

Ramification of multiple eigenvalues for the Dirichlet-Laplacian in perforated domains

Laura Abatangelo*, Corentin Léna†, and Paolo Musolino‡

August 24, 2022

Abstract. Taking advantage from the so-called *Lemma on small eigenvalues* by Colin de Verdière, we study ramification for multiple eigenvalues of the Dirichlet Laplacian in bounded perforated domains. The asymptotic behavior of multiple eigenvalues turns out to depend on the asymptotic expansion of suitable associated eigenfunctions.

We treat the case of planar domains in details, thanks to the asymptotic expansion of a generalization of the so-called *u-capacity* which we compute in dimension 2. In this case multiple eigenvalues are proved to split essentially by different rates of convergence of the perturbed eigenvalues or by different coefficients in front of their expansion if the rate of two eigenbranches turns out to be the same.

Keywords. Dirichlet-Laplacian; multiple eigenvalues; small capacity sets; asymptotic expansion; perforated domain

2010 MSC. 35P20; 31C15; 31B10; 35B25; 35C20

1 Introduction

The present paper deals with multiple eigenvalues for Dirichlet Laplacian in bounded domains with small holes. Let Ω be a bounded open set in $\mathbb{R}^d$, $d \geq 2$, the considered eigenvalue problem is

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1)$$

In what follows, $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, \dots\}$ and $\mathbb{N}^* \equiv \mathbb{N} \setminus \{0\}$. From classical results in spectral theory, the Dirichlet Laplacian admits a sequence of real eigenvalues tending to infinity

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots \leq \lambda_N(\Omega) \leq \dots$$

*Dipartimento di Matematica, Politecnico di Milano, P.zza Leonardo da Vinci 32, 20133 Milano, Italy
laura.abatangelo@polimi.it

†Institut de Mathématiques, Bâtiment UniMail Rue Emile-Argand 11, 2000 Neuchâtel, Switzerland
corentin.lena@unine.ch

‡Dipartimento di Scienze Molecolari e Nanosistemi, Università Ca' Foscari Venezia, via Torino 155, 30172 Venezia Mestre, Italy
paolo.musolino@unive.it

where every eigenvalue is repeated as many times as its multiplicity and in particular the first one is simple if Ω is connected. The dependence of Laplace operator's spectrum upon domain perturbations has been long investigated, with particular attention to the case of sets with small perforations.

Samarskiĭ [43] considered the variation $\delta\lambda_N$ of an eigenvalue λ_N for the Dirichlet-Laplacian when a small set ω_ε is removed from a subset Ω of $\mathbb{R}^3$ and obtained the estimate

$$\delta\lambda_N \leq 4\pi\kappa_N^2 \text{Cap}_\Omega(\omega_\varepsilon) + O(\text{Cap}_\Omega(\omega_\varepsilon)^2),$$

where $\text{Cap}_\Omega(\omega_\varepsilon)$ is the *standard capacity* of ω_ε in Ω , κ_N is the maximum value of the N -th normalized eigenfunction on $\overline{\omega_\varepsilon}$ (see Maz'ya, Nazarov, and Plamenevskii [32]). Here we recall that, if we consider a bounded, connected open set Ω of $\mathbb{R}^d$, then for every compact subset K of Ω , the *standard capacity* of K in Ω is defined as

$$\text{Cap}_\Omega(K) \equiv \inf \left\{ \int_\Omega |\nabla f|^2 dx : f \in H_0^1(\Omega) \text{ and } f - \eta_K \in H_0^1(\Omega \setminus K) \right\},$$

where η_K is a fixed smooth function such that $\text{supp } \eta_K \subseteq \Omega$ and $\eta_K \equiv 1$ in a neighborhood of K . Rauch and Taylor [42] investigated the behavior of the eigenvalues and eigenfunctions of the Laplacian in a domain Ω where a “thin” set is removed. In a series of papers (see, e.g., [37, 38, 39, 40, 41]), Ozawa computed several asymptotic expansions for the eigenvalues of the Laplacian, under many different boundary conditions, when a small perforation is made. For example, Ozawa has shown in [39] that if $n = 2$ then

$$\lambda_N(\Omega \setminus (\varepsilon \overline{\mathbb{B}_2(0, 1)})) = \lambda_N(\Omega) - 2\pi(\log \varepsilon)^{-1}(u_N(0))^2 + O((\log \varepsilon)^{-2}) \quad \text{as } \varepsilon \rightarrow 0^+,$$

where $\mathbb{B}_2(0, 1)$ is the unit ball in $\mathbb{R}^2$, $\lambda_N(\Omega)$ is a simple eigenvalue for the Dirichlet-Laplacian in Ω and u_N a corresponding $L^2(\Omega)$ -normalized eigenfunction.

A detailed analysis of boundary value problem in singularly perturbed domain has been performed by Maz'ya, Nazarov, and Plamenevskii, who considered also eigenvalue problems (see, e.g., [32] and [33, Chapter 9]). For example, in the three dimensional case, they have shown in [32] that for the first eigenvalue of the Laplacian with Dirichlet condition we have

$$\begin{aligned} \lambda_1(\Omega \setminus (\varepsilon \overline{\omega})) &= \lambda_1(\Omega) + 4\pi \text{Cap}(\omega)(u_1(0))^2 \varepsilon + [4\pi u_1(0) \text{Cap}(\omega)]^2 \\ &\quad \times \left\{ \Gamma(0) + \frac{u_1(0)}{4\pi} \int_\Omega u_1(x) |x|^{-1} dx \right\} \varepsilon^2 + O(\varepsilon^3) \quad \text{as } \varepsilon \rightarrow 0^+, \end{aligned}$$

where u_1 is a corresponding $L^2(\Omega)$ -normalized eigenfunction in Ω , $\text{Cap}(\omega)$ the harmonic capacity of ω and Γ is a function defined through an auxiliary boundary value problem.

Moreover, techniques based on potential theory and integral operators for the study of eigenvalues of the Laplacian in perforated domains have been exploited for example in Ammari, Kang, and Lee [3] and in Lanza de Cristoforis [30].

The behavior of the spectrum of the Laplacian under removal of “small” sets in the Riemannian setting has been studied by many authors: as an example, we mention the works by Besson [6], Chavel [9], Chavel and Feldman [10], Colbois and Courtois [11], Courtois [12].

If on one hand, most of the above mentioned results deals with simple eigenvalues, less is known for the case of multiple eigenvalues.

A general study on perturbation problems for eigenvalues can be found in the monograph by Kato [22], which contains also some results for multiple eigenvalues.

In Nguyen [34], the author investigated the asymptotic expansion of the multiple eigenvalues and eigenfunctions for boundary value problems in a domain with a small hole. More precisely, he studied the bifurcation of a double or triple eigenvalue in a smooth bounded domain in $\mathbb{R}^3$ where a small ball of radius ε is removed.

Asymptotic expansions for eigenvalues, both simple and multiple, and eigenfunctions of the Neumann Laplacian in a three-dimensional domain with a small hole are obtained in Nazarov and Sokolowski [35], Laurain, Nazarov, and Sokolowski [31], and Novotny and Sokolowski [36] in terms of the eigenvalues of an auxiliary matrix.

Asymptotic expansions for multiple eigenvalues of regularly perturbed domains are obtained in Bruno and Reitich [8]. Concerning regular perturbations, analyticity results for symmetric functions of possibly multiple eigenvalues upon regular domain perturbations can be found in Lamberti and Lanza de Cristoforis [24, 25, 26].

We note that a fundamental tool in the study of perturbation problems for the eigenvalues of the Laplacian is the standard capacity. Indeed, as is well known, if we remove a compact subset K of zero capacity, the spectrum of the Dirichlet-Laplacian on the bounded domain Ω does not change (see, *e.g.*, Rauch and Taylor [42]). Moreover, if we denote by

$$0 < \lambda_1(\Omega \setminus K) < \lambda_2(\Omega \setminus K) \leq \dots \leq \lambda_N(\Omega \setminus K) \leq \dots$$

the sequences of the eigenvalues of the Dirichlet-Laplacian in $\Omega \setminus K$, then Rauch and Taylor [42] also proved that the N -th eigenvalue $\lambda_N(\Omega \setminus K)$ of the Dirichlet-Laplacian in $\Omega \setminus K$ is close to $\lambda_N(\Omega)$ if the capacity $\text{Cap}_\Omega(K)$ of K in Ω is small.

On one hand, the result by Rauch and Taylor [42] can be seen as a continuity result for eigenvalues with respect to the capacity. On the other hand, Courtois [12] has obtained a higher regularity: he has shown that if $K \subseteq \Omega$ is compact and $\text{Cap}_\Omega(K)$ is small then the function

$$\lambda_N(\Omega \setminus K) - \lambda_N(\Omega)$$

is in some sense differentiable with respect to $\text{Cap}_\Omega(K)$.

A first improvement of Courtois' results has been achieved by [2] and later on by [1]. In the first paper, for simple eigenvalues the authors establish a sharp relation between the vanishing order of a Dirichlet eigenfunction at a point and the leading term of the asymptotic expansion of the corresponding Dirichlet eigenvalue variation, as a removed compact set concentrates at that point. We recall indeed that a point has zero capacity. If, for example, $\lambda_N(\Omega)$ is simple, then one can replace the capacity $\text{Cap}_\Omega(K)$ by the so-called u_N -capacity $\text{Cap}_\Omega(K, u_N)$, u_N being a suitable associated eigenfunction (see equation (3) below). In this way one can obtain more refined asymptotic expansions of the difference $\lambda_N(\Omega \setminus K) - \lambda_N(\Omega)$ (on this topic, see also [5]). In the second cited paper [1], the authors provide the asymptotic behavior of u -capacities of small sets of type $\varepsilon\bar{\omega}$, with ω a sufficiently regular open bounded connected set of $\mathbb{R}^2$ and u a function analytic in a neighborhood of the origin. Consequently, they are able to deduce asymptotic behaviors for simple eigenvalues when a small set of this type is removed from the original domain. In particular, this analysis sheds some light on the dependence of these asymptotics on the hole's shape ω .

In the present paper we rather consider multiple eigenvalues. We aim at investigating whether multiple eigenvalues can be splitted in different branches if we perturb the initial domain by re-

moving a small hole around a point, namely the origin. In [12, Theorem 1.2] Courtois gave a first response in this direction. We recall his result in the following

Theorem 1.1. [12, Theorem 1.2] *Let X be a compact Riemannian manifold with or without boundary of dimension greater than or equal to 2. Let $\lambda := \lambda_N = \dots = \lambda_{N+k-1}$ be a Dirichlet eigenvalue of X with multiplicity k . There exist a function $r : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{t \rightarrow 0} r(t) = 0$ and a positive constant ε_N , such that, for any compact subset A of X , if $\text{Cap}_X(A) \leq \varepsilon_N$, then*

$$|\lambda_{N+j}(X \setminus A) - \lambda_{N+j} - \text{Cap}_X(A) \cdot \mu_A(u_{N+j}^2)| \leq \text{Cap}_X(A) \cdot r(\text{Cap}_X(A))$$

where μ_A is a finite positive probability measure supported in A defined as the renormalized singular part of $-\Delta V_A$ and $\{u_N, \dots, u_{N+k-1}\}$ is an orthonormal basis of the eigenspace of λ which diagonalises the quadratic form $\mu_A(u^2)$ according to the increasing order of its eigenvalues.

Theorem 1.1 above provides a sharp asymptotic expansion of $\lambda_{N+j}(X \setminus A) - \lambda_{N+j}$ as $\text{Cap}_X(A) \rightarrow 0$ only if $\mu_A(u_{N+j}^2) \not\rightarrow 0$, but in general it is just an estimate for $\lambda_{N+j}(X \setminus A) - \lambda_{N+j}$ when $\mu_A(u_{N+j}^2) \rightarrow 0$. Moreover, even when $\mu_A(u_{N+j}^2) \not\rightarrow 0$ the possible splitting of a multiple eigenvalue is hidden in the involved quadratic form $\mu_A(u^2)$. (We note that in the present paper, with an abuse of notation, we denote by the same symbol a bilinear form and the corresponding quadratic form.)

