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Free Covers and Minimal Sets of Generators

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Abstract. We continue our study of the relation between existence of projective covers and existence of minimal sets of generators for right modules over an arbitrary ring.

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1. Introduction

In our previous papers [3,4], we studied the relations between the existence of projective covers and the existence of minimal sets of generators for right modules over a local ring. Right perfect rings appear to have a particularly good behaviour for the existence of minimal sets of generators. In this paper, we extend part of the results obtained in those two papers to the case of right modules over an arbitrary ring, not necessarily local.

To this end, it is necessary to replace the notion of projective cover with that of free cover, that is, a projective cover that is a free module (Proposition 2.1(1)). In fact, we prove that over an arbitrary ring, the relation is between the existence of *free* covers and the existence of minimal sets of generators. This explains why right perfect rings appear so naturally in the study of existence of minimal sets of generators in the local case, in which projective modules are free, and free covers always exist. As in our previous papers [3,4], our results are heavily based on results by Hrbek and Růžička [7,8] and on the article [9].

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Similarly to [3,4], we consider three classes of right R-modules: the class $\mathcal{A}$ of all right R-modules with a free cover; the class $\mathcal{B}$ of right R-modules for which every set of generators contains a minimal set of generators; and the class $\mathcal{C}$ of all right R-modules with a minimal set of generators. One has that $\mathcal{A} \cup \mathcal{B} \subseteq \mathcal{C}$. The class $\mathcal{A} \cap \mathcal{B}$ consists of all finitely generated right R-modules M_R with $M/MJ(R)$ a free right $R/J(R)$-module if the ring R is not right perfect (Theorem 5.4). If R is local and right perfect, then $\mathcal{A} = \mathcal{B} = \mathcal{C} = \text{Mod-}R$ [3, Theorem 4.3]. In the case of R right perfect but not local, little is known [8, Problem 5.1]. We still even don't know whether every set of generators of a free right module over a semisimple artinian ring always contains a minimal set of generators.

The rings we consider are associative rings R with identity 1_R. The Jacobson radical of R is denoted by $J(R)$.

2. Free Covers

2.1. Elementary Facts

A submodule N of a module M_R is *superfluous* (or *small*, or *inessential*) in M_R if, for every submodule L of M_R, $N + L = M_R$ implies $L = M_R$. An epimorphism $g\colon M_R \to N_R$ is *superfluous* if $\ker g$ is a superfluous submodule of M_R. It is easy to see that an epimorphism $g\colon M \to N$ is superfluous if and only if for every module L and every homomorphism $h\colon L \to M$, if gh is onto, then h is onto. A *projective cover* of a module M_R is a pair (P_R, p) where P_R is a projective right R-module and $p\colon P \to M$ is a superfluous epimorphism.

Let (P, p) be a projective cover of a right R-module M. If Q is a projective module and $q\colon Q \to M$ is an epimorphism, then Q has a direct-sum decomposition $Q = P' \oplus P''$ where $P' \cong P$, $P'' \subseteq \ker(q)$ and $(P', q|_{P'}\colon P' \to M)$ is a projective cover. Projective covers don't exist in general, but when they exist, they are unique up to isomorphism in the following sense. If (P, p), (Q, q) are any two projective covers of a right R-module M, there is an isomorphism $h\colon Q \to P$ such that $p \circ h = q$.

The notion of projective cover has been extended to that of $\mathcal{C}$-cover, replacing the class of projective right R-modules with another class $\mathcal{C}$ of right R-modules. More precisely, in the standard definition of $\mathcal{C}$-precover and $\mathcal{C}$-cover ([2], [6, Section 5.1], [10, Section 1.2]), it is usually required that the class $\mathcal{C}$ of right R-modules is closed under isomorphism and direct summands. In this paper, we need *free* covers and precovers, in which $\mathcal{C}$ is the class of all free right R-modules, and therefore some care is needed.

Hence, for us, a *free precover* of a right module M_R is any module epimorphism $\varphi\colon F_R \to M_R$ with F_R a free right R-module. Equivalently, a module morphism $\varphi\colon F_R \to M_R$ with F_R a free right R-module is a free precover if and only if for every free R-module F'_R the group morphism $\text{Hom}(F'_R, \varphi)\colon \text{Hom}(F'_R, F_R) \to \text{Hom}(F'_R, M_R)$ is a surjective mapping.

A *free cover* of a right module M_R is any free precover $\varphi\colon F_R \to M_R$ such that, for every endomorphism f of F_R, the equality $\varphi f = \varphi$ implies that f is an automorphism of F_R.

