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**STATISTICAL ANALYSIS OF
PROCESS CAPABILITY INDICES
WITH MEASUREMENT ERRORS**

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Statistical Analysis of Process Capability Indices with Measurement Errors

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Abstract: Process capability indices (PCIs) have been widely used in manufacturing industries to provide a quantitative measure of process potential and performance. While some efforts have been dedicated in the literature to the statistical properties of PCIs estimators, scarce attention has been given to the evaluation of these properties when sample data are affected by measurement errors. In this work we deal with the problem of measurement errors effects on the performance of PCIs. The analysis is illustrated with reference to C_p and C_{pk} , i.e. the two most common measures suggested to evaluate process capability.

Key Words: Process capability indices, measurement errors, C_p , C_{pk} , statistical properties of PCIs estimators.

1 Introduction

Process capability indices (PCIs) are widely used in manufacturing industries to measure the ability of a process to realize items that meet the specification limits. The analytical formulation of these indices is generally easy to understand and straightforward to apply. Further, they provide management with a single-number summary of what is happening on the production floor. These are the main reasons of their extensive use in quality assurance and quality improvement activities of many companies.

However, since PCIs are calculated on sampling observations, a certain amount of uncertainty, due to the sampling error, is necessarily present in the evaluation of the process performances. In the operative context this fact is rather neglected and conclusions about the capability of the process are often based only on the single numerical value of the index provided by sampling data.

Clearly, this approach is not reliable, since sampling errors are ignored. This uncritical use of PCIs and the scarce attention given to accuracy problems, have caused an increasing interest toward the statistical properties of PCIs estimators in the literature of the last years.

Evidence of this attention is confirmed by the recent book of Kotz and Lovelace [1] and other numerous articles as, for example, Bissel [2], Chan *et al* [3], Chou and Owen [4], Chou *et al* [5], Gunter [6], Kane [7].

A further source of uncertainty in the evaluation of the PCIs is given by the frequent presence of measurement errors in the sampling observations. The causes of measurement errors are numerous as, for example, insufficient gauge calibration and external influences on the measuring devices.

The analysis of the effects of measurement errors on PCIs has been received scarce attention in the literature. An exception is represented by the paper of Mittag [8]. This author quantified the percentage error on PCIs evaluation in presence of measurement errors. In this work our goal is to extend the analysis of Mittag [8] to the inferential properties of the estimators of C_p and C_{pk} . The paper is organized as follows. In section 2 the main results on the effects of measurement errors on some theoretical capability indices are briefly reported. Section 3 extends the analysis considering the statistical properties of the C_p estimator in the measurement error case. The analysis and the results relative to C_{pk} are discussed in Section 4. Some concluding remarks and directions for future work are reported in Section 5.

2 Process capability indices and measurement errors: some general remarks

Let $X \sim N(\mu, \sigma^2)$ denote the relevant quality characteristic of a manufacturing process. Given upper and lower specification limits, LSL and USL respectively, two capability indices, frequently used to describe the performances of a process relative to the specification limits, are (Montgomery [9])

$$C_p = \frac{USL - LSL}{6\sigma} = \frac{d}{3\sigma} \quad (1)$$

and

$$C_{pk} = \frac{d - |\mu - \frac{1}{2}(USL + LSL)|}{3\sigma} = \frac{d - |\mu - m|}{3\sigma}, \quad (2)$$

where $d = (USL - LSL)/2$ is the half-length of the specification interval and $m = (USL + LSL)/2$ is the midpoint of the specification interval.

C_p compares the 6σ spread of the process to the tolerance spread. It does not require knowledge of the process location μ and for this reason it can be seen as a measure for the process capability of an optimally centered process. C_{pk} was introduced to give μ some influence on the value of the index. Both C_p

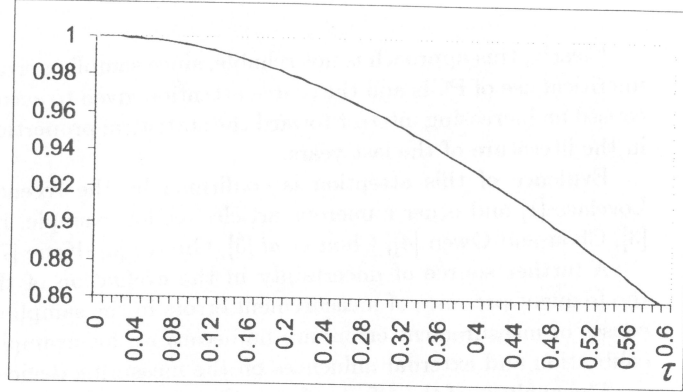


Figure 1: $\frac{C_p^e}{C_p^c} = \frac{C_{pk}^e}{C_{pk}^c}$ as functions of τ

and C_{pk} are designed to reflect changes in the amount of product beyond the specification limits. Neither index considers the target value of the process.

Let us consider now the case of random measurement errors. In particular we assume that measurement errors are described by a random variable $V \sim N(0, \sigma_V^2)$. It is further assumed that X and V are additively linked according to

$$X^e = X + V \quad (3)$$

and that X and V are stochastically independent. Thus, according to (3), one observes instead of the $N(\mu, \sigma^2)$ -distributed variable X the $N(\mu, \sigma_e^2)$ -distributed empirical variable X^e , with $\sigma_e^2 = \sigma^2 + \sigma_V^2$. Therefore the observable PCIs C_p^e and C_{pk}^e are obtained by formulae (1) and (2) after substituting σ_e for σ .

The effects of constant and random measurement errors on the performances of theoretical capability indices are examined in Mittag [8]. In the case of random measurement errors this author showed that the ratios C_p^e/C_p^c and C_{pk}^e/C_{pk}^c are functions of the contamination degree $\tau = \sigma_V/\sigma$, according to the following relationship

$$\frac{C_p^e}{C_p^c} = \frac{C_{pk}^e}{C_{pk}^c} = \frac{1}{\sqrt{1 + \tau^2}}. \quad (4)$$

It is clear from (4) that the ratios between observable and true PCIs are decreasing functions of τ (Figure 1) and consequently C_p^e and C_{pk}^e , calculated using the observable variable X^e , systematically understate the true capability of the process. The results provided by Mittag [8] are quite interesting, since they emphasize that the accuracy of a capability analysis could be significantly influenced by the accuracy of the gauges and, consequently, measurement errors

should receive greater attention. However, the analysis of Mittag [8] is confined to considering the effects of measurement errors only on the behavior of theoretical capability indices, while such effects, when PCIs are estimated from sample data, are not taken into account. In the next sections we deal with this problem with reference to C_p and C_{pk} .