To introduce our main results, we need to specify the functional context where we are working.

1.1 Functional setting and basic problem

Let us give a more convenient reformulation of the eigenvalue problem (1). We consider the quadratic form defined by

$$q_\Omega(u) \equiv \int_\Omega |\nabla u|^2 dx \quad (2)$$

for $u \in H_0^1(\Omega)$. We denote by $\langle \cdot, \cdot \rangle$ the usual scalar product in $L^2(\Omega)$ and by $\|\cdot\|$ the associated norm. It is well known that the spectrum of q_Ω with respect to $\langle \cdot, \cdot \rangle$ consists of positive eigenvalues of finite multiplicity tending to infinity. We denote the sequence of these eigenvalues, repeated according to their multiplicity, by $\{\lambda_i(\Omega)\}_{i \geq 1}$.

We are interested in the effect on the eigenvalues of imposing an additional Dirichlet condition on a small compact set $K \subseteq \Omega$. This can intuitively be described as creating a small hole K in Ω . To make our meaning clear, let us recall the following definition from [2].

Definition 1.2. *Let $K \subseteq \Omega$ be compact and let $\{K_\varepsilon\}_{\varepsilon > 0}$ be a family of compact sets contained in Ω . We say that $\{K_\varepsilon\}$ concentrates to K when, for every open set U satisfying $K \subseteq U \subseteq \Omega$, there exists $\varepsilon_U > 0$ such that, for all $\varepsilon < \varepsilon_U$, $K_\varepsilon \subseteq U$.*

We now fix $K \subseteq \Omega$ a compact set, such that $\text{Cap}_\Omega(K) = 0$, and $\{K_\varepsilon\}_{\varepsilon > 0}$ a family of compact sets concentrating to K . We define, for $\varepsilon > 0$, the perforated domain $\Omega_\varepsilon \equiv \Omega \setminus K_\varepsilon$. The following result is easy to prove.

Proposition 1.3. *For all $i \geq 1$, $\lambda_i(\Omega_\varepsilon) \rightarrow \lambda_i(\Omega \setminus K)$ as $\varepsilon \rightarrow 0$ and $\lambda_i(\Omega \setminus K) = \lambda_i(\Omega)$.*

To simplify the notation, we set $\lambda_i \equiv \lambda_i(\Omega)$ and $\lambda_i^\varepsilon \equiv \lambda_i(\Omega_\varepsilon)$. We wish to estimate the difference $\lambda_i^\varepsilon - \lambda_i$. To that end, we recall a definition from [2], inspired by [5, 12].

Definition 1.4. Given a function $u \in H_0^1(\Omega)$, the u -capacity of a compact set $K \subseteq \Omega$ is

$$\text{Cap}_\Omega(K, u) \equiv \inf \left\{ \int_\Omega |\nabla f|^2 dx : f \in H_0^1(\Omega) \text{ and } f - u \in H_0^1(\Omega \setminus K) \right\}. \quad (3)$$

The infimum in (3) is achieved by a unique function $V_{K,u} \in H_0^1(\Omega)$, so that

$$\text{Cap}_\Omega(K, u) = \int_\Omega |\nabla V_{K,u}|^2 dx.$$

We call $V_{K,u}$ the potential associated with u and K .

Furthermore, $V_{K,u}$ is the unique weak solution of the Dirichlet problem

$$\begin{cases} \Delta V_{K,u} = 0 & \text{in } \Omega \setminus K, \\ V_{K,u} = 0 & \text{on } \partial\Omega, \\ V_{K,u} = u & \text{on } K, \end{cases} \quad (4)$$

where, by weak solution, we mean that $V_{K,u} \in H_0^1(\Omega)$, $u - V_{K,u} \in H_0^1(\Omega \setminus K)$ and $\int_\Omega \nabla V_{K,u} \cdot \nabla \varphi = 0$ for all $\varphi \in H_0^1(\Omega \setminus K)$.

Remark 1.5. Let us note that the Sobolev space $H_0^1(\Omega \setminus K)$ can be seen as a subspace of $H_0^1(\Omega)$. Indeed, the quadratic form q_Ω of Equation (2) endows $H_0^1(\Omega)$ with a Hilbert space norm. The space $H_0^1(\Omega \setminus K)$ can then be identified with the closure of the subspace $C_c^\infty(\Omega \setminus K)$, consisting of smooth functions compactly supported in $\Omega \setminus K$. More explicitly, this identification associates to $f \in H_0^1(\Omega \setminus K)$ its extension g to Ω obtained by setting $g = f$ in $\Omega \setminus K$ and $g = 0$ in K . We systematically perform this identification in the paper, and in particular we use the same notation for an element of $H_0^1(\Omega \setminus K)$ and the element of $H_0^1(\Omega)$ obtained after extension.

Remark 1.6. Definition 1.4 has a geometric interpretation. According to the previous remark, $H_0^1(\Omega \setminus K)$ can be seen as a subspace of $H_0^1(\Omega)$, closed for the Hilbert space norm given by the quadratic form q_Ω . Then, $\text{Cap}_\Omega(K, u)$ is the square of the q_Ω -distance of u to $H_0^1(\Omega \setminus K)$ and $u - V_{K,u}$ is the q_Ω -orthogonal projection of u . The existence and uniqueness of $V_{K,u}$, as well as its characterization as the unique weak solution of (4), follow immediately from the properties of the orthogonal projection on a closed subspace of a Hilbert space.

Note also that Definition (3) can be extended to $H^1(\Omega)$ functions, by setting, for any $u \in H^1(\Omega)$,

$$\text{Cap}_\Omega(K, u) \equiv \text{Cap}_\Omega(K, \eta_K u)$$

where η_K is a fixed smooth function such that $\text{supp } \eta_K \subseteq \Omega$ and $\eta_K \equiv 1$ in a neighborhood of K .

In order to study multiple eigenvalues, we will need the following generalization of the u -capacity.

Definition 1.7. Given $u, v \in H_0^1(\Omega)$, the (u, v) -capacity of a compact set $K \subseteq \Omega$ is

$$\text{Cap}_\Omega(K, u, v) \equiv \int_\Omega \nabla V_{K,u} \cdot \nabla V_{K,v} dx.$$

Remark 1.8. As it is done in Remark 1.6, we extend Definition 1.7 of $\text{Cap}_\Omega(K, u, v)$ to functions $u, v \in H^1(\Omega)$, by setting

$$\text{Cap}_\Omega(K, u, v) \equiv \text{Cap}_\Omega(K, \eta_K u, \eta_K v)$$

where η_K is a fixed smooth function such that $\text{supp } \eta_K \subseteq \Omega$ and $\eta_K \equiv 1$ in a neighborhood of K .

Let us now fix an eigenvalue λ_N , of multiplicity m , for Problem (1). Let us denote by $E(\lambda_N)$ the associated eigenspace. According to Proposition 1.3, for $i \in \{1, \dots, m\}$,

$$\lambda_{N+i-1}^\varepsilon \rightarrow \lambda_N \text{ as } \varepsilon \rightarrow 0.$$

We therefore have m eigenvalue branches departing from the multiple eigenvalue λ_N . We will prove the following result about their asymptotic behavior.

Theorem 1.9. *For $i \in \{1, \dots, m\}$,*

$$\lambda_{N+i-1}^\varepsilon = \lambda_N + \mu_i^\varepsilon + o(\chi_\varepsilon^2) \text{ as } \varepsilon \rightarrow 0, \quad (5)$$

where

$$\chi_\varepsilon^2 \equiv \sup\{\text{Cap}_\Omega(K_\varepsilon, u) : u \in E(\lambda_N) \text{ and } \|u\| = 1\}$$

and $\{\mu_i^\varepsilon\}_{i=1}^m$ are the eigenvalues of the quadratic form r_ε defined, for $u, v \in E(\lambda_N)$, by

$$r_\varepsilon(u, v) \equiv \int_\Omega \nabla V_{K_\varepsilon, u} \cdot \nabla V_{K_\varepsilon, v} dx - \lambda_N \int_\Omega V_{K_\varepsilon, u} V_{K_\varepsilon, v} dx.$$

Theorem 1.9 is a generalization of [2, Theorem 1.4], the latter applying only to simple eigenvalues. It is similar to [12, Theorem 1.2], which involves the standard capacity rather than the u -capacity. However, Theorem 1.9 suffers from several limitations. First of all, the quantities χ_ε , $\{\mu_i^\varepsilon\}$ can be difficult to compute explicitly. Furthermore, it can happen that, for some i , $\mu_i^\varepsilon = o(\chi_\varepsilon^2)$, in which case (5) reduces to the estimate $\lambda_{N+i-1}^\varepsilon - \lambda_N = o(\chi_\varepsilon^2)$. To progress further, we will focus on regular compact sets concentrating to a point, describe the local behavior of eigenfunctions, and determine asymptotic expansions for the (u, v) -capacity. For definiteness, and in view of concrete applications, we will give the final result in the two-dimensional case ($d = 2$). However, our method could be extended to higher dimensions.

1.2 Refinement using analyticity

In this section, we fix a point x_0 in $\Omega \subseteq \mathbb{R}^d$. Without loss of generality, we can assume $x_0 = 0$. Let us recall some basic facts about Laplacian eigenfunctions. Let λ be an eigenvalue of Problem (1) and u an associated eigenfunction. Since the differential operator $-\Delta - \lambda$ is elliptic with analytic coefficients, u is real-analytic in Ω (see, e.g., [21, Theorem 7.5.1]). Using the standard multi-index notation, this means in particular that

$$u(x) = \sum_{\beta \in \mathbb{N}^d} \frac{1}{\beta!} D^\beta u(0) x^\beta$$

for $|x|$ small enough. This can be rewritten as

$$u(x) = \sum_{k \in \mathbb{N}} P_k(x)$$

where

$$P_k(x) \equiv \sum_{|\beta|=k} \frac{1}{\beta!} D^\beta u(0) x^\beta$$

is a homogeneous polynomial of degree k in d variables. We call the smallest k such that $P_k \neq 0$ the *order of vanishing* (or more briefly the *order*) of u and we denote it by $\kappa(u)$. We denote the polynomial $P_{\kappa(u)}$ by $u_{\#}$ and call it the *principal part* of u . Note that these definitions make sense whenever u is real-analytic in a neighborhood of 0, and we will use them as soon as this condition is satisfied. When u is an eigenfunction, we have an additional property: it follows immediately from the eigenvalue equation $\Delta u + \lambda u = 0$ that $\Delta u_{\#} = 0$ (i.e. the homogeneous polynomial $u_{\#}$ is harmonic).

We now return to the situation described in the previous section, with λ_N an eigenvalue of multiplicity m for Problem (1) and $E(\lambda_N)$ the associated eigenspace. We will use the following results, proved in Appendix A.

Proposition 1.10. *There exists a decomposition of $E(\lambda_N)$ into a sum of orthogonal subspaces*

$$E(\lambda_N) = E_1 \oplus \cdots \oplus E_p$$

and an associated finite decreasing sequence of integers

$$k_1 > \cdots > k_p \geq 0$$

such that, for all $1 \leq j \leq p$, a function in $E_j \setminus \{0\}$ has the order of vanishing k_j at 0. In addition, such a decomposition is unique. We call it the order decomposition of $E(\lambda_N)$.