It is easily seen ([1, Section 27], [10, Theorem 1.2.12]) that:

Proposition 2.1. (1) *A free cover of a module M_R is exactly a projective cover $\varphi\colon F_R \to M_R$ of M_R with F_R a free R-module.*

(2) *A right R-module has a free cover if and only if it is isomorphic to a module F_R/S, where F_R is a free R-module and S is a superfluous submodule of F_R.*

(3) *Every right R-module has a free cover if and only if R is a right perfect ring over which every projective right R-module is free, if and only if $R = 0$ or R is a local right perfect ring.*

Proof. The statement is trivial, except perhaps for the fact that if $R \neq 0$ is a right perfect ring over which every projective right R-module is free, then R is local. Now if $R \neq 0$ is a right perfect ring over which every projective right R-module is free, R has at least a maximal right ideal M, and the simple module R_R/M has a free cover $R^{(X)} \to R/M$. This is in particular a projective cover, and the canonical projection $R_R \to R/M$ is an epimorphism. Hence $R^{(X)}$ is a direct summand of R_R, so that $R^{(X)}$ is isomorphic to eR for some non-zero idempotent $e \in R$. The non-zero R-module eR has a maximal superfluous submodule because it is a projective cover of the simple module R/M. Since it is a finitely generated projective module, and it must be free, it must be isomorphic to R^n_R for some $n \geq 0$. But if R^n_R has a maximal superfluous submodule, then $n = 1$, and R is a local ring. □

A finitely generated right R-module has a free cover if and only if it is isomorphic to a module R^n_R/S, where S is a submodule of $J(R)^n$.

2.2. Superfluous Submodules of Free Modules

We now recall a result due to Sexauer and Warnock [9]. Let R be a ring with identity and X be a set. Consider the endomorphism ring of the free right R-module $F_R = R^{(X)}_R = \bigoplus_{x \in X} xR$. The endomorphism ring $E := \mathrm{End}(F_R)$ of the module F_R is isomorphic to the ring of all column-finite $X \times X$ matrices with entries in R. For any such matrix A, let

$\mathcal{A}(A, x)$ denote the right ideal of R generated by all entries on the x-th row of A. Let $\pi_x\colon R^{(X)}_R \to R_R$ denote the canonical projection of F_R onto the x-th direct summand of $R^{(X)}_R$, and, for any column-finite matrix $A \in E$, let φ_A indicate the endomorphism of F_R corresponding to the matrix A. Then $\mathcal{A}(A, x) = \pi_x(\varphi_A(F_R))$.

An indexed family of right ideals $\mathcal{I}_x$, $x \in X$, of R is *left vanishing* [9] if for every sequence $x_1, x_2, x_3, \dots$ of distinct elements of X and every sequence a_i of elements of $\mathcal{I}_{x_i}$ there exists a positive integer m such that $a_m a_{m-1} \cdots a_2 a_1 = 0$. When the set X is finite, all families of right ideals indexed in X are left vanishing.

Theorem 2.2. (Sexauer and Warnock *[9]) Let R be a ring, X be a set, and F_R the free right R-module $R^{(X)}_R = \bigoplus_{x \in X} xR$. Then an endomorphism φ*

of F_R belongs to the Jacobson radical $J(\mathrm{End}(F_R))$ if and only if the indexed family of right ideals $\pi_x(\varphi(F_R))$, $x \in X$, is a left vanishing family of right ideals of R and $\pi_x(\varphi(F_R)) \subseteq J(R)$ for every $x \in X$.

We thus obtain a characterization of superfluous submodules of F_R:

Proposition 2.3. *A submodule of the free right module $F_R = R_R^{(X)} = \bigoplus_{x\in X} xR$ over a ring R is a superfluous submodule of F_R if and only if it is contained in a submodule of F_R of the form $\bigoplus_{x\in X} x\mathcal{I}_x$ for some left vanishing indexed family $\{\, \mathcal{I}_x \mid x \in X \,\}$ of right ideals of R with $\mathcal{I}_x \subseteq J(R)$ for every $x \in X$.*

The proof of Proposition 2.3 is the same as the proof of [3, Proposition 2.3].

Proposition 2.4. *If R is an integral domain, an infinitely generated right R-module has a free cover if and only if it is the direct sum of a free module and a finitely generated module with a free cover. More generally, if R is a ring, P is a completely prime ideal of R and $\{\, \mathcal{I}_x \mid x \in X \,\}$ is a left vanishing indexed family of right ideals of R, then $\mathcal{I}_x \subseteq P$ for almost all $x \in X$.*

Proof. If R is a ring and $\mathcal{I}_x \nsubseteq P$ for infinitely many indices $x \in X$, it is easy to construct a sequence $a_n \in \mathcal{I}_x \setminus P$ with $a_n \dots a_1 \notin P$ for every $n \geq 1$.

