{-2\pi i}\bar{z}) = -\bar{\psi}^{\alpha\dot{A}}_{(r)}(\bar{z})
\end{aligned}
\tag{3.1.18}
$$

that gives the mode expansion

$$
\begin{aligned}
\partial X^{A\dot{A}}_{(r)}(z) &= \sum_{n\in\mathbb{Z}} \alpha^{A\dot{A}}_{(r)n} z^{-n-1}, \quad \bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{\alpha}^{A\dot{A}}_{(r)n} \bar{z}^{-n-1} \\
\psi^{\alpha\dot{A}}_{(r)}(z) &= \sum_{n\in\mathbb{Z}} \psi^{\alpha\dot{A}}_{(r)n} z^{-n-\frac{1}{2}}, \quad \bar{\psi}^{\alpha\dot{A}}_{(r)}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{\psi}^{\alpha\dot{A}}_{(r)n} \bar{z}^{-n-\frac{1}{2}}
\end{aligned}
\tag{3.1.19}
$$

The bosonic vacuum is still defined in the same way as in the NS sector while we define the R vacuum as the set of degenerate ground states coming from the non trivial action of the zero modes of the fermions $\psi^{\alpha\dot{A}}_{(r)0}$. The total of sixteen R vacua with $h = \frac{1}{4}$ coming from all the possible non zero combinations of $(\alpha, \dot{A})$ indices in the left and in the right sector can be represented by

$$
|\alpha\dot{\alpha}\rangle, \quad |\alpha\dot{A}\rangle, \quad |\dot{A}\dot{\alpha}\rangle, \quad |\dot{A}\dot{A}\rangle \tag{3.1.20}
$$

As example, that will be important in the next chapters, we write down the fermionic highest weight R vacuum state with $\alpha = \dot{\alpha} = +$

$$
|++\rangle_{(r)}, \quad (h, m) = \left(\frac{1}{4}, \frac{1}{2}\right) \tag{3.1.21}
$$

with the condition

$$
\psi^{+\dot{A}}_{(r)0}\bar{\psi}^{+\dot{A}}_{(r)0}|++\rangle_{(r)} = 0, \quad \forall\, \dot{A} \tag{3.1.22}
$$

In the orbifold theory, the R vacuum reads

$$
|++\rangle \equiv \prod_{r=1}^{N} |++\rangle_{(r)} = |++\rangle^N, \quad (h, m) = \left(\frac{N}{4}, \frac{N}{2}\right) \tag{3.1.23}
$$

Twisted sector

In a twisted sector, the boundary conditions mix different copies of the CFT to form a strand of length k, which means that we have the following periodicites for the bosons

$$
\partial X^{A\dot{A}}_{(r)}(e^{2\pi i}z) = \partial X^{A\dot{A}}_{(r+1)}(z), \quad \bar{\partial} X^{A\dot{A}}_{(r)}(e^{-2\pi i}\bar{z}) = \bar{\partial} X^{A\dot{A}}_{(r+1)}(\bar{z}) \tag{3.1.24}
$$

with the identification $\partial X^{A\dot{A}}_{(k+1)} = \partial X^{A\dot{A}}_{(1)}$. We see that the boundary condition are not diagonal but it is still possible to diagonalize the boundary conditions by taking linear combinations of the fields on different copies: it's essentially a change of basis and we label the independent fields of this new basis with the index $\rho = 0, \ldots, k-1$,

$$\partial X_\rho^{1\dot{1}}(z) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{-2\pi \mathrm{i} \frac{r\rho}{k}}\, \partial X_{(r)}^{1\dot{A}}(z), \quad \partial X_\rho^{2\dot{2}}(z) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{2\pi \mathrm{i} \frac{r\rho}{k}}\, \partial X_{(r)}^{2\dot{2}}(z) \tag{3.1.25a}$$

$$\partial X_\rho^{1\dot{2}}(z) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{2\pi \mathrm{i} \frac{r\rho}{k}}\, \partial X_{(r)}^{1\dot{2}}(z), \quad \partial X_\rho^{2\dot{1}}(z) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{-2\pi \mathrm{i} \frac{r\rho}{k}}\, \partial X_{(r)}^{2\dot{1}}(z) \tag{3.1.25b}$$

$$\bar\partial X_\rho^{1\dot{1}}(\bar z) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{2\pi \mathrm{i} \frac{r\rho}{k}}\, \bar\partial X_{(r)}^{1\dot{1}}(\bar z), \quad \bar\partial X_\rho^{2\dot{1}}(\bar z) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{-2\pi \mathrm{i} \frac{r\rho}{k}}\, \bar\partial X_{(r)}^{2\dot{1}}(\bar z) \tag{3.1.25c}$$

$$\bar\partial X_\rho^{1\dot{2}}(\bar z) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{-2\pi \mathrm{i} \frac{r\rho}{k}}\, \bar\partial X_{(r)}^{1\dot{2}}(\bar z), \quad \bar\partial X_\rho^{2\dot{1}}(\bar z) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{2\pi \mathrm{i} \frac{r\rho}{k}}\, \bar\partial X_{(r)}^{2\dot{1}}(\bar z) \tag{3.1.25d}$$

$$\tag{3.1.25e}$$

with the (diagonalized) monodromy conditions in the ρ basis now being

$$\partial X_\rho^{1\dot{1}}\left(e^{2\pi\mathrm{i}} z\right) = e^{2\pi\mathrm{i}\frac{\rho}{k}}\, \partial X_\rho^{1\dot{1}}(z), \qquad \partial X_\rho^{2\dot{2}}\left(e^{2\pi\mathrm{i}} z\right) = e^{-2\pi\mathrm{i}\frac{\rho}{k}}\, \partial X_\rho^{2\dot{2}}(z) \tag{3.1.26a}$$

$$\partial X_\rho^{1\dot{2}}\left(e^{2\pi\mathrm{i}} z\right) = e^{-2\pi\mathrm{i}\frac{\rho}{k}}\, \partial X_\rho^{1\dot{2}}(z), \qquad \partial X_\rho^{2\dot{1}}\left(e^{2\pi\mathrm{i}} z\right) = e^{2\pi\mathrm{i}\frac{\rho}{k}}\, \partial X_\rho^{2\dot{1}}(z) \tag{3.1.26b}$$

$$\bar\partial X_\rho^{1\dot{1}}(e^{-2\pi\mathrm{i}}\, \bar z) = e^{-2\pi\mathrm{i}\frac{\rho}{k}}\, \bar\partial X_\rho^{1\dot{1}}(\bar z), \qquad \bar\partial X_\rho^{2\dot{2}}(e^{-2\pi\mathrm{i}}\, \bar z) = e^{2\pi\mathrm{i}\frac{\rho}{k}}\, \bar\partial X_\rho^{2\dot{2}}(\bar z) \tag{3.1.26c}$$

$$\bar\partial X_\rho^{1\dot{2}}(e^{-2\pi\mathrm{i}}\, \bar z) = e^{2\pi\mathrm{i}\frac{\rho}{k}}\, \bar\partial X_\rho^{1\dot{2}}(\bar z), \qquad \bar\partial X_\rho^{2\dot{1}}(e^{-2\pi\mathrm{i}}\, \bar z) = e^{-2\pi\mathrm{i}\frac{\rho}{k}}\, \bar\partial X_\rho^{2\dot{1}}(\bar z) \tag{3.1.26d}$$

$$\tag{3.1.26e}$$

Then the standard mode expansion following from (3.1.26) are

$$\partial X_\rho^{1\dot{1}}(z) = \sum_{n\in\mathbf{Z}} \alpha^{1\dot{1}}_{\rho, n-\frac{\rho}{k}} z^{-n-1+\frac{\rho}{k}}, \qquad \partial X_\rho^{2\dot{2}}(z) = \sum_{n\in\mathbf{Z}} \alpha^{2\dot{2}}_{\rho, n+\frac{\rho}{k}} z^{-n-1-\frac{\rho}{k}} \tag{3.1.27a}$$

$$\partial X_\rho^{1\dot{2}}(z) = \sum_{n\in\mathbf{Z}} \alpha^{1\dot{2}}_{\rho, n+\frac{\rho}{k}} z^{-n-1-\frac{\rho}{k}}, \qquad \partial X_\rho^{2\dot{1}}(z) = \sum_{n\in\mathbf{Z}} \alpha^{2\dot{A}}_{\rho, n-\frac{\rho}{k}} z^{-n-1+\frac{\rho}{k}} \tag{3.1.27b}$$

$$\bar\partial X_\rho^{1\dot{1}}(\bar z) = \sum_{n\in\mathbf{Z}} \tilde\alpha^{1\dot{1}}_{\rho, n-\frac{\rho}{k}} \bar z^{-n-1+\frac{\rho}{k}}, \qquad \bar\partial X_\rho^{2\dot{2}}(\bar z) = \sum_{n\in\mathbf{Z}} \tilde\alpha^{2\dot{2}}_{\rho, n-\frac{\rho}{k}} \bar z^{-n-1+\frac{\rho}{k}} \tag{3.1.27c}$$

$$\bar\partial X_\rho^{1\dot{2}}(\bar z) = \sum_{n\in\mathbf{Z}} \tilde\alpha^{1\dot{2}}_{\rho, n+\frac{\rho}{k}} \bar z^{-n-1-\frac{\rho}{k}}, \qquad \bar\partial X_\rho^{2\dot{1}}(\bar z) = \sum_{n\in\mathbf{Z}} \tilde\alpha^{2\dot{1}}_{\rho, n-\frac{\rho}{k}} \bar z^{-n-1+\frac{\rho}{k}} \tag{3.1.27d}$$

$$\tag{3.1.27e}$$

For the fermions we still have the distinction between the R and the NS sectors.

In the NS sector the monodromies of the fermions in the (r) basis are

$$\psi^{\alpha\dot{A}}_{(r)}(e^{2\pi i} z) = \psi^{\alpha\dot{A}}_{(r+1)}(z), \quad \bar\psi^{\dot\alpha\dot{A}}_{(r)}(e^{-2\pi i}\bar z) = \psi^{\dot\alpha\dot{A}}_{(r+1)}(\bar z) \tag{3.1.28}$$

with the identification $\psi^{\alpha\dot{A}}_{(k+1)} = (-1)^{k+1}\psi^{\alpha\dot{A}}_{(1)}$ and $\bar\psi^{\dot\alpha\dot{A}}_{(k+1)} = (-1)^{k+1}\bar\psi^{\dot\alpha\dot{A}}_{(1)}$. In the diagonalized basis we have

$$\begin{aligned} \psi_\rho^{+\dot{A}}(z) &= \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{2\pi\mathrm{i}\frac{r\rho}{k}}\ \psi^{+\dot{A}}_{(r)}(z), \qquad & \tilde\psi_\rho^{\dot{+}\dot{A}}(\bar z) &= \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{-2\pi\mathrm{i}\frac{r\rho}{k}}\ \tilde\psi^{\dot{+}\dot{A}}_{(r)}(\bar z), \\ \psi_\rho^{-\dot{A}}(z) &= \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{-2\pi\mathrm{i}\frac{r\rho}{k}}\ \psi^{-\dot{A}}_{(r)}(z), \qquad & \tilde\psi_\rho^{\dot{-}\dot{A}}(\bar z) &= \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{2\pi\mathrm{i}\frac{r\rho}{k}}\ \tilde\psi^{\dot{-}\dot{A}}_{(r)}(\bar z), \end{aligned} \tag{3.1.29}$$

which behave like

$$\psi_\rho^{+\dot{A}}(e^{2\pi\mathrm{i}}\ z) = e^{-2\pi\mathrm{i}\frac{\rho}{k}}\ \psi_\rho^{+\dot{A}}(z), \qquad \tilde{\psi}_\rho^{+\dot{A}}(e^{-2\pi\mathrm{i}}\ \bar{z}) = e^{2\pi\mathrm{i}\frac{\rho}{k}}\ \tilde{\psi}_\rho^{+\dot{A}}(\bar{z}), \tag{3.1.30}$$

where the behaviour of $\psi_\rho^{-\dot{2}}$ and $\psi_\rho^{-\dot{1}}$ are obtained respectively from $\psi_\rho^{+\dot{1}}$ and $\psi_\rho^{+\dot{2}}$ by complex conjugation. Analogous relations hold for the antiholomorphic fermions. Vacuum in the twisted sector is analogous to the vacuum in the untwisted one, apart from having the monodromy conditions discussed above for the fields. We define the bosonic vacuum on a strand of length k as

$$\alpha_{\rho,n}^{A\dot{A}}|0\rangle_k^X = 0, \quad \bar{\alpha}_{\rho,n}^{A\dot{A}}|0\rangle_k^X = 0, \quad \forall n \geq 0\, \forall A, \dot{A} \tag{3.1.31}$$

Then we define the fermionic NS vacuum in the k-twisted sector in the following way

$$\psi_{\rho,n}^{\alpha\dot{A}}|0\rangle_k = 0, \quad \bar{\psi}_{\rho,n}^{\dot{\alpha}\dot{A}}|0\rangle_k = 0, \quad \forall n \geq 0,\, \forall \alpha, \dot{\alpha}, \dot{A} \tag{3.1.32}$$

In the R sector the monodromies of the fermions in the (r) basis are

$$\psi_{(r)}^{\alpha\dot{A}}(e^{2\pi i}z) = \psi_{(r+1)}^{\alpha\dot{A}}(z), \quad \bar{\psi}_{(r)}^{\dot{\alpha}\dot{A}}(\mathrm{e}^{-2\pi i}\bar{z}) = \psi_{(r+1)}^{\dot{\alpha}\dot{A}}(\bar{z}) \tag{3.1.33}$$

with $\psi_{(k+1)}^{\alpha\dot{A}} = \psi_{(1)}^{\alpha\dot{A}}$ and $\bar{\psi}_{(k+1)}^{\dot{\alpha}\dot{A}} = \bar{\psi}_{(1)}^{\dot{\alpha}\dot{A}}$. In the diagonalized basis we have

$$\begin{aligned} \psi_\rho^{+\dot{A}}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{2\pi\mathrm{i}\frac{r\rho}{k}}\ \psi_{(r)}^{+\dot{A}}(z), \qquad & \tilde{\psi}_\rho^{\dot{+}\dot{A}}(\bar{z}) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi\mathrm{i}\frac{r\rho}{k}}\ \tilde{\psi}_{(r)}^{\dot{+}\dot{A}}(\bar{z}), \\ \psi_\rho^{-\dot{A}}(z) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi\mathrm{i}\frac{r\rho}{k}}\ \psi_{(r)}^{-\dot{A}}(z), \qquad & \tilde{\psi}_\rho^{\dot{-}\dot{A}}(\bar{z}) &= \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{2\pi\mathrm{i}\frac{r\rho}{k}}\ \tilde{\psi}_{(r)}^{\dot{-}\dot{A}}(\bar{z}), \end{aligned} \tag{3.1.34}$$

which behave like

$$\psi_\rho^{+\dot{A}}(e^{2\pi\mathrm{i}}\ z) = e^{-2\pi\mathrm{i}\frac{\rho}{k}}\ \psi_\rho^{+\dot{A}}(z), \qquad \tilde{\psi}_\rho^{+\dot{A}}(e^{-2\pi\mathrm{i}}\ \bar{z}) = e^{2\pi\mathrm{i}\frac{\rho}{k}}\ \tilde{\psi}_\rho^{+\dot{A}}(\bar{z}), \tag{3.1.35}$$

where the behaviour of $\psi_\rho^{-\dot{2}}$ and $\psi_\rho^{-\dot{1}}$ are obtained respectively from $\psi_\rho^{+\dot{1}}$ and $\psi_\rho^{+\dot{2}}$ by complex conjugation.

The bosonic vacuum state on a strand of length k is the same of the one in the NS sector. The fermionic vacuum state in the R sector is given by

$$|\alpha\dot{\alpha}\rangle_k, \quad |\alpha\dot{A}\rangle_k, \quad |\dot{A}\dot{\alpha}\rangle_k, \quad |\dot{A}\dot{A}\rangle_k \tag{3.1.36}$$

In analogy with the untwisted sector the spin highest weight state $|++\rangle_k$ is annihilated by all the fermions' positive modes in the ρ basis as usual and by the following zero modes

$$\psi_{\rho,0}^{+\dot{A}}|++\rangle_k = 0, \quad \bar{\psi}_{\rho,0}^{+\dot{A}}|++\rangle_k = 0 \tag{3.1.37}$$

We introduced the field content of the theory and the sectors of the Hilbert space. For each sector we gave the boundary conditions on the fundamental operators with the corresponding mode expansion. In this way we can construct other operators in all sectors using the given rules. In particular in the twisted sector we saw that exists a change of basis that brings boundary conditions in a diagonal form and hence we can still construct the orbifold operators in this sector simply by changing basis. For instance, the expression for the conserved currents written in (3.1.7) is still valid even in the k-twisted sector provide we perform the change of basis we used for the fundamental fields. For instance, for the R-current operator, we have

$$J^a(z) = \frac{1}{4}\sum_{r=1}^{k} \epsilon_{\dot{A}\dot{B}}\psi^{\alpha\dot{A}}_{(r)}\epsilon_{\alpha\beta}(\sigma^{*a})^{\beta}_{\gamma}\psi^{\gamma\dot{B}}_{(r)} = \frac{1}{4}\sum_{\rho=0}^{k-1} \epsilon_{\dot{A}\dot{B}}\psi^{\alpha\dot{A}}_{\rho}\epsilon_{\alpha\beta}(\sigma^{*a})^{\beta}_{\gamma}\psi^{\gamma\dot{B}}_{\rho} \tag{3.1.38}$$

and a similar expression for the stress energy tensor and for the supercurrents. The mode expansion in the twisted sector follows form the mode expansion for the fundamental field in the ρ basis.

Twist operators

An important class of operators we are going to discuss comprise the twist operators which are operators acting on a tensor product of k untwisted strand to give a single strand of length k obtained by sewing the copies togheter. As before we will have twist operators in each sector.

In the bosonic sector we define the twist operators σ^X_k, $\bar{\sigma}^X_k$ as

$$\lim_{z,\bar{z}\to 0} \sigma^X_k(z),\, \bar{\sigma}^X_k(\bar{z}) \left[\otimes^k_{r=1}|0\rangle_{(r)}\right] = |0\rangle_k \tag{3.1.39}$$

with conformal dimension and spin (for the left sector)

$$(h,m) = \left(\frac{k^2-1}{6k}, 0\right) \tag{3.1.40}$$

In the NS fermionic sector we define the twist field $\Sigma_k(z,\bar{z})$ as

$$\lim_{z,\bar{z}\to 0} \Sigma_k(z,\bar{z}) \left[\otimes^k_{r=1}|0\rangle_{(r)}\right] = |0\rangle_k \tag{3.1.41}$$

with

$$(h,m) = \left(\frac{k^2-1}{12k}, 0\right) \tag{3.1.42}$$

The twist field in the R sector $\Sigma^{\alpha_1\dot{\alpha}_2}_k(z,\bar{z})$ are defined in a similar way and the highest weight has

$$(h,m) = \left(\frac{(k-1)(2k-1)}{6k}, \frac{k-1}{2}\right) \tag{3.1.43}$$

and with the index α_1, $\dot{\alpha}_2$ transforming in a $((k-1)/2,(k-1)/2)$ representation of $SU(2)_L \times SU(2)_R$.

3.1.3 Chiral primary spectrum

Chiral primaries in D1D5 CFT are in the NS sector of the theory and they play a fundamental role in the context of AdS/CFT. Here we limit ourselves to define and to find these particular states/operators. In the NS sector we have the global subalgebra

$$\left(L_0, L_{\pm 1}, J^a_0, G^{\alpha A}_{\pm\frac{1}{2}}\right) \tag{3.1.44}$$

State	J_0^3	L_0	Degeneracy
$\|\chi\rangle$	h	h	$2h+1$
$G^{-1}_{-\frac{1}{2}}\|\chi\rangle$	$h-1/2$	$h+1/2$	$2h$
$G^{-2}_{-\frac{1}{2}}\|\chi\rangle$	$h-1/2$	$h+1/2$	$2h$
$G^{-1}_{-\frac{1}{2}}G^{-2}_{-\frac{1}{2}}\|\chi\rangle$	$h-1$	$h+1$	$2h-1$

Table 3.1: Structure of a short multiplet

that generates the $SU(1,1|2)$ group. Chiral primaries are contained in *short multiplets* of representation of the algebra of $SU(1,1|2)$. A short representation of $SU(1,1|2)$ is basically a representation that is shorter than others, because it has been truncated by a condition imposed by the algebra. We define a *chiral primary* state $|\chi\rangle$ a state annihilated also by half of the lowering operator supercharges

$$G^{+A}_{-\frac{1}{2}}|\chi\rangle = 0 \qquad A = 1,2 \tag{3.1.45}$$

This implies, thanks to the algebra

$$h_\chi = m_\chi = j_\chi \tag{3.1.46}$$

By using the $G^{+A}_{\pm 3/2}$ anti-commutator we can derive the general bound, valid for each chiral primary

$$h_\chi \leq N \tag{3.1.47}$$

We want to stress the fact that the seed states in each row are the highest weight component of the $SU(2)$ representation and are also primaries of $SL(2,R)$, and that's also why we write the degeneracy of each row, that is basically the dimension of the $SU(2)$ representation. Constructing the similar multiplet for right handed side we denote the generic short multiplet in Table 3.1 of the $SU_L(1,1|2)\times SU_R(1,1|2)$ as

$$(\mathbf{2h+1}, \mathbf{2h'+1})_S \tag{3.1.48}$$

The $SU(2)$ decomposition of the supermultiplet reads

$$(\mathbf{h}, \mathbf{h'})_S = (\mathbf{h}, \mathbf{h'}) \oplus (\mathbf{h-1/2}, \mathbf{h'-1/2}) \oplus (\mathbf{h-1}, \mathbf{h'-1}) \tag{3.1.49}$$

where we denote with $(\mathbf{j}, \mathbf{j'})$ the $SU(2)_L \times SU(2)_R$ of dimension $(2j+1)\times(2j'+1)$. However, it's common to denote the representation and the decomposition by their dimension, so to get contact with the literature we use

$$(\mathbf{2h+1}, \mathbf{2h'+1})_S = (\mathbf{2h+1}, \mathbf{2h'+1}) \oplus 2(\mathbf{2h}, \mathbf{2h'}) \oplus (\mathbf{2h-1}, \mathbf{2h'-1}) \tag{3.1.50}$$

The fact that we denote with text bold also the $SU(2)$'s irreps is because each state in a given $SU(2)$ representation of given spin actually is also a primary state of $SL(2)$ subalgebra and so it generates a Verma module and therefore each state is actually a tower of $SL(2,R)$ descendants.
From the above analysis it is clear that once we fix the short representation the $SL(2,R)\times SU(2)$ decomposition is also fixed basically in terms of the seed chiral primary that generate all the supermultiplet. Thus it is crucial to find all the chiral primaries of the theories. Note that what should match with sugra modes are chiral primary single trace or single particle states. In orbifold theory this corresponds to a chiral primary excitation

Chiral primary	J_0^3	L_0
$\|0\rangle$	0	0
$\psi^{+\dot{1}}_{-\frac{1}{2}}\|0\rangle$	1/2	1/2
$\psi^{+\dot{2}}_{-\frac{1}{2}}\|0\rangle$	1/2	1/2
$\psi^{+\dot{1}}_{-\frac{1}{2}}\psi^{+\dot{2}}_{-\frac{1}{2}}\|0\rangle$	1	1

TABLE 3.2: Chiral primaries in untwisted sector

on only one strand or on single k-cycle, if we are considering the k-twisted sector.
It is useful to start with the analysis in one copy of the CFT, then extending to the orbifold CFT.

Untwisted sector On one copy of the SCFT we have 4 chiral primaries on the left sector, listed in the table below On the right sector we have the same and this give rise to $4 \times 4 = 16$ chiral primaries and so to 16 short multiplets. Combining left and right in all possible ways we can write the short multiplets organized, for future reasons, in terms of the difference of weight $h - h'$. In the notation we introduced above these chiral primaries give rise to the following short multiplets

$$\begin{gathered}(\mathbf{1},\mathbf{1})_S \oplus 4(\mathbf{2},\mathbf{2})_S \oplus (\mathbf{3},\mathbf{3})_S \\ 2(\mathbf{1},\mathbf{2})_S \oplus 2(\mathbf{2},\mathbf{1})_S \\ (\mathbf{1},\mathbf{3})_S \oplus (\mathbf{3},\mathbf{1})_S \\ (\mathbf{2},\mathbf{3})_S \oplus (\mathbf{3},\mathbf{2})_S\end{gathered} \tag{3.1.51}$$

where the number behind a given multiplet the number of combinations left-right that give rise to the same weights. The decomposition under $SU(2)$ follows from the previous arguments taking care of the physical bounds of $SU(2)$ irreps.

The single particle chiral primaries follow, straightforwardly, from the ones discussed for a single copy since the single particle chiral primary of the orbifold CFT in the untwisted sector is the excitation of only one strand, and so is in one-to-one correspondence to the chiral primaries on a single copy. Thus, the spectrum is again

$$\begin{gathered}(\mathbf{1},\mathbf{1})_S \oplus 4(\mathbf{2},\mathbf{2})_S \oplus (\mathbf{3},\mathbf{3})_S \\ 2(\mathbf{1},\mathbf{2})_S \oplus 2(\mathbf{2},\mathbf{1})_S \\ (\mathbf{1},\mathbf{3})_S \oplus (\mathbf{3},\mathbf{1})_S \\ (\mathbf{2},\mathbf{3})_S \oplus (\mathbf{3},\mathbf{2})_S\end{gathered} \tag{3.1.52}$$

Twisted sector The twist operator that implements these boundary conditions is denoted by

$$|\Sigma_k^{\alpha_1\dot{\alpha}_2}\rangle, \qquad (h,h') = ((k-1)/2,(k-1)/2) = (j,j') \tag{3.1.53}$$

This is the first chiral primary of the k twisted sector. The others are produced by acting on this state with, as in the untwisted sector, with fermions and it produce the following chiral primary Organizing into short multiplet we get

Chiral primary	$\bar{J}^2$	J_0^3	L_0
$\lvert\Sigma_k^{\alpha_1\dot{\alpha}_2}\rangle$	$(k-1)/2$	$(k-1)/2$	$(k-1)/2$
$\psi^{+\dot{1}}_{-\frac{1}{2}}\lvert\Sigma_k^{\alpha_1\dot{\alpha}_2}\rangle$	$k/2$	$k/2$	$k/2$
$\psi^{+\dot{2}}_{-\frac{1}{2}}\lvert\Sigma_k^{\alpha_1\dot{\alpha}_2}\rangle$	$k/2$	$k/2$	$k/2$
$\psi^{+\dot{1}}_{-\frac{1}{2}}\psi^{+\dot{2}}_{-\frac{1}{2}}\lvert\Sigma_k^{\alpha_1\dot{\alpha}_2}\rangle$	$(k+1)/2$	$(k+1)/2$	$(k+1)/2$

TABLE 3.3: Chiral primaries in k-twisted sector

$$\begin{gathered}
5(\mathbf{2},\mathbf{2})_S \oplus_{\mathbf{m}\geq 3} 6(\mathbf{m},\mathbf{m})_S \\
2(\mathbf{1},\mathbf{2})_S \oplus 2(\mathbf{2},\mathbf{1})_S \oplus (\mathbf{1},\mathbf{3})_S \oplus (\mathbf{3},\mathbf{1})_S \\
\oplus_{\mathbf{m}\geq 2}[(\mathbf{m},\mathbf{m+2})_S + (\mathbf{m+2},\mathbf{m})_S + 4(\mathbf{m},\mathbf{m+1})_S + 4(\mathbf{m+1},\mathbf{m})_S]
\end{gathered} \tag{3.1.54}$$

The weight and spin of chiral primaries of short multiplets is bounded by $(1+N)/2$, therefore the sum should stop at a certain point. This decomposition exactly matches the decomposition of short multiplets of the group of symmetry of the near horizon limit of the asymptotically $\mathrm{AdS}_3 \times S^3 \times T^4$ supergravity solutions [25].

3.1.4 Spectral flow

Spectral flow in CFT is a particular local transformation on operators that leaves the $\mathcal{N}=4$ algebra invariant. It's defined as a finite transformation generated by $J^3(z)$ of $SU(2)_L$ with an angle given by

$$\eta(z) = i\alpha \log z \tag{3.1.55}$$

where $\alpha \in \mathbb{R}$ is the amount of spectral flow. There is an analogous transformation in the right sector generated by $\bar{J}^3(\bar{z})$ of $SU(2)_R$ and with $\bar{\alpha}$ as units of spectral flow. In general, one expects that an operator with charge m under $SU(2)_L$ transforms as

$$\mathcal{O}(z) \;\rightarrow\; z^{-\alpha m}\mathcal{O}(z) \tag{3.1.56}$$

One finds that the the currents transform under spectral flow as follows

$$\begin{aligned}
J^3(z) &\rightarrow J^3(z) - \frac{c\alpha}{12z} \\
J^{\pm}(z) &\rightarrow z^{\mp\alpha}J^{\pm}(z) \\
G^{\pm A}(z) &\rightarrow z^{\mp\frac{\alpha}{2}}G^{\pm A}(z) \\
T(z) &\rightarrow T(z) - \frac{\alpha}{z}J^3(z) + \frac{c\alpha^2}{24z^2}
\end{aligned} \tag{3.1.57}$$

which gives rise to the transformation of the modes

$$\begin{aligned}
J^3_m &\rightarrow J^3_m - \frac{c\alpha}{12}\delta_{m,0} \\
J^{\pm}_m &\rightarrow J^{\pm}_{m\mp\alpha} \\
G^{\pm A}_m &\rightarrow G^{\pm A}_{m\mp\frac{\alpha}{2}} \\
L_m &\rightarrow L_m - \alpha J^3_m + \frac{c\alpha^2}{24}\delta_{m,0}
\end{aligned} \tag{3.1.58}$$

Therefore the charges of the states change in the following way

$$\begin{aligned} h \to h' &= h + \alpha m + \frac{c\alpha^2}{24} \\ m \to m' &= m + \frac{c\alpha}{12} \end{aligned} \tag{3.1.59}$$

Of particular interest is the case of spectral flow with $\alpha = -1$. Indeed in this case the fermions transforms as

$$\psi^{\pm\dot{A}}(z) \to z^{\pm\frac{1}{2}}\psi^{\pm\dot{A}}(z) \tag{3.1.60}$$

changing so the periodicity boundary conditions. If we started with a periodic fermion (NS sector) on the plane we end up with an antiperiodic fermion (R sector). This fact can also be seen from the transformation rules on the supercharge modes which go from half integers to integers. Since the NS sector and the R sector are related by spectral flow, we can map states in one sector into states in the other. In particular there is a one-to-one correspondence between the R vacua and the NS chiral primary states in each twisted sector. Let's take as example the four chiral primaries in the NS untwisted left sector

$$|0\rangle, \quad \psi^{+\dot{A}}_{-\frac{1}{2}}|0\rangle, \quad \psi^{+\dot{1}}_{-\frac{1}{2}}\psi^{+\dot{2}}_{-\frac{1}{2}}|0\rangle \tag{3.1.61}$$

Using (3.1.59) and the transformation rule on the operators (3.1.56), we can see that the four states above get mapped into four Ramond states with $h = \frac{c}{24}$ (or $1/4$ on a single strand) and with $SU(2)_L$ charge given by the formula in (3.1.59). Joining left and right sector in all possible way we map 16 chiral primary states in the NS into 16 Ramond ground states in (3.1.20)

$$|\alpha\dot{\alpha}\rangle, \quad |\alpha\dot{A}\rangle, \quad |\dot{A}\dot{\alpha}\rangle, \quad |\dot{A}\dot{A}\rangle \tag{3.1.62}$$

It worth noticing that the NS vacuum is mapped into the maximally spinning R ground state

$$|0\rangle \to |++\rangle \tag{3.1.63}$$

3.1.5 Two-charge microstates

The two-charge microstates for the D1D5 black hole are states of the orbifold D1D5 CFT preserving 1/4 of the total supersymmetries and are constructed using as seeds the Ramond ground states found in each twisted sector and combining them in a coherent state of the orbifold theory. As we already saw, on each strand a state is characterized by the length k of the strand, its weight h and the spin m and in general is denoted by

$$|m\rangle_k, \quad m = (0,0)^{\dot{A}\dot{B}}, \ (\pm\pm) \tag{3.1.64}$$

where we denote with the states with $m = (0,0)^{\dot{A}\dot{B}}$ those states with excitation on the torus. The quartet splits into a singlet, invariant on the torus and usually written omitting the indices $\dot{A}$, and a triplet non invariant on the torus.

The most general two-charge microstate is obtained by taking the tensor product of $N_k^{(m)}$ copies of the strand $|m\rangle_k$, with a constraints that the total winding number be $N = N_1 N_5$. Thus a ground state is specified by a partition $\{N_k^{(m)}\}$ of N:

$$\psi_{\{N_k^{(m)}\}} \equiv \prod_{k,m} (|m\rangle_k)^{N_k^{(m)}}, \quad \sum_{m,k} k N_k^{(m)} = N \tag{3.1.65}$$

By convention we relate the norm of these states to the number of ways, $\mathcal{N}\left(\{N_k^{(m)}\}\right)$, the strand configuration determined by the partition $\{N_k^{(m)}\}$ can be obtained starting from the state $|++\rangle^N$:

$$\langle\psi_{\{N_k^{(m)}\}}|\psi_{\{N_k^{\prime(m)}\}}\rangle = \delta_{\{N_k^{(m)}\},\{N_k^{\prime(m)}\}}\mathcal{N}\left(\{N_k^{(m)}\}\right) \tag{3.1.66}$$

To compute the combinatoric factor consider the action of the twist field $\Sigma_k^{\pm\pm}$ on N copies of the CFT, to produce a strand of length k: there are $\frac{N!}{(N-k)!k}$ ways in which the twist field can act. The full state is obtained by acting repeatedly with twist fields, so that the total number of terms produced is

$$\frac{N!}{(N-k_1)!k_1}\frac{(N-k_1)!}{(N-k_1-k_2)!k_2}\cdots = \frac{N!}{\prod_{k,m} k^{N_k^{(m)}}} \tag{3.1.67}$$

and therefore one finds

$$\mathcal{N}\left(\{N_k^{(m)}\}\right) = \frac{N!}{\prod_{k,m} N_k^{(m)}!k^{N_k^{(m)}}} \tag{3.1.68}$$

The quantum numbers (h,m) of the generic state $|\psi\rangle$ can be obtained by acting on it with the respective orbifold operators L_0, J^3 and they give

$$(h,m) = \left(\frac{N}{4}, \sum_{k,m} mN_k^{(m)}\right) \tag{3.1.69}$$

In section 3.2 we will provide the gravity solutions corresponding to these two-charge microstates and in 3.3 we give the dictionary between gravity and CFT for these two-charge states.

3.2 Gravity side

On the gravity side we have a type IIB supergravity compactified on $T^4 \times S^1$. Since we are looking for microstate solutions for the D1D5P black hole we have to seek solutions with the same number of supersymmetries and with equal conserved charges. Solutions for the two-charge D1D5 black hole preserves 1/4 supersymmetries of Type IIB while the three-charge D1D5P microstates preserve 1/8 supersymmetries. The charges entering in the solutions have been computed in (2.1.28),(2.1.29) and read

$$Q_1 = \frac{(2\pi)^4}{V_4} g_s \alpha'^3 n_1, \quad Q_5 = g_s \alpha' n_5, \quad Q_p = \frac{(2\pi)^4}{V_4} g_s^2 \frac{\alpha'^4}{R_y^2} n_p \tag{3.2.1}$$

We will give, in the following, the general ansatz for the generic three-charge solutions and we will provide the recipe for constructing two-charge microstates in 3.2.3 with a dictionary that map them in the CFT side.

3.2.1 General solutions

The general solution of type IIB supergravity compactified on $T^4 \times S^1$ preserving the same supersymmetries as the D1D5P system was found in [26] under the assumption that the geometry is invariant under rotations of T^4. The solution can be written as

$$ds^2_{(10)} = -\frac{2\alpha}{\sqrt{Z_1 Z_2}}(dv+\beta)\left[du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\right] + \sqrt{Z_1 Z_2}\, ds_4^2 + \sqrt{\frac{Z_1}{Z_2}}\, d\hat{s}_4^2 \tag{3.2.2a}$$

$$e^{2\Phi} = \alpha \frac{Z_1}{Z_2} \tag{3.2.2b}$$

$$B_2 = -\frac{\alpha Z_4}{Z_1 Z_2}(du+\omega)\wedge(dv+\beta) + a_4 \wedge (dv+\beta) + \delta_2 \tag{3.2.2c}$$

$$C_0 = \frac{Z_4}{Z_1} \tag{3.2.2d}$$

$$C_2 = -\frac{\alpha}{Z_1}(du+\omega)\wedge(dv+\beta) + a_1 \wedge (dv+\beta) + \gamma_2 \tag{3.2.2e}$$

$$C_4 = \frac{Z_4}{Z_2}\hat{\mathrm{vol}}_4 - \frac{\alpha Z_4}{Z_1 Z_2}\gamma_2 \wedge (du+\omega)\wedge(dv+\beta) + x_3 \wedge (dv+\beta) \tag{3.2.2f}$$

where

$$\alpha = \frac{Z_1 Z_2}{Z_1 Z_2 - Z_4^2} \tag{3.2.3}$$

The ten-dimensional space-time is split into the compact manifold T^4, endowed with a flat metric $d\hat{s}_4^2$, the four non-compact spatial directions, diffeomorphic to $\mathbb{R}^4$, over which we define a generically non-trivial Euclidean metric ds_4^2, the time and S^1 directions, t and y, that we parametrize with light-cone coordinates

$$u = \frac{t-y}{\sqrt{2}}, \quad v = \frac{t+y}{\sqrt{2}} \tag{3.2.4}$$

The remaining ingredients defining the ansatz are: the 0-forms on $\mathbb{R}^4$ Z_1, Z_2, Z_4 and $\mathcal{F}$; the 1-forms β, ω, a_1 and a_4; the 2-forms γ_2 and δ_2; the 3-form x_3. One can also introduce a 1-form a_2 and a 2-form γ_1 that appear in C_6, the 6-form dual to C_2, in a way analogous to how a_1 and γ_2 appear in C_2. All these objects, including ds_4^2, depend in general on the coordinate v and the $\mathbb{R}^4$ coordinates x_i. The constraints that these geometric data have to satisfy in order to preserve supersymmetry and satisfy the equations of motion have been derived in [27]. Defining the objects

$$\mathcal{D} \equiv d - \beta \wedge \frac{d}{dv} \tag{3.2.5}$$

with d the differential operator on $\mathbb{R}^4$. The equations has to satisfy Z_1, a_2 and γ_1 read

$$*_4 \mathcal{D} Z_1 = \mathcal{D}\gamma_1 - a_2 \wedge d\beta \tag{3.2.6a}$$
$$\Theta_2 = *_4\Theta_2, \quad \Theta_2 = \mathcal{D}a_2 + \dot{\gamma}_1 \tag{3.2.6b}$$

For Z_1, a_1 and γ_2 we have:

$$*_4 \mathcal{D} Z_2 = \mathcal{D}\gamma_2 - a_1 \wedge d\beta \tag{3.2.7a}$$
$$\Theta_1 = *_4\Theta_1, \quad \Theta_1 = \mathcal{D}a_1 + \dot{\gamma}_2 \tag{3.2.7b}$$

For Z_4, a_4 and δ_2 we have:

$$*_4 \mathcal{D} Z_4 = \mathcal{D}\delta_2 - a_4 \wedge d\beta \tag{3.2.8a}$$
$$\Theta_4 = *_4\Theta_4, \quad \Theta_4 = \mathcal{D}a_4 + \dot{\delta}_2 \tag{3.2.8b}$$

Equations for ω and $\mathcal{F}$ read:

$$\mathcal{D}\omega + *_4\mathcal{D}\omega + \mathcal{F}d\beta = Z_1\Theta_1 + Z_2\Theta_2 - 2Z_4\Theta_4 \tag{3.2.9a}$$
$$\begin{aligned} *_4\mathcal{D} *_4 \left(\dot{\omega} - \frac{1}{2}\mathcal{D}\mathcal{F}\right) &= \dot{Z}_1\dot{Z}_2 + Z_1\ddot{Z}_2 + \ddot{Z}_1 Z_2 - (\dot{Z}_4)^2 - 2Z_4\ddot{Z}_4 \\ &\quad - \frac{1}{2} *_4 [\Theta_1 \wedge \Theta_2 - \Theta_4 \wedge \Theta_4] \end{aligned} \tag{3.2.9b}$$

Equation for x_3 is:

$$\mathcal{D}x_3 - \Theta_4 \wedge \gamma_2 + a_1 \wedge (\mathcal{D}\delta_2 - a_4 \wedge d\beta) = Z_2^2 *_4 \frac{d}{dv}\left(\frac{Z_4}{Z_2}\right) \tag{3.2.10a}$$

3.2.2 Black hole solution

The simplest solution of the equations summarised in the previous section is the naive solution reproducing the black hole solution given in chapter 2, which corresponds to setting all functions to zero, except Z_1, Z_2 and $\mathcal{F}$:

$$Z_1 = 1 + \frac{Q_1}{r^2}, \quad Z_2 = 1 + \frac{Q_5}{r^2}, \quad \frac{\mathcal{F}}{2} = -\frac{Q_p}{r^2} \tag{3.2.11}$$

with charges Q_i given by (3.2.1). Setting $\mathcal{F} = 0$ we will find the two-charge black hole while keeping a non vanishing $\mathcal{F}$ we get the three-charge black hole solution found in chapter 2. In the following we are interested in finding less trivial solutions that correspond to bound states of D-branes. In particular, we describe a class of two-charge microstates specified by the set of shape functions we gave above and resulting in supergravity solution regular and horizonless.

3.2.3 Two-charge microstates solution

All the two-charge solutions corresponding to a D1D5 bound state were constructed in [28, 29] by going to a duality frame where the system is described in terms of a fundamental string with a momentum (the F1P frame). In this case the corresponding supergravity geometries are parametrized by a curve $g_A(v)$ in $\mathbb{R}^4 \times T^4$ describing the profile of the string. After applying a duality transformation on the known solution in the F1P frame, it was possible to write the solution for the D1D5 configuration in terms of profiles $g_A(v)$, that do not have any direct geometric meaning in the new duality frame. The two-charge geometries invariant on the torus[6] can be written in terms of the ansatz (B.4.1)

$$Z_1 = 1 + \frac{Q_5}{L}\int_0^L \frac{|\dot g_i(v')|^2 + |\dot g(v')|^2}{|x_i - g_i(v')|^2} dv', \quad Z_2 = 1 + \frac{Q_5}{L}\int_0^L \frac{1}{|x_i - g_i(v')|^2} dv' \tag{3.2.12a}$$

$$Z_4 = -\frac{Q_5}{L}\int_0^L \frac{\dot g(v')}{|x_i - g_i(v')|^2} dv', \quad d\gamma_2 = *_4 dZ_2, \quad d\delta_2 = *_4 dZ_4 \tag{3.2.12b}$$

$$A = -\frac{Q_5}{L}\int_0^L \frac{\dot g_j(v') dx^j}{|x_i - g_i(v')|^2} dv', \quad dB = - *_4 dA \tag{3.2.12c}$$

$$\beta = \frac{-A + B}{\sqrt{2}}, \quad \omega = \frac{-A - B}{\sqrt{2}}, \quad \mathcal{F} = 0, \quad a_1 = a_4 = x_3 = 0 \tag{3.2.12d}$$

with the profiles $g_i(v)$ defined on $\mathbb{R}^4$. We parametrize $\mathbb{R}^4$ with coordinates x_i and we define

$$z_1 = x_1 + i x_2 = \tilde r \sin\tilde\theta e^{i\phi}, \quad z_2 = x_3 + i x_4 = \tilde r \cos\tilde\theta e^{i\psi} \tag{3.2.13}$$

If we define the coordinates

$$\tilde r^2 = r^2 + a^2 \sin^2\theta, \quad \cos^2\tilde\theta = \frac{r^2\cos^2\theta}{r^2 + a^2\sin^2\theta} \tag{3.2.14}$$

the metric on $\mathbb{R}^4$ reads

$$ds_4^2 = (r^2 + a^2\cos^2\theta)\left(\frac{dr^2}{r^2+a^2} + d\theta^2\right) + (r^2+a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta\psi^2 \tag{3.2.15}$$

The supergravity geometries dual to coherent states containing states with zero momentum non invariant on the torus are constructed in [29] and adapted with our convention in [30][7].

As we said, once we have the profiles $g_A(v)$ we are able in principle to construct all the geometries. The next step is to identify the dual CFT states corresponding to these solutions and a precise map is provided for a particular class of two-charge states that we will discuss in 3.3.

3.2.4 Anatomy of a microstate

One of the features of the microstate solutions is that they have different regions that can be decoupled. In particular we have an *asymptotically flat* region valid for

$$r \gg \sqrt{Q_i} \tag{3.2.16}$$

where if we either consider the black hole or a microstate, in this regime we still get Minkowski spacetime $\mathbb{R}^{1,4}$.

As r decreases we encounter a region called the *neck* where the defining functions Z_i do not differ from the naive solution and therefore the black hole and the microstate

[6]The ones containing profiles with only component on $\mathbb{R}^4$ or profiles invariant on torus.

[7]See section 3.1.1 for detailed discussion of the solutions

solutions are indistinguishable.

In the regime of parameters[8]

$$a \ll r \ll \sqrt{Q_i} \tag{3.2.17}$$

we can forget the 1's in the shape function definitions and we are in the region of the so called decoupling limit or near-horizon limit where we have decoupled the asymptotic flat physics and we are left with a region called *throat*. This region is fundamental because is where the microstate starts to differ from the black hole solution and more important is the regime of validity where we can use AdS/CFT. We will discuss more in detail this region in the next section.

As the radius r approaches zero the geometry ends in a cap whose shape is given by the shape function. The shape of the cap is what distinguishes one microstates from another.

3.3 $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence

We resume the discussion of the beginning of this chapter and we define the gauge/gravity duality in the D1D5 system after we gave the necessary ingredients. The AdS/CFT correspondence that is going to be proposed in this setup is that type IIB string theory compactified on T^4 in the near horizon limit is dual to two-dimensional $\mathcal{N} = (4,4)$ SCFT with central charge $c = 6N = 6N_1N_5$. The CFT lives at the boundary of the AdS_3 factor appearing in the solutions in the decoupling limit.

3.3.1 Near horizon limit

In order for us to make use of the AdS/CFT correspondence, we must take a near-horizon limit that decouples the asymptotic flat physics from the AdS physics as we already explained in section 3.2.4. This decoupling limit corresponds to going to the IR fixed point of the D-brane description, and is in effect a low-energy limit. In this limit, the supergravity description becomes $\mathrm{AdS}_3 \times S^3 \times T^4$ where at boundary of the AdS_3 factor lives the CFT_2. The matching of the symmetries in both sides happens in this limit, in particular in the fermionic sector we saw in the CFT that we have eight real supersymmetries that are enhanced to 16 fermionic symmetries when we take the IR limit. In the gravity side the IR limit corresponds to the near horizon limit where the metric becomes $\mathrm{AdS}_3 \times S^3 \times T^4$ which preserves exactly 16 supersymetries. In the bosonic sector the metric has an $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ isometry of AdS_3, dual to the global subalgebra of the Virasoro algebra in the CFT, an $SO(4)_E$ isometry of S^3 playing the role of R-symmetry, and $SO(4)_I$ of the torus. Gathering bosonic and fermionic symmetries one can show that also both CFT and gravity side share the supergroup $SU(1,1|2) \times SU(1,1|2)$.

