

# Differentiability properties of the flow of 2d autonomous vector fields

Elio Marconi

EPFL B, Station 8, CH-1015 Lausanne, Switzerland

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## Abstract

We investigate under which assumptions the flow associated to autonomous planar vector fields inherits the Sobolev or BV regularity of the vector field. We consider nearly incompressible and divergence-free vector fields, taking advantage in both cases of the underlying Hamiltonian structure. Finally, we provide an example of an autonomous planar Sobolev divergence-free vector field, such that the corresponding regular Lagrangian flow has no bounded variation.

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**Keywords:** Transport equation; Regular Lagrangian flow; Sobolev regularity; Hamiltonian structure; Nearly incompressible vector fields

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## 1. Introduction

We consider bounded vector fields  $b \in L^\infty((0, T) \times \mathbb{R}^d; \mathbb{R}^d)$ . Although the analysis of this paper is limited to the case of autonomous vector fields with  $d = 2$ , we introduce the relevant notions and the related results in the general setting. The following notion of *regular Lagrangian flow* is an appropriate extension for merely locally integrable vector fields of the classical flow associated to Lipschitz vector fields.

**Definition 1.1.** Given  $b \in L^1_{\text{loc}}((0, T) \times \mathbb{R}^d; \mathbb{R}^d)$ , we say that  $X : [0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a *regular Lagrangian flow* of the vector field  $b$  if

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E-mail address: [elio.marconi@epfl.ch](mailto:elio.marconi@epfl.ch).

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- (1) for  $\mathcal{L}^d$ -a.e.  $x \in \mathbb{R}^d$  the map  $t \mapsto X(t, x)$  is absolutely continuous,  $X(0, x) = x$  and for  $\mathcal{L}^1$ -a.e.  $t \in (0, T)$  it holds  $\partial_t X(t, x) = b(t, X(t, x))$ ;  
 (2) for every  $t \in [0, T)$  it holds

$$X(t, \cdot)_\# \mathcal{L}^d \leq L \mathcal{L}^d,$$

for some  $L > 0$ .

Regular Lagrangian flows have been introduced in a different form in [17], where the authors proved their existence and uniqueness for vector fields  $b \in L_t^1 W_x^{1,p}$  with  $p \geq 1$  and bounded divergence. The theory has been extended to vector fields  $b \in L_t^1 BV_x$  with bounded divergence in [8]. In [9], uniqueness of regular Lagrangian flows was finally achieved in the more general class of *nearly incompressible* vector fields with bounded variation, which was introduced in the study of the hyperbolic system of conservation laws named after Keyfitz and Kranzer (see [18]).

**Definition 1.2.** A vector field  $b \in L_{\text{loc}}^1((0, T) \times \mathbb{R}^d; \mathbb{R}^d)$  is called *nearly incompressible* if there exist  $C > 0$  and  $\rho \in C^0([0, T]; L_w^\infty(\mathbb{R}^d))$  solving the continuity equation

$$\partial_t \rho + \operatorname{div}_x(\rho b) = 0 \quad (1.1)$$

with  $\rho(t, x) \in [C^{-1}, C]$  for  $\mathcal{L}^{d+1}$ -a.e.  $(t, x) \in (0, T) \times \mathbb{R}^d$ .

Several results about the differentiability properties of regular Lagrangian flows are available now. From the contributions in [20,7], it follows that regular Lagrangian flows associated to vector fields  $b \in L_t^1 W_x^{1,1}$  are *differentiable in measure* (see [7] for the definition of this notion). The same regularity property has been obtained recently in [11] for nearly incompressible vector fields with bounded variation. The stronger property of *approximate differentiability* was obtained in [6] for regular Lagrangian flows associated to vector fields  $b \in L_t^1 W_x^{1,p}$  with  $p > 1$ . A quantitative version of the same regularity property was provided in [15], where the authors proved a quantitative Lusin-Lipschitz regularity of the flow, see also [21] for some recent developments for vector fields in  $L_t^1 BV_x$ .

The optimality of the regularity estimates obtained in [15] is discussed in [19,22,23] and more recently in [14]. In particular the authors provided an example of time dependent divergence-free Sobolev vector field in  $\mathbb{R}^2$  such that the regular Lagrangian flow has not Sobolev regularity, even of fractional order.

### 1.1. 2d autonomous vector fields

The analysis in the setting of 2d autonomous vector fields is facilitated by the following Hamiltonian structure (see [3]): if  $b \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$  with  $\operatorname{div} b = 0$ , then there exists a Lipschitz Hamiltonian  $H : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$b = \nabla^\perp H = (-\partial_2 H, \partial_1 H). \quad (1.2)$$

At least formally, the Hamiltonian is preserved by the flow, indeed given an integral curve  $\gamma : [0, T] \rightarrow \mathbb{R}^2$  of the vector field  $b = \nabla^\perp H$ , we can formally compute

$$\frac{d}{dt}H(\gamma(t)) = \nabla H(\gamma(t)) \cdot \dot{\gamma}(t) = \nabla H(\gamma(t)) \cdot \nabla^\perp H(\gamma(t)) = 0,$$

so that the trajectories of the flow are contained in the level sets of  $H$ . In the series of papers [3, 1, 2], the authors reduced the uniqueness problem for the continuity equation to a family of one-dimensional problems on the level sets of  $H$ . With this approach, they were able to characterize the Hamiltonians for which the uniqueness for (1.1) holds in the class of  $L^\infty$  solutions, and therefore the uniqueness for the regular Lagrangian flow, including in particular the case of BV vector fields.

It is worth mentioning that, before the general result in [9] was available, the approach introduced above allowed to obtain in [10] a simpler and more direct proof of the uniqueness of regular Lagrangian flow for nearly incompressible vector fields with bounded variation; see also [12] for the intermediate step of steady nearly incompressible vector fields, namely vector fields satisfying Definition 1.2 with  $\rho$  constant in time.

The Lusin-Lipschitz regularity estimate of the flow associated to Sobolev vector fields obtained in [15] has been extended in [13] to the case of autonomous divergence-free vector fields  $b \in \text{BV}(\mathbb{R}^2; \mathbb{R}^2)$ , although with a less explicit dependence on the regularity of  $b$ . As in the Sobolev setting, this implies that the flow associated to autonomous divergence-free vector fields  $b \in \text{BV}(\mathbb{R}^2; \mathbb{R}^2)$  is approximately differentiable.

In the present paper, we investigate under which assumptions the regular Lagrangian flow inherits the Sobolev or BV regularity of the vector field. The first result is a local estimate for nearly incompressible vector fields.

**Proposition 1.3.** *Let  $b \in \text{BV}(\mathbb{R}^2; \mathbb{R}^2)$  be a bounded nearly incompressible vector field and let  $\Omega \subset \mathbb{R}^2$  be an open ball of radius  $R > 0$  such that there exist  $\delta > 0$  and  $e \in \mathbb{S}^1$  for which  $b \cdot e > \delta$  a.e. in  $\Omega$ . Let  $\Omega' \subset \Omega$  be an open set and  $\bar{t} > 0$  be such that  $\text{dist}(\Omega', \partial\Omega) > \|b\|_{L^\infty} \bar{t}$ . Then*

$$X(\bar{t}) \in \text{BV}(\Omega').$$

Moreover, if  $b \in W^{1,p}(\mathbb{R}^2; \mathbb{R}^2)$  for some  $p \geq 1$ , then

$$X(\bar{t}) \in W^{1,p}(\Omega').$$

We now comment on the assumptions of the above proposition: from the analysis in [3], it emerges that the uniqueness of regular Lagrangian flows associated to 2d autonomous divergence-free vector fields is determined by the properties of the associated Hamiltonian  $H$  in a neighborhood of the set where  $b = \nabla^\perp H$  vanishes. Accordingly, we will see in Section 4 that the regularity estimates of the regular Lagrangian flow deteriorate as  $|b|$  approaches 0. The assumption in Proposition 1.3 excludes points where  $b$  vanishes and is locally satisfied by non zero and continuous vector fields. At a more technical level, this assumption allows to reduce the case of general nearly incompressible vector fields to the case of steady nearly incompressible vector fields, see Section 2 for more details.

In the next result, we deduce a global estimate from the local one in Proposition 1.3, under the assumption that the vector field  $b \in \text{BV}(\mathbb{R}^2; \mathbb{R}^2)$  is continuous and divergence-free. Since we are going to consider bounded vector fields, by finite speed of propagation, it is not restrictive to assume that  $b$  has compact support. In particular, there exists a unique Hamiltonian  $H \in C_c^1(\mathbb{R}^2)$  satisfying (1.2) and it is straightforward to check that the set of critical values

$$\mathcal{S} := \{h \in \mathbb{R} : \exists x \in \mathbb{R}^2 (H(x) = h \text{ and } b(x) = 0)\}$$

is closed. Therefore the set of regular values  $\mathcal{R} := H(\mathbb{R}) \setminus \mathcal{S}$  and  $\Omega = H^{-1}(\mathbb{R} \setminus \mathcal{S}) = H^{-1}(\mathcal{R})$  are open.