Now suppose R an integral domain. Let M_R be an infinitely generated right R-module with a free cover. Then $M_R \cong R_R^{(X)}/S$ for some infinite set X and a superfluous submodule S of $R_R^{(X)}$. Therefore $S \subseteq \bigoplus_{x\in X} x\mathcal{I}_x$ for some left vanishing indexed family $\{\, \mathcal{I}_x \mid x \in X \,\}$ of right ideals of R with $\mathcal{I}_x \subseteq J(R)$ for every $x \in X$. From the previous paragraph (with $P = 0$), $\mathcal{I}_x = 0$ for almost all $x \in X$. So $F := \{\, x \in X \mid \mathcal{I}_x \neq 0 \,\}$ is a finite set. Then M_R is isomorphic to the direct sum of the free right R-module $R_R^{(X\setminus F)}$ and the finitely generated module with a free cover $R_R^{(F)}/S$. □

3. Minimal Sets of Generators

A *minimal set of generators* for a module M_R is a set of generators X for M_R with the property that, for every $x \in X$, the subset $X \setminus \{x\}$ generates a proper submodule of M_R.

Let us show that superfluous epimorphisms are exactly the module morphisms that behave well as far as sets of generators and minimal sets of generators are concerned. Of course, free covers are exactly the superfluous epimorphisms $f \colon A_R \to B_R$ with A_R free.

Proposition 3.1 [4, Proposition 6.6]. *The following conditions are equivalent for a morphism $f \colon A_R \to B_R$ between right modules A_R, B_R over an arbitrary ring R:*

(a) *$f \colon A_R \to B_R$ is a superfluous epimorphism.*

(b) *For every subset X of A_R, X generates A_R if and only if $f(X)$ generates B_R.*

By $\mathrm{rad}(A_R)$ we denote the *radical* of a module A_R, that is, the intersection of all its maximal submodules.

Corollary 3.2. *Let $f\colon A_R \to B_R$ be a superfluous epimorphism between two right modules A_R, B_R over an arbitrary ring R. Then:*

(a) *The sets of generators of A_R are the subset X of A_R such that $f(X)$ generates B_R.*

(b) *The minimal sets of generators of A_R are the subset X of A_R such that $f(X)$ is a minimal sets of generators of B_R and $f(x) \neq f(x')$ for every pair x, x' of distinct elements of X.*

(c) *There is a one-to-one correspondence between the set of all maximal submodules M of A_R and the set of all maximal submodules N of B_R given by $M \mapsto f(M)$.*

(d) $f(\mathrm{rad}(A_R)) = \mathrm{rad}(B_R)$ *and* $f^{-1}(\mathrm{rad}(B_R)) = \mathrm{rad}(A_R)$.

Proof. (a) and (b) appear in [4, Corollary 6.7]. (c) and (d) follow from the fact that any maximal submodule contains all superfluous submodules [5, proof of Lemma 2.15]. □

By (b), if $f\colon A_R \to B_R$ is a superfluous epimorphism, the minimal sets of generators of A_R are the subset $g(Y)$, where Y is a minimal set of generators of B_R and $g\colon Y \to A_R$ is any mapping such that $fg\colon Y \to B_R$ is the inclusion of Y into B_R.

The inverse of the one-to-one correspondence in (c) is $N \mapsto f^{-1}(N)$.

Let R be any ring. There are three classes of right R-modules interconnected with each other:

(a) The class $\mathcal{A}$ of right R-modules with a free cover.

(b) The class $\mathcal{B}$ of right R-modules for which every set of generators contains a minimal set of generators.

(c) The class $\mathcal{C}$ of right R-modules with a minimal set of generators.

Trivially, $\mathcal{B} \subseteq \mathcal{C}$. Also, $\mathcal{A} \subseteq \mathcal{C}$, as the next Proposition shows. Its proof is easy.