Proposition 1.11. *Let $E = E_1 \oplus \cdots \oplus E_p$, be the order decomposition of Proposition 1.10. Then the dimension of E_j is at most the dimension of the space of spherical harmonics in d variables of degree k_j (see, e.g., [4, pp. 159–165]). Explicitly,*

$$\dim(E_j) \leq \binom{k_j + d - 2}{k_j} + \binom{k_j + d - 3}{k_j - 1}.$$

Remark 1.12. *As a consequence of Propositions 1.10 and 1.11,*

- *if $k_p = 0$ (that is to say if E contains an eigenfunction which does not vanish at 0), $\dim(E_p) = 1$, whatever the dimension d is;*
- *in the case $d = 2$, $\dim(E_j) \leq 2$ for all $1 \leq j \leq p$.*

Our analysis of the (u, v) -capacity is of independent interest and will yield stronger results than strictly needed for studying the behavior of eigenvalues. We will need some additional regularity of the data. We fix an $\alpha \in]0, 1[$ and make the following assumptions.

(A1) The dimension d equals 2;

(A2) Ω is an open, bounded and connected set in $\mathbb{R}^2$, of class $C^{1,\alpha}$, such that $\mathbb{R}^2 \setminus \overline{\Omega}$ is connected and $0 \in \Omega$;

(A3) $K_\varepsilon = \varepsilon \overline{\omega}$, where $\omega \subseteq \mathbb{R}^2$ satisfies Assumption (A2).

Let us insist on the fact that we will prove the subsequent results in the two-dimensional case (Assumption (A1)). Let us also note that (A2) and (A3) imply the existence of a positive number ε_0 such that $K_\varepsilon = \varepsilon \overline{\omega} \subseteq \Omega$ for all $\varepsilon \in]-\varepsilon_0, \varepsilon_0[$. Finally, we consider functions u, v defined in Ω , satisfying

(A4) u and v belong to $H_0^1(\Omega)$ and are real-analytic in a neighborhood of 0.

Thus, in the case of dimension $d = 2$, we will be able to show the following theorem on the representation of $\text{Cap}_\Omega(\varepsilon\bar{\omega}, u, v)$ as the sum of a convergent series (see Theorem 2.13).

Theorem 1.13. *Under Assumptions (A1)–(A4), $\text{Cap}_\Omega(\varepsilon\bar{\omega}, u, v)$ is the sum of a convergent series. More precisely, there exist $r_0 \in \mathbb{R}$, a family of real numbers $\{c_{(n,l)}\}_{\substack{(n,l) \in \mathbb{N}^2 \\ l \leq n+1}}$, and ε_c positive and small enough, such that the series*

$$\sum_{n=0}^{\infty} \varepsilon^n \sum_{l=0}^{n+1} \frac{c_{(n,l)} \eta^l}{(r_0 \eta + (2\pi)^{-1})^l}$$

converges absolutely for all $(\varepsilon, \eta) \in]-\varepsilon_c, \varepsilon_c[\times]1/\log \varepsilon_c, -1/\log \varepsilon_c[$ and that

$$\text{Cap}_\Omega(\varepsilon\bar{\omega}, u, v) = \sum_{n=0}^{\infty} \varepsilon^n \sum_{l=0}^{n+1} \frac{c_{(n,l)}}{(r_0 + (2\pi)^{-1} \log |\varepsilon|)^l} \quad (6)$$

for all $\varepsilon \in]-\varepsilon_c, \varepsilon_c[\setminus \{0\}$. Moreover, $c_{(0,0)} = 0$ and $c_{(0,1)} = -u(0)v(0)$.

We will need more precise information on the coefficients $\{c_{(n,l)}\}_{\substack{(n,l) \in \mathbb{N}^2 \\ l \leq n+1}}$ in the series (6). In order to state our results, we introduce the following notation. Given u satisfying assumption (A4) and such that $u(0) = 0$, we define u as the unique function, continuous in $\mathbb{R}^2 \setminus \omega$, satisfying

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus \bar{\omega}, \\ u = u_{\#} & \text{on } \partial\omega, \\ \sup_{\mathbb{R}^2 \setminus \omega} |u| < \infty, \end{cases} \quad (7)$$

where $u_{\#}$ is, as before, the principal part of u (see problem (33)). Then, for u, v satisfying (A4), with $u(0) = v(0) = 0$, we define

$$\mathcal{Q}(u, v) \equiv \int_{\mathbb{R}^2 \setminus \bar{\omega}} \nabla u \cdot \nabla v \, dx + \int_{\omega} \nabla u_{\#} \cdot \nabla v_{\#} \, dx. \quad (8)$$

We observe that the expression in (8) is well defined. Indeed, the functions $u_{\#}$ and $v_{\#}$ are C^∞ in the whole of $\mathbb{R}^2$ (and thus their gradients are bounded in $\bar{\omega}$). Moreover, u and v are harmonic in $\mathbb{R}^2 \setminus \bar{\omega}$ and harmonic at infinity and thus the Divergence Theorem and the decay of their radial derivatives imply that $\int_{\mathbb{R}^2 \setminus \bar{\omega}} \nabla u \cdot \nabla v \, dx$ is bounded. Let us note that $\mathcal{Q}$ is *not* a bilinear form, since the principal part $u_{\#}$ does not depend linearly on u (and, as a consequence, neither does u). However, the restriction of $\mathcal{Q}$ to suitable subspaces of $E(\lambda_N)$ defines bilinear forms, as we will see in Definition 1.16 below.

Under vanishing assumptions on u and v at 0, we deduce the validity of the following proposition on the coefficients of the series in Equation (6) (see Theorem 2.15).

Proposition 1.14. *If u, v satisfy (A4), with $u(0) = v(0) = 0$, the coefficients in the series (6) satisfy*

$$1. \quad c_{(n,l)} = 0 \text{ if } n \leq \kappa(u) + \kappa(v) - 1;$$

2. $c_{(\kappa(u)+\kappa(v),0)} = \mathcal{Q}(u, v)$.

From Theorem 1.13 and Proposition 1.14, we immediately deduce the asymptotic behavior of some (u, v) -capacities (see also Remark 2.16).

Corollary 1.15. *Let us fix u, v satisfying Assumption (A4). Then,*

1. *if $u(0)$ and $v(0)$ are non-zero,*

$$\text{Cap}_\Omega(\varepsilon\bar{\omega}, u, v) = \frac{2\pi u(0)v(0)}{|\log(\varepsilon)|} + o\left(\frac{1}{|\log(\varepsilon)|}\right) \text{ as } \varepsilon \rightarrow 0^+;$$

2. *if $u(0) = v(0) = 0$,*

$$\text{Cap}_\Omega(\varepsilon\bar{\omega}, u, v) = \varepsilon^{\kappa(u)+\kappa(v)} \mathcal{Q}(u, v) + o\left(\varepsilon^{\kappa(u)+\kappa(v)}\right) \text{ as } \varepsilon \rightarrow 0^+.$$

Let us now consider again an eigenvalue λ_N of multiplicity m and $E(\lambda_N)$ the associated eigenspace. We use the order decomposition defined in Proposition 1.10.

Definition 1.16. *For all $1 \leq j \leq p$, we define $\mathcal{Q}_j$ on E_j in the following way. If $k_j = 0$, $u, v \in E_j$,*

$$\mathcal{Q}_j(u, v) \equiv 2\pi u(0)v(0).$$

If $k_j \geq 1$, $u, v \in E_j$

$$\mathcal{Q}_j(u, v) \equiv \mathcal{Q}(u, v).$$

It is a strictly positive (in particular non-degenerate) symmetric bilinear form on E_j .

Using the previous definitions and results, we can describe the behavior of the eigenvalues $(\lambda_i^\varepsilon)_{N \leq i \leq N+m-1}$, and more specifically give the principal part of the spectral shift $\lambda_i^\varepsilon - \lambda_N$ for each eigenvalue branch departing from λ_N . Let us formulate the result precisely.

In order to unify the notation, we define, for $k \in \mathbb{N}$ and $\varepsilon > 0$,

$$\rho_k^\varepsilon \equiv \begin{cases} \frac{1}{|\log(\varepsilon)|} & \text{if } k = 0, \\ \varepsilon^{2k} & \text{if } k \geq 1. \end{cases} \quad (9)$$

The functions $\{\varepsilon \mapsto \rho_k^\varepsilon\}_{k \geq 0}$ form a so-called *asymptotic scale*: they are continuous and positive in $]0, +\infty[$, and as $\varepsilon \rightarrow 0^+$, $\rho_0^\varepsilon \rightarrow 0$ and $\rho_{k+1}^\varepsilon = o(\rho_k^\varepsilon)$. We will find an equivalent of $\lambda_i^\varepsilon - \lambda_N$ on that scale. Note that according to Lemma 3.2, the error term χ_ε^2 (defined by Equation (35)) is of the same order as $\rho_{k_p}^\varepsilon$ (recall that k_p is the smallest possible order for an eigenfunction in $E(\lambda_N)$). Note also that the particular form of the functions $\{\rho_k^\varepsilon\}$ depends on Assumption (A1), namely $d = 2$.

Theorem 1.17. *Let us assume that (A1)–(A3) are satisfied. For $1 \leq j \leq p$, we write*

$$m_j \equiv \dim(E_j),$$

so that

$$m = m_1 + \cdots + m_j + \cdots + m_p,$$

and we denote by

$$0 < \mu_{j,1} \leq \cdots \leq \mu_{j,\ell} \leq \cdots \leq \mu_{j,m_j}$$

the eigenvalues of the quadratic form $\mathcal{Q}_j$. Then, for all $1 \leq j \leq p$ and $1 \leq \ell \leq m_j$,

$$\lambda_{N-1+m_1+\cdots+m_{j-1}+\ell}^\varepsilon = \lambda_N + \mu_{j,\ell} \rho_{k_j}^\varepsilon + o(\rho_{k_j}^\varepsilon) \text{ as } \varepsilon \rightarrow 0^+. \quad (10)$$

As an illustration, let us consider a particular case.

Corollary 1.18. *Let us assume that (A1)–(A3) are satisfied and that λ_N has multiplicity 2 (i.e. $m = 2$). Then one of the following alternative holds.*

1. *There exist two normalized eigenfunctions $u_1, u_2 \in E(\lambda_N) \setminus \{0\}$, with respective order of vanishing k_1, k_2 such that $k_1 > k_2$. In that case,*

$$\lambda_N^\varepsilon = \lambda_N + \mathcal{Q}(u_1, u_1)\varepsilon^{2k_1} + o(\varepsilon^{2k_1})$$

and

$$\lambda_{N+1}^\varepsilon = \lambda_N + \begin{cases} \frac{2\pi u_2(0)^2}{|\log(\varepsilon)|} + o\left(\frac{1}{|\log(\varepsilon)|}\right) & \text{if } k_2 = 0, \\ \mathcal{Q}(u_2, u_2)\varepsilon^{2k_2} + o(\varepsilon^{2k_2}) & \text{if } k_2 \geq 1. \end{cases}$$

2. *All eigenfunctions in $E(\lambda_N)$ have the same order of vanishing, which we denote by k . Let us note that necessarily $k \geq 1$. In that case, let us choose eigenfunctions u_1, u_2 forming an orthonormal basis of $E(\lambda_N)$ and let us denote by $0 < \mu_1 \leq \mu_2$ the eigenvalues of the symmetric and positive matrix*

$$\begin{pmatrix} \mathcal{Q}(u_1, u_1) & \mathcal{Q}(u_1, u_2) \\ \mathcal{Q}(u_1, u_2) & \mathcal{Q}(u_2, u_2) \end{pmatrix}.$$

Then

$$\begin{aligned} \lambda_N^\varepsilon &= \lambda_N + \mu_1 \varepsilon^{2k} + o(\varepsilon^{2k}); \\ \lambda_{N+1}^\varepsilon &= \lambda_N + \mu_2 \varepsilon^{2k} + o(\varepsilon^{2k}). \end{aligned}$$

We note that in the preceding Corollary 1.18, item 2, the splitting of the two branches does not necessarily occur since the two eigenvalues μ_1 and μ_2 are not necessarily different. In the subsequent result we exhibit a particular case where this splitting in fact takes place.