The moduli space is parameterized by the 25 fields:

$$h_{ij}, \quad B_{ij}, \quad \Phi, \quad C_{ijkl}, \quad C_{ij}, \quad C_0 \tag{3.3.1}$$

[8] We use the parameter a to indicate a generic paramter depending on the particular microstate solution we are considering that estimates the order of the shape functions.

Field	$SO(4)_E \simeq SU(2)_L \times SU(2)_R$	$SO(4)_I \simeq SU(2)_1 \times SU(2)_2$	d.o.f.
$h_{ij} - \frac{1}{4} h\delta_{ij}$	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{3}, \mathbf{3})$	9
b^+_{ij}	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{3}, \mathbf{1})$	3
c_{ij}	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$	6
Ξ	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{1})$	1
v	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{1})$	1
			20

TABLE 3.4: The 20 free sugra moduli and their representation under the symmetry groups.

Sugra	CFT	$SO(4)_I \simeq SU(2)_1 \times SU(2)_2$	d.o.f.
$h_{ij} - \frac{1}{4} h\delta_{ij}$	$\partial X^{(i} \bar{\partial} X^{j)} - \frac{1}{4}\delta^{ij} \partial X^i \bar{\partial} X_i$	$(\mathbf{3}, \mathbf{3})$	9
b^+_{ij}	$\mathcal{T}^1$	$(\mathbf{3}, \mathbf{1})$	3
c_{ij}	$\partial X^{[i} \bar{\partial} X^{j]}$	$(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$	6
Ξ	$\mathcal{T}^0$	$(\mathbf{1}, \mathbf{1})$	1
v	$\partial X^i \bar{\partial} X_i$	$(\mathbf{1}, \mathbf{1})$	1
			20

TABLE 3.5: The sugra/CFT dictionary. $\mathcal{T}$ is the bosonic superdescendant of the twist operator, i.e. $\mathcal{T}^{AB} = G^A_{-1/2} \bar{G}^B_{-1/2} \Sigma^{++}$; $\mathcal{T}^1$ is the triplet while $\mathcal{T}^0$ is the singlet.

where the indices run over the four torus directions. For the D1D5 system, the 25-dimensional moduli space is attracted to a twenty-dimensional subspace in the near-horizon limit five of the scalars get fixed by the constraints

$$v B_{ij} h^{ik} h^{jl} = \frac{1}{2} B_{ij} \epsilon^{ijkl} \tag{3.3.2a}$$

$$v C_0 = C_{6789} - \frac{1}{8} \epsilon^{ijkl} B_{ij} C_{kl} \tag{3.3.2b}$$

$$v + \frac{1}{8} \epsilon^{ijkl} B_{ij} B_{kl} = \frac{n_1}{n_5} \tag{3.3.2c}$$

$$v = \frac{n_1}{n_5} \tag{3.3.2d}$$

A basis for the twenty-dimensional near-horizon moduli solving the above constraints is given by the Table 3.4 where we classified the fields in terms of their group symmetries.

The 20 moduli in gravity correspond to 20 marginal deformation operators in CFT. To find these marginal deformations we have to look for operators preserving the $\mathcal{N} = (4, 4)$ supersymmetry and with conformal weight $h = \bar{h} = 1$. A generic operator of weight 1 is not guaranteed that its dimension will not be corrected moving away from the orbifold point. Therefore it should be in a short multiplet and it must be a singlet of R-symmetry. We are led to start from chiral primaries with $h = m = 1/2$ and apply supercharges. There are five chiral primaries with the correct weight and they are

$$\psi^{+\dot{A}} \bar{\psi}^{+\dot{B}}, \quad \Sigma_2^{++} \tag{3.3.3}$$

Acting with the supercharges we get the 20 marginal deformations listed in Table 3.5.

3.3.2 Two-charge microstate: gravity-CFT map

We provided here a dictionary in a the 1/4 BPS sector for two-charges microstates described in the CFT side in section 3.1. The geometries dual to these coherent superpositions of RR ground states have been con- structed in [31, 29, 32] they are completely specified in terms of a closed curve in $\mathbb{R}^5$, $g_A(v?)$ $(A = 1, ..., 5)$. The parameter along the curve, $v?$, has periodicity $L = 2\pi\frac{Q_5}{R}$. The map between geometries and states can however be expressed solely in terms of the profile: the general idea is that the 5 spin states m are related with the 5 components of $g_A(v?)$, the length of each strand is related with the harmonic number in the Fourier expansion of $g_A(v?)$, and the magnitude of each harmonic mode specifies the number of strands of each type. More precisely, define the Fourier expansions

$$
\begin{aligned}
g_1(v') + ig_2(v') &= \sum_{n\neq 0} \frac{a_n^{(1)}}{n} \mathrm{e}^{\frac{2\pi i n v'}{L}}, \quad g_3(v') + ig_4(v') = \sum_{n\neq 0} \frac{a_n^{(2)}}{n} \mathrm{e}^{\frac{2\pi i n v'}{L}} \\
g_5(v') &= -\mathrm{Im}\left[\sum_{k=1}^{\infty} \frac{a_k^{(00)}}{k} \mathrm{e}^{\frac{2\pi i k v'}{L}}\right]
\end{aligned}
\tag{3.3.4}
$$

where, for convenience, we rename

$$
a_{k>0}^{(1)} = a_k^{(++)}, \quad a_{k<0}^{(1)} = -a_k^{(--)}, \quad a_{k>0}^{(2)} = a_k^{(+-)}, \quad a_{k<0}^{(2)} = -a_k^{(-+)} \tag{3.3.5}
$$

The Fourier coefficients $a_k^{(m)}$ are in general complex and satisfy the constraint

$$
\sum_k \left[|a_k^{(++)}|^2 + |a_k^{(--)}|^2 + |a_k^{(+-)}|^2 + |a_k^{(-+)}|^2 + \frac{1}{2}|a_k^{(00)}|^2\right] = \frac{Q_1 Q_5}{R_y^2} \tag{3.3.6}
$$

The dual CFT state is more naturally expressed in terms of dimensionless coefficients $A_k^{(s)}$:

$$
A_k^{(\pm\pm)} \equiv R_y \sqrt{\frac{N}{Q_1 Q_5}} a_k^{(\pm\pm)}, \quad A_k^{(00)} \equiv R_y \sqrt{\frac{N}{2Q_1 Q_5}} a_k^{(00)} \tag{3.3.7}
$$

with

$$
\sum_{k,m} |A_k^{(s)}|^2 = N \tag{3.3.8}
$$

A given set of Fourier coefficients $\{A_k^{(s)}\}$ specifies a profile $g_A(v')$ and hence a geometry; the CFT state dual to this geometry is

$$
\psi(\{A_k^{(m)}\}) = \sum_{\{N_k^{(m)}\}} \prod_{k,m} \left(A_k^{(m)} |m\rangle_k\right)^{N_k^{(m)}} \tag{3.3.9}
$$

where the sum is restricted to

$$
\sum_{m,k} k N_k^{(m)} = N \tag{3.3.10}
$$

Chapter 4

Four-point functions: protected case

In this chapter we start the study of the 1/4 and 1/8-BPS microstates in the D1D5 CFT and their dual asymptotically $\mathrm{AdS}_3 \times S^3 \times \mathcal{M}$ geometries by studying the holographic correlators of two light operators in a heavy state. On the CFT side, the light operators have conformal dimension of order one while the heavy states have conformal dimension scaling with the central charge c. On the gravity side heavy operators are dual to a microstates solution while the light ones are described by a fluctuation around this particular microstate solution. The studied object is a correlator with four operators[1] that can be seen either as a four-point function in the vacuum or as a two-point function between heavy states because of state/operator correspondence of the CFT.

In some way is an extension of the analysis done in [33] where the authors found the gauge/gravity dictionary for a class of microstates and they studied the three-point function as the vev of one light operator between two heavy states. In that case the correlators were extracted from the normalizable modes behaviour of the corresponding dual field, as general AdS/CFT argument prescribes. Indeed, if O_L is light operator of dimension h and ϕ is its dual gravity field we have, schematically[2]

$$\phi(t, y, r) \sim \langle s|O_L(t, y)|s\rangle\, r^{h-2}, \quad r \to \infty$$

where (t, y, r) are the coordinates of AdS_3 and $\phi(t, y, r)$ takes value in the particular microstate solution dual to the heavy state $|s\rangle$. In order to generalize to four-point functions, we need to turn on a source mode (non-normalizable mode) $J_L(t, y)$ of the dual gravity field, implemented by allowing this field to fluctuate around the background solutions. After solving the linearized equations of motion for the fluctuations $\delta\phi$, we will have to set boundary conditions for the source term $J_L(t, y) = \delta(t, y)$, so we can read off the two-point function from the normalizable mode

$$\delta\phi(t, y, r) \sim \langle s|O_L(0, 0)O_L(t, y)|s\rangle\, r^{h-2} + \delta(t, y)r^{-h}\,, \quad r \to \infty$$

Once we have computed the correlators, one of the analysis that can be conducted is the OPE analysis thanks to which it is possible to extract informations about the exchanged operators and singularities in two different points of the moduli space: the orbifold point and the gravity point. In order to clarify these aspects it turns out to be important to perform the block expansion of the four-point functions that highlights the physics of the correlators both from the free CFT and gravity point of view and gives also a strong confirmation of the results obtained.

[1]Since there are two light (L) and two heavy (H) operators we will refer to these correlators to HHLL correllators.

[2]We consider the boundary of AdS to be at $r \to \infty$.

Despite the schematic picture we give above, where we refer only on the AdS_3, one of the main features of our analysis is that the full higher dimensional geometry is important in the bulk calculation, and is reflected on the CFT side by the OPE analysis of the exchanged operators into the correlators. The example presented in this chapter show that pure heavy states are not directly described by the three-dimensional geometry of the BTZ solution and that, on the CFT side, Virasoro primaries different from the identity can play an important role also in the large c limit. In particular, in the correlators we consider, the singularities due to the large c Virasoro block of the identity are resolved by the contributions of new primaries that are non-trivial already at the leading order in the limit of large central charge. In the simple cases we investigate here, this mechanism is visible already at the supergravity level as the relevant new Virasoro primaries are actually affine descendants of the identity.

The structure of the chapter is the following: in section 4.1 we describe the CFT computation of the correlators giving the ingredients at the orbifold CFT useful for doing that. Even if this is appropriate for a point in the moduli space that is far from the regime where supergravity is valid, the free orbifold description provides a simple way to characterize the operators we use and calculate the correlators we are interested in. In section 4.2 we also analyse the same correlators in terms of Virasoro blocks in order to highlight that non-trivial primaries contribute also in the large c limit. For the examples under analysis, it turns out that these new Virasoro primaries are actually part of the identity affine block of the R-symmetry and, in particular, the full answer is captured by the $U(1)$ affine blocks. This shows that the correlators we focus on are fully determined by protected quantities and so it should be possible to reproduce the same results by a gravity calculation. This is discussed in section 4.3 where the geometries dual to the heavy states are introduced. Then, we extract holographically the four-point correlators discussed on the CFT side and show the matching of the two results. Section 4.4 contains a another particular example of a correlator that turns out to be protected by the fact of being an extremal correlator and sharing the correct properties of extremal correlators in AdS/CFT. The chapter ends with a discusssion of the results.

4.1 The CFT picture

In this section we discuss some simple examples of four-point correlators in the D1D5 CFT. In particular we are interested in correlators with two heavy (O_H) operators, which have conformal dimension of order c, and two light (O_L) operators, which have conformal dimension of order one. Thus the structure of the correlators we consider is (see appendix A)

$$\langle O_H(z_1)\bar{O}_H(z_2)O_L(z_3)\bar{O}_L(z_4)\rangle = \frac{1}{z_{12}^{2h_H}\, z_{34}^{2h_L}}\frac{1}{\bar{z}_{12}^{2\bar{h}_H}\, \bar{z}_{34}^{2\bar{h}_L}}\mathcal{G}(z,\bar{z}) \;, \tag{4.1.1}$$

where, as usual, $z_{ij} = z_i - z_j$ and the cross ratio is defined as

$$z = \frac{z_{14}z_{23}}{z_{13}z_{24}} \,, \tag{4.1.2}$$

while $(h_H, \bar{h}_H)$ and $(h_L, \bar{h}_L)$ are the holomorphic/anti-holomorphic conformal dimensions of the heavy and light operators respectively.

We will take two main simplifying assumptions. First we focus on highly supersymmetric operators. The light operators we use are chiral primaries both in the left and in the right sector of the CFT. Instead the heavy operators are in the Ramond-Ramond sector of the CFT, but are related to chiral primaries by a chiral algebra transformation that acts only on the left sector (hence they generically preserve half of the CFT supercharges). Second we work at the free orbifold point of the CFT moduli space, where the theory, which has central charge $c = 6N$, is described by a collection of N copies of free fields. We remind the discussion on the orbiifold CFT in chapter 4 and we recall the collection of elementary fields

$$\left(X^{A\dot{A}}_{(r)}(z, \bar{z}) \, , \; \psi^{\alpha\dot{A}}_{(r)}(z) \, , \; \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z}) \right) , \tag{4.1.3}$$

where $r = 1, \ldots, N$ labels the copy.

Before introducing the operators entering in the four-point function, we give additional ingredients for the CFT computations. Since we consider correlators both in the untwisted and twisted sector we divide the discussion and we start from the untwisted sector of the theory. The holomorphic fermions on the r-th strand $\psi^{\alpha\dot{A}}_{(r)}$ and the antiholomorphic ones $\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}$ have the nontrivial OPEs

$$\psi^{+\dot{A}}_{(r)}(z)\psi^{-\dot{B}}_{(r)}(w) = \frac{\epsilon^{\dot{A}\dot{B}}}{z-w} + [\text{Reg.}] \, , \qquad \tilde{\psi}^{+\dot{A}}_{(r)}(\bar{z})\tilde{\psi}^{-\dot{B}}_{(r)}(\bar{w}) = \frac{\epsilon^{\dot{A}\dot{B}}}{\bar{z}-\bar{w}} + [\text{Reg.}] \, , \tag{4.1.4}$$

where our convention is $\epsilon_{\dot{1}\dot{2}} = -\epsilon^{\dot{1}\dot{2}} = 1$. The indices $\alpha, \dot{\alpha}$ are in the fundamental representation of $SU(2)$ and will take values $\alpha, \dot{\alpha} = \pm$ or $\alpha, \dot{\alpha} = 1, 2$ depending on what's more convenient case by case. Through bosonization, the fermions can be written in terms of bosons $H(z), K(z)$ as

$$\psi^{+\dot{1}}_{(r)} = \mathrm{i}\; e^{\mathrm{i}H_{(r)}}, \quad \psi^{-\dot{2}}_{(r)} = \mathrm{i}\; e^{-\mathrm{i}H_{(r)}}, \quad \psi^{+\dot{2}}_{(r)} = e^{\mathrm{i}K_{(r)}}, \quad \psi^{-\dot{1}}_{(r)} = e^{-\mathrm{i}K_{(r)}}, \tag{4.1.5}$$

and an analogous dictionary holds for the right fermions, with bosons $\tilde{H}_{(r)}(\bar{z}), \tilde{K}_{(r)}(\bar{z})$. The bosons have the nontrivial OPEs

$$H_{(r)}(z)H_{(r)}(w) = -\log{(z-w)} + [\text{Reg.}] \, , \qquad K_{(r)}(z)K_{(r)}(w) = -\log{(z-w)} + [\text{Reg.}] \, , \tag{4.1.6}$$

and the rule for contractions of bosonized fields is

$$: e^{\mathrm{i}\alpha H_{(r)}(z)} :: e^{\mathrm{i}\beta H_{(r)}(w)} := (z-w)^{-\alpha\beta} \; : \exp\Big(\alpha H_{(r)}(z) + \beta H_{(r)}(w)\Big) : . \tag{4.1.7}$$

A further ingredient we need is given by the current operators. In the untwisted sector, these are written as

$$J^a(z) = \sum_{r=1}^{N} J^a_{(r)}(z) \, , \tag{4.1.8}$$

with

$$J^3_{(r)} = -\frac{1}{2}\left(\psi^{+\dot{A}}_{(r)}\psi^{-\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}\right) , \tag{4.1.9a}$$

$$J^+_{(r)} = J^1_{(r)} + \mathrm{i}J^2_{(r)} = \frac{1}{2}\psi^{+\dot{A}}_{(r)}\psi^{+\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}} \, , \tag{4.1.9b}$$

$$J^-_{(r)} = J^1_{(r)} - \mathrm{i}J^2_{(r)} = -\frac{1}{2}\psi^{-\dot{A}}_{(r)}\psi^{-\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}} \, . \tag{4.1.9c}$$

The current $J^3_{(r)}$ can also be written in terms of the bosons H and K, noticing that

$$\psi^{+\dot{1}}_{(r)}\psi^{-\dot{2}}_{(r)} = -\mathrm{i}\partial H_{(r)}, \qquad \psi^{+\dot{2}}_{(r)}\psi^{-\dot{1}}_{(r)} = \mathrm{i}\partial K_{(r)}, \tag{4.1.10}$$

as

$$J^3_{(r)} = \frac{\mathrm{i}}{2}\Big(\partial H_{(r)} + \partial K_{(r)}\Big), \qquad J^+_{(r)} = \mathrm{i}e^{\mathrm{i}\left(H_{(r)}+K_{(r)}\right)}, \qquad J^-_{(r)} = \mathrm{i}e^{-\mathrm{i}\left(H_{(r)}+K_{(r)}\right)}. \tag{4.1.11}$$

4.1.1 Four-point function in the untwisted sector

We first focus on operators in the untwisted sector of the symmetric orbifold, which means that they are written as combinations of operators acting on each copy. The symmetry under permutations among the copies is realised differently in the light and the heavy operators: the light operators act trivially on all the strands but one, while the heavy ones are constructed by multiplying N copies of the same operator, each copy acting on a different strand:

$$O_L = \frac{1}{\sqrt{N}}\sum_{r=1}^{N} O^L_{(r)}\ , \quad O_H = \otimes^N_{r=1} O^H_{(r)}\ . \tag{4.1.12}$$

Here we concentrate on light operators of dimension $h_L = \bar{h}_L = 1/2$ constructed with the fermions; concretely we take

$$O^L_{(r)} = -\frac{\mathrm{i}}{\sqrt{2}}\psi^{1\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\tilde{\psi}^{\dot{1}\dot{B}}_{(r)} \equiv O^{++}_{(r)}\ . \tag{4.1.13}$$

All the operators $O^H_{(r)}$ we are going to consider in the untwisted sector have right conformal dimension $\bar{h}_{(r)} = 1/4$ and right spin $\bar{\jmath}_{(r)} = 1/2$, which gives a total right conformal dimension for the heavy operators $\bar{h}_H = N/4$, so we can distinguish the heavy operators by their left conformal dimension and left spin. The heavy operators we choose in the untwisted sector are characterised by an integer s determining the number of J^+ excitations acting on a ground state in each copy; their explicit expression is more easily written in the bosonized language (see (4.1.16)), and their left conformal dimension and spin are given by

$$h_H = N\left(s+\frac{1}{2}\right)^2, \qquad j_H = N\left(s+\frac{1}{2}\right). \tag{4.1.14}$$

We therefore denote the single copy operators making up the heavy states as $O^H_{(r)}(s)$ and the same notation will be adopted for the correlators, which are denoted as $\mathcal{G}\left(s; z, \bar{z}\right)$.

As a first concrete example we consider the heavy operator corresponding to $s = 0$; it is written in terms of the spin fields $S^{\dot{A}}_{(r)}$ twisting the elementary fermions $\psi^{\alpha\dot{A}}_{(r)}$ (and $\tilde{S}^{\dot{A}}_{(r)}$ twisting $\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}$)

$$O^H_{(r)}(s=0) = S^{\dot{1}}_{(r)} S^{\dot{2}}_{(r)} \tilde{S}^{\dot{1}}_{(r)} \tilde{S}^{\dot{2}}_{(r)}\ . \tag{4.1.15}$$

Let us comment on the AdS-dual interpretation of the operators entering this correlator. The heavy state is the Ramond-Ramond ground state with the highest value for the left (J^3_0) and right ($\tilde{J}^3_0$) spins allowed by unitarity. This state can be obtained by starting from the $SL(2,\mathbb{C})$ invariant vacuum and performing a spectral flow to the Ramond-Ramond sector, which means that the dual supergravity solution[3] is locally isometric to $\mathrm{AdS}_3 \times S^3$. We can calculate the correlator at the orbifold point of the CFT moduli space by using the standard bosonization approach and the free field contractions in the bosonic language.

[3]It is possible to extend this solution to an asymptotically flat type IIB supergravity background, which then represents a (very special) microstate for the Strominger-Vafa black hole [34, 35].

The heavy operators that we want to consider for the correlators in the untwisted sector are obtained from (4.1.12) and (4.1.15) with

$$S^{\dot{1}}_{s,(r)} = \mathrm{e}^{\mathrm{i}(s+\frac{1}{2})H_{(r)}}\ ,\quad S^{\dot{2}}_{s,(r)} = \mathrm{e}^{\mathrm{i}(s+\frac{1}{2})K_{(r)}}\ . \tag{4.1.16}$$

The corresponding states are

$$\begin{aligned} |s\rangle &\equiv \lim_{z,\bar{z}\to 0} O^H(s;z,\bar{z})|0\rangle \\ &= \otimes_r \left[(J^+_{-2s})_{(r)} \dots (J^+_{-2})_{(r)} \lim_{z,\bar{z}\to 0} O^H_{(r)}(s=0;z,\bar{z}) \right] |0\rangle\ . \end{aligned} \tag{4.1.17}$$

The left and right parts of the four-point function (4.1.12) factorize and we need to evaluate correlators of the form

$$\begin{aligned} F^{\dot{A}\dot{C}}_{s,(r)}(z_i) \equiv \langle e^{\mathrm{i}\left(s+\frac{1}{2}\right)\left(H_{(r)}(z_1)+K_{(r)}(z_1)\right)} e^{-\mathrm{i}\left(s+\frac{1}{2}\right)\left(H_{(r)}(z_2)+K(z_2)_{(r)}\right)} \psi^{+\dot{A}}_{(r)}(z_3)\psi^{-\dot{C}}_{(r)}(z_4)\rangle \times \\ \times \prod_{r'\neq r} \langle e^{-\mathrm{i}\left(s+\frac{1}{2}\right)\left(H_{(r)}(z_1)+K_{(r)}(z_1)\right)} e^{\mathrm{i}\left(s+\frac{1}{2}\right)\left(H_{(r)}(z_2)+K_{(r)}(z_2)\right)}\rangle. \end{aligned} \tag{4.1.18}$$

The right part is completely analogous, with the exception that in the right sector we always have $s = 0$. Notice that in principle the light operators acting on the product theory bring two sums over strands. Despite this, by spin conservation, the only nonzero contributions come from the cases in which both light operators act on the same strand, which reduces the full correlator to just one sum over copies. Moreover, since the heavy operators are product over copies, the term relative to the r-th copy is multiplied by the two-point functions of the heavy operators on all the copies $r' \neq r$. The full correlation function reads

$$\langle O_H(z_1)\bar{O}_H(z_2)O_L(z_3)\bar{O}_L(z_4)\rangle = \sum_{r=1}^{N} \frac{1}{2}\ F^{\dot{A}\dot{C}}_{s,(r)}(z_i) F^{\dot{B}\dot{D}}_{0,(r)}(\bar{z}_i)\ \epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}}. \tag{4.1.19}$$

$F^{\dot{A}\dot{C}}_{s,(r)}(z_i)$ is nonzero only if the two fermions can have a nontrivial contraction, which selects the cases $(\dot{A},\dot{B}) = (\dot{1},\dot{2})$ and $(\dot{A},\dot{B}) = (\dot{2},\dot{1})$. In the first case, using (4.1.7) to contract each possible pair of fields, we get

$$F^{\dot{1}\dot{2}}_{s,(r)}(z_i) = -\frac{z_{13}^{s+\frac{1}{2}}\, z_{24}^{s+\frac{1}{2}}}{z_{12}^{2h}\, z_{14}^{s+\frac{1}{2}}\, z_{23}^{s+\frac{1}{2}}\, z_{34}} = -\frac{1}{z_{12}^{2h} z_{34}}\, z^{-s-\frac{1}{2}}\ , \tag{4.1.20}$$

where $h = \left(s+\frac{1}{2}\right)^2$. The second case is analogous, giving $F^{\dot{2}\dot{1}}_{s,(r)}(z_i) = -F^{\dot{1}\dot{2}}_{s,(r)}(z_i)$. The antiholomorphic parts are obtained from these setting $s = 0$ and replacing $z_i \to \bar{z}_i$ and $h \to \bar{h} = 1/4$. Putting everything together we get

$$\langle O_H(z_1)\bar{O}_H(z_2)O_L(z_3)\bar{O}_L(z_4)\rangle = \frac{1}{z_{12}^{2h_H}\, \bar{z}_{12}^{2\bar{h}_H}\, |z_{34}|^2}\, |z|^{-1}\, z^{-s}\, ; \tag{4.1.21}$$

a factor N would come from the fact that each term of the sum over r gives the same contribution, but this is cancelled by the normalization (4.1.12) of O_L.

The first correlator we compute in the untwisted sector corresponds to two-charge heavy states and is found by setting $s = 0$.

$$\mathcal{G}\Big(s=0;z,\bar{z}\Big) = \frac{1}{|z|}\ . \tag{4.1.22}$$

The simple generalization of (4.1.22) is to consider the correlator with the same light states, but three-charge heavy states corresponding to generic s, which contain excited spin fields in the holomorphic sector

$$O^{H}_{(r)}\left(s;z,\bar{z}\right)=S^{\dot{1}}_{s(r)}S^{\dot{2}}_{s(r)}\tilde{S}^{\dot{1}}_{(r)}\tilde{S}^{\dot{2}}_{(r)}\ , \tag{4.1.23}$$

where $S^{\dot{A}}_{s(r)}$ has conformal weight $(s+1/2)^2/2$. Again by using the bosonized language it is straightforward to calculate the correlator

$$\mathcal{G}\left(s;z,\bar{z}\right)=\frac{1}{z^{s+\frac{1}{2}}\bar{z}^{\frac{1}{2}}} \tag{4.1.24}$$

4.1.2 Four-point function in the twisted sector

We now consider correlators in the twisted sector of the CFT, meaning that the N copies are divided into N/k bunches, each made of k copies glued together. We call each bunch a strand of length k. Within a strand of length k, we have k elementary bosons and fermions with non-trivial periodicities, as already explained in Chapter 3. As usual we can take the linear combinations (3.1.34) and define new fields which diagonalize the boundary conditions. We label these twisted sectors with ρ: for instance the fermions $\psi^{\alpha\dot{A}}_{\rho}$, with $\rho=0,\ldots,k-1$, have the standard twisted boundary conditions (3.1.35). In analogy to what we did in the previous section, the heavy operators are constructed by taking N/k identical strands of length k. The anti-holomorphic conformal dimension of our heavy operators on each strand is always $\bar{h}_H=k/4$ and their right spin is $\bar{j}_H=1/2$. As before, we consider s momentum-carrying excitations in the holomorphic sector, so we characterize the heavy operators by two integers s and k, and their left conformal dimension and spin read

$$h_H=\frac{N}{k}\left(\frac{k}{4}+\frac{s(s+1)}{k}\right),\qquad j=\frac{N}{k}\left(s+\frac{1}{2}\right). \tag{4.1.25}$$

Using (3.1.34) the light operators are rewritten as

$$\sum_{r=1}^{k}O^{++}_{(r)}=\sum_{\rho=0}^{k-1}O^{++}_{\rho},\qquad O^{++}_{\rho}\equiv-\frac{\mathrm{i}}{\sqrt{2}}\psi^{+\dot{A}}_{\rho}\epsilon_{\dot{A}\dot{B}}\tilde{\psi}^{+\dot{B}}_{\rho}, \tag{4.1.26}$$

where O^{--} is the complex conjugate of this.

The heavy operators are denoted as $O_H(s,k)$ and the correlators as $\mathcal{G}(s,k;z,\bar{z})$. We divide the case where $s=pk$ and $s=pk-1$.

Our choice for the heavy operators in the $s=pk$ case is

$$S^{\dot{1}}_{k,pk,\rho}=e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}+\frac{s}{k}\right)H_\rho},\qquad S^{\dot{2}}_{k,pk,\rho}=e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}+\frac{s}{k}\right)K_\rho}, \tag{4.1.27}$$

with the right part given by analogous definitions with $s=0$. The states generated by these operators are

$$|s,k\rangle\equiv\left[\left(J^{+}_{-2s/k}\ldots J^{+}_{-2/k}\right)\lim_{z,\bar{z}\to 0}\Sigma_k\,\tilde{\Sigma}_k\otimes^{k-1}_{\rho=0}\ S^{\dot{1}}_{k,\rho}S^{\dot{2}}_{k,\rho}\tilde{S}^{\dot{1}}_{k,\rho}\tilde{S}^{\dot{2}}_{k,\rho}\right]^{N/k}|0\rangle\,. \tag{4.1.28}$$

Following the same logic as in the untwisted sector, the correlator is given in terms of functions

$$F^{\dot{A}\dot{C}}_{pk,k,\rho}(z_i) \equiv \langle e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}+p\right)(H_\rho(z_1)+K_\rho(z_1))} e^{-\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}+p\right)(H_\rho(z_2)+K(z_2)_\rho)} \psi^{+\dot{A}}_\rho(z_3)\psi^{-\dot{C}}_\rho(z_4)\rangle \tag{4.1.29}$$

$$\times \prod_{\rho'\neq\rho} \langle e^{\mathrm{i}\left(-\frac{\rho'}{k}+\frac{1}{2}+p\right)\left(H_{\rho'}(z_1)+K_{\rho'}(z_1)\right)} e^{-\mathrm{i}\left(-\frac{\rho'}{k}+\frac{1}{2}+p\right)\left(H_{\rho'}(z_2)+K_{\rho'}(z_2)\right)}\rangle\ \langle\Sigma_k(z_1)\Sigma_k(z_2)\rangle$$

as

$$\langle O_H(z_1)\bar{O}_H(z_2)O_L(z_3)\bar{O}_L(z_4)\rangle = \frac{1}{k}\sum_{\rho=0}^{k-1}\frac{1}{2}\ F^{\dot{A}\dot{C}}_{pk,k,\rho}(z_i)F^{\dot{B}\dot{D}}_{0,k,\rho}(\bar{z}_i)\ \epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}}, \tag{4.1.30}$$

where the $1/k$ factor takes care of the fact that we have the same contribution for each length k strand (it would be N/k, but N cancels out because of the normalization of the light operators). As in the untwisted sector, $F^{\dot{A}\dot{C}}_{s,k,\rho}(z_i)$ is nonzero only if $(\dot{A},\dot{C})$ take values $(\dot{1},\dot{2})$ or $(\dot{2},\dot{1})$, and we have

$$F^{\dot{1}\dot{2}}_{pk,k,\rho}(z_i) = -\frac{z_{13}^{-\frac{\rho}{k}+\frac{1}{2}+p}\, z_{24}^{-\frac{\rho}{k}+\frac{1}{2}+p}}{z_{12}^{2h}\, z_{14}^{-\frac{\rho}{k}+\frac{1}{2}+p}\, z_{23}^{-\frac{\rho}{k}+\frac{1}{2}+p}\, z_{34}} = -\frac{1}{z_{12}^{2h} z_{34}}\, z^{\frac{\rho}{k}-\frac{1}{2}-p}\,, \tag{4.1.31}$$

where $h = \frac{k}{4} + \frac{s(s+1)}{k}$, $F^{\dot{2}\dot{1}}_{s,k,\rho}(z_i) = -F^{\dot{1}\dot{2}}_{s,k,\rho}(z_i)$ and z is defined in (4.1.2). The antiholomorphic part is again obtained taking the holomorphic one, setting $s = 0$ (i.e. $p = 0$) and replacing $z_i \to \bar{z}_i$ and $h \to \bar{h} = k/4$. Putting everything together we get

$$\langle O_H(z_1)\bar{O}_H(z_2)O_L(z_3)\bar{O}_L(z_4)\rangle = \frac{1/k}{z_{12}^{2h_H}\,\bar{z}_{12}^{2\bar{h}_H}\,|z_{34}|^2}\ \frac{z^{-p}}{|z|}\ \frac{1-|z|^2}{1-|z|^{\frac{2}{k}}}. \tag{4.1.32}$$

The heavy operator in the $s = pk-1$ case is obtained from the one in the $s = pk$ case by acting on it with J^-_{2p} (the mode $2p$). This only changes the operator in the $\rho = 0$ sector: it has the form (4.1.41), where the action on the $\rho = 0$ part of the left sector is

$$\hat{S}^{\dot{1}}_{k,0} = e^{\mathrm{i}\left(-\frac{1}{2}+p\right)H_{\rho=0}}, \qquad \hat{S}^{\dot{2}}_{k,0} = e^{\mathrm{i}\left(-\frac{1}{2}+p\right)K_{\rho=0}}\,. \tag{4.1.33}$$

With the same procedure as before, we have

$$F^{\dot{A}\dot{C}}_{pk-1,k,0}(z_i) = \langle e^{\mathrm{i}\left(-\frac{1}{2}+p\right)(H_0(z_1)+K_0(z_1))} e^{-\mathrm{i}\left(-\frac{1}{2}+p\right)(H_0(z_2)+K_0(z_2))} \psi^{+\dot{A}}_0(z_3)\psi^{-\dot{C}}_0(z_4)\rangle \tag{4.1.34}$$

$$\times \prod_{\rho'=1}^{k-1} \langle e^{\mathrm{i}\left(-\frac{\rho'}{k}+\frac{1}{2}+p\right)\left(H_{\rho'}(z_1)+K_{\rho'}(z_1)\right)} e^{-\mathrm{i}\left(-\frac{\rho'}{k}+\frac{1}{2}+p\right)\left(H_{\rho'}(z_2)+K_{\rho'}(z_2)\right)}\rangle\ \langle\Sigma_k(z_1)\Sigma_k(z_2)\rangle,$$

while for $\rho \neq 0$ (and in the whole right sector) we have the same functions as in (4.1.29), i.e. $F^{\dot{A}\dot{C}}_{pk-1,k,\rho\neq 0} = F^{\dot{A}\dot{C}}_{pk,k,\rho\neq 0}$. The correlator takes again the form (4.1.30) and the only new object to compute is

$$F^{\dot{1}\dot{2}}_{pk-1,k,0}(z_i) = -\frac{z_{13}^{-\frac{1}{2}+p}\, z_{24}^{-\frac{1}{2}+p}}{z_{12}^{2h}\, z_{14}^{-\frac{1}{2}+p}\, z_{23}^{-\frac{1}{2}+p}\, z_{34}} = -\frac{1}{z_{12}^{2h} z_{34}}\, z^{\frac{1}{2}-p}, \tag{4.1.35}$$

where again $h = \frac{k}{4} + \frac{s(s+1)}{k}$ and $F^{\dot{2}\dot{1}}_{pk-1,k,0}(z_i) = -F^{\dot{1}\dot{2}}_{pk-1,k,0}(z_i)$. The full correlator in the $s = pk - 1$ case reads

$$\langle O_H(z_1)\bar{O}_H(z_2)O_L(z_3)\bar{O}_L(z_4)\rangle = \frac{1/k}{z_{12}^{2h_H}\bar{z}_{12}^{2\bar{h}_H}|z_{34}|^2}\, z^{-p}\left(\left(\frac{z}{\bar{z}}\right)^{\frac{1}{2}} + \frac{1}{|z|}\frac{|z|^{\frac{2}{k}} - |z|^2}{1 - |z|^{\frac{2}{k}}}\right). \tag{4.1.36}$$

The first correlator considered in the twisted sector corresponds to choosing $s = 0$, while the second and the third correspond respectively to the $s = pk$ and the $s = pk - 1$ case.

The first kind of heavy operators we consider corresponds to $s = 0$ and generic k and is a generalization to strands of length k of (4.1.15): on each strand we have k operators $S^{\dot{A}}_{k,\rho}$ and k operators $\tilde{S}^{\dot{A}}_{k,\rho}$ and the total heavy operator is

$$O_H(s=0,k) = \left[\Sigma_k\,\tilde{\Sigma}_k \otimes_{\rho=0}^{k-1}\; S^{\dot{1}}_{k,\rho}S^{\dot{2}}_{k,\rho}\tilde{S}^{\dot{1}}_{k,\rho}\tilde{S}^{\dot{2}}_{k,\rho}\right]^{N/k}, \tag{4.1.37}$$

where Σ_k ($\tilde{\Sigma}_k$) is the twist field inducing on the bosonic fields $\partial X^{A\dot{A}}$ ($\bar{\partial}X^{A\dot{A}}$) the same identification specified in the fermionic sector.. The correlator is obtained again through bosonization in the twisted sector and reads

$$\mathcal{G}\left(s=0,k;z,\bar{z}\right) = \frac{1/k}{|z|}\,\frac{1-|z|^2}{1-|z|^{2/k}}, \tag{4.1.38}$$

where the $1/k$ factor comes from having the same contribution from each of the N/k strands and from the normalization chosen for the light operators in (4.1.12).

The second kind of heavy operator we consider corresponds to nonzero s and k and is a generalization to strands of length k of (4.1.23). These states have $s(s+1)/k$ units of momentum on each strand, and since the number of momentum units must be integer, assuming k is a prime number for simplicity, we have that either $s = pk$ or $s = pk - 1$, with $p \in \mathbb{N}$. In the $s = pk$ case, in the left sector of each strand we have k operators $S^{\dot{A}}_{k,s,\rho}$, and another k operators $\tilde{S}^{\dot{A}}_{k,\rho}$ live in the right sector. The total heavy operator is

$$O_H(s=pk,k) = \left[\Sigma_k\,\tilde{\Sigma}_k \otimes_{\rho=0}^{k-1}\; S^{\dot{1}}_{k,pk,\rho}S^{\dot{2}}_{k,pk,\rho}\tilde{S}^{\dot{1}}_{k,\rho}\tilde{S}^{\dot{2}}_{k,\rho}\right]^{N/k}. \tag{4.1.39}$$

Notice that since h_H depends on s, for $s > 0$ we have $h_H \neq \bar{h}_H$ and so heavy states carry non-vanishing momentum; the correlator reads

$$\mathcal{G}\left(s=pk,k;z,\bar{z}\right) = \frac{1/k}{|z|}\,\frac{1-|z|^2}{1-|z|^{2/k}}\,z^{-p}. \tag{4.1.40}$$

When $s = pk - 1$ the heavy operator differs from the previous case only in the $\rho = 0$ sector, and has the form

$$O_H(s=pk-1,k) = \left[\Sigma_k\tilde{\Sigma}_k\hat{S}^{\dot{1}}_{k,0}\hat{S}^{\dot{2}}_{s,0} \otimes_{\rho=1}^{k-1} S^{\dot{1}}_{k,pk,\rho}S^{\dot{2}}_{k,pk,\rho}\tilde{S}^{\dot{1}}_{k,\rho}\tilde{S}^{\dot{2}}_{k,\rho}\right]^{N/k}, \tag{4.1.41}$$

and the correlator reads

$$\mathcal{G}(s=kp-1,k;z,\bar{z}) = \frac{1/k}{|z|}\,z^{-p}\left(z + \frac{|z|^{2/k}-|z|^2}{1-|z|^{2/k}}\right). \tag{4.1.42}$$

4.2 Conformal blocks decomposition

In this section we analyze the correlators obtained above in terms of Virasoro and affine conformal blocks, exploiting the underlying $SU(2)$ R-symmetry. To do this and in order to use the results of appendix A we look at the correlators with

$$z_1 = 0, \quad z_2 = \infty, \quad z_3 = 1, \quad z_4 = z \tag{4.2.1}$$

and the four-point function turns out to be a two-point function between heavy states. In the channel where the two light operators approach each other ($z_3 \to z_4$), the cross-ratio z tends to 1 and we can expand the function $\mathcal{G}$ in (4.1.1) to extract the Virasoro or affine primary operators entering in the decomposition:

$$\mathcal{G} = (1-z)^{2h_L}(1-\bar{z})^{2\bar{h}_L} \sum_{O_p} C_{HHO_p} C_{LLO_p} \mathcal{V}_{V,A}(h_p, h_H, h_L, z) \bar{\mathcal{V}}_{V,A}(\bar{h}_p, \bar{h}_H, \bar{h}_L, \bar{z}) \ , \tag{4.2.2}$$

where the sum is over all Virasoro or affine primaries O_p, $\mathcal{V}_V$ and $\mathcal{V}_A$ are the Virasoro or affine blocks, and C_{HHO_p} (C_{LLO_p}) are the structure constants between O_p and the heavy (light) operators.

4.2.1 Virasoro blocks decomposition

For the description in terms of the Virasoro blocks we focus on the large c limit where it is possible to use the results of appendix A. In this limit the contribution of the Virasoro descendents of a primary of weight h_p is captured by the block whose holomorphic part is given by (A.3.43) [4]

$$\mathcal{V}_V^{(0)}(h_p, h_H, h_L, z) = z^{h_L(\alpha-1)} \left(\frac{1-z^\alpha}{\alpha}\right)^{h_p - 2h_L} {}_2F_1\left(h_p, h_p; 2h_p; 1-z^\alpha\right) \ , \tag{4.2.3}$$

where $\alpha = \sqrt{1 - \frac{24 h_H}{c}}$. Some of the heavy states we consider have conformal dimension $h_H = c/24$; in this case the large c limit of the Virasoro block is captured by the $\alpha \to 0$ limit of (4.2.3)

$$\mathcal{V}_V^{(0)}(h_p, h_H \to c/24, h_L, z) = z^{-h_L} \left(-\ln z\right)^{h_p - 2h_L} \ . \tag{4.2.4}$$

In all amplitudes analyzed in the previous section, the first primary entering the $z \to 1$ decomposition is the identity. If we consider only the contribution of its Virasoro block, for instance in the simplest case (4.1.22), we have

$$\mathcal{G}\Big(s = 0; z, \bar{z}\Big) = \frac{1}{|z|} \frac{|1-z|^2}{|\ln z|^2} + \dots \ , \tag{4.2.5}$$

where we used (4.2.2) and (4.2.4) with $h_p = 0$, $h_L = 1/2$, and the analogous expression for the anti-holomorphic sector with $\bar{h}_p = 0$, $\bar{h}_L = 1/2$. Focusing on the holomorphic dependence, there is a mismatch between (4.2.5) and (4.1.22) already at the order $(1-z)$, which signals that primaries of conformal dimension $(h_p, \bar{h}_p) = (1, 0)$ must contribute to the correlator (4.1.22). It is straightforward to see that in the OPE of the two light

[4] With respect to the result in appendix A we sent $z \to 1 - z$ to reshuffle the operators and to have the two light ones between the heavies. We also normalize the conformal block so that the first term of the $z \to 1$ expansion is $(1-z)^{h_p - 2h_L}$.

operators O_L, $\bar{O}_L$ the first (normalized) Virasoro primaries are

$$
\begin{aligned}
O_{(1,0)} &= \sqrt{\frac{2}{N}} \sum_{r=1}^{N} J^3_{(r)} \,, \\
O_{(2,0)} &= \frac{1}{\sqrt{6N}} \sum_{r=1}^{N} \left(-\partial\psi^{\alpha\dot{A}}_{(r)} \psi^{\beta\dot{B}}_{(r)} \epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}} + \frac{1}{2}\partial X^{A\dot{A}}_{(r)} \partial X^{B\dot{B}}_{(r)} \epsilon_{AB}\epsilon_{\dot{A}\dot{B}} \right) .
\end{aligned}
\tag{4.2.6}
$$

We can straightforwardly compute the three-point correlators between these primaries and the heavy or the light operators so to extract the structure constants entering in the decomposition (4.2.3). For later convenience, we summarize the results involving the light and the heavy operators in (4.1.23) for generic s:

$$
\begin{aligned}
C_{LLO_{(1,0)}} &= \frac{1}{\sqrt{2}} \,, \quad C_{HHO_{(1,0)}} = \sqrt{2}\left(s + \frac{1}{2}\right) , \\
C_{LLO_{(2,0)}} &= \frac{1}{\sqrt{6}} \,, \quad C_{HHO_{(2,0)}} = \frac{(1+2s)^2}{2\sqrt{6}} \,.
\end{aligned}
\tag{4.2.7}
$$

Thus one can improve on the decomposition (4.2.5) by adding the Virasoro blocks for the operators in (4.2.6)

$$
\mathcal{G}\Big(s=0; z, \bar{z}\Big) = \frac{1}{|z|} \frac{|1-z|^2}{|\ln z|^2} \left(1 - \frac{1}{2}\ln z + \frac{1}{12}(\ln z)^2 + \dots \right) , \tag{4.2.8}
$$

which reproduces (4.1.22) to the leading order in the $\bar{z} \to 1$ and to second order in $z \to 1$ limits.

We can proceed with the same analysis for the remaining correlator (4.1.24) in the untwisted sector. One now has $h_H = \frac{c}{6}\left(s + \frac{1}{2}\right)^2$, $\bar{h}_H = \frac{c}{24}$, and we have to use the large c Virasoro blocks (4.2.3) for the holomorphic part and (4.2.4) for the anti-holmorphic one. The contribution of the identity gives

$$
\mathcal{G}\Big(s; z, \bar{z}\Big) = -\frac{|1-z|^2}{\sqrt{\bar{z}}\log(\bar{z})} \frac{\alpha\, z^{\frac{\alpha-1}{2}}}{1-z^\alpha} + \cdots , \tag{4.2.9}
$$

where $\alpha = \sqrt{1 - 4\left(s + \frac{1}{2}\right)^2}$. Again, the expansion of the expression above for $z \to 1$ already disagrees with the exact result (4.1.24) at order $(1-z)$ and, as before, we need to add the Virasoro blocks of other primaries. By using the s-dependent structure constants in (4.2.7), we have

$$
\mathcal{G}\Big(s; z, \bar{z}\Big) = \frac{|1-z|^2}{\sqrt{\bar{z}}\log(\bar{z})} \frac{\alpha\, z^{\frac{\alpha-1}{2}}}{z^\alpha - 1} \left[1 - \frac{1+2s}{2}\log z - \frac{(1+2s)^2}{2\alpha^2}\left(2 + \frac{1+z^\alpha}{1-z^\alpha}\log z^\alpha \right) + \dots \right] . \tag{4.2.10}
$$

As in the $s = 0$ case, the expression above agrees with the exact result (4.1.24) up to order $(1-z)^2(1-\bar{z})^0$ in the $z \to 1$ expansion.

4.2.2 Affine blocks decomposition

In all our examples the light operator (4.1.13) used to probe the heavy states is written just in terms of the elementary fermions of the orbifold CFT. This suggests that it is convenient to study the decomposition of this type of correlators in terms of affine blocks related to the $SU(2)_L$ current algebra (4.1.8). As this symmetry is part of the chiral superalgebra we can use this analysis to argue that the correlators considered in the previous section are protected by supersymmetry, and then, in the next section, to match the free CFT result with supergravity calculations. Also, in contrast to the pure Virasoro case, the results for the affine blocks are exact in c and so we can use them to understand the effect of resumming the large c limit of the blocks of all Virasoro primaries: we will see that the singularities due to each Virasoro block [10] disappear even at large c. This is reminiscent of what happens in some out-of-time-ordered correlators in $SU(N)_k$ WZW models [36].