Proposition 3.3. *A right R-module M_R has a free cover if and only if it is has a minimal set X of generators and the kernel of the canonical mapping $\varphi\colon R_R^{(X)} \to M_R$ is a superfluous submodule of $R_R^{(X)}$.*

As far as the modules in the class $\mathcal{B}$ are concerned, we have the following Proposition:

Proposition 3.4. *Let M_R be a module over an arbitrary ring R, and S be a superfluous submodule of M_R. If the R-module M_R has the property that every set of generators contains a minimal set of generators, then the R-module M_R/S has the same property.*

Proof. Apply Proposition 3.1 and Corollary 3.2 to the canonical projection $M_R \to M_R/S$, which is a superfluous epimorphism. □

Similarly:

Theorem 3.5. *Let M_R be a module over an arbitrary ring R. If the R-module M_R has the property that every set of generators contains a minimal set of generators, then the $R/J(R)$-module $M_R/M_RJ(R)$ has the same property.*

Proof. Let M_R be a right R-module with the property that every set of generators of M_R contains a minimal set of generators. Let $\mathcal{F} \subseteq M_R/M_RJ(R)$ be a set of generators of the $R/J(R)$-module $M_R/M_RJ(R)$ (equivalently, of the $R/J(R)$-module $M_R/M_RJ(R)$). Let $g\colon M_R/M_RJ(R) \to M_R$ be a mapping with $\pi g = 1_{M/MJ(R)}$, where $\pi\colon M_R \to M_R/M_RJ(R)$ denotes the canonical projection. Then the disjoint union $g(\mathcal{F} \setminus \{0_{M/MJ(R)}) \cup MJ(R)$ generates M_R, so that there exist subsets $\mathcal{G} \subseteq (\mathcal{F} \setminus \{0_{M/MJ(R)})$ and $Y \subseteq MJ(R)$ such that $\mathcal{G} \cup Y$ is a minimal set of generators of M_R. Then $\pi(\mathcal{G})$ generates $M_R/M_RJ(R)$. Let us show that $\pi(\mathcal{G})$ is a minimal set of generators for $M_R/M_RJ(R)$. Fix $y_0 \in \pi(\mathcal{G})$. Then $y_0 = \pi(g_0)$ for some $g_0 \in \mathcal{G}$. It remains to show that $\pi(\mathcal{G}) \setminus \{y_0\}$ generates a proper subset of $M_R/M_RJ(R)$. Now $(\mathcal{G} \setminus \{y_0\}) \cup Y$ generates a proper submodule of M_R, because $\mathcal{G} \cup Y$ is a minimal set of generators of M_R. Thus $T := \sum_{g\in\mathcal{G}\setminus\{y_0\}} gR + \sum_{y\in Y} yR$ is a proper submodule of M_R, and M_R/T is a non-zero cyclic R-module. By Nakayama's Lemma, $(M_R/T)J(R)$ is properly contained in M_R/T. Hence $M_RJ(R) + T$ is properly contained in M_R. But $M_RJ(R) + T = M_RJ(R) + \sum_{g\in\mathcal{G}\setminus\{y_0\}} gR$. Applying the canonical projection $\pi\colon M_R \to M_R/M_RJ(R)$ to the inclusions $M_RJ(R) \subseteq M_RJ(R) + \sum_{g\in\mathcal{G}\setminus\{y_0\}} gR \subset M_R$, we get that $\pi(M_RJ(R) + \sum_{g\in\mathcal{G}\setminus\{y_0\}} gR)$ is properly contained in $M_R/M_RJ(R)$, that is, $\sum_{g\in\mathcal{G}\setminus\{y_0\}} \pi(g)R$ is properly contained in $M_R/M_RJ(R)$. This proves that $\pi(\mathcal{G})$ is a minimal set of generators for $M_R/M_RJ(R)$. □

Propositions 2.1(2) and 3.3, which describe modules with a free cover, can be weakened to describe modules with a minimal set of generators, as follows.

Proposition 3.6. *A right R-module has a minimal set of generators if and only if it is isomorphic to a module F_R/S, where $F_R = R_R^{(X)}$ is a free R-module and S is a submodule of F_R such that $S + R_R^{(X\setminus\{x\})} \neq R_R^{(X)}$ for every $x \in X$.*

Notice that $R_R^{(X\setminus\{x\})} = \ker(\pi_x)$, where $\pi_x\colon R_R^{(X)} \to R_R$ is the canonical projection onto the x-th direct summand of $R_R^{(X)}$. Thus $S + R_R^{(X\setminus\{x\})} \neq R_R^{(X)}$ if and only if $\pi_x(S) \neq R_R$. That is:

Corollary 3.7. *A right R-module has a minimal set of generators if and only if it is isomorphic to a module F_R/S, where $F_R = R_R^{(X)}$ is a free R-module and S is a submodule of $\bigoplus_{x\in X} x\mathcal{I}_x$ for some indexed family $\{\,\mathcal{I}_x \mid x \in X\,\}$ of proper right ideals of R.*

In the following, we need two well known lemmas:

Lemma 3.8. *Let $A_R \leq B_R \leq C_R$ be R-modules. Then B_R is superfluous in C_R if and only if A_R is superfluous in C_R and B_R/A_R is superfluous in C_R/A_R.*