Corollary 1.19. *Let us assume that (A1)–(A3) are satisfied and that λ_N has multiplicity 2 (i.e. $m = 2$). Moreover, let us assume that ω is a disk and that $\{u_N, u_{N+1}\}$ is an orthogonal basis of $E(\lambda_N)$. If, at the point $x = 0$, the nodal lines of u_N are not tangent to any bisector between two nodal lines of u_{N+1} , then $\mu_1 \neq \mu_2$ in Corollary 1.18, item 2. Thus, the double eigenvalue λ_N splits into two different branches λ_N^ε and $\lambda_{N+1}^\varepsilon$.*

The proof of the preceding corollary is contained in Section 5. It is deduced from the general case of elliptic holes in dimension 2. Even in this general case, we find sufficient conditions for eigenvalues splitting involving angles and coefficients appearing in the asymptotic behavior of eigenfunctions.

2 Asymptotic behavior of (u^a, u^b) -capacities

The aim of this section is to study the asymptotic behavior of $\text{Cap}_\Omega(\varepsilon\bar{\omega}, u^a, u^b)$ as $\varepsilon \rightarrow 0$. To reach this goal, we consider the case of dimension $d = 2$ and we assume some smoothness on the sets and some regularity on the functions u^a and u^b (see assumptions (A1)–(A4)). We recall here these assumptions. We will work in the frame of Schauder classes: we take

$$\alpha \in]0, 1[,$$

and we assume that

$$\begin{aligned} \Omega \text{ and } \omega \text{ are open bounded connected subsets of } \mathbb{R}^2 \text{ of} \\ \text{class } C^{1,\alpha} \text{ such that } \mathbb{R}^2 \setminus \overline{\Omega} \text{ and } \mathbb{R}^2 \setminus \overline{\omega} \text{ are connected,} \\ \text{and such that the origin } 0 \text{ of } \mathbb{R}^2 \text{ belongs both to } \Omega \text{ and } \omega. \end{aligned} \quad (11)$$

For the definition of functions and sets of the Schauder classes $C^{0,\alpha}$ and $C^{1,\alpha}$ we refer for example to Gilbarg and Trudinger [19, §6.2]. Condition (11) implies that there exists a real number ε_0 such that

$$\varepsilon_0 > 0 \text{ and } \varepsilon \overline{\omega} \subseteq \Omega \text{ for all } \varepsilon \in]-\varepsilon_0, \varepsilon_0[.$$

Then we set

$$\Omega_\varepsilon \equiv \Omega \setminus K_\varepsilon \quad \text{where} \quad K_\varepsilon \equiv \varepsilon \overline{\omega} \quad \forall \varepsilon \in]-\varepsilon_0, \varepsilon_0[.$$

Clearly, Ω_ε is an open bounded connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ for all $\varepsilon \in]-\varepsilon_0, \varepsilon_0[\setminus \{0\}$. The boundary $\partial\Omega_\varepsilon$ of Ω_ε is the union of the two connected components $\partial\Omega$ and $\partial(\varepsilon\omega) = \varepsilon\partial\omega$, for all $\varepsilon \in]-\varepsilon_0, \varepsilon_0[$. We also note that $\Omega_0 = \Omega \setminus \{0\}$. Moreover, $\{\varepsilon\overline{\omega}\}_{\varepsilon>0}$ is a family of compact sets concentrating to the origin in the sense of [2, Definition 1.2].

As we have mentioned, we also need some regularity on the functions u^a, u^b : namely, we ask that

$$u^a, u^b \in H^1(\Omega) \text{ are analytic in a neighborhood of } 0. \quad (12)$$

Our aim is twofold. On one side, we wish to obtain accurate and explicit expansions for $\text{Cap}_\Omega(\varepsilon\overline{\omega}, u^a, u^b)$ in terms of the parameter ε . On the other side, we also wish to emphasize the dependence on the geometric data of the problem (i.e., Ω and ω) and on the functions u^a and u^b on $\text{Cap}_\Omega(\varepsilon\overline{\omega}, u^a, u^b)$.

We observe that in this section we confine to the two-dimensional case. We exploit tools from potential theory and, as it happens often in this framework, the two-dimensional case and the one of dimension equal to or greater than three require a different analysis. This is mainly due to the different aspect of the fundamental solution of the Laplace equation: a logarithmic function of the $|x|$ if the dimension is two and a multiple of $|x|^{2-d}$ if the dimension d is equal to or greater than three.

2.1 Our strategy: the functional analytic approach

In order to study the asymptotic behavior of $\text{Cap}_\Omega(\varepsilon\overline{\omega}, u^a, u^b)$ as $\varepsilon \rightarrow 0$, we proceed as in [1] and we adopt the Functional Analytic Approach proposed by Lanza de Cristoforis [27, 28] for the analysis of singular perturbation problems in perforated domains (see also the monograph [14] for a detailed presentation of the method). By applying this approach, one can deduce the possibility to represent the solution or related functionals as convergent power series.

To analyze $\text{Cap}_\Omega(\varepsilon\overline{\omega}, u^a, u^b)$, we modify the results of [1], where we considered the u -capacity $\text{Cap}_\Omega(\varepsilon\overline{\omega}, u)$. Although the modifications are quite straightforward, they require some attention to write out explicitly all the coefficients in the asymptotic expansions. For this reason, for the sake of clarity and completeness, we decided to include in the present paper all the modified statements and some of the proofs.

By assumption (12) on the analyticity of u^a and u^b together with analyticity results for the composition operator (see Böhme and Tomi [7, p. 10], Henry [20, p. 29], Valent [46, Thm. 5.2,

p. 44]), we deduce that, possibly shrinking ε_0 , there exists two real analytic maps $U_{\#}^a, U_{\#}^b$ from $] - \varepsilon_0, \varepsilon_0[$ to $C^{1,\alpha}(\partial\omega)$ such that

$$u^a(\varepsilon t) = U_{\#}^a[\varepsilon](t), \quad u^b(\varepsilon t) = U_{\#}^b[\varepsilon](t), \quad \forall t \in \partial\omega, \forall \varepsilon \in] - \varepsilon_0, \varepsilon_0[\quad (13)$$

(see Deimling [17, §15] for the definition and properties of analytic maps). Then for all $\varepsilon \in] - \varepsilon_0, \varepsilon_0[\setminus \{0\}$, we denote by u_{ε}^a and u_{ε}^b the unique solutions in $C^{1,\alpha}(\overline{\Omega_{\varepsilon}})$ of the problems

$$\begin{cases} \Delta u_{\varepsilon}^a = 0 & \text{in } \Omega_{\varepsilon}, \\ u_{\varepsilon}^a(x) = 0 & \text{for all } x \in \partial\Omega, \\ u_{\varepsilon}^a(x) = U_{\#}^a[\varepsilon](x/\varepsilon) & \text{for all } x \in \varepsilon\partial\omega \end{cases} \quad (14)$$

and

$$\begin{cases} \Delta u_{\varepsilon}^b = 0 & \text{in } \Omega_{\varepsilon}, \\ u_{\varepsilon}^b(x) = 0 & \text{for all } x \in \partial\Omega, \\ u_{\varepsilon}^b(x) = U_{\#}^b[\varepsilon](x/\varepsilon) & \text{for all } x \in \varepsilon\partial\omega, \end{cases}$$

respectively. Clearly,

$$\begin{aligned} V_{\varepsilon\overline{\omega}, u^a}(x) &= u_{\varepsilon}^a(x), & V_{\varepsilon\overline{\omega}, u^b}(x) &= u_{\varepsilon}^b(x), & \forall x \in \Omega_{\varepsilon}, \forall \varepsilon \in] - \varepsilon_0, \varepsilon_0[\setminus \{0\}, \\ V_{\varepsilon\overline{\omega}, u^a}(x) &= u^a(x), & V_{\varepsilon\overline{\omega}, u^b}(x) &= u^b(x), & \forall x \in \varepsilon\overline{\omega}, \forall \varepsilon \in] - \varepsilon_0, \varepsilon_0[\setminus \{0\}. \end{aligned}$$

Accordingly, by the Divergence Theorem, we have that

$$\begin{aligned} \text{Cap}_{\Omega}(\varepsilon\overline{\omega}, u^a, u^b) &= \int_{\Omega_{\varepsilon}} \nabla u_{\varepsilon}^a \cdot \nabla u_{\varepsilon}^b dx + \int_{\varepsilon\omega} \nabla u^a \cdot \nabla u^b dx \\ &= - \int_{\partial(\varepsilon\omega)} \frac{\partial u_{\varepsilon}^a}{\partial \nu_{\varepsilon\omega}} u_{\varepsilon}^b d\sigma + \varepsilon^2 \int_{\omega} (\nabla u^a)(\varepsilon t) \cdot (\nabla u^b)(\varepsilon t) dt \\ &= - \int_{\partial\omega} \nu_{\omega}(t) \cdot \nabla(u_{\varepsilon}^a(\varepsilon t)) u^b(\varepsilon t) d\sigma_t + \varepsilon^2 \int_{\omega} (\nabla u^a)(\varepsilon t) \cdot (\nabla u^b)(\varepsilon t) dt, \end{aligned} \quad (15)$$

for all $\varepsilon \in] - \varepsilon_0, \varepsilon_0[\setminus \{0\}$. Here above the symbols ν_{ω} and $\nu_{\varepsilon\omega}$ denote the outward unit normal to $\partial\omega$ and to $\partial(\varepsilon\omega)$, respectively.

As we have mentioned, our goal is to provide a fully constructive and complete asymptotic expansion for $\text{Cap}_{\Omega}(\varepsilon\overline{\omega}, u^a, u^b)$ as $\varepsilon \rightarrow 0$. As in [1], to obtain an asymptotic expansion of $\text{Cap}_{\Omega}(\varepsilon\overline{\omega}, u^a, u^b)$, we follow the method developed in [16] for the solution of the Dirichlet problem in a planar perforated domain. By the computation in (15), the quantity $\text{Cap}_{\Omega}(\varepsilon\overline{\omega}, u^a, u^b)$ can be expressed as the sum of two integrals:

$$\varepsilon^2 \int_{\omega} (\nabla u^a)(\varepsilon t) \cdot (\nabla u^b)(\varepsilon t) dt$$

and the opposite of the integral on $\partial\omega$ of the function

$$t \mapsto \nu_{\omega}(t) \cdot \nabla(u_{\varepsilon}^a(\varepsilon t)) u^b(\varepsilon t). \quad (16)$$

2.2 Classical notions of potential theory

Formula (15) shows that $\text{Cap}_\Omega(\varepsilon\bar{\omega}, u^a, u^b)$ can be expressed in terms of the gradient of the function u_ε^a which solve problem (14) and of the traces of the (given) functions u^a and u^b . As a consequence, we need to understand the behavior of the solution to problem (14) as $\varepsilon \rightarrow 0$ and, in order to do so, we shall exploit the approach of [1] based on integral operators, which allows to convert a boundary value problem into a set of integral equations defined on the boundary of the domain.

These operators are integral operators whose kernel is the fundamental solution of the Laplacian or its normal derivative. Therefore, we introduce the fundamental solution S of $\Delta \equiv \sum_{j=1}^2 \partial_j^2$ in $\mathbb{R}^2$ as the function from $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S(x) \equiv \frac{1}{2\pi} \log |x| \quad \forall x \in \mathbb{R}^2 \setminus \{0\}.$$

Now let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$.