**Theorem 1.4.** *Let  $b \in \text{BV}(\mathbb{R}^2; \mathbb{R}^2)$  be a continuous divergence-free vector field with bounded support and let  $\Omega$  be defined as above. Then, for every  $t > 0$  the regular Lagrangian flow has a representative*

$$X(t) \in C^0(\Omega) \cap \text{BV}_{\text{loc}}(\Omega).$$

*If moreover  $b \in W^{1,p}(\mathbb{R}^2; \mathbb{R}^2)$ , then  $X(t) \in W^{1,p}_{\text{loc}}(\Omega)$ .*

The last result is an example that shows that the existence of  $\delta > 0$  as in Proposition 1.3 cannot be dropped, as well as the restriction to  $\Omega$  in Theorem 1.4. More precisely it follows from the proof of Proposition 1.5 that Theorem 1.4 fails also if we restrict the flow to the set  $\tilde{\Omega}_c := \{x : |b(x)| \geq c\}$  for some  $c > 0$ , instead of restricting to  $\Omega$ . The main reason is that  $\tilde{\Omega}_c$  is not invariant by the action of the flow, as opposed to  $\Omega$ .

**Proposition 1.5.** *There exists a divergence-free vector field  $b : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $b \in W^{1,p}_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$  for every  $p \in [1, \infty)$ ,  $b(z) \cdot e_1 > 0$  for  $\mathcal{L}^2$ -a.e.  $z \in \mathbb{R}^2$  and for every time  $t > 0$  the regular Lagrangian flow*

$$X(t) \notin \text{BV}_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2).$$

The construction of the Hamiltonian  $H$  associated to  $b$  in Proposition 1.5 is a suitable modification of the construction in [1] of a Lipschitz Hamiltonian for which the uniqueness of the corresponding regular Lagrangian flow fails. As opposed to the already mentioned results in [19,14], the proposed construction disproves the Sobolev regularity of the regular Lagrangian flow also for autonomous vector fields.

We finally mention that the question about the Sobolev or BV regularity of the regular Lagrangian flow associated to autonomous planar vector fields was posed to the author by M. Colombo and R. Tione, motivated by the study of the commutativity property of the flows associated to vector fields with vanishing Lie bracket [16].

## 2. Local estimate for nearly incompressible vector fields

In this section, we prove Proposition 1.3. We begin with two preliminary lemmas about autonomous nearly incompressible vector fields in  $\mathbb{R}^d$ . In the first lemma, we show that in the case of autonomous nearly incompressible vector fields we can assume without loss of generality that the existence time  $T$  of  $\rho$  in Definition 1.2 is arbitrarily large.

**Lemma 2.1.** *Let  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be an autonomous nearly incompressible vector field and let  $\rho : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $C > 0$  be as in Definition 1.2. Then there exists  $\tilde{\rho} \in C^0([0, +\infty); L^\infty_w(\mathbb{R}^d))$  solving (1.1) such that for  $\mathcal{L}^{d+1}$ -a.e.  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$  it holds*

$$\tilde{C}^{-1} \leq \tilde{\rho}(t, x) \leq \tilde{C}, \quad \text{with} \quad \tilde{C} = \tilde{C}(t) := C^{\frac{2t}{T}+1}. \quad (2.1)$$

**Proof.** By Ambrosio's superposition principle (see [4]), there exists a Radon measure  $\eta$  on  $\Gamma_T := C([0, T]; \mathbb{R}^d)$  such that for every  $t \in [0, T]$  it holds

$$(e_t)_\# \eta = \rho(t, \cdot) \mathcal{L}^d,$$

where  $e_t(\gamma) := \gamma(t)$  denotes the evaluation map at time  $t$  defined on  $\Gamma_T$ . We denote by  $\{\eta_x\}_{x \in \mathbb{R}^d} \subset \mathcal{P}(\Gamma_T)$  its disintegration with respect to the evaluation map at time 0, so that

$$\eta = \int_{\mathbb{R}^d} \rho(0, x) \eta_x dx$$

and we define

$$\eta' = \int_{\mathbb{R}^d} \rho(T, x) \eta_x dx.$$

Since  $\rho \in [C^{-1}, C]$ , it holds  $C^{-2}\rho(0, x) \leq \rho(T, x) \leq C^2\rho(0, x)$  for  $\mathcal{L}^d$ -a.e.  $x \in \mathbb{R}^d$ . In particular  $C^{-2}(e_t)_\# \eta \leq (e_t)_\# \eta' \leq C^2(e_t)_\# \eta$  for every  $t \in [0, T]$ , therefore

$$(e_t)_\# \eta' = \rho'(t, \cdot) \mathcal{L}^d, \quad \text{with} \quad C^{-2}\rho(t, \cdot) \leq \rho'(t, \cdot) \leq C^2\rho(t, \cdot). \quad (2.2)$$

Let  $\tilde{\rho} : (0, 2T) \times \mathbb{R}^d \rightarrow \mathbb{R}$  be defined by

$$\tilde{\rho}(t, z) = \begin{cases} \rho(t, z) & \text{if } t \in (0, T], \\ \rho'(t - T, z) & \text{if } t \in (T, 2T). \end{cases}$$

Since  $\tilde{\rho}$  solves (1.1) in  $\mathcal{D}'((0, T) \times \mathbb{R}^d)$  and  $\mathcal{D}'((T, 2T) \times \mathbb{R}^d)$  separately and  $t \mapsto \tilde{\rho}(t)$  is continuous with respect to the weak\* topology in  $L^\infty(\mathbb{R})$ , then  $\tilde{\rho}$  solves (1.1) in  $\mathcal{D}'((0, 2T) \times \mathbb{R}^d)$ . By (2.2) it follows that for every  $t \in [T, 2T]$  it holds

$$C^{-2}\tilde{\rho}(t - T, \cdot) \mathcal{L}^d \leq \tilde{\rho}(t, \cdot) \mathcal{L}^d \leq C^2\tilde{\rho}(t - T, \cdot) \mathcal{L}^d. \quad (2.3)$$

Iterating the construction above we obtain a solution  $\tilde{\rho} : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}$  of (1.1) such that (2.3) holds for every  $t \geq T$ . In particular, for every  $N \in \mathbb{N}$  and for every  $t \in [NT, (N + 1)T]$  it holds

$$C^{-2N}\rho(t - NT, \cdot) \mathcal{L}^d \leq \tilde{\rho}(t, \cdot) \mathcal{L}^d \leq C^{2N}\rho(t - NT, \cdot) \mathcal{L}^d,$$

which immediately implies (2.1) since  $\rho \in [C^{-1}, C]$ .  $\square$

The vector fields for which the function  $\rho$  in Definition 1.2 can be chosen independent of  $t$  are called *steady nearly incompressible*. Although not every nearly incompressible autonomous vector field is steady nearly incompressible, we can reduce to the latter case under the assumptions of Proposition 1.3. The proof of the following lemma is an adaptation of the argument in [10] and it is based on the following observation: since  $b$  is autonomous, the integral in time over  $[0, T]$  of the function  $\rho$  as in Definition 1.2 is a steady solution of the continuity equation (1.1) with a

source which arises taking into account the boundary terms in the integration. In particular, this source term vanishes if  $\rho$  is identically 0 in  $\Omega$  both at time 0 and time  $T$ .

**Lemma 2.2.** *Let  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be an autonomous, bounded, nearly incompressible vector field and let  $\Omega \subset \mathbb{R}^d$  be an open ball of radius  $R > 0$ . Assume that there exist  $\delta > 0$  and  $e \in \mathbb{S}^{d-1}$  for which for  $\mathcal{L}^d$ -a.e.  $x \in \Omega$  it holds  $b(x) \cdot e \geq \delta$ . Then  $b \llcorner \Omega$  is steady nearly incompressible, namely there exists  $r : \Omega \rightarrow \mathbb{R}$  and  $\tilde{C} > 0$  such that*

$$\tilde{C}^{-1} \leq r \leq \tilde{C} \quad \text{and} \quad \operatorname{div}(rb) = 0 \quad \text{in } \Omega.$$

**Proof.** Let  $\rho : (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $C > 0$  be as in Definition 1.2. Let  $\eta \in \mathcal{M}(\Gamma_T)$  the Radon measure provided by Ambrosio's superposition principle. In particular if we denote

$$\begin{aligned} \tilde{e} : \Gamma_T \times [0, T] &\rightarrow [0, T] \times \mathbb{R}^d \\ (\gamma, t) &\mapsto (t, \gamma(t)) \end{aligned}$$

it holds

$$\tilde{e}_\#(\eta \times \mathcal{L}^1) = \rho \left( \mathcal{L}^1 \times \mathcal{L}^d \right).$$

For every  $\gamma \in \Gamma$  we set

$$I_\gamma := \{t \in [0, T] : \gamma(t) \in \Omega\}.$$

Let  $I_{\gamma,0} = [0, t_\gamma^-)$  be the (possibly empty) connected component of  $I_\gamma$  containing 0 and similarly let  $I_{\gamma,T} = (t_\gamma^+, T]$  be the connected component of  $I_\gamma$  containing  $T$ . We use the notation

$$\tilde{I}_\gamma = [0, T] \setminus (I_{\gamma,0} \cup I_{\gamma,T})$$

and

$$\Gamma^- := \{\gamma \in \Gamma_T : \gamma(0) \in \Omega\}, \quad \Gamma^+ := \{\gamma \in \Gamma_T : \gamma(T) \in \Omega\}.$$