We start from the simplest example discussed in (4.1.22) and analyze it in two slightly different ways. First we observe that the correlator is purely fermionic and that it is given by a sum over the N strands of correlators that involve non-trivially only the fields on one strand at a time. We can then effectively restrict to two free complex fermions on a length one strand, which realize a $SU(2)_{k=1} \times U(1)$ WZW model[5] (see for instance [37]). Note that the $SU(2)_{k=1}$ factor is identified with the R-symmetry $SU(2)_L$, and is thus a symmetry of the CFT at a generic point in the moduli space; the $U(1)$ symmetry, instead, disappears away from the free orbifold point. The non-trivial four-point function to compute is the one appearing in the first line of (4.1.18) for $s = 0$; with respect to the $SU(2)_{k=1}$ subsector of the WZW model, all the four operators involved are $SU(2)_{k=1}$ primaries of spin 1/2. Though the light operators also carry a $U(1)$ charge, the heavy states are scalars under this $U(1)$, and thus the correlator reduces to a trivial 2-point function in the $U(1)$ sector. This means that it should be possible to write the amplitude (4.1.22) by using the classic result of [38] for the affine blocks of $SU(N)_k$ WZW models in the special case where $N = 2$ and $k = 1$. This model has only two primaries (the identity and the spin 1/2 primary) and so the only $SU(2)_{k=1}$ primary appearing in the OPE of two spin 1/2 operators has to be the identity. So in this case the affine decomposition (4.2.2) contains just one term, given by the $SU(2)_{k=1}$ block of the identity: since, as we said, $SU(2)_{k=1}$ is part of the superconformal algebra, this shows that the amplitude (4.1.22) can be written in terms of protected quantities.

It is straightforward to check that the hypergeometric describing the $SU(N)_k$ blocks reduce to elementary functions for the identity block with $N = 2$ and $k = 1$; by adapting the results summarized in [37] to our notations we have[6]

$$\mathcal{V}_{SU(2)_1} = (1-z)^{-2h_L} \begin{pmatrix} F_1^- \\ F_2^- \end{pmatrix} = (1-z)^{-2h_L} \begin{pmatrix} z^{-\frac{1}{2}} \\ z^{\frac{1}{2}} \end{pmatrix} , \tag{4.2.11}$$

where the component F_1^- (F_2^-) contributes if the operators in z_1 and z_4 (z_2 and z_4) have opposite spin. In our case (4.1.18) F_1^- enters in the decomposition of (4.1.22) and reproduces directly the whole amplitude.

The simple result in (4.2.11) suggests that only a subsector of the full $SU(2)_{k=1}$ affine blocks contributes to our correlator. This is indeed the case and the amplitude is saturated just considering the affine descendants obtained by acting with the modes of the currents J^3 (and $\tilde{J}^3$) on the identity. Focusing on this $U(1)_L$ subgroup, the affine

[5]This approach is similar to one adopted in [36] in the study of quantum chaos in rational CFT. Notice however that in that analysis the large central charge limit is obtained by studying the WZW model $SU(N)_k$ in the limit $N, k \to \infty$ with N/k fixed, instead of using the symmetric orbifold of many copies of $SU(2)_{k=1}$, as relevant for our case.

[6]In order to translate the choice of the z_i^D's of [37] into ours it is sufficient to take $z_{i=1,3}^D = z_{i=1,3}$, $z_2^D = z_4$, and $z_4^D = z_2$; notice also that the blocks in [37] have a different normalization and that the hypergeometric appearing in Eq.(15.170) of [37] should read ${}_2F_1\left(\frac{\kappa+1}{\kappa}, \frac{\kappa-1}{\kappa}, \frac{2\kappa-N}{\kappa}, x\right)$.

block of the identity reads (see appendix A)[7]

$$\mathcal{V}_{U(1)}(q_H, q_L, z) = (1-z)^{-2h_L} z^{2q_H q_L} \ , \tag{4.2.12}$$

where the q_H and q_L are identified with the J_0^3 quantum numbers (j) of the operators $\bar{O}^H_{(r)}(z_2)$ and $O^L_{(r)}(z_3)$ (note that, with this identification, the level of the $U(1)_L$ current algebra is $k = 1/2$, in the conventions of [39]). Then, by using $q_H = -1/2 - s$ and $q_L = 1/2$, we immediately reproduce not just (4.1.22) but also (4.1.24).

The correlators involving states in the twisted sector can also be described in terms of $U(1)_L$ affine blocks. The generator J^3 on a strand of length k splits into the sum of k $U(1)$'s labelled by $\rho = 0, \dots, k-1$. While the charge of the light operator is still $q_L = 1/2$ for any ρ, the charge of the heavy operators is ρ-dependent, as can be seen from (4.1.27) and (4.1.33). So the contribution to the block decomposition of each ρ-sector is given by (4.2.12) with the values for the q's that can be read off from (4.1.27) and (4.1.33); after performing the sum over ρ, one can check that the correlators (4.1.40) and (4.1.42) are reproduced by (4.2.2) with only the inclusion of the $U(1)_L$ affine block of the identity.

4.3 The gravity picture

Let $|s, k\rangle$ denote the pure states generated by the action of the heavy operators on the conformal invariant vacuum:

$$|s,k\rangle \equiv \lim_{z,\bar{z}\to 0} O_H(s,k;z,\bar{z})|0\rangle \, . \tag{4.3.1}$$

Since operators of conformal dimension of order c backreact strongly on the geometry and generate a non-trivial gravity background, these states admit a dual gravity description. The four-point correlators computed in the previous section can thus be thought as two-point functions of light correlators in a non-trivial geometry:

$$\langle s,k|O_L(1)\bar{O}_L(z)|s,k\rangle = \frac{1}{|1-z|^{4h_L}}\, \mathcal{G}(z,\bar{z}) \, . \tag{4.3.2}$$

In the limit of large central charge this geometry is well approximated by a solution in supergravity. In this section we will compute this two-point function at the point in the CFT moduli space where supergravity is weakly coupled, i.e. higher curvature corrections are negligible.

This point in moduli space differs from the free orbifold point, where the CFT correlators have been computed. While the light operators we consider are chiral primaries both in the left and right sector and the heavy operators are chiral at least in the right sector, their four-point correlators are generically expected to receive corrections when one deforms the free orbifold theory towards the point in moduli space corresponding to weakly coupled supergravity. This is made evident by the decomposition (4.2.2), which generically contains also non-chiral primaries (and their descendants). For the particular correlators we consider in this paper, we have however shown in Section 4.2.2 that the expansion (4.2.2) only contains the identity operator and its super-descendants with respect to a $U(1)$ subgroup of the superconformal algbera. This implies that CFT and gravity results must agree. In this section we verify this expectation.

[7] See [39] for a recent discussion of the $U(1)$ blocks in the context of the heavy-light large c limit.

4.3.1 The six-dimensional geometries

The D1D5 CFT is dual to a gravity theory on spaces that are asymptotically[8] AdS$_3 \times S^3$: the S^3 factor is necessary to geometrically implement the $SU(2)_L \times SU(2)_R$ R-symmetry of the CFT. The geometries generated by generic heavy operators are complicated six-dimensional spaces, which only asymptotically factorize into the product of AdS$_3$ and S^3. All these geometries are known when the heavy operators are chiral primaries both on the left and the right sector [40, 41, 29]; a subset of geometries is known for heavy operators that are chiral only on the right sector [22, 42, 43, 44, 45, 46, 27, 47], or are not chiral on either sector [48, 49].

We concentrate on a particularly simple set of BPS states, whose dual geometries are *locally* isometric to AdS$_3 \times S^3$ via a diffeomorphism that does not vanish at the boundary. The solution in terms six-dimensional fields defined in appendix B.3 reads, for the metric in Einstein frame

$$ds_6^2 = \sqrt{Q_1 Q_5}\,(ds^2_{AdS_3} + ds^2_{S^3})\,, \tag{4.3.3a}$$
$$ds^2_{AdS_3} = \frac{dr^2}{a^2k^{-2}+r^2} - \frac{a^2k^{-2}+r^2}{Q_1Q_5}dt^2 + \frac{r^2}{Q_1Q_5}dy^2\,, \tag{4.3.3b}$$
$$ds^2_{S^3} = d\theta^2 + \sin^2\theta\, d\hat\phi^2 + \cos^2\theta\, d\hat\psi^2\,. \tag{4.3.3c}$$

The gravity solution also includes a RR two-form, whose field strength is

$$F = 2\,Q_5\,(-\mathrm{vol}_{AdS_3} + \mathrm{vol}_{S^3})\,, \tag{4.3.4a}$$
$$\mathrm{vol}_{AdS_3} = \frac{r}{Q_1Q_5}\,dr\wedge dt\wedge dy\,, \quad \mathrm{vol}_{S^3} = \sin\theta\cos\theta\, d\theta\wedge d\hat\phi\wedge d\hat\psi\,. \tag{4.3.4b}$$

The three-form field strength is anti-self-dual in the 6D Einstein metric

$$*_6\, F = -F\,, \tag{4.3.5}$$

where $*_6$ is the Hodge star with respect to ds^2 and we choose the orientation $\epsilon_{rty\theta\hat\phi\hat\psi} = +1$.

The coordinates t, y are identified with the time and space coordinates of the CFT, and we take y to parametrize an S^1 of radius R; $\hat\phi$ and $\hat\psi$ are some linear combinations of the S^3 Cartan's angles ϕ, ψ and the CFT coordinates t, y; the particular linear combination depends on the state and will be given below. We recall here Q_1 and Q_5 computed in previous chapter and encoding the numbers of D1 and D5 charges, n_1 and n_5 (with $N = n_1 n_5$):

$$Q_1 = \frac{(2\pi)^4 n_1 g_s (\alpha')^3}{V_4}\,, \quad Q_5 = g_s n_5 \alpha'\,, \tag{4.3.6}$$

where g_s is the string coupling and V_4 is the volume of the compact space M. The parameter a is linked to the D-brane charges and the S^1 radius by

$$a = \frac{\sqrt{Q_1Q_5}}{R}\,. \tag{4.3.7}$$

Finally k is a positive integer which introduces a conical defect in the geometry $ds^2_{AdS_3}$: indeed this space represents a $\mathbb{Z}_k$ orbifold of AdS$_3$.

[8]To describe generic states one should consider the full ten-dimensional geometry, which asymptotes AdS$_3 \times S^3 \times M$, with M either T^4 or $K3$. For the class of states we consider, the M factor is irrelevant and we restrict to the six-dimensional part of the geometry.

The states $|s=0,k\rangle$ have $h_H=\bar h_H=\frac{c}{24}=\frac{N}{4}$ and thus carry D1 and D5 charges but no momentum charge. The geometries dual to these states were found in [40] and can be written in the form (4.3.3) with

$$\hat\phi=\phi-\frac{t}{R\,k}\,,\quad \hat\psi=\psi-\frac{y}{R\,k}\,. \tag{4.3.8}$$

Note that the original set of coordinates (t,y,ϕ,ψ) is subject to the identifications

$$(t,y,\phi,\psi)\sim(t,\,y+2\pi\,l\,R,\,\phi+2\pi\,m,\,\psi+2\pi\,n)\,, \tag{4.3.9}$$

with $l,m,n\in\mathbb{Z}$. Only when $k=1$ eq. (4.3.8) defines a new set of coordinates $(t,y,\hat\phi,\hat\psi)$ which satisfy analogous identifications

$$(t,y,\hat\phi,\hat\psi)\sim(t,\,y+2\pi\,l\,R,\,\hat\phi+2\pi\,m,\,\hat\psi+2\pi\,n)\,,\quad (k=1)\,. \tag{4.3.10}$$

In this case the coordinate transformation $(t,y,\phi,\psi)\to(t,y,\hat\phi,\hat\psi)$ realizes the spectral flow from the state $|s=0,k=1\rangle$ to the SL(2,$\mathbb{C}$)-invariant vacuum, whose dual geometry is (4.3.3) with the identifications (4.3.10), i.e. global $\mathrm{AdS}_3\times S^3$. For $k>1$ the identifications induced on the $(t,y,\hat\phi,\hat\psi)$ coordinates are more complicated:

$$(t,y,\hat\phi,\hat\psi)\sim\Big(t,\,y+2\pi\,l\,R,\,\hat\phi+2\pi\,m,\,\hat\psi-2\pi\,\frac{l}{k}+2\pi\,n\Big)\,. \tag{4.3.11}$$

The geometry dual to the state $|s=0,k\rangle$ is given by (4.3.3) expressed in the (t,y,ϕ,ψ) coordinate system via (4.3.8): geometrically it represents a $\mathbb{Z}_k$ orbifold of $\mathrm{AdS}_3\times S^3$. For $k>1$ there is no state in the D1D5 CFT dual to the geometry (4.3.3) with the identifications (4.3.10).

The states $|s,k\rangle$ have $h_H=\frac{N}{4}+\frac{N\,s(s+1)}{k^2}$, $\bar h_H=\frac{N}{4}$ and thus carry momentum $n_p=h-\bar h=\frac{N\,s(s+1)}{k^2}$. The dual geometries have been found in [46] and are of the form (4.3.3) with

$$\hat\phi=\phi-\frac{t}{R\,k}-s\,\frac{t+y}{R\,k}\,,\quad \hat\psi=\psi-\frac{y}{R\,k}-s\,\frac{t+y}{R\,k}\quad (s\in\mathbb{Z})\,. \tag{4.3.12}$$

As in the previous example, this coordinate redefinition preserves the simple periodic identifications only for $k=1$. For $k>1$ the geometry is again a $\mathbb{Z}_k$ orbifold of $\mathrm{AdS}_3\times S^3$, though the orbifold group, determined by the coordinate redefinition (4.3.12), acts differently than in the previous example. It is important to keep in mind that the integers s and k must be such that the momentum on each strand $s(s+1)/k$ be integer[9]. This allows for non-integer s/k; states with s/k integer are particularly simple, as they are obtained from the 2-charge states with $s=0$ by a global chiral algebra transformation.

We note that setting $s=0$ the D1D5P states specified by eq. (4.3.12) reduce to the D1D5 states corresponding to (4.3.8). In the following we will thus work with the more general class of states described by (4.3.12).

4.3.2 The holographic two-point function

We want to compute the correlator of the light operators $O_L\equiv O^{++}$ and $\bar O_L\equiv O^{--}$ in the states $|s,k\rangle$, whose dual geometries are specified by (4.3.3,4.3.4) and (4.3.12). We will do this by computing the vev of the operator $\bar O_L$ in the presence of a source for the operator O_L, and then differentiating the vev with respect to the source to obtain the

[9] This condition only holds when n_1 and n_5 are coprime and a more general condition applies if n_1 and n_5 share a common divisor [46]; for simplicity we will assume n_1 and n_5 coprime in this article.

two-point correlator:

$$\langle s,k|O_L(0,0)\bar{O}_L(t,y)|s,k\rangle = \mathrm{i}\frac{\delta\langle\bar{O}_L(t,y)\rangle_J}{\delta\bar{J}_L(0,0)}\bigg|_{J=0}, \tag{4.3.13}$$

where $\bar{J}_L$ is the source coupling to O_L and the correlator is computed on the cylinder parametrized by t and y. The vev $\langle\bar{O}_L(t,y)\rangle_J$ is extracted from the supergravity field dual to $\bar{O}_L$.

In 6D[10] the fields dual to the chiral primary operators $O^{\pm\pm}$ are the scalar w, the perturbation of χ_2 (see appendix B.3) and a two-form B_2 (whose field strength is h), which satisfy a coupled system of differential equations. The linearization of these equations around the background (4.3.3,4.3.4) gives [50, 51]

$$h - *_6 h = 2\,\mathrm{w}\,F\,, \quad d *_6 d\mathrm{w} = \frac{Q_1}{Q_5}\,h\wedge F\,. \tag{4.3.14}$$

The factorised form of the background (when expressed in $\hat{\phi}$, $\hat{\psi}$ coordinates) allows to reduce the six-dimensional equations (4.3.14) to two sets of decoupled equations on AdS_3 and S^3. To this purpose one can make the ansatz [52]

$$\mathrm{w} = Y\,B\,, \quad B_2 = \gamma\,(Y *_{AdS_3} dB - B *_{S^3} dY)\,, \tag{4.3.15}$$

where Y is a function of $\theta, \hat{\phi}, \hat{\psi}$, B is a function of r, t, y, $*_{AdS_3}$ and $*_{S^3}$ are the Hodge duals with respect to $ds^2_{AdS_3}$ and $ds^2_{S^3}$ and γ is a constant that will be determined shortly. It is straightforward to verify that this ansatz satisfies (4.3.14) if Y and B are eigenfunctions of the respective Laplacians:

$$\square_{AdS_3} B = \ell(\ell-2)\,B\,, \quad \square_{S^3} Y = -\ell(\ell+2)\,Y\,, \tag{4.3.16}$$

and if $\gamma = \frac{Q_5}{\ell}$. Then Y is a scalar harmonic on S^3 of order ℓ, with ℓ a positive integer; B is a minimally coupled scalar in AdS_3 with mass $m^2 = \ell(\ell-2)$.

As the CPO's $O^{\pm\pm}$ form a multiplet with $SU(2)_L\times SU(2)_R$ charges $j=\bar{j}=1/2$, the gravity dual field must have spin 1, and hence we should look for solutions for B and Y with $\ell=1$. The vev of O^{--} is encoded in the component of the field w proportional to the spherical harmonic $Y_1^{++} = \sin\theta\, e^{\mathrm{i}\phi}$ (see eqs. (4.9), (4.10) in [33]). Thus we seek for a solution of the form

$$\mathrm{w} = B(t,y,r)\,\sin\theta\, e^{\mathrm{i}\hat{\phi}} = B(t,y,r)\, e^{-\mathrm{i}\,\frac{t}{Rk} - \mathrm{i}\,s\,\frac{t+y}{Rk}}\,\sin\theta\, e^{\mathrm{i}\phi}\,, \tag{4.3.17}$$

where $B(t,y,r)$ solves the AdS_3 Laplace equation (4.3.16) with $\ell=1$. Note that the phase $e^{-\mathrm{i}\,s\,\frac{y}{Rk}}$ is not globally well-defined on the circle $y\sim y+2\pi\,R$ when s/k is fractional. Thus, for w to be a globally defined field, we need to require that the function $B(t,y,r)$ has an appropriate monodromy when going around the S^1 to cancel that of the phase:

$$B(y, y+2\pi R, r) = B(t,y,r)\, e^{\mathrm{i}\,\frac{\hat{s}}{k}\,2\pi}\,, \tag{4.3.18}$$

where $\hat{s} = s \bmod k$ and we choose $0\le\hat{s}<k$.

Since the non-normalizable and normalizable solutions of the AdS_3 wave equation go like $r^{-1}\log r$ and r^{-1}, the usual AdS/CFT prescription implies that the asymptotic

[10] When lifted to the ten-dimensional IIB duality frame, B_2 is the NSNS two-form and w is the component of the RR four-form along the compact space M.

behaviour of the field w has the form

$$\mathrm{w} \approx \frac{\bar{J}_L(t,y)\,\log r + \langle \bar{O}_L(t,y)\rangle_J}{r} \sin\theta\, e^{\mathrm{i}\phi}\,. \tag{4.3.19}$$

Requiring that w is finite in the interior of space links the normalizable and non-normalizable terms of the solution. In accordance with (4.3.13), the two point function of $O_L(0,0)$ and $\bar{O}_L(t,y)$ is given by the vev $\langle \bar{O}_L(t,y)\rangle_J$ when the source for O_L is a delta-function: $\bar{J}_L(t,y) = \delta(t,y)$.

In summary, one looks for a solution of the equation (4.3.16) for B with $\ell = 1$ which is *regular* in the bulk, has the monodromy (4.3.18), and its leading behavior at large r is

$$B(t,y,r) \approx \delta(t,y)\,\frac{\log r}{r} + b_1(t,y)\,\frac{1}{r}\,. \tag{4.3.20}$$

AdS solutions with monodromies like in (4.3.18) are not usually considered in the literature.

4.3.3 Wave equation in $\mathrm{AdS}_3/\mathbb{Z}_k$

In this section we solve the wave equation (4.3.16) for a scalar field of dimension 1, in the geometry written in (4.3.3), with the monodromy (4.3.18) and the boundary condition (4.3.20). We will follow a route similar to the one employed in [53, 54], and our result generalises the one obtained in the previous works to the case with non-trivial monodromy ($\hat{s} \neq 0$). The boundary CFT lives on the cylinder and to induce the appropriate geometry on the boundary we will work in global AdS coordinates; we will keep careful track of the periodicity of the spatial circle, which is crucial to distinguish geometries with different values of the conical defect and to properly implement the monodromy condition. More general discussions about the dynamics of a scalar field in Lorentzian AdS of general conformal dimension, the interpretation of the normalizable modes solution, and the difference between different choice of patch can be found in [55].

The AdS part of the geometry in (4.3.3b) can be simplified by the redefinitions:

$$t = k\,\frac{\sqrt{Q_1 Q_5}}{a}\tau \qquad y = k\,\frac{\sqrt{Q_1 Q_5}}{a}\sigma\,, \quad r = \frac{a}{k}\tan\rho\,, \tag{4.3.21}$$

where the new coordinates τ, σ, ρ have the following domains

$$\rho \in \left[0, \frac{\pi}{2}\right]\,, \quad \sigma \in \left[0, \frac{2\pi}{k}\right]\,, \quad \tau \in [0, +\infty)\,. \tag{4.3.22}$$

After this change the metric takes the form

$$ds^2_{AdS_3} = \frac{1}{\cos^2\rho}\left(-d\tau^2 + d\rho^2 + \sin^2\!\rho\, d\sigma^2\right) \tag{4.3.23}$$

with the boundary located at $\rho = \frac{\pi}{2}$.
The most general solution with the prescribed monodromies involves an arbitrary sum over Fourier modes:

$$B(\tau,\sigma,\rho) = \frac{1}{(2\pi)^2}\, e^{\mathrm{i}\hat{s}\sigma} \sum_{l\in\mathbb{Z}} \int d\omega\, e^{\mathrm{i}\omega\tau} e^{\mathrm{i}lk\sigma} g(l,\omega)\chi_{l,\omega}(\rho)\,, \tag{4.3.24}$$

where the choice of the function $g(l,\omega)$ encodes a particular boundary data and we assume $0 \le \hat{s} < k$. Substituting into the wave equation we obtain a differential equation for $\chi_{l,\omega}(\rho)$ that reads

$$\chi''_{l,\omega}(\rho) + \csc\rho\sec\rho\,\chi'_{l,\omega}(\rho) + \Big(\omega^2 - (lk+\hat{s})^2\csc^2\rho + \ell(\ell-2)\Big)\,\chi_{l,\omega}(\rho) = 0\,. \tag{4.3.25}$$

This is an hypergeometric equation, as it is made evident by the change $x = \sin^2\rho$:

$$\chi''_{l,\omega}(x) + \frac{1}{x}\chi'_{l,\omega}(x) + \frac{1}{4}\left(\frac{\omega^2}{x(1-x)} - \frac{(lk+\hat{s})^2}{x^2(1-x)} + \frac{1}{x(1-x)^2}\right)\chi_{l,\omega}(x) = 0\,. \tag{4.3.26}$$

The solution that is finite everywhere in the bulk[11] is

$$\chi_{l,\omega}(x) = x^{\frac{|lk+\hat{s}|}{2}}(1-x)^{\frac{1}{2}}\,{}_2F_1\left(\frac{1}{2}(1+|lk+\hat{s}|-\omega), \frac{1}{2}(1+|lk+\hat{s}|+\omega), 1+|lk+\hat{s}|, x\right)\,. \tag{4.3.27}$$

From the expansion of this solution near the boundary ($x = 1$) one can extract the non-normalizable and the normalizable modes

$$\begin{aligned}\chi_{l,\omega}(x) \approx\; & \frac{\Gamma(1+|lk+\hat{s}|)}{\Gamma(\frac{1}{2}(1+|lk+\hat{s}|-\omega))\Gamma(\frac{1}{2}(1+|lk+\hat{s}|+\omega))}\times \\ & \left\{\left[2\gamma_E + \psi(\frac{1}{2}(1+|lk+\hat{s}|-\omega)) + \psi(\frac{1}{2}(1+|lk+\hat{s}|+\omega))\right](1-x)^{\frac{1}{2}}\right. \\ & \left. + \left[\log(1-x)\right](1-x)^{\frac{1}{2}}\right\},\end{aligned} \tag{4.3.28}$$

with the digamma function defined as $\psi(z) \equiv \frac{d}{dz}\log(\Gamma(z))$, and γ_E the Euler constant. The non-normalizable mode (the source) is the coefficient of the $[\log(1-x)]\,(1-x)^{\frac{1}{2}}$ term and the normalizable mode (the VEV) is the term proportional to $(1-x)^{\frac{1}{2}}$. Reverting to the original coordinates, these two terms correspond to the ones shown in (4.3.20). A delta function source at the boundary is obtained by tuning the function $g(l,\omega)$ in (4.3.24) in such a way that the non-normalizable term has constant Fourier transform; this is achieved setting

$$g(l,\omega) = \frac{\Gamma(\frac{1}{2}(1+|lk+\hat{s}|-\omega))\Gamma(\frac{1}{2}(1+|lk+\hat{s}|+\omega))}{\Gamma(1+|lk+\hat{s}|)}\,. \tag{4.3.29}$$

The coefficient of the normalizable term, denoted as $b_1(\tau,\sigma)$ in (4.3.20), is then found from (4.3.28) to be

$$b_1(\tau,\sigma) = \sum_{l\in\mathbb{Z}}\int\frac{d\omega}{(2\pi)^2}\,e^{\mathrm{i}\omega\tau+\mathrm{i}(lk+\hat{s})\sigma}\left[\psi(\frac{1}{2}(1+|lk+\hat{s}|-\omega)) + \psi(\frac{1}{2}(1+|lk+\hat{s}|+\omega)) + 2\gamma_E\right]. \tag{4.3.30}$$

In order to perform the sum we use the series representation of the digamma function

$$\psi(z) = -\gamma_E + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right)\,. \tag{4.3.31}$$

[11] The form of the other independent solution can be found, for example, in [55]. It can be shown to contain divergences for $x \to 0$ (i.e. $r \to 0$).

Separating the term with $l = 0$ in the sum, and forgetting contact terms coming from summation over constants Fourier modes we have

$$b_1(\tau,\sigma) = \sum_{n=0}^{\infty}\Bigg[\sum_{l=0}^{\infty}\int\frac{d\omega}{(2\pi)^2}\,e^{\mathrm{i}\omega\tau+\mathrm{i}(lk+\hat{s})\sigma}\left(\frac{2}{\omega-(lk+\hat{s})-1-2n}-\frac{2}{\omega+(lk+\hat{s})+1+2n}\right) + \sum_{l=1}^{\infty}\int\frac{d\omega}{(2\pi)^2}\,e^{\mathrm{i}\omega\tau-\mathrm{i}(lk-\hat{s})\sigma}\left(\frac{2}{\omega-(lk-\hat{s})-1-2n}-\frac{2}{\omega+(lk-\hat{s})+1+2n}\right)\Bigg]\,. \tag{4.3.32}$$

As usual, to define the ω-integral one has to pick the integration contour: we choose the Feynman prescription, which allows the Wick rotation to Euclidean space and hence comparison with the CFT correlator, which is evaluated on the Euclidean plane. The integral is thus readily computed and yields

$$\begin{aligned} b_1(\tau,\sigma) &= -\frac{\mathrm{i}}{2\pi}\sum_{n=0}^{\infty}\Bigg[\sum_{l=0}^{\infty}e^{\mathrm{i}(lk+\hat{s})\sigma}e^{-\mathrm{i}(lk+\hat{s}+1+2n)\tau}+\sum_{l=1}^{\infty}e^{-\mathrm{i}(lk-\hat{s})\sigma}e^{-\mathrm{i}(lk-\hat{s}+1+2n)\tau}\Bigg] \\ &= -\frac{\mathrm{i}}{2\pi}\frac{e^{\mathrm{i}\hat{s}\sigma}}{e^{\mathrm{i}\tau}-e^{-\mathrm{i}\tau}}\left[\frac{e^{-\mathrm{i}\hat{s}\tau}}{1-e^{\mathrm{i}k(\sigma-\tau)}}+\frac{e^{\mathrm{i}\hat{s}\tau}}{e^{\mathrm{i}k(\sigma+\tau)}-1}\right]. \end{aligned} \tag{4.3.33}$$

Re-expressing the result in the original physical coordinates defined in (4.3.21), and suppressing the overall numerical coefficient (which is not meaningful as we did not keep track of the normalization of the operators), we finally obtain

$$\begin{aligned} b_1(t,y) &= -\mathrm{i}\frac{e^{\mathrm{i}\hat{s}\frac{y}{Rk}}}{e^{\mathrm{i}\frac{t}{Rk}}-e^{-\mathrm{i}\frac{t}{Rk}}}\left[\frac{e^{\mathrm{i}\frac{t-y}{R}}}{e^{\mathrm{i}\frac{t-y}{R}}-1}e^{-\mathrm{i}\hat{s}\frac{t}{Rk}}+\frac{1}{e^{\mathrm{i}\frac{t+y}{R}}-1}e^{\mathrm{i}\hat{s}\frac{t}{Rk}}\right] \\ &= -\mathrm{i}\left(\frac{z}{\bar{z}}\right)^{\frac{\hat{s}}{2k}}\frac{1}{|z|^{\frac{1}{k}}-|z|^{-\frac{1}{k}}}\left[\frac{\bar{z}}{\bar{z}-1}|z|^{-\frac{\hat{s}}{k}}+\frac{1}{z-1}|z|^{\frac{\hat{s}}{k}}\right]. \end{aligned} \tag{4.3.34}$$

The two-point correlator of the light operators in the state $|s,k\rangle$ is given by

$$\langle s,k|O_L(0,0)\bar{O}_L(t,y)|s,k\rangle = \mathrm{i}\,b_1(t,y)\,e^{-\mathrm{i}\,\frac{t}{Rk}-\mathrm{i}\,s\,\frac{t+y}{Rk}}\,. \tag{4.3.35}$$

To compare the bulk result (4.3.35) with the CFT, one should transform from the cylinder coordinates t and y to the Euclidean plane coordinates[12] $z,\bar{z}$:

$$z = e^{\mathrm{i}\,\frac{t+y}{R}}\,,\quad \bar{z} = e^{\mathrm{i}\,\frac{t-y}{R}}\,, \tag{4.3.36}$$

and remember that

$$O_L(z,\bar{z}) = (z\bar{z})^{-1/2}\,O_L(t,y)\,, \tag{4.3.37}$$

(and the same for $\bar{O}_L$) since $O_L(z,\bar{z})$ is a primary of dimension $h_L = \bar{h}_L = 1/2$. The gravity result for the correlator on the plane is then

$$\langle s,k|O_L(1)\bar{O}_L(z,\bar{z})|s,k\rangle = \frac{z^{\frac{\hat{s}-s}{k}}}{|z|\,|1-z|^2}\,\frac{1-|z|^{2(1-\frac{\hat{s}}{k})}+\bar{z}\left(|z|^{-2\frac{\hat{s}}{k}}-1\right)}{1-|z|^{\frac{2}{k}}}\,. \tag{4.3.38}$$

[12] This is different from what is done when the thermal results are extracted from the Euclidean correlators. Of course in the thermal case, one needs to perform the Wick rotation so as to identify the compact coordinate with time and, on the bulk side, the four point correlators are compared with the wave equation on a BTZ black hole.

One can check that when $s = kp$ (and thus $\hat{s} = 0$) the previous result reduces to the CFT expression (4.1.40), and when $s = kp - 1$ (and thus $\hat{s} = k - 1$) one recovers (4.1.42), up to overall numerical coefficients that have not been kept in the gravity derivation.

4.4 Extremal correlators

The structure of the correlators we consider is again

$$\langle O_H(z_1)\bar{O}_H(z_2)O_L(z_3)O_L(z_4)\rangle = \frac{1}{z_{12}^{2h_H} z_{34}^{2h_L}} \frac{1}{\bar{z}_{12}^{2\bar{h}_H} \bar{z}_{34}^{2\bar{h}_L}} \mathcal{G}(z,\bar{z}) \; , \tag{4.4.1}$$

where, as usual, $z_{ij} = z_i - z_j$ and

$$z = \frac{z_{14}z_{23}}{z_{13}z_{24}} \, , \tag{4.4.2}$$

while $(h_H, \bar{h}_H)$ and $(h_L, \bar{h}_L)$ are the holomorphic/anti-holomorphic conformal dimensions of the heavy and light operators respectively. In order to easily isolate $\mathcal{G}$ from the correlators one can take as usual $z_2 \to \infty$, $z_1 = 0$ and $z_3 = 1$, which implies $z = z_4$:

$$\langle \bar{O}_H | O_L(1)\bar{O}_L(z,\bar{z}) | O_H\rangle \equiv \mathcal{C}(z,\bar{z}) = \frac{1}{(1-z)^{2h_L}} \frac{1}{(1-\bar{z})^{2\bar{h}_L}} \mathcal{G}(z,\bar{z}) \tag{4.4.3}$$

4.4.1 CFT picture

We first focus on operators in the untwisted sector of the symmetric orbifold, which means that they are written as combinations of operators acting on each copy. The symmetry under permutations among the copies is realised differently in the light and the heavy operators: the light operators act trivially on all the strands but one, while the heavy ones are constructed by multiplying N copies of the same operator, each copy acting on a different strand:

$$O_L = \frac{1}{\sqrt{N}} \sum_{r=1}^{N} O^L_{(r)}, \quad O_H = \otimes_{r=1}^{N} O^H_{(r)} \tag{4.4.4}$$

In this chapter we concentrate on light operators of dimension $h_L = \bar{h}_L = 1/2$ constructed with the fermions

$$O^L_{(r)} = -\frac{\mathrm{i}}{\sqrt{2}} \psi^{1\dot{A}}_{(r)} \epsilon_{\dot{A}\dot{B}} \tilde{\psi}^{\dot{1}\dot{B}}_{(r)} \equiv O^{++}_{(r)} \tag{4.4.5}$$

The heavy operator is defined as follows

$$\lim_{z\to 0} O_H|0\rangle = |t_B\rangle = \frac{1}{N^{\frac{N}{2}}} \sum_{p=0}^{N} A^{N-p} \, |++\rangle^{N-p} \, B^p \, |--\rangle^p \; , \quad \text{with} \quad |A|^2 + |B|^2 = N \; , \tag{4.4.6}$$

with

$$|--\rangle_{(r)} = J^-_{0(r)} \tilde{J}^-_{0(r)} |++\rangle_{(r)} \tag{4.4.7}$$

In order to calculate this correlator at the free orbifold point we notice that there are two type of contributions: the diagonal terms where O_L and $\bar{O}_L$ are non-trivial on the same copy and the off-diagonal ones where the two light operators act on different copies. By spin conservation we can conclude that the only non vanishing contribution comes from the terms where the two operators act on the same strand $|--\rangle$ giving a combinatorial factor, times the analytic factor. In this case, the calculation has the structure of a product of two 3-point correlators and we can derive the contribution proportional to AB in $\mathcal{G}(z,\bar{z})$ by choosing $z_2 \to \infty$, $z_1 \to 0$, $z_3 = 1$ and $z_4 = z$

$$\frac{1}{|1-z|^2}\mathcal{G}(z,\bar{z}) = \sum_{r=s} \langle t_B | O^{++}_{(r)}(1,1) O^{++}_{(s)}(z,\bar{z}) | t_B \rangle \tag{4.4.8}$$

To get the correct combinatorial factor we have that the action of $O^{++}O^{++}$ on a strand $|--\rangle$ give a term proportional to $|++\rangle$. Since the state $|p\rangle$ contain p of $|--\rangle$ states with $\binom{N}{p}$ different configuration coming from the reshuffling, the action of $O^{++}O^{++}$ on $|p\rangle$ gives a total of $p\binom{N}{p}$ states that contain one less state $|--\rangle$. In order to reconstruct the state $|p-1\rangle$ we need $\binom{N}{p-1}$ different configuration that should be contained in the $p\binom{N}{p}$ states that the action of the two operators produced. The quotient of the total number of states $p\binom{N}{p}$ and the number of inequivalent configuration $\binom{N}{p-1}$ gives us the factor

$$O^{++}O^{++}|p\rangle = (N-p+1)|p-1\rangle \tag{4.4.9}$$

Adding the analytic factor due to the zero modes of the operators and resembling what we have to compute we get

$$\begin{aligned}
\langle t_B | O^{++}(1) O^{++}(z,\bar{z}) | t_B \rangle &= \frac{1}{|z|} \sum_{q=0}^{N} \sum_{p=1}^{N} \bar{A}^{N-q} A^{N-p} \bar{B}^q B^p (N-p+1) \langle q | p-1 \rangle \\
&= \frac{1}{|z|} \sum_{q=0}^{N} \bar{A}^{N-q} A^{N-q-1} \bar{B}^q B^{q+1} (N-q) \binom{N}{q} \\
&= \frac{\bar{A}B}{|z|} \frac{\partial}{\partial(|A|^2)} \sum_{q=0}^{N} |A|^{2(N-q)} |B|^{2q} \binom{N}{q} \\
&= \frac{\bar{A}B}{|z|} \frac{\partial}{\partial(|A|^2)} (|A|^2 + |B|^2)^N \\
&= \frac{\bar{A}B}{|z|} N^N
\end{aligned} \tag{4.4.10}$$

From the normalized correlator we obtain the free orbifold point result

$$\mathcal{G}(z,\bar{z}) = |1-z|^2 \frac{AB}{|z|} \tag{4.4.11}$$

4.4.2 Gravity picture

The background dual to the heavy states in (6.2.6) is obtained from the general ansatz in (B.4.1) and from the dictionary in Section 3.3.2. In particular we construct the solution from the following profiles

$$\begin{aligned}
g_1(v') + i g_2(v') &= a\, e^{\frac{2\pi i v'}{L}}, \quad g_3(v') + i g_4(v') = b\, e^{\frac{-2\pi i v'}{L}} \\
g_5(v') &= 0
\end{aligned} \tag{4.4.12}$$

By performing the integrals in section 3.2.3 we can write the six-dimensional background solution in series of the parameter b around $b = 0$, and up to $O(b)$ we have

$$
\begin{aligned}
ds_6^2 &= \sqrt{Q_1 Q_5}\left[ds^2_{AdS_3} + ds^2_{S^3} + b\,\delta g_{MN}\,dx^M dx^N\right] \\
\delta g_{MN} &= ab(\nabla_\mu \nabla_\nu \hat{B}\hat{Y} - g_{\mu\nu}\hat{B}\hat{Y}) + 3ab\hat{B}\hat{Y} g_{ab} \\
C &= C_{(0)} - abQ_5\left(\hat{Y} \star_{\mathrm{AdS}_3} d\hat{B} + 2\hat{B} \star_{\mathrm{S}^3} d\hat{Y}\right) && (4.4.13a) \\
e^{2\phi_1} &= e^{-2\phi_1} = \frac{a^2 R^2}{Q_5^2}\left(1 - 2ab\hat{B}\hat{Y}\right) && (4.4.13b) \\
\chi_1 &= \chi_1 = 0 && (4.4.13c) \\
B &= 0 && (4.4.13d)
\end{aligned}
$$

with the zero order defined as

$$
\begin{aligned}
ds^2_{AdS_3} &= \frac{dr^2}{a_0^2 + r^2} - \frac{a_0^2 + r^2}{Q_1 Q_5} dt^2 + \frac{r^2}{Q_1 Q_5} dy^2\,, \quad a_0 \equiv \frac{\sqrt{Q_1 Q_5}}{R} && (4.4.14a) \\
ds^2_{S^3} &= d\theta^2 + \sin^2\theta\, d\hat{\phi}^2 + \cos^2\theta\, d\hat{\psi}^2\,, \quad \hat{\phi} \equiv \phi - \frac{t}{R}\,,\ \hat{\psi} \equiv \psi - \frac{y}{R} && (4.4.14b) \\
C &= C_0 = -\frac{r^2}{Q_1}\, dt \wedge dy - Q_5 \cos^2\theta\, d\hat{\phi} \wedge d\hat{\psi} && (4.4.14c)
\end{aligned}
$$

with all other fields vanishing. We defined the functions

$$
\hat{B} = \frac{e^{\frac{-2it}{R}}}{r^2 + a_0^2} \qquad \hat{Y} = e^{-2i\hat{\phi}} \sin^2\theta \tag{4.4.15}
$$

satisfying

$$
\Box \hat{B} = 0, \quad \Box \hat{Y} + 8\hat{Y} = 0 \tag{4.4.16}
$$

The four point function is computed in a similar way to the one in the previous Chapter. we need to switch on perturbations dual to the operator O^{++} and solve the equations with suitable boundary condition for the non-normalizable mode, and then the four point function is extracted by the normalizable modes with the correct spherical harmonic. The equations to solve are (B.5.13) with the perturbations given by, up to order b

$$
\mathrm{w} = \mathrm{w}_0 + b\,\mathrm{w}_1 + \mathcal{O}(b^2)\,, \quad h = h_0 + b\,h_1 + \mathcal{O}(b^2) \tag{4.4.17}
$$

with

$$
\mathrm{w}_0 = B_0(r,t,y)\,Y^{++}(\theta,\hat{\phi})\,, \quad h_0 = Q_5\, d\left[Y^{++}(\theta,\hat{\phi}) *_{AdS_3} dB_0 - B_0 *_{S^3} dY^{++}(\theta,\hat{\phi})\right] \tag{4.4.18}
$$

The equations can be written in the form similar to ((5.2.22a), (5.2.22b))

$$
\begin{aligned}
h_1 - *_0 h_1 - 2\,\mathrm{w}_1\, dC_0 &= \mathcal{F}\,, && (4.4.19a) \\
\mathcal{F} &\equiv \mathrm{w}_0\left(F_1 - *_0 F_1 - *_1 dC_0\right) + *_1 h_0 && (4.4.19b)
\end{aligned}
$$

and

$$\frac{Q_5}{Q_1}\, d *_0 d\mathrm{w}_1 - h_1 \wedge dC_0 = \widehat{\mathcal{F}} \tag{4.4.20a}$$

$$\widehat{\mathcal{F}} \equiv h_0 \wedge dC_1 - \frac{Q_5}{Q_1}\, d *_1 d\mathrm{w}_0 - d[(e^{2\phi_2})_1 *_0 d\mathrm{w}_0]\,. \tag{4.4.20b}$$

The zero-order source functions, obtained from terms containing the perturbations (4.4.19b) and the order $O(b)$ background contain in principle a sum over many spherical harmonics. Since we are interested in operator O^{++} we have to project on Y^{--} harmonic (see discussion in Section 5.2.2) and we get the contributions

$$\mathcal{F} = Q_5\,[f_0\,(\mathrm{vol}_{S^3} - \mathrm{vol}_{AdS_3})\,Y^{--}(\theta,\hat{\phi}) + f_1 \wedge *_{S^3} dY^{--}(\theta,\hat{\phi}) + *_{AdS_3} f_1 \wedge dY^{--}(\theta,\hat{\phi})] \tag{4.4.21a}$$

$$f_0 = \frac{44}{3} a_0 B_0 \hat{B} \tag{4.4.21b}$$

$$f_1 = \frac{2}{3}\, a_0 \left[-4\, B_0\, d\hat{B} + \frac{2}{3}\,\hat{B}\, dB_0 + \frac{1}{3}\, dx^\mu\, \nabla_\mu \nabla^\nu \hat{B}\, \partial_\nu B_0\right] \tag{4.4.21c}$$

$$\widehat{\mathcal{F}} = Q_5^2\, \hat{f}_0\, Y^{--}(\theta,\hat{\phi})\, \mathrm{vol}_{AdS_3} \wedge \mathrm{vol}_{S^3}\,, \tag{4.4.21d}$$

$$\hat{f}_0 = \frac{2a_0}{3}\left[20\hat{B}B_0 + (\partial_\mu B_0)\Box(\partial^\mu \hat{B}) + \left(\nabla^\nu \partial^\mu \hat{B}\right)(\nabla_\nu \partial_\mu B_0) - 18(\partial_\mu \hat{B})(\partial^\mu B_0)\right] \tag{4.4.21e}$$

where operations on the AdS$_3$ indices μ, ν are performed using the unperturbed AdS$_3$ metric as usual.