3 Statistical analysis of C_p

Denoting with $\{X_j, j = 1, 2, \dots, n\}$ the random sample of size n from the quality characteristics X , the most commonly employed estimator of C_p is

$$\hat{C}_p = \frac{USL - LSL}{6\hat{\sigma}}, \quad (5)$$

where $\hat{\sigma}^2 = \frac{1}{n-1} \sum_{j=1}^n (X_j - \bar{X})^2$ and $\bar{X} = \frac{1}{n} \sum_{j=1}^n X_j$. Kotz and Johnson [10] derived the following expressions for the expected value and the variance of \hat{C}_p

$$E(\hat{C}_p) = C_p \frac{1}{b_f}, \quad (6)$$

$$Var(\hat{C}_p) = \left(\frac{f}{f-2} - b_f^{-2} \right) (C_p)^2, \quad (7)$$

where

$$b_f = \left(\frac{2}{f} \right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}f)}{\Gamma(\frac{1}{2}(f-1))},$$

$f = n - 1$ and $\Gamma(\cdot)$ is the Gamma function. Expression (6) shows that \hat{C}_p is a biased estimator of C_p and, since $b_f < 1$, the resulting bias is positive. However, this bias goes to zero as $n \rightarrow \infty$. As $Var(\hat{C}_p)$ goes to zero as $n \rightarrow \infty$, \hat{C}_p is also mean square consistent.

3.1 Bias of the estimator in the measurement error case

In the measurement error case we can observe $X^e = X + V$ instead of the true variable X . Denoting with $\{X_j^e, j = 1, 2, \dots, n\}$ the random sample of size n from X^e , C_p is estimated by the quantity

$$\hat{C}_p^e = \frac{USL - LSL}{6\hat{\sigma}_e}, \quad (8)$$

with $\hat{\sigma}_e^2 = \frac{1}{n-1} \sum_{j=1}^n (X_j^e - \bar{X}^e)^2$ and $\bar{X}^e = \frac{1}{n} \sum_{j=1}^n X_j^e$.

By the same arguments of Kotz and Johnson [10] we obtain

$$E(\hat{C}_p^e) = C_p^e \frac{1}{b_f} \quad (9)$$

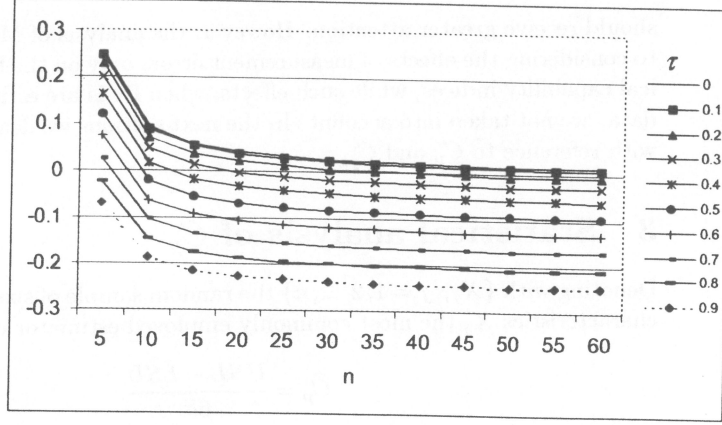


Figure 2: $\frac{1}{b_f} \frac{1}{\sqrt{1+\tau^2}} - 1$ as a function of n for different values of τ

and, consequently, the bias of \hat{C}_p^e with respect to the true process capability index C_p is given by

$$\begin{aligned} B(\hat{C}_p^e) &= E(\hat{C}_p^e) - C_p = C_p^e \frac{1}{b_f} - C_p = \\ &= C_p \left(\frac{1}{b_f} \frac{1}{\sqrt{1+\tau^2}} - 1 \right). \end{aligned} \quad (10)$$

From the analysis of $B(\hat{C}_p^e)$ we can find some interesting results. The bias arises from two sources, the sampling error, which leads to overestimate C_p , and the measurement error, which leads to underestimate C_p . From the combination of these two effects we have that $B(\hat{C}_p^e)$ can assume either positive or negative values. In particular, the behavior of the term $\frac{1}{b_f} \frac{1}{\sqrt{1+\tau^2}} - 1$ as a function of n is shown in Figure 2 for different values of τ , while the 3-D graph of $\frac{1}{b_f} \frac{1}{\sqrt{1+\tau^2}} - 1$, as a function of n and τ , is reported in Figure 3. It can be seen that when τ and n increase the bias becomes negative.

Let $\tau > 0$ be fixed. Then, the asymptotic behavior of $B(\hat{C}_p^e)$ is given by

$$\lim_{n \rightarrow \infty} B(\hat{C}_p^e) = C_p \left(\frac{1}{\sqrt{1+\tau^2}} - 1 \right), \quad (11)$$

which implies that \hat{C}_p^e is asymptotically biased, while it is worth reminding that \hat{C}_p is an asymptotically unbiased estimator.

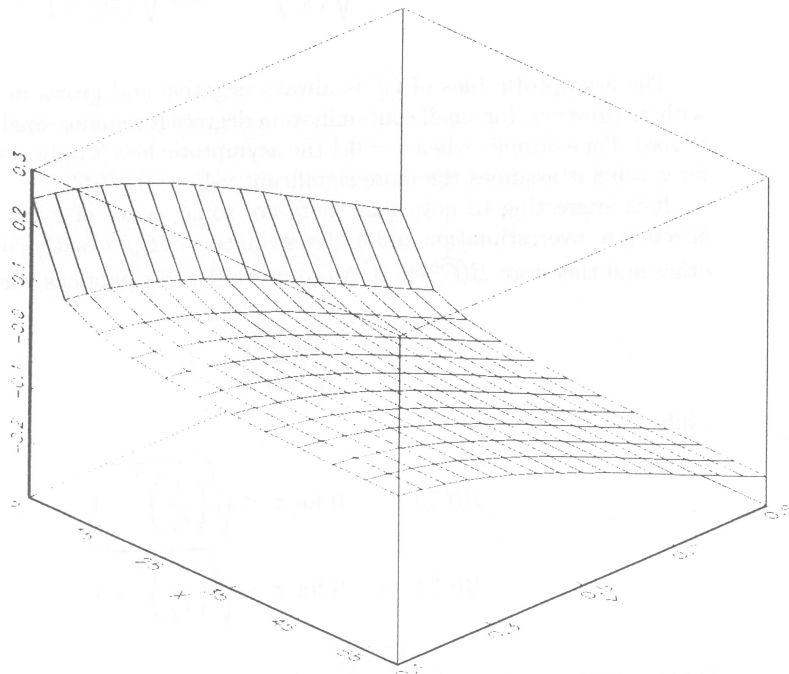


Figure 3: $\frac{1}{b_f} \frac{1}{\sqrt{1+\tau^2}} - 1$ as a function of n (x -axis) and τ (y -axis)

n	$\sqrt{\left(\frac{1}{b_f}\right)^2 - 1}$	$\sqrt{\left(\frac{1}{2b_f-1}\right)^2 - 1}$
10	0.44426	0.67786
20	0.29201	0.42654
30	0.23318	0.33660
40	0.19977	0.28680
50	0.17754	0.25408
60	0.16139	0.23049
100	0.12398	0.17635

Table 1: Values of $\sqrt{\left(\frac{1}{b_f}\right)^2 - 1}$ and $\sqrt{\left(\frac{1}{2b_f-1}\right)^2 - 1}$

The asymptotic bias of \hat{C}_p^e is always negative and grows in absolute value with τ . However, for small contamination degrees it remains small and very near to zero. For example, when $\tau = 0.1$ the asymptotic bias is only $-0.0049C_p$, while for $\tau = 0.8$ it assumes the more significant value $-0.2191C_p$.