Moreover we consider

$$\tilde{\eta} = \eta \otimes (\mathcal{L}^1 \llcorner \tilde{I}_\gamma).$$

By definition  $\tilde{\eta} \leq \eta \times \mathcal{L}^1$  therefore there exists  $\tilde{\rho} \in L^\infty((0, T) \times \mathbb{R}^d)$  such that

$$\tilde{e}_\# \tilde{\eta} = \tilde{\rho}(\mathcal{L}^1 \times \mathcal{L}^d),$$

with  $0 \leq \tilde{\rho} \leq \rho$ . The following standard computation shows that the density  $\tilde{\rho}$  satisfies the continuity equation

$$\partial_t \tilde{\rho} + \operatorname{div}_x(\tilde{\rho}b) = \mu \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^d),$$

where

$$\mu = \int_{\Gamma^-} \delta_{t_\gamma^-, \gamma(t_\gamma^-)} d\eta(\gamma) - \int_{\Gamma^+} \delta_{t_\gamma^+, \gamma(t_\gamma^+)} d\eta(\gamma).$$

Given  $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^d)$  it holds

$$\begin{aligned} \langle \partial_t \tilde{\rho} + \operatorname{div}_x(\tilde{\rho}b), \varphi \rangle &= - \int \tilde{\rho}(t, x) (\varphi_t(t, x) + b(t, x) \cdot \nabla_x b(t, x)) dx dt \\ &= - \int (\varphi_t(t, \gamma(t)) + b(t, \gamma(t)) \cdot \nabla_x b(t, \gamma(t))) \chi_{\tilde{I}_\gamma}(t) dt d\eta(\gamma) \\ &= - \int (\varphi_t(t, \gamma(t)) + \dot{\gamma}(t) \cdot \nabla_x b(t, \gamma(t))) \chi_{\tilde{I}_\gamma}(t) dt d\eta(\gamma) \\ &= - \int \frac{d}{dt} (\varphi(t, \gamma(t)) \chi_{\tilde{I}_\gamma}(t)) dt d\eta(\gamma) \\ &= \int (\varphi(t_\gamma^-, \gamma(t_\gamma^-)) - \varphi(t_\gamma^+, \gamma(t_\gamma^+))) d\eta(\gamma) \\ &= \int \varphi d\mu, \end{aligned}$$

where in the last equality we used that  $\varphi(0, \cdot) = \varphi(T, \cdot) \equiv 0$ . In particular  $\mu$  is concentrated on  $\partial\Omega$  so that

$$\partial_t \tilde{\rho} + \operatorname{div}_x(\tilde{\rho}b) = 0 \quad \text{in } \mathcal{D}'((0, T) \times \Omega). \quad (2.4)$$

Since  $b \cdot e > \delta$  in  $\Omega$ , every connected component of  $I_\gamma$  has length at most  $2R/\delta$ . Up to change the constant  $C > 0$ , by Lemma 2.1 we can assume that

$$T \geq \frac{6R}{\delta},$$

therefore it follows that  $\tilde{\rho}(t, z) = \rho(t, z)$  for  $\mathcal{L}^1 \times \mathcal{L}^d$ -a.e.  $(t, z) \in [T/3, 2T/3] \times \Omega$ , in particular

$$\tilde{\rho}(t, z) \geq C^{-1} \quad \text{in } [T/3, 2T/3] \times \Omega. \quad (2.5)$$

Being  $\tilde{\rho}(0, x) = \tilde{\rho}(T, x) = 0$  for  $\mathcal{L}^d$ -a.e.  $x \in \Omega$ , by integrating (2.4) with respect to  $t$ , it follows that

$$r(x) := \frac{1}{T} \int_0^T \tilde{\rho}(t, x) dx$$

satisfies  $\operatorname{div}(rb) = 0$  in  $\mathcal{D}'(\Omega)$ . From (2.5) and the definition of  $r$ , it follows that for  $\mathcal{L}^d$ -a.e.  $x \in \mathbb{R}^2$  it holds

$$\frac{1}{3C} \leq r(x) \leq \|\rho\|_{L^\infty} \leq C$$

and this proves the claim with  $\tilde{C} = 3C$ .  $\square$

In the remainder of the paper we restrict to the case  $d = 2$  and in the remaining part of this section we will always assume that the hypothesis in Proposition 1.3 is satisfied. In particular, there exists a Lipschitz Hamiltonian  $H : \Omega \rightarrow \mathbb{R}$  such that

$$rb = \nabla^\perp H \quad \mathcal{L}^2\text{-a.e. in } \Omega. \quad (2.6)$$

The generic point in  $\mathbb{R}^2$  will be denoted by  $z = (x, y)$  and we assume without loss of generality that  $e = e_1$ . Being  $b \cdot e_1 > \delta$  and  $r \in [\tilde{C}^{-1}, \tilde{C}]$  for every  $h \in H(\Omega)$  there exist an open set  $O_h \subset \mathbb{R}$  and a Lipschitz function  $f_h : O_h \rightarrow \mathbb{R}$  such that

$$\{z \in \Omega : H(z) = h\} = \{(x, y) : x \in O_h, y = f_h(x)\}.$$

We will also denote

$$\tilde{f}_h(x) := (x, f_h(x)) \quad (2.7)$$

for every  $x \in O_h$ . The Lipschitz constant  $L$  of  $f_h$  can be estimated by

$$L \leq \frac{\|rb\|_{L^\infty}}{\inf_{\Omega}(rb \cdot e_1)} \leq \frac{C^2 \|b\|_{L^\infty}}{\delta}. \quad (2.8)$$

In the following we consider vector fields  $b$  with bounded variation. As already mentioned in the introduction, the uniqueness problem for the regular Lagrangian flow associated to  $rb$  was solved in [10], where in particular it is proven that the local Hamiltonian is preserved by the flow, namely

$$H(t, X(t, z)) = H(z) \quad \forall z \in \Omega \text{ and } t < \text{dist}(\partial\Omega, z).$$

We will consider the representative of the regular Lagrangian flow defined as follows: for every  $z = (x, y) \in \Omega$  and  $t < \text{dist}(\partial\Omega, z)$ , we set  $X(t, z) = (X_1(t, z), f_{H(z)}(X_1(t, z)))$  where  $X_1(t, z)$  is uniquely determined by

$$\int_x^{X_1(t, z)} \frac{1}{\tilde{b}_1(s, f_{H(z)}(s))} ds = t, \quad (2.9)$$

and where  $\tilde{b}$  denotes the precise representative of  $b$ , defined at  $\mathcal{H}^1$ -a.e.  $z \in \mathbb{R}^2$  (see for example [5]). In particular, if we denote

$$R := \{h \in H(\Omega) : |Db|(H^{-1}(h) \cap \Omega) = 0\},$$

it holds that  $\mathcal{H}^1$ -a.e.  $z \in H^{-1}(h) \cap \Omega$  is a Lebesgue point of  $b$  with value  $\tilde{b}(z)$ . In the following we will still denote by  $b$  the precise representative  $\tilde{b}$ .



**Proposition 2.3.** Let  $b \in \text{BV}(\mathbb{R}^2; \mathbb{R}^2)$  be a bounded autonomous nearly incompressible vector field. Let  $\Omega \subset \mathbb{R}^2$  be an open ball of radius  $R > 0$  such that there exist  $\delta > 0$  and  $e \in \mathbb{S}^1$  for which  $b \cdot e > \delta$  a.e. in  $\Omega$ . Then there exists  $g \in \text{BV}_{\text{loc}}(\mathbb{R})$  and a constant  $C' = C'(R, \|b\|_{L^\infty}, \delta, C) > 0$  such that for every  $z, z' \in \Omega$  with  $H(z), H(z') \in R$  and every  $\bar{t} > 0$  for which

$$\text{dist}(z, \partial\Omega), \text{dist}(z', \partial\Omega) > \|b\|_{L^\infty} \bar{t},$$

it holds

$$|X(\bar{t}, z) - X(\bar{t}, z')| \leq C'(|z - z'| + |g(H(z)) - g(H(z'))|), \quad (2.10)$$

where  $C$  is the compressibility constant in Definition 1.2 and  $H$  is the Hamiltonian introduced in (2.6).

If moreover  $b \in W^{1,p}(\mathbb{R}^2; \mathbb{R}^2)$  for some  $p \in [1, \infty)$ , then (2.10) holds with  $g \in W_{\text{loc}}^{1,p}(\mathbb{R})$ .