Lemma 3.9 [1, Lemma 28.3]. *The following conditions are equivalent for a right ideal I of a ring R:*

(a) *I is right T-nilpotent.*

(b) *M_RI is superfluous in M_R for every right R-module M_R.*

(c) *There exists an infinite set X for which $R_R^{(X)}I = I^{(X)}$ is a superfluous submodule of $R_R^{(X)}$.*

Proposition 3.10. *Let R be a ring, I a two-sided ideal of R contained in the Jacobson radical $J(R)$ of R, M_R an R-module that is not finitely generated, and suppose $MI = 0$. Then the following conditions are equivalent:*

(1) *The module M_R has a free cover.*

(2) *The module $M_{R/I}$ has a free cover and I is a right T-nilpotent ideal of R.*

Proof. Suppose that (1) holds. Let $\pi\colon R_R^{(X)} \to M_R$ be a free cover for M_R. The set X is infinite because M_R is not finitely generated. Then $\pi(I^{(X)}) = \pi(R^{(X)}I) = M_RI = 0$. Hence we have an induced mapping $\overline{\pi}\colon (R/I)^{(X)} \to M_R$ and three right R-modules $I^{(X)} \leq \ker(\pi) \leq R_R^{(X)}$. As $\ker(\pi)$ is superfluous in $R_R^{(X)}$, by Lemma 3.8 we have that $I^{(X)}$ is superfluous in $R_R^{(X)}$ and $\ker(\pi)/I^{(X)} = \ker(\overline{\pi})$ is superfluous in $R_R^{(X)}/I^{(X)}$. In particular, $\overline{\pi}$ is a free cover of the module $M_{R/I}$. Since $I^{(X)} = R^{(X)}I$ is superfluous in $R_R^{(X)}$, we get that I is right T-nilpotent by Lemma 3.9.

Conversely, suppose (2) holds. Let $\pi'\colon (R/I)^{(X)} \to M_R$ be a free cover of $M_{R/I}$. Let $\pi''\colon R_R^{(X)} \to (R/I)^{(X)}$ be the canonical projection. The kernel of the composite mapping $\pi'\pi''$ contains $I^{(X)}$ and is such that $\ker(\pi'\pi'')/I^{(X)} = \ker(\pi')$, which is a superfluous submodule of $(R/I)^{(X)}$. By Lemma 3.9, from I right T-nilpotent we get that $I^{(X)}$ is superfluous in $R_R^{(X)}$. From Lemma 3.8, we obtain that $\ker(\pi'\pi'')$ is superfluous in $R_R^{(X)}$. Hence $\pi'\pi''$ is the required free cover for M_R. □

4. Modules for Which Every Set of Generators Contains a Minimal Set of Generators

Lemma 4.1. *Let R be a ring with Jacobson radical $J(R)$ and M_R a right R-module. Assume that every set of generators of M_R contains a minimal set of generators. Let $\mathcal{F} := \{\, x_\lambda \mid \lambda \in \Lambda \,\}$ be an indexed family of elements of M_R. If $\{\, x_\lambda + MJ(R) \mid \lambda \in \Lambda \,\}$ is a minimal set of generators for $M/MJ(R)$, then $\mathcal{F}$ is a minimal set of generators for M_R.*

Proof. Set $N_R := \sum_{\lambda\in\Lambda} x_\lambda R$, so that $N_R + MJ(R) = M_R$. The disjoint union $\mathcal{F} \cup MJ(R)$ generates M_R, so that there exist subsets $\mathcal{G} \subseteq \mathcal{F}$ and $Y \subseteq MJ(R)$ such that $\mathcal{G} \cup Y$ is a minimal set of generators of M_R. Then $\mathcal{G} = \mathcal{F}$, because if $\mathcal{G} \subset \mathcal{F}$, then $\mathcal{G} \cup Y$ generates M_R implies $\mathcal{G} \cup MJ(R)$ generates M_R, so $\overline{\mathcal{G}} := \{\, x_\lambda + MJ(R) \mid \lambda \in \Lambda,\ x_\lambda \in \mathcal{G} \,\}$ generates $M/MJ(R)$, which is not. Thus $\mathcal{F} \cup Y$ is a minimal set of generators for M_R. Let us show that $\overline{\mathcal{Y}} := \{\, y + N_R \mid y \in Y \,\}$ is a minimal set of generators for M_R/N. For every $y_0 \in Y$, we know that $\sum_{y\in Y\setminus\{y_0\}} yR + N_R \neq M_R$ because $\mathcal{F} \cup Y$ is a minimal set of generators for M_R. It follows that $\overline{\mathcal{Y}}$ is a minimal set of generators for M_R/N. Moreover $(M_R/N)J(R) = M_R/N_R$ because $N_R + MJ(R) = M_R$.