We begin by introducing the single layer potential. If $\phi \in C^{0,\alpha}(\partial\mathcal{O})$, then the single layer potential $v[\partial\mathcal{O}, \phi]$ with density ϕ is the function defined by

$$v[\partial\mathcal{O}, \phi](x) \equiv \int_{\partial\mathcal{O}} \phi(y) S(x-y) d\sigma_y \quad \forall x \in \mathbb{R}^2,$$

where $d\sigma$ denotes the arc length element on $\partial\mathcal{O}$. The function $v[\partial\mathcal{O}, \phi]$ is continuous from $\mathbb{R}^2$ to $\mathbb{R}$. The restriction $v^+[\partial\mathcal{O}, \phi] \equiv v[\partial\mathcal{O}, \phi]|_{\bar{\mathcal{O}}}$ belongs to $C^{1,\alpha}(\bar{\mathcal{O}})$. Moreover, if we denote by $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^2 \setminus \mathcal{O})$ the space of functions on $\mathbb{R}^2 \setminus \mathcal{O}$ whose restrictions to $\bar{\mathcal{U}}$ belong to $C^{1,\alpha}(\bar{\mathcal{U}})$ for all open bounded subsets $\mathcal{U}$ of $\mathbb{R}^2 \setminus \mathcal{O}$, then $v^-[\partial\mathcal{O}, \phi] \equiv v[\partial\mathcal{O}, \phi]|_{\mathbb{R}^2 \setminus \mathcal{O}}$ belongs to $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^2 \setminus \mathcal{O})$.

Instead, for a function $\psi \in C^{1,\alpha}(\partial\mathcal{O})$, we denote by $w[\partial\mathcal{O}, \psi]$ the double layer potential with density ψ , namely

$$w[\partial\mathcal{O}, \psi](x) \equiv - \int_{\partial\mathcal{O}} \psi(y) \nu_{\mathcal{O}}(y) \cdot \nabla S(x-y) d\sigma_y \quad \forall x \in \mathbb{R}^2,$$

where $\nu_{\mathcal{O}}$ denotes the outer unit normal to $\partial\mathcal{O}$ and the symbol $\cdot$ denotes the scalar product in $\mathbb{R}^2$. Then the restriction $w[\partial\mathcal{O}, \psi]|_{\mathcal{O}}$ extends to a function $w^+[\partial\mathcal{O}, \psi]$ of $C^{1,\alpha}(\bar{\mathcal{O}})$ and the restriction $w[\partial\mathcal{O}, \psi]|_{\mathbb{R}^2 \setminus \bar{\mathcal{O}}}$ extends to a function $w^-[\partial\mathcal{O}, \psi]$ of $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^2 \setminus \mathcal{O})$.

In order to describe the properties of the trace of the double layer potential on $\partial\mathcal{O}$ and of the normal derivative of the single layer potential, we introduce the boundary integral operators $W_{\mathcal{O}}$ and $W_{\mathcal{O}}^*$ as follows:

$$W_{\mathcal{O}}[\psi](x) \equiv - \int_{\partial\mathcal{O}} \psi(y) \nu_{\mathcal{O}}(y) \cdot \nabla S(x-y) d\sigma_y \quad \forall x \in \partial\mathcal{O},$$

for all $\psi \in C^{1,\alpha}(\partial\mathcal{O})$, and

$$W_{\mathcal{O}}^*[\phi](x) \equiv \int_{\partial\mathcal{O}} \phi(y) \nu_{\mathcal{O}}(x) \cdot \nabla S(x-y) d\sigma_y \quad \forall x \in \partial\mathcal{O},$$

for all $\phi \in C^{0,\alpha}(\partial\mathcal{O})$. Then $W_{\mathcal{O}}$ is a compact operator from $C^{1,\alpha}(\partial\mathcal{O})$ to itself and $W_{\mathcal{O}}^*$ is a compact operator from $C^{0,\alpha}(\partial\mathcal{O})$ to itself (see Schauder [44] and [45]). The operators $W_{\mathcal{O}}$ and $W_{\mathcal{O}}^*$ are adjoint one to the other with respect to the duality on $C^{1,\alpha}(\partial\mathcal{O}) \times C^{0,\alpha}(\partial\mathcal{O})$ induced by the

inner product of the Lebesgue space $L^2(\partial\mathcal{O})$ (see, *e.g.*, Kress [23, Chap. 4]). For the theory of dual systems and the corresponding Fredholm Alternative Principle, we refer the reader to Kress [23] and Wendland [47, 48]. By means of the operators $W_{\mathcal{O}}$ and $W_{\mathcal{O}}^*$, we can describe the traces $w^\pm[\partial\mathcal{O}, \psi]_{|\partial\mathcal{O}}$ and the normal derivatives $\nu_{\mathcal{O}} \cdot \nabla v^\pm[\partial\mathcal{O}, \phi]_{|\partial\mathcal{O}}$:

$$\begin{aligned} w^\pm[\partial\mathcal{O}, \psi]_{|\partial\mathcal{O}} &= \pm \frac{1}{2} \psi + W_{\mathcal{O}}[\psi] & \forall \psi \in C^{1,\alpha}(\partial\mathcal{O}), \\ \nu_{\mathcal{O}} \cdot \nabla v^\pm[\partial\mathcal{O}, \phi]_{|\partial\mathcal{O}} &= \mp \frac{1}{2} \phi + W_{\mathcal{O}}^*[\phi] & \forall \phi \in C^{0,\alpha}(\partial\mathcal{O}) \end{aligned}$$

(see, *e.g.*, Folland [18, Chap. 3]).

We shall need to consider subspaces of $C^{0,\alpha}(\partial\mathcal{O})$ and of $C^{1,\alpha}(\partial\mathcal{O})$, consisting of functions with zero integral on $\partial\mathcal{O}$. Therefore, we set

$$C^{k,\alpha}(\partial\mathcal{O})_0 \equiv \left\{ f \in C^{k,\alpha}(\partial\mathcal{O}) : \int_{\partial\mathcal{O}} f \, d\sigma = 0 \right\} \quad \text{for } k = 0, 1.$$

2.3 An integral formulation of the boundary value problem

To convert problem (14) into a system of integral equations, we follow the strategy of Lanza de Cristoforis [29] and of [16]. As in [1], we divide the problem in a part which can be solved in terms of the double layer potential and a part which will be represented by a single layer potential. Now we proceed as in [16] and we introduce the map $M \equiv (M^o, M^i, M^c)$ from $]-\varepsilon_0, \varepsilon_0[\times C^{0,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\omega)$ to $C^{0,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\omega)_0 \times \mathbb{R}$ by setting

$$\begin{aligned} M^o[\varepsilon, \rho^o, \rho^i](x) &\equiv \frac{1}{2} \rho^o(x) + W_{\Omega}^*[\rho^o](x) + \int_{\partial\omega} \rho^i(s) \, \nu_{\Omega}(x) \cdot \nabla S(x - \varepsilon s) \, d\sigma_s & \forall x \in \partial\Omega, \\ M^i[\varepsilon, \rho^o, \rho^i](t) &\equiv \frac{1}{2} \rho^i(t) - W_{\omega}^*[\rho^i](t) - \varepsilon \int_{\partial\Omega} \rho^o(y) \, \nu_{\omega}(t) \cdot \nabla S(\varepsilon t - y) \, d\sigma_y & \forall t \in \partial\omega, \\ M^c[\varepsilon, \rho^o, \rho^i] &\equiv \int_{\partial\omega} \rho^i \, d\sigma - 1, \end{aligned}$$

for all $(\varepsilon, \rho^o, \rho^i) \in]-\varepsilon_0, \varepsilon_0[\times C^{0,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\omega)$. Then we state the following result of Lanza de Cristoforis [29, §3] (see also [16, Prop. 4.1]).

Proposition 2.1. *The following statements hold.*

- (i) *The map M is real analytic.*
- (ii) *If $\varepsilon \in]-\varepsilon_0, \varepsilon_0[$, then there exists a unique pair $(\rho^o[\varepsilon], \rho^i[\varepsilon]) \in C^{0,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\omega)$ such that $M[\varepsilon, \rho^o[\varepsilon], \rho^i[\varepsilon]] = 0$.*
- (iii) *The map from $]-\varepsilon_0, \varepsilon_0[$ to $C^{0,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\omega)$ which takes ε to $(\rho^o[\varepsilon], \rho^i[\varepsilon])$ is real analytic.*

Remark 2.2. *For each $\varepsilon \in]-\varepsilon_0, \varepsilon_0[\setminus \{0\}$, let τ_ε be defined by $\tau_\varepsilon(x) \equiv \rho^o[\varepsilon](x)$ for all $x \in \partial\Omega$ and $\tau_\varepsilon(x) \equiv |\varepsilon|^{-1} \rho^i[\varepsilon](x/\varepsilon)$ for all $x \in \partial(\varepsilon\omega)$. Then*

$$\frac{1}{2} \tau_\varepsilon + W_{\Omega_\varepsilon}^*[\tau_\varepsilon] = 0, \quad \int_{\partial(\varepsilon\omega)} \tau_\varepsilon \, d\sigma = 1,$$

for all $\varepsilon \in]-\varepsilon_0, \varepsilon_0[\setminus \{0\}$.

We now consider the part which can be solved by the double layer potential. For $\varepsilon \in] - \varepsilon_0, \varepsilon_0[\setminus \{0\}$, we define the boundary datum g_ε^a by setting

$$g_\varepsilon^a(x) \equiv 0 \quad \forall x \in \partial\Omega, \quad g_\varepsilon^a(x) = U_\#^a[\varepsilon](x/\varepsilon) - \int_{\partial(\varepsilon\omega)} U_\#^a[\varepsilon](y/\varepsilon) \tau_\varepsilon(y) d\sigma_y \quad \forall x \in \partial(\varepsilon\omega),$$

Standard Fredholm theory and classical potential theory imply that for $\varepsilon \in] - \varepsilon_0, \varepsilon_0[\setminus \{0\}$ the function g_ε^a belongs to the image of the trace of the double layer potential (for the definition of $U_\#^a$ see (13)). As in [16], we then define the map $\Lambda \equiv (\Lambda^o, \Lambda^i)$ from $] - \varepsilon_0, \varepsilon_0[\times C^{1,\alpha}(\partial\Omega) \times C^{1,\alpha}(\partial\omega)_0$ to $C^{1,\alpha}(\partial\Omega) \times C^{1,\alpha}(\partial\omega)$ by

$$\begin{aligned} \Lambda^o[\varepsilon, \theta^o, \theta^i](x) &\equiv \frac{1}{2} \theta^o(x) + W_\Omega[\theta^o](x) \\ &\quad + \varepsilon \int_{\partial\omega} \theta^i(s) \nu_\omega(s) \cdot \nabla S(x - \varepsilon s) d\sigma_s \quad \forall x \in \partial\Omega, \\ \Lambda^i[\varepsilon, \theta^o, \theta^i](t) &\equiv \frac{1}{2} \theta^i(t) - W_\omega[\theta^i](t) + w[\partial\Omega, \theta^o](\varepsilon t) \\ &\quad - U_\#^a[\varepsilon](t) + \int_{\partial\omega} U_\#^a[\varepsilon] \rho^i[\varepsilon] d\sigma \quad \forall t \in \partial\omega, \end{aligned}$$

for all $(\varepsilon, \theta^o, \theta^i) \in] - \varepsilon_0, \varepsilon_0[\times C^{1,\alpha}(\partial\Omega) \times C^{1,\alpha}(\partial\omega)_0$. Then we have the following result of Lanza de Cristoforis [29, §4] on the regularity of Λ (see [16, Prop. 4.3]).