**Proof.** We denote by  $h = H(z)$  and  $h' = H(z')$ . By (2.9) it follows that

$$\int_x^{X_1(\bar{t}, z)} \frac{1}{b_1(\tilde{f}_h(s))} ds = \bar{t} = \int_{x'}^{X_1(\bar{t}, z')} \frac{1}{b_1(\tilde{f}_{h'}(s))} ds, \quad (2.11)$$

where  $\tilde{f}_h$  and  $\tilde{f}_{h'}$  are defined in (2.7). Without loss of generality we assume  $x \leq x'$  and we also suppose that  $X_1(\bar{t}, z) \leq X_1(\bar{t}, z')$ , being the opposite case analogous. We first estimate the distance of the horizontal components of the flows. We denote

$$I_1 = (x, x'), \quad I_2 = (x', X_1(\bar{t}, z)), \quad I_3 = (X(\bar{t}, z), X(\bar{t}, z')).$$

If  $I_2 = \emptyset$ , since  $b \cdot e_1 > \delta$  in  $\Omega$ , then

$$|z' - z| \geq |x' - x| \geq |X_1(\bar{t}, z) - x| \geq \bar{t}\delta,$$

therefore

$$\begin{aligned} |X_1(\bar{t}, z') - X_1(\bar{t}, z)| &\leq |X_1(\bar{t}, z') - x'| + |x' - x| + |x - X_1(\bar{t}, z)| \\ &\leq \|b\|_{\infty} \bar{t} + |x' - x| + \|b\|_{\infty} \bar{t} \\ &\leq \left( \frac{2\|b\|_{\infty}}{\delta} + 1 \right) |x' - x|. \end{aligned}$$

If  $I_2 \neq \emptyset$ , it follows by (2.11) that

$$\begin{aligned}
|X_1(\bar{t}, z') - X_1(\bar{t}, z)| &\leq \|b\|_\infty \int_{I_3} \frac{1}{b_1(\tilde{f}_{h'}(s))} ds \\
&= \|b\|_\infty \left( \int_{I_1} \frac{1}{b_1(\tilde{f}_h(s))} ds + \int_{I_2} \frac{1}{b_1(\tilde{f}_h(s))} ds - \int_{I_2} \frac{1}{b_1(\tilde{f}_{h'}(s))} ds \right) \\
&\leq \frac{\|b\|_\infty}{\delta} |x' - x| + \|b\|_\infty \int_{I_2} \left| \frac{1}{b_1(\tilde{f}_h(s))} - \frac{1}{b_1(\tilde{f}_{h'}(s))} \right| ds \\
&\leq \frac{\|b\|_\infty}{\delta} |x' - x| + \frac{\|b\|_\infty}{\delta^2} |Db|(H^{-1}(I(h, h'))),
\end{aligned}$$

where  $I(h, h')$  denotes the closed interval with endpoints  $h$  and  $h'$ ; in the last inequality we used that the function  $v \mapsto 1/v$  is  $\delta^{-2}$ -Lipschitz on  $(\delta, +\infty)$ . Then we estimate the difference of the vertical components:

$$|X_2(\bar{t}, z') - X_2(\bar{t}, z)| \leq |X_2(\bar{t}, z') - f_{h'}(X_1(\bar{t}, z))| + |f_{h'}(X_1(\bar{t}, z)) - X_2(\bar{t}, z)|. \quad (2.12)$$

By definition of  $f_{h'}$  it holds

$$|X_2(\bar{t}, z') - f_{h'}(X_1(\bar{t}, z))| = |f_{h'}(X_1(\bar{t}, z')) - f_{h'}(X_1(\bar{t}, z))| \leq L |X_1(\bar{t}, z') - X_1(\bar{t}, z)|, \quad (2.13)$$

where  $L$  denotes the Lipschitz constant of the function  $f_{h'}$  and is bounded by  $\frac{C^2 \|b\|_{L^\infty}}{\delta}$  as in (2.8). By definition of  $f_h$  we have that

$$\begin{aligned}
|f_{h'}(X_1(\bar{t}, z)) - X_2(\bar{t}, z)| &= |f_{h'}(X_1(\bar{t}, z)) - f_h(X_1(\bar{t}, z))| \\
&\leq \frac{|h' - h|}{\inf_{\Omega} |\partial_2 H|} \\
&\leq \frac{C|h' - h|}{\delta} \\
&\leq \frac{C^2 \|b\|_{L^\infty} |z' - z|}{\delta}
\end{aligned} \quad (2.14)$$

where we used  $b_1 \geq \delta$ ,  $\partial_y H = r b_1$  and  $\|\nabla H\|_{L^\infty} \leq C \|b\|_{L^\infty}$ . Plugging (2.13) and (2.14) into (2.12), we finally obtain

$$|X_2(\bar{t}, z') - X_2(\bar{t}, z)| \leq \frac{C^2 \|b\|_{L^\infty}}{\delta} \left[ \left( 1 + \frac{2\|b\|_{L^\infty}}{\delta} \right) |z' - z| + \frac{\|b\|_{L^\infty}}{\delta^2} |Db|(H^{-1}(I(h, h'))) \right]$$

so that (2.10) holds with

$$C' = \frac{C^2 \|b\|_{L^\infty}}{\delta^2} \left( 1 + \frac{2\|b\|_{L^\infty}}{\delta} + \frac{\|b\|_{L^\infty}}{\delta^2} \right), \quad g(h) = |Db|(\{H \leq h\}).$$

Notice that  $g \in \text{BV}_{\text{loc}}(\mathbb{R})$  by construction, since  $Dg = H_{\sharp} |Db|$  is a finite Radon measure.

If  $b \in W^{1,p}(\mathbb{R}^2; \mathbb{R}^2)$  the same computation leads to (2.10) with

$$g(h) = \int_{\{H \leq h\}} |Db|(z) dz.$$

It only remains to check that  $g \in W_{\text{loc}}^{1,p}(\mathbb{R})$ . Denoting by  $E_h = \{z \in \Omega : H(z) = h\}$ , by the coarea formula we have that

$$|\nabla H| \mathcal{L}^2 \llcorner \Omega = \int_{\mathbb{R}} \mathcal{H}^1 \llcorner E_h dh \quad \text{so that} \quad \mathcal{L}^2 = \int_{\mathbb{R}} \frac{1}{|\nabla H|} \mathcal{H}^1 \llcorner E_h dh$$

and therefore

$$g'(h) = \int_{E_h} \frac{|Db|}{|\nabla H|} d\mathcal{H}^1.$$

Being  $|\nabla H| = |rb| > \delta/C$ , then by Jensen's inequality and co-area formula we get

$$\begin{aligned} \int_{\mathbb{R}} |g'|^p &= \int_{\mathbb{R}} \left| \int_{E_h} \frac{|Db|}{|\nabla H|} d\mathcal{H}^1 \right|^p dh \\ &\leq \int_{\mathbb{R}} \frac{(C \mathcal{H}^1(E_h))^{p-1}}{\delta^{p-1}} \int_{E_h} \frac{|Db|^p}{|\nabla H|} d\mathcal{H}^1 dh \\ &\leq \left( \frac{2C\sqrt{1+L^2}R}{\delta} \right)^{p-1} \int_{\Omega} |Db|^p, \end{aligned} \tag{2.15}$$

where  $L$  is as above. This concludes the proof of the proposition.  $\square$

In order to conclude the proof of Proposition 1.3, we deduce in the following two lemmas the BV and Sobolev regularity of the flow from the pointwise estimate obtained in Proposition 2.3.

**Corollary 2.4.** *In the same setting as in Proposition 2.3 let  $\Omega' \subset \Omega$  be an open set and  $\bar{t} > 0$  be such that  $\text{dist}(\Omega', \partial\Omega) > \|b\|_{L^\infty} \bar{t}$ . Then*

$$X(\bar{t}) \in \text{BV}(\Omega').$$

**Proof.** From Proposition 2.3 it is sufficient to check that  $g \circ H \in \text{BV}(\Omega')$ . Let  $g_n$  be a sequence of smooth functions converging to  $g$  in  $L^1(\mathbb{R})$  with  $\text{Tot.Var.}_{\mathbb{R}}(g_n) \leq \text{Tot.Var.}_{\mathbb{R}}(g)$ . By coarea formula

$$H_{\#}(\mathcal{L}^2 \llcorner \Omega') = \rho \mathcal{L}^1, \quad \text{with} \quad \rho(h) = \int_{E_h} \frac{1}{|\nabla H|} d\mathcal{H}^1.$$

In particular

$$\rho(h) \leq \frac{C\mathcal{H}^1(E_h)}{\delta} \leq \frac{2RC\sqrt{1+L^2}}{\delta}$$

is uniformly bounded. Hence  $g_n \circ H$  converges in  $L^1(\Omega')$  to  $g \circ H$  and

$$\begin{aligned} \text{Tot.Var.}_{\Omega'}(g \circ H) &\leq \liminf_{n \rightarrow \infty} \text{Tot.Var.}_{\Omega'}(g_n \circ H) \\ &= \liminf_{n \rightarrow \infty} \int_{\Omega'} |g'_n(H(z)) \nabla H(z)| dz \\ &\leq \liminf_{n \rightarrow \infty} \|\nabla H\|_{L^\infty} \int |g'_n(h)| \rho(h) dh \\ &\leq \|\nabla H\|_{L^\infty} \|\rho\|_{L^\infty} \text{Tot.Var.}_{\mathbb{R}}(g). \quad \square \end{aligned}$$

**Corollary 2.5.** *Let us consider the same setting as in Proposition 2.3 with  $b \in W^{1,p}(\mathbb{R}^2; \mathbb{R}^2)$ . Let  $\Omega' \subset \Omega$  be an open set and  $\bar{t} > 0$  be such that  $\text{dist}(\Omega', \partial\Omega) > \|b\|_{L^\infty} \bar{t}$ . Then*

$$X(\bar{t}) \in W^{1,p}(\Omega').$$

**Proof.** From Proposition 2.3 it is sufficient to check that  $g \circ H \in W^{1,p}(\Omega')$ . By chain rule and coarea formula we have

$$\begin{aligned} \int_{\Omega'} |D(g \circ H)|^p dz &\leq \int_{\Omega'} |g' \circ H|^p |\nabla H|^p dz \\ &\leq (C\|b\|_\infty)^{p-1} \int_{\Omega'} |g' \circ H|^p |\nabla H| dz \\ &\leq (C\|b\|_\infty)^{p-1} \int_{\mathbb{R}} \int_{E_h} |g' \circ H|^p d\mathcal{H}^1 dh \\ &= (C\|b\|_\infty)^{p-1} \int_{\mathbb{R}} |g'|^p(h) \mathcal{H}^1(E_h) dh \\ &\leq 2\sqrt{1+L^2} R (C\|b\|_\infty)^{p-1} \int_{\mathbb{R}} |g'|^p(h) dh. \quad \square \end{aligned}$$

### 3. Global estimate for divergence free vector fields

In this section we prove Theorem 1.4. In the next lemma, we show that we can cover  $\Omega$  with countably many open sets invariant for the flow and such that  $|b|$  is uniformly bounded from below far from 0.