The general ansatz for (w_1, h_1) includes now the scalar spherical harmonic Y^{--}

$$\mathrm{w}_1 = B_1\, Y^{--}(\theta,\hat{\phi})\,, \quad h_1 = Q_5\, d[S_1 *_{S^3} dY^{--}(\theta,\hat{\phi}) + *_{AdS_3} V_1\, Y^{--}(\theta,\hat{\phi})]\,, \tag{4.4.22}$$

where B_1, S_1 are scalars and V_1 is a 1-form on AdS$_3$. Equations for perturbations then reduce to

$$-3\,S_1 - \nabla_\mu V_1^\mu - 4\,B_1 = f_0\,, \quad dS_1 + V_1 = f_1\,, \quad \Box B_1 - 3B_1 + 2\nabla_\mu V_1^\mu - 6S_1 = -\hat{f}_0\,. \tag{4.4.23}$$

One can solve the middle equation for V_1 and substitute it in the remaining two equations. These become coupled differential equations for the two scalars B_1 and S_1, which can be decoupled by introducing the combinations

$$s = B_1 - (\ell+2)S_1\,, \quad t = B_1 + \ell S_1 \tag{4.4.24}$$

We then obtain the equations

$$\Box s + \ell s = -(\hat{f}_0 - \ell f_0 - (\ell+2)\nabla_\mu f_1^\mu) \equiv J_s \tag{4.4.25}$$

$$\Box t - (\ell+4)(\ell+2)t = -\hat{f}_0 + (\ell+2) f_0 + \ell \nabla_\mu f_1^\mu \equiv J_t \tag{4.4.26}$$

We see that s is a field dual to an operator of dimension 1, while the operator dual to t has dimension 5 and is a super-descendant of the chiral primary O_L. To obtain the correlator we are interested in, we should then set $t = 0$, which gives

$$B_1(r,t,y) = \frac{s(r,t,y)}{4} = -\frac{i}{4} \int d^3\mathbf{r}'\, \sqrt{-g_{AdS_3}}\, G_1^{\mathrm{Glob}}(\mathbf{r}'|r,t,y)\, J_s(\mathbf{r}')\,, \tag{4.4.27}$$

Keeping the explicit dependence on the dimension of the operator we get, for the field s the source

$$J_s = \frac{1}{2(\ell+1)}\left[\frac{8}{3}(\ell-3)(\partial_\mu \hat{B})(\partial^\mu B_0) - \frac{128}{9}(\ell-1)B_0\hat{B} - \frac{2}{9}(\ell-1)\left(\nabla^\nu\partial^\mu\hat{B}\right)(\nabla_\nu\partial_\mu B_0)\right] \tag{4.4.28}$$

We can redefine the field s without changing the behavior of the normalizable modes

$$s \to s + \alpha_s B_0\hat{B} + \beta_s(\partial_\mu\hat{B})(\partial^\mu B_0) \tag{4.4.29}$$

requiring the source to be free of derivatives term we fix the coefficients to be

$$\alpha_s = \frac{5\ell-17}{9(\ell+1)} \qquad \beta_s = -\frac{\ell-1}{18(\ell+1)} \tag{4.4.30}$$

after the redefinition the equation for s reduces to

$$\Box s - \ell(\ell-2)s = \lambda_s B_0\hat{B}, \quad \lambda_s = -\frac{128}{18(\ell+1)}(\ell-1) \tag{4.4.31}$$

or thinking of the field B_1 we can write

$$B_1(r,t,y) = \frac{s(r,t,y)}{4} = -\frac{i}{4}\int d^3\mathbf{r}'\,\sqrt{-g_{AdS_3}}\,G_1^{\text{Glob}}(\mathbf{r}'|r,t,y)\,J_s(\mathbf{r}')\,, \tag{4.4.32}$$

with

$$J_s(\mathbf{r}') = (\ell-1)B_0\hat{B} \tag{4.4.33}$$

We notice that setting $\ell = 1$ now we get a vanishing source giving a vanishing correlators. This zero is however compensate by a infinite factor coming from the integral when the sum of two dimensions of the operators involved is equal to the third one. Indeed, after including the factor originating from the spectral flow relation and continuing to Euclidian signature ($t \to -it_e$), one finds the order b contribution to the $\mathcal{O}(N^0)$ correlator on the Euclidean cylinder:

$$\begin{aligned}\langle O_H(t_e\!=\!-\infty)\bar{O}_H(t_e\!=\!\infty)O_L(0,0)\bar{O}_L(t_e,y)\rangle_b^{(0)} &= -\frac{b\,e^{\frac{t_e}{R}}}{8\pi}\int d^3\mathbf{r}'_e\,\sqrt{\bar{g}}\,K_1^{\text{Glob}}(\mathbf{r}'_e|t_e,y)J_s(\mathbf{r}'_e)\\ &= (\ell-1)\frac{b\,e^{\frac{t_e}{R}}}{8\pi}I_{\ell\,1\,2}\end{aligned} \tag{4.4.34}$$

with

$$I_{\Delta_1\Delta_2\Delta_3}(z_i) = \int d^2w\,\sqrt{\bar{g}}\,K_{\Delta_1}(w,\vec{z}_1)K_{\Delta_2}(w,\vec{z}_2)K_{\Delta_3}(w,\vec{z}_3) \tag{4.4.35}$$

That integral is performed in literature and it is given by

$$I(z_i) = \frac{C}{|\bar{x}_1-\bar{x}_2|^{h_1+h_2-h_3}|\bar{x}_2-\bar{x}_3|^{h_2+h_3-h_1}|\bar{x}_3-\bar{x}_1|^{h_3+h_1-h_2}} \tag{4.4.36}$$

with

$$C = \frac{\Gamma\left[\frac{1}{2}(h_1+h_2-h_3)\right]\Gamma\left[\frac{1}{2}(h_2+h_3-h_1)\right]\Gamma\left[\frac{1}{2}(h_3+h_1-h_2)\right]}{2\pi^2\Gamma\left[h_1-1\right]\Gamma\left[h_2-1\right]\Gamma\left[h_3-1\right]}\Gamma\left[\frac{1}{2}(h_1+h_2+h_3-2)\right] \tag{4.4.37}$$

Setting $h_1 = \ell$, $h_2 = 1$ and $h_3 = 2$ we get the result

$$I(z_i) = \frac{1}{\ell - 1} \frac{1}{|\bar{z}_1 - \bar{z}_2|^{\ell-1} |\bar{z}_2 - \bar{z}_3|^{3-\ell} |\bar{z}_3 - \bar{z}_1|^{1+\ell}} \tag{4.4.38}$$

and we can see that the two zeros, one in the numerator of the source and one in the denominator of the integral, cancel out, giving the result

$$\mathcal{G}_{grav}(z, \bar{z}) = |1 - z|^2 \frac{ab}{|z|} \tag{4.4.39}$$

after setting $z_1 = z$, $z_2 = 0$ $z_3 = \infty$.

We see that the bulk result for the correlator is given by an integral with 3 bulk to boundary propagator encoding a three-point function, fixed by conformal invariance. The reason why we obtain our four-point function in terms of a three-point function is that the only non-trivial contribution comes from the diagonal term

$$\langle - - | O^{++} O^{++} | + + \rangle \tag{4.4.40}$$

At the supergravity point and after going in the NS sector, when the bulk computation is performed, this translates into a three-point function with the operator $| - - \rangle$ implemented into the bulk-to-boundary operator with dimension 2, the two light operators, dual to $h = 1$ propagators, and with the $| + + \rangle$ flowed into the vacuum. The fact that the bulk correlator is given by a three point function explains also the matching of the result at the free point and at the supergravity point.

Due to the particular dimensions of the operators involved, we also expected to have an extremal correlator and indeed this fact is confirmed by the source term proportional to $(\ell - 1)$ fundamental for the cancellation of the divergent term in the integral (4.4.36).

4.5 Discussion

We focused here on the best known example of such orbifold theories, the D1D5 CFT at the free point. In section 4.1 we calculated on the CFT side a very special class of four-point correlators among BPS operators, where two states are heavy (i.e. have conformal dimension of order c), while the other two are light (i.e. their conformal dimension is of order 1). These correlators are essentially combination of the free-fermion result and, in the $(O_H O_H)(O_L O_L)$ OPE, are completed saturated by the affine identity block of a $U(1)$ subgroup of the $SU(2)$ symmetry of the theory. This suggests that they are protected by supersymmetry and motivates the supergravity analysis of section 4.3. Again thanks to the simplicity of our external states, also the gravity calculation is easy and, in this case, the basic ingredient is obtained by studying the scalar wave equation in $\mathrm{AdS}_3/\mathbb{Z}_k$. Then in order to obtain the full correlator it is important to know how the 3D result is uplifted to the full ten-dimensional geometry. In all examples under analysis, we find agreement with the free CFT result, even if this description is valid in different point of the moduli space, thus confirming the expectations based on supersymmetry as mentioned above.

Of course, in the Euclidean case, the correlators we studied are singular only in the OPE limits. One of the main features of our result is that, for the whole correlator, this holds even at the leading order in the large c limit, while, in the same limit, the contribution of the Virasoro identity block in the $(O_H O_H)(O_L O_L)$ OPE develops spurious singularities [56, 10]. In other words, the $c \to \infty$ limit of the correlators studied here is not captured by the contribution of the identity Virasoro block in the heavy-light channel. This is reflected by the gravity calculations: the two-point functions of the light operators in the near-horizon limit of the Strominger-Vafa black hole (which is the extremal BTZ) captures just the identity Virasoro block, while the same calculation in the microstate geometry dual to the heavy state reproduces the whole four-point correlators, including the contributions of the higher order Virasoro primaries. This supports the intuition that the black hole geometry describes the correlators in a statistical ensemble, while each individual microstate yields correlators that deviate from the statistical answer.

In our case, due to the simple form of the heavy states, these deviations are present even at distances larger than the Schwarzschild radius. On the CFT side, this means that, in the $(HH)(LL)$ OPE, there are contributions of non-trivial Virasoro primaries with conformal dimension of order 1. The pattern discussed above is different from the one advocated in [10], where it is suggested that quantum (i.e. $1/c$ corrections) are needed to resolve the spurious singularities of the statistical/black-hole result. Thus it is natural to ask whether the regularity of our Euclidean correlators in the large c regime is due to some peculiar feature of the D1D5 CFT under analysis and/or is a consequence of the very special operators considered. We believe that this is actually a general property as argued below.

The absence of spurious singularities at finite values of the central charge c is a direct consequence of the convergence of the OPE expansion in unitary CFT and of the basic properties of the Hilbert space structure of the spectrum [57]. In a nutshell, in the radial quantization, one can separate the four operators in the correlator by a sphere of radius r, with $|z_4| < |z_3| < r < |z_2| < |z_1|$. Then the convergence of the OPE ensures that the operators O_1 and O_2 in the external region produce a new state $|\phi_e\rangle$ on the sphere and the same happens, in the internal region, for the operators O_3 and O_4 that produce $|\phi_i\rangle$ (of course if $z_1 \to \infty$, $z_2 = 1 > z_3 > z_4 = 0$, $|\phi_i\rangle$ depends on $z = 1 - z_3$). So the four-point correlator reduces to the scalar product $\langle \phi_e|\phi_i(z)\rangle$ which is finite for any value of z in the interval $0 < |z| < 1$. In [10] it was noted that it is not straightforward to take the $c \to \infty$ limit in this argument if one identifies O_1, O_2 with the heavy operators and O_3, O_4 with the light ones. We can see this directly in the simplest one of our examples, i.e. the correlator with the operators (4.1.13) and (4.1.15). The OPE between the light operators reads

$$O_L(w)\bar{O}_L(0) = \frac{1}{|w|^2} + \frac{1}{N}\sum_r \left(\frac{J^3_{(r)}}{\bar{w}} + \frac{\tilde{J}^3_{(r)}}{w}\right) + \frac{1}{N}\sum_{r\neq s} O^L_{(r)} O^L_{(s)} + \dots \tag{4.5.1}$$

In the large c limit, normally one would discard the contribution of the terms with the currents, as their norm is of order $1/N$. However the OPE between the heavy operators produces terms, again proportional to the currents, that are non-normalizable in the $N \to \infty$ limit

$$O_H(w)\bar{O}_H(0) = \frac{1}{|w|^{2h_H}}\left(1 + w\sum_r J^3_{(r)} + \bar{w}\sum_r \tilde{J}^3_{(r)} + \dots\right). \tag{4.5.2}$$

Such non-normalizable terms can combine with the currents that appear in (4.5.1) to give non-negligible contributions to the block decomposition of the correlator; moreover their presence invalidates the regularity argument based on the existence of a well-defined scalar product, and is probably responsible for the singular behaviour of the heavy-light Virasoro blocks.

At the level of the correlators one can repeat the same derivation focusing on the OPE channel where the light operators are close to the heavy ones. In this case the intermediate states are normalizable even in the $c \to \infty$ limit and so the argument discussed above shows that the large c Euclidean correlators should not have spurious singularities. Of course this does not provide any information on the identity Virasoro block nor other $(HH)(LL)$ blocks because they do not appear in the $(HL)(HL)$ decomposition. However once the regularity of the large c limit of the correlators is established, we know that there is an infinite number of Virasoro primaries contributing to the $(HH)(LL)$ OPE. In the simple cases considered in this paper, it turns out that these primaries are protected, as they are affine descendants of the identity operator. Thus the correlator we compute at the CFT orbifold point reproduces the one extracted from the dual geometry: in these instances then correlators are regular already at the level of supergravity. In general the OPE argument in the $(HL)(HL)$ channel predicts that correlators be regular in the large c limit *at a generic point* in the CFT moduli space. We do not expect, however, that all the operators ensuring the absence of spurious singularities at large c will be captured in the supergravity approximation. It would be an important progress to identify explicitly the CFT operators that are relevant to the $(HH)(LL)$ decomposition of a more general correlator. This could help to understand from a CFT prospective what contributions survive in the large c limit beside those that reproduce the thermal behaviour.

In the next chapter we focus on the same type of calculations but with different heavy states and we will see that the results in CFT and in gravity don't match anymore, suggesting a different dynamic of the correlator away from the free point.

Chapter 5

Four-point functions: non protected case

In this chapter we continue the study of holographic four-point function of the same type of the ones in the previous chapter with two light and two heavy states, i.e.

$$\langle O_H(z_1, \bar{z}_1)\bar{O}_H(z_2, \bar{z}_2)O_L(z_3, \bar{z}_3)\bar{O}_L(z_4, \bar{z}_4)\rangle \, , \tag{5.0.1}$$

where the heavy state was composed of equal single trace constituents, and the dual geometry was factorised as the product of an orbifold of AdS_3 times S^3. Here we consider a more generic heavy state composed of two different types of strands.

In the orbifold CFT, each constituent is characterised by a winding number, which specifies the twist sector, and by the R-charge. In the heavy state of chapter 4 each constituent had winding one and maximal R-charge; here we add a new type of constituent, with the same winding number but vanishing R-charge. Since the total winding number is fixed in terms of the CFT central charge, the heavy state depends on a single parameter (B) which controls the relative number of the two types of single trace constituents; the state preserves the same (eight) supercharges for any value of this parameter and it reduces to the state of chapter 4 for $B = 0$. When $B \neq 0$, the dual geometry is a more complicated space which cannot be factorised in AdS_3 and S^3 factors: of course, the majority of the bulk microstates are of this type.

The gravity computation of the correlator of two of these heavy states and two light states involves a perturbation around the D1D5 non-factorised geometry dual to the heavy state; the analysis of this perturbation requires some non-trivial calculations. First one needs to find the linearized equations of motion for the perturbation around a background, which, apart from being non-factorized, displays non-trivial values for all type IIB fields (see appendix B). Then one has to reduce the six-dimensional linearised equations to a system of three-dimensional equations (in the asymptotically AdS_3 part of the space) describing the field dual to the light operator. This step is obviously complicated by the non-factorised form of the background. Here we simplify this task by performing a perturbative expansion that is motivated by the result of the dual correlator at the orbifold point. The orbifold result has a simple polynomial dependence on the parameter B, which, on the gravity side, controls the deviation from the factorised geometry considered in the previous chapter. This suggests to expand the gravity equations in this parameter: we will keep here only the first non-trivial order. In this way we can organise the computation using the basis of spherical harmonics on the S^3 of the factorised $B = 0$ background and obtain a set of solvable three-dimensional bulk equations.

We will not provide a detailed comparison between the free orbifold and supergravity correlators, but some simple features are immediately visible. First the correlator analysed is not protected, contrary to what happens to the $B = 0$. This fact is not surprising, and can be understood already from the analysis of the single trace operators exchanged between the two light operators. The chapter is organised similarly to the previous one, splitting the CFT side and the gravity side of the computation. In the end of the chapter we will discuss the results.

5.1 The CFT picture

We start by repeating some general facts leading to our correalator, in order to have a self-contained chapter. We focus here on the untwisted sector where we have N groups of bosons and fermions[1] labelled by an index $r = 1, \ldots, N$

$$\Big(X^{A\dot{A}}_{(r)}(z,\bar{z})\ ,\ \psi^{\alpha\dot{A}}_{(r)}(z)\ ,\ \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z})\Big)\ , \tag{5.1.1}$$

As standard in an orbiford description, we have to keep only states invariant under the orbifold group, so the operators involved in the correlators must be invariant under the S_N transformations permuting the copies of $\mathcal{M}$. In the untwisted sector this is achieved simply by symmetries over the index r; for instance we will consider the following operators in the NS-NS sector

$$O^{\alpha\dot{\beta}} = \sum_{r=1}^{N} O^{\alpha\dot{\beta}}_{(r)} = \sum_{r=1}^{N} \frac{-i}{\sqrt{2N}} \psi^{\alpha\dot{A}}_{(r)}\, \epsilon_{\dot{A}\dot{B}}\, \tilde{\psi}^{\dot{\beta}\dot{B}}_{(r)}\ , \quad J^3 = -\frac{1}{2}\sum_{r=1}^{N} \psi^{+\dot{A}}_{(r)}\, \epsilon_{\dot{A}\dot{B}}\, \psi^{-\dot{B}}_{(r)}\ . \tag{5.1.2}$$

These operators are protected also away from the orbifold point since they are part chiral-primary multiplets, i.e. the highest weight state conformal dimension is equal to the R-symmetry spin j (defined as the eigenvalue under J^3): $h = j = \bar{h} = \bar{j} = 1/2$ for O^{++} and $h = j = 1$, $\bar{h} = \bar{j} = 0$ for J^+. These operators are light since their conformal dimension $\Delta = h + \bar{h}$ remains fixed when the central charge c is scaled to infinity. On the contrary the R-R ground states are heavy since they have $h = \bar{h} = c/24$.

We are interested in four-point correlators with two light NS-NS operators and two R-R ground states

$$\langle O_H(z_1,\bar{z}_1)\bar{O}_H(z_2,\bar{z}_2)O_L(z_3,\bar{z}_3)\bar{O}_L(z_4,\bar{z}_4)\rangle = \frac{1}{z_{12}^{2h_H}\,\bar{z}_{12}^{2\bar{h}_H}}\frac{1}{z_{34}^{2h_L}\,\bar{z}_{34}^{2\bar{h}_L}}\mathcal{G}(z,\bar{z})\ , \tag{5.1.3}$$

where $z_{jk} = z_j - z_k$ and $\mathcal{G}$ is a function of the projective-invariant ratio

$$z = \frac{z_{14}z_{23}}{z_{13}z_{24}}\ , \quad \bar{z} = \frac{\bar{z}_{14}\bar{z}_{23}}{\bar{z}_{13}\bar{z}_{24}}\ . \tag{5.1.4}$$

We can characterise the R-R insertions in terms of states (by sending $z_2 \to \infty$ and $z_1 \to 0$). A simple example of a correlator of the type (5.1.3) is obtained by taking as the heavy operator O_H the R-R state that has maximum value of the spin $j = N/2$. This state is related to the $SL(2,C)$-invariant vacuum by a spectral flow transformation and, in the orbifold language, correspond to the product of the R-R ground state $|++\rangle_{(r)}$ of spin $(j = 1/2, \bar{j} = 1/2)$ in each copy of the CFT. If we choose the heavy and light operator as follows

$$O_L = O^{++}\ , \quad \lim_{z\to 0} O_H|0\rangle = \prod_{r=1}^{N} |++\rangle_{(r)} \equiv |++\rangle^N\ , \tag{5.1.5}$$

[1] We follow the conventions of [14], which are based on [23].

the correlator (5.1.3) takes the following form $\mathcal{G}(z,\bar{z}) = \frac{1}{|z|}$. We saw that this correlator was analysed both within the CFT and the dual supergravity description, and it was shown that the two results agree. This non-renormalisation property can be understood by decomposing the result in the channel where the two light operators approach each other and by showing that the correlator is saturated by considering the $U(1)$-affine descendants of $(J^3, \tilde{J}^3)$. The same result holds for a more general class of correlators, where the heavy states are different from the ones in (5.1.5) but share a key property: as in (5.1.5) they are constructed by multiplying the same building block which acts on different copies of the CFT. The building blocks considered in chapter 4 live in the k^{th} twisted sector of the orbifold CFT and can also carry a (holomorphic) momentum obtained by taking a spectral flow of level s (in the holomorphic sector): again the correlators (5.1.3) constructed with these heavy operators are protected and the orbifold CFT results match the corresponding supergravity expressions.

The heavy operators we consider here can be constructed by taking $N-p$ copies in the state $|++\rangle_{(r)}$ as in (5.1.5) and the remaining copies in a RR ground state $|00\rangle_{(r)}$ of spin $(j,\bar{j}) = (0,0)$

$$|00\rangle_{(r)} = \lim_{z,\bar{z}\to 0} O^{--}_{(r)}(z,\bar{z})\,|++\rangle_{(r)}\ . \tag{5.1.6}$$

In order to obtain a heavy state that has semiclassical dual description as a smooth geometry, one needs to take a linear combination of such states with a different number p of $|00\rangle_{(r)}$ constituents [32]

$$|s_B\rangle = \frac{1}{N^{\frac{N}{2}}}\sum_{p=0}^{N} A^{N-p}\,|++\rangle^{N-p}\,B^p\,|00\rangle^p\ , \quad \text{with} \quad |A|^2+|B|^2 = N\ , \tag{5.1.7}$$

where A and B are complex parameters. Here we follow the conventions of [33]: we understand a full symmetrization between the N copies in (5.1.7) and the norm of the ket $|++\rangle^{N-p}|00\rangle^p$ is equal to $\binom{N}{p}$, i.e. the number of distinct permutations of the constituents. When A and B are of order $\sqrt{N}$, then the sum over p is peaked, in the large N limit, around $p \sim |B|^2$ and this semiclassical state is dual to a smooth 1/4-BPS geometry. In this paper we study in detail the correlator (5.1.3) where the light operator is as in (5.1.5), while the heavy one creates the ket (5.1.8); in summary we choose

$$O_L = O^{++}\ , \quad \lim_{z\to 0} O_H|0\rangle = |s_B\rangle\ . \tag{5.1.8}$$

In order to calculate this correlator at the free orbifold point and, with the choice (5.1.8) we notice that there are two type of contributions: the "diagonal" terms where O_L and $\bar{O}_L$ are non-trivial on the same copy and the "off-diagonal" ones where the two light operators act on different copies. We start from the second type of contributions, which produces the last term of (5.1.16). In this case, the calculation has the structure of a product of two 3-point correlators and we can derive the contribution proportional to $|A|^2|B|^2$ in (5.1.16) (which we indicate by $\mathcal{G}_{off}$) by choosing $z_2\to\infty$, $z_1\to 0$, $z_3 = 1$ and $z_4 = z$

$$\frac{1}{|1-z|^2}\mathcal{G}_{off}(z,\bar{z}) = \sum_{r\neq s}\langle s_B|O^{++}_{(r)}(1,1)O^{--}_{(s)}(z,\bar{z})|s_B\rangle\ . \tag{5.1.9}$$

By using (5.1.7) and the fact that the zero-mode of $O^{++}_{(r)}$ turns the state $|00\rangle_{(r)}$ into $|++\rangle_{(r)}$ and viceversa for $O^{--}_{(r)}$ we have

$$\sum_{r\neq s} O^{++}_{(r)}(1,1) O^{--}_{(s)}(z,\bar z)|s_B\rangle = \frac{1}{|z|}\sum_{p=0}^{N}\frac{A^{N-p}B^p}{N^{\frac{N}{2}}}\Big[p(N-p)|++\rangle^{N-p}|00\rangle^p + \ldots\Big] \ , \tag{5.1.10}$$

where the dots stand for terms that contain copies of the CFT that are not in the $|++\rangle$ or $|00\rangle$ state and that we can ignore as they do not give any contribution to (5.1.9). The factors of p and $(N-p)$ in (5.1.10) follow from the action of O^{++} and O^{--} on the different $|00\rangle$ and $|++\rangle$ copies respectively. Then, since the norm of the $|++\rangle^{N-p}|00\rangle^p$ is $\binom{N}{p}$, we have

$$\begin{aligned}\mathcal{G}_{off}(z,\bar z) &= \frac{|1-z|^2}{|z|}\sum_{p=0}^{N} p(N-p)\,\frac{|A^2|^{N-p}|B^2|^p}{N^N}\binom{N}{p}\\ &= \frac{|1-z|^2}{|z|}\frac{N(N-1)|A|^2|B|^2(|A|^2+|B|^2)^{N-2}}{N^N}\ ,\end{aligned} \tag{5.1.11}$$

which, as anticipated, yields the last term of (5.1.16) after using the normalisation condition (5.1.7) $|A|^2+|B|^2=N$.

The diagonal contribution follows from the building blocks

$${}_{(r)}\langle ++|O^{++}_{(r)}(1,1)O^{--}_{(r)}(z,\bar z)|++\rangle_{(r)} = \frac{1}{|1-z|^2}\frac{1}{|z|}\ , \tag{5.1.12}$$

$${}_{(r)}\langle 00|O^{++}_{(r)}(1,1)O^{--}_{(r)}(z,\bar z)|00\rangle_{(r)} = \frac{1}{|1-z|^2}\frac{1}{2}\frac{1}{|z|}\Big(1+|z|^2+|1-z|^2\Big)\ . \tag{5.1.13}$$

These results can be derived explicitly by using the bosonisation formulae as done in the previous chapter or equivalently by using the RR mode expansion for the fermions. Alternatively, one can reconstruct the correlators from their behaviour as $z\to 0,1,\infty$. In both cases there should be a simple pole in $1-z$ and $1-\bar z$ due to the fusion of $O^{++}_{(r)}$ and $O^{--}_{(r)}$ on the identity. Then the zero-modes of $O^{--}_{(r)}$ act non-trivially both on $|00\rangle_{(r)}$ and $|++\rangle_{(r)}$, so again the two equations should have the same $z\to 0$ limit proportional to $1/|z|$. In the $z\to\infty$ limit there is a difference: the zero-modes of $O^{--}_{(r)}$ act non-trivially on ${}_{(r)}\langle 00|$ and so in this limit (5.1.13) should be proportional to $1/|z|$. On the contrary, the first modes of $O^{--}_{(r)}$ acting non-trivially on ${}_{(r)}\langle ++|$ are at level one, so (5.1.12) should go as $1/|z|^3$ as $z\to\infty$. These constraints determine uniquely the sphere correlators above. Then the diagonal contributions to (5.1.7) are obtained by counting how many times the building blocks above appear:

$$\begin{aligned}\mathcal{G}^{A}_{diag}(z,\bar z) =& \frac{1}{|z|}\sum_{p=0}^{\infty}(N-p)\,\frac{|A^2|^{N-p}|B^2|^p}{N^N}\binom{N}{p}\\ =& \frac{1}{|z|}\frac{N|A|^2(|A|^2+|B|^2)^{N-1}}{N^N}\ ,\end{aligned} \tag{5.1.14}$$

which is the contribution following from (5.1.12) and

$$\begin{aligned}\mathcal{G}^B_{diag}(z,\bar z) =& \frac{1+|z|^2+|1-z|^2}{2|z|}\sum_{p=0}^{\infty} p\,\frac{|A^2|^{N-p}|B^2|^p}{N^N}\binom{N}{p}\\ =& \frac{1+|z|^2+|1-z|^2}{2|z|}\,\frac{N|B|^2(|A|^2+|B|^2)^{N-1}}{N^N}\,,\end{aligned} \tag{5.1.15}$$

which is obtained from (5.1.13). Summing $\mathcal{G}^A_{diag}$, $\mathcal{G}^B_{diag}$ and $\mathcal{G}_{off}$, we obtain

$$\mathcal{G}(z,\bar z) = \frac{1}{|z|} + \frac{|B|^2}{2N}\,\frac{|z|^2+|1-z|^2-1}{|z|} + \frac{|A|^2|B|^2}{N}\left(1-\frac{1}{N}\right)\frac{|1-z|^2}{|z|}\,. \tag{5.1.16}$$

Notice that the last term scales, in the large N limit, as $\mathcal{O}(N)$, while the first two terms are of order $\mathcal{O}(N^0)$. In the free CFT calculation they have two different combinatoric origins: the leading term in N is due to the contributions from the terms where the light operators O_L and $\bar O_L$ are non trivial in different copies, while the remaining terms are due to the "diagonal" contribution where both O_L and $\bar O_L$ act on the same copy.

5.2 Gravity picture

In this section we describe the holographic computation of the correlator (5.1.16). We first introduce the general background geometries and the linearised equations satisfied by the perturbation describing the light operator O_L, and then specialise to the geometry dual to the state $|s_B\rangle$ in (5.1.7).

5.2.1 The background

The supergravity solution dual to the state (5.1.7) can be found firstly by specifying the profile functions $g_A(v)$, then computing the harmonic functions and then using the general form of the solution in B.4.1. The dictionary constructed in 3.3.2 then provide the precise map between CFT and gravity. The profile that gives the background dual to the states in (5.1.7) reads

$$g_a(v') + i g_2(v') = a\,e^{\frac{2\pi i v'}{L}}\,, \quad g_5(v') = b\,e^{\frac{2\pi i v'}{L}} \tag{5.2.1}$$

Therefore the complete solution is encoded in the following quantities

$$ds_4^2 = (r^2 + a^2\cos^2\theta)\left(\frac{dr^2}{r^2+a^2} + d\theta^2\right) + (r^2+a^2)\sin^2\theta\, d\phi^2 + r^2\cos^2\theta\, d\psi^2\,, \tag{5.2.2a}$$

$$\beta = \frac{R\,a^2}{\sqrt{2}\,(r^2+a^2\cos^2\theta)}\,(\sin^2\theta\, d\phi - \cos^2\theta\, d\psi)\,, \tag{5.2.2b}$$

$$Z_1 = 1 + \frac{R^2}{Q_5}\frac{a^2+\frac{b^2}{2}}{r^2+a^2\cos^2\theta} + \frac{R^2\,a^2\,b^2}{2\,Q_5}\,\frac{\cos 2\phi\,\sin^2\theta}{(r^2+a^2\cos^2\theta)(r^2+a^2)}\,, \tag{5.2.2c}$$

$$Z_2 = 1 + \frac{Q_5}{r^2+a^2\cos^2\theta}\,,\quad \gamma_2 = -Q_5\,\frac{(r^2+a^2)\,\cos^2\theta}{r^2+a^2\cos^2\theta}\,d\phi\wedge d\psi\,, \tag{5.2.2d}$$

$$Z_4 = R\,a\,b\,\frac{\cos\phi\,\sin\theta}{\sqrt{r^2+a^2}\,(r^2+a^2\cos^2\theta)}\,, \tag{5.2.2e}$$

$$\delta_2 = \frac{-R\,a\,b\,\sin\theta}{\sqrt{r^2+a^2}}\left[\frac{r^2+a^2}{r^2+a^2\cos^2\theta}\cos^2\theta\cos\phi\; d\phi\wedge d\psi + \sin\phi\,\frac{\cos\theta}{\sin\theta}\,d\theta\wedge d\psi\right], \tag{5.2.2f}$$

$$\omega = \frac{R\,a^2}{\sqrt{2}\,(r^2+a^2\cos^2\theta)}\,(\sin^2\theta\, d\phi + \cos^2\theta\, d\psi)\,. \tag{5.2.2g}$$

and can be written in the form (B.4.3a). The geometry depends on two parameters a and b, that are related to the CFT parameters A and B via (3.3.7)

$$A = R\sqrt{\frac{N}{Q_1Q_5}}\,a\,,\quad B = R\sqrt{\frac{N}{2\,Q_1Q_5}}\,b\,. \tag{5.2.3}$$

where, as usual, Q_1 and Q_5 are the supergravity D1 and D5 charges. The constraint $|A|^2 + |B|^2 = N$ translates into

$$\frac{Q_1Q_5}{R^2} = a^2 + \frac{b^2}{2}\,. \tag{5.2.4}$$

When $b = 0$ as we already said the geometry is just $\text{AdS}_3 \times S^3$:

$$ds_6^2 = \sqrt{Q_1Q_5}\,(ds^2_{AdS_3} + ds^2_{S^3})\,, \tag{5.2.5a}$$

$$ds^2_{AdS_3} = \frac{dr^2}{a_0^2+r^2} - \frac{a_0^2+r^2}{Q_1Q_5}dt^2 + \frac{r^2}{Q_1Q_5}dy^2\,,\quad a_0 \equiv \frac{\sqrt{Q_1Q_5}}{R}\,, \tag{5.2.5b}$$

$$ds^2_{S^3} = d\theta^2 + \sin^2\theta\, d\hat\phi^2 + \cos^2\theta\, d\hat\psi^2\,,\quad \hat\phi \equiv \phi - \frac{t}{R}\,,\ \hat\psi \equiv \psi - \frac{y}{R}\,, \tag{5.2.5c}$$

$$C = C_0 = -\frac{r^2}{Q_1}\,dt\wedge dy - Q_5\,\cos^2\theta\, d\hat\phi\wedge d\hat\psi\,,\quad e^{2\phi_1} = e^{-2\phi_2} = \frac{Q_1}{Q_5}\,, \tag{5.2.5d}$$

with all other fields vanishing.

Since we are interesting in the perturbative expansion around $AdS_3 \times S^3$, in particular up to $O(b^2)$ we can rewrite the solution in the following form, keeping Q_1, Q_5 and R fixed, yields

$$\frac{ds_6^2}{\sqrt{Q_1Q_5}} = V^{-2}\left[ds^2_{AdS_3} + b^2\,\delta g_{\mu\nu}\,dx^\mu dx^\nu\right]$$
$$+ \left(1 - \frac{b^2}{4\,a_0^2}\,B_+B_-\,\sin^2\theta\right)d\theta^2 + \sin^2\theta\left(1 + \frac{b^2}{4\,a_0^2}\,B_+B_-\,\sin^2\theta\right)(d\hat\phi + b^2A^\phi)^2$$
$$+ \cos^2\theta\left(1 - \frac{b^2}{4\,a_0^2}\,B_+B_-\,(\cos^2\theta+1)\right)(d\hat\psi + b^2A^\psi)^2\,, \tag{5.2.6a}$$

$$C = C_0 + Q_5\, b^2 \left[\sin^2\theta\, d\hat\phi \wedge A^\psi + \cos^2\theta\, d\hat\psi \wedge A^\phi - \frac{B_+ B_-}{2\, a_0^2}\, \sin^2\theta\, \cos^2\theta\, d\hat\phi \wedge d\hat\psi\right], \tag{5.2.6b}$$

$$e^{2\phi_1} = \frac{Q_1}{Q_5}\left[1 + \frac{b^2}{2\, a_0^2}\, \left[(B_+ Y^{++})^2 + (B_- Y^{--})^2 + B_+ B_- Y^{++} Y^{--}\right]\right], \tag{5.2.6c}$$

$$e^{2\phi_2} = \frac{Q_5}{Q_1}\left[1 + \frac{b^2}{2\, a_0^2}\, B_+ B_- Y^{++} Y^{--}\right], \tag{5.2.6d}$$

$$\chi_1 = \sqrt{\frac{Q_5}{Q_1}}\, \frac{b}{2\, a_0}\, (B_+ Y^{++} + B_- Y^{--}), \quad \chi_2 = \sqrt{\frac{Q_1}{Q_5}}\, \frac{b}{2\, a_0}\, (B_+ Y^{++} + B_- Y^{--}), \tag{5.2.6e}$$

$$\frac{B}{\sqrt{Q_1 Q_5}} = \frac{b}{2\, a_0}\, (Y^{++} *_{AdS_3} dB_+ - B_+ *_{S^3} dY^{++} + Y^{--} *_{AdS_3} dB_- - B_- *_{S^3} dY^{--}), \tag{5.2.6f}$$

where

$$V = 1 - \frac{b^2}{8\, a_0^2}\, B_+ B_-\, (\cos^2\theta + 1), \quad a_0^2 = \frac{Q_1 Q_5}{R^2} \tag{5.2.7}$$

and $B_\pm$, $Y^{\pm\pm}$, $\delta g_{\mu\nu}$, A^ϕ and A^ψ are defined as follows

$$B_\pm = \frac{a_0}{\sqrt{r^2 + a_0^2}}\, e^{\pm i\, t/R}, \quad Y^{\pm\pm} = \sin\theta\, e^{\pm i\hat\phi}. \tag{5.2.8}$$

The $\mathcal{O}(b^2)$ terms in the metric, in the RR 2-form and in the scalars ϕ_1, ϕ_2 represent the backreaction, at second order in b, of this linear perturbation. They are encoded in the AdS_3 metric fluctuation $\delta g_{\mu\nu}$ and in the vectors A^ϕ and A^ψ

$$\delta g_{\mu\nu}\, dx^\mu dx^\nu = \frac{dt^2}{Q_1 Q_5}, \quad A^\phi = \frac{R}{2\, Q_1 Q_5}\, dt, \quad A^\psi = \frac{R}{2\, Q_1 Q_5}\, \frac{r^2}{r^2 + a_0^2}\, dy, \tag{5.2.9}$$

which are linked to the first order perturbation fields $B_\pm$ by covariant differential identities:

$$\delta g^\mu_\mu = -a_0^{-2}\, B_+ B_-, \quad \nabla^\mu\left(\delta g_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, \delta g^\rho_\rho\right) = 0, \tag{5.2.10a}$$

$$(\Box_{AdS_3} + 2)\, \delta g_{\mu\nu} = -a_0^{-2}\, (\partial_\mu B_+ \partial_\nu B_- + \partial_\nu B_+ \partial_\mu B_-), \tag{5.2.10b}$$

$$dA^\phi = 0, \quad dA^\psi = -\frac{i}{2\, a_0^2}\, (B_- *_{AdS_3} dB_+ - B_+ *_{AdS_3} dB_-), \tag{5.2.10c}$$

where to raise and lower indices and to define covariant derivatives one uses the unperturbed AdS_3 metric $ds^2_{AdS_3}$.

5.2.2 Calculation of the four-point function

As we see from the CFT result (5.1.16), the correlator comprises a term of order N, which dominates the large N expansion, and a subleading term of order N^0:

$$\begin{aligned}\langle O_H(t=-\infty)\bar O_H(t=\infty) O_L(0,0)\bar O_L(t,y)\rangle \equiv{}& \frac{e^{-i\frac{t}{R}}}{N}\, |\langle O_H(t=-\infty)\bar O_H(t=\infty) O_L(0,0)\rangle|^2 \\ &+ \langle O_H(t=-\infty)\bar O_H(t=\infty) O_L(0,0)\bar O_L(t,y)\rangle^{(0)}.\end{aligned} \tag{5.2.11}$$

The leading term, which represents the disconnected part of the correlator, is proportional to the modulus square of the 3-point function [29, 33]

$$\langle O_H(t=-\infty)\bar{O}_H(t=\infty)O_L(0,0)\rangle = B\,\bar{A}\,, \tag{5.2.12}$$

a protected quantity which can be computed both in supergravity and at the free orbifold point. The order N^0 term, which is denoted with the subscript (0), is the sum of the subleading part of the disconnected correlator and of the connected correlator, and is, in general, a non-protected quantity. To compute this term on the gravity side, one needs to solve the linearized equations (B.5.13) for (w, h) in the background (5.2.6), with the following boundary condition for w at large r:

$$\mathrm{w} \approx \delta(t,y)\,\frac{\log r}{r}\,Y^{++}(\theta,\phi) + \frac{b(t,y)}{r}\,Y^{++}(\theta,\phi) + \ldots\,; \tag{5.2.13}$$

the term proportional to $\log r/r$ is the source for O_L localized at the point $t = y = 0$ on the boundary, the term proportional to $1/r$ is the normalizable term proportional to the vev of $\bar{O}_L$, and the dots represent terms proportional to spherical harmonics other than Y^{++}, which do not contribute to the correlator of interest here. We further require w and h to be regular in the interior of the space, and this determines uniquely the normalizable term. The vev of $\bar{O}_L$ in the presence of a source for O_L gives the 2-point function in the geometry sourced by O_H, and hence the correlator at order N^0 is given by

$$\langle O_H(t=-\infty)\bar{O}_H(t=\infty)O_L(0,0)\bar{O}_L(t,y)\rangle^{(0)} = b(t,y)\,, \tag{5.2.14}$$

up to a proportionality factor of which we do not keep track here. The correlator above is computed on the cylinder with coordinates t, y. We can transform from the cylinder to the Minkowski plane with the usual transformation

$$z = e^{i\frac{t+y}{R}}\,,\quad \bar{z} = e^{i\frac{t-y}{R}}\,,\quad \bar{O}_L(z,\bar{z}) = (z\bar{z})^{-1/2}\,\bar{O}_L(t,y)\,, \tag{5.2.15}$$

and we can further analytically continue to the Euclidean complex plane by sending $t \to -it_e$, with t_e the Euclidean time. Then the $\mathcal{O}(N^0)$ correlator on the plane is

$$\langle O_H(0)\bar{O}_H(\infty)O_L(1,1)\bar{O}_L(z,\bar{z})\rangle^{(0)} = \frac{1}{|1-z|^2}\,\mathcal{G}^{(0)}(z,\bar{z}) = (z\bar{z})^{-1/2}\,b(z,\bar{z})\,, \tag{5.2.16}$$

where $\bar{O}_H(\infty)$, $O_H(0)$ denote the heavy operators evaluated at $z = \infty$ and $z = 0$.

The b-expansion of the background induces a corresponding expansion for the perturbation (w, h):

$$\mathrm{w} = \mathrm{w}_0 + b^2\,\mathrm{w}_1 + \mathcal{O}(b^4)\,,\quad h = h_0 + b^2\,h_1 + \mathcal{O}(b^4)\,. \tag{5.2.17}$$

The order zero term (w_0, h_0) is the solution of (B.5.13) in the $\mathrm{AdS}_3 \times S^3$ background (5.2.5) and hence it admits the factorized form as in (4.3.15)

$$\mathrm{w}_0 = B_0(r,t,y)\,Y^{++}(\theta,\hat{\phi})\,,\quad h_0 = Q_5\,d\,[Y^{++}(\theta,\hat{\phi})*_{AdS_3}dB_0 - B_0*_{S^3}dY^{++}(\theta,\hat{\phi})]\,, \tag{5.2.18}$$

where $*_{AdS_3}$ and $*_{S^3}$ are the Hodge duals with respect to the AdS_3 and S^3 metrics (5.2.5b), (5.2.5c). The AdS_3 function $B_0(r,t,y)$ satisfies $\square_{AdS_3}B_0 + B_0 = 0$, has the boundary behaviour $B_0 \approx \delta(t,y)\log r/r$ for large r and is regular at any finite value of r: it is thus the usual bulk-to-boundary propagator for a field of dimension $\Delta = 1$ in global

AdS_3, with the boundary point set to $t' = y' = 0$:

$$B_0(r,t,y) = K_1^{\mathrm{Glob}}(r,t,y|t'=0,y'=0) = \frac{1}{2}\,\frac{a_0}{\sqrt{r^2+a_0^2}\,\cos(t/R) - r\,\cos(y/R)}\,. \tag{5.2.19}$$

To extract the order b^0 contribution to the correlator, we have to transform from the coordinate $\hat{\phi}$ to ϕ (which is equivalent to spectrally flowing from the NSNS to the RR sector of the CFT): this is easily done by using

$$Y^{++}(\theta,\hat{\phi}) = e^{-i\frac{t}{R}}\,Y^{++}(\theta,\phi)\,. \tag{5.2.20}$$

Taking the large r limit of (6.2.36), including the $e^{-i\frac{t}{R}}$ factor coming from (5.2.20), and switching to Euclidian plane coordinates, we find the $b=0$ value of the correlator

$$\mathcal{G}(z,\bar{z})|_{b=0} = \frac{1}{|z|}\,. \tag{5.2.21}$$

This result was already obtained in [14], and coincides with the $b=0$ term of the CFT correlator.