Proposition 2.3. *The following statements hold.*

- (i) *The map Λ is real analytic.*
- (ii) *If $\varepsilon \in] - \varepsilon_0, \varepsilon_0[$, then there exists a unique pair $(\theta^o[\varepsilon], \theta^i[\varepsilon]) \in C^{1,\alpha}(\partial\Omega) \times C^{1,\alpha}(\partial\omega)_0$ such that $\Lambda[\varepsilon, \theta^o[\varepsilon], \theta^i[\varepsilon]] = 0$.*
- (iii) *The map from $] - \varepsilon_0, \varepsilon_0[$ to $C^{1,\alpha}(\partial\Omega) \times C^{1,\alpha}(\partial\omega)_0$ which takes ε to $(\theta^o[\varepsilon], \theta^i[\varepsilon])$ is real analytic.*

Remark 2.4. *For each $\varepsilon \in] - \varepsilon_0, \varepsilon_0[\setminus \{0\}$, let μ_ε be defined by $\mu_\varepsilon(x) \equiv \theta^o[\varepsilon](x)$ for all $x \in \partial\Omega$ and $\mu_\varepsilon(x) \equiv \theta^i[\varepsilon](x/\varepsilon)$ for all $x \in \partial(\varepsilon\omega)$. Then*

$$\frac{1}{2} \mu_\varepsilon + W_{\Omega_\varepsilon}[\mu_\varepsilon] = g_\varepsilon^a,$$

for all $\varepsilon \in] - \varepsilon_0, \varepsilon_0[\setminus \{0\}$.

We can recover the solution u_ε^a (and in particular the rescaled function $t \mapsto u_\varepsilon^a(\varepsilon t)$) by summing the double layer potential with density μ_ε (see Remark 2.4) and a convenient multiple of the single layer potential with density τ_ε (see Remark 2.2). By arguing as in [16, Prop. 4.5], we show in the following Proposition 2.5 how to represent the rescaled function $u_\varepsilon^a(\varepsilon t)$ by means of the functions $\rho^o[\varepsilon]$, $\rho^i[\varepsilon]$, $\theta^o[\varepsilon]$, and $\theta^i[\varepsilon]$ introduced in Propositions 2.1 and 2.3 (see also Lanza de Cristoforis [29, §5] and [15, §2.4]).

Proposition 2.5. *Let $\varepsilon \in] - \varepsilon_0, \varepsilon_0[\setminus \{0\}$. Then*

$$\begin{aligned} u_\varepsilon^a(\varepsilon t) &\equiv w^+[\partial\Omega, \theta^o[\varepsilon]](\varepsilon t) - w^-[\partial\omega, \theta^i[\varepsilon]](t) \\ &+ \int_{\partial\omega} U_\#^a[\varepsilon] \rho^i[\varepsilon] d\sigma \left(v^+[\partial\Omega, \rho^o[\varepsilon]](\varepsilon t) + v^-[\partial\omega, \rho^i[\varepsilon]](t) + \frac{\log|\varepsilon|}{2\pi} \right) \\ &\times \left(\frac{1}{\int_{\partial\omega} d\sigma} \int_{\partial\omega} v[\partial\Omega, \rho^o[\varepsilon]](\varepsilon s) + v[\partial\omega, \rho^i[\varepsilon]](s) d\sigma_s + \frac{\log|\varepsilon|}{2\pi} \right)^{-1} \end{aligned}$$

for all $t \in \overline{(\varepsilon^{-1}\Omega)} \setminus \omega$.

2.4 Power series expansions of the auxiliary functions $(\rho^o[\varepsilon], \rho^i[\varepsilon])$ and $(\theta^o[\varepsilon], \theta^i[\varepsilon])$ around $\varepsilon = 0$

As in [1], the plan is first to construct an expansion for $\nu_\omega(t) \cdot \nabla \left(u_\varepsilon^a(\varepsilon t) \right) u^b(\varepsilon t)$ and then to integrate such an expansion on $\partial\omega$. We note that $u_\varepsilon^a(\varepsilon t)$ is represented by means of the auxiliary density functions $(\rho^o[\varepsilon], \rho^i[\varepsilon])$ and $(\theta^o[\varepsilon], \theta^i[\varepsilon])$. Thus the plan is to obtain an expansion for those densities and then to get the one for $u_\varepsilon^a(\varepsilon t)$ by exploiting the representation formula of Proposition 2.5.

The following Proposition 2.6 of [16, Prop. 5.1] provides a power series expansion around 0 of $(\rho^o[\varepsilon], \rho^i[\varepsilon])$. Throughout the paper, if $j \in \{1, 2\}$, then $(\partial_j F)(y)$ denotes the partial derivative with respect to x_j of the function $F(x) \equiv F(x_1, x_2)$ evaluated at $y \equiv (y_1, y_2) \in \mathbb{R}^2$.

Proposition 2.6. *Let $(\rho^o[\varepsilon], \rho^i[\varepsilon])$ be as in Proposition 2.1 for all $\varepsilon \in] - \varepsilon_0, \varepsilon_0[$. Then there exist $\varepsilon_\rho \in]0, \varepsilon_0[$ and a sequence $\{(\rho_k^o, \rho_k^i)\}_{k \in \mathbb{N}}$ in $C^{0,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\omega)$ such that*

$$\rho^o[\varepsilon] = \sum_{k=0}^{+\infty} \frac{\rho_k^o}{k!} \varepsilon^k \quad \text{and} \quad \rho^i[\varepsilon] = \sum_{k=0}^{+\infty} \frac{\rho_k^i}{k!} \varepsilon^k \quad \forall \varepsilon \in] - \varepsilon_\rho, \varepsilon_\rho[,$$

where the two series converge normally in $C^{0,\alpha}(\partial\Omega)$ and in $C^{0,\alpha}(\partial\omega)$, respectively, for all $\varepsilon \in] - \varepsilon_\rho, \varepsilon_\rho[$. Moreover, the pair of functions (ρ_0^o, ρ_0^i) is the unique solution in $C^{0,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\omega)$ of the following system of integral equations

$$\begin{aligned} \frac{1}{2} \rho_0^o(x) + W_\Omega^*[\rho_0^o](x) &= -\nu_\Omega(x) \cdot \nabla S(x) & \forall x \in \partial\Omega, \\ \frac{1}{2} \rho_0^i(t) - W_\omega^*[\rho_0^i](t) &= 0 & \forall t \in \partial\omega, \\ \int_{\partial\omega} \rho_0^i d\sigma &= 1, \end{aligned}$$

and for each $k \in \mathbb{N} \setminus \{0\}$ the pair (ρ_k^o, ρ_k^i) is the unique solution in $C^{0,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\omega)$ of the following system of integral equations which involves $\{(\rho_j^o, \rho_j^i)\}_{j=0}^{k-1}$,

$$\begin{aligned} &\frac{1}{2} \rho_k^o(x) + W_\Omega^*[\rho_k^o](x) \\ &= \sum_{j=0}^k \binom{k}{j} (-1)^{j+1} \sum_{h=0}^j \binom{j}{h} \nu_\Omega(x) \cdot (\nabla \partial_1^h \partial_2^{j-h} S)(x) \int_{\partial\omega} \rho_{k-j}^i(s) s_1^h s_2^{j-h} d\sigma_s \quad \forall x \in \partial\Omega, \\ &\frac{1}{2} \rho_k^i(t) - W_\omega^*[\rho_k^i](t) \end{aligned}$$

$$\begin{aligned}
&= k \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^{j+1} \sum_{h=0}^j \binom{j}{h} t_1^h t_2^{j-h} \nu_\omega(t) \cdot \int_{\partial\Omega} \rho_{k-1-j}^o (\nabla \partial_1^h \partial_2^{j-h} S) d\sigma \quad \forall t \in \partial\omega, \\
&\int_{\partial\omega} \rho_k^i d\sigma = 0.
\end{aligned}$$

The coefficients in the power series expansion of $(\theta^o[\varepsilon], \theta^i[\varepsilon])$ are instead determine in the following Proposition 2.7 (see [1, Prop. 3.2]).

Proposition 2.7. *Let $(\theta^o[\varepsilon], \theta^i[\varepsilon])$ be as in Proposition 2.3 for all $\varepsilon \in]-\varepsilon_0, \varepsilon_0[$. Then there exist $\varepsilon_\theta \in]0, \varepsilon_0[$ and a sequence $\{(\theta_k^o, \theta_k^i)\}_{k \in \mathbb{N}}$ in $C^{1,\alpha}(\partial\Omega) \times C^{1,\alpha}(\partial\omega)_0$ such that*

$$\theta^o[\varepsilon] = \sum_{k=0}^{\infty} \frac{\theta_k^o}{k!} \varepsilon^k \quad \text{and} \quad \theta^i[\varepsilon] = \sum_{k=0}^{\infty} \frac{\theta_k^i}{k!} \varepsilon^k \quad \forall \varepsilon \in]-\varepsilon_\theta, \varepsilon_\theta[,$$

$$(\theta_0^o, \theta_0^i) = (0, 0), \quad \theta_1^o = 0,$$

and θ_1^i is the unique solution in $C^{1,\alpha}(\partial\omega)_0$ of

$$\begin{aligned}
&\frac{1}{2} \theta_1^i(t) - W_\omega[\theta_1^i](t) \\
&= \sum_{h=0}^1 t_1^h t_2^{1-h} (\partial_1^h \partial_2^{1-h} u^a)(0) - \sum_{l=0}^1 \sum_{h=0}^l \int_{\partial\omega} s_1^h s_2^{l-h} (\partial_1^h \partial_2^{l-h} u^a)(0) \rho_{1-l}^i(s) d\sigma_s \quad \forall t \in \partial\omega,
\end{aligned}$$

and for each $k \in \mathbb{N} \setminus \{0, 1\}$ the pair (θ_k^o, θ_k^i) is the unique solution in $C^{1,\alpha}(\partial\Omega) \times C^{1,\alpha}(\partial\omega)_0$ of the following system of integral equations which involves $\{(\theta_j^o, \theta_j^i)\}_{j=0}^{k-1}$,

$$\begin{aligned}
&\frac{1}{2} \theta_k^o(x) + W_\Omega[\theta_k^o](x) \\
&= k \sum_{j=0}^{k-2} \binom{k-1}{j} (-1)^{j+1} \sum_{h=0}^j \binom{j}{h} (\nabla \partial_1^h \partial_2^{j-h} S)(x) \cdot \int_{\partial\omega} \theta_{k-1-j}^i(s) \nu_\omega(s) s_1^h s_2^{j-h} d\sigma_s \\
&\quad \forall x \in \partial\Omega, \\
&\frac{1}{2} \theta_k^i(t) - W_\omega[\theta_k^i](t) = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{j+1} \sum_{h=0}^j \binom{j}{h} t_1^h t_2^{j-h} \int_{\partial\Omega} \theta_{k-j}^o \nu_\Omega \cdot \nabla \partial_1^h \partial_2^{j-h} S d\sigma \\
&+ \sum_{h=0}^k \binom{k}{h} t_1^h t_2^{k-h} (\partial_1^h \partial_2^{k-h} u^a)(0) - \sum_{l=0}^k \sum_{h=0}^l \binom{k}{l} \binom{l}{h} \int_{\partial\omega} s_1^h s_2^{l-h} (\partial_1^h \partial_2^{l-h} u^a)(0) \rho_{k-l}^i(s) d\sigma_s \\
&\quad \forall t \in \partial\omega.
\end{aligned}$$

2.5 Series expansion of $\nu_\omega(\cdot) \cdot \nabla(u_\varepsilon^a(\cdot)) u^b(\varepsilon \cdot)$ around $\varepsilon = 0$

In order to compute an asymptotic expansion of

$$- \int_{\partial\omega} \nu_\omega(t) \cdot \nabla(u_\varepsilon^a(\varepsilon t)) u^b(\varepsilon t) d\sigma_t$$

as $\varepsilon \rightarrow 0$, we now turn to construct a series expansion for $\nu_\omega(\cdot) \cdot \nabla(u_\varepsilon^a(\cdot))u^b(\varepsilon \cdot)$ for ε in a neighborhood of 0. The coefficients of such expansion will be defined by means of the derivatives of the functions u^a and u^b and of the sequences $\{(\rho_k^o, \rho_k^i)\}_{k \in \mathbb{N}}$ and $\{(\theta_k^o, \theta_k^i)\}_{k \in \mathbb{N}}$ introduced in Section 2.4. As in [16, Prop. 6.1], in the following Proposition 2.8, we prove a representation formula which can be easily obtained by Propositions 2.5, 2.6 and 2.7, and by standard properties of real analytic maps (see also Lanza de Cristoforis [29, Theorem 5.3] and [15, Theorem 3.1]).