It is interesting to note that there are some values of τ for which the two effects (i.e. overestimation and underestimation of C_p) exactly compensate each other and therefore $B(\hat{C}_p^e) = 0$. More precisely, this happens when

$$\tau = \sqrt{\left(\frac{1}{b_f}\right)^2 - 1}, \quad (12)$$

while

$$\begin{aligned} B(\hat{C}_p^e) &> 0 \text{ for } \tau < \sqrt{\left(\frac{1}{b_f}\right)^2 - 1}, \\ B(\hat{C}_p^e) &< 0 \text{ for } \tau > \sqrt{\left(\frac{1}{b_f}\right)^2 - 1}. \end{aligned}$$

Some numerical values of $\sqrt{\left(\frac{1}{b_f}\right)^2 - 1}$ corresponding to different sample sizes n are reported in the first column of Table 1. These values represent thresholds for τ : under these thresholds the overestimation effect due to the sampling error prevails, while the negative effect of bias due to measurement errors becomes more prominent for values of τ greater than the thresholds.

For example, if $n = 50$, then $\sqrt{\left(\frac{1}{b_f}\right)^2 - 1} = 0.17754$. For this value of τ , corresponding to $\sigma_V = (17.754\%)\sigma$, $B(\hat{C}_p^e) = 0$. Only values of τ in the

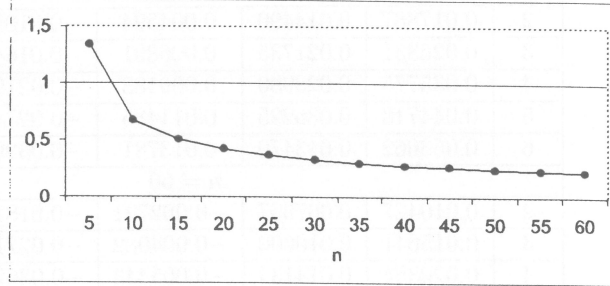


Figure 4: $\sqrt{\left(\frac{1}{2b_f-1}\right)^2 - 1}$ as a function of n

interval $0 < \tau < 0.17754$ can provide the positive bias $0 < B(\hat{C}_p^e) < B(\hat{C}_p) = 0.01523C_p$. For $\tau > 0.17754$ the underestimating effect due to measurement errors definitively prevails and becomes more prominent as τ increases.

In order to complete the picture on the behavior of $B(\hat{C}_p^e)$, a formal comparison between $B(\hat{C}_p^e)$ and the bias in the measurement error free case, $B(\hat{C}_p)$, is provided through the following inequalities

$$|B(\hat{C}_p^e)| \leq B(\hat{C}_p) \text{ for } \tau \leq \sqrt{\left(\frac{1}{2b_f-1}\right)^2 - 1}, \quad (13)$$

$$|B(\hat{C}_p^e)| > B(\hat{C}_p) \text{ for } \tau > \sqrt{\left(\frac{1}{2b_f-1}\right)^2 - 1}. \quad (14)$$

Some numerical values of $\sqrt{\left(\frac{1}{2b_f-1}\right)^2 - 1}$ are shown in the second column of Table 1, while the graph of $\sqrt{\left(\frac{1}{2b_f-1}\right)^2 - 1}$ as a function of n is shown in Figure 4. As an example we report in Table 2 some numerical values of $B(\hat{C}_p^e)$ for $n = 30, 50$, $\tau = 0, 0.1, 0.2, 0.3, 0.4$ and different values of C_p .

Table 2 should be examined jointly with the second column of Table 1. Picking out from Table 1 the values of $\sqrt{\left(\frac{1}{2b_f-1}\right)^2 - 1}$ for $n = 30$ and $n = 50$, 0.33660 and 0.25408 respectively, one can observe that for values of τ not greater than these values $|B(\hat{C}_p^e)| \leq B(\hat{C}_p)$ according to (13) and (14).

d/σ	$B(\hat{C}_p)$	$B(\hat{C}_p^e)$ $\tau=0.1$	$B(\hat{C}_p^e)$ $\tau=0.2$	$B(\hat{C}_p^e)$ $\tau=0.3$	$B(\hat{C}_p^e)$ $\tau=0.4$
$n = 30$					
2	0.017887	0.014490	0.004594	-0.010983	-0.031074
3	0.026831	0.021735	0.006891	-0.016474	-0.046611
4	0.035775	0.028980	0.009188	-0.021966	-0.062148
5	0.044718	0.036225	0.011485	-0.027457	-0.077685
6	0.053662	0.043470	0.013781	-0.032948	-0.093223
$n = 50$					
2	0.010427	0.007067	-0.002721	-0.018128	-0.038001
3	0.015641	0.010600	-0.004082	-0.027192	-0.057001
4	0.020854	0.014134	-0.005443	-0.036257	-0.076002
5	0.026068	0.017667	-0.006804	-0.045321	-0.095002
6	0.031282	0.021201	-0.008164	-0.054385	-0.114002

Table 2: Numerical values of $B(\hat{C}_p^e)$ for different values of n , τ , and $\frac{d}{\sigma} = 3C_p$

3.2 Mean square error of the estimator in the measurement error case

Examining now the variance of \hat{C}_p^e , by the same arguments of Kotz and Johnson [10], we can obtain the following expression

$$\begin{aligned}
Var(\hat{C}_p^e) &= \left(\frac{f}{f-2} - b_f^{-2} \right) (C_p^e)^2 = \\
&= \left(\frac{f}{f-2} - b_f^{-2} \right) \frac{(USL - LSL)^2}{36(\sigma^2 + \sigma_1^2)} = \\
&= \left(\frac{f}{f-2} - b_f^{-2} \right) C_p^2 \frac{1}{1 + \tau^2}.
\end{aligned} \tag{15}$$

Since $\tau \geq 0$, it follows that

$$Var(\hat{C}_p) \geq Var(\hat{C}_p^e).$$

Let us consider now the mean square errors ($MSEs$) of the two estimators (5) and (8) respectively

$$MSE(\hat{C}_p) = B(\hat{C}_p)^2 + Var(\hat{C}_p), \tag{16}$$

$$MSE(\hat{C}_p^e) = B(\hat{C}_p^e)^2 + Var(\hat{C}_p^e). \tag{17}$$

Comparing (16) and (17), we obtain

$$\begin{aligned}
MSE(\hat{C}_p) &> MSE(\hat{C}_p^e) \text{ for } \tau < \tau_{01}, \\
MSE(\hat{C}_p) &= MSE(\hat{C}_p^e) \text{ for } \tau = \tau_{01}, \\
MSE(\hat{C}_p) &< MSE(\hat{C}_p^e) \text{ for } \tau > \tau_{01},
\end{aligned} \tag{18}$$

n	τ_{01}
10	1.01406
20	0.58219
30	0.44971
40	0.37954
50	0.33444
60	0.30234
100	0.22979
200	0.16026

Table 3: Values of τ_{01}

where

$$\tau_{01} = \sqrt{\left\{ \frac{-\frac{f}{f-2}}{\frac{f}{f-2} - \frac{2}{b_f}} \right\}^2 - 1}. \quad (19)$$

Table 3 gives few values of τ_{01} and Figure 5 shows the graph of τ_{01} as a function of n , while the ratio $MSE(\hat{C}_p)/MSE(\hat{C}_p^c)$ as a function of both τ and n is shown in Figure 6.