**Lemma 3.1.** *Let  $b$  and  $\Omega$  as in Theorem 1.4 and let  $H \in C_c^1(\mathbb{R}^2)$  be the Hamiltonian associated to  $b$  as in (1.2). For every  $k \in \mathbb{N}$  let  $\Omega_k := H^{-1}(\{h : \min_{H^{-1}(h)} |b| > 1/k\})$ . Then  $\Omega_k$  is open,*

$$\overline{\Omega}_k \subset \Omega_{k+1}, \quad \text{and} \quad \Omega = \bigcup_{k \in \mathbb{N}} \Omega_k.$$

**Proof.** The last equality follows immediately from the definition of  $\Omega$  and the continuity of  $b$ . In order to complete the proof it is sufficient to check that the map

$$h \mapsto \min_{H^{-1}(h)} |b|$$

is continuous on the set  $\mathcal{R}$  of regular values of  $H$ . Both the lower semicontinuity and the upper semicontinuity are straightforward consequences of the continuity of  $b$  and the compactness of the level sets  $H^{-1}(h)$  with regular value  $h$ .  $\square$

The main estimate in the proof of Theorem 1.4 is proven in the following lemma.

**Lemma 3.2.** *Let  $H \in C_c^1(\mathbb{R}^2)$  be such that  $b := \nabla^\perp H \in \text{BV}(\mathbb{R}^2)$  and let  $k \in \mathbb{N}$  and  $\Omega_k \subset \mathbb{R}^2$  be as above. Then there exist a representative of the regular Lagrangian flow  $X$ ,  $g \in \text{BV}_{\text{loc}} \cap C^0(\mathbb{R})$  and  $r > 0$  such that for every  $t > 0$  the following holds: there exist  $c_1 > 0$  and  $c_2 > 0$  such that  $\forall \bar{z} \in \Omega_k$  and every  $z \in B_r(\bar{z})$  there exists  $s > 0$  such that*

- (1)  $|X(t, \bar{z}) - X(s, z)| \leq c_1 |H(\bar{z}) - H(z)|$
- (2)  $|t - s| \leq c_2 (|g(H(\bar{z})) - g(H(z))| + |\bar{z} - z|).$

**Proof.** The proof is divided in several steps.

*Step 1.* By Lemma 3.1  $\Omega_k$  is compactly contained in the open set  $\Omega_{k+1}$ . Since  $b$  is continuous and uniformly bounded from below on  $\Omega_{k+1}$ , for every  $L > 0$  there exist  $\bar{r} > 0$  and a finite covering  $(B_{\bar{r}}(z_i))_{i=1}^N$  of  $\Omega_k$  such that

- (1) for every  $i = 1, \dots, N$  it holds  $B_{4\bar{r}}(z_i) \subset \Omega_{k+1}$ ;
- (2) for every  $i = 1, \dots, N$  there exists  $e_i \in \mathbb{S}^1$  such that

$$b(z) \cdot e_i \geq |b(z)| \cos(\tan^{-1}(L)) \quad \forall z \in B_{4\bar{r}}(z_i). \quad (3.1)$$

We take  $L > 0$  sufficiently small so that  $\cos(\tan^{-1}(L)) > 1/2$  and such that for every  $i = 1, \dots, N$  and for every  $h \in H(B_{3\bar{r}}(z_i))$  there exist an open interval  $I_{i,h} \subset \mathbb{R}$  and a  $L$ -Lipschitz function  $f_{i,h} : I_{i,h} \rightarrow \mathbb{R}$  such that

$$H^{-1}(h) \cap B_{4\bar{r}}(z_i) = \left\{ z \in \mathbb{R}^2 : z \cdot e_i \in I_{i,h}, \quad z \cdot e_i^\perp = f_{i,h}(z \cdot e_i) \right\}.$$

*Step 2.* We show that the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$g(h) := |Db|(\{H \leq h\} \cap \Omega_{k+1})$$

is continuous and with bounded variation.

Since  $Dg = H_{\sharp}^+ |Db| \llcorner \Omega_{k+1}$  is a finite measure, the function  $g$  has bounded variation. In order to prove that  $g$  is continuous it is sufficient to check that for every  $h \in \mathbb{R}$  it holds

$$|Db|(H^{-1}(h) \cap \Omega_{k+1}) = 0.$$

By Step 1, the set  $H^{-1}(h) \cap \Omega_{k+1}$  is the union of finitely many Lipschitz curve of finite length. Being  $b$  continuous, the measure  $|Db|$  vanishes on all sets with finite  $\mathcal{H}^1$  measure (see for example [5]), and this proves the continuity of  $g$ .

*Step 3.* Given  $T > 0$  and  $\bar{z} \in \Omega_k$ , we denote

$$\tilde{N} := \left\lceil \frac{T \|b\|_{L^\infty}}{\bar{r}} \right\rceil, \quad \text{and} \quad I_j := \left[ \frac{j-1}{\tilde{N}} T, \frac{j}{\tilde{N}} T \right] =: [t_{j-1}, t_j] \quad \text{for } j = 1, \dots, \tilde{N}.$$

Moreover for every  $j \in 1, \dots, \tilde{N}$  we consider  $i = i(j) \in 1, \dots, N$  such that

$$X(t_j, \bar{z}) \in B_{\bar{r}}(z_{i(j)}).$$

We set

$$r := \min \left\{ \bar{r}, \frac{\bar{r}}{2(k+1)\|b\|_{L^\infty}}, \frac{T}{2\tilde{N}(k+1)} \right\}. \quad (3.2)$$

For every  $z \in \mathbb{R}^2$  with  $|z - \bar{z}| \leq r$  we prove that for every  $j = 1, \dots, \tilde{N}$  there exists  $s_j > 0$  such that

$$|X(t_j, \bar{z}) - X(s_j, z)| \leq 2(k+1)|\bar{h} - h|$$

and

$$|t_j - s_j| \leq 2(k+1)|z - \bar{z}| + j(k+1)^2|g(\bar{h}) - g(h)| + 2(j-1)(k+1)^2|\bar{h} - h|. \quad (3.3)$$

By (3.2) and the definition of  $(z_i)_{i=1}^N$ , for every  $j = 1, \dots, \tilde{N}$  there exists a unique point  $\tilde{z}_j \in \Omega_{k+1}$  in  $z \in B_{2\bar{r}}(z_{i(j)})$  such that

$$H(\tilde{z}_j) = h \quad \text{and} \quad \tilde{z}_j \cdot e_i = X(t_j, \bar{z}) \cdot e_i.$$

We immediately have

$$|X(t_j, \bar{z}) - \tilde{z}_j| = |(X(t_j, \bar{z}) - \tilde{z}_j) \cdot e_i^\perp| \leq \frac{|\bar{h} - h|}{\min_{B_{2\bar{r}}(z_{i(j)})} b \cdot e_j} \leq 2(k+1)|\bar{h} - h|.$$

Notice in particular that  $|X(t_j, \bar{z}) - \tilde{z}_j| \leq \bar{r}$  by (3.2). In order to prove the claim, it is sufficient to show that for every  $j = 1, \dots, \tilde{N}$ , there exists  $s_j \geq 0$  satisfying (3.3) and such that  $X(s_j) = \tilde{z}_j$ . We prove this by induction on  $j$ .