Suppose $M_R/N_R \neq 0$. Then M_R/N_R has a maximal submodule, A_R/N_R say [7, Lemma 3.1]. Hence $J(R)$ annihilates the simple module M_R/A_R, so $MJ(R) \subseteq A_R$. which contradicts $(M_R/N)J(R) = M_R/N_R$. The contradiction proves that $M_R/N_R = 0$, so $Y = \emptyset$. Thus $\mathcal{F}$ is a minimal set of generators for M_R. □

Theorem 4.2. *The following conditions are equivalent for a right module M_R over a ring R:*

(1) *Every set of generators of M_R contains a minimal set of generators.*

(2) *Every set of generators of $M_R/MJ(R)$ contains a minimal set of generators, and the submodule $MJ(R)$ of M_R is superfluous.*

Proof. (1) ⇒ (2) Suppose that (1) holds. Let X' be a set of generators of $M_R/MJ(R)$. Let $g\colon M_R/MJ(R) \to M_R$ be a mapping such that $\pi g = 1_{M/MJ(R)}$, where $\pi\colon M_R \to M_R/MJ(R)$ is the canonical projection. Then $X' \setminus \{0_{M/MJ(R)}\}$ is a set of generators of $M_R/MJ(R)$ and the disjoint union $g(X' \setminus \{0_{M/MJ(R)}\}) \cup MJ(R)$ is a set of generators of M_R. By (1), there exist subsets $\mathcal{G} \subseteq g(X' \setminus \{0_{M/MJ(R)}\})$ and $Y \subseteq MJ(R)$ such that $\mathcal{G} \cup Y$ is a minimal set of generators of M_R. Then $\pi(\mathcal{G})$ is a minimal set of generators of $M/MJ(R)$ contained in X'.

Let us prove that $MJ(R)$ is superfluous in M_R. Let A be a submodule of M_R such that $A + MJ(R) = M_R$. By what we have proved in the previous paragraph, $\pi(A)$ contains a minimal set $\mathcal{G}'$ of generators of M/MP. The restriction $\pi'\colon A_R \to M/MJ(R)$ of the canonical projection $\pi\colon M_R \to M_R/MJ(R)$ is an epimorphism. Let $g'\colon M_R/MJ(R) \to A_R$ be a mapping such that $\pi' g' = 1_{M/MJ(R)}$. By Lemma 4.1 applied to $\mathcal{F} := g'(\mathcal{G}')$, we know that $\mathcal{F}$ is a minimal set of generators for M_R contained in A_R.

(2) ⇒ (1) Assume that (2) holds. Let X be any set of generators for M_R. Then $\{\, x + MJ(R) \mid x \in X \,\}$ is a set of generators for $M_R/MJ(R)$, hence contains a minimal set of generators for $M_R/MJ(R)$. That is, there exists a subset Y of X such that $\{\, y + MJ(R) \mid y \in Y \,\}$ is a minimal set of generators for $M_R/MJ(R)$. Therefore $\sum_{y\in Y} yR + MJ(R) = M_R$. By (2), $\sum_{y\in Y} yR = M_R$. It follows that Y is a minimal set of generators for M_R. □

The next proposition generalizes and completes Lemma 3.9 for the Jacobson radical.

Proposition 4.3. *The following conditions are equivalent for a ring R:*

(a) *$J(R)$ is right T-nilpotent.*

(b) *There exists an infinitely generated module M_R with a free cover and with $M_R J(R)$ superfluous in M_R.*

Proof. (a) ⇒ (b) follows from Lemma 3.9((a) ⇒ (c)).

(b) ⇒ (a). Let M_R be an infinitely generated module with a free cover and with $M_R J(R)$ superfluous in M_R. Then $M_R \cong R_R^{(X)}/S$ for some infinite set X, S a superfluous submodule of $R_R^{(X)}$, $S \subseteq J(R)_R^{(X)}$ (Proposition 2.3). From $M_R J(R)$ superfluous in M_R we get that $J(R)_R^{(X)}/S$ is superfluous in

$R_R^{(X)}/S$. Thus $J(R)_R^{(X)}$ is superfluous in $R_R^{(X)}$ (Lemma 3.8), so $J(R)$ is right T-nilpotent by Lemma 3.9. □

Now let R be a fixed ring. Let us go back to the previous three classes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of right R-modules, with $\mathcal{A} \cup \mathcal{B} \subseteq \mathcal{C}$.