The interesting new information is contained in the $\mathcal{O}(b^2)$ terms (w_1, h_1). The perturbation equations (B.5.13) at order b^2 give:

$$h_1 - *_0\, h_1 - 2\,\mathrm{w}_1\, dC_0 = \mathcal{F}\,, \tag{5.2.22a}$$

$$\mathcal{F} \equiv \mathrm{w}_0\,(F_1 - *_0 F_1 - *_1 dC_0) + *_1 h_0 - 2\,(e^{-(\phi_1+\phi_2)})_1\,\mathrm{w}_0\, dC_0\,, \tag{5.2.22b}$$

and

$$\frac{Q_5}{Q_1}\, d *_0 d\mathrm{w}_1 - h_1 \wedge dC_0 = \widehat{\mathcal{F}} \tag{5.2.23a}$$

$$\widehat{\mathcal{F}} \equiv h_0 \wedge dC_1 - \frac{Q_5}{Q_1}\, d *_1 d\mathrm{w}_0 - d[(e^{2\phi_2})_1 *_0 d\mathrm{w}_0]\,. \tag{5.2.23b}$$

The left-hand sides of (5.2.22a) and (5.2.23a) are obtained by keeping the second order terms of (w, h) and the zeroth order background ($*_0$ is the Hodge dual with respect to the $\mathrm{AdS}_3 \times S^3$ metric (5.2.5)); viceversa the sources in (5.2.22b) and (5.2.23b) originate from the zeroth order perturbation (5.2.18) and the second order corrections to the background (5.2.6), which we denote with the subscript 1. The sources $\mathcal{F}$ and $\widehat{\mathcal{F}}$ contain scalar and vector spherical harmonics of various orders; we need to keep only terms containing the scalar harmonic Y^{++} (or its derivative), which are the only ones contributing to our correlator. A laborious computation gives:

$$\mathcal{F} = Q_5\,[f_0\,(\mathrm{vol}_{S^3} - \mathrm{vol}_{AdS_3})\,Y^{++}(\theta,\hat{\phi}) + f_1 \wedge *_{S^3} dY^{++}(\theta,\hat{\phi}) + *_{AdS_3} f_1 \wedge dY^{++}(\theta,\hat{\phi})]\,, \tag{5.2.24a}$$

$$f_0 = \frac{1}{6a_0^2} B_0 B_+ B_- - i\,\partial^\mu B_0\, A^\phi_\mu\,, \tag{5.2.24b}$$

$$f_1 = \frac{1}{9a_0^2} B_0 B_- dB_+ - \frac{2}{9a_0^2} B_0 B_+ dB_- - \frac{1}{18a_0^2} B_+ B_- dB_0 - i\,B_0\, A^\phi + \frac{2\,i}{3}\, *_{AdS_3} (dB_0 \wedge A^\psi) - \delta g_{\mu\nu} \partial^\mu B_0 dx^\nu\,, \tag{5.2.24c}$$

$$\widehat{\mathcal{F}} = Q_5^2\, \hat{f}_0\, Y^{++}(\theta,\hat{\phi})\, \mathrm{vol}_{AdS_3} \wedge \mathrm{vol}_{S^3}\,, \tag{5.2.24d}$$

$$\hat{f}_0 = -\frac{1}{3a_0^2} B_0 B_+ B_- + \frac{1}{a_0^2} B_-\, \partial_\mu B_+\, \partial^\mu B_0 - \nabla_\mu \partial_\nu B_0\, \delta g^{\mu\nu} - 4i\, \partial^\mu B_0\, A^\phi_\mu\,, \tag{5.2.24e}$$

where operations on the AdS$_3$ indices μ, ν are performed using the unperturbed AdS$_3$ metric (5.2.5b) and

$$\mathrm{vol}_{AdS^3} = \frac{r}{Q_1 Q_5}\, dr \wedge dt \wedge dy\,, \quad \mathrm{vol}_{S^3} = \sin\theta\, \cos\theta\, d\theta \wedge d\hat{\phi} \wedge d\hat{\psi}\,. \tag{5.2.25}$$

The general ansatz for (w_1, h_1) includes now the scalar spherical harmonic Y^{++}

$$\mathrm{w}_1 = B_1\, Y^{--}(\theta, \hat{\phi})\,, \quad h_1 = Q_5\, d[S_1 *_{S^3} dY^{--}(\theta, \hat{\phi}) + *_{AdS_3} V_1\, Y^{--}(\theta, \hat{\phi})]\,, \tag{5.2.26}$$

where B_1, S_1 are scalars and V_1 is a 1-form on AdS$_3$. Equations for perturbations then reduce to

$$-3\, S_1 - \nabla_\mu V_1^\mu - 4\, B_1 = f_0\,, \quad dS_1 + V_1 = f_1\,, \quad \Box B_1 - 3B_1 + 2\nabla_\mu V_1^\mu - 6 S_1 = -\hat{f}_0\,. \tag{5.2.27}$$

One can solve the middle equation for V_1 and substitute it in the remaining two equations. These become coupled differential equations for the two scalars B_1 and S_1, which can be decoupled by introducing the combinations

$$s = B_1 - (\ell + 2) S_1\,, \quad t = B_1 + \ell S_1 \tag{5.2.28}$$

We then obtain the equations

$$\Box s + \ell s = -(\hat{f}_0 - \ell f_0 - (\ell+2)\nabla_\mu f_1^\mu) \equiv J_s \tag{5.2.29}$$

$$\Box t - (\ell+4)(\ell+2) t = -\hat{f}_0 + (\ell+2) f_0 + \ell \nabla_\mu f_1^\mu \equiv J_t \tag{5.2.30}$$

To obtain the correlator we are interested in, we should then set $t = 0$, which gives

$$B_1(r,t,y) = \frac{s(r,t,y)}{4} = -\frac{i}{4} \int d^3\mathbf{r}'\, \sqrt{-g_{AdS_3}}\, G_1^{\mathrm{Glob}}(\mathbf{r}'|r,t,y)\, J_s(\mathbf{r}')\,, \tag{5.2.31}$$

where $\mathbf{r}' \equiv \{r', t', y'\}$ is a point in AdS$_3$ and $G_1^{\mathrm{Glob}}(\mathbf{r}'|r,t,y)$ is the bulk-to-bulk propagator for a scalar field of mass $m^2 = -1$ in global AdS$_3$, normalized such that $(\Box + 1) G_1^{\mathrm{Glob}} = i/\sqrt{-g_{AdS_3}}\, \delta$. In the $r \to \infty$ limit G_1^{Glob} is related with the bulk-to-boundary propagator $K_1^{\mathrm{Glob}}(\mathbf{r}'|t,y)$ normalized as (see for example eq. (6.12) in [58])

$$G_1^{\mathrm{Glob}}(\mathbf{r}'|r,t,y) \to \frac{a_0}{2\pi\, r}\, K_1^{\mathrm{Glob}}(\mathbf{r}'|t,y)\,. \tag{5.2.32}$$

After including the factor originating from the spectral flow relation (5.2.20) and continuing to Euclidian signature ($t \to -it_e$), one finds the order b^2 contribution to the $\mathcal{O}(N^0)$ correlator on the Euclidean cylinder:

$$\langle O_H(t_e\!=\!-\infty) \bar{O}_H(t_e\!=\!\infty) O_L(0,0) \bar{O}_L(t_e,y) \rangle_{b^2}^{(0)} = -\frac{b^2 e^{-\frac{t_e}{R}}}{8\pi} \int d^3\mathbf{r}'_e\, \sqrt{\bar{g}}\, K_1^{\mathrm{Glob}}(\mathbf{r}'_e|t_e,y) J_s(\mathbf{r}'_e), \tag{5.2.33}$$

with $\mathbf{r}'_e \equiv \{r', t'_e, y'\}$ and $\bar{g}$ the metric of Euclidean AdS$_3$. The source $J_s(\mathbf{r}'_e)$ follows from (5.2.29), (5.2.24) and (5.2.10a):

$$\begin{aligned} J_s(\mathbf{r}'_e) =& -\frac{1}{3a_0^2}\, B_0 B_+ B_- + \frac{1}{3a_0^2}\, B_0\, \partial'_\mu B_+\, \partial'^\mu B_- - \frac{11}{3a_0^2}\, B_-\, \partial'_\mu B_+\, \partial'^\mu B_0 \\ &+ \frac{1}{12}\, B_+\, \partial'_\mu B_-\, \partial'^\mu B_0 + 4\delta g^{\mu\nu}\, \nabla'_\mu \partial'_\nu B_0 + 8i\, \partial'^\mu B_0\, A^\phi_\mu \\ =& -\frac{1}{3a_0^2}\, B_0 B_+ B_- + \frac{1}{3a_0^2}\, B_0\, \partial'_\mu B_+\, \partial'^\mu B_- - \frac{5}{3a_0^2}\, B_-\, \partial'_\mu B_+\, \partial'^\mu B_0 \\ &+ \frac{7}{3a_0^2}\, B_+\, \partial'_\mu B_-\, \partial'^\mu B_0 - 4\frac{a_0^2\, R^2}{(r'^2+a_0^2)^2}\, \partial^2_{t'_e} B_0 + 4\frac{R}{r'^2+a_0^2}\, \partial_{t'_e} B_0\,, \end{aligned} \tag{5.2.34}$$

with ∂'_μ the derivative with respect to $\mathbf{r}'_e$. With manipulations similar to the ones used to simplify Witten's diagrams [59, 60, 61], we can rewrite the correlator (5.2.33) in terms of the integrals defined in appendix C

$$\begin{aligned} \hat{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} &\equiv \lim_{z_2\to\infty} |z_2|^{2\Delta_2}\, D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z_1=0, z_2=\infty, z_3=1, z_4=z) \\ &= \lim_{z_2\to\infty} |z_2|^{2\Delta_2} \int d^3\mathbf{w}\, \sqrt{\bar{g}}\, K_{\Delta_1}(\mathbf{w}|0)\, K_{\Delta_2}(\mathbf{w}|z_2,\bar{z}_2)\, K_{\Delta_3}(\mathbf{w}|1)\, K_{\Delta_4}(\mathbf{w}|z,\bar{z})\,. \end{aligned} \tag{5.2.35}$$

Reduction to D-integrals

We can rewrite the r.h.s. of eq. (5.2.33), with the source given in (5.2.34), as

$$-\frac{b^2 e^{-\frac{t_e}{R}}}{8\pi}\left[-\frac{1}{3}I_1 + \frac{1}{3}I_2 - \frac{5}{3}I_3 + \frac{7}{3}I_4 - 4I_5 + 4I_6\right]. \tag{5.2.36}$$

The integrals I_i are defined as

$$I_1 = \int d^3\mathbf{r}'_e\, \sqrt{\bar{g}}\, B_0(\mathbf{r}'_e|t_e,y)\, B_0(\mathbf{r}'_e|0,0)\, B_+(\mathbf{r}'_e)\, B_-(\mathbf{r}'_e)\,, \tag{5.2.37a}$$

$$I_2 = \int d^3\mathbf{r}'_e\, \sqrt{\bar{g}}\, B_0(\mathbf{r}'_e|t_e,y)\, B_0(\mathbf{r}'_e|0,0)\, \partial'_\mu B_+(\mathbf{r}'_e)\, \partial'^\mu B_-(\mathbf{r}'_e)\,, \tag{5.2.37b}$$

$$I_3 = \int d^3\mathbf{r}'_e\, \sqrt{\bar{g}}\, B_0(\mathbf{r}'_e|t_e,y)\, \partial'^\mu B_0(\mathbf{r}'_e|0,0)\, B_-(\mathbf{r}'_e)\, \partial'_\mu B_+(\mathbf{r}'_e)\,, \tag{5.2.37c}$$

$$I_4 = \int d^3\mathbf{r}'_e\, \sqrt{\bar{g}}\, B_0(\mathbf{r}'_e|t_e,y)\, \partial'^\mu B_0(\mathbf{r}'_e|0,0)\, B_+(\mathbf{r}'_e)\, \partial'_\mu B_-(\mathbf{r}'_e)\,, \tag{5.2.37d}$$

$$I_5 = \int d^3\mathbf{r}'_e\, \sqrt{\bar{g}}\, B_0(\mathbf{r}'_e|t_e,y)\, R^2 \partial^2_{t'_e} B_0(\mathbf{r}'_e|0,0)\, \frac{a_0^4}{(r'^2+a_0^2)^2}\,, \tag{5.2.37e}$$

$$I_6 = \int d^3\mathbf{r}'_e\, \sqrt{\bar{g}}\, B_0(\mathbf{r}'_e|t_e,y)\, R\, \partial_{t'_e} B_0(\mathbf{r}'_e|0,0)\, \frac{a_0^2}{r'^2+a_0^2}\,, \tag{5.2.37f}$$

$$I_7 = \int d^3\mathbf{r}'_e\, \sqrt{\bar{g}}\, B_0(\mathbf{r}'_e|t_e,y)\, i\, R\, \partial_{y'} B_0(\mathbf{r}'_e|0,0)\, \frac{a_0^2}{r'^2+a_0^2}\,, \tag{5.2.37g}$$

where in the last line we have also introduced the integral I_7 for later convenience. We defined

$$B_0(\mathbf{r}'_e|t_e,y) \equiv K_1^{\text{Glob}}(\mathbf{r}'_e|t_e,y) = \frac{1}{2}\, \frac{a_0}{\sqrt{r'^2+a_0^2}\, \cosh((t'_e-t_e)/R) - r'\, \cos((y'-y)/R)}\,; \tag{5.2.38}$$

with this notation we have $B_0(\mathbf{r}'_e) = B_0(\mathbf{r}'_e|0,0)$. Our goal is to rewrite the integrals I_i in terms of the D-functions evaluated at the boundary points $0, \infty, 1, z$ as defined in (5.2.35). To simplify these integrals it is useful to switch to Poincaré coordinates $\mathbf{w} \equiv \{w_0, w, \bar{w}\}$ via the coordinate transformation

$$w_0 = \frac{a_0}{\sqrt{r'^2 + a_0^2}} e^{\frac{t'_e}{R}}\,, \quad w = \frac{r'}{\sqrt{r'^2 + a_0^2}} e^{\frac{t'_e + iy'}{R}}\,, \quad \bar{w} = \frac{r'}{\sqrt{r'^2 + a_0^2}} e^{\frac{t'_e - iy'}{R}}\,. \tag{5.2.39}$$

The bulk-to-boundary propagator of global AdS$_3$ $K_1^{\text{Glob}}(\mathbf{r}'_e|t_e, y)$ is related to the bulk-to-boundary propagator in Poincaré coordinates $K_1(\mathbf{w}|z, \bar{z})$ as

$$K_\Delta^{\text{Glob}}(\mathbf{r}'_e|t_e, y) = |z|^\Delta\, K_\Delta(\mathbf{w}|z, \bar{z}) \quad \text{with} \quad K_\Delta(\mathbf{w}|z, \bar{z}) = \frac{w_0^\Delta}{(w_0^2 + |w - z|^2)^\Delta}\,; \tag{5.2.40}$$

the factor $|z|^\Delta$ in this relation is precisely the factor that appears in the transformation from the cylinder to the plane in (5.2.16). The propagator B_0 in (5.2.33) can be rewritten in Poincaré coordinates by using (5.2.40) with $\Delta = 1$. Similarly, for B_+ and B_-, we use the identification for the boundary points $z = \infty$, $z = 0$; for a general Δ the relations are

$$B_+^\Delta(\mathbf{r}'_e) = \lim_{z \to \infty} |z|^{2\Delta} K_\Delta(\mathbf{w}|z, \bar{z}) = \left(\frac{a_0 e^{\frac{t'_e}{R}}}{\sqrt{r'^2 + a_0^2}} \right)^\Delta\,, \tag{5.2.41a}$$

$$B_-^\Delta(\mathbf{r}'_e) = K_\Delta(\mathbf{w}|0) = \left(\frac{a_0 e^{-\frac{t'_e}{R}}}{\sqrt{r'^2 + a_0^2}} \right)^\Delta\,. \tag{5.2.41b}$$

With the above relations one immediately finds

$$|z|^{-1} I_1 = \hat{D}_{1111}\,. \tag{5.2.42}$$

The relation

$$\partial'_\mu B_+(\mathbf{r}'_e)\, \partial'^\mu B_-(\mathbf{r}'_e) = \frac{a_0^2}{r'^2 + a_0^2} - \frac{2\, a_0^4}{(r'^2 + a_0^2)^2} = K_1(\mathbf{w}|\infty) K_1(\mathbf{w}|0) - 2\, K_2(\mathbf{w}|\infty) K_2(\mathbf{w}|0) \tag{5.2.43}$$

yields

$$|z|^{-1} I_2 = \hat{D}_{1111} - 2\hat{D}_{2211}\,. \tag{5.2.44}$$

I_3 can be computed by explicitly writing the integral in Poincaré coordinates

$$\begin{aligned} |z|^{-1} I_3 &= \int d^3\mathbf{w}\, w_0^{-1} \frac{w_0}{w_0^2 + |w - z|^2}\, \partial_{w_0} \left(\frac{w_0}{w_0^2 + |w - 1|^2} \right) \frac{w_0}{w_0^2 + |z|^2} \\ &= \int d^3\mathbf{w} \frac{w_0}{w_0^2 + |w - z|^2} \left(\frac{1}{w_0^2 + |w - 1|^2} - \frac{2\, w_0^2}{(w_0^2 + |w - 1|^2)^2} \right) \frac{1}{w_0^2 + |z|^2} \\ &= \hat{D}_{1111} - 2\, \hat{D}_{1221}\,. \end{aligned} \tag{5.2.45}$$

Moreover the fact that $B_0(\mathbf{r}'_e|t_e, y)$ is an even function of $t'_e - t_e$ and $y' - y$ implies that

$$\begin{aligned} &\int d^3\mathbf{r}'_e\, \sqrt{\bar{g}}\, B_0(\mathbf{r}'_e|t_e, y)\, \partial'^\mu B_0(\mathbf{r}'_e|0,0)\, B_-(\mathbf{r}'_e)\, \partial'_\mu B_+(\mathbf{r}'_e) = \\ &\quad = \int d^3\mathbf{r}'_e\, \sqrt{\bar{g}}\, B_0(\mathbf{r}'_e|0,0)\, \partial'^\mu B_0(\mathbf{r}'_e|t_e, y)\, B_+(\mathbf{r}'_e)\, \partial'_\mu B_-(\mathbf{r}'_e)\,, \end{aligned} \tag{5.2.46}$$

since the change of integration variables $(t'_e, y') \to (-t'_e, -y')$ followed by $(t'_e, y') \to (t'_e - t_e, y' - y)$ exchanges $B_0(\mathbf{r}'_e|t_e, y)$ with $B_0(\mathbf{r}'_e|0,0)$ and $B_+(\mathbf{r}'_e)$ with $B_-(\mathbf{r}'_e)$. Then

$$\frac{I_3 + I_4}{|z|} = \frac{1}{2}\int d^3\mathbf{w}\,\sqrt{\bar{g}}\,\partial^\mu[K_1(w|z)K_1(w|1)][K_1(w|0)\partial_\mu K_1(w|\infty) + K_1(w|\infty)\partial_\mu K_1(w|0)]$$
$$= -\frac{I_2}{|z|} + \hat{D}_{1111} = 2\,\hat{D}_{2211}\,, \tag{5.2.47}$$

where in the last line we have integrated by parts and used $\Box K_1 + K_1 = 0$. From (5.2.45) and (5.2.47) we then deduce I_4

$$|z|^{-1} I_4 = -\hat{D}_{1111} + 2\,\hat{D}_{2211} + 2\,\hat{D}_{1221}\,. \tag{5.2.48}$$

To compute I_5 one notes that

$$\frac{a_0^4\,R}{(r'^2 + a_0^2)^2}\,\partial_{t'_e} B_0(\mathbf{r}'_e|0,0) = \frac{1}{2}[B_-(\mathbf{r}'_e)\,\partial'_\mu B_+(\mathbf{r}'_e) - B_+(\mathbf{r}'_e)\,\partial'_\mu B_-(\mathbf{r}'_e)]\partial'^\mu B_0(\mathbf{r}'_e|0,0)\,, \tag{5.2.49}$$

which implies

$$\int d^3\mathbf{r}'_e\,\sqrt{\bar{g}}\,B_0(\mathbf{r}'_e|t_e,y)\,R\,\partial_{t'_e} B_0(\mathbf{r}'_e|0,0)\,\frac{a_0^4}{(r'^2 + a_0^2)^2} = \frac{1}{2}(I_3 - I_4) =$$
$$= |z|(\hat{D}_{1111} - 2\,\hat{D}_{1221} - \hat{D}_{2211})\,. \tag{5.2.50}$$

Then

$$\begin{aligned} I_5 &= -\int d^3\mathbf{r}'_e\,\sqrt{\bar{g}}\,R\,\partial_{t'_e} B_0(\mathbf{r}'_e|t_e,y)\,R\,\partial_{t'_e} B_0(\mathbf{r}'_e|0,0)\,\frac{a_0^4}{(r'^2 + a_0^2)^2} \\ &= R\,\partial_{t_e}\int d^3\mathbf{r}'_e\,\sqrt{\bar{g}}\,B_0(\mathbf{r}'_e|t_e,y)\,R\,\partial_{t'_e} B_0(\mathbf{r}'_e|0,0)\,\frac{a_0^4}{(r'^2 + a_0^2)^2} \\ &= (z\partial_z + \bar{z}\partial_{\bar{z}})\,(|z|(\hat{D}_{1111} - 2\,\hat{D}_{1221} - \hat{D}_{2211})) \\ &= |z|\,\left(2\,\hat{D}_{1122} + \hat{D}_{2121} - \hat{D}_{1221} - \frac{\pi}{|1-z|^2}(1 - \log|z|)\right)\,, \end{aligned} \tag{5.2.51}$$

where we have first integrated by parts, exploited the fact that $B_0(\mathbf{r}'_e|t_e, y)$ is a function of $t'_e - t_e$ and then used (5.2.50); the last line follows by substituting the explicit expressions for the functions $\hat{D}$ given in appendix C. Finally I_6 and I_7 follow from similar manipulations

$$\begin{aligned} I_6 &= -\int d^3\mathbf{r}'_e\,\sqrt{\bar{g}}\,R\,\partial_{t'_e} B_0(\mathbf{r}'_e|t_e,y)\,B_0(\mathbf{r}'_e|0,0)\,\frac{a_0^2}{r'^2 + a_0^2} \\ &= R\,\partial_{t_e}\int d^3\mathbf{r}'_e\,\sqrt{\bar{g}}\,B_0(\mathbf{r}'_e|t_e,y)\,B_0(\mathbf{r}'_e|0,0)\,\frac{a_0^2}{r'^2 + a_0^2} \\ &= (z\partial_z + \bar{z}\partial_{\bar{z}})\,(|z|\,\hat{D}_{1111}) = -\frac{\pi\,|z|}{|1-z|^2}\,\log|z|\,, \end{aligned} \tag{5.2.52}$$

$$
\begin{aligned}
I_7 &= -\int d^3\mathbf{r}'_e\,\sqrt{\bar g}\,i\,R\,\partial_{y'}B_0(\mathbf{r}'_e|t_e,y)\,B_0(\mathbf{r}'_e|0,0)\,\frac{a_0^2}{r'^2+a_0^2}\\
&= i\,R\,\partial_y\int d^3\mathbf{r}'_e\,\sqrt{\bar g}\,B_0(\mathbf{r}'_e|t_e,y)\,B_0(\mathbf{r}'_e|0,0)\,\frac{a_0^2}{r'^2+a_0^2}\\
&= -(z\partial_z-\bar z\partial_{\bar z})\,(|z|\,\hat D_{1111}) = -|z|\,\frac{z-\bar z}{|1-z|^2}\,\hat D_{2211}\,,
\end{aligned}
\tag{5.2.53}
$$

where the last steps we have used (C.2.2).

Substituting the simplified expressions for the integrals I_i in (5.2.36), using the identities (C.2.12) and performing the transformation from the cylinder to the plane correlator yields the result

$$
\mathcal{G}^{(0)}(z,\bar z)|_{b^2} = \frac{b^2}{a_0^2}\,\frac{1}{|z|}\left[\frac{|z|^2}{\pi}\,\hat D_{2211}-\frac{1}{2}\right]. \tag{5.2.54}
$$

Including the $b=0$ and the $\mathcal{O}(N)$ terms, one finds the total correlator up to order b^2:

$$
\mathcal{G}_{\rm grav}(z,\bar z) = \frac{1}{|z|}\left[1+\frac{b^2}{a_0^2}\left(\frac{|z|^2}{\pi}\,\hat D_{2211}-\frac{1}{2}+\frac{N}{2}\,|1-z|^2\right)\right]. \tag{5.2.55}
$$

Clearly the gravity result differs from the free correlator (5.1.16), and so this correlator is not protected for $b\neq 0$.

By using the same supergravity solutions it is possible to derive the correlator with a different choice of the light states while keeping the heavy states unchanged.

It is straightforward to repeat our computations to derive a different four-point correlator where the light operator O^{++} is replaced by O^{+-}, while the heavy operators are left unchanged. The orbifold point result reads

$$
\mathcal{G}(z,\bar z) = \sqrt{\frac{\bar z}{z}}+\frac{|B|^2}{2N}\sqrt{\frac{\bar z}{z}}\left[z+\frac{1}{\bar z}-2+\frac{|1-z|^2}{\bar z}\right]. \tag{5.2.56}
$$

Note that the term of order N is now absent.

The gravity computation follows the lines of section 5.2.2, where now one looks for a perturbation of the form (5.2.13) with the spherical harmonic Y^{++} replaced by Y^{+-}. The r.h.s. of eq. (5.2.33) becomes

$$
-\frac{b^2e^{-i\frac{y}{R}}}{8\pi}\left[\frac{1}{3}I_1-\frac{1}{3}I_2+\frac{5}{3}I_3+\frac{5}{3}I_4-4I_5+4I_7\right] = b^2\frac{|z|}{|1-z|^2}\sqrt{\frac{\bar z}{z}}\left[\frac{z}{\pi}\,\hat D_{2211}-\frac{1}{2}\right], \tag{5.2.57}
$$

having used the results of appendix C. The gravity correlator with $O_L=O^{+-}$ up to order b^2 is then

$$
\mathcal{G}_{\rm grav}(z,\bar z) = \sqrt{\frac{\bar z}{z}}\left[1+\frac{b^2}{a_0^2}\left(\frac{z}{\pi}\,\hat D_{2211}-\frac{1}{2}\right)\right]. \tag{5.2.58}
$$

5.3 Discussion

The correlator studied in this chapter has been computed, as in the previous case, at the free orbifold point and at the supergravity point. On the CFT side, it is straightforward to calculate the correlator at a special point of the moduli space where the CFT reduces to a free orbifold and the result is given by (5.1.16). On the bulk side, the calculation is more challenging and the result (5.2.55) is obtained in the limit $b^2\ll a^2$. Notice that the approximation on b is performed after taking the large N limit, so we are taking a double scaling limit where N is large and b is small but constant; in other words the parameter B^2 (5.2.3) on the CFT side is always of order N.

On the bulk side, the gravity result has a more intricate structure and should display several features such as Landau singularities, anomalous dimensions and couplings of double trace operators that are absent in the free orbifold result.

The exchange of the operators O_L and $\bar{O}_L$ should act on the correlator as $\mathcal{G}(z,\bar{z}) \to \mathcal{G}(z^{-1},\bar{z}^{-1})$, as can be seen from the definitions (5.1.3) and (5.1.4). On the gravity side, the correlator with exchanged O_L and $\bar{O}_L$ is computed by replacing $Y^{++} \to Y^{--}$ in (5.2.13), which is equivalent to sending $\phi \to -\phi$. As the background is invariant if one reverses at the same time the signs of ϕ and t, this exchange produces a result for the correlator of the form of eq. (5.2.36) with $t \to -t$, I_3 and I_4 interchanged and $I_6 \to -I_6$. Using the symmetry (C.2.11) of the function $\hat{D}_{2211}$, one can verify that the final result indeed equals $\mathcal{G}_{\rm grav}(z^{-1},\bar{z}^{-1})$.

Of course an important consistency check is provided by the study of the OPE degeneration, as done for the AdS_5 case [62, 63]. As usual, the expansion of the gravity correlator in the various channels contains logarithmic terms, which capture the anomalous dimensions of the exchanged multiparticle states. In the $z \to 1$ limit the two light operators are close and the leading multiparticle state is $:\bar{O}_L O_L:$. Since the single particle constituents are a chiral and an anti-chiral operator, this multiparticle state is expected to gain an anomalous dimension at subleading order in $1/N$, encoded in the coefficient of the term $|1-z|^2 \log|1-z|^2$ in the expansion of $\mathcal{G}$. By looking at the sign of the relevant logarithm, it is clear that the anomalous dimension of the state $:\bar{O}_L O_L:$ will be positive. This is a somewhat surprising result since, to the best of our knowledge, all the previous holographic computations, mostly performed in 4D CFT's, have produced negative anomalous dimensions [64, 65]. It is interesting to study the logarithms also in the $z \to 0$ limit: in this channel the multiparticle operators exchanged between $\bar{O}_L$ and O_H are composed by constituents of the same chirality, unless they contain both holomorphic and anti-holomorphic derivatives. This implies the absence of terms of the form $z^n \log|z|^2$ and $\bar{z}^n \log|z|^2$, for any integer $n \geq 0$: one can verify that this expectation is fulfilled by our result.

It is possible to extract some interesting information also from the non-logarithmic terms of the OPE expansions. For instance one can check that the contributions due to the exchange of the lowest order protected operators in the various OPE channels are equal at the orbifold and the gravity point: in the $z \to 1$ channel one can match the contributions from the identity and the R-currents J^3 and $\bar{J}^3$, and in the $z \to 0$ channel the contribution of the lowest order anti-chiral multiparticle operator exchanged between $\bar{O}_L$ and O_H. Notice also that in the $z \to 0$ and $z \to \infty$ limits the coefficient of the OPE expansion are positive as unitarity requires, since in those cases the expansion involves the modulus square of the three-point correlators. Beyond the leading order the analysis is more complicated. In the OPE channel where the two light operators are close, the gravity result contains holomorphic corrections that can not be explained by the exchange of the affine descendants of the identity (and of course the same holds in the antiholomorphic sector). This suggests that some primary operators, that at the orbifold point can be constructed by using affine descendants on different CFT copies, have a fixed conformal dimension, with $\bar{h} = 0$, at large N. It would be of course interesting to verify or disprove this conjecture, since it is relevant to determine the singularity structure of the large c correlators and can shed some light on the problem of information loss in the dual gravitational description, see [10, 14, 66].

Chapter 6

Minimal coupled scalar and Ward identities

In the two previous chapters we presented a class of four-point functions that we divided into two cases, related to the dynamic of the correlators when computed into different point of moduli space: the protected case and non protected case. In particular, we used the same light operators, constructed with fermions as fundamental fields of the CFT, and varying the heavy states, we saw different behaviors for the two cases, explained in terms of operators exchanged in the OPE analysis.

We present here a class of correlators slightly different from the ones studied in the two previous chapters but having a their own interest. We consider here the change of the light operators as well the heavy operators. For the heavy states we remind that this is accounted, in the gravity picture, by a different background solution while changing the light operators implies the study of a fluctuation dual to a different fields from the ones dual to the light operators used so far. Since the light operators used in this chapter are constructed with bosonic fundamental fields that are superdescendants of fermions we seek for a relation between the correlators with those two different kind of light operators. This relation turns out to be a Ward Identity (WI) for the supersymmetric current algebra and it will be a powerful tools to test all the result obtained .

In section 6.1 we introduce the supersymmetric WI and the action on the four-point functions considered. In section 6.2 we study a four-point function involving bosonic light operators whose dual field in gravity is described by a minimal coupled scalar coming form reduction of ten-dimensional two-form with legs on T^4. The heavy operators involve a generalization of the ones used in chapter 5 and in the limit in which they reduce to those states some exact result can be extracted. This correlator provides a very non-trivial check for the previous results. Indeed the two four-point functions are computed independently solving two different equations, on the gravity side. However, as already said, on the CFT side they are related to supersymmetric Ward Identities valid for every value of the coupling and regime we are focusing on, and the consistency of these identities for our bulk results is not obvious at all.

6.1 Supersymmetric Ward identities

It is possible to map the correlators studied in chapters 4 and 5 with light operators constructed with fermions, to correlators with light operators written in terms of the bosons. To do this we need to know the action of the supercurrents, which maps bosons into fermions and viceversa. This will allow us to write a supersymmetric Ward identity between four-point functions with different light operators.

Let's consider the left and right supercurrents on single strand, part of the current algebra

$$G^{\alpha}_{A}(z) = \left(\partial X_{A\dot{A}}\psi^{\alpha\dot{A}}\right)(z), \qquad \tilde{G}^{\dot{\alpha}}_{A}(z) = \left(\bar{\partial} X_{A\dot{A}}\tilde{\psi}^{\dot{\alpha}\dot{A}}\right)(\bar{z}), \tag{6.1.1}$$

where the indices of the bosons have been lowered[1] using ϵ_{AB} and $\epsilon_{\dot{A}\dot{B}}$,

$$\partial X_{A\dot{A}} = \epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\,\partial X^{B\dot{B}}, \qquad \bar{\partial} X_{A\dot{A}} = \epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\,\bar{\partial} X^{B\dot{B}}. \tag{6.1.2}$$

We follow the convention $\epsilon_{12} = -\epsilon_{21} = \epsilon^{21} = -\epsilon^{12} = +1$, and the same for the dotted indices. The action of the supercurrent $G^{\alpha}_{A}(w)$ on a fermion $\psi^{\beta\dot{A}}(z)$ is a contour integral in which w is taken on a counter-clockwise path around z,

$$\begin{aligned}
\oint_z \frac{dw}{2\pi\mathrm{i}}\, G^{\alpha}_{A}(w)\psi^{\beta\dot{A}}(z) &= \oint_z \frac{dw}{2\pi\mathrm{i}}\, \partial X_{A\dot{B}}(w)\psi^{\alpha\dot{B}}(w)\psi^{\beta\dot{A}}(z) \\
&= \oint_z \frac{dw}{2\pi\mathrm{i}}\, \partial X_{A\dot{B}}(w)\left(-\frac{\epsilon^{\alpha\beta}\,\epsilon^{\dot{B}\dot{A}}}{w-z} + [\text{reg.}]\right) \\
&= -\epsilon^{\alpha\beta}\,\partial X_{A\dot{B}}(z)\,\epsilon^{\dot{B}\dot{A}},
\end{aligned} \tag{6.1.3}$$

where we used the OPE between fermions and bosons and the result follows from the fact that $\partial X_{A\dot{B}}(w)$ has no singularities at $w = z$, so the only singular term is the one brought by the OPE of the fermions. An analogous fact holds for the antiholomorphic supercurrent and fermions,

$$\oint_{\bar{z}} \frac{d\bar{w}}{2\pi\mathrm{i}}\, \tilde{G}^{\dot{\alpha}}_{A}(\bar{w})\,\tilde{\psi}^{\dot{\beta}\dot{A}}(\bar{z}) = -\epsilon^{\dot{\alpha}\dot{\beta}}\,\bar{\partial} X_{A\dot{B}}(\bar{z})\,\epsilon^{\dot{B}\dot{A}}. \tag{6.1.4}$$

The action of the supercurrents on the bosons is defined in the same way,

$$\begin{aligned}
\oint_z \frac{dw}{2\pi\mathrm{i}}\, G^{\alpha}_{A}(w)\,\partial X^{B\dot{B}}(z) &= \oint_z \frac{dw}{2\pi\mathrm{i}}\, \epsilon_{AC}\epsilon_{\dot{A}\dot{C}}\,\psi^{\alpha\dot{A}}(w)\,\partial X^{C\dot{C}}(w)\,\partial X^{B\dot{B}}(w) \\
&= \epsilon_{AC}\epsilon_{\dot{A}\dot{C}} \oint_z \frac{dw}{2\pi\mathrm{i}} \left(\psi^{\alpha\dot{A}}(z) + (w-z)\partial\psi^{\alpha\dot{A}}(z) + O\left((w-z)^2\right)\right) \times \\
&\quad \times \left(\frac{\epsilon^{CB}\epsilon^{\dot{C}\dot{B}}}{(w-z)^2} + O\left((w-z)^0\right)\right) \\
&= \epsilon_{AC}\epsilon_{\dot{A}\dot{C}}\epsilon^{CB}\epsilon^{\dot{C}\dot{B}}\,\partial\psi^{\alpha\dot{A}}(z) \\
&= \delta^{B}_{A}\,\partial\psi^{\alpha\dot{A}}(z),
\end{aligned} \tag{6.1.5}$$

where we used the OPE rule, expanded $\psi^{\alpha\dot{A}}(w)$ around $w = z$ and used that $\epsilon_{AC}\epsilon^{CB} = \delta^{B}_{A}$. Analogously, for the antiholomorphic bosons we have

$$\oint_{\bar{z}} \frac{d\bar{w}}{2\pi\mathrm{i}}\, \tilde{G}^{\dot{\alpha}}_{A}(\bar{w})\,\bar{\partial} X^{B\dot{B}}(\bar{z}) = \delta^{B}_{A}\,\partial\tilde{\psi}^{\dot{\alpha}\dot{B}}(\bar{z}). \tag{6.1.6}$$

The action of the supercurrents can be used to define a Ward identity among correlators. In the untwisted sector, working on a single strand of length 1, we want to compute

[1] For convenience, notice that the operation of raising the indices works the same way,

$$\partial X^{A\dot{A}} = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\,\partial X_{B\dot{B}}, \qquad \bar{\partial} X^{A\dot{A}} = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\,\bar{\partial} X_{B\dot{B}}.$$

a Ward identity relating the four-point function

$$\mathcal{C}_{\text{fer}}(z_i,\bar z_i) \equiv \langle O_H | O_L(z_1,\bar z_1)\, \bar O_L(z_2,\bar z_2) | O_H\rangle \tag{6.1.7}$$

to a 4-point function with light operators written in terms of the free bosons. The light operators are defined as

$$O_L \to O_{\text{fer}} = \sum_{r=1}^{N} \frac{-i\epsilon_{\dot A\dot B}}{\sqrt{2N}} \psi^{1\dot A}_{(r)}\, \tilde\psi^{1\dot B}_{(r)}\ , \qquad \bar O_L \to \bar O_{\text{fer}} = \sum_{r=1}^{N} \frac{-i\epsilon_{\dot A\dot B}}{\sqrt{2N}} \psi^{2\dot A}_{(r)}\, \tilde\psi^{2\dot B}_{(r)}\ . \tag{6.1.8}$$

while the heavy states are considered to be a product of a Ramond ground state and a bosonic vacuum.

It is useful to consider slightly different integrals with respect to (6.1.5) and (6.1.6), including a factor $\sqrt{w}$ and $\sqrt{\bar w}$ in the integrand: the reason will be clarified in the following. We have

$$\begin{aligned}
\oint_z \frac{dw}{2\pi \mathrm{i}} \sqrt{w}\, G^\alpha_A(w)\, \partial X^{B\dot B}(w) &= \oint_z \frac{dw}{2\pi \mathrm{i}} \sqrt{w}\, \epsilon_{AC}\epsilon_{\dot A\dot C}\, \psi^{\alpha\dot A}(w)\, \partial X^{C\dot C}(w)\, \partial X^{B\dot B}(w) \\
&= \epsilon_{AC}\epsilon_{\dot A\dot C} \oint_{w\sim z} \frac{dw}{2\pi \mathrm{i}} \left(\sqrt{z} + \frac{1}{2\sqrt{z}}(w-z) + O\left(((w-z)^2)\right)\right) \times \\
&\quad \times \left(\psi^{\alpha\dot A}(z) + (w-z)\partial\psi^{\alpha\dot A}(z) + O\left((w-z)^2\right)\right) \times \\
&\quad \times \left(\frac{\epsilon^{CB}\epsilon^{\dot C\dot B}}{(w-z)^2} + O\left((w-z)^0\right)\right) \\
&= \epsilon_{AC}\epsilon_{\dot A\dot C}\epsilon^{CB}\epsilon^{\dot C\dot B}\left(\sqrt{z}\,\partial\psi^{\alpha\dot A}(z) + \frac{1}{2\sqrt{z}}\,\psi^{\alpha\dot A}\right) \\
&= \delta^B_A\, \partial_z\left(\sqrt{z}\,\psi^{\alpha\dot A}(z)\right),
\end{aligned} \tag{6.1.9}$$

where we also had to expand $\sqrt{w}$ around $w = z$. For the antiholomorphic bosons we have the similar relation

$$\oint_{\bar z} \frac{d\bar w}{2\pi \mathrm{i}} \sqrt{\bar w}\, \tilde G^{\dot\alpha}_A(\bar w)\, \bar\partial X^{B\dot B}(\bar w) = \delta^B_A\, \partial_{\bar z}\left(\sqrt{\bar z}\,\tilde\psi^{\dot\alpha\dot A}(\bar z)\right). \tag{6.1.10}$$

Knowing this we can write

$$\begin{aligned}
&\delta^B_A\delta^D_C\, \partial_z\partial_{\bar z}\left\{\left(-\frac{\mathrm{i}}{\sqrt{2}}\right) |z|\, \psi^{1\dot A}(z)\tilde\psi^{1\dot B}(\bar z)\epsilon_{\dot A\dot B}\right\} = \\
&\qquad = \delta^B_A\delta^D_C\, \partial_z\partial_{\bar z}\Big(|z|\, O_L(z,\bar z)\Big) \\
&\qquad = \left(-\frac{\mathrm{i}}{\sqrt{2}}\right) \oint_z \frac{dw}{2\pi \mathrm{i}} \sqrt{w}\, G^1_A(w)\partial X^{B\dot A}(z) \oint_{\bar z} \frac{d\bar w}{2\pi \mathrm{i}} \sqrt{\bar w}\, \tilde G^{\dot 1}_C(\bar w)\bar\partial X^{D\dot B}(\bar z)\epsilon_{\dot A\dot B}.
\end{aligned} \tag{6.1.11}$$

We then choose $A = B = C = D = 1$, multiply by $\bar{O}_L$ and consider the four-point function with heavy states $|O_H\rangle$, getting

$$\begin{aligned}
\partial_{z_1}\partial_{\bar{z}_1}\Big\{|z_1|\langle O_H|O_{\text{fer}}(z_1,\bar{z}_1)\bar{O}_{\text{fer}}(z_2,\bar{z}_2)|O_H\rangle\Big\} &= \\
&= \partial_{z_1}\partial_{\bar{z}_1}\Big(|z_1|\,\mathcal{C}_{\text{fer}}(z_i,\bar{z}_i)\Big) \\
&= \Big(-\frac{\mathrm{i}}{\sqrt{2}}\Big)^2 \oint_{w\sim z_1}\frac{dw}{2\pi\mathrm{i}}\oint_{\bar{w}\sim\bar{z}_1}\frac{d\bar{w}}{2\pi\mathrm{i}}\,|w|\,G^1_1(w)\tilde{G}^{\dot{1}}_1(\bar{w})\langle O_H|\partial X^{1\dot{A}}(z_1)\bar{\partial}X^{1\dot{B}}(\bar{z}_1)\epsilon_{\dot{A}\dot{B}}\times \\
&\quad\times\psi^{2\dot{C}}(z_2)\tilde{\psi}^{\dot{2}\dot{D}}(\bar{z}_2)\epsilon_{\dot{C}\dot{D}}|O_H\rangle.
\end{aligned} \tag{6.1.12}$$

In order to compute the integrals above we could deform the contour of the dw integral, picking contributions for all the points other than z_1 where we can have poles of order 1 (we also do the same for the $d\bar{w}$ integral). Naively, these points will be where the supercurrent $G^1_1(w)$ can have nontrivial contractions with other operators, i.e. at $w = 0, \infty, z_2$, while for the antiholomorphic integral we have to look at $\tilde{G}^{\dot{1}}_1(\bar{w})$ and the points will be $\bar{w} = 0, \infty, \bar{z}_2$. If the heavy states in the fermionic sector are Ramond vacua, though, a branch cut is introduced corresponding to the antiperiodic boundary conditions of the fermions. The branch cut has the nature of a square root, as going around the origin once the fermions in the R sector get a minus sign. The factors $\sqrt{w}$ and $\sqrt{\bar{w}}$ in the integrand were introduced to cancel the branch cut. We can obtain a Ward identity relating the correlators with bosonic and fermionic light operators by computing the integrals, which can now be done pushing the contour without worrying about the presence of branch cuts.

Let's consider the holomorphic term. As we said before, pushing the contour we get a contour integral with a path going around the only possible points where we could have singularities coming from the contraction of $G^1_1(w)$ with other operators: in total the integral becomes a sum of three integrals, with w going around $0, \infty$ and z_2. The paths go around these points clockwise, so we get an extra minus sign to bring them back into counter-clockwise orientation. If w goes around 0, singularities can arise from the contraction of G^1_1 with $|O_H\rangle$ and we have

$$\begin{aligned}
-\oint_0\frac{dw}{2\pi\mathrm{i}}\,\sqrt{w}\,G^1_1(w)|O_H\rangle &= -\sum_{n\in\mathbb{Z}}\oint_0\frac{dw}{2\pi\mathrm{i}}\,\sqrt{w}\,G^1_{1,n}\,w^{-n-3/2}|O_H\rangle \\
&= -G^1_{1,0}|O_H\rangle \\
&= 0,
\end{aligned} \tag{6.1.13}$$

where we expanded the supercurrent in modes $G^2_{2,n}$ and assumed the fact that the heavy state is invariant under supersymmetry (which is true if we choose it to be the product of a Ramond vacuum and of the bosonic vacuum)[2].

[2] The modes $G^{\alpha}_{A,n}$ of the supercurrents on a strand of length 1 are written in terms of the modes of the bosons and fermions as

$$G^{\alpha}_{A,n} = \sum_{m\in\mathbb{Z}}\alpha_{A\dot{A},m}\,\psi^{\alpha\dot{A}}_{n-m}\,,$$

so we immediately see that if $|O_H\rangle$ is a Ramond vacuum all the nonzero modes of G^1_1 certainly annihilate it.

Analogously, from the term in which w goes around ∞ we get

$$\begin{aligned} -\oint_\infty \frac{dw}{2\pi \mathrm{i}}\,\sqrt{w}\,\langle O_H|G^1_1(w) &= -\sum_{n\in\mathbb{Z}}\oint_\infty \frac{dw}{2\pi \mathrm{i}}\,\sqrt{w}\,\langle O_H|G^1_{1,n}\,w^{-n-3/2} \\ &= \oint_0 \frac{du}{2\pi \mathrm{i}}\,u^{-2}\langle O_H|G^1_{1,n}\,u^{n+1} \\ &= \langle O_H|G^1_{1,0} \\ &= 0, \end{aligned} \tag{6.1.14}$$

where we changed variables in the integral as $w \to u = w^{-1}$ and again we assumed the heavy state is invariant under supersymmetry.

The same results hold for the $d\bar{w}$ integral, so the only possible contributions come from w going around z_2 and $\bar{w}$ going around $\bar{z}_2$. The last line of (6.1.12) can be rewritten as

$$\begin{aligned} \partial_{z_1}\partial_{\bar{z}_1}\Big(|z_1|\,\mathcal{C}_{\rm fer}(z_i,\bar{z}_i)\Big) = -\left(-\frac{\mathrm{i}}{\sqrt{2}}\right)^2 \epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}}\,\langle O_H|\partial X^{1\dot{A}}(z_1)\oint_{z_2}\frac{dw}{2\pi\mathrm{i}}\,\sqrt{w}\,G^1_1(w)\,\psi^{2\dot{C}}(z_2)|O_H\rangle\times \\ \times\langle \bar{O}_H|\bar{\partial}X^{1\dot{B}}(\bar{z}_1)\oint_{\bar{z}_2}\frac{d\bar{w}}{2\pi\mathrm{i}}\,\sqrt{\bar{w}}\,\tilde{G}^{\dot{1}}_1(\bar{w})\,\tilde{\psi}^{2\dot{D}}(\bar{z}_2)|\bar{O}_H\rangle, \end{aligned} \tag{6.1.15}$$

where with an abuse of notation we denoted $|O_H\rangle$ as the holomorphic part of the asymptotic state and $|\bar{O}_H\rangle$ as its antiholomorphic part. An extra minus sign comes from the fact that $\tilde{G}^{\dot{\alpha}}_{A}$ and $\psi^{\beta\dot{B}}$ anticommute.

The presence of the square roots in the integrals in (6.1.15) does not alter the structure of the poles of the integrands, so the results are just obtained as in (6.1.3) and (6.1.4), with extra factors obtained by evaluating $\sqrt{w}$ and $\sqrt{\bar{w}}$ in $w = z_2$ and $\bar{w} = \bar{z}_2$ respectively,

$$\oint_{z_2}\frac{dw}{2\pi\mathrm{i}}\,\sqrt{w}\,G^1_1(w)\,\psi^{2\dot{C}}(z_2) = \sqrt{z_2}\,\partial X_{1\dot{E}}(z_2)\,\epsilon^{\dot{E}\dot{C}} = -\sqrt{z_2}\,\partial X^{2\dot{C}}(z_2), \tag{6.1.16a}$$

$$\oint_{\bar{z}_2}\frac{d\bar{w}}{2\pi\mathrm{i}}\,\sqrt{\bar{w}}\,\tilde{G}^{\dot{1}}_1(\bar{w})\,\tilde{\psi}^{2\dot{D}}(\bar{z}_2) = \sqrt{\bar{z}_2}\,\bar{\partial}X_{1\dot{F}}(\bar{z}_2)\,\epsilon^{\dot{F}\dot{D}} = -\sqrt{\bar{z}_2}\,\bar{\partial}X^{2\dot{D}}(\bar{z}_2). \tag{6.1.16b}$$

Inserting (6.1.16) into (6.1.15) we get a Ward identity for correlators involving O^{++} and O^{--},

$$\partial_{z_1}\partial_{\bar{z}_1}\Big(|z_1|\,\mathcal{C}_{\rm fer}(z_i,\bar{z}_i)\Big) = \frac{|z_1|}{2}\langle O_H|\left(\partial X^{1\dot{A}}\bar{\partial}X^{1\dot{B}}\epsilon_{\dot{A}\dot{B}}\right)(z_1,\bar{z}_1)\left(\partial X^{2\dot{C}}\bar{\partial}X^{2\dot{D}}\epsilon_{\dot{C}\dot{D}}\right)(z_2,\bar{z}_2)|O_H\rangle. \tag{6.1.17}$$

In summary we obtain the relation

$$\langle \bar{O}_H|O_{\rm bos}(1)\bar{O}_{\rm bos}(z,\bar{z})|O_H\rangle = \partial\bar{\partial}\Big[|z|\langle\bar{O}_H|O_{\rm fer}(1)\bar{O}_{\rm fer}(z,\bar{z})|O_H\rangle\Big]\,. \tag{6.1.18}$$

This is clearly satisfied by the orbifold point results but since this relation uses only the superconformal algebra, it holds at a generic point of the CFT moduli space and in the next section we will check its validity in the supergravity limit.

6.2 Correlator with bosonic light operators

The structure of the correlators we consider is again

$$\langle O_H(z_1)\bar{O}_H(z_2)O_L(z_3)O_L(z_4)\rangle = \frac{1}{z_{12}^{2h_H}\, z_{34}^{2h_L}}\frac{1}{\bar{z}_{12}^{2\bar{h}_H}\, \bar{z}_{34}^{2\bar{h}_L}}\mathcal{G}(z,\bar{z})\ , \tag{6.2.1}$$

where, as usual, $z_{ij} = z_i - z_j$ and

$$z = \frac{z_{14}z_{23}}{z_{13}z_{24}}\,, \tag{6.2.2}$$

while $(h_H, \bar{h}_H)$ and $(h_L, \bar{h}_L)$ are the holomorphic/anti-holomorphic conformal dimensions of the heavy and light operators respectively.