*Case  $j = 1$ .* First we observe that

$$\{\bar{z} \cdot e_{i(1)}, z \cdot e_{i(1)}, X(t_1, \bar{z}) \cdot e_{i(1)}\} \subset I_{i(1),h} \cap I_{i(1),\bar{h}}.$$

In particular,  $z$  and  $\tilde{z}_1$  belong to the same connected component of  $H^{-1}(h)$ . Since  $|z - \bar{z}| < r$  it trivially holds

$$\bar{z} \cdot e_{i(1)} - r < z \cdot e_{i(1)} < \bar{z} \cdot e_{i(1)} + r.$$

Moreover

$$X(t_1, \bar{z}) \cdot e_{i(1)} \geq \bar{z} \cdot e_{i(1)} + t_1 \min_{B_{4\bar{r}}(z_{i(1)})} b \cdot e_{i(1)} \geq \bar{z} \cdot e_{i(1)} + \frac{t_1}{2(k+1)} > \bar{z} \cdot e_{i(1)} + r.$$

We assume  $z \cdot e_{i(1)} \geq \bar{z} \cdot e_{i(1)}$ , being the opposite case analogous. We denote by  $\tilde{t}_0 \geq 0$  the unique  $t \in [0, t_1]$  such that  $X(t, \bar{z}) \cdot e_{i(1)} = z \cdot e_{i(1)}$ . We have

$$\tilde{t}_0 \leq \frac{|z - \bar{z}|}{\min_{B_{4\bar{r}}(z_{i(1)})} b \cdot e_{i(1)}} \leq 2(k+1)|z - \bar{z}|.$$

Moreover

$$t_1 - \tilde{t}_0 = \int_{z \cdot e_{i(1)}}^{\tilde{z}_1 \cdot e_{i(1)}} \frac{1}{b \cdot e_{i(1)}} (\bar{f}_{i, \bar{h}}(x)) dx$$

and similarly

$$s_1 = \int_{z \cdot e_{i(1)}}^{\tilde{z}_1 \cdot e_{i(1)}} \frac{1}{b \cdot e_{i(1)}} (\bar{f}_{i, h}(x)) dx.$$

We use the notation

$$S := \{z' \in \mathbb{R}^2 : z' \cdot e_{i(1)} \in (z \cdot e_{i(1)}, \tilde{z}_1 \cdot e_{i(1)}), z' \cdot e_{i(1)} \in (f_{i(1), h}(z' \cdot e_{i(1)}), f_{i(1), \bar{h}}(z' \cdot e_{i(1)}))\}.$$

Therefore

$$\begin{aligned} |t_1 - s_1| &\leq |\tilde{t}_0| + |t_1 - \tilde{t}_0 - s_1| \\ &\leq 2(k+1)|z - \bar{z}| + \int_{z \cdot e_{i(1)}}^{\tilde{z}_1 \cdot e_{i(1)}} \left| \frac{1}{b \cdot e_{i(1)}} (\bar{f}_{i, \bar{h}}(x)) - \frac{1}{b \cdot e_{i(1)}} (\bar{f}_{i, h}(x)) \right| dx \\ &\leq 2(k+1)|z - \bar{z}| + \left( \frac{1}{\inf_S b \cdot e_{i(1)}} \right)^2 |Db|(S) \\ &\leq 2(k+1)|z - \bar{z}| + 4(k+1)^2 |g(\bar{h}) - g(h)|, \end{aligned}$$

where, in the third inequality we used that the maps  $v \mapsto 1/v$  are  $1/\delta^2$ -Lipschitz on  $[\delta, +\infty)$  with  $\delta = \inf_S b \cdot e_{i(1)}$ . This proves (3.3) for  $j = 1$ .

Case  $j > 1$ . We assume

$$|t_{j-1} - s_{j-1}| \leq 2(k+1)|z - \bar{z}| + (j-1)(k+1)^2|g(\bar{h}) - g(h)| + 2(j-2)(k+1)^2|\bar{h} - h| \quad (3.4)$$

and we prove (3.3). We observe that  $|X(t_j, \bar{z}) - X(t_{j-1}, \bar{z})| \leq \bar{r}$ , therefore (3.1) implies that  $b \cdot e_{i(j)} > 1/(k+1)$  in  $B_{2\bar{r}}(X(t_{j-1}, \bar{z}))$ . In particular, we can define  $\tilde{w}_{j-1}$  as the unique  $z \in B_{2\bar{r}}(X(t_{j-1}, \bar{z}))$  such that

$$H(z) = h \quad \text{and} \quad z \cdot e_{i(j)} = X(t_{j-1}, \bar{z}) \cdot e_{i(j)}.$$

We have

$$|\tilde{w}_{j-1} - \tilde{z}_{j-1}| \leq |\tilde{w}_{j-1} - X(t_{j-1}, \bar{z})| + |X(t_{j-1}, \bar{z}) - \tilde{z}_{j-1}| \leq 2(k+1)|h - \bar{h}|.$$

Since  $\tilde{w}_{j-1} \cdot e_{i(j)}, \tilde{z}_{j-1} \cdot e_{i(j)} \in I_{i(j-1), h}$ , then there exists  $s_{j-1} > 0$  such that  $\tilde{w}_{j-1} = X(\tilde{s}_{j-1}, z)$  with

$$|\tilde{s}_{j-1} - s_{j-1}| \leq \frac{2(k+1)}{\min_{B_{4\bar{r}}(z_{i(j-1)})} b \cdot e_{i(j-1)}} |\bar{h} - h| \leq 2(k+1)^2 |\bar{h} - h|. \quad (3.5)$$

By the triangular inequality

$$\begin{aligned} |t_j - s_j| &\leq |t_j - t_{j-1} + t_{j-1} - s_{j-1} + s_{j-1} - \tilde{s}_{j-1} + \tilde{s}_{j-1} - s_j| \\ &\leq |t_{j-1} - s_{j-1}| + |t_j - t_{j-1} - s_j + \tilde{s}_{j-1}| + |\tilde{s}_{j-1} - s_{j-1}|. \end{aligned} \quad (3.6)$$

The same computation as in the case  $j = 1$  gives

$$\begin{aligned} |t_j - t_{j-1} - s_j + \tilde{s}_{j-1}| &= \left| \int_{X(t_{j-1}, \bar{z}) \cdot e_{i(j)}}^{X(t_j, \bar{z}) \cdot e_{i(j)}} \left( \frac{1}{b \cdot e_{i(j)}} (\bar{f}_{i, \bar{h}}(x)) - \frac{1}{b \cdot e_{i(j)}} (\bar{f}_{i, h}(x)) \right) dx \right| \\ &\leq (k+1)^2 |g(\bar{h}) - g(h)|. \end{aligned} \quad (3.7)$$

By plugging (3.4), (3.5) and (3.7) into (3.6), we finally get (3.3).

The statement is therefore proven with  $s = s_{\tilde{N}}$ ,  $c_1 = 2(k+1)$  and  $c_2 = \tilde{N}(k+1)^2(1 + 2\|b\|_{L^\infty}) + 2(k+1)$ .  $\square$

**Remark 3.3.** If we additionally assume that  $b \in W^{1,p}(\mathbb{R}^2)$  in Lemma 3.2, then the statement holds true with

$$g(h) := \int_{\{H \leq h\} \cap \Omega_{k+1}} |\nabla H| dz.$$

In particular, we showed in the proof of Proposition 1.3 that  $g \in W_{\text{loc}}^{1,p}(\mathbb{R})$ .



**Proof of Theorem 1.4.** By Lemma 3.1, it is sufficient to prove that  $X(t) \in \text{BV}(\Omega_k)$  for every  $k \in \mathbb{N}$ . In the same setting as in Lemma 3.2, if  $\bar{z} \in \Omega_k$  and  $z \in B_r(\bar{z})$ , then

$$\begin{aligned} |X(t, \bar{z}) - X(t, z)| &\leq |X(t, \bar{z}) - X(s, z)| + |X(s, z) - X(t, z)| \\ &\leq c_1 |H(\bar{z}) - H(z)| + \|b\|_{L^\infty} |t - s| \\ &\leq c_1 |H(\bar{z}) - H(z)| + c_2 \|b\|_{L^\infty} (|g(H(\bar{z})) - g(H(z))| + |\bar{z} - z|) \\ &\leq \|b\|_{L^\infty} (c_1 + c_2) |\bar{z} - z| + c_2 \|b\|_{L^\infty} |g(H(\bar{z})) - g(H(z))|. \end{aligned} \quad (3.8)$$

The argument in the proof of Corollary 2.4 shows that  $g \circ H \in \text{BV}(\Omega_k)$  therefore it follows from (3.8) that  $X(t) \in \text{BV}(\Omega_k \cap B_r(z))$  for every  $z \in \Omega_k$ . Being  $\Omega_k$  bounded this proves that  $X(t) \in \text{BV}(\Omega_k)$ . Finally the continuity of  $X(t)$  follows immediately from (3.8) and the continuity of  $g$ . If moreover we assume that  $b \in W^{1,p}(\mathbb{R}^2)$ , then the same argument proves that  $X(t) \in W^{1,p}(\Omega_k)$  thanks to Remark 3.3 and Corollary 2.5.  $\square$

**Remark 3.4.** By inspection in the argument used to prove Lemma 3.2, we observe that  $\|X(t)\|_{\text{BV}(\Omega)}$  (or  $\|X(t)\|_{W^{1,p}(\Omega)}$ ) is locally bounded for  $t \in [0, +\infty)$  and it diverges at most linearly in  $t$  as  $t \rightarrow \infty$ .