Proposition 2.8. *Let $\{(\rho_k^o, \rho_k^i)\}_{k \in \mathbb{N}}$ and $\{(\theta_k^o, \theta_k^i)\}_{k \in \mathbb{N}}$ be as in Propositions 2.6 and 2.7, respectively. Let*

$$\begin{aligned} u_{m,0}^a(t) &\equiv 0 & \forall t \in \mathbb{R}^2 \setminus \omega, \\ u_{m,1}^a(t) &\equiv -w^-[\partial\omega, \theta_1^i](t) & \forall t \in \mathbb{R}^2 \setminus \omega, \\ u_{m,k}^a(t) &\equiv \frac{1}{k!} \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j \sum_{h=0}^j \binom{j}{h} t_1^h t_2^{j-h} \int_{\partial\Omega} \theta_{k-j}^o \nu_\Omega \cdot (\nabla \partial_1^h \partial_2^{j-h} S) d\sigma \\ &\quad - \frac{1}{k!} w^-[\partial\omega, \theta_k^i](t) & \forall t \in \mathbb{R}^2 \setminus \omega, \quad \forall k \geq 2 \end{aligned}$$

and

$$\begin{aligned} v_{m,k}(t) &\equiv \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^j \sum_{h=0}^j \binom{j}{h} t_1^h t_2^{j-h} \int_{\partial\Omega} \rho_{k-j}^o \partial_1^h \partial_2^{j-h} S d\sigma + \frac{1}{k!} v^-[\partial\omega, \rho_k^i](t) \\ & \quad \forall t \in \mathbb{R}^2 \setminus \omega, \end{aligned}$$

$$\begin{aligned} g_k^a &\equiv \frac{1}{k!} \sum_{l=0}^k \sum_{h=0}^l \binom{k}{l} \binom{l}{h} \int_{\partial\omega} s_1^h s_2^{l-h} (\partial_1^h \partial_2^{l-h} u^a)(0) \rho_{k-l}^i(s) d\sigma_s, \\ r_k &\equiv \frac{1}{k! \int_{\partial\omega} d\sigma} \sum_{j=0}^k \binom{k}{j} (-1)^j \sum_{h=0}^j \binom{j}{h} \int_{\partial\omega} s_1^h s_2^{j-h} d\sigma_s \int_{\partial\Omega} \rho_{k-j}^o \partial_1^h \partial_2^{j-h} S d\sigma \\ &\quad + \frac{1}{k! \int_{\partial\omega} d\sigma} \int_{\partial\omega} v[\partial\omega, \rho_k^i] d\sigma, \end{aligned}$$

for all $k \in \mathbb{N}$. Then the following statements hold.

- (i) *There exists $\varepsilon^* \in]0, \varepsilon_0]$ such that the series $\sum_{k=0}^\infty g_k^a \varepsilon^k$ and $\sum_{k=0}^\infty r_k \varepsilon^k$ converge absolutely for all $\varepsilon \in]-\varepsilon^*, \varepsilon^*[$. Moreover,*

$$g_0^a = u^a(0).$$

- (ii) *If $\Omega_m \subseteq \mathbb{R}^2 \setminus \bar{\omega}$ is open and bounded, then there exists $\varepsilon_m \in]0, \varepsilon^*] \cap]0, 1[$ such that $\varepsilon \bar{\Omega}_m \subseteq \Omega$ for all $\varepsilon \in]-\varepsilon_m, \varepsilon_m[$ and such that*

$$u_\varepsilon^a(\varepsilon \cdot)_{|\bar{\Omega}_m} = \sum_{k=1}^\infty u_{m,k}^a|_{\bar{\Omega}_m} \varepsilon^k + \left(\sum_{k=0}^\infty g_k^a \varepsilon^k \right) \frac{\sum_{k=0}^\infty v_{m,k}|_{\bar{\Omega}_m} \varepsilon^k + (2\pi)^{-1} \log |\varepsilon|}{\sum_{k=0}^\infty r_k \varepsilon^k + (2\pi)^{-1} \log |\varepsilon|} \quad (17)$$

for all $\varepsilon \in]-\varepsilon_m, \varepsilon_m[\setminus \{0\}$. Moreover, the series $\sum_{k=1}^\infty u_{m,k}^a|_{\bar{\Omega}_m} \varepsilon^k$ and $\sum_{k=0}^\infty v_{m,k}|_{\bar{\Omega}_m} \varepsilon^k$ converge normally in $C^{1,\alpha}(\bar{\Omega}_m)$ for all $\varepsilon \in]-\varepsilon_m, \varepsilon_m[$.

We are now ready to obtain an expansion for the map in (16) by exploiting Proposition 2.8.

Proposition 2.9. *With the notation introduced in Proposition 2.8, let*

$$\begin{aligned}
u_{\#,k}^l(t) &\equiv \sum_{\substack{(h,j) \in \mathbb{N}^2 \\ h+j=k}} \frac{\partial_1^h \partial_2^j u^l(0)}{h!j!} t_1^h t_2^j & \forall t \in \mathbb{R}^2, \quad l = a, b, \\
\tilde{u}_k(t) &\equiv \sum_{l=0}^k \nu_\omega(t) \cdot \nabla u_{\mathbf{m},l|\partial\omega}^a(t) u_{\#,k-l}^b(t) & \forall t \in \partial\omega, \\
\tilde{v}_k(t) &\equiv \nu_\omega(t) \cdot \nabla v_{\mathbf{m},k|\partial\omega}(t) & \forall t \in \partial\omega, \\
\tilde{g}_k(t) &\equiv \sum_{l=0}^k g_l^a u_{\#,k-l}^b(t) & \forall t \in \partial\omega,
\end{aligned}$$

for all $k \in \mathbb{N}$. Then there exists $\tilde{\varepsilon} \in]0, \varepsilon^*] \cap]0, 1[$ such that

$$\nu_\omega(t) \cdot \nabla(u_\varepsilon^a(\varepsilon t)) u^b(\varepsilon t) = \sum_{k=1}^{\infty} \tilde{u}_k(t) \varepsilon^k + \left(\sum_{k=0}^{\infty} \tilde{g}_k(t) \varepsilon^k \right) \frac{\sum_{k=0}^{\infty} \tilde{v}_k(t) \varepsilon^k}{\sum_{k=0}^{\infty} r_k \varepsilon^k + (2\pi)^{-1} \log |\varepsilon|} \quad \forall t \in \partial\omega, \quad (18)$$

for all $\varepsilon \in]-\tilde{\varepsilon}, \tilde{\varepsilon}[\setminus \{0\}$. Moreover, the series $\sum_{k=0}^{\infty} \tilde{g}_k \varepsilon^k$, $\sum_{k=0}^{\infty} \tilde{u}_k \varepsilon^k$, and $\sum_{k=0}^{\infty} \tilde{v}_k \varepsilon^k$ converge normally in $C^{0,\alpha}(\partial\omega)$ for all $\varepsilon \in]-\tilde{\varepsilon}, \tilde{\varepsilon}[$.

Proof. Taking $\tilde{\varepsilon} \in]0, \varepsilon^*[$ small enough, we have that

$$\begin{aligned}
u^l(\varepsilon t) &= \sum_{(i,j) \in \mathbb{N}^2} \varepsilon^{i+j} \frac{\partial_1^i \partial_2^j u^l(0)}{i!j!} t_1^i t_2^j \\
&= \sum_{h=0}^{\infty} \left(\sum_{\substack{(i,j) \in \mathbb{N}^2 \\ i+j=h}} \frac{\partial_1^i \partial_2^j u^l(0)}{i!j!} t_1^i t_2^j \right) \varepsilon^h = \sum_{h=0}^{\infty} u_{\#,h}^l(t) \varepsilon^h \quad \forall t \in \partial\omega, \quad l = a, b,
\end{aligned}$$

for $\varepsilon \in]-\tilde{\varepsilon}, \tilde{\varepsilon}[$, and that the power series $\sum_{h=0}^{\infty} u_{\#,h}^l \varepsilon^h$ converges normally in $C^{0,\alpha}(\partial\omega)$ for all $\varepsilon \in]-\tilde{\varepsilon}, \tilde{\varepsilon}[$, for $l = a, b$. We observe that, possibly taking a smaller $\tilde{\varepsilon}$,

$$\left(\sum_{k=1}^{\infty} \nu_\omega \cdot \nabla u_{\mathbf{m},k|\partial\omega}^a \varepsilon^k \right) \left(\sum_{h=0}^{\infty} u_{\#,h}^b \varepsilon^h \right) = \sum_{k=0}^{\infty} \tilde{u}_k \varepsilon^k, \quad \left(\sum_{k=0}^{\infty} g_k^a \varepsilon^k \right) \left(\sum_{h=0}^{\infty} u_{\#,h}^b \varepsilon^h \right) = \sum_{k=0}^{\infty} \tilde{g}_k \varepsilon^k$$

where the series converge normally in $C^{0,\alpha}(\partial\omega)$ for all $\varepsilon \in]-\tilde{\varepsilon}, \tilde{\varepsilon}[$ and we have set

$$\tilde{u}_k \equiv \sum_{l=0}^k \nu_\omega \cdot \nabla u_{\mathbf{m},l|\partial\omega}^a u_{\#,k-l}^b, \quad \tilde{g}_k \equiv \sum_{l=0}^k g_l^a u_{\#,k-l}^b.$$

Then by Proposition 2.8 (see formula (17)), we deduce the validity of (18). $\square$

The next step is to represent $\nu_\omega(\cdot) \cdot \nabla(u_\varepsilon^a(\varepsilon \cdot))|_{\partial\omega} u^b(\varepsilon \cdot)|_{\partial\omega}$ as a convergent series of the type

$$\sum_{n=0}^{\infty} \varphi_\varepsilon(\cdot) \varepsilon^n.$$

By exploiting exactly the same argument of [16, Thm. 6.3], we can prove Theorem 2.10 below where we obtain a series expansion for the map

$$\varepsilon \mapsto \nu_\omega(\cdot) \cdot \nabla(u_\varepsilon^a(\cdot))|_{\partial\omega} u^b(\varepsilon \cdot)|_{\partial\omega}.$$

Theorem 2.10. *With the notation introduced in Proposition 2.8, let $\{\tilde{a}_n\}_{n \in \mathbb{N}}$ be the sequence of functions from $\partial\omega$ to $\mathbb{R}$ defined by*

$$\tilde{a}_n \equiv \sum_{k=0}^n \tilde{g}_{n-k} \tilde{v}_k \quad \forall n \in \mathbb{N}.$$

Let $\{\tilde{\lambda}_{(n,l)}\}_{(n,l) \in \mathbb{N}^2, l \leq n+1}$ be the family of functions from $\partial\omega$ to $\mathbb{R}$ defined by

$$\tilde{\lambda}_{(n,0)} \equiv \tilde{u}_n, \quad \tilde{\lambda}_{(n,1)} \equiv \tilde{a}_n,$$

for all $n \in \mathbb{N}$, and

$$\tilde{\lambda}_{(n,l)} \equiv (-1)^{l-1} \sum_{k=l-1}^n \tilde{a}_{n-k} \sum_{\beta \in (\mathbb{N} \setminus \{0\})^{l-1}, |\beta|=k} \prod_{h=1}^{l-1} r_{\beta_h}$$

for all $n, l \in \mathbb{N}$ with $2 \leq l \leq n+1$. Then there exists $\tilde{\varepsilon}' \in]0, \varepsilon_0] \cap]0, 1[$ such that

$$\nu_\omega(t) \cdot \nabla(u_\varepsilon^a(\varepsilon t)) u^b(\varepsilon t) = \sum_{n=0}^{\infty} \varepsilon^n \sum_{l=0}^{n+1} \frac{\tilde{\lambda}_{(n,l)}(t)}{(r_0 + (2\pi)^{-1} \log |\varepsilon|)^l} \quad \forall t \in \partial\omega, \quad (19)$$

for all $\varepsilon \in]-\tilde{\varepsilon}', \tilde{\varepsilon}'[\setminus \{0\}$. Moreover, the series

$$\sum_{n=0}^{\infty} \varepsilon^n \sum_{l=0}^{n+1} \frac{\tilde{\lambda}_{(n,l)} \eta^l}{(r_0 \eta + (2\pi)^{-1})^l}$$

converges normally in $C^{1,\alpha}(\partial\omega)$ for all $(\varepsilon, \eta) \in]-\tilde{\varepsilon}', \tilde{\varepsilon}'[\times]1/\log \tilde{\varepsilon}', -1/\log \tilde{\varepsilon}'[$.