From the previous results, one can observe that there exists a considerable range of values of τ , for which $MSE(\hat{C}_p^c) \leq MSE(\hat{C}_p)$, specially for small and moderate sample sizes. As n increases, τ_{01} goes to zero. Thus, for large n $MSE(\hat{C}_p^c)$ tends to be definitively greater than $MSE(\hat{C}_p)$, for almost every value of $\tau > 0$ and the difference between the two $MSEs$ increases with τ . However, this difference is practically indistinguishable if τ is very small. As an example we report in Table 4 some numerical values of $MSE(\hat{C}_p^c)$ for $n = 50, 100, 200$ and $\tau = 0, 0.2, 0.3, 0.4$.

Reading from Table 3 the values of τ_{01} for $n = 50, 100, 200$, $\tau_{01} = 0, 33444, 0.22979, 0.16026$ respectively one can observe that for values of τ not greater than these values $MSE(\hat{C}_p) > MSE(\hat{C}_p^c)$ according to (18).

4 Statistical analysis of C_{pk}

If $\{X_j, j = 1, 2, \dots, n\}$ is the usual random sample of size n from the quality characteristics X , an estimator of C_{pk} is

$$\hat{C}_{pk} = \frac{d - |\bar{X} - \frac{1}{2}(USL + LSL)|}{3\hat{\sigma}}. \quad (20)$$

d/σ	$MSE(\hat{C}_p)$	$MSE(\hat{C}_p^e)$ $\tau=0.2$	$MSE(\hat{C}_p^e)$ $\tau=0.3$	$MSE(\hat{C}_p^e)$ $\tau=0.4$
$n = 50$				
2	0.005010	0.004720	0.004825	0.005669
3	0.011271	0.010619	0.010856	0.012755
4	0.020038	0.018879	0.019299	0.022676
5	0.031310	0.029498	0.030155	0.035431
6	0.045086	0.042477	0.043423	0.051020
$n = 100$				
2	0.002357	0.002304	0.002677	0.003853
3	0.005302	0.005184	0.006024	0.008669
4	0.009426	0.009215	0.010710	0.015412
5	0.014728	0.014399	0.016734	0.024081
6	0.021209	0.020735	0.024097	0.034677
$n = 200$				
2	0.001148	0.001208	0.001708	0.003040
3	0.002584	0.002717	0.003843	0.006840
4	0.004593	0.004831	0.006833	0.012161
5	0.007177	0.007548	0.010676	0.019001
6	0.010335	0.010869	0.015373	0.027361

Table 4: MSE values of \hat{C}_p and \hat{C}_p^e for different values of n and τ

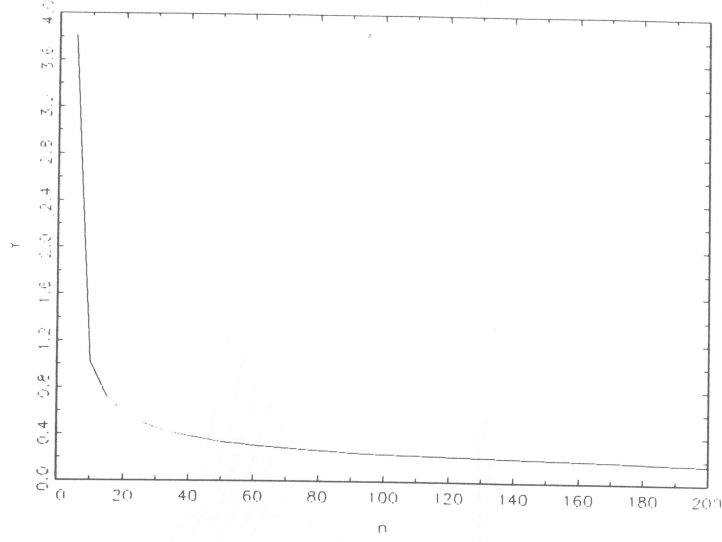


Figure 5: $\tau_{01}(y)$ as a function of n

Kotz and Johnson [10] derived the following expressions for the expected value and the variance of \hat{C}_{pk} :

$$E(\hat{C}_{pk}) = \frac{1}{3}b_f^{-1} \left[\frac{d}{\sigma} - \left(\frac{2}{\pi n} \right)^{\frac{1}{2}} \times \exp \left\{ -\frac{n}{2}A^2 \right\} - A \times \{1 - 2\Phi(-\sqrt{n}A)\} \right], \quad (21)$$

$$\begin{aligned} Var(\hat{C}_{pk}) = & \frac{f}{9(f-2)} \times \left\{ \left(\frac{d}{\sigma} \right)^2 - 2 \left(\frac{d}{\sigma} \right) \times \right. \\ & \left[\left(\frac{2}{\pi n} \right)^{\frac{1}{2}} \times \exp \left(-\frac{n}{2}A^2 \right) + A \times \{1 - 2\Phi(-\sqrt{n}A)\} \right] \\ & \left. + A^2 + \frac{1}{n} \right\} - \left\{ E(\hat{C}_{pk}) \right\}^2, \end{aligned} \quad (22)$$

where

$$A = \frac{|\mu - m|}{\sigma}.$$

Expression (21) shows that \hat{C}_{pk} is a biased estimator of C'_{pk} . A detailed study

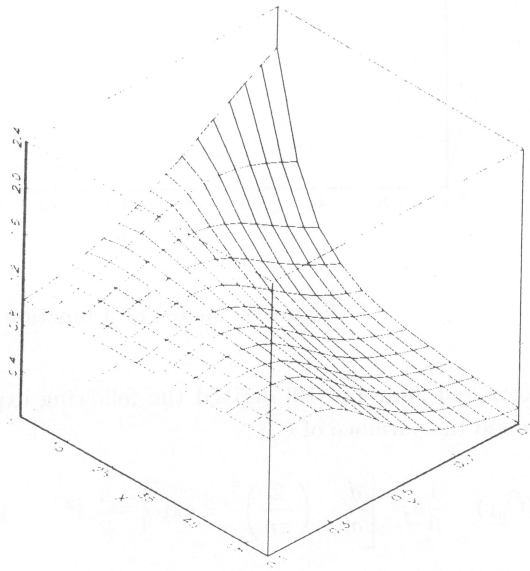


Figure 6: $MSE(\hat{C}_p)/MSE(\hat{C}_p^e)$ as a function of n (x -axis) and τ (y -axis)

of the bias of \hat{C}_{pk} , given by

$$B(\hat{C}_{pk}) = \frac{1}{3}b_f^{-1} \left\{ \frac{d}{\sigma} - \left(\frac{2}{\pi n} \right)^{\frac{1}{2}} \times \exp \left[-\frac{n}{2}A^2 \right] - A \times [1 - 2\Phi(-\sqrt{n}A)] \right\} - C_{pk}, \quad (23)$$

is reported in Kotz and Johnson [10]. They found that the bias is positive when $\mu \neq m$ (the specification range midpoint). When $\mu = m$ the bias is positive when $n = 10$, but becomes negative for larger values. As $n \rightarrow \infty$ the bias tends to zero.