#### 4. Non BV flow of a 2d autonomous Sobolev divergence-free vector field

In this section we prove Proposition 1.5. The proof is based on the construction of a suitable Lipschitz Hamiltonian with Sobolev gradient, obtained by modifying the construction proposed in [1] in order to produce a Hamiltonian for which the uniqueness of the associated regular Lagrangian flow fails. The building blocks of the construction are introduced in Sections 4.1 and 4.2. In Section 4.3, we provide the regularity bounds that will ensure that the constructed Hamiltonian is Lipschitz. In Sections 4.4 and 4.5, we disprove the BV regularity of the flow associated to the Hamiltonian built in the previous sections. This proves the weaker version of Proposition 1.5 where the vector field is only required to have bounded variation. In order to complete the proof, we need to improve the regularity of the vector field from BV to  $W^{1,p}$  for every  $p \in [1, \infty)$  and this is achieved in Section 4.6.

##### 4.1. Construction of $C_n$ , $D_n$ , $E_n$ and $F_n$

We consider the following parameters:

$$c_n = \frac{1}{n^2 2^n}, \quad a_n = \frac{n-1}{2n} \left( \frac{c_n}{2} - c_{n+1} \right) \sim \frac{1}{2^{n-1} n^3}, \quad r_n = \frac{1}{2n} \left( \frac{c_n}{2} - c_{n+1} \right) \sim \frac{1}{2^{n-1} n^4}. \quad (4.1)$$

We set  $C_1 = [0, 1/2]^2 \subset \mathbb{R}^2$  and we inductively define  $C_{n+1}$  for  $n \geq 1$  as follows:  $C_{n+1} \subset C_n$  and every connected component  $R'$  of  $C_n$  contains two connected components of  $C_{n+1}$ , which are squares of side  $c_n$  as in Fig. 1. For every  $n \in \mathbb{N}$  we also consider the sets  $D_n, E_n, F_n \subset C_n$  as in Fig. 1.

We observe that for every  $n \geq 1$  it holds

$$C_{n+1} = \{z \in D_n : \text{dist}(z, \partial D_n) \geq r_n\}.$$

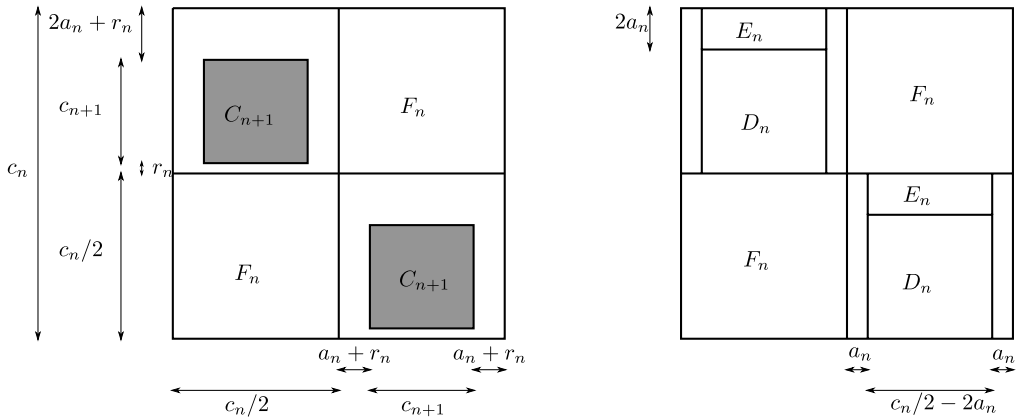


Fig. 1. Construction of the set  $C_{n+1}, D_n, E_n, F_n \subset C_n$ . The two pictures represent a connected component of  $C_n$ .

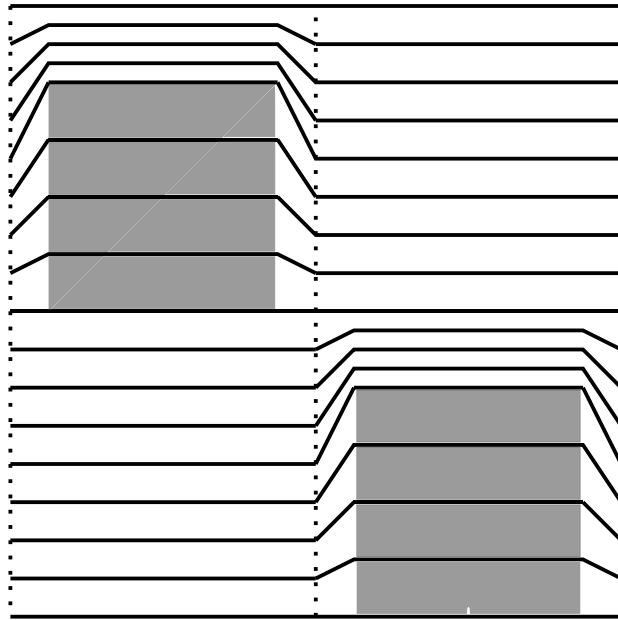


Fig. 2. Level sets of  $f_n$  on a connected component  $R'$  of  $C_n$ . The set  $D_n \cap R'$  is colored in gray.

#### 4.2. Construction of $f_n$ and $h_n$

The function  $f_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$  is defined by  $f_0(x, y) = y$ . The function  $f_n$  coincides with  $f_{n-1}$  on  $\mathbb{R}^2 \setminus C_n$  and its level lines in  $C_n$  are as in Fig. 2. In particular,  $f_n$  coincides with  $f_{n-1}$  on  $F_n$ .

Let  $R$  be a connected component of  $D_n$ , then  $f_n$  is affine on  $R$  and depends only on  $y$ , therefore  $\nabla f_n = (0, v_n)$ . Let  $R'$  be a connected component of  $C_n$  and denote by  $s_n = \text{Osc}(f_n, R')$ .

$$v_n = \frac{\text{Osc}(f_n, R)}{\text{height}(R)} = \frac{\text{Osc}(f_n, R')}{\text{height}(R')},$$

where  $\text{height}(R) = c_{n+1} + 2r_n$ ,  $\text{height}(R') = c_{n+1}$ ,  $\text{Osc}(f_n, R) = s_n/4$  and  $\text{Osc}(f_n, R') = s_{n+1}$  so that

$$4s_{n+1} = \frac{c_{n+1}}{c_{n+1} + 2r_n} s_n.$$

In particular

$$4^n s_n = 4c_1 \prod_{l=2}^n \frac{c_l}{c_l + 2r_{l-1}} \searrow 4c_1 \prod_{l=2}^{\infty} \frac{c_l}{c_l + 2r_{l-1}}.$$

From the choice (4.1) it follows that  $r_{l-1}c_l^{-1} = O(l^{-2})$  therefore

$$\log \left( \frac{c_l}{c_l + 2r_{l-1}} \right) = O(l^{-2}).$$

In particular, the infinite product is strictly positive and we denote it by  $\sigma$ . We finally get

$$v_n = \frac{s_{n+1}}{c_{n+1}} \sim c_1 \sigma \frac{n^2}{2^n}.$$

Similarly we compute the speed  $\nabla f_n = (0, v'_n)$  in the region  $E_n$  as in the picture. Denoting by  $R''$  one of its components, we have

$$v'_n = \frac{\text{Osc}(f_n, R'')}{\text{height}(R'')} = \frac{s_n}{8a_n} \sim \frac{c_1 \sigma}{4} \frac{n^3}{2^n}.$$

#### 4.3. Estimates on the norms of $\nabla f_n$ and $\nabla h_n$

We first estimate  $\|\nabla f_n\|_{L^\infty(C_n)}$ . From Fig. 2, we observe that  $\|\partial_2 f_n\|_{L^\infty(C_n)} = v'_n$  and the maximal slope of the level sets of  $f_n$  in  $C_n$  is  $\frac{c_n - 8a_n}{4a_n}$ . Therefore

$$\|\nabla f_n\|_{L^\infty(C_n)} \leq v'_n \left( \frac{c_n - 8a_n}{4a_n} + 1 \right) \sim \frac{c_1 \sigma}{16} \frac{n^4}{2^n}.$$

Since  $f_n$  and  $f_{n-1}$  coincide outside  $C_n$ , it holds

$$\|\nabla h_n\|_{L^\infty} \leq \|\nabla f_n\|_{L^\infty(C_n)} + \|\nabla f_{n-1}\|_{L^\infty(C_{n-1})} = O\left(\frac{n^4}{2^n}\right).$$

This proves that

$$f = \lim_{n \rightarrow \infty} f_n = f_0 + \sum_{l=1}^{\infty} h_l$$

is a Lipschitz function.