Proposition 4.4. *Let $f\colon M_R \to N_R$ be a superfluous epimorphism between right modules M_R, N_R over an arbitrary ring R. Then:*

(a) *$M_R \in \mathcal{A}$ if and only if $N_R \in \mathcal{A}$.*

(b) *$M_R \in \mathcal{B}$ if and only if $N_R \in \mathcal{B}$.*

(c) *$M_R \in \mathcal{C}$ if and only if $N_R \in \mathcal{C}$.*

Proof. (a) is an easy exercise that follows from the characterization of superfluous epimorphisms $\alpha\colon X_R \to Y_R$ as the epimorphisms $\alpha\colon X_R \to Y_R$ such that αg epi implies g epi for every morphism $g\colon Z_R \to X_R$.

(b) and (c) are essentially a restatement of Corollary 3.2. □

5. The Special Case of Rings R with $J(R)$ T-Nilpotent

Lemma 5.1. *Let M_R be a right module over a ring R.*

(a) *If M_R has a free cover, then $M_R/M_RJ(R)$ is a free right $R/J(R)$-module.*

(b) *If M_R has a projective cover, then $M_R/M_RJ(R)$ is a projective right $R/J(R)$-module and there exists a projective right R-module P_R with*

$$P_R/P_RJ(R) \cong M_R/M_RJ(R).$$

Proof. (a) If M_R has a free cover, then $M_R \cong F_R/S$ for some free R-module F_R and some superfluous submodule S of F_R (Proposition 2.1(2)). Hence $M_R/M_RJ(R) \cong (F_R/S)/(F_R/S)J(R) \cong F_R/(F_RJ(R)+S) = F_R/F_RJ(R)$, because $F_RJ(R) \supseteq S$ (Proposition 2.3). Thus $M_R/M_RJ(R)$ is isomorphic to the free right $R/J(R)$-module $F_R/F_RJ(R)$.

(b) Let $P_R \to M_R$ be a projective cover. There exists a right R-module Q_R such that $P_R \oplus Q_R$ is a free right R-module. Thus there is a free cover $P_R \oplus Q_R \to M_R \oplus Q_R$. Since $M_R \oplus Q_R$ has a free cover, by (a) $M_R/M_RJ(R) \oplus Q_R/Q_RJ(R)$ is a free right $R/J(R)$-module. Therefore its direct summand $M_R/M_RJ(R)$ is a projective right $R/J(R)$-module.

Moreover, like in (a), $M_R \cong P_R/S$ for some superfluous submodule S of P_R. Hence $M_R/M_RJ(R) \cong (P_R/S)/(P_R/S)J(R) \cong P_R/(P_RJ(R)+S) = P_R/P_RJ(R)$, because $P_RJ(R) \supseteq S$ ([5, Lemma 2.15 and Proposition 2.160]). □

Theorem 5.2. *Let R be a ring with $J(R)$ T-nilpotent, and let M_R be a right R-module. Then:*

(a) *The R-module M_R has a free cover if and only if $M_R/M_RJ(R)$ is a free right $R/J(R)$-module.*

(b) *The R-module M_R has a projective cover if and only if $M_R/M_RJ(R)$ is a projective right $R/J(R)$-module and there exists a projective right R-module P_R with $P_R/P_RJ(R) \cong M_R/M_RJ(R)$.*

Proof. In view of the previous lemma, it suffices to prove one of the two implications, both in (a) and in (b).

(a) Assume that $M_R/M_RJ(R)$ is a free right $R/J(R)$-module, $M_R/M_R J(R) \cong (R/J(R))^{(X)}$ say. Then there is an epimorphism $\pi\colon R_R^{(X)} \to M_R/M_R J(R)$ with kernel $(J(R))^{(X)}$. By the projectivity of $R_R^{(X)}$, there is a morphism $f\colon R_R^{(X)} \to M_R$ such that the composite mapping of f with the canonical projection $M_R \to M_R/M_RJ(R)$ is π. Then $f(R_R^{(X)}) + M_RJ(R) = M_R$ and $\ker(f) \subseteq \ker(\pi) = (J(R))^{(X)} = R_R^{(X)}J(R)$. By Lemma 3.9((a) $\Rightarrow$ (b)), it follows that f is an epimorphism, and its kernel is superfluous in $R_R^{(X)}$ by Lemma 3.8. Hence f is a free cover for M_R.