4.1 Bias of the estimator in the measurement error case

If $\{X_j^e, j = 1, 2, \dots, n\}$ is the random sample of size n from the observable quality characteristics X^e , C_{pk} is estimated by the quantity

$$\hat{C}_{pk}^e = \frac{d - |\bar{X}_e - \frac{1}{2}(USL + LSL)|}{3\hat{\sigma}_e}. \quad (24)$$

Consequently the bias of \hat{C}_{pk}^e with respect to the true capability index C_{pk} results

$$B(\hat{C}_{pk}^e) = \frac{1}{3}b_f^{-1} \left\{ \frac{d}{\sigma_e} - \left(\frac{2}{\pi n} \right)^{\frac{1}{2}} \times \exp \left[-\frac{n}{2}A_e^2 \right] - A_e \times [1 - 2\Phi(-\sqrt{n}A_e)] \right\} - C_{pk}, \quad (25)$$

where

$$A_e = A \frac{1}{\sqrt{1 + \tau^2}}.$$

From the analysis of (25) it turns out that $B(\hat{C}_{pk}^e)$ is a function of n and τ . Let τ be fixed. Then

$$\lim_{n \rightarrow \infty} B(\hat{C}_{pk}^e) = \frac{1}{3} \left[\frac{d}{\sigma_e} - A_e \right] - C_{pk} = C_{pk} \left(\frac{1}{\sqrt{1 + \tau^2}} - 1 \right). \quad (26)$$

Thus, the asymptotic behavior of $B(\hat{C}_{pk}^e)$ is the same of that of \hat{C}_p^e (see expression (11)). Consequently also the same results hold, i.e.: a) \hat{C}_{pk}^e is asymptotically biased, while \hat{C}_{pk} is an asymptotically unbiased estimator; b) the asymptotic bias of \hat{C}_{pk}^e is always negative and grows in absolute value with τ ; c) for small contamination degrees it remains small and very near to zero.

In order to study the bias as function of τ for some fixed n , it is useful to distinguish the situation $A = 0$ (i.e. $\mu = m$) from $A \neq 0$ (i.e. $\mu \neq m$).

For processes centered on the midpoint of the specification interval ($A = 0$) the bias results

$$\begin{aligned} B(\hat{C}_{pk}^e) &= \frac{1}{3}b_f^{-1} \left[\frac{d}{\sigma_e} - \left(\frac{2}{\pi n} \right)^{\frac{1}{2}} \right] - C_{pk} = \\ &= \frac{1}{3}b_f^{-1} \left[\frac{d}{\sigma\sqrt{1+\tau^2}} - \left(\frac{2}{\pi n} \right)^{\frac{1}{2}} \right] - \frac{d}{3\sigma}. \end{aligned} \quad (27)$$

The first derivative of $B(\hat{C}_{pk}^e)$ with respect to τ is given by

$$-\frac{1}{2} \frac{d}{3\sigma} b_f^{-1} (1+\tau^2)^{-\frac{3}{2}} (2\tau). \quad (28)$$

Since (28) is always negative, $B(\hat{C}_{pk}^e)$ is a decreasing function of τ and reaches its maximum value in $\tau = 0$. As for C_p there are some values of τ for which the measurement error effect compensates the sampling error effect and $B(\hat{C}_{pk}^e) = 0$. More precisely it can be shown that

$$\begin{aligned} B(\hat{C}_{pk}^e) &> 0 \text{ for } 0 \leq \tau < \tau_{02}, \\ B(\hat{C}_{pk}^e) &= 0 \text{ for } \tau = \tau_{02}, \\ B(\hat{C}_{pk}^e) &< 0 \text{ for } \tau > \tau_{02}, \end{aligned} \quad (29)$$

where

$$\tau_{02} = \sqrt{\left(\frac{b_f^{-1} \frac{d}{\sigma}}{b_f^{-1} \left(\frac{2}{\pi n} \right)^{\frac{1}{2}} + \frac{d}{\sigma}} \right)^2 - 1}. \quad (30)$$

We observe that τ_{02} is a function of n and d/σ and real values of τ_{02} exist only if the term $\left(\frac{b_f^{-1} \frac{d}{\sigma}}{b_f^{-1} \left(\frac{2}{\pi n} \right)^{\frac{1}{2}} + \frac{d}{\sigma}} \right)$ is greater than one. This means that some combinations of n and d/σ can lead to not real values for τ_{02} . In these cases the bias results always negative.

Some numerical values of τ_{02} are shown in Table 5. We can note that τ_{02} increases as $\frac{d}{\sigma}$ increases, but decreases as n increases. Further, if $n \leq 30$, then combinations of n and $\frac{d}{\sigma}$ for which $B(\hat{C}_{pk}^e) = 0$ do exist.

The comparison of $B(\hat{C}_{pk})$ with $B(\hat{C}_{pk}^e)$ should be performed considering the sign of the two biases. In the following we will examine the possible cases.

- Case 1: $B(\hat{C}_{pk}) > 0$ and $B(\hat{C}_{pk}^e) > 0$. $B(\hat{C}_{pk}^e)$ and $B(\hat{C}_{pk})$ are both positive for $0 \leq \tau < \tau_{02}$. For combinations of n and d/σ leading to real values of τ_{02} according to (30) (some examples are shown in Table 5) it turns out that $B(\hat{C}_{pk}^e) < B(\hat{C}_{pk})$.

d/σ	$MSE(\hat{C}_p)$	$MSE(\hat{C}_p^e)$ $\tau=0.2$	$MSE(\hat{C}_p^e)$ $\tau=0.3$	$MSE(\hat{C}_p^e)$ $\tau=0.4$
$n = 50$				
2	0.005010	0.004720	0.004825	0.005669
3	0.011271	0.010619	0.010856	0.012755
4	0.020038	0.018879	0.019299	0.022676
5	0.031310	0.029498	0.030155	0.035431
6	0.045086	0.042477	0.043423	0.051020
$n = 100$				
2	0.002357	0.002304	0.002677	0.003853
3	0.005302	0.005184	0.006024	0.008669
4	0.009426	0.009215	0.010710	0.015412
5	0.014728	0.014399	0.016734	0.024081
6	0.021209	0.020735	0.024097	0.034677
$n = 200$				
2	0.001148	0.001208	0.001708	0.003040
3	0.002584	0.002717	0.003843	0.006840
4	0.004593	0.004831	0.006833	0.012161
5	0.007177	0.007548	0.010676	0.019001
6	0.010335	0.010869	0.015373	0.027361

Table 5: MSE values of \hat{C}_p and \hat{C}_p^e for different values of n and τ

n	10	20	30	40	50	60
$\frac{d}{\sigma}$						
2	—	—	—	—	—	—
3	0.09017	—	—	—	—	—
4	0.31461	—	—	—	—	—
5	0.39937	0.13364	—	—	—	—
6	0.45020	0.20674	0.08620	—	—	—

Table 6: Values of τ_{02S}

- Case 2: $B(\hat{C}_{pk}) < 0$ and $B(\hat{C}_{pk}^e) < 0$. The two biases are both negative for the combinations of n and d/σ for which a real value of τ_{02} does not exist. In these cases if $\tau \geq 0$, then $|B(\hat{C}_{pk}^e)| \leq |B(\hat{C}_{pk})|$.
- Case 3: $B(\hat{C}_{pk}) > 0$ and $B(\hat{C}_{pk}^e) < 0$. For combinations of n and d/σ leading to real values of τ_{02} according to (30) if $\tau_{02} < \tau \leq \tau_{02S}$, then $|B(\hat{C}_{pk}^e)| \leq B(\hat{C}_{pk})$ where

$$\tau_{02S} = \sqrt{\frac{\left(\frac{d}{3\sigma b_f}\right)^2 - \left(\frac{d}{3\sigma b_f} - \frac{2d}{3b_f} \left(\frac{2}{\pi n}\right)^{\frac{1}{2}} - \frac{2d}{3\sigma}\right)^2}{\left(\frac{d}{3\sigma b_f} - \frac{d}{3b_f} \left(\frac{2}{\pi n}\right)^{\frac{1}{2}} - \frac{2d}{3\sigma}\right)^2}}.$$

Some numerical values of τ_{02S} are displayed in Table 6.