#### 4.4. Estimate on crossing time

Let  $T_n^1$  be the amount of time needed by an integral curve of the vector field  $-\nabla^\perp f_{n-1}$  to cross a connected component of  $C_n$ . Then

$$T_n^1 = \frac{c_n}{v_{n-1}} \sim \frac{1}{2c_1\sigma n^4}.$$

Let moreover  $T_n^s$  be the amount of time needed by an integral curve of the vector field  $-\nabla^\perp f_n$  intersecting  $D_n$  to cross a connected component of  $C_n$ . Since  $a_n + r_n = o(c_n)$ , then  $T_n^s$  is asymptotically equivalent at the sum of the amounts of time needed to cross a connected component of  $D_n$  and a connected component of  $F_n$ , namely

$$T_n^s \sim \frac{1}{2c_1\sigma n^4} + \frac{1}{4c_1\sigma n^4}.$$

Finally, let  $T_n^f$  be the amount of time needed by an integral curve of the vector field  $-\nabla^\perp f_n$  intersecting  $E_n$  to cross a connected component of  $C_n$ : similarly as above we have the  $T_n^f$  is asymptotically equivalent to the amount of time needed to cross a connected component of  $F_n$ , namely

$$T_n^f \sim \frac{1}{4c_1\sigma n^4}.$$

Let us denote by  $X_n$  the flow of  $-\nabla^\perp f_n$  and by  $X$  the flow of  $-\nabla^\perp f$ . For every  $z = (x, y) \in \mathbb{R}^2$  with  $x < 0$  we define  $t_1(z)$  as the unique  $t > 0$  such that  $X(t_1(x)) \cdot e_1 = 0$  and  $t_2(z)$  as the unique  $t > 0$  such that  $X(t_1(x)) \cdot e_1 = 1$ . Since  $f(x, y) = y$  for every  $(x, y) \in \mathbb{R}^2$  with  $x < 0$ , then the function  $z \mapsto t_2(z) - t_1(z)$  depends only on  $y$ . We therefore set

$$T(y) := t_2(-1, y) - t_1(-1, y)$$

and we observe that for every  $x < 0$  it holds  $T(y) = t_2(x, y) - t_1(x, y)$ . Let  $z = (x, y) \in \mathbb{R}^2$  be such that  $x < 0$  and there exists  $t > 0$  for which  $X_n(t, z) \in E_n$ . Then, by construction,  $X(t, z) = X_n(t, z)$  for every  $t > 0$  and therefore

$$T(y) = T_1 + \sum_{l=2}^{n-1} (T_l^s - T_l^1) + T_n^f - T_n^1 =: T_n$$

By construction, there exist  $0 < y_1 < y_2 < \dots < y_{2^{n-1}} < 1$  such that for every  $x < 0$  and every  $k \in [1, 2^{n-1}] \cap \mathbb{N}$  there exists  $t = t(k) > 0$  for which

$$X(t, -1, y_k) \in E_n \quad \text{if } k \text{ is even} \quad \text{and} \quad X(t, -1, y_{2k-1}) \in E_{n-1} \quad \text{if } k \text{ is odd}.$$

In particular

$$\text{Tot.Var.}_{(0,1)} T \geq 2^{n-1} (T_n - T_{n-1}) = 2^{n-1} (T_n^f - T_n^1 + T_{n-1}^s - T_{n-1}^1) \sim \frac{2^n}{8c_1\sigma n^4}.$$

This shows that  $T$  has not bounded variation.

#### 4.5. Regularity of the flow

We observe that the function  $T$  constructed in the previous section is bounded: indeed

$$\sup_n T = \sup_n T_n \leq T_0 + \sum_{n=2}^{\infty} |T_n - T_{n-1}| \leq 1 + C \sum_{n=1}^{\infty} \frac{1}{4c_1 \sigma n^4} < \infty$$

for some universal constant  $C > 0$ . Since  $f(x, y) = y$  for every  $(x, y) \in \mathbb{R}^2 \setminus [0, 1/2]^2$ , for every  $t > \sup T$  it holds

$$X(t, x, y) \cdot e_1 = x + 1 + t - T(y) \quad \forall (x, y) \in (\sup T - t, 0) \times \mathbb{R}.$$

Since  $T$  has not bounded variation, then  $X(t) \notin \text{BV}((-\varepsilon, 0) \times (0, 1/2))$  for every  $\varepsilon > 0$  and every  $t > \sup T$ . If  $R = [a, b] \times [c, d]$  denotes a connected component of  $C_n$ , the same argument as above shows that  $X(t) \notin \text{BV}((a - \varepsilon, a) \times (c, d))$  for every  $\varepsilon > 0$  and every  $t > t_n$  for some  $t_n \rightarrow 0$  as  $n \rightarrow \infty$ . In particular  $X(t) \notin \text{BV}_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$  for every  $t > 0$ .

#### 4.6. More regular vector field

The example constructed above does not prove Proposition 1.5 since the vector field  $b = -\nabla^\perp f$  has no Sobolev regularity. In order to make the vector field more regular, we consider

$$\tilde{f} = f_0 + \sum_{l=1}^{\infty} h_l * \rho_l,$$

where

$$\rho_l(z) = r_l^{-2} \rho(z/r_l)$$

and  $\rho: \mathbb{R}^2 \rightarrow \mathbb{R}$  is a positive smooth function such that

- (1)  $\text{supp } \rho \subset B_{1/2}(0)$ ;
- (2)  $\int_{\mathbb{R}^2} \rho = 1$  and  $\int_{\mathbb{R}^2} z \rho(z) dz = 0$ .

Let us first check that  $\tilde{f} \in W_{\text{loc}}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$  for every  $p \in [1, \infty)$ : indeed, it follows by Young and Hölder inequalities that

$$\begin{aligned} \|\nabla^2(h_l * \rho_l)\|_{L^p} &\leq \|\nabla h_l * \nabla \rho_l\|_{L^p} \\ &\leq \|\nabla h_l\|_{L^p} \|\nabla \rho_l\|_{L^1} \\ &\leq \|\nabla h_l\|_{L^\infty} \left( \mathcal{L}^2(\text{supp}(\nabla h_l)) \right)^{1/p} \|\nabla \rho_l\|_{L^1}. \end{aligned}$$

Recalling the estimate on  $\|\nabla h_l\|_{L^\infty}$  obtained in Section 4.3 and the choice of the parameter  $r_l$  and of the size of  $C_l$  in Section 4.1, we continue the chain of inequalities above:

$$\begin{aligned}
\|\nabla^2(h_l * \rho_l)\|_{L^p} &\leq \|\nabla h_l\|_{L^\infty} \left( \mathcal{L}^2(\text{supp}(\nabla h_l)) \right)^{1/p} \|\nabla \rho_l\|_{L^1} \\
&\leq O(l^4 2^{-l}) \left( \mathcal{L}^2(C_{l-1}) \right)^{1/p} O(r_l^{-1}) \\
&= O(l^4 2^{-l}) O(2^{-l/p} l^{-4/p}) O(2^l l^4) \\
&= O(2^{-l/p} l^{8-\frac{4}{p}}).
\end{aligned}$$

Being  $\|\nabla^2(h_l * \rho_l)\|_{L^p}$  summable, the sequence

$$\tilde{f}_n := f_0 + \sum_{l=1}^n h_l * \rho_l$$

converges to  $\tilde{f}$  in  $W_{\text{loc}}^{2,p}(\mathbb{R}^2)$  for every  $p \in [1, +\infty)$ . We now prove that the same argument of Sections 4.4 and 4.5 for  $f$  can be applied to  $\tilde{f}$ .

Being  $f_n$  affine on each connected component of  $D_n$ , it follows from the properties of the convolution kernel that  $h_n * \rho_n(z) = h_n(z)$  for every  $z \in D_n$  such that  $\text{dist}(z, \partial D_n) > r_n$ . We denote

$$\begin{aligned}
\tilde{D}_n &:= \{x \in D_n : \text{dist}(x, \partial D_n) > r_n\}, \\
\tilde{E}_n &:= \{x \in E_n : \text{dist}(x, \partial E_n) > r_n\}, \\
\tilde{F}_n &:= \{x \in D_n : \text{dist}(x, \partial F_n) > r_n\}.
\end{aligned}$$

Observe that all the sets above are non-empty by the choice of the parameters (4.1). Since  $\tilde{D}_n = C_{n+1}$ , we have in particular that  $f_n = \tilde{f}_n$  on the set  $C_{n+1}$ . Similarly  $h_n * \rho_n = h_n$  on  $\tilde{E}_n \cup \tilde{F}_n$ . As in Section 4.4, we denote by  $\tilde{T}_n^1$  the total amount of time needed by an integral curve of the vector field  $-\nabla^\perp \tilde{f}_{n-1}$  to cross a connected component of  $C_n$ . Since  $f_{n-1} = \tilde{f}_{n-1}$  on  $C_n$ , then  $\tilde{T}_n^1 = T_n^1$ . We moreover denote by  $\tilde{T}_n^s$  the amount of time needed by an integral curve of the vector field  $-\nabla^\perp \tilde{f}_n$  intersecting  $\tilde{D}_n$  to cross a connected component of  $C_n$ . Since  $\tilde{f}_n = f_n$  in  $\tilde{D}_n \cup \tilde{F}_n$  it is straightforward to check that  $\tilde{T}_n^s \sim T_n^s$ . Similarly, we denote  $\tilde{T}_n^f$  the amount of time needed by an integral curve of the vector field  $-\nabla^\perp \tilde{f}_n$  intersecting  $\tilde{E}_n$  to cross a connected component of  $C_n$ . Since  $\tilde{f}_n = f_n$  in  $\tilde{E}_n \cup \tilde{F}_n$ , it is straightforward to check that  $\tilde{T}_n^f \sim T_n^f$ .

We are now in position to repeat the argument in Sections 4.4 and 4.5 and this proves that for every  $t > 0$  the regular Lagrangian flow  $\tilde{X}$  of the vector field  $-\nabla^\perp \tilde{f}$  satisfies

$$\tilde{X}(t) \notin \text{BV}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2).$$

This concludes the proof of Proposition 1.5.

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