As usual, in order to easily isolate $\mathcal{G}$ from the correlators one can take $z_2 \to \infty$, $z_1 = 0$ and $z_3 = 1$, which implies $z = z_4$:

$$\mathcal{C}(z,\bar{z}) \equiv \langle \bar{O}_H|O_L(1)\bar{O}_L(z,\bar{z})|O_H\rangle = \frac{1}{(1-z)^{2h_L}}\frac{1}{(1-\bar{z})^{2\bar{h}_L}}\mathcal{G}(z,\bar{z})\ . \tag{6.2.3}$$

6.2.1 CFT picture

The light and heavy operators we consider are of the form

$$O_L = \frac{1}{\sqrt{N}}\sum_{r=1}^{N} O^L_{(r)}, \quad O_H = \otimes_{r=1}^{N} O^H_{(r)} \tag{6.2.4}$$

In particular, we choose now light operators to be constructed with bosonic operators with $h_L = \bar{h}_L = 1$

$$O_L \to O_{\text{bos}} = \sum_{r=1}^{N} \frac{\epsilon_{\dot{A}\dot{B}}}{\sqrt{2N}}\partial X^{1\dot{A}}_{(r)}\bar{\partial} X^{1\dot{B}}_{(r)}\ , \quad \bar{O}_L \to \bar{O}_{\text{bos}} = \sum_{r=1}^{N} \frac{\epsilon_{\dot{A}\dot{B}}}{\sqrt{2N}}\partial X^{2\dot{A}}_{(r)}\bar{\partial} X^{2\dot{B}}_{(r)}\ . \tag{6.2.5}$$

The heavy operator is defined as follows

$$\lim_{z\to 0} O_H(z)|0\rangle = |O_H\rangle = (|++\rangle_1)^{N_1^{(++)}}\prod_k (|00\rangle_k)^{N_k^{(0)}}, \quad \text{with} \quad N_1^{(++)} + \sum_k k N_k^{(0)} = N \tag{6.2.6}$$

To compute the correlator at the orbifold point we diagonalize the boundary conditions and then we take linear combination of the contributions of each strand. We recall the change of basis for the bosons

$$\sum_{r=1}^{k} \partial X^{A\dot{B}}_{(r)}(z)\bar{\partial} X^{A\dot{C}}_{(r)}(\bar{z}) = \sum_{\rho=0}^{k-1} \partial X^{A\dot{B}}_{\rho}(z)\bar{\partial} X^{A\dot{C}}_{\rho}(\bar{z})\ . \tag{6.2.7}$$

and the commutation relations of the modes in the twisted sector

$$\left[\alpha^{A\dot{A}}_{\rho_1,n}, \alpha^{B\dot{B}}_{\rho_2,m}\right] = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\, n\, \delta_{n+m,0}\, \delta_{\rho_1,\rho_2}\ , \tag{6.2.8}$$

The non trivial contribution to the correlator comes from the action of the modes of the bosonic operators on the twisted vacuum state.

$$\begin{aligned}\mathcal{C}_{\rm bos}(z_i,\bar z_i) &= \sum_{\rho_1,\rho_2=0}^{k-1} {}_k\langle 0|\partial X^{1\dot 1}_{\rho_1}(z_1)\partial X^{2\dot 2}_{\rho_2}(z_2)|0\rangle_k\, {}_k\langle 0|\bar\partial X^{1\dot 1}_{\rho_1}(\bar z_1)\bar\partial X^{2\dot 2}_{\rho_2}(\bar z_2)|0\rangle_k \\ &= \sum_{\rho=0}^{k-1} {}_k\langle 0|\partial X^{1\dot 1}_{\rho}(z_1)\partial X^{2\dot 2}_{\rho}(z_2)|0\rangle_k\, {}_k\langle 0|\bar\partial X^{1\dot 1}_{\rho}(\bar z_1)\bar\partial X^{2\dot 2}_{\rho}(\bar z_2)|0\rangle_k \\ &\equiv \sum_{\rho=0}^{k-1} \mathcal{C}_\rho(z_i)\,\mathcal{C}_\rho(\bar z_i),\end{aligned} \tag{6.2.9}$$

where

$$\mathcal{C}_\rho(z_i) = {}_k\langle 0|\partial X^{1\dot 1}_{\rho}(z_1)\partial X^{2\dot 2}_{\rho}(z_2)|0\rangle_k, \tag{6.2.10a}$$

$$\mathcal{C}_\rho(\bar z_i) = {}_k\langle 0|\bar\partial X^{1\dot 1}_{\rho}(\bar z_1)\bar\partial X^{2\dot 2}_{\rho}(\bar z_2)|0\rangle_k, \tag{6.2.10b}$$

where we exploited the fact that in the ρ basis the correlators are diagonal, i.e. that $\partial X^{1\dot 1}_{\rho_1}$ and $\partial X^{2\dot 2}_{\rho_2}$ have nontrivial contractions only if $\rho_1 = \rho_2$ (and the same holds for their antiholomorphic counterparts). The correlator is insensitive to which states are present in the fermionic sector, as they are factorized away and we assume the normalization

$${}_k\langle s|s\rangle_k = 1. \tag{6.2.11}$$

The correlator $\mathcal{C}_\rho(z_i)$ is computed using the mode expansions and the commutation relations (6.2.8) and knowing that the bosonic vacuum is annihilated by the positive modes as usual,

$$\begin{aligned}\mathcal{C}_\rho(z_i) &= {}_k\langle 0|\partial X^{1\dot 1}_{\rho}(z_1)\,\partial X^{2\dot 2}_{\rho}(z_2)|0\rangle_k = \\ &= \sum_{\substack{n_1\in\mathbf{Z}\\ n_1-\frac{\rho}{k}>0}}\ \sum_{\substack{n_2\in\mathbf{Z}\\ n_2+\frac{\rho}{k}<0}} z_1^{-n_1-1+\frac{\rho}{k}} z_2^{-n_2-1-\frac{\rho}{k}}\ {}_k\langle 0|\alpha^{1\dot 1}_{\rho,n_1-\frac{\rho}{k}},\alpha^{2\dot 2}_{\rho,n_2+\frac{\rho}{k}}|0\rangle_k \\ &= \sum_{\substack{n_1\in\mathbf{Z}\\ n_1-\frac{\rho}{k}>0}}\ \sum_{\substack{n_2\in\mathbf{Z}\\ n_2+\frac{\rho}{k}<0}} z_1^{-n_1-1+\frac{\rho}{k}} z_2^{-n_2-1-\frac{\rho}{k}}\ {}_k\langle 0|\left[\alpha^{1\dot 1}_{\rho,n_1-\frac{\rho}{k}},\alpha^{2\dot 2}_{\rho,n_2+\frac{\rho}{k}}\right]|0\rangle_k \\ &= (z_1z_2)^{-1}\left(\frac{z_1}{z_2}\right)^{\rho/k}\sum_{n=1}^{+\infty}\left(\frac{z_2}{z_1}\right)^n\left(n-\frac{\rho}{k}\right) \\ &= (z_1z_2)^{-1}\left(\frac{z_1}{z_2}\right)^{\rho/k}\left\{\frac{\left(\frac{z_2}{z_1}\right)}{\left[1-\left(\frac{z_2}{z_1}\right)\right]^2}-\frac{\rho}{k}\frac{\left(\frac{z_2}{z_1}\right)}{1-\left(\frac{z_2}{z_1}\right)}\right\},\end{aligned} \tag{6.2.12}$$

where we used that

$$\sum_{n=1}^{+\infty} A^n = \frac{A}{1-A}, \tag{6.2.13a}$$

$$\sum_{n=1}^{+\infty} n\,A^n = A\,\partial_A\left[\sum_{n=1}^{+\infty} A^n\right] = \frac{A}{(1-A)^2}. \tag{6.2.13b}$$

the antiholomorphic part is obtained taking the above and replacing $z_i \to \bar z_i$.

From the above relation we can easily calculate the two-point correlator on strand of length k

$$ {}_k\langle 0|\partial X^{1\dot{1}}_\rho(z_1)\,\partial X^{2\dot{2}}_\rho(z_2)|0\rangle_k = \frac{1}{(z_1-z_2)^2}\left(\frac{z_1}{z_2}\right)^{-\frac{\rho}{k}}\left\{1-\frac{\rho}{k}\left(1-\frac{z_1}{z_2}\right)\right\}, \tag{6.2.14} $$

with similar formulae holding for the antiholomorphic sector. Then the contribution from such strand to the correlator with the light operators in (6.2.5) is

$$ \begin{aligned} \mathcal{C}^{\rm bos}_k(z,\bar z) =&\frac{1}{(1-z)^2(1-\bar z)^2}\sum_{\rho=0}^{k-1}|z|^{\frac{2\rho}{k}}\left|1-\frac{\rho}{k}\left(1-\frac{1}{z}\right)\right|^2 \\ =&\partial\bar\partial\left[\frac{1-z\bar z}{(1-z)(1-\bar z)\left(1-(z\bar z)^{\frac{1}{k}}\right)}\right]. \end{aligned} \tag{6.2.15} $$

Summing over contributions we get

$$ \mathcal{C}^{\rm bos} = \frac{1}{N}\sum_{k=1}^{N} N_k \mathcal{C}^{\rm bos}_k = \frac{1}{N}\sum_{k=1}^{N} N_k\partial\bar\partial\left[\frac{1-z\bar z}{(1-z)(1-\bar z)\left(1-(z\bar z)^{\frac{1}{k}}\right)}\right], \tag{6.2.16} $$

6.2.2 Gravity picture

Similarly to previous cases, we focus on the states that are invariant under the $SU(2)$'s acting on the coordinates of $\mathcal{M}_4$, which ensures that the dual solutions are invariant under rotations of the four stringy-sized compact directions. Then we focus on the case where the RR ground states are made of a large number $N_1^{(++)}$ of strands of the type $|++\rangle_1$ while the remaining strands have arbitrary winding $k \geq 1$ but are in the unique RR state $s=0$ that is a scalar of all $SU(2)$'s; we denote strands of this type as $|00\rangle_k$ and their numbers as $N_k^{(0)}$. On the bulk side the restriction to this subset of states simplifies the six-dimensional metric (6.2.17). The family of D1D5 geometries dual to these states has in fact played an important role in some recent supergravity developments [27, 47, 67]. At some point of our analysis we will also assume that the numbers of $|00\rangle_k$ strands are parametrically smaller than the number of $|++\rangle_1$ strands ($N_k^{(0)} \ll N_1^{(++)}$): this will allow the perturbative approach in b_k discussed in section 6.2.3.

Also in this case, the heavy operators O_H are described in the gravity regime by six-dimensional geometries that asymptotically approximate AdS$_3\times S^3$ and are everywhere regular and horizonless. Operators that are Ramond ground states both in the left and in the right sector are dual to geometries carrying D1 and D5 charges but no momentum charge. The six-dimensional Einstein metric dual to RR ground states that are invariant under rotations in the four compact dimensions is given by (B.4.3a)

$$ ds_6^2 = -\frac{2}{\sqrt{\mathcal{P}}}(dv+\beta)(du+\omega)+\sqrt{\mathcal{P}}\,ds_4^2\,,\mathcal{P}=Z_1Z_2-Z_4^2 \tag{6.2.17} $$

Apart from the metric, all other fields of type IIB supergravity are non-trivial in the solution: their expressions are given in (B.4.3a)-(B.4.3e), but will not be relevant for the correlator we compute here.

The form of the supergravity data Z_1, Z_2, Z_4, β and ω depends on the RR ground state and is generically complicated. As mentioned above, we focus on the family of D1D5 states described in (**??**). The dual gravity solutions depend on some continuous parameters: a, whose square is proportional to $N_1^{(++)}$, and b_k, whose square is proportional to

$kN_k^{(0)}$ [33]:

$$N_1^{(++)} = N\,\frac{a^2}{a_0^2}\,,\quad kN_k^{(0)} = N\,\frac{b_k^2}{2a_0^2}\quad \text{with}\quad a_0^2 \equiv \frac{Q_1 Q_5}{R^2}\,. \tag{6.2.18}$$

Here R is the radius of the CFT circle and Q_1, Q_5 are the supergravity D1 and D5 charges, related to the numbers n_1, n_5 of D1 and D5 branes by standard expressions found in the previous chapters

$$Q_1 = \frac{(2\pi)^4\, n_1\, g_s\, \alpha'^4}{V_4}\,,\quad Q_5 = n_5\, g_s\, \alpha'\,, \tag{6.2.19}$$

with g_s the string coupling and V_4 the volume of T^4. The condition that the total number of strands be N implies the constraint

$$a^2 + \sum_k \frac{b_k^2}{2} = a_0^2\,, \tag{6.2.20}$$

which turns out to be also the regularity condition for the metric. We also have the usual form of the flat $\mathbb{R}^4$ metric

$$ds_4^2 = \Sigma\left(\frac{dr^2}{r^2+a^2} + d\theta^2\right) + (r^2+a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2\,,\quad \Sigma \equiv r^2 + a^2\cos^2\theta\,. \tag{6.2.21}$$

The remaining data encoding the metric are

$$\begin{aligned} Z_1 = \frac{R^2}{Q_5\,\Sigma}\Bigg[a_0^2 &+ \sum_{k,k'} \frac{b_k b_{k'}}{2}\,\frac{a^{k+k'}}{(r^2+a^2)^{\frac{k+k'}{2}}}\,\sin^{k+k'}\theta\cos((k+k')\phi) \\ &+ \sum_{k>k'} b_k b_{k'}\,\frac{a^{k-k'}}{(r^2+a^2)^{\frac{k-k'}{2}}}\,\sin^{k-k'}\theta\cos((k-k')\phi)\Bigg]\,,\quad Z_2 = \frac{Q_5}{\Sigma}\,, \end{aligned} \tag{6.2.22a}$$

$$Z_4 = \frac{R}{\Sigma}\sum_k b_k \frac{a^k}{(r^2+a^2)^{\frac{k}{2}}}\,\sin^k\theta\cos(k\phi)\,, \tag{6.2.22b}$$

$$\beta = \frac{R\,a^2}{\sqrt{2}\,\Sigma}\,(\sin^2\theta d\phi - \cos^2\theta d\psi)\,,\quad \omega = \frac{R\,a^2}{\sqrt{2}\,\Sigma}\,(\sin^2\theta d\phi + \cos^2\theta d\psi)\,. \tag{6.2.22c}$$

For generic values of b_k the geometry is complicated, but it can be shown to be regular and without horizon for any values of the parameters, as far as the constraint (6.2.20) is satisfied.

6.2.3 Four-point functions computation

To compute the correlator of two light and two heavy operators one should consider the wave equation for a perturbation in the background (6.2.17). The bosonic light operator $O_L = O_{\text{bos}}$ is described by a minimally coupled scalar in the six-dimensional Einstein metric ds_6^2. Such a scalar is actually a two-form with indices on T^4. However it can be shown that thanks to dualities this scalar can be mapped into another scalar given by dimensional reduction of traceless perturbations of the metric on T^4, and thus having the right quantum numbers to be dual to the CFT operators $\partial X^{(i}\bar\partial X^{j)}$, with $i, j = 1, \ldots, 4$.

Following the logic of the previous cases, the gravity computation of the correlator requires solving the wave equation (B.5.22)

$$\Box_6 B = 0\,, \tag{6.2.23}$$

where $\Box_6$ is the scalar Laplace operator with respect to ds_6^2

$$\Box_6 \cdot \equiv \frac{1}{\sqrt{g_6}}\,\partial_M\big(\sqrt{g_6}\, g_6^{MN}\partial_N\cdot\big)\,, \tag{6.2.24}$$

with the boundary condition

$$B \sim \delta(t,y) + \frac{b(t,y)}{r^2} \tag{6.2.25}$$

for large r. Since the background metric is regular everywhere, one should also require that B have no singularities at any finite value of r. As the operator O_L is an R-charge singlet, only the projection of B on the trivial scalar spherical harmonic on S^3 contributes to our correlator. The four-point function computed on the Euclidean plane is encoded in the function $b(t,y)$ via

$$\langle O_H(0)\bar O_H(\infty)O_L(1,1)\bar O_L(z,\bar z)\rangle = \frac{1}{|1-z|^4}\mathcal{G}^{\rm bos}(z,\bar z) = (z\bar z)^{-1}\, b(z,\bar z)\,, \tag{6.2.26}$$

where

$$z = e^{i\frac{t+y}{R}} = e^{\frac{t_e+iy}{R}}\,,\quad \bar z = e^{i\frac{t-y}{R}} = e^{\frac{t_e-iy}{R}}\,, \tag{6.2.27}$$

with $t_e \equiv it$ the Euclidean time. The factor $(z\bar z)^{-1}$ on the r.h.s. of (6.2.26) comes from the transformation of the primary field $\bar O_L(z,\bar z) = (z\bar z)^{-1}\,\bar O_L(t,y)$ from the cylinder to the plane coordinates.

The laplacian in (6.2.24) is most easily derived if one writes the six-dimensional metric as if one were performing a dimensional reduction on S^3 [68, 33, 69]:

$$ds_6^2 = V^{-2} g_{\mu\nu}dx^\mu dx^\nu + G_{\alpha\beta}(dx^\alpha + A^\alpha_\mu dx^\mu)(dx^\beta + A^\beta_\nu dx^\nu)\,, \tag{6.2.28}$$

where

$$V^2 \equiv \frac{\det G}{(Q_1Q_5)^{3/2}\sin^2\theta\cos^2\theta}\,. \tag{6.2.29}$$

We have split the six-dimensional coordinates in the AdS$_3$ coordinates $x^\mu, x^\nu, \ldots \equiv (r,t,y)$ and the S^3 coordinates $x^\alpha, x^\beta, \ldots \equiv (\theta,\phi,\psi)$. The definition of $g_{\mu\nu}$, $G_{\alpha\beta}$, A^α_μ depends of course on the choice of coordinates: the coordinates are fixed at the boundary by the requirement that the metric looks like AdS$_3\times S^3$ asymptotically, but one is free to redefine the coordinates in the space-time interior. For lack of a better choice, we will stick to the coordinates defined in (6.2.21).

If one takes the solution in (6.2.22) and sets $b_k = 0$ for any k, one finds that $g_{\mu\nu}$ becomes the metric of global AdS$_3$

$$g_{\mu\nu}dx^\mu dx^\nu\Big|_{b_k=0} = \sqrt{Q_1Q_5}\left[\frac{dr^2}{r^2+a_0^2} - \frac{r^2+a_0^2}{Q_1Q_5}dt^2 + \frac{r^2}{Q_1Q_5}dy^2\right] \equiv \sqrt{Q_1Q_5}\, ds^2_{AdS_3} \tag{6.2.30}$$

and $G_{\alpha\beta}$ the metric of the round S^3. When, like in this case, the metric $g_{\mu\nu}$ does not depend on the coordinates of S^3, the six-dimensional Laplace equation (6.2.23) admits an S^3-independent solution which satisfies the simpler equation

$$\Box_3 B = 0\,, \tag{6.2.31}$$

with $\Box_3$ the laplacian of $g_{\mu\nu}$:

$$\Box_3 \cdot \equiv \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g}\, g^{\mu\nu} \partial_\nu \cdot) \,. \tag{6.2.32}$$

In general however the six-dimensional metric does not factorise and $g_{\mu\nu}$ and $G_{\alpha\beta}$ depend on both AdS$_3$ and S^3 coordinates. In this situation solving the six-dimensional equation (6.2.23) exactly seems hard. When this happens one can resort to an approximation scheme that was used already in chapter 5: we solve the wave equation perturbatively in b_k, keeping only the first non-trivial order $\mathcal{O}(b_k^2)$. In the following we will apply this perturbative method to compute the correlator for generic b_k's. In the particular example in which b_1 is the only non-vanishing mode, we will be able to do better and perform the computation exactly in b_1.

Four-point function: perturbative computation for generic b_k's

We consider here a generic state in the ensemble (6.2.6) and compute the correlator in the limit $N_k^{(0)} \ll N_1^{(++)}$, keeping the first non-trivial term in an expansion in b_k/a_0. This contribution already depends on the CFT moduli and hence it contains non-trivial dynamical informations. We perform the b_k-expansion keeping Q_1, Q_5 and R (and hence a_0) fixed: on the CFT side this means we are not varying the central charge nor the size of the circle on which the CFT is defined. At zero-th order in b_k the metric is AdS$_3 \times S^3$, and we will expand the terms of order b_k^2 in the basis of spherical harmonics of this unperturbed S^3. We thus write the solution of (6.2.23) as

$$B = B_0 + B_1 + \mathcal{O}(b_k^4)\,, \tag{6.2.33}$$

where B_1 quadratic in b_k. The terms of order zero and two of the wave equation give

$$\Box_0 B_0 = 0\,, \quad \Box_0 B_1 = -\Box_1 B_0\,, \tag{6.2.34}$$

where $\Box_0$ is the laplacian of global AdS$_3$

$$\Box_0 \cdot \equiv \frac{1}{r}\partial_r (r(r^2+a_0^2)\partial_r \cdot) - \frac{a_0^2\, R^2}{r^2+a_0^2}\, \partial_t^2 \cdot + \frac{a_0^2\, R^2}{r^2}\, \partial_y^2 \cdot \,, \tag{6.2.35}$$

and $\Box_1$ is the order b_k^2 contribution to the laplacian $\Box_3$ defined in (6.2.31). The first equation in (6.2.34), together with the asymptotic boundary condition (6.2.25) and the regularity condition, implies that B_0 is the usual bulk-to-boundary propagator of dimension $\Delta = 2$ in global AdS$_3$:

$$B_0(r,t,y) = K_2^{\text{Glob}}(r,t,y|t'=0,y'=0) = \left[\frac{1}{2}\, \frac{a_0}{\sqrt{r^2+a_0^2}\, \cos(t/R) - r\, \cos(y/R)}\right]^2 . \tag{6.2.36}$$

The second equation in (6.2.34) is an equation for B_1. If the metric $g_{\mu\nu}$ is a non-trivial function on S^3, the B_1 that solves this equation has components along non-trivial S^3 spherical harmonics, which we should project away for the purpose of extracting the bosonic correlator. In particular all terms in the solution (6.2.22) that are proportional to $b_k b_{k'}$ for $k \neq k'$ depend non-trivially on ϕ as $\cos((k-k')\phi)$ and source non-trivial spherical harmonics in B_1: hence they do not contribute to the correlator at quadratic order in b_k. We can thus simplify the computation by focusing on a single k-mode at a time. The metric $g_{\mu\nu}$ derived from the solution where a single b_k is non-vanishing is

$$\frac{g_{tt}^{(k)}}{\sqrt{Q_1Q_5}} = -\frac{r^2+a^2}{R^2a_0^4}\left(a^2+\frac{b_k^2}{2}\frac{r^2}{\Sigma}F_k\right)\,, \quad \frac{g_{yy}^{(k)}}{\sqrt{Q_1Q_5}} = \frac{r^2}{R^2a_0^4}\left(a^2+\frac{b_k^2}{2}\frac{r^2+a^2}{\Sigma}F_k\right)\,, \tag{6.2.37a}$$

$$\frac{g_{rr}^{(k)}}{\sqrt{Q_1Q_5}} = \frac{1}{a_0^4(r^2+a^2)}\left(a^2+\frac{b_k^2}{2}\frac{r^2}{\Sigma}F_k\right)\left(a^2+\frac{b_k^2}{2}\frac{r^2+a^2}{\Sigma}F_k\right)\,, \tag{6.2.37b}$$

with

$$F_k \equiv 1-\left(\frac{a^2\sin^2\theta}{r^2+a^2}\right)^k\,. \tag{6.2.38}$$

We see that, unless $k=1$, even for a single mode $g_{\mu\nu}$ depends non-trivially on the S^3 coordinate θ. To compute B_1, one should expand the laplacian of $g_{\mu\nu}^{(k)}$ up to order b_k^2 ($\Box^{(k)} = \Box_0 + b_k^2\,\Box_1^{(k)} + \mathcal{O}(b_k^4)$) and project on the trivial spherical harmonic. One finds

$$\langle J_k\rangle \equiv -\langle \Box_1^{(k)} B_0\rangle = -\frac{r}{(r^2+a_0^2)}\partial_r B_0 + \frac{a_0^2\,R^2}{(r^2+a_0^2)^2}\,\partial_t^2 B_0 + \frac{R^2}{2a_0^2}\,S_k\,(\partial_t^2 B_0 - \partial_y^2 B_0)\,, \tag{6.2.39}$$

where

$$S_k \equiv \sum_{p=2}^{k}\left(\frac{a_0^2}{r^2+a_0^2}\right)^p \langle \sin^{2p-2}\theta\rangle = \sum_{p=2}^{k}\frac{1}{p}\left(\frac{a_0^2}{r^2+a_0^2}\right)^p\,, \tag{6.2.40}$$

and the bracket $\langle\cdot\rangle$ denotes the average on S^3. In deriving (6.2.39) we have also used that $\Box_0 B_0 = 0$. The second equation in (6.2.34) is then easily integrated using the AdS_3 bulk-to-bulk propagator $G_2^{\text{Glob}}(\mathbf{r}'|r,t,y)$, and summing over all the modes:

$$B_1(r,t,y) = -i\sum_k b_k^2\int d^3\mathbf{r}'\,\sqrt{-g_{AdS_3}}\,G_2^{\text{Glob}}(\mathbf{r}'|r,t,y)\,\langle J_k(\mathbf{r}')\rangle\,, \tag{6.2.41}$$

where $\mathbf{r}'\equiv\{r',t',y'\}$ is a point in AdS_3 and g_{AdS_3} the metric of global AdS_3.

According to (6.2.26), the correlator is determined by the large r limit of B_1, which follows from the asymptotic limit of $G_2^{\text{Glob}}(\mathbf{r}'|r,t,y)$: $G_2^{\text{Glob}}(\mathbf{r}'|r,t,y) \to \frac{a_0^2}{2\pi r^2}K_2^{\text{Glob}}(\mathbf{r}'|t,y)$. Moving from Lorentzian cylinder to Euclidean plane, one finds that the order b_k^2 contribution to the 4-point function is

$$\langle O_H(t_e=-\infty)\bar{O}_H(t_e=\infty)O_L(0,0)\bar{O}_L(t_e,y)\rangle|_{b_k^2} =$$
$$= -\sum_k\frac{b_k^2}{2\pi}\int d^3\mathbf{r}'_e\sqrt{\bar g}\,K_2^{\text{Glob}}(\mathbf{r}'_e|t_e,y)\,\langle J_k(\mathbf{r}'_e)\rangle = -\sum_k\frac{b_k^2}{2\pi a_0^2}\left(\frac{I_1+I_2}{2} - I_3 - \sum_{p=2}^{k}\frac{1}{2\,p}\tilde{I}_p\right)\,, \tag{6.2.42}$$

where

$$I_1 \equiv \int d^3\mathbf{r}'_e\sqrt{\bar g}\,B_0(\mathbf{r}'_e|t_e,y)\,\partial'^{\mu}B_0(\mathbf{r}'_e|0,0)\,B_-(\mathbf{r}'_e)\,\partial'_{\mu}B_+(\mathbf{r}'_e)\,, \tag{6.2.43a}$$

$$I_2 \equiv \int d^3\mathbf{r}'_e\sqrt{\bar g}\,B_0(\mathbf{r}'_e|t_e,y)\,\partial'^{\mu}B_0(\mathbf{r}'_e|0,0)\,B_+(\mathbf{r}'_e)\,\partial'_{\mu}B_-(\mathbf{r}'_e)\,, \tag{6.2.43b}$$

$$I_3 \equiv \int d^3\mathbf{r}'_e\sqrt{\bar g}\,B_0(\mathbf{r}'_e|t_e,y)\,R^2\partial_{t'_e}^2 B_0(\mathbf{r}'_e|0,0)\,\frac{a_0^4}{(r'^2+a_0^2)^2}\,, \tag{6.2.43c}$$

$$\tilde{I}_p \equiv \int d^3\mathbf{r}'_e\sqrt{\bar g}\,B_0(\mathbf{r}'_e|t_e,y)\,R^2(\partial_{t'_e}^2+\partial_{y'}^2)B_0(\mathbf{r}'_e|0,0)\,\frac{a_0^{2p}}{(r'^2+a_0^2)^p}\,. \tag{6.2.43d}$$

These integrals can be written in terms of the same D-functions $D_{p_1p_2p_3p_4}$ as happened in the fermionic cases.

Following the same steps of chapter 5, I_1 can be computed by writing the integral in Poincaré coordinates $\mathbf{w} \equiv \{w_0, w, \bar{w}\}$:

$$\begin{aligned}|z|^{-2}I_1 &= \int d^3\mathbf{w}\, w_0^{-1}\left(\frac{w_0}{w_0^2+|w-z|^2}\right)^2 \partial_{w_0}\left(\frac{w_0}{w_0^2+|w-1|^2}\right)^2 \frac{w_0}{w_0^2+|z|^2}\\ &= \int d^3\mathbf{w}\, w_0^{-1}\left(\frac{w_0}{w_0^2+|w-z|^2}\right)^2 \left[\frac{2w_0}{(w_0^2+|w-1|^2)^2} - \frac{4w_0^3}{(w_0^2+|w-1|^2)^3}\right]\frac{w_0}{w_0^2+|z|^2}\\ &= 2\hat{D}_{1122} - 4\hat{D}_{1232}\,.\end{aligned} \tag{6.2.44}$$

Therefore

$$|z|^{-2}I_2 = 2\hat{D}_{2222} - 2\hat{D}_{1122} + 4\hat{D}_{1232}\,. \tag{6.2.45}$$

therefore we get

$$I_1 + I_2 = 2|z|^2\hat{D}_{2222}\,. \tag{6.2.46}$$

The computation of I_3 follows from the intergal I_5 in (5.2.37)

$$\begin{aligned}I_3 &= R\partial_{t_e}\frac{I_1-I_2}{2} = (z\partial + \bar{z}\bar{\partial})\Big(|z|^2(2\hat{D}_{1122} - 4\hat{D}_{1232} - \hat{D}_{2222})\Big)\\ &= \frac{2|z|^2}{|1-z|^4}\Big(2(1+|z|^2)\hat{D}_{3311} - \pi\Big)\,,\end{aligned} \tag{6.2.47}$$

where the last identity follows from a computation that uses the explicit expression of the $\hat{D}$-functions. Finally

$$\begin{aligned}\tilde{I}_p &= R^2(\partial_{t_e}^2 + \partial_y^2)\int d^3\mathbf{r}'_e\sqrt{\bar{g}}\, B_0(\mathbf{r}'_e|t_e,y)B_0(\mathbf{r}'_e|0,0)\,\frac{a_0^{2p}}{(r'^2+a_0^2)^p}\\ &= 4\partial\bar{\partial}(|z|^2\hat{D}_{pp22})\,.\end{aligned} \tag{6.2.48}$$

Including also the free contribution at $b_k = 0$, the final result for the strong coupling limit of the bosonic correlator up to order b_k^2 can be written in the suggestive form

$$\mathcal{C}^{\text{bos}}_{\mathcal{O}(b^2)}(z,\bar{z}) = \partial\bar{\partial}\left[\frac{1}{|1-z|^2} + \sum_k \frac{b_k^2}{a_0^2}\left(-\frac{1}{2}\frac{1}{|1-z|^2} + \sum_{p=1}^{k}\frac{|z|^2\hat{D}_{pp22}}{\pi\, p}\right)\right]\,. \tag{6.2.49}$$

One can check that the bosonic correlator (6.2.49) has the expected symmetry under the exchange of the points z_3 and z_4. This transformation permutes O_L with $\bar{O}_L$ and, according to the definition (6.2.5), amounts to exchange the $\mathcal{M}_4$ index $A = 1$ with $A = 2$; since the heavy operators we consider are invariant under transformations of the compact space $\mathcal{M}_4$, the correlator should be left invariant. From the definition of the ratios z one sees that the transformation $z_3 \to z_4$ is equivalent to $z \to 1/z$ and thus one should have that

$$\mathcal{G}^{\text{bos}}(z,\bar{z}) = \mathcal{G}^{\text{bos}}(z^{-1},\bar{z}^{-1})\,. \tag{6.2.50}$$

That the result (6.2.49) has this property follows from the symmetry of the $\hat{D}$-functions

$$\hat{D}_{pp22}(z^{-1},\bar{z}^{-1}) = |z|^4\hat{D}_{pp22}(z,\bar{z})\,. \tag{6.2.51}$$

Four-point function: exact computation for $b_k = b\,\delta_{k,1}$

The solution in which only the mode $b_1 \equiv b$ is non-vanishing is particularly simple: one sees indeed from (6.2.37) and (6.2.38) that $F_1 = \Sigma/(r^2+a^2)$ and thus the 3D metric $g_{\mu\nu}$ is θ-independent. One can thus look for an exact solution of the 3D Laplace equation (6.2.31):

$$\frac{r^2+a^2}{r(r^2+a^4/a_0^2)}\,\partial_r[r(r^2+a^2)\partial_r B] - \frac{a_0^2}{r^2+a^4/a_0^2}\,\partial_\tau^2 B + \frac{a_0^2}{r^2}\,\partial_\sigma^2 B = 0\,, \tag{6.2.52}$$

where we have defined

$$\tau \equiv \frac{t}{R}\,, \quad \sigma \equiv \frac{y}{R}\,. \tag{6.2.53}$$

Our analysis here will follow the one in appendix B of [14]. The solution of (6.2.52) that is regular at $r=0$ and that has the asymptotic behaviour (6.2.25) for large r is

$$B = \frac{1}{(2\pi)^2}\sum_{l\in\mathbb{Z}}\int d\omega\, e^{i\omega\tau+il\sigma}\, g(\omega,l)\,\left(\frac{r}{\sqrt{r^2+a^2}}\right)^{|l|}\,{}_2F_1\left(\frac{|l|+\gamma}{2},\frac{|l|-\gamma}{2},1+|l|;\frac{r^2}{r^2+a^2}\right)\,, \tag{6.2.54}$$

where

$$g(\omega,l) = \frac{\Gamma\left(1+\frac{|l|+\gamma}{2}\right)\Gamma\left(1+\frac{|l|-\gamma}{2}\right)}{\Gamma(1+|l|)} \tag{6.2.55}$$

and

$$\gamma \equiv \frac{\sqrt{a_0^2\,\omega^2-\frac{1}{2}\,b^2\,l^2}}{a}\,. \tag{6.2.56}$$

The function $b(t,y)$ defined in (6.2.25) is extracted from the large r limit of B:

$$b(\tau,\sigma) = \frac{a^2}{a_0^2}\sum_{l\in\mathbb{Z}}\int\frac{d\omega}{(2\pi)^2}\,e^{i\omega\tau+il\sigma}\left[-\frac{|l|}{2}+\frac{l^2-\gamma^2}{4}\left(H\left(\frac{|l|+\gamma}{2}\right)+H\left(\frac{|l|-\gamma}{2}\right)-1\right)\right], \tag{6.2.57}$$

where $H(z)$ is the harmonic number, which is related to the digamma function $\psi(z)$ as

$$H(z) = \psi(z+1)+\gamma_E = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right)\,. \tag{6.2.58}$$

Discarding contact terms proportional to $\delta(\tau)$ and/or $\delta(\sigma)$ and their derivatives, and using the identity

$$l^2-\gamma^2 = \frac{a_0^2}{a^2}\,(l^2-\omega^2)\,, \tag{6.2.59}$$

one can write

$$b(\tau,\sigma) = \frac{\partial_\tau^2-\partial_\sigma^2}{4}\,b_F(\tau,\sigma)\,, \tag{6.2.60}$$

where

$$b_F(\tau,\sigma) = \sum_{l\in\mathbb{Z}}\int\frac{d\omega}{(2\pi)^2}\,e^{i\omega\tau+il\sigma}\sum_{n=1}^{\infty}\left(\frac{2}{\gamma-|l|-2n}-\frac{2}{\gamma+|l|+2n}\right)\,. \tag{6.2.61}$$

The ω-integral is performed along Feynman's contour; assuming $\tau > 0$ the contour has to be closed on the upper half plane, so we pick the poles on the negative real axis:

$$\omega_n = -\frac{a}{a_0}\sqrt{(|l|+2n)^2 + \frac{b^2 l^2}{2a^2}}\,. \tag{6.2.62}$$

The correlator on the plane is found by transforming from the (τ, σ) coordinates to the $(z = e^{i(\tau+\sigma)}, \bar{z} = e^{i(\tau-\sigma)})$ coordinates and using (6.2.26). Dropping an irrelevant overall normalization one finds

$$\mathcal{C}^{\rm bos}(z,\bar{z}) = \partial\bar{\partial}\left(|z|\,\mathcal{C}^{\rm fer}(z,\bar{z})\right)\,, \tag{6.2.63}$$

with $\mathcal{C}^{\rm fer}(z,\bar{z}) = \mathcal{C}^{\rm fer}(\tau,\sigma)/|z|$, where the factor $1/|z|$ follows from the transformation of the operator in z, and

$$\mathcal{C}^{\rm fer}(\tau,\sigma) = \frac{a}{a_0}\sum_{l\in\mathbb{Z}} e^{il\sigma}\sum_{n=1}^{\infty}\frac{\exp\left[-i\frac{a}{a_0}\sqrt{(|l|+2n)^2+\frac{b^2l^2}{2a^2}}\,\tau\right]}{\sqrt{1+\frac{b^2}{2a^2}\frac{l^2}{(|l|+2n)^2}}}\,. \tag{6.2.64}$$

In our computation the fermionic correlator $\mathcal{C}^{\rm fer}(\tau,\sigma)$ is determined only up to terms that are annihilated by the derivatives in (6.2.60). We have chosen these ambiguous terms such that $\mathcal{C}^{\rm fer}(\tau,\sigma)$ agrees[3] up to terms of order $O(b^2)$ with the correlator computed in [15]. In order to verify that the $O(b^2)$ expansion of the $\mathcal{C}^{\rm bos}(z,\bar{z})$ and $\mathcal{C}^{\rm fer}(z,\bar{z})$ above agrees with the result obtained via the perturbative method in (6.2.49) and (6.2.66) one can start by expanding each term of the series for small b at fix a_0 up to order b^2

$$\mathcal{C}^{\rm fer}(\tau,\sigma) \sim \sum_{l\in\mathbb{Z}} e^{il\sigma}\sum_{n=1}^{\infty} e^{il\sigma}e^{i(|l|+2n)\tau}\left[1+\frac{b^2}{2a_0^2}\left(-\frac{1}{2}-\frac{l^2}{2(|l|+2n)^2}+\frac{2i\tau(|l|+n)n}{|l|+2n}\right)\right]\,. \tag{6.2.65}$$

The terms in the round parenthesis can be written as ratios of polynomials in the combinations l and $|l|+2n$ that appear in the exponentials. Then it is possible to reduce the sums over l and n in terms of derivative or integrals (with respect to τ and σ) of the geometric series. In particular, the presence in the denominator of a factor of $(|l|+2n)^2$ implies that we have to integrate twice with respect to τ. It is easy to see that the first integration yields logarithms and the second one dilogarithms, producing exactly the terms proportional to Li_2 in the $\hat{D}$ function present in (6.2.66). With some patience it is possible to check that also all other terms of (6.2.66) are reproduced by performing the sums for the remaining terms in (6.2.65).

Using the Ward identity (6.1.18) linking bosonic and fermionic correlators, one is lead to the following natural guess for the correlator with fermionic light operators

$$\mathcal{C}^{\rm fer}_{\mathcal{O}(b^2)}(z,\bar{z}) = \frac{1}{|z|}\left[\frac{1}{|1-z|^2}+\frac{b_1^2}{a_0^2}\frac{N}{2}+\sum_k\frac{b_k^2}{a_0^2}\left(-\frac{1}{2}\frac{1}{|1-z|^2}+\sum_{p=1}^{k}\frac{|z|^2\hat{D}_{pp22}}{\pi\,p}\right)\right]\,. \tag{6.2.66}$$

The term of order N is the disconnected contribution to the correlator, which cannot be predicted by the Ward identity since it is annihilated by the operator $\partial\bar{\partial}(|z|\cdot)$.

Specialising (6.2.66) to the heavy state considered in chapter 3, which has $b_1 = b \neq 0$ and $b_k = 0$ for $k > 1$, one can verify that the above result is in perfect agreement with

[3] Note that in (6.2.63) we have not included the disconnected contribution to the correlator; this contribution can be computed in the free theory and is given by the $O(N)$ term in (6.2.66) at all values of b^2/a_0^2.

(5.2.55) this checks that the Ward identity is satisfied for this particular heavy state, and provides a quite non-trivial validation of our computations.

6.3 Discussion

In this chapter we used the supergravity approximation of type IIB string theory to derive, via the AdS_3/CFT_2, the strong coupling expression for the HHLL correlators where the two light operators are the bosonic states and the heavy operators belong to the ensemble of RR ground states. At the orbifold point in the superconformal moduli space, it is straightforward to calculate these correlators in full generality. Of course in order to study the problem in a regime where weakly coupled AdS gravity is a valid approximation, one needs to deform the orbifold description and move to a region where the CFT is strongly coupled. Here we bypassed this challenging task by working directly with the supergravity description, and to make the computation feasible we restricted to the regime ($N_k^{(0)} \ll N_1^{(++)}$) where the states are close to the RR ground state with maximal R-charge. For a particular family of states (with $N_k^{(0)} = 0$ for $k \geq 2$) we were able to compute the correlator at strong coupling for all values of the R-charge (even if only in the form of a Fourier series), including the limit in which the R-charge becomes vanishingly small. To make contact between the gravity results ((6.2.49), (6.2.66) and (6.2.63), (6.2.64)) and the CFT point of view, we have to look at different OPE limits of the correlator.

For instance the leading terms of the z, $\bar z \to 1$ limit (corresponding to the OPE where the two light operators are close) do not receive contributions[4] from the $\hat D_{pp22}$ with $p > 1$. It is straightforward to check that, in this OPE limit, the singular terms obtained from the round parenthesis in (6.2.49) and (6.2.66) are

$$\left(-\frac{1}{2}\frac{1}{|1-z|^2} + \sum_{p=1}^{k} \frac{|z|^2 \hat D_{pp22}}{\pi\, p}\right) \sim -\frac{1}{4(1-z)} - \frac{1}{4(1-\bar z)} \tag{6.3.1}$$

and so do not contribute to the bosonic correlator (6.2.49). The two singular terms above capture the contributions to the fermionic correlator of the $SU(2)_R$ and $SU(2)_L$ currents. After substituting the result (6.3.1) in (6.2.66), we can easily extract the contribution due to the exchange of the $SU(2)_L$ current by focusing on the term proportional to $1/(1-\bar z)$

$$\mathcal{C}^{\text{fer}}_{\mathcal{O}(b^2)} \sim \frac{1}{1-\bar z}\left[\frac{1}{2} - \frac{1}{4}\sum_k \frac{b_k^2}{a_0^2}\right] = \frac{a^2}{2a_0^2}\,\frac{1}{1-\bar z}\ , \tag{6.3.2}$$

where in the last line we used (6.2.20). This provides a check of the relative normalization between the free contribution and the terms proportional to b_k^2: at order $1/(1-\bar z)$ the two combine to produce a result proportional to a^2 which is related to the number of strands with $j = 1/2$. This is the only type of strands in the state considered that can contribute to the exchange of the $SU(2)_L$ currents; in particular, the OPE (6.3.2) is saturated by the exchange of J^3 and, since the correlator factorizes into two protected three-point functions $\langle O_H \bar O_H J^3\rangle$ $\langle J^3 O_L \bar O_L\rangle$, it is straightforward to check also the overall normalization just by using the free theory result for the 3-point building blocks.

It is possible to extend the result above and focus on the leading term in the $(1-\bar z)$ expansion, but keep all corrections in $(1-z)$. In Minkowskian signature this corresponds

[4] It is easy to see this from (C.2.8) by rewriting $\partial_{|z_{12}|^2}$ in terms of ∂_z and $\partial_{\bar z}$ and checking that each Jacobian brings a factor of $|1-z|^2$.

to a light-cone OPE where $y \to t$. Also in this case, only the terms proportional to $\hat{D}_{1122}$ are relevant and we obtain

$$\mathcal{C}^{\rm bos}_{\mathcal{O}(b^2)} \sim \frac{1}{|1-z|^4}\left\{1-\sum_k \frac{b_k^2}{a_0^2}\left[1+\frac{1}{2}\frac{1+z}{1-z}\ln z\right]\right\} . \tag{6.3.3}$$

It is interesting to compare this result with the contribution of the (holomorphic) Virasoro block of the identity, but this has to be done with some care. While the heavy operators have conformal weight $h_H = \bar{h}_H = c/24$ (being RR ground state), it is convenient to factor out the contribution of the Sugawara part of the stress tensor that is due to the $SU(2)_L \times SU(2)_R$ R-currents. The reason for doing this is the following: it is possible to take linear combinations of a Virasoro descendant (such as $L_{-2}|0\rangle$) and an affine descendant constructed with the Sugawara stress-tensor (such as $L^{\rm Sug}_{-2}|0\rangle$) to construct a Virasoro primary (i.e. a state annihilated by L_n for $n > 0$). So, if we try to interpret the correlators (6.2.49) and (6.2.66) in terms of the full Virasoro blocks, primaries such as the ones mentioned above would appear as new "dynamical" contributions. However, their contributions is completely fixed by the symmetries of the theory and so it is more convenient to analyze the bulk results above in terms of the Virasoro blocks generated by $L^{[0]} = L - L^{\rm Sug}$ times the blocks generated by the R-symmetry currents. This approach is particularly apt for the bosonic correlator (6.2.49), since it is not constrained by the R-symmetry at all. By indicating with a superscript [0] all quantities after factoring out the Sugawara contributions, we have $h_L^{[0]} = \bar{h}_L^{[0]} = 1$ and[5]

$$h_H^{[0]} = \bar{h}_H^{[0]} = \frac{N}{4} - \frac{\langle J^2\rangle}{N} = \frac{N}{4}\left[1-\left(\frac{N_1^{(++)}}{N}\right)^2\right] , \tag{6.3.4}$$

where J^2 is the Casimir operator of the $SU(2)_L$ algebra and in our case, is sensitive just to the strands with $j, \bar{j} \neq 0$. Thus we should compare (6.3.3) with the contribution of the HHLL identity Virasoro block with the $h_H^{[0]}$ and $h_L^{[0]}$ above, and $c \sim 6N$ (since subtracting the Sugawara sector does not change the leading N contribution of the D1D5 CFT). By using the results of [39], we have that the leading term in $(1-\bar{z})$ expansion of the leading N contribution of such Virasoro block reads

$$\mathcal{C}^{\rm bos}_{\rm Id} \sim \frac{1}{(1-\bar{z})^2}\left[z^{\alpha-1}\left(\frac{\alpha}{1-z^\alpha}\right)^2\right] \sim \frac{1}{|1-z|^4}\left\{1-\sum_k \frac{b_k^2}{a_0^2}\left[1+\frac{1}{2}\frac{1+z}{1-z}\ln z\right]\right\} , \tag{6.3.5}$$

where in the second step we used

$$\alpha = \sqrt{1-\frac{24 h_H^{[0]}}{c}} = \frac{N_1^{(++)}}{N} = \frac{a^2}{a_0^2} = 1 - \sum_k \frac{b_k^2}{2a_0^2} \tag{6.3.6}$$

and took the approximation $b_k^2 \ll a_0^2$ up to the order b_k^2/a_0^2. This shows that the light-cone OPE (6.3.3) of the strong coupling correlator (6.2.49) is entirely saturated by the $L^{[0]}$ Virasoro descendants of the identity (6.3.5), at least in the $\mathcal{O}(b^2)$ approximation. Of course the full correlator away from the light-cone limit receives contributions from other $L^{[0]}$ Virasoro blocks. By expanding (6.2.49) for $z \to 1$ and $\bar{z} \to 1$ and comparing with

[5] To be precise, the heavy operators dual to the two-charge geometries are linear combinations of terms with different values of $h_H^{[0]}$ and $\bar{h}_H^{[0]}$ [32, 29]. It is possible to calculate the contribution of each term to the correlator as done for instance in [33] for the three-point functions, but the result at order b^2 coincides with that of the term with the average number of $j = \bar{j} = 1/2$ strands.

the same expansion of the (left times right) identity Virasoro block, one sees that the first primaries beyond the identity that appear in the OPE have conformal dimension $h = \bar{h} = 2$. As we argued in the introduction these primaries should be multi-particle operators.

In the case of the heavy state discussed in section 6.2.3, it is possible to show that light-cone OPE reproduces the $L^{[0]}$ identity Virasoro block even at finite values of b. Consider first the fermionic correlator in (6.2.64). The light-cone OPE is captured by the modes with $l \gg n$, so we can approximate each term in the series (6.2.64) by expanding the square roots and by neglecting all terms proportional to $1/l$; then, when z^α is not too close to 1, the leading contribution in the $\bar{z} \to 1$ limit is captured by

$$\mathcal{C}^{\rm fer}(\tau,\sigma) \sim \frac{a^2}{a_0^2} \sum_{l=0}^{\infty} e^{il\,(\sigma-\tau)} \sum_{n=1}^{\infty} e^{-2\,i\,\frac{a^2}{a_0^2} n\tau} = \alpha\, \frac{1}{1-\bar{z}}\, \frac{1}{1-|z|^{2\alpha}}\,. \tag{6.3.7}$$

By inserting this approximation in (6.2.63) we have

$$\mathcal{C}^{\rm bos}(z,\bar{z}) \sim \partial\bar{\partial} \left(\frac{1}{1-\bar{z}} \frac{\alpha}{1-|z|^{2\alpha}}\right) \sim \frac{1}{(1-\bar{z})^2}\, z^{\alpha-1} \left(\frac{\alpha}{1-z^\alpha}\right)^2 , \tag{6.3.8}$$

where we focused just on the leading contribution in the limit $\bar{z} \to 1$. As mentioned above, this result agrees with (6.3.5) even at finite values of b_1.