From the previous results we can briefly conclude that for small sample size ($n \geq 30$) generally $|B(\hat{C}_{pk}^e)| \leq |B(\hat{C}_{pk})|$ if $\tau \leq \tau_{02S}$, while for $n > 30$ measurement errors lead to a more biased estimator.

As an illustration, computed values of $B(\hat{C}_{pk})$ and $B(\hat{C}_{pk}^e)$ for some values of τ and n are reported in Table 7. The joint examination of Tables 5-7 allows one to check the results illustrated in the previous different cases.

Let us examine now the situation with $A \neq 0$. If $A \geq 1$ and $n \geq 10$, then the quantities $\Phi(-\sqrt{n}A)$ and $\frac{1}{3} \left(\frac{2}{\pi n}\right)^{\frac{1}{2}} \times \exp\left\{-\frac{n}{2}A^2\right\}$ in expression (23) are approximately null¹. Therefore, $B(\hat{C}_{pk})$ can be approximated by

$$B(\hat{C}_{pk}) \simeq b_f^{-1} \left[\left(\frac{d}{3\sigma} - \frac{A}{3}\right) \right] - C_{pk} = b_f^{-1} C_{pk} - C_{pk} = C_{pk}(b_f^{-1} - 1) \quad (31)$$

¹For example with $n = 10$ and $A = 1$ $\Phi(-\sqrt{n}A) = 0.000783$ and $\frac{1}{3} \left(\frac{2}{\pi n}\right)^{\frac{1}{2}} \times \exp\left\{-\frac{n}{2}A^2\right\} = 0.000567$

d/σ	$B(\hat{C}_{pk})_{\tau=0}$	$B(\hat{C}_{pk}^e)_{\tau=0.1}$	$B(\hat{C}_{pk}^e)_{\tau=0.2}$	$B(\hat{C}_{pk}^e)_{\tau=0.4}$
$n = 10$				
2	-0.02920	-0.03282	-0.04337	-0.08138
3	0.00221	-0.00322	-0.01904	-0.07605
4	0.03363	0.02638	0.00529	-0.07073
5	0.06504	0.05599	0.02962	-0.06540
6	0.09645	0.08559	0.05395	-0.06007
$n = 20$				
2	-0.03411	-0.03756	-0.04760	-0.08379
3	-0.02019	-0.02536	-0.04042	-0.09470
4	-0.00627	-0.01316	-0.03324	-0.10562
5	0.00765	-0.00096	-0.02607	-0.11653
6	0.02157	0.01123	-0.01889	-0.12745

Table 7: Values of $B(\hat{C}_{pk}^e)$ for different values of n and τ

and likewise

$$B(\hat{C}_{pk}^e) \simeq C_{pk} \left(\frac{1}{\sqrt{1+\tau^2}} b_f^{-1} - 1 \right). \quad (32)$$

Comparing (32) and (31), the same results already obtained for C_p still hold, i.e.:

$$\begin{aligned}
|B(\hat{C}_{pk}^e)| &\leq B(\hat{C}_{pk}) \text{ for } \tau \leq \sqrt{\left(\frac{1}{2b_f - 1}\right)^2 - 1}, \\
|B(\hat{C}_{pk}^e)| &> B(\hat{C}_{pk}) \text{ for } \tau > \sqrt{\left(\frac{1}{2b_f - 1}\right)^2 - 1}.
\end{aligned} \quad (33)$$

As an example in Table 8 we report for $\tau = 0.3$ and $n = 20, 50$ the values of $B(\hat{C}_{pk}^e)$ calculated using the exact formula (25), the values of $B(\hat{C}_{pk}^e)$ calculated using the approximation (32) and the values of $B(\hat{C}_{pk})$.

The reader is encouraged to examine Table 8 jointly with the second column of Table 1 to verify the relationships (33) and to check the goodness of the approximation. It is worth noting that the approximation is not longer accurate for small sample size and $A < 1$.

$\frac{d}{\sigma}$	A			
	0.5	1	1.5	2
$B(\hat{C}_{pk}^e) (25) \ n = 20$				
2	0.00364	-0.00072	-0.00036	0.0000
3	0.00292	-0.00144	-0.00109	-0.00072
4	0.00219	-0.00216	-0.00181	-0.00145
5	0.00147	-0.00289	-0.00253	-0.00217
6	0.0075	-0.00361	-0.00326	-0.00289
$B(\hat{C}_{pk}^e) (32) \ n = 20$				
2	-0.00109	-0.00072	-0.00036	0.0000
3	-0.00181	-0.00145	-0.00109	-0.00072
4	-0.00253	-0.00217	-0.00181	-0.00145
5	-0.00326	-0.00289	-0.00253	-0.00217
6	-0.00398	-0.00362	-0.00326	-0.00289
$B(\hat{C}_{pk}^e) (23) \ n = 20$				
2	0.02487	0.01392	0.00696	0.0000
3	0.03879	0.02785	0.02088	0.01392
4	0.05271	0.04177	0.03480	0.02784
5	0.06663	0.05569	0.04872	0.04176
6	0.08055	0.06961	0.06265	0.05569
$\frac{d}{\sigma}$	A			
	0.5	1	1.5	2
$B(\hat{C}_{pk}^e) (25) \ n = 50$				
2	-0.01348	-0.00906	-0.00453	0.0000
3	-0.02255	-0.01813	-0.01360	-0.00906
4	-0.03161	-0.02719	-0.02266	-0.01813
5	-0.04068	-0.03626	-0.03173	-0.02719
6	-0.04974	-0.04532	-0.04079	-0.03626
$B(\hat{C}_{pk}^e) (32) \ n = 50$				
2	-0.01360	-0.00906	-0.00453	0.0000
3	-0.02266	-0.01813	-0.01360	-0.00906
4	-0.03173	-0.02719	-0.02266	-0.01813
5	-0.04079	-0.03626	-0.03173	-0.02719
6	-0.04986	-0.04532	-0.04079	-0.03626
$B(\hat{C}_{pk}^e) (23) \ n = 50$				
2	0.00789	0.00521	0.00261	0.0000
3	0.01310	0.01043	0.00782	0.00521
4	0.01831	0.01564	0.01303	0.01043
5	0.02353	0.02085	0.01825	0.01564
6	0.02874	0.02606	0.02346	0.02085