(b) The proof is very similar to the proof of (a). Suppose that $M_R/M_RJ(R)$ is a projective right $R/J(R)$-module and that there exists a projective right R-module P_R with $P_R/P_RJ(R) \cong M_R/M_RJ(R)$. Then there is an epimorphism $\pi\colon P_R \to M_R/M_RJ(R)$ with kernel $P_RJ(R)$. There is a morphism $f\colon P_R \to M_R$ such that the composite mapping of f with the canonical projection $M_R \to M_R/M_RJ(R)$ is π. Then $f(P_R)+M_RJ(R) = M_R$ and $\ker(f) \subseteq P_RJ(R)$. It follows that f is an epimorphism, and its kernel is superfluous in $R_R^{(X)}$. Hence f is a projective cover. □

Remark 5.3. (a) and (b) in Theorem 5.2 do not necessarily hold if $J(R)$ is not T-nilpotent. For instance, let R be a local ring with $J(R)$ not T-nilpotent (e.g., let R be a DVR). In this case, projective modules are free, so (a) and (b) coincide. In fact, for every module M_R, if X is any basis of the $R/J(R)$-vector space $M_R/M_RJ(R)$, the free R-module $P_R := R_R^{(X)}$ is such that $P_R/P_RJ(R) \cong M_R/M_RJ(R)$. For such a ring R, the module $M_R/M_RJ(R)$ is always a free (=projective) right $R/J(R)$-module. But R is not right perfect, and therefore there exist modules M_R without projective covers.

We are ready for our last Theorem:

Theorem 5.4. *If a ring R is not right perfect, then the class $\mathcal{A} \cap \mathcal{B}$ consists of all finitely generated right R-modules M_R with $M/MJ(R)$ a free right $R/J(R)$-module.*

Notice that the finitely generated right R-modules M_R with $M/MJ(R)$ a free right $R/J(R)$-module are those isomorphic to R_R^n/S for some integer $n \geq 0$ and some submodule S of $(J(R))^n$.

Proof. Let R be a ring that is not right perfect. There are two cases: either $J(R)$ is not right T-nilpotent, or $J(R)$ is right T-nilpotent but $R/J(R)$ is not semisimple artinian.

Suppose $J(R)$ is not right T-nilpotent. Assume by contradiction that the class $\mathcal{A}\cap\mathcal{B}$ properly contains the class of all finitely generated right R-modules with a free cover. Then there exists an infinitely module M_R with a free cover such that every set of generators of M_R contains a minimal set of generators. By Theorem 4.2, $MJ(R)$ is superfluous in M_R. By Proposition 4.3, $J(R)$ is T-nilpotent, a contradiction.

Suppose now $J(R)$ right T-nilpotent and $R/J(R)$ not semisimple artinian. There is the canonical functor

$$F = - \otimes_R R/J(R)\colon \text{Mod-}R \to \text{Mod-}(R/J(R)).$$

For every module M_R, the canonical epimorphism $M_R \to F(M_R) = M \otimes_R R/J(R) = M_R/M_RJ(R)$ is always a superfluous epimorphism (Lemma 3.9). Therefore, by Proposition 4.4, $M_R \in \mathcal{A}_R$ if and only if $F(M_R) \in \mathcal{A}_{R/J(R)}$, $M_R \in \mathcal{B}$ if and only if $F(M_R) \in \mathcal{B}_{R/J(R)}$, and $M_R \in \mathcal{C}$ if and only if $F(M_R) \in \mathcal{C}_{R/J(R)}$. In other words, F preserves the classes $\mathcal{A}$, $\mathcal{B}$ (this implies [8, Proposition 2.4(2)]) and $\mathcal{C}$. It also preserves the class mod$-R$ of finitely generated modules. Moreover, $\mathcal{A}_{R/J(R)}$ coincides with the class $\mathcal{F}_{R/J(R)}$ of all free right $R/J(R)$-modules (Theorem 5.2(a)). This reduces the study of the classes $\mathcal{A} = \mathcal{F}$, $\mathcal{B}$ and $\mathcal{C}$ to the case of the rings R with $J(R) = 0$.

Hence we suppose that $J(R) = 0$ but R is not a semisimple artinian ring. In this case, $\mathcal{A} \cap \mathcal{B} = \mathcal{F} \cap \mathcal{B}$ is the class of all finitely generated free right R-modules by [8, Lemma 2.2]. □

If a ring R is local and right perfect, then $\mathcal{A} = \mathcal{B} = \mathcal{C} = \text{Mod-}R$. See [3, Theorem 4.3]. In the case of R right perfect but not local, little is known. For example let R be a semisimple artinian ring that is not a division ring. In this case, we don't know what $\mathcal{A} \cap \mathcal{B} = \mathcal{F} \cap \mathcal{B}$ is. In this case, it is also unknown whether $\mathcal{B} = \text{Mod-}R$ [8, Problem 5.1]. We even don't know whether every set of generators of a free right module over a semisimple artinian ring always contains a minimal set of generators.

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