Chapter 7

Black hole correlators and late time behavior

7.1 Two-point function in BTZ

We saw in the previous chapter that tor finite b we were not able to resum the series in (6.2.64). However it is still possible to extract useful informations already from (6.2.64), and in particular one can analyze the behaviour of the correlator for large values of the Lorentzian time τ. The aim is to compare the late-time behaviour of the correlator in a pure heavy state with that of the correlator in the naive D1D5 geometry

$$ds^2 = \sqrt{Q_1 Q_5}\left[\frac{dr^2}{r^2} + \frac{r^2}{a_0^2}(-d\tau^2 + d\sigma^2)\right] , \tag{7.1.1}$$

which is the limit of the BTZ black hole when both the left and right temperatures are vanishing, and represents the dual of the statistical ensemble of the RR ground states. In particular, we want to apply our method to compute the two-point function the a black hole background given by the massless BTZ black hole, with two light operators given by bosonic operators whose dual field is described by a minimal couple scalar in the black hole background.

In analogy to the previous notations, we defined the correlators to compute

$$\mathcal{C}^{\text{bos}}_{\text{BTZ}}(z,\bar z) \equiv \langle O_L(1)\,\bar O_L(z,\bar z)\rangle_{\text{BTZ}} = \frac{1}{(1-z)^{2h_L}}\frac{1}{(1-\bar z)^{2\bar h_L}}\mathcal{G}^{\text{bos}}_{\text{BTZ}}(z,\bar z) \tag{7.1.2}$$

with

$$O_L \to O_{\text{bos}} = \sum_{r=1}^{N} \frac{\epsilon_{\dot A\dot B}}{\sqrt{2N}}\partial X^{1\dot A}_{(r)}\bar\partial X^{1\dot B}_{(r)} \;, \quad \bar O_L \to \bar O_{\text{bos}} = \sum_{r=1}^{N} \frac{\epsilon_{\dot A\dot B}}{\sqrt{2N}}\partial X^{2\dot A}_{(r)}\bar\partial X^{2\dot B}_{(r)} \;. \tag{7.1.3}$$

Since we found in the previous chapter a powerful tools to relate correlators with light bosonic operator with light fermionic operator we will also be able, thanks to supersymmetric WI (6.1.18) to recover the correlator

$$\mathcal{C}^{\text{fer}}_{\text{BTZ}}(z,\bar z) \equiv \langle O_L(1)\,\bar O_L(z,\bar z)\rangle_{\text{BTZ}} = \frac{1}{(1-z)^{2h_L}}\frac{1}{(1-\bar z)^{2\bar h_L}}\mathcal{G}^{\text{fer}}_{\text{BTZ}}(z,\bar z) \tag{7.1.4}$$

with

$$O_L \to O_{\text{fer}} = \sum_{r=1}^{N} \frac{-i\epsilon_{\dot A\dot B}}{\sqrt{2N}}\psi^{1\dot A}_{(r)}\,\tilde\psi^{\dot 1\dot B}_{(r)} \;, \quad \bar O_L \to \bar O_{\text{fer}} = \sum_{r=1}^{N} \frac{-i\epsilon_{\dot A\dot B}}{\sqrt{2N}}\psi^{2\dot A}_{(r)}\,\tilde\psi^{\dot 2\dot B}_{(r)} \;. \tag{7.1.5}$$

We start to find the easier correlator (7.1.2) by solving the linearized equation of motion for a scalar, whose normalizable mode will give us the two-point function. The minimally coupled scalar equation reads

$$\Box_3 B(\tau,\sigma,r) = 0 \tag{7.1.6}$$

where the $\Box_3$ is the laplacian computed with the metric (7.1.1). Fourier transforming

$$B(\tau,\sigma,\rho) = \frac{1}{(2\pi)^2}\sum_{l\in\mathbb{Z}}\int d\omega\, e^{\mathrm{i}\omega\tau} e^{\mathrm{i}l\sigma} g(l,\omega)\chi_{l,\omega}(r)\,, \tag{7.1.7}$$

we obtain the equation in r

$$\frac{1}{r}\partial_r(r^3\partial_r\chi_{l,\omega}) + \frac{\omega^2-l^2}{r^2}\,\chi_{l,\omega} = 0\,. \tag{7.1.8}$$

This can be recast in a *Modified Bessel equation* via the change

$$\chi_{l,\omega}(r) = \frac{\sqrt{l^2-\omega^2}}{r}\,\psi(r)\,,\quad x = \frac{\sqrt{l^2-\omega^2}}{r}\,, \tag{7.1.9}$$

becoming

$$x^2\psi'' + x\psi' - (x^2+1)\psi = 0 \tag{7.1.10}$$

that has solution

$$\psi(x) = c_1 I_1(x) + c_2 K_1(x)\,. \tag{7.1.11}$$

Going back in the r coordinate and f function

$$\chi_{l,\omega}(r) = \frac{\sqrt{l^2-\omega^2}}{r}\left[c_1 I_1\left(\frac{\sqrt{l^2-\omega^2}}{r}\right) + c_2 K_2\left(\frac{\sqrt{l^2-\omega^2}}{r}\right)\right]. \tag{7.1.12}$$

Here we have a geometry with naked singularity at $r=0$; this means that we cannot require regularity of the solution everywhere inside the bulk; we have to require the standard boundary conditions, purely ingoing wave at $r=0$ and purely outgoing wave at $r=\infty$. Using propoerties of Bessel funcitons it is immediate to see that we must kill the I_1 in order to have a well defined ingoing wave at $r=0$. At $r=\infty$ we have

$$\chi_{l,\omega}(r) \simeq 1 - (1-2\gamma_E)\,\frac{l^2-\omega^2}{4r^2} + \frac{l^2-\omega^2}{2r^2}\log\left(\frac{l^2-\omega^2}{2}\right) + \cdots\,. \tag{7.1.13}$$

In order to extract the two-point function we follow the method we used so far, thus by imposing a non-normalizable mode going as a delta function, imposing

$$g(l,\omega) = 1\,, \tag{7.1.14}$$

and this gives the normalizable mode

$$b(\tau,\sigma) = \frac{1}{(2\pi)^2}\sum_{l\in\mathbb{Z}}\int d\omega\, e^{\mathrm{i}\omega\tau} e^{\mathrm{i}l\sigma}\left\{(1-2\gamma_E)\,\frac{l^2-\omega^2}{4} - \frac{l^2-\omega^2}{2}\log\left(\frac{l^2-\omega^2}{2}\right)\right\} \tag{7.1.15}$$

The first piece is a pure contact term. We should also recall that $l^2-\omega^2 = i(\partial_\tau^2-\partial_\sigma^2)$ so

$$b(\tau,\sigma) = \frac{\mathrm{i}}{(2\pi)^2}(\partial_\tau^2-\partial_\sigma^2)\int\frac{d\omega}{2\pi}\sum_l e^{i(\omega\tau+l\sigma)}\left\{\frac{1}{2}(1-2\gamma_E) - \log\left(\frac{l^2-\omega^2}{2}\right)\right\}\,. \tag{7.1.16}$$

The first term is a contact term, and we can drop it. The second term is indeed interesting since it has no poles but a branch cut, and neglecting contact terms we have

$$b(\tau,\sigma) = \frac{\mathrm{i}}{(2\pi)^2}(\partial_\tau^2 - \partial_\sigma^2)\int \frac{d\omega}{2\pi}\sum_l e^{i(\omega\tau + l\sigma)}\log\left(l^2-\omega^2\right). \tag{7.1.17}$$

We now assume $\tau > 0$ so we can integrate over the upper half of the complex plane. The prescription is the Feynman prescription, so that only the branch cut on the negative real axis matter. To perform the computation we split in $l = 0$, $l > 1$ and $l < 1$.

For $l > 1$ we have[1]

$$\begin{aligned}
&\int \frac{d\omega}{2\pi}\sum_{l=1}^{\infty} e^{i(\omega\tau+l\sigma)}\log\left(l^2-\omega^2\right) = \sum_{l=1}^{\infty} e^{il\sigma}\left[\int\frac{d\omega}{2\pi}\,e^{i\omega\tau}\log(l-\omega) + \int \frac{d\omega}{2\pi}\,e^{i\omega\tau}\log(l+\omega)\right]\\
&\simeq \sum_{l=1}^{\infty} e^{il\sigma}\int\frac{d\omega}{2\pi}\,e^{i\omega\tau}\log(l-\omega) = -\sum_{l=1}^{\infty} e^{il(\tau+\sigma)}\frac{1}{2\pi}\left[-\frac{\pi}{|\tau|}+2\pi\gamma_E\delta(\tau)\right]\\
&\simeq -\frac{1}{2\tau}\frac{1}{1-e^{i(\sigma+\tau)}}
\end{aligned} \tag{7.1.20}$$

where we have neglected contact terms and dropped the $\log(\omega + l)$ whose branch cut is outside the region of integration.

When case $l < 1$, we simply shift the $l \in (-\infty, -1)$ to $l \in (1, \infty)$ and recast it as the integral done above

$$\begin{aligned}
&\int \frac{d\omega}{2\pi}\sum_{l=-\infty}^{-1} e^{i(\omega\tau+l\sigma)}\log\left(l^2-\omega^2\right) = \int \frac{d\omega}{2\pi}\sum_{l=1}^{\infty} e^{i(\omega\tau-l\sigma)}\log\left(l^2-\omega^2\right)\\
&\simeq -\frac{1}{2\tau}\frac{1}{1-e^{i(\tau-\sigma)}}
\end{aligned} \tag{7.1.21}$$

where, again, we have neglected contact terms and dropped the $\log(\omega + l)$ whose branch cut is outside the region of integration.

Finally in the case of $l = 0$, we obtain the result as the limit with $l \to 0$ of the above result in order to do not count twice the contribute in $l = 0$:

$$\lim_{\epsilon\to 0}\int\frac{d\omega}{2\pi}\sum_l e^{i\omega\tau}\log\left(\epsilon-\omega\right) \simeq \frac{1}{2\tau} \tag{7.1.22}$$

where we neglected contact terms.

Gathering all the pieces and summing over l we get

$$b(\tau,\sigma) = (\partial_\tau^2-\partial_\sigma^2)\left[\frac{1}{2i\tau}\left(\frac{1}{1-e^{i(\tau+\sigma)}}+\frac{1}{1-e^{i(\tau-\sigma)}}-1\right)\right] \tag{7.1.23}$$

[1] Recalling that

$$FT\left[\mathcal{P}\frac{1}{|x|}\right] = \int dx\, e^{i\omega x}\,\frac{1}{|x|} = -2\gamma_E - 2\log|\omega|\,. \tag{7.1.18}$$

$$\int d\omega\, e^{i\omega\tau}\log|\omega| = -\frac{\pi}{|\tau|} - 2\pi\gamma_E\delta(\tau)\,, \tag{7.1.19}$$

that gives the correlator, in (σ, τ) coordinates,

$$\mathcal{C}^{\text{bos}}_{\text{BTZ}} = ib(\tau,\sigma) = \frac{1}{4(\sigma_+ - \sigma_-)^2}\left[\frac{1}{\sin^2\frac{\sigma_+}{2}} + \frac{1}{\sin^2\frac{\sigma_-}{2}} - \frac{4\sin\frac{\sigma_+ - \sigma_-}{2}}{(\sigma_+ - \sigma_-)\sin\frac{\sigma_+}{2}\sin\frac{\sigma_-}{2}}\right] \tag{7.1.24}$$

$\sigma_\pm \equiv \sigma \pm \tau$. Since we wrote the result (7.1.23) in the suggestive form where it appears explicitly the derivatives $(\partial^2_\tau - \partial^2_\sigma)$ we can extract also the fermionic correlators from supersymmetric WI (6.1.18) after taking into account factors coming from the change of coordinates from cylinder to plane

$$\mathcal{C}^{\text{fer}}_{\text{BTZ}} = \frac{1}{2(\sigma_+ - \sigma_-)}\left[\cot\frac{\sigma_+}{2} - \cot\frac{\sigma_-}{2}\right] \tag{7.1.25}$$

7.2 Late time behavior

What we want to analyse now is the behaviour of the correlators computed at large Lorentzian time. Indeed, the large-time decay is a signal of information loss [9] and we would like to see if there or not a pattern of this behavior that differentiate the black hole from the microstate states.

From the correlator in (7.1.24) in the naive geometry we can extract the object

$$\mathcal{G}^{\text{bos}}_{\text{BTZ}}(\tau,\sigma) = \frac{1}{4(\sigma_+ - \sigma_-)^2}\left[\sin^2\frac{\sigma_+}{2} + \sin^2\frac{\sigma_-}{2} - \frac{4\sin\frac{\sigma_+ - \sigma_-}{2}\sin\frac{\sigma_+}{2}\sin\frac{\sigma_-}{2}}{(\sigma_+ - \sigma_-)}\right], \tag{7.2.1}$$

For large τ this correlator vanishes like

$$\mathcal{G}^{\text{bos}}_{\text{BTZ}}(\tau,\sigma) \sim \frac{1}{\tau^2}, \quad \mathcal{G}^{\text{fer}}_{\text{BTZ}}(\tau,\sigma) \sim \frac{1}{\tau} \tag{7.2.2}$$

This large-time decay is polynomial rather than exponential, because the naive geometry (7.1.1) is a degenerate zero-temperature limit of a regular finite-temperature black hole.

Let us now consider and recall the correlator (6.2.64) in the pure heavy state characterized by $b_k = b\delta_{k,1}$ studied in previous chapter

$$\mathcal{C}^{\text{fer}}(\tau,\sigma) = \frac{a}{a_0}\sum_{l\in\mathbb{Z}} e^{il\sigma}\sum_{n=1}^{\infty}\frac{\exp\left[-i\frac{a}{a_0}\sqrt{(|l|+2n)^2 + \frac{b^2 l^2}{2a^2}}\,\tau\right]}{\sqrt{1+\frac{b^2}{2a^2}\frac{l^2}{(|l|+2n)^2}}}\,. \tag{7.2.3}$$

In order to study the late time behavior it would be necessary to sum the series above in the limit in which the microstate gets closer to the black hole naive solution, namely, in the range of parameters when $a \ll b$. We can extract informations, by defining different regimes of the parameters controlling the the correlator, in which the sum is feasible and the late time behavior emerges. By defining the parameter

$$\eta^2 \equiv \frac{b_1^2}{2a^2}, \quad a_0^2 = a^2 + \frac{b_1^2}{2} \tag{7.2.4}$$

it is possible to rewrite the function (7.2.3) in the equivalent form

$$\mathcal{C}^{\text{fer}}(\tau,\sigma) = \sum_{l\in\mathbb{Z}} e^{il\sigma} \sum_{n=1}^{\infty} \mathcal{C}_{l,n} \tag{7.2.5}$$

with

$$\mathcal{C}_{l,n} = \frac{1}{\sqrt{1+\eta^2}}\, e^{il\sigma} \frac{\exp\left[-i\frac{|l|+2n}{\sqrt{1+\eta^2}}\sqrt{1+\eta^2\frac{l^2}{(|l|+2n)^2}}\,\tau\right]}{\sqrt{1+\eta^2\frac{l^2}{(|\lambda|+2n)^2}}} \tag{7.2.6}$$

We can split the sum into two regimes defined by

$$l\eta \gg 2n \;:\quad \mathcal{C}_{l,n} \simeq \frac{1}{\eta^2}\left(1+\frac{2n}{l}\right) e^{il(\sigma-\tau)}\,, \tag{7.2.7a}$$

$$1 \ll l\eta \ll 2n \;:\quad \mathcal{C}_{l,n} \simeq \frac{1}{\eta} e^{il(\sigma-\frac{\tau}{\eta})}\, e^{-2in\frac{\tau}{\eta}} \tag{7.2.7b}$$

Dividing the sum over n into a sum of those two disconnected pieces we get

$$\mathcal{C}^{\text{fer}}(\tau,\sigma) = \frac{1}{\eta}\frac{1}{1-e^{-2i\frac{\tau}{\eta}}}\left[\frac{1}{1-e^{i(\sigma-\tau)}} + \frac{1}{1-e^{-i(\sigma+\tau)}} - 1\right] \tag{7.2.8}$$

Taking the limit for which $\tau \ll \eta$ we can expand the factor $\frac{1}{\eta}\frac{1}{1-e^{-2i\frac{\tau}{\eta}}}$ obtaining

$$\frac{1}{\eta}\frac{1}{1-e^{-2i\frac{\tau}{\eta}}} \sim \frac{1}{2i\tau} \tag{7.2.9}$$

and therefore we recover the naive decaying behavior of the correlator in BTZ in (7.1.25) in suitable coordinates. Otherwise if we consider the Lorentzian time to be of order $\tau \sim \eta$ (or larger), we cannot get rid of the oscillating factor in front of the square parenthesis in (7.2.8), and the correlator doesn't decay even at large time. This is of course something different from the pattern shared by the probes in black hole background.

This behavior in the regime of microstate approximating the naive geometry can be also viewed from the Virasoro block analysis point of view. Indeed (7.2.3) implies that, for generic values of $\sigma = \sigma_0$, the correlator given above has the same singularities at $\tau_k = \sigma_0 + 2\pi k$ as the vacuum correlator. Indeed in this regime the leading contribution to the sum comes from the modes with $l \gg n$ and so, close to τ_k the fermionic and bosonic correlators are well approximated by (6.3.7) and (6.3.8), reproducing the identity Virasoro block. Then, as expected for a pure state, we have that $\mathcal{G}^{\text{bos}}_{b_1}$ or $\mathcal{G}^{\text{fer}}_{b_1}$ tend to a finite value when $\tau \to \tau_k$ for every k:

$$\mathcal{G}^{\text{fer}}_{b_1} \sim \alpha\frac{1-e^{2i\sigma_0}}{1-e^{2i\alpha\sigma_0}e^{2\pi i\alpha k}}\,, \quad \mathcal{G}^{\text{bos}}_{b_1} \sim \alpha^2 e^{2i\sigma_0(\alpha-1)}e^{2\pi i\alpha k}\left(\frac{1-e^{2i\sigma_0}}{1-e^{2i\alpha\sigma_0}e^{2\pi i\alpha k}}\right)^2\,. \tag{7.2.10}$$

with

$$\alpha = \frac{a^2}{a_0^2} = 1 - \frac{b_1^2}{2a_0^2} \tag{7.2.11}$$

This is in contrast with what happens in the case of the naive geometry (7.2.2) where $\mathcal{G}^{\text{bos}}_{\text{BTZ}}$ goes to zero at late times.

Chapter 8

Conclusions

Black holes have represented the best theoretical laboratory in which to test ideas about quantum gravity. String theory in particular has performed well in this direction, with an unparalleled richness of concepts and techniques that revealed to be useful to tackle one of the most difficult problems theoretical physics has ever had to face. On the other hand, black hole physics has been a constant theme within string theory for the last twenty years and has guided string theorists towards new discoveries, in an an enriching feedback loop of ideas. In particular, the description of black holes, the understanding of the origin of their thermodynamic properties and the resolution of the information paradox are perhaps the most important and ambitious topics on which a theory of quantum gravity must be tested. This becomes even more pressing if we think that these problems lead to theoretical inconsistencies between two theories, GR and quantum field theory, that are well established and incredibly successful experimentally, in their domains of validity. Fuzzball proposal is a genuine description of black holes, and it's well motivated from fundamental principles in string theory. It incorporates successfully the most important tools at our disposal: the supergravity black hole solutions are understood in terms of D-branes configuration and AdS/CFT is naturally implemented in the near-horizon limit, with a CFT description given by the D1D5 CFT. This last point allows to shed a light on the thermodynamics of black holes, as statistical field theory is rigorously established, as opposed to the thermodynamic description emerging in the gravity description. We finally seem to have an explicit form for classes of black hole microstates, both as CFT states and as bulk geometries.

One of the study one can carry out is the probing of these microstates. Indeed, four-point function with two light and two heavy operators (HHLL correlators) have recently been connected to the black hole information loss problem, seen from a dual CFT point of view.

In particular, information loss signals from correlators are encoded in their behavior in certain limits of the two-dimensional coordinates involved: from Lorentzian point of view information is lost if the correlators decay at large time. In Euclidean signature a connection was made between spurious singularities in Euclidean time appearing in the Virasoro block of the identity in the large c limit and how one expects the correlator to behave. HHLL four-point functions can be seen two-point functions computed in the background given by the heavy operators, and there must a qualitative differences between a thermal two-point function, corresponding holographically to probes moving in a black hole background, and a two-point function computed in a pure state. In a thermal two-point function Euclidean time is periodic, and in the CFT computation this creates infinitely many images of the OPE singularities. This is exactly what one gets from the Virasoro block of the identity as $c \to \infty$. The puzzle appears when we consider the same two-point function in a pure state: with pure states time is not periodic anymore and a CFT correlator in Euclidean signature should have only singularities corresponding to the different OPE limits that can be taken. If the correlator is saturated by the Virasoro block of the identity, then it develops infinitely many singularities in the $c \to \infty$ limit, as in the thermal case, which is not acceptable. Mechanisms to avoid this considering each Virasoro block separately have been suggested, referring to sub-leading corrections in the $1/c$ expansion and to non-perturbative features. Our results, on the contrary, moves in the other direction: having a specific CFT and dual supergravity theory to work with, we know exactly which Virasoro blocks appear in the correlator, and, with our choice of light and heavy operators, we observe that spurious singularities cancel out among the different blocks even in the large c limit. This results also in the large time behavior of the correlators that turn out to have a very different decay from the thermal correlators.

Even though fifteen years of research have brought much progress towards the explanation of black hole physics through the fuzzball proposal, a lot remains to be done. The known classes of microstates do not account for the totality of the entropy of three-charge black hole, and an active branch of research consists in finding bulk microstate geometries and their dual CFT states. Four-point function analysis on these three-charge microstates has to be done and it would be another important piece in understanding the microscopic structure of the black holes.

Appendix A

2D CFT tools

In this appendix we give the basic ingredients of a 2D CFT focusing in particular on the tools we mostly use to extract the results we discuss in the body of the thesis. We start with generalities and we will move to correlators and to the techniques developed to study them. For the most part of the following discussion we refer to [37] adapting conventions for our purposes and introducing definitions and notations we adopt.

A.1 Generalities

A 2D Conformal Field Theory possess an infinite dimensional symmetry algebra, generating infinitesimal conformal transformations in an Euclidean two-dimensional space and it is given by two copies of Virasoro algebra

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\
&[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12}n(n^2-1)\delta_{n+m,0} \\
&[\bar{L}_n, L_m] = 0
\end{aligned} \tag{A.1.1}$$

whose generators are the Laurent expansion modes of the stress energy tensors defined on a complex plane $(z, \bar{z})$

$$\begin{aligned}
T(z) &= \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z) \\
\bar{T}(\bar{z}) &= \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}, \quad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{n+1} \bar{T}(\bar{z})
\end{aligned} \tag{A.1.2}$$

From now on we concentrate only on the holomorphic (Left) sector with straightforward generalization on the antiholomorphic (Right) sector. The algebra (A.1.1) can be extracted from (A.1.2) by using the Operator Product Expansion (OPE)

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \text{Reg} \tag{A.1.3}$$

The subalgebra $\{L_0, L_{\pm 1}\}$ is called *global* or *anomaly free* algebra and it generates conformal transformations defined globally.

We define a *Virasoro primary* state $|h\rangle$ of conformal dimension h as

$$\begin{aligned}
&L_n|h\rangle = 0, \quad \forall n > 0 \\
&L_0|h\rangle = h|h\rangle
\end{aligned} \tag{A.1.4}$$

The Hilbert space of the theory $\mathcal{H}$ is a direct sum of Verma moduli $\mathcal{H}_h$

$$\mathcal{H} = \oplus_h \mathcal{H}_h \tag{A.1.5}$$

where each module $\mathcal{H}_h$ is the subspace obtained by acting on $|h\rangle$ with the Virasoro generators with $n < 0$. The generic state obtained n this way is called *Virasoro descendants* of $|h\rangle$. Since in a CFT we have a state/operator correspondence is useful to think about primary operators O_h defined as

$$|h\rangle = \lim_{z\to 0} O_h(z)|0\rangle \tag{A.1.6}$$

with $|0\rangle$ being the vacuum of the theory. Primary operators transforms under conformal transformation

$$z \to w(z) \tag{A.1.7}$$

with the following rule

$$O_h(z) \to O'_h(w) = \left(\frac{dw}{dz}\right)^{-h} O_h(z) \tag{A.1.8}$$

and the stress energy tensor transform as

$$T(w) = \left(\frac{dw}{dz}\right)^{-2} T(z) + \frac{c}{12} S(z,w) \tag{A.1.9}$$

where $S(z,w)$ is the Schwarzian and is defined as

$$S(w,z) \equiv -2\left(\frac{dw}{dz}\right)^{1/2} \frac{d^2}{dz^2}\left[\left(\frac{dw}{dz}\right)^{-1/2}\right] \tag{A.1.10}$$

The OPE between a primary operator and the stress energy tensor is given by

$$T(z)O_h(w) = \frac{h}{(z-w)^2} O_h(w) + \frac{\partial_w O_h(w)}{(z-w)} + \mathrm{Reg} \tag{A.1.11}$$

that gives the action of the Virasoro generators

$$[L_n, O_h(z)] = h(n+1)z^n O_h(z) + z^{n+1}\partial O_h(z) \tag{A.1.12}$$

A.2 Correlators

Correlators in a CFT are constrained by conformal symmetry. In particular the two-point function of two primaries is fixed and given by

$$\langle O_{h_i}(z_i) O_{h_j}(z_j)\rangle = \frac{\delta_{h_i,h_j}}{(z_{ij})^{2h_i}}, \quad z_{ij} = z_i - z_j \tag{A.2.1}$$

Three-point function reads

$$\langle O_{h_i}(z_i) O_{h_j}(z_j) O_{h_k}(z_k)\rangle = \frac{C_{h_i h_j h_k}}{(z_{ij})^{h_i+h_j-h_k}(z_{ik})^{h_i+h_k-h_j}(z_{jk})^{h_j+h_k-h_i}} \tag{A.2.2}$$

with $C_{h_i h_j h_k}$ being the structure constants.

Four-point function is less contrained and can be written as

$$\langle O_{h_i}(z_i)O_{h_j}(z_j)O_{h_k}(z_k)O_{h_l}(z_l)\rangle = \frac{1}{z_{ij}^{h_i+h_j}z_{kl}^{h_k+h_l}}\mathcal{G}(z), \quad z = \frac{z_{il}z_{jk}}{z_{ik}z_{jl}} \tag{A.2.3}$$

Since the cross ratio z is invariant under global transformations we shall perform such a transformation in order to set

$$z_i = \infty, \quad z_2 = 1, \quad z_3 = z, \quad z_4 = 0 \tag{A.2.4}$$

and the above four-point function may be related to a matrix element between two asymptotic states of a two field product

$$\mathcal{C}(z) = \langle O_{h_1}(\infty)O_{h_2}(1)O_{h_3}(z)O_{h_4}(0)\rangle = \langle O_{h_i}|O_{h_j}(1)O_{h_k}(z)|O_{h_l}\rangle = \frac{1}{(1-z)^{h_i+h_j}}\mathcal{G}(z) \tag{A.2.5}$$

where we used

$$\langle O_{h_i}| = \lim_{z_i\to\infty} z_i^{2h_i}\langle 0|O_{h_i}(z_i), \quad |O_{h_l}\rangle = O_{h_l}(0)|0\rangle \tag{A.2.6}$$

Higher-point functions are less constrained and they satisfy the conformal Ward Identity (WI)

$$\langle T(z)O_{h_1}(z_1)\cdots O_{h_n}(z_n)\rangle = \sum_{i=1}^{n}\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_{z_i}\right)\langle O_{h_1}(z_1)\cdots O_{h_n}(z_n)\rangle \tag{A.2.7}$$

A.3 Conformal block decomposition

Let's consider conformal blocks approach to four-point functions. We introduce a resolution of the identity written as a sum over the projectors of all the conformal families, and given by

$$\begin{aligned}\mathbb{I} &= \sum_h \sum_{\{m_i,k_i\},\{m'_i,k'_i\}} L_{-m_1}^{k_1}\cdots L_{-m_n}^{k_n}|h\rangle\mathcal{N}^{-1}_{\{m_i,k_i\},\{m'_i,k'_i\}}\langle h|L_{m'_s}^{k'_s}\cdots L_{m'_1}^{k'_1} \\ &\equiv \sum_h P_h\end{aligned} \tag{A.3.1}$$

with

$$\mathcal{N}_{\{m_i,k_i\},\{m'_i,k'_i\}} = \langle h|L_{m'_s}^{k'_s}\cdots L_{m'_1}^{k'_1}L_{-m_1}^{k_1}\cdots L_{-m_n}^{k_n}|h\rangle \tag{A.3.2}$$

Inserting the identity into the four point function we have

$$\begin{aligned}\mathcal{C}(z) &= \sum_h \langle O_{h_1}(\infty)O_{h_2}(1)P_hO_{h_3}(z)O_{h_4}(0)\rangle \\ &\equiv \sum_h C_{h_1h_2h}C_{hh_3h_4}\mathcal{V}_V(z,c,h,h_i)\end{aligned} \tag{A.3.3}$$

where $\mathcal{V}_V(z,c,h,h_i)$ is the Virasoro block containing all the information about the conformal family of the primary of weight h and $C_{h_1h_2h}$ and $C_{hh_3h_4}$ are the structure constant defined by means expression (A.2.2). In general the block $\mathcal{V}_V(z,c,h,h_i)$ will be a holomorphic function of z and it has a dependence on the central charge c. Since we work in the large c limit it useful to write the block as a power series

$$\mathcal{V}_V(z,c,h,h_i) = \sum_n \frac{1}{c^n}\mathcal{V}_V^{(n)}(z,h,h_i) \tag{A.3.4}$$

Here we only concentrate on the leading term of the series $\mathcal{V}_V^{(0)}(z,h,h_i)$. To compute the explicit form of the leading block we have to give some definitions about the primaries we are considering in the four-point function, to differentiate the various case. We define a *light* operator O_L a primary with conformal dimension of order $O(1)$, while we call *heavy* operator O_H a primary with weight of order $O(c)$. The four point functions we consider in this work contain two heavy and two light operators and they are schematically indicated by HHLL correlator. In order to find the leading conformal block for the HHLL function we start with the simplest case, namely the LLLL four-point function.

A.3.1 Conformal blocks for LLLL

In the case of a four-point function made of four light operators, the leading Virasoro block can be extracted by considering only the following contribution

$$\sum_h \langle O_{L_1}(\infty)O_{L_2}(1)\left(\sum_{k=0}^{\infty}\frac{L_{-1}^k|h\rangle\langle h|L_1^k}{\langle h|L_1^kL_{-1}^k|h\rangle}\right)O_{L_3}(z)O_{L_4}(0)\rangle \tag{A.3.5}$$

since the only part bringing powers of the central charge c is contained in the denominator, and, using the algebra, it is easy to see that only norms of states of the form $L_{-1}^k|h\rangle$ are of order $O(1)$. Therefore the block is given by

$$\mathcal{V}_V^{(0)}(z,h,h_i)=\frac{1}{C_{L_1L_2h}C_{hL_3L_4}}\langle O_{L_1}(\infty)O_{L_2}(1)\left(\sum_{k=0}^{\infty}\frac{L_{-1}^k|h\rangle\langle h|L_1^k}{\langle h|L_1^kL_{-1}^k|h\rangle}\right)O_{L_3}(z)O_{L_4}(0)\rangle \tag{A.3.6}$$

In order to compute the block we need to compute these three objects

$$\langle O_{L_1}(\infty)O_{L_2}(1)L_{-1}^k|h\rangle,\quad \langle h|L_1^kO_{L_3}(z)O_{L_4}(0)\rangle,\quad \langle h|L_1^kL_{-1}^k|h\rangle \tag{A.3.7}$$

The first one can be computed by considering the differential representation of the operator L_{-1} given by the WI in (A.2.7), and by the mode definition (A.1.2) that gives

$$\langle O_{L_1}(\infty)O_{L_2}(1)L_{-1}^k|h\rangle=\lim_{z\to0}\partial_z^k\langle O_{L_1}(\infty)O_{L_2}(1)O_h(z)\rangle=C_{L_1L_2h}(h-h_{L_1}+h_{L_2})_k \tag{A.3.8}$$

where the Pochammer symbol is defined as follow

$$(k)_n=\begin{cases}1, & \text{if } n=0\\ k(k+1)\cdots(k+n-1), & \text{if } n>0\end{cases} \tag{A.3.9}$$

The second building block can be computed by using the action in (A.1.12) iteratively and it gives the result

$$\langle h|L_1^kO_{L_3}(z)O_{L_4}(0)\rangle=C_{hL_3L_4}z^{h-h_3-h_4+k}\Big(h+h_{L_3}-h_{L_4}\Big)_k \tag{A.3.10}$$

The last piece is given by the norm of the states in the projector. In order to compute these object it's useful to consider a representation of L_{-1},L_0,L_1 in terms of $SL(2,\mathbb{C})$ matrices acting on two-component vectors of which we only consider the ratio between the components (so that the matrices act *projectively*).

L_{-1} generates infinitesimal translations, so the action of $e^{\alpha L_{-1}}$ is the finite translation $z \to z + \alpha$. This can be seen as acting on a vector $\begin{pmatrix} z \\ 1 \end{pmatrix}$ as the matrix

$$e^{\alpha L_{-1}} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}. \tag{A.3.11}$$

Notice that, in this case, the fact that the action is projective is trivial, since the second component remains 1.
In an analogous way, L_0 generates infinitesimal dilatations, so $e^{\gamma L_0}$ generates finite dilatations $z \to e^{\gamma} z$. The representation is

$$e^{\gamma L_0} = \begin{bmatrix} e^{\gamma/2} & 0 \\ 0 & e^{-\gamma/2} \end{bmatrix}, \tag{A.3.12}$$

Finally, L_1 generates infinitesimal special conformal transformations and thus $e^{\beta L_1}$ generate finite ones,

$$z \to z' = \frac{az + b}{cz + d}. \tag{A.3.13}$$

However, we can restrict to special conformal transformations of the form

$$z \to z' = \frac{z}{1 - \beta z} \tag{A.3.14}$$

and represent

$$e^{\beta L_1} = \begin{bmatrix} 1 & 0 \\ -\beta & 1 \end{bmatrix}. \tag{A.3.15}$$

In order to have a simpler matrix form for dilatations, we can also use $\log(\gamma)$ as the parameter and put

$$e^{\log(\gamma) L_0} = \begin{bmatrix} \sqrt{\gamma} & 0 \\ 0 & 1/\sqrt{\gamma} \end{bmatrix}. \tag{A.3.16}$$

In order to compute the norm it's useful to ask whether exist constants c_1, c_2, c_3 such that a special conformal transformation followed by a translation can be put in the form

$$e^{\beta L_1} e^{\alpha L_{-1}} = e^{c_1 L_{-1}} e^{\log(c_2) L_0} e^{c_3 L_1}. \tag{A.3.17}$$

When taking the expectation value of the expression above on a primary state $|h\rangle$ we get

$$\begin{aligned} \langle h| e^{\beta L_1} e^{\alpha L_{-1}} |h\rangle &= \langle h| e^{c_1 L_{-1}} e^{\log(c_2) L_0} e^{c_3 L_1} |h\rangle \\ &= \langle h| e^{\log(c_2) L_0} |h\rangle \\ &= e^{h \log(c_2)} \\ &= c_2^h, \end{aligned} \tag{A.3.18}$$

because L_1 annihilates primary states acting from the left and L_{-1} does it acting from the right.

Using the representations above for the global conformal transformations, (A.3.17) becomes

$$\begin{bmatrix} 1 & \alpha \\ -\beta & 1 - \alpha\beta \end{bmatrix} = \begin{bmatrix} \sqrt{c_2} - \frac{c_1 c_3}{\sqrt{c_2}} & \frac{c_1}{\sqrt{c_2}} \\ -\frac{c_3}{\sqrt{c_2}} & \frac{1}{\sqrt{c_2}} \end{bmatrix}, \tag{A.3.19}$$

which gives

$$c_1 = \frac{\alpha}{1-\alpha\beta}, \qquad c_2 = \frac{1}{(1-\alpha\beta)^2}, \qquad c_3 = \frac{\beta}{1-\alpha\beta}. \tag{A.3.20}$$

Armed with this result, we can rewrite (A.3.18) and expand it in powers of α and β,

$$\begin{aligned}\langle h|e^{\beta L_1}e^{\alpha L_{-1}}|h\rangle &= \sum_{m,n=0}^{+\infty} \frac{\beta^m\alpha^n}{m!n!}\langle h|L_1^m L_{-1}^n|h\rangle \\ &= \Big(1-\alpha\beta\Big)^{-2h} \\ &= \sum_p \frac{\Big(\alpha\beta\Big)^p}{p!}\Big(2h\Big)_p.\end{aligned} \tag{A.3.21}$$

The coefficient of $(\alpha\beta)^k$ give what we were looking for

$$\langle h|L_1^k L_{-1}^k|h\rangle = k!\,\Big(2h\Big)_k \tag{A.3.22}$$

Putting everything together we get the Virasoro block

$$\mathcal{V}_V^{(0)}(z,h,h_i) = \sum_{k=0}^{+\infty} \frac{(h-h_{L_1}+h_{L_2})_k\Big(h+h_{L_3}-h_{L_4}\Big)_k z^{h-h_{L_3}-h_{L_4}+k}}{k!\,\Big(2h\Big)_k} \tag{A.3.23}$$

$$= z^{h-h_{L_3}-h_{L_4}}\ {}_2F_1\Big(h-h_{L_1}+h_{L_2}, h+h_{L_3}-h_{L_4}, 2h; z\Big), \tag{A.3.24}$$

A.3.2 Conformal blocks for HHLL

When considering the four-point function with two heavy and two light operators of the form

$$\langle O_{H_1}(\infty)O_{H_2}(1)O_{L_1}(z)O_{L_2}(0)\rangle \tag{A.3.25}$$

we should, in principle, insert the same projector as in the LLLL case to obtain the leading contribution to the conformal block. The difference now is that we also have a contribution of powers of c in the numerator, coming from the first piece of (A.3.7), since the commutator between the Virasoro modes and the part containing the two heavy operators gives a term proportional to the conformal dimension of order $O(c)$. To avoid this, the authors of [39] proposed to perform a conformal transformation that maps the complex plane into a curved background and where the OPE of the stress energy tensor with the heavy operators doesn't contain the dependence on h_H.

Concretely, let's consider (A.2.7)

$$\begin{aligned}\langle O_{H_1}(z_1)O_{H_2}(z_2)T(z)O_h(z_3)\rangle = \Bigg(&\frac{h_{H_1}}{(z-z_1)^2} + \frac{1}{z-z_1}\partial_{z_1} + \frac{h_{H_2}}{(z-z_2)^2} + \frac{1}{z-z_2}\partial_{z_2} \\ &+\frac{h}{(z-z_3)^2} + \frac{1}{z-z_3}\partial_{z_3}\Bigg)\langle O_{H_1}(z_1)O_{H_2}(z_2)O_h(z_3)\rangle\end{aligned} \tag{A.3.26}$$

with the three-point function given by (A.2.2). Let's now perform a conformal transformation

$$z_i \to w_i(z_i) \tag{A.3.27}$$

and thanks to (A.1.9) and (A.1.10), we have

$$\langle O_{H_1}(w_1)O_{H_2}(w_2)T(w)O_h(w_3)\rangle = \left(\frac{dw_1}{dz_1}\right)^{-h_{H_1}} \left(\frac{dw_2}{dz_2}\right)^{-h_{H_2}} \left(\frac{dw}{dz}\right)^{-2} \left(\frac{dw_3}{dz_3}\right)^{-h} \times\left[\langle O_{H_1}(z_1)O_{H_2}(z_2)T(z)O_h(z_3)\rangle - \frac{c}{12}S(w,z)\langle O_{H_1}(z_1)O_{H_2}(z_2)O_h(z_3)\rangle\right] \tag{A.3.28}$$

Taking now the limit in z coordinates

$$z_1 = \infty, \quad z_2 = 1, \quad z_3 = 0 \tag{A.3.29}$$

we need to cancel the term depending on the heavy weights in the square parenthesis of (A.3.28) and so we have to impose[1]

$$\frac{h_H}{(1-z)^2} - \frac{c}{12}S(w,z) = 0 \tag{A.3.30}$$

If we take the change of variables defined as

$$(1-w) = (1-z)^{\alpha} \tag{A.3.31}$$

the condition (A.3.30) reduces to

$$\alpha = \sqrt{1 - \frac{24h_H}{c}} \tag{A.3.32}$$

For $\alpha = 0$ ($h = \frac{c}{24}$) the solution has a different functional form and the correct coordinate transormation reads

$$w = \log(1-z) \tag{A.3.33}$$

In summary, in z coordinates the key object reads

$$\langle O_{H_1}(\infty)O_{H_2}(1)T(z)O_h(0)\rangle = C_{HHh}\left(\frac{h_H}{(1-z)^2} + \frac{h}{(1-z)z^2}\right) \tag{A.3.34}$$

and contains an explicit dependence on the heavy conformal weight, while in the w coordinates we have

$$\langle O_{H_1}(\infty)O_{H_2}(1)T(w)O_h(0)\rangle = C_{HHh}\left(h\frac{1-z(w)}{z^2(w)}\right) \tag{A.3.35}$$

with no dependence on h_H. As explained in [39], we can expand $T(w)$ in the new coordinates w

$$T(w) = \sum_n w^{-2-n}\mathcal{L}_n \tag{A.3.36}$$

The Virasoro algebra can be derived entirely from the singular terms in the $T(z)T(0)$ OPE, and these terms are preserved by conformal transformations. An important consequence is that the new generators $\mathcal{L}_n$ still satisfy the usual Virasoro algebra and the relation (A.1.12). The $\mathcal{L}_n$ are a complete basis of Virasoro generators so one can write L_n as a linear combination of the $\mathcal{L}_n$.

[1] We define $h_{H,L} = \frac{h_{H_1,L_1}+h_{H_2,L_2}}{2}$ and for future purpose we also define $\delta h_{H,L} = \frac{h_{H_1,L_1}-h_{H_2,L_2}}{2}$.

We start the computation for the leading block for HHLL with $\alpha \neq 0$. The Virasoro block in w coordinates then reduces to

$$\mathcal{V}_V^{(0)}(w,h,h_i) = \frac{1}{C_{H_1H_2h}C_{hL_1L_2}} \langle O_{H_1}(\infty) O_{H_2}(1) \left(\sum_{k=0}^{\infty} \frac{\mathcal{L}_{-1}^k |h\rangle\langle h| \mathcal{L}_1^k}{\langle h|\mathcal{L}_1^k \mathcal{L}_{-1}^k|h\rangle} \right) O_{L_1}(w) O_{L_2}(0)\rangle \tag{A.3.37}$$

Thanks to the fact that the Virasoro generators satisfy the same algebra, two of the three elements in $\mathcal{V}_h(w)$ are the same as LLLL case, just in the new coordinates,

$$\langle h|\mathcal{L}_1^k O_{L_1}(w) O_{L_2}(0)\rangle = C_{hL_1L_2} w^{h-h_{L_1}-h_{L_2}+k} \Big(h + h_{L_1} - h_{L_2}\Big)_k \tag{A.3.38a}$$

$$\langle h|\mathcal{L}_{-1}^k \mathcal{L}_1^k|h\rangle = k!\, \Big(2h\Big)_k \tag{A.3.38b}$$

The only new thing we need to compute is

$$\langle O_{H_1}(\infty) O_{H_2}(1) \mathcal{L}_{-1}^k|h\rangle = \lim_{w\to 0} \partial_w^k \langle O_{H_1}(\infty) O_{H_2}(1) O_h(w)\rangle \tag{A.3.39}$$

The only difference with respect the LLLL case is that in the curved background the three-point function doesn't have the usual form imposed by conformal invariance: we need to express this in term of the three-point function in the z coordinates, where we know its form, and then take the derivatives

$$\begin{aligned}\langle O_{H_1}(\infty) O_{H_2}(1) O_h(w)\rangle &= \left(\frac{\partial w}{\partial z}\right)^{-h} \langle O_{H_1}(\infty) O_{H_2}(1) O_h(z)\rangle \\ &= \Big[\alpha(1-z)^{\alpha-1}\Big]^{-h} C_{H_1H_2h}\ (1-z)^{h_{H_1}-h_{H_2}-h} \\ &= \alpha^{-h} C_{H_1H_2h}\ (1-w)^{-h+\delta_H/\alpha}\end{aligned} \tag{A.3.40}$$

we get

$$\begin{aligned}\langle O_{H_1}(\infty) O_{H_2}(1) \mathcal{L}_{-1}^k|h\rangle &= \alpha^{-h} C_{H_1H_2h}\ \lim_{w\to 0} \partial_w^k (1-w)^{-h+\delta_H/\alpha} \\ &= \alpha^{-h} C_{H_1H_2h} \left(h - \frac{\delta_H}{\alpha}\right)_k\end{aligned} \tag{A.3.41}$$

The full conformal block then reads

$$\begin{aligned}\mathcal{V}_V^{(0)}(w,h,h_i) &= \alpha^{-h} w^{h-h_{L_1}-h_{L_2}} \sum_{k=0}^{+\infty} \frac{\Big(h + h_{L_1} - h_{L_2}\Big)_k \Big(h - \delta_H/\alpha\Big)_k\ w^k}{k!\Big(2h\Big)_k} \\ &= \alpha^{-h} w^{h-h_{L_1}-h_{L_2}}\ {}_2F_1\Big(h - \delta_H/\alpha, h + \delta_L, 2h; w\Big)\end{aligned} \tag{A.3.42}$$

In order to obtain the conformal block in the z coordinates, we use the conformal transformation (A.3.31) backwards, always including the Jacobians for the light operators,

$$
\begin{aligned}
\mathcal{V}_V^{(0)}(z,h,h_i) &= \lim_{\substack{w_1\to w\\ w_2\to 0}} \left(\frac{\partial w_1}{\partial z_1}\right)^{h_{L_1}} \left(\frac{\partial w_2}{\partial z_2}\right)^{h_{L_2}} \langle O_{H_1}(\infty)O_{H_2}(1)\Big[\mathcal{P}_h\Big]O_{L_1}(w_1)\mathcal{O}_{L_2}(w_2)\rangle \\
&= \lim_{\substack{z_1\to z\\ z_2\to 0}} \Big[\alpha(1-z_1)^{\alpha-1}\Big]^{h_{L_1}}\Big[\alpha(1-z_2)^{\alpha-1}\Big]^{h_{L_2}}\alpha^{-h}(w_1(z_1))^{h-h_{L_1}-h_{L_2}}\times \\
&\quad\times\ {}_2F_1\Big(h-\delta_H/\alpha, h+\delta_L, 2h; w_1(z_1)\Big) \\
&= \alpha^{-h+h_{L_1}+h_{L_2}}(1-w(z))^{h_{L_1}(1-1/\alpha)}(w(z))^{h-h_{L_1}-h_{L_2}}\times \\
&\quad\times\ {}_2F_1\Big(h-\delta_H/\alpha, h+\delta_L, 2h; w(z)\Big)
\end{aligned}
\tag{A.3.43}
$$

Result for $\alpha = 0$ follows straightforwardly from above computation with the difference that now we have to use the transformation (A.3.33) and the result for the leading block reads

$$
\mathcal{V}_V^{(0)}(z,h,h_i) = (1-z)^{-h_{L_1}}\Big(\log(1-z)\Big)^{h-h_{L_1}-h_{L_2}}\ {}_2F_1\Big(h+\delta_L, 2h; \log(1-z)\Big) \tag{A.3.44}
$$

A.4 Affine block decomposition

In the case where the theory contains additional conserved currents, the symmetry algebra is enlarged beyond the Virasoro algebra, and in the case of an additional $SU(2)_k$ conserved current the algebra is enhanced to a Kac-Moody algebra:

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \\
\Big[J^a_m, J^b_n\Big] &= km\,\delta^{ab}\delta_{m+n,0} + i\epsilon^{ab}_c J^c_{m+n} \\
[L_m, J^a_n] &= -nJ^a_{m+n}
\end{aligned}
\tag{A.4.1}
$$

where k is the *level* of the affine current. It is convenient to choose another basis for the generators introuducing the Sugawara stress energy tensor T_{sug}

$$
\begin{aligned}
T_{\text{sug}}(z) &= \sum_n z^{-n-2}L^{\text{sug}}_n \\
L^{\text{sug}}_n &= \frac{1}{2k}\left(\sum_{m\geq -1} J^a_m J^a_{n-m} + \sum_{m\leq 0} J^a_{n-m}J^a_m\right)
\end{aligned}
\tag{A.4.2}
$$

Defining

$$
L^{(0)}_n \equiv L_n - L^{\text{sug}}_n \tag{A.4.3}
$$

it can be shown that $L^{(0)}_n$ and J^a_n generators provide a basis that factors the starting algebra into separate Virasoro and $SU(2)_k$ sector:

$$
\begin{aligned}
\Big[L^{(0)}_n, L^{(0)}_m\Big] &= (n-m)L^{(0)}_{n+m} + \frac{c-1}{12}n(n^2-1)\delta_{n+m,0}, \\
\Big[J^a_m, J^b_n\Big] &= km\,\delta^{ab}\delta_{m+n,0} + i\epsilon^{ab}_c J^c_{m+n} \\
\Big[L^{(0)}_n, J^a_m\Big] &= 0
\end{aligned}
\tag{A.4.4}
$$

In this basis the Hilbert space is still given by a sum of moduli each of them is generated now by an *affine primary*, defined as

$$\begin{aligned} L_n\,, J_n^a|h\rangle &= 0, \quad \forall n>0,\, \forall a \\ J_0^3|h\rangle = q_h|h\rangle, \quad L_0|h\rangle &= h|h\rangle \end{aligned} \tag{A.4.5}$$

Primaries with respect to the L_n's and J_n's with weight h and charge q_h, are primaries under $L_n^{(0)}$ as well, with conformal weight

$$h^{(0)} = h - \frac{q_h^2}{2k} \tag{A.4.6}$$

In the conformal block decomposition, all descendants of the exchanged state $|h\rangle$ can be written in basis (A.4.4) and thanks to the factorization of the algebra we expect a factorization between the Virasoro block generated by $L_n^{(0)}$ and the affine block generated by J_n^a. Focusing only on the $U(1) \in SU(2)_k$ subgroup of the entire Kac-Moody algebra, and following [39], the factorized conformal blocks read

$$\mathcal{V}_{V+A}(c,z,h,h_i) = \mathcal{V}_V(c-1,z,h^{(0)},h_i^{(0)})\mathcal{V}_A(z,q_{h_i}) \tag{A.4.7}$$

Working in the large c limit we can still consider the leading Virasoro block and following the previous discussion we can write

$$\mathcal{V}_{V+A}^{(0)}(z,h,h_i) = \mathcal{V}_V^{(0)}(z,h^{(0)},h_i^{(0)})\mathcal{V}_A(z,q_{h_i}) \tag{A.4.8}$$

with

$$\mathcal{V}_A(z,q_{h_i}) = z^{-q_L^2}(1-z)^{q_H q_L} \tag{A.4.9}$$

and $\mathcal{V}_V^{(0)}(z,h^{(0)},h_i^{(0)})$ the blocks computed in (A.3.42) or (A.3.44), and q_L, q_H are the $U(1)$ charges of the light and heavy operators.