Table 8: Example with $\tau = 0.3$

4.2 Mean square error of the estimators in the measurement error case

The variance of \hat{C}_{pk}^e is readily obtained as

$$\begin{aligned} Var(\hat{C}_{pk}^e) = & \frac{f}{9(f-2)} \times \left\{ \left(\frac{d}{\sigma_c} \right)^2 - 2 \left(\frac{d}{\sigma_c} \right) \times \right. \\ & \left[\left(\frac{2}{\pi n} \right)^{\frac{1}{2}} \times \exp \left\{ -\frac{n}{2} A_c^2 \right\} + A_c \times \{ 1 - 2\Phi(-\sqrt{n} A_c) \} \right] \\ & \left. + A_c^2 + \frac{1}{n} \right\} - \left\{ E(\hat{C}_{pk}^e) \right\}^2. \end{aligned} \quad (34)$$

Then we can compare the mean square errors of the estimators \hat{C}_{pk} and \hat{C}_{pk}^e :

$$MSE(\hat{C}_{pk}) = B(\hat{C}_{pk})^2 + Var(\hat{C}_{pk}), \quad (35)$$

$$MSE(\hat{C}_{pk}^e) = B(\hat{C}_{pk}^e)^2 + Var(\hat{C}_{pk}^e), \quad (36)$$

as functions of the contamination degree τ .

Examining first the case $A = 0$, we obtain

$$\begin{aligned} MSE(\hat{C}_{pk}) &= MSE(\hat{C}_{pk}^e) \text{ for } \tau = \tau_{03} \\ MSE(\hat{C}_{pk}) &> MSE(\hat{C}_{pk}^e) \text{ for } \tau < \tau_{03} \\ MSE(\hat{C}_{pk}) &< MSE(\hat{C}_{pk}^e) \text{ for } \tau > \tau_{03} \end{aligned} \quad (37)$$

where

$$\begin{aligned} \tau_{03} = & \left\{ \left\{ \left\{ -b_f^{-1} \frac{d}{\sigma} - \frac{f}{f-2} \left(\frac{2}{\pi n} \right)^{\frac{1}{2}} + \right. \right. \right. \\ & - \left[\left(b_f^{-1} \right)^2 \left(\frac{d}{\sigma} \right)^2 + \left(\frac{f}{f-2} \right)^2 \left(\frac{2}{\pi n} \right) + \right. \\ & + \frac{2}{b_f} \frac{d}{\sigma} \left(\frac{f}{f-2} \right) \left(\frac{2}{\pi n} \right)^{\frac{1}{2}} + \left(\frac{d}{\sigma} \right)^2 \left(\frac{f}{f-2} \right)^2 + \\ & \left. \left. \left. - \frac{2}{b_f} \left(\frac{d}{\sigma} \right)^2 \left(\frac{f}{f-2} \right) - 2 \frac{d}{\sigma} \left(\frac{f}{f-2} \right)^2 \left(\frac{2}{\pi n} \right)^{\frac{1}{2}} \right]^{1/2} \right\} \right. \\ & \left. \left[\frac{d}{\sigma} \frac{f}{f-2} - \frac{2}{b_f} \frac{d}{\sigma} - 2 \frac{f}{f-2} \left(\frac{2}{\pi n} \right)^{\frac{1}{2}} \right]^{-1} \right\}^2 - 1 \right\}^{1/2}. \end{aligned} \quad (38)$$

n	10	20	30	40	50
$\frac{d}{\sigma}$					
2	0.31255	—	—	—	—
3	0.56575	0.18593	—	—	—
4	0.67640	0.31640	0.17603	0.06326	—
5	0.74198	0.37745	0.24935	0.17284	0.11455
6	0.78587	0.41482	0.28942	0.21836	0.16923
n	60	70	80	90	100
$\frac{d}{\sigma}$					
2	—	—	—	—	—
3	—	—	—	—	—
4	—	—	—	—	—
5	0.05647	—	—	—	—
6	0.13089	0.09782	0.06558	0.02185	—

Table 9: Values of τ_{03}

τ_{03} is a function of n and d/σ and some combinations of n and d/σ can leads to not real values for τ_{03} : in these cases $MSE(\hat{C}_{pk}) < MSE(\hat{C}_{pk}^e)$. Table 9 displays some numerical values of τ_{03} .

It should be noted that τ_{03} is an increasing function of $\frac{d}{\sigma}$ and a decreasing function of n . If $n \leq 90$, then combinations of sample size and $\frac{d}{\sigma}$ for which $MSE(\hat{C}_{pk}) \geq MSE(\hat{C}_{pk}^e)$ can be found.

As an example, in Table 10 we report some values of $MSE(\hat{C}_{pk}^e)$ and $MSE(\hat{C}_{pk})$ calculated for $n = 20, 30$ and few values of τ .

The reader is encouraged to examine Tables 10 and 9 jointly in order verify the relationships (37)

Let us consider now the case $A \neq 0$. For $A \geq 1$ and $n > 10$ the expected values of \hat{C}_{pk} and \hat{C}_{pk}^e can be approximated² by

$$E(\hat{C}_{pk}) \simeq \frac{1}{3} b_f^{-1} \left[\frac{d}{\sigma} - A \right],$$

and likewise

$$E(\hat{C}_{pk}^e) \simeq \frac{1}{3} b_f^{-1} \left[\frac{d}{\sigma_c} - A_c \right].$$

² $\Phi(-\sqrt{n}A) \simeq 0$ and $\frac{1}{3} \left(\frac{2}{\pi n} \right)^{\frac{1}{2}} \times \exp \left\{ -\frac{n}{2} A^2 \right\} \simeq 0$

d/σ	$MSE(\hat{C}_{pk})$ ($\tau=0$)	$MSE(\hat{C}_{pk}^e)$ ($\tau=0.2$)	$MSE(\hat{C}_{pk}^e)$ ($\tau=0.3$)	$MSE(\hat{C}_{pk}^e)$ ($\tau=0.4$)
$n = 20$				
2	0.01536	0.01595	0.01713	0.01941
3	0.03130	0.03136	0.03250	0.03567
4	0.05483	0.05378	0.05446	0.05838
5	0.08594	0.08323	0.08303	0.08752
6	0.12463	0.11969	0.11820	0.12310
$n = 30$				
2	0.01000	0.01071	0.01201	0.01441
3	0.01981	0.02041	0.02214	0.02601
4	0.03417	0.03433	0.03634	0.04187
5	0.05306	0.05246	0.05436	0.06198
6	0.07649	0.07482	0.07699	0.08635

Table 10: Values of $MSE(\hat{C}_{pk}^e)$ for different values of n and τ

The variances of the estimators are approximated by

$$Var(\hat{C}_{pk}) \simeq \frac{f}{9(f-2)} \left[\left(\frac{d}{\sigma} \right)^2 - 2 \left(\frac{d}{\sigma} \right) A + A^2 + \frac{1}{n} \right] - [E(\hat{C}_{pk})]^2,$$

$$Var(\hat{C}_{pk}^e) \simeq \frac{f}{9(f-2)} \left[\left(\frac{d}{\sigma_e} \right)^2 - 2 \left(\frac{d}{\sigma_e} \right) A_e + A_e^2 + \frac{1}{n} \right] - [E(\hat{C}_{pk}^e)]^2.$$