Appendix B

Supergravity tools

B.1 Type IIA/B Supergravity

The equations of motion for Type IIA reads

$$e^{-2\Phi}\left(R_{MN}+2\nabla_M\nabla_N\Phi-\frac{1}{4}H^{(3)}_{MPQ}H^{(3)PQ}_N\right)-\frac{1}{2}F^{(2)}_{MP}F^{(2)P}_N-\frac{1}{12}F^{(4)}_{MPQR}F^{(4)PQR}_N$$
$$+\frac{1}{4}G_{MN}\left(\frac{1}{2}F^{(2)}_{PQ}F^{(2)PQ}+\frac{1}{24}F^{(4)}_{PQRS}F^{(4)PQRS}\right)=0 \qquad \text{(B.1.1a)}$$
$$4d\star d\Phi-4d\Phi\wedge\star d\Phi+\star R-\frac{1}{2}H^{(3)}\wedge\star H^{(3)}=0 \qquad \text{(B.1.1b)}$$
$$d\star(e^{-2\Phi}H^{(3)})-F^{(2)}\wedge\star F^{(4)}-\frac{1}{2}F^{(4)}\wedge F^{(4)}=0 \qquad \text{(B.1.1c)}$$
$$d\star F^{(2)}-H^{(3)}\wedge\star F^{(4)}=0 \qquad \text{(B.1.1d)}$$
$$d\star F^{(4)}-H^{(3)}\wedge F^{(4)}=0 \qquad \text{(B.1.1e)}$$

The field strengths are given by:

$$H^{(3)}=dB_2,\quad F^{(2)}=dC_1,\quad F^{(4)}=dC_3-H^{(3)}\wedge C_1 \qquad \text{(B.1.2)}$$

The Bianchi identities implied by the above definitions are

$$dH^{(3)}=0,\quad dF^{(2)}=0,\quad dF^{(4)}=H^{(3)}\wedge F^{(2)} \qquad \text{(B.1.3)}$$

The equations of motion for Type IIB reads

$$e^{-2\Phi}\left(R_{MN}+2\nabla_M\nabla_N\Phi-\frac{1}{4}H^{(3)}_{MPQ}H^{(3)PQ}_N\right)-\frac{1}{2}F^{(1)}_{MP}F^{(1)P}_N-\frac{1}{4}F^{(3)}_{MPQ}F^{(3)PQ}_N$$
$$-\frac{1}{4}\frac{1}{4!}F^{(5)}_{MPQRS}F^{(5)PQRS}_M+\frac{1}{4}G_{MN}\left(F^{(1)}_PF^{(1)P}+\frac{1}{24}F^{(3)}_{PQR}F^{(3)PQR}\right)=0$$
$$\text{(B.1.4a)}$$
$$4d\star d\Phi-4d\Phi\wedge\star d\Phi+\star R-\frac{1}{2}H^{(3)}\wedge\star H^{(3)}=0 \qquad \text{(B.1.4b)}$$
$$d\star(e^{-2\Phi}H^{(3)})-F^{(2)}\wedge\star F^{(4)}-\frac{1}{2}F^{(4)}\wedge F^{(4)}=0 \qquad \text{(B.1.4c)}$$
$$d\star F^{(1)}+H^{(3)}\wedge\star F^{(3)}=0 \qquad \text{(B.1.4d)}$$
$$d\star F^{(3)}+H^{(3)}\wedge F^{(5)}=0 \qquad \text{(B.1.4e)}$$
$$F^{(5)}=\star F^{(5)} \qquad \text{(B.1.4f)}$$

The field strengths are given by:

$$H^{(3)} = dB_2, \quad F^{(1)} = dC_0, \quad F^{(3)} = dC_2 - H^{(3)}C_0, \quad F^{(5)} = dC_4 - H^{(3)} \wedge C_2 \tag{B.1.5}$$

The Bianchi identities are

$$dH^{(3)} = 0, \quad dF^{(1)} = 0, \quad dF^{(3)} = H^{(3)} \wedge F^{(1)}, \quad dF^{(5)} = H^{(3)} \wedge F^{(3)} \tag{B.1.6}$$

B.2 Duality rules

T-duality is a symmetry coming from perturbative string theory on worldsheet when one or more directions are compact and it acts exchanging Neumann and Dirichlet boundary conditions along the compact direction. Since Dirichlet boundary conditions imply that the string end points are attached to a D-brane, if we have p Neumann boundary conditions for a superstring in ten dimensions, it means that the strings is free to move along these p-dimensions, namely along a Dp-brane, while is constrained on the transverse directions. If we perform a T-duality along one of the p directions the string will be free to move freely in $p+1$ dimension and it will be constrained on $p-1$ directions, a Dp-1- brane. If the T-duality is performed along transverse direction the strings will be constrained on a Dp+1- brane. Schematically, T duality acts on objects in string theory as

$$\begin{aligned} T_y &: P_y \longrightarrow F1_y \\ T_y &: D_{y12\cdots} \longrightarrow D_{12\cdots} \end{aligned} \tag{B.2.1}$$

Being a symmetry of a perturbative string the spectrum of fields doesn't change after T-duality but it is just reshuffled. Because of the action of T-duality on Dp-branes, we have non trivial action on the Ramond fields at the level of effective low energy action. In particular it acts as a map from solutions in Type IIA to solutions in Type IIB and viceversa.

Let us denote by y the compact direction of radius R, along which one performs T-duality and by x^M the ten-dimensional coordinates.

The transformations of the NS fields are given by

$$\begin{aligned} &\tilde{G}_{yy} = \frac{1}{G_{yy}}, \quad e^{2\tilde{\Phi}} = \frac{e^{2\Phi}}{G_{yy}}, \quad \tilde{G}_{My} = \frac{B_{My}}{G_{yy}}, \quad \tilde{B}_{My} = \frac{G_{My}}{G_{yy}} \\ &\tilde{G}_{MN} = G_{MN} - \frac{G_{My}G_{Ny} - B_{My}B_{Ny}}{G_{yy}}, \quad \tilde{B}_{MN} = B_{MN} - \frac{B_{My}G_{Ny} - G_{My}B_{Ny}}{G_{yy}} \end{aligned} \tag{B.2.2}$$

while for RR potentials we have

$$\begin{aligned} \tilde{C}^{(n)}_{M\cdots NPy} &= C^{(n-1)}_{M\cdots NP} - (n-1)\frac{C^{(n-1)}_{[M\cdots Ny}G_{P]y}}{G_{yy}} \\ \tilde{C}^{(n)}_{M\cdots NPQ} &= C^{(n+1)}_{M\cdots NPQy} + n\frac{C^{(n-1)}_{[M\cdots NP}G_{Q]y}}{G_{yy}} + n(n-1)\frac{C^{(n-1)}_{[M\cdots Ny}B_{Py}G_{Q]y}}{G_{yy}} \end{aligned} \tag{B.2.3}$$

On the moduli, T-duality acts in the following way

$$\tilde{R} = \frac{\alpha'}{R}, \quad \tilde{g}_s = \frac{g_s}{R}\sqrt{\alpha'} \tag{B.2.4}$$

S-duality is a symmetry of Type IIB and it maps solutions into solutions within Type IIB. By defining

$$\tau = C_0 + ie^{-\Phi} \tag{B.2.5}$$

S-duality transformed fields are

$$\begin{aligned} d\tilde{s}^2 &= e^{-\Phi} ds^2, \quad \tilde{B}_2 = C_2 \\ \tilde{C}_2 &= -B_2, \quad \tilde{\tau} = -\frac{1}{\tau} \end{aligned} \tag{B.2.6}$$

S-duality acts on fundamental objects as

$$\begin{aligned} S &: P_y \longrightarrow P_y \\ S &: F1 \longrightarrow D1 \\ S &: D3 \longrightarrow D3 \\ S &: D5 \longrightarrow NS5 \end{aligned} \tag{B.2.7}$$

B.3 Type IIB on T^4

Since we mostly concentrate on Type IIB compactified on a four torus, it's useful to fix here the conventions and notations and to find the reduced six-dimensional equations of motions. We denote the ten-dimensional coordinates with capital letters $M, N = 0, \cdots, 9$

$$X^M = \left(x^\mu,\, z^i\right) \tag{B.3.1}$$

the six dimensional directions with $\mu, \nu = 0, \cdots, 5$ and the coordinates along T^4 with $i, j = 6, 7, 8, 9$. In the following, we will consider ten-dimensional fields as functions of only the six-dimensional coordinates. We define the six-dimensional and torus metric as

$$ds_6^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad d\hat{s}_4^2 = \delta_{ij} dz^i dz^j \tag{B.3.2}$$

The generic ten-dimensional metric solution $ds_{10}^2 = G_{MN} dX^M dX^N$ in the string frame has the form

$$ds_{10}^2 = ds_6^2 + \hat{X}(x) d\hat{s}_4^2 \tag{B.3.3}$$

where $\hat{X}$ is a scalar field dependent on the six-dimensional coordinates and parametrizing the volume of the torus.

In order to find the six-dimensional metric in the Einstein frame $g^E_{\mu\nu}$ we assume the following transformation rules

$$g^E_{\mu\nu} = y\, g_{\mu\nu}, \quad R^E_6 = y^{-1} R_6 \tag{B.3.4}$$

where $R_6^{S/E}$ is scalar of curvature entering in the respective actions and y is a scalar to be fixed by imposing

$$\int d^{10}X\, e^{-2\Phi} R^S_{(10)} \sqrt{G^S} = \int d^6x\, R^E_6 \sqrt{G^E_6} + \cdots \tag{B.3.5}$$

where the dots represents additional terms coming from the rescaling. To find the equation that fixes the scalar y we use (B.3.3) and (B.3.4) to obtain

$$\int d^{10}X\, e^{-2\Phi} R^S_{10} \sqrt{G} = \int d^6x\, e^{-2\Phi} y^{-2} \hat{X}^2 R^E_6 \sqrt{g^E} + \cdots \tag{B.3.6}$$

Imposing (B.3.5) we find

$$y = e^{-\Phi}\hat{X} \tag{B.3.7}$$

In order to find the six-dimensional equation in the Einstein frame it is useful to find the expression of the reduced ten-dimensional string frame Hodge operator acting on a generic p-form A_p with legs only on six-dimensional part. We formally define the p-form as

$$A_p = A_{M_1\cdots,M_p} dX^{M_1} \wedge \cdots \wedge dX^{M_p} \tag{B.3.8}$$

and the Hodge operator, in string frame [1], as

$$\begin{aligned} \star A_p = & \frac{A_{M_1\cdots M_p}}{p!(10-p)!}\sqrt{|G^S_{10}|}\left(G^S_{10}\right)^{M_1 M_1'}\cdots\left(G^S_{(10)}\right)^{M_p M_p'} \\ & \times \epsilon_{M_{p+1}\cdots M_{10} M_1'\cdots M_p'} dX^{M_{p+1}} \wedge \cdots \wedge dX^{M_{10}} \end{aligned} \tag{B.3.9}$$

By restrict the form only on μ, ν coordinates and using (B.3.3), (B.3.4), (B.3.5) we obtain

$$\star A_p = e^{(3-p)\phi}\hat{X}^{(p-1)}\left[\star^E_{(6)} A_p\right] \wedge \hat{\text{vol}}_4 \tag{B.3.10}$$

with

$$\hat{\text{vol}}_4 \equiv \frac{1}{4!}\epsilon_{ijkl}\, dz^i \wedge dz^j \wedge dz^k \wedge dz^l, \quad \epsilon_{6789} = +1 \tag{B.3.11}$$

We define the six-dimensional fields in terms of the ten-dimensional ones and we establish the following dictionary for the scalars

$$e^{\phi_1} \equiv e^{\Phi}, \quad e^{\phi_2} \equiv e^{\Phi}\hat{X}^{-2}, \quad \chi_1 \equiv C_0, \quad C_4 \equiv \chi_2 \hat{\text{vol}}_4 + \cdots \tag{B.3.12}$$

The B_2 and C_2 fields are simply defined as the ten-dimensional ones restricted to the six-dimensional coordinates, with the field strength given by

$$H \equiv dB_2, \quad F \equiv dC_2 - \chi_1 H \tag{B.3.13}$$

[1] We use the notation $\star$ to denote the ten-dimensional Hodge operator in string frame, while we use $*$ to denote the Hodge operator in the six-dimensional reduced theory. When necessary, we use $\hat{*}_4$ to denote the Hodge operator on the torus and $*_4$ for the Hodge on R^4.

Starting from equations of motion in (B.1.4a)-(B.1.4f) and using the relations (B.3.3), (B.3.4), (B.3.5) and the fields defintition in (B.3.12) we get the six dimensional equations

$$d(F+\chi_1 H)=0, \quad dH=0 \tag{B.3.14a}$$
$$d(e^{\phi_1-\phi_2}*F-\chi_2 H)=0\,, \tag{B.3.14b}$$

$$d(e^{-(\phi_1+\phi_2)}*H+\chi_1\chi_2 H-e^{\phi_1-\phi_2}\chi_1*F+\chi_2 F)=0 \tag{B.3.14c}$$

$$d(e^{2\phi_1}*d\chi_1)-e^{\phi_1-\phi_2}*F\wedge H=0 \tag{B.3.14d}$$

$$d(e^{2\phi_2}*d\chi_2)+F\wedge H=0 \tag{B.3.14e}$$

$$d(*d\phi_1)+e^{2\phi_1}*d\chi_1\wedge d\chi_1-\frac{1}{2}e^{-(\phi_1+\phi_2)}*H\wedge H+\frac{1}{2}e^{\phi_1-\phi_2}*F\wedge F=0 \tag{B.3.14f}$$

$$d(*d\phi_2)+e^{2\phi_2}*d\chi_2\wedge d\chi_2-\frac{1}{2}e^{-(\phi_1+\phi_2)}*H\wedge H-\frac{1}{2}\mathrm{e}^{\phi_1-\phi_2}*F\wedge F=0 \tag{B.3.14g}$$

B.4 D1D5 background

The two-charge D1D5 solutions read

$$ds^2_{(10)}=-\frac{2\alpha}{\sqrt{Z_1Z_2}}(dv+\beta)\,[du+\omega]+\sqrt{Z_1Z_2}ds^2_4+\sqrt{\frac{Z_1}{Z_2}}d\hat{s}^2_4 \tag{B.4.1a}$$
$$\mathrm{e}^{2\Phi}=\alpha\frac{Z_1}{Z_2} \tag{B.4.1b}$$
$$B_2=-\frac{\alpha Z_4}{Z_1Z_2}(du+\omega)\wedge(dv+\beta)+\delta_2 \tag{B.4.1c}$$
$$C_0=\frac{Z_4}{Z_1} \tag{B.4.1d}$$
$$C_2=-\frac{\alpha}{Z_1}(du+\omega)\wedge(dv+\beta)+\gamma_2 \tag{B.4.1e}$$
$$C_4=\frac{Z_4}{Z_2}\hat{\mathrm{vol}}_4-\frac{\alpha Z_4}{Z_1Z_2}\gamma_2\wedge(du+\omega)\wedge(dv+\beta) \tag{B.4.1f}$$

where

$$\alpha=\frac{Z_1Z_2}{Z_1Z_2-Z_4^2} \tag{B.4.2}$$

In terms of these six-dimensional fields the D1D5 solutions can be written as

$$ds_6^2 = -\frac{2}{\sqrt{\mathcal{P}}}(dv+\beta)\,(du+\omega) + \sqrt{\mathcal{P}}\,ds_4^2\,, \quad \mathcal{P} = Z_1 Z_2 - Z_4^2 \tag{B.4.3a}$$

$$d\beta = *_4 d\beta\,, \quad d\omega = -*_4\,d\omega \tag{B.4.3b}$$

$$e^{2\phi_1} = \frac{Z_1^2}{\mathcal{P}}\,, \quad e^{2\phi_2} = \frac{Z_2^2}{\mathcal{P}}\,, \quad \chi_1 = \frac{Z_4}{Z_1}\,, \quad \chi_2 = \frac{Z_4}{Z_2} \tag{B.4.3c}$$

$$B_2 = -\frac{Z_4}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + \delta_2\,, \quad *_4 dZ_4 = d\delta_2 \tag{B.4.3d}$$

$$C_2 = -\frac{Z_2}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + \gamma_2\,, \quad *_4 dZ_2 = d\gamma_2 \tag{B.4.3e}$$

In order to find the linearized equations of motion of interest, it is convenient to list a set of definitions and useful relations: We define

$$d\hat{u} \equiv du + \omega\,, \quad d\hat{v} \equiv dv + \beta\,. \tag{B.4.4}$$

The field strengths can be written as

$$H = \frac{Z_4 d(Z_1 Z_2) - (Z_1 Z_2 + Z_4^2) dZ_4}{\mathcal{P}^2} \wedge d\hat{u}\wedge d\hat{v} - \frac{Z_4}{\mathcal{P}}\left[d\omega\wedge d\hat{v} - d\beta\wedge d\hat{u}\right] + *_4 dZ_4 \tag{B.4.5a}$$

$$*_4\,H = -\frac{dZ_4}{\mathcal{P}}\wedge d\hat{u}\wedge d\hat{v} + \frac{Z_4}{\mathcal{P}}\left[d\omega\wedge d\hat{v} - d\beta\wedge d\hat{u}\right] - \frac{Z_4 *_4 d(Z_1 Z_2) - (Z_1 Z_2 + Z_4^2) *_4 dZ_4}{\mathcal{P}} \tag{B.4.5b}$$

$$F = \left[\frac{Z_2}{Z_1}dZ_1 - \frac{Z_4}{Z_1}dZ_4\right]\wedge\frac{d\hat{u}\wedge d\hat{v}}{\mathcal{P}} - \frac{1}{Z_1}\left[d\omega\wedge d\hat{v} - d\beta\wedge d\hat{u}\right] + *_4 dZ_2 - \frac{Z_4}{Z_1} *_4 dZ_4 \tag{B.4.5c}$$

$$*\,F = -\left[dZ_2 - \frac{Z_4}{Z_1}dZ_4\right]\wedge\frac{d\hat{u}\wedge d\hat{v}}{\mathcal{P}} + \frac{1}{Z_1}\left[d\omega\wedge d\hat{v} - d\beta\wedge d\hat{u}\right] - \frac{Z_2}{Z_1} *_4 dZ_1 + \frac{Z_4}{Z_1} *_4 dZ_4 \tag{B.4.5d}$$

Useful relations are

$$\frac{\chi_1}{\chi_2} = e^{\phi_2-\phi_1}\,, \quad \chi_1\chi_2 = 1 - e^{-(\phi_1+\phi_2)}\,, \tag{B.4.6a}$$

$$e^{-(\phi_1+\phi_2)}(H - *H) = \chi_2(F - *F) \tag{B.4.6b}$$

$$e^{\phi_1-\phi_2} * F - \chi_2 H = -d\tilde{C}\,, \quad \tilde{C} = -\frac{Z_1}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + \gamma_1\,, \quad *_4 dZ_1 = d\gamma_1 \tag{B.4.6c}$$

with the following orientation

$$\epsilon_{ty1234} = \epsilon_{uv1234} = +1 \tag{B.4.7}$$

B.5 Linearized equations of motion

In this section we provide a list of equations for the perturbations of fields dual to the light operators studied in CFT. The light operators analyzed in CFT are basically of two kinds: operators constructed with fermions as fundamental fields and operators constructed with bosons. The fermionic ones are dual to perturbations of fields in the six-dimensional reduced theory, while the bosonic operators are dual to forms with indices along T^4. We start with perturbations dual to fermionic operators around the most generic two-charge background solution.

B.5.1 Equations for perturbation dual to O_F

The six-dimensional fields dual to the chiral primary operators O_L, $\bar{O}_L$ are a scalar w and a closed three-form h which are respectively the fluctuations of the χ_2 and H six-dimensional fields.

The linearized perturbation equations around the $\mathrm{AdS}_3 \times S^3$ background were derived in [50]. The $\mathrm{AdS}_3 \times S^3$ geometry is a special case of ((B.4.3a)-(B.4.3e)) with

$$Z_4 = \delta_2 = 0, \quad \phi_1 = -\phi_2, \quad *F = -F \tag{B.5.1}$$

Around such a background the perturbation equations for (w, h) are

$$h - *h = 2\,\mathrm{w}F\,, \quad e^{2\phi_2}\, d * d\mathrm{w} = h \wedge F\,. \tag{B.5.2}$$

We need to generalize the perturbation equations to a generic D1D5 background. So we are looking for deformations of the solution ((B.4.3a)-(B.4.3e)) controlled by a scalar w and a three-form h that satisfy the equations of motion at linear order, but generically break supersymmetry. It is immediate to see that in the presence of non-vanishing background values for χ_1, χ_2 and H, perturbing χ_1, χ_2 and H induces at first order a perturbation of all other fields, so the task of constructing a consistent deformation is considerably more involved in this more general setting. Even if the perturbed solution does not need to be supersymmetric, we can use the supersymmetric solution as a guide to understand which fields will be excited by the perturbation. In particular we can consider the effect of varying at first order Z_4 by $\delta Z_4 \equiv \mathrm{w}\, Z_2$. This motivates the following ansatz for the perturbation:

$$\delta\chi_1 = e^{\phi_2-\phi_1}\mathrm{w}\,, \quad \delta\chi_2 = \mathrm{w}\,, \quad \delta e^{2\phi_2} = 2e^{4\phi_2}\chi_2\mathrm{w}\,, \quad \delta e^{2\phi_1} = 2e^{2(\phi_1+\phi_2)}\chi_2\mathrm{w} \tag{B.5.3}$$

Given the form of B_2 in ((B.4.3d)), it is also natural to define h as

$$\delta H \equiv h + dy\,, \quad y \equiv -2\frac{Z_2 Z_4^2}{\mathcal{P}^2}\,\mathrm{w}\, d\hat{u}\wedge d\hat{v}\,. \tag{B.5.4}$$

The Bianchi identity $dH = 0$ implies $dh = 0$. The Bianchi identity for F:

$$d(F + \chi_1 H) = 0 \tag{B.5.5}$$

implies the form of the F-variation:

$$\delta F = -e^{\phi_2-\phi_1}\mathrm{w}H - \chi_1 h - \chi_1 dy + dx\,. \tag{B.5.6}$$

The two-form x is not fixed by the Bianchi identity, but the supersymmetric solution suggests the ansatz

$$x = -2\frac{Z_2^2 Z_4}{\mathcal{P}^2}\,\mathrm{w}\, d\hat{u}\wedge d\hat{v} = \chi_2^{-1} y\,, \tag{B.5.7}$$

which is what one would obtain by varying Z_4 in the expression for C in (B.4.3e). Since Z_4 also appears in the six-dimensional Einstein metric, ds_6^2 should also fluctuate. Instead of guessing the full form of the metric perturbation, we will determine its effects on the

Hodge star by consistency with the equations of motion. The Maxwell's equation for F

$$d(e^{\phi_1-\phi_2} * F - \chi_2 H) = 0 \tag{B.5.8}$$

implies

$$\delta(*F) = e^{\phi_2-\phi_1}\mathrm{w}H + \chi_1 h + \chi_1 dy - e^{\phi_2-\phi_1} d\tilde{x}\,; \tag{B.5.9}$$

the 2-form $\tilde{x}$, which represents the variation of the dual potential $\tilde{C}$, can be inferred from (B.4.6c):

$$\tilde{x} = -2\frac{Z_1 Z_2 Z_4}{\mathcal{P}^2}\,\mathrm{w} d\hat{u}\wedge d\hat{v} = e^{\phi_1-\phi_2} x\,. \tag{B.5.10}$$

If one assumes that (B.3.14c) is preserved by the perturbation, one deduces

$$\begin{aligned} e^{-(\phi_1+\phi_2)}\delta(*H) &= (2-e^{-(\phi_1+\phi_2)})h - \mathrm{w}(1+2e^{2\phi_2}\chi_2^2)(F-*F) \\ &\quad + 2\chi_2 e^{\phi_2-\phi_1}\mathrm{w}H + (2-e^{-(\phi_1+\phi_2)})dy + \chi_2(dx + d\tilde{x})\,. \end{aligned} \tag{B.5.11}$$

Finally, we need to know how the Hodge operator acting on one-form on R^4 is deformed: since in the supersymmetric ansatz one does not get any factor of Z_4 when the star acts on one-forms with legs only along the spatial ditrections, we assume that $\delta(*\omega_1) = *\,\delta\omega_1$ for any one-form ω_1 on R^4.

We can now apply these deformation rules on the remaining equations of motion and require that they are preserved at first order in the deformation. If one looks at the variation of the equations for the RR scalars:

$$d(e^{2\phi_1} * d\chi_1) - e^{\phi_1-\phi_2} * F\wedge H = 0\,, \quad d(e^{2\phi_2} * d\chi_2) + F\wedge H = 0\,, \tag{B.5.12}$$

one finds that w and h must satisfy the differential constraints

$$e^{-(\phi_1+\phi_2)}(h - *h) = \mathrm{w}(F - *F)\,, \quad d(e^{2\phi_2} * d\mathrm{w}) + dC\wedge h = 0\,. \tag{B.5.13}$$

These are the perturbation equations that generalize (B.5.2) around a general D1D5 background.

As a further consistency check, one can verify that the identity

$$H\wedge *F + *H\wedge F = 0\,, \tag{B.5.14}$$

is preserved by our deformation rules: this checks that the deformation ansatz for the Hodge start operation, that we have derived somewhat indirectly (see eqs. (B.5.9) and (B.3.14c)), is actually consistent.

B.5.2 Equations for perturbation dual to O_B

The CFT operator $\partial X^{(i}\bar{\partial}X^{j)}$, with $i,j = 1,\dots,4$, after a chain of dualitites, is dual to a deformation h_{ij} of the T^4 metric. For simplicity we restrict here to a traceless deformation $\delta^{ij}h_{ij} = 0$. We derive here the linearized equation satisfied by h_{ij} in the background of a generic two-charge microstate. When the background is that of the naive D1D5 geometry, it is know that h_{ij} is a minimally coupled scalar (see for example [24]). We show that this remains true for a generic D1D5 microstate.

The deformed string metric is

$$ds^2_{10} = \sqrt{\frac{Z_1 Z_2}{\mathcal{P}}}\, ds^2_6 + \sqrt{\frac{Z_1}{Z_2}}\,(\delta_{ij} + h_{ij})\, dz^i dz^j\,, \tag{B.5.15}$$

We would like to derive the equations of motion at first order in h_{ij}. The only non-trivial equation is Einstein's equation:

$$
\begin{aligned}
&e^{-2\Phi}\left(R_{MN}+2\nabla_M\nabla_N\Phi\right)+\frac{1}{4}g_{MN}\left(F_PF^P+\frac{1}{3!}F_{PQR}F^{PQR}\right)-\frac{1}{4}\frac{1}{4!}F_{MPQRS}F_N{}^{PQRS}\\
&-\frac{1}{2}F_MF_N-\frac{1}{4}e^{-2\Phi}H_{MPQ}H_N{}^{PQ}-\frac{1}{2}\frac{1}{2!}F_{MPQ}F_N{}^{PQ}=0\,,
\end{aligned}
\tag{B.5.16}
$$

where the Ricci tensor R_{MN}, the covariant derivatives and the raising of indices are referred to the string metric; we have omitted to write the subscripts indicating the form degree since the explicit presence of the indices leaves no space to confusion. One finds the only non trivial contributions are

$$
\begin{aligned}
\delta R_{ij}=&-\frac{1}{2}\frac{\sqrt{\mathcal{P}}}{Z_2}\Bigg[\Box_6 h_{ij}+\frac{\mathcal{P}}{Z_1^2}\partial^\mu\left(\frac{Z_1^2}{\mathcal{P}}\right)\partial_\mu h_{ij}\\
&+\frac{1}{2}\left(\frac{Z_2}{Z_1}\Box_6\left(\frac{Z_1}{Z_2}\right)+\frac{\mathcal{P}}{Z_1^2}\partial^\mu\left(\frac{Z_1Z_2}{\mathcal{P}}\right)\partial_\mu\left(\frac{Z_1}{Z_2}\right)\right)h_{ij}\Bigg]\,,
\end{aligned}
\tag{B.5.17}
$$

$$
\delta(\nabla_i\nabla_j\Phi)=\frac{1}{4}\frac{\mathcal{P}^{3/2}}{Z_1^2Z_2}\partial^\mu\left(\frac{Z_1^2}{\mathcal{P}}\right)\left[\partial_\mu h_{ij}+\frac{1}{2}\frac{Z_2}{Z_1}\partial_\mu\left(\frac{Z_1}{Z_2}\right)h_{ij}\right]\,,
\tag{B.5.18}
$$

$$
\begin{aligned}
F_PF^P+\frac{1}{3!}F_{PQR}F^{PQR}=&\frac{\sqrt{\mathcal{P}}}{Z_1Z_2^2}\Bigg[\partial^\mu Z_2\partial_\mu Z_2-\frac{\mathcal{P}Z_2}{Z_1^3}\partial^\mu Z_1\partial_\mu Z_1\\
&+\frac{Z_2}{Z_1}\partial^\mu Z_4\partial_\mu Z_4-2\frac{Z_4}{Z_1}\partial^\mu Z_2\partial_\mu Z_4\Bigg]\,,
\end{aligned}
\tag{B.5.19}
$$

$$
\frac{1}{4!}\delta(F_{iPQRS}F_j{}^{PQRS})=\frac{\sqrt{\mathcal{P}}Z_2}{Z_1^2}\,\partial^\mu\left(\frac{Z_4}{Z_2}\right)\partial_\mu\left(\frac{Z_4}{Z_2}\right)h_{ij}\,,
\tag{B.5.20}
$$

and of course $\delta g_{ij}=\sqrt{\frac{Z_1}{Z_2}}\,h_{ij}$. Here $\Box_6$ is the scalar laplacian of the six-dimensional Einstein metric ds_6^2 and the six-dimensional indices μ are raised and lowered with ds_6^2. The warp factors Z_1 and Z_2 of a generic two-charge microstate are harmonic: $\Box_6 Z_1=\Box_6 Z_2=0$. Exploiting this property, the variation of the first two terms of (B.5.16) can be simplified to

$$
e^{-2\Phi}\left[\delta R_{ij}+2\delta(\nabla_i\nabla_j\Phi)\right]=-\frac{1}{4}\frac{\mathcal{P}^{3/2}}{Z_1^3Z_2}\left[2Z_1\Box_6 h_{ij}+\left(\frac{Z_1}{Z_2^2}\,\partial_\mu Z_2\partial^\mu Z_2-\frac{1}{Z_1}\,\partial_\mu Z_1\partial^\mu Z_1\right)h_{ij}\right].
\tag{B.5.21}
$$

Substituting (B.5.21), (B.5.19) and (B.5.20) in the first line of (B.5.16) one can verify that at first order in h_{ij} the equation reduces to

$$
\Box_6 h_{ij}=0\,,
\tag{B.5.22}
$$

i.e. h_{ij} is a minimally coupled scalar in six dimensions.

Appendix C

D-integrals

C.1 Schwinger representation of D-integrals

The unnormalised boundary-to-bulk propagator for a scalar field propagating in Euclidean AdS$_{d+1}$ is

$$K_\Delta(w,\vec{z}) = \left[\frac{w_0}{w_0^2+(\vec{w}-\vec{z})^2}\right]^\Delta = \frac{1}{\Gamma(\Delta)}\int_0^\infty dt\, w_0^\Delta\, t^{\Delta-1} e^{-t(w_0^2+(\vec{w}-\vec{z})^2)}\;, \tag{C.1.1}$$

where Δ is the conformal dimension of the dual operator (related to the mass as usual: $m^2 = \Delta(\Delta-d)$). The D-integrals arising from a four-point contact vertex in the bulk take the following form

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4} = \int d^{d+1}w\,\sqrt{\bar{g}}\prod_{i=1}^{4} K_{\Delta_i}(w,\vec{z}_i)\;, \tag{C.1.2}$$

where the AdS$_{d+1}$ metric in the Euclidean Poincaré coordinates is

$$d\bar{s}^2 = \frac{1}{w_0^2}\left(dw_0^2 + \sum_{i=1}^{d} dw_i^2\right)\;. \tag{C.1.3}$$

By using the representation of the propagator in terms of Schwinger parameters given in (C.1.1) it is straightfoward to perform the integration over the interaction point $(w_0,\vec{w})$.

$$\begin{aligned} D_{\Delta_1\Delta_2\Delta_3\Delta_4} &= \prod_i\left[\int_0^\infty dt_i \frac{t_i^{\Delta_i-1}}{\Gamma(\Delta_i)}\right]\int d^d\vec{w}\,dw_0\,w_0^{-d-1-\hat{\Delta}} e^{-\sum_{i=1}^4 t_i\left(w_0^2+(\vec{w}-\vec{z}_i)^2\right)} \\ &= \prod_i\left[\int_0^\infty dt_i \frac{t_i^{\Delta_i-1}}{\Gamma(\Delta_i)}\right]\frac{T^{\frac{d-\hat{\Delta}}{2}}}{2}\Gamma\left(\frac{\hat{\Delta}-d}{2}\right)\int d^d\vec{w}e^{-\sum_{i=1}^4 t_i(\vec{w}-\vec{z}_i)^2} \end{aligned} \tag{C.1.4}$$

with $T=\sum_i t_i$, $\hat{\Delta}=\sum_i \Delta_i$ and where z_j are the standard complex coordinates. now let's perform the d-dimensional Gaussian integration by writing the argument of the exponential in the following way[1]

$$\sum_{i=1}^{4} t_i\left(\vec{w}-\vec{z}_i\right)^2 = \left(\sum_{i=1}^{4} t_i\right)\sum_{j,k=1}^{d}(\vec{w})_j(\vec{w})_k\delta_{jk} + \sum_{i=1}^{4} t_i(z_i)_j(z_i)_k\delta_{jk} - 2\sum_{i=1}^{4} t_i(w)_k(z_i)_k$$
$$= \frac{1}{2}\sum_{j,k=1}^{d} 2T\delta_{jk}(\vec{w})_j(\vec{w})_k - \sum_{k=1}^{d} 2\sum_{i=1}^{4} t_i(z_i)_k(w)_k + \sum_{k=1}^{d}\sum_{i=1}^{4} t_i(z_i)_k(z_i)_k$$
$$= \frac{1}{2}\sum_{j,k=1}^{d} A_{jk}(\vec{w})_j(\vec{w})_k - \sum_{k=1}^{d} B_k(w)_k + \sum_{k=1}^{d}\sum_{i=1}^{4} t_i(z_i)_k(z_i)_k \tag{C.1.5}$$

with

$$A_{jk} \equiv 2T\delta_{jk} \qquad B_k \equiv 2\sum_{i=1}^{4} t_i(z_i)_k \tag{C.1.6}$$

The Gaussian integral gives

$$\int d^d\vec{w}e^{-\sum_{i=1}^{4} t_i(\vec{w}-\vec{z}_i)^2} = \pi^{d/2}T^{-d/2}e^{-\sum_{k=1}^{d}\sum_{i=1}^{4} t_i(z_i)_k(z_i)_k}e^{\frac{1}{2}B_kA^{-1}_{kj}B_j} \tag{C.1.7}$$

Then the argument of the exponential reads

$$-\sum_{k=1}^{d}\sum_{i=1}^{4} t_i(z_i)_k(z_i)_k + \frac{1}{2}B_kA^{-1}_{kj}B_j = -\sum_{i=1}^{4} t_i(z_i)_k(z_i)_k + \frac{1}{T}\sum_{i,i'=1}^{4}(z_i)_k(z_{i'})_k t_i t_{i'}$$
$$= \frac{1}{T}\sum_{k=1}^{d}\left(-\sum_{i,i'=1}^{4} t_i t_{i'}(z_i)_k(z_i)_k + \sum_{i,i'=1}^{4}(z_i)_k(z_{i'})_k t_i t_{i'}\right)$$
$$= -\frac{1}{T}\sum_{i<i'} t_i t_{i'}(z_i - z_{i'})^2 = -\frac{1}{T}\sum_{i<i'} t_i t_{i'}(z_{ii'})^2 \tag{C.1.8}$$

Thus the final result reads

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4} = \Gamma\left(\frac{\hat{\Delta}-d}{2}\right)\int_0^\infty \prod_i \left[dt_i\frac{t_i^{\Delta_i-1}}{\Gamma(\Delta_i)}\right]\frac{\pi^{d/2}}{2T^{\frac{\hat{\Delta}}{2}}}e^{-\sum_{i,j=1}^{4}|z_{ij}|^2\frac{t_it_j}{2T}} \tag{C.1.9}$$

C.2 Useful properties of D-integrals

Once written in terms of Schwinger parameter, one can see that D_{1111} is proportional to the massless box-integral in four dimensions with external massive state the result can be written in term of logarithms and dilogarithms

$$D_{1111} = \frac{\pi}{2|z_{13}|^2|z_{24}|^2(z-\bar{z})}\left[2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \ln(z\bar{z})\ln\frac{1-z}{1-\bar{z}}\right], \tag{C.2.1}$$

where z is the crossratio defined in (5.1.4).

[1] We use j,k index for the space time components of the vectors and i,i' for the external points.

The result in (C.1.9) is proportional to the Bloch-Wigner dilogarithm $D(z,\bar{z})$

$$D_{1111} = \frac{2\pi i}{|z_{13}|^2|z_{24}|^2(z-\bar{z})} D(z,\bar{z}) \tag{C.2.2}$$

where

$$\begin{aligned} D(z,\bar{z}) &= \mathrm{Im}[\mathrm{Li}_2(z)] + \mathrm{Arg}[\ln(1-z)]\ln|z| \\ &= \frac{1}{2i}\left[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\ln(z\bar{z})\ln\frac{1-z}{1-\bar{z}}\right] . \end{aligned} \tag{C.2.3}$$

The function $D(z,\bar{z})$ is a real-analytic function[2] except in $z=0,1$. It is continuous also in those two points, but not differentiable (since it has singularities of the type $y\log(y)$, where $y=\mathrm{Im}[z]$ or $y=\mathrm{Im}[1-z]$ and $y\to 0$). Moreover we have the following useful identities

$$D(z,\bar{z}) = -D\left(\frac{1}{z},\frac{1}{\bar{z}}\right) = -D\left(1-z,1-\bar{z}\right) \ , \tag{C.2.4}$$

which implies

$$D(z,\bar{z}) = D\left(1-\frac{1}{z},1-\frac{1}{\bar{z}}\right) = D\left(\frac{1}{1-z},\frac{1}{1-\bar{z}}\right) = -D\left(\frac{-z}{1-z},\frac{-\bar{z}}{1-\bar{z}}\right) \ . \tag{C.2.5}$$

Our correlator involves also the D-integrals of the type D_{1122} and permutations, i.e. we have two Δ_i equal to two and the other two equal to one. By using the expression in terms of Schwinger parameters (C.1.9) it is easy to write these integrals as derivatives of D_{1111}. It is useful therefore to list some useful relations for derivatives. Firstly to derive higher D-functions, we will use

$$\frac{\partial}{\partial z_{ij}^2} = \frac{\partial z}{\partial z_{ij}^2}\partial + \frac{\partial \bar{z}}{\partial z_{ij}^2}\bar{\partial} \tag{C.2.6}$$

and thus we need the following useful relations

$$\begin{aligned} \frac{\partial z}{\partial z_{34}^2} &= -\frac{z}{z-\bar{z}}\frac{|1-z|^2}{z_{34}^2}\,, & \frac{\partial \bar{z}}{\partial z_{34}^2} &= +\frac{\bar{z}}{z-\bar{z}}\frac{|1-z|^2}{z_{34}^2} \\ \frac{\partial z}{\partial z_{23}^2} &= -\frac{z\bar{z}}{z_{23}^2}\frac{1-z}{z-\bar{z}}\,, & \frac{\partial \bar{z}}{\partial z_{23}^2} &= +\frac{z\bar{z}}{z_{23}^2}\frac{1-\bar{z}}{z-\bar{z}} \\ \frac{\partial z}{\partial z_{12}^2} &= -\frac{z}{z-\bar{z}}\frac{|1-z|^2}{|z_{12}|^2}\,, & \frac{\partial \bar{z}}{\partial z_{12}^2} &= +\frac{\bar{z}}{z-\bar{z}}\frac{|1-z|^2}{z_{12}^2} \end{aligned} \tag{C.2.7}$$

Each pair (kl) of subscripts can be increased by one by taking the derivative with respect to the corresponding $|z_{kl}|^2$; hence one has

$$D_{p_1+1\,p_2+1\,p_3\,p_4} = -\frac{\hat{p}-d}{2p_1p_2}\frac{\partial}{\partial|z_{12}|^2}D_{p_1\,p_2\,p_3\,p_4} \tag{C.2.8}$$

and its permutations (with $\hat{p}=\sum_i p_i$ and, in our case, $d=2$). It is also convenient to introduce the rescaled functions

$$\hat{D}_{p_1\,p_2\,p_3\,p_4} = \lim_{z_2\to\infty}|z_2|^{2p_2}D_{p_1\,p_2\,p_3\,p_4}(0,z_2,1,z)\,. \tag{C.2.9}$$

[2] In order for this to hold, $\bar{z}$ has to be the complex conjugate of z, so the correlator (C.2.2) has a more complicated analytic structure in Minkowski space where $\bar{z}\neq z^*$, see [70] and references therein: in that case it has non-trivial monodromies around $z=1$ and $z=\infty$ with $\bar{z}$ fixed (and vice-versa).

Some useful relations between the $\hat{D}$'s appearing in the intermediate results are

$$\hat{D}_{1122}(z,\bar{z}) = \frac{\hat{D}_{2211}(z,\bar{z})}{|1-z|^2}\,, \quad \hat{D}_{1212}(z,\bar{z}) = \hat{D}_{2121}(z,\bar{z}) = \frac{1}{|z|^2}\hat{D}_{2211}\left(\frac{z-1}{z},\frac{\bar{z}-1}{\bar{z}}\right)\,, \tag{C.2.10a}$$

$$\hat{D}_{1221}(z,\bar{z}) = |z|^2\,\hat{D}_{2112}(z,\bar{z}) = \hat{D}_{2211}(1-z,1-\bar{z})\,. \tag{C.2.10b}$$

The symmetry under exchange of the first two points in D_{2211} implies

$$\hat{D}_{2211}\left(\frac{1}{z},\frac{1}{\bar{z}}\right) = |z|^2\,\hat{D}_{2211}(z,\bar{z})\,. \tag{C.2.11}$$

All the $\hat{D}$ functions are linear combinations of $D(z,\bar{z})$, $\log|1-z|^2$ and $\log|z|^2$ with coefficients that are ratios of polynomials in z and $\bar{z}$. Inverting the relations between the functions $\hat{D}$ and $D(z,\bar{z})$, $\log|1-z|^2$, $\log|z|^2$ one finds the useful identities

$$\hat{D}_{1111} = \hat{D}_{2211} + \hat{D}_{1221} + \hat{D}_{2121}\,, \quad \pi\log|z| = |1-z|^2(\hat{D}_{1221} - \hat{D}_{2121}) + (|z|^2-1)\,\hat{D}_{2211}\,. \tag{C.2.12}$$

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