Therefore, the approximated mean square errors are

$$MSE(\hat{C}_{pk}) \simeq \frac{f}{9(f-2)} \left[\left(\frac{d}{\sigma} \right)^2 - 2 \left(\frac{d}{\sigma} \right) A + A^2 + \frac{1}{n} \right] - \frac{2}{3} b_f^{-1} \left[\frac{d}{\sigma} - |A| \right] \left[\frac{\frac{d}{\sigma} - A}{3} \right] + \left[\frac{\frac{d}{\sigma} - A}{3} \right]^2 \quad (39)$$

and

$$MSE(\hat{C}_{pk}^e) \simeq \frac{f}{9(f-2)} \left[\left(\frac{d}{\sigma_e} \right)^2 - 2 \left(\frac{d}{\sigma_e} \right) A_e + A_e^2 + \frac{1}{n} \right] - \frac{2}{3} b_f^{-1} \left[\frac{d}{\sigma_e} - A_e \right] \left[\frac{\frac{d}{\sigma_e} - A_e}{3} \right] + \left[\frac{\frac{d}{\sigma_e} - A_e}{3} \right]^2 \quad (40)$$

Finally we obtain the following relationship

$$MSE(\hat{C}_{pk}) = MSE(\hat{C}_{pk}^e) \text{ for } \tau = \tau_{04},$$

$\frac{d}{\sigma}$	A			
	0.5	1	1.5	2
$MSE(\hat{C}_{pk}^e) (40)$				
2	0.00497	0.00349	0.00261	0.00232
3	0.00968	0.00703	0.00497	0.00349
4	0.01674	0.01291	0.00968	0.00703
5	0.02616	0.02116	0.01674	0.01291
6	0.03794	0.03176	0.02616	0.02116
$MSE(\hat{C}_{pk}^e) (36)$				
2	0.00500	0.00349	0.00261	0.00232
3	0.00971	0.00703	0.00497	0.00349
4	0.01667	0.01291	0.00968	0.00703
5	0.02619	0.02116	0.01674	0.01291
6	0.03797	0.03176	0.02616	0.02116
$MSE(\hat{C}_{pk})$				
2	0.00516	0.00357	0.00263	0.00232
3	0.01017	0.00733	0.00514	0.00357
4	0.01769	0.01359	0.01015	0.00733
5	0.02772	0.02236	0.01766	0.01359
6	0.04025	0.03364	0.02769	0.02236

Table 11: Values of $MSE(\hat{C}_{pk}^e)$ for $n = 50$, $\tau = 0.25$

$$\begin{aligned}
MSE(\hat{C}_{pk}) &> MSE(\hat{C}_{pk}^e) \text{ for } \tau < \tau_{04}, \\
MSE(\hat{C}_{pk}) &< MSE(\hat{C}_{pk}^e) \text{ for } \tau > \tau_{04},
\end{aligned} \tag{41}$$

where

$$\tau_{04} = \sqrt{\left(\frac{-\frac{f}{f-2}}{\frac{f}{f-2} - \frac{2}{b_f}}\right)^2 - 1}. \tag{42}$$

Since τ_{04} coincides with τ_{01} (see (19)), the same considerations provided in the previous section about the comparison between $MSE(\hat{C}_p)$ and $MSE(\hat{C}_p^e)$ still hold. As an illustration in Table 11 we report for $n = 50$ and $\tau = 0.25$ the values of $MSE(\hat{C}_{pk}^e)$ calculated using the approximation (40), the values of $MSE(\hat{C}_{pk}^e)$ computed using the exact formula (36) and the values of $MSE(\hat{C}_{pk})$. From the Table one can check the accuracy of the approximations and that since $\tau = 0.25$ is less than τ_{01} for $n = 50$ we have $MSE(\hat{C}_{pk}) > MSE(\hat{C}_{pk}^e)$.

5 Concluding remarks

In this work we have been concerned with the problem of measurement errors when dealing with process capability indices. In particular, the pioneering analysis of Mittag [8] has been extended by turning our attention to the inferential side of the problem, i.e. by considering the effects of measurement errors on the properties of capability indices estimated from sample data.

In the following we summarize the main results obtained in this paper.

First, the C_p estimator, that one obtained from sample data contaminated by random measurement errors, is generally biased. The bias has two sources, the sampling error whose effect is overestimating C_p and the measurement error which causes the underestimating effect on C_p . The combination of these two effects involves that the overall bias can be either positive or negative, according to which effect is predominant. Further, if the two effects offset each other, a null bias could occur. The bias of \hat{C}_p^e tends towards steady negative values as $n \rightarrow \infty$ and increases with the contamination degree τ . It should be noted that this behavior is rather different from that displayed by the usual estimator of C_p in the measurement error free case, where the bias is always positive and goes to zero as $n \rightarrow \infty$. Thus, for suitable combinations of not very large n and small τ a relatively small absolute bias can occur, even smaller than the bias of \hat{C}_p . But, as soon as n becomes sufficiently large, \hat{C}_p^e underestimates the true C_p .

As far as the variability of \hat{C}_p^e is concerned, we have shown that $Var(\hat{C}_p^e)$ is never greater than $Var(\hat{C}_p)$. Therefore, when comparing the mean square errors of the two estimators, the bias component plays a more prominent role. In particular, for increasing n , $MSE(\hat{C}_p^e)$ converges to a finite quantity greater than zero, while $MSE(\hat{C}_p)$ goes to zero. Further, for small and moderate sample sizes, it could happen that $MSE(\hat{C}_p^e) < MSE(\hat{C}_p)$, but for large n $MSE(\hat{C}_p^e)$ tends to be definitively greater than $MSE(\hat{C}_p)$. However the difference between the two $MSEs$ is practically insignificant if τ remains very small.

Substantially similar results have been obtained for the C_{pk} index, although more difficult computational efforts have been required for the analysis of the statistical properties of the measurement error affected estimator \hat{C}_{pk}^e . In particular, we have been able to derive analytical expressions for quantities such as bias, asymptotic bias and mean square error, allowing us to study in detail the behavior of the estimator for different values of n and/or τ , and to compare this behavior with that of the usual estimator in the measurement error free case.

In the whole, it seems to us that the results obtained in this paper have some interest for the following reasons: (a) they focus one's attention on an often neglected problem in industrial practice, that one represented by the presence of errors when measuring a quality characteristics; (b) they give a first picture about the effects of measurement errors when dealing with process capability indices; (c) they provide the practitioner with useful tools to evaluate the reli-

ability of his capability index, taking into account both the sampling error and the measurement error. In this connection such results state that if n is not too large and τ is sufficiently small, measurement errors do not represent a serious problem in the evaluation of a process capability index, while in the remaining situations it is reasonable to take in account its effects, since they lead to a systematical undervaluation of the index. Clearly, the previous analysis represents only a first step towards supplying a more useful and complete picture. In this connection further extensions are needed and we are currently undertaking this task turning our efforts towards two directions, concerning the investigation of other process capability indices, as C_{pm} , and different specifications of the measurement error.

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