Remark 2.11. *With the notation of Theorem 2.10, a straightforward computation shows that*

$$\begin{aligned} \tilde{\lambda}_{(0,0)} &= \tilde{u}_0 = 0, \\ \tilde{\lambda}_{(0,1)} &= \tilde{a}_0 = u^a(0) u^b(0) \frac{\partial}{\partial \nu_\omega} v^-[\partial\omega, \rho_0^i]. \end{aligned}$$

2.6 Series expansion of $\text{Cap}_\Omega(\varepsilon \bar{\omega}, u^a, u^b)$

We now wish to compute a series expansion of the (u^a, u^b) -capacity $\text{Cap}_\Omega(\varepsilon \bar{\omega}, u^a, u^b)$. Since $\text{Cap}_\Omega(\varepsilon \bar{\omega}, u^a, u^b)$ is represented as a combination of $\int_{\Omega_\varepsilon} \nabla u_\varepsilon^a \cdot \nabla u_\varepsilon^b dx$ and of $\int_{\varepsilon \omega} \nabla u^a \cdot \nabla u^b dx$, as a first step, we provide an expansion for $\int_{\varepsilon \omega} \nabla u^a \cdot \nabla u^b dx$ around $\varepsilon = 0$. The term $\int_{\varepsilon \omega} \nabla u^a \cdot \nabla u^b dx$ depends analytically on ε . As a consequence, it can be expanded in a power series and we compute such a power series in the following lemma.

Lemma 2.12. *Let $\{\xi_n\}_{n \in \mathbb{N}}$ be the sequence of real numbers defined by*

$$\xi_0 \equiv 0, \quad \xi_1 \equiv 0, \quad \xi_n \equiv \sum_{j=1}^2 \sum_{l=0}^{n-2} \int_{\omega} \partial_j u_{\#,l+1}^a(t) \partial_j u_{\#,n-l-1}^b(t) dt \quad \forall n \geq 2.$$

Then there exists $\varepsilon_{\xi} \in]0, \varepsilon_0]$ such that

$$\int_{\varepsilon\omega} \nabla u^a \cdot \nabla u^b dx = \sum_{n=2}^{\infty} \xi_n \varepsilon^n$$

for all $\varepsilon \in]-\varepsilon_{\xi}, \varepsilon_{\xi}[\setminus \{0\}$. Moreover,

$$\xi_2 = \nabla u^a(0) \cdot \nabla u^b(0) m_2(\omega),$$

and the series

$$\sum_{n=2}^{\infty} \xi_n \varepsilon^n$$

converges absolutely for all $\varepsilon \in]-\varepsilon_{\xi}, \varepsilon_{\xi}[$. (The symbol $m_2(\dots)$ denotes the two-dimensional Lebesgue measure of a set).

Proof. By the Theorem of change of variable in integrals, we have

$$\int_{\varepsilon\omega} \nabla u^a \cdot \nabla u^b dx = \varepsilon^2 \int_{\omega} (\nabla u^a)(\varepsilon t) \cdot (\nabla u^b)(\varepsilon t) dt,$$

for all $\varepsilon \in]-\varepsilon_0, \varepsilon_0[\setminus \{0\}$. Let $j \in \{1, 2\}$. Assumption (12) on the analyticity of u^a and u^b and analyticity results for the composition operator (see Böhme and Tomi [7, p. 10], Henry [20, p. 29], Valent [46, Thm. 5.2, p. 44]), imply that there exists $\varepsilon_{\xi} \in]0, \varepsilon_0]$ such that the maps from $] -\varepsilon_{\xi}, \varepsilon_{\xi}[$ to $C^{0,\alpha}(\overline{\omega})$ which take ε to $(\partial_j u^l)(\varepsilon \cdot)|_{\overline{\omega}}$, $l = a, b$ are real analytic. Also, possibly shrinking ε_{ξ} , one can verify that for $\varepsilon \in]-\varepsilon_{\xi}, \varepsilon_{\xi}[\setminus \{0\}$,

$$(\partial_j u^l)(\varepsilon t) = \sum_{h=0}^{\infty} \partial_j u_{\#,h+1}^l(t) \varepsilon^h \quad \forall t \in \overline{\omega}, \quad l = a, b,$$

where the series $\sum_{h=0}^{\infty} \partial_j u_{\#,h+1}^l|_{\overline{\omega}} \varepsilon^h$ converges normally in $C^{0,\alpha}(\overline{\omega})$ for all $\varepsilon \in]-\varepsilon_{\xi}, \varepsilon_{\xi}[$, for $l = a, b$. Accordingly,

$$(\partial_j u^a)(\varepsilon t) (\partial_j u^b)(\varepsilon t) = \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \partial_j u_{\#,l+1}^a(t) \partial_j u_{\#,n-l+1}^b(t) \right) \varepsilon^n \quad \forall t \in \overline{\omega}, \forall \varepsilon \in]-\varepsilon_{\xi}, \varepsilon_{\xi}[\setminus \{0\}.$$

Moreover, by the continuity of the linear operator from $C^{0,\alpha}(\overline{\omega})$ to $\mathbb{R}$ which takes a function h to its integral $\int_{\omega} h dt$, by summing on $j \in \{1, 2\}$, possibly taking a smaller ε_{ξ} , we have that

$$\int_{\omega} (\nabla u^a)(\varepsilon t) \cdot (\nabla u^b)(\varepsilon t) dt = \sum_{n=0}^{\infty} \left(\sum_{j=1}^2 \sum_{l=0}^n \int_{\omega} \partial_j u_{\#,l+1}^a(t) \partial_j u_{\#,n-l+1}^b(t) dt \right) \varepsilon^n, \quad (20)$$

for all $\varepsilon \in]-\varepsilon_\xi, \varepsilon_\xi[\setminus \{0\}$. Also,

$$\sum_{j=1}^2 \int_{\omega} \partial_j u_{\#,1}^a(t) \partial_j u_{\#,1}^b(t) dt = \nabla u^a(0) \cdot \nabla u^b(0) m_2(\omega).$$

Finally, by multiplying equation (20) by ε^2 , we deduce the validity of the lemma. $\square$

We are now ready to deduce the validity of our main result of this section on the asymptotic behavior of $\text{Cap}_{\Omega}(\varepsilon\bar{\omega}, u^a, u^b)$.

Theorem 2.13. *With the notation introduced in Proposition 2.8, Theorem 2.10 and Lemma 2.12, let $\{c_{(n,l)}\}_{(n,l) \in \mathbb{N}^2, l \leq n+1}$ be the family of real numbers defined by*

$$c_{(n,l)} \equiv - \int_{\partial\omega} \tilde{\lambda}_{(n,l)} d\sigma + \delta_{0,l} \xi_n,$$

for all $n, l \in \mathbb{N}$ with $l \leq n+1$ (where $\delta_{0,l} = 1$ if $l = 0$ and $\delta_{0,l} = 0$ if $l \neq 0$). Then there exists $\varepsilon_c \in]0, \varepsilon_0] \cap]0, 1[$ such that the series

$$\sum_{n=0}^{\infty} \varepsilon^n \sum_{l=0}^{n+1} \frac{c_{(n,l)} \eta^l}{(r_0 \eta + (2\pi)^{-1})^l}$$

converges absolutely for all $(\varepsilon, \eta) \in]-\varepsilon_c, \varepsilon_c[\times]1/\log \varepsilon_c, -1/\log \varepsilon_c[$ and that

$$\text{Cap}_{\Omega}(\varepsilon\bar{\omega}, u^a, u^b) = \sum_{n=0}^{\infty} \varepsilon^n \sum_{l=0}^{n+1} \frac{c_{(n,l)}}{(r_0 + (2\pi)^{-1} \log |\varepsilon|)^l}$$

for all $\varepsilon \in]-\varepsilon_c, \varepsilon_c[\setminus \{0\}$.

Proof. By integrating over $\partial\omega$ formula (19) and adding the coefficients of Lemma 2.12 and by Theorem 2.10, we immediately deduce the validity of the statement. $\square$

Remark 2.14. *With the notation of Theorem 2.13, we observe that Remark 2.11 and a straightforward computation based on Folland [18, Lem. 3.30] imply that*

$$c_{(0,0)} = 0, \quad c_{(0,1)} = -u^a(0)u^b(0).$$

Moreover, if we denote by H_0^o the unique solution in $C^{1,\alpha}(\bar{\Omega})$ of

$$\begin{cases} \Delta H_0^o = 0 & \text{in } \Omega, \\ H_0^o(x) = S(x) & \text{for all } x \in \partial\Omega, \end{cases}$$

and by H_0^i the unique solution in $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^2 \setminus \omega)$ of

$$\begin{cases} \Delta H_0^i = 0 & \text{in } \mathbb{R}^2 \setminus \bar{\omega}, \\ H_0^i(t) = S(t) & \text{for all } t \in \partial\omega, \\ \sup_{t \in \mathbb{R}^2 \setminus \omega} |H_0^i(t)| < +\infty, \end{cases}$$

then by [16, Prop. 7.3] we have

$$r_0 = \lim_{t \rightarrow \infty} H_0^i(t) - H_0^o(0).$$

Accordingly,

$$\begin{aligned} \text{Cap}_\Omega(\varepsilon \bar{\omega}, u^a, u^b) &= - \frac{u^a(0)u^b(0)}{\lim_{t \rightarrow \infty} H_0^i(t) - H_0^o(0) + (2\pi)^{-1} \log |\varepsilon|} \\ &+ \varepsilon \left(\sum_{n=1}^{\infty} \varepsilon^{n-1} \sum_{l=0}^{n+1} \frac{c_{(n,l)}}{(\lim_{t \rightarrow \infty} H_0^i(t) - H_0^o(0) + (2\pi)^{-1} \log |\varepsilon|)^l} \right) \end{aligned} \quad (21)$$

for all $\varepsilon \in]-\varepsilon_c, \varepsilon_c[\setminus \{0\}$.

2.7 Asymptotic behavior of $\text{Cap}_\Omega(\varepsilon \bar{\omega}, u^a, u^b)$ under vanishing assumption for u^a and u^b

The aim of this subsection is to discuss the behavior of $\text{Cap}_\Omega(\varepsilon \bar{\omega}, u^a, u^b)$ under the assumption that u^a and u^b vanish together with their derivatives up to a certain order. We work as in [1, §5.1] and we assume that there exist $\bar{k}^a, \bar{k}^b \in \mathbb{N} \setminus \{0\}$ such that

$$D^\gamma u^l(0) = 0 \quad \forall |\gamma| < \bar{k}^l, \quad D^{\beta^l} u^l(0) \neq 0 \quad \text{for some } \beta^l \in \mathbb{N}^2 \text{ with } |\beta^l| = \bar{k}^l, \quad l = a, b. \quad (22)$$

By condition (22) and Proposition 2.7 we have that

$$(\theta_k^o, \theta_k^i) = (0, 0) \quad \forall k < \bar{k}^a, \quad \theta_{\bar{k}^a}^o = 0, \quad (23)$$

and that $\theta_{\bar{k}^a}^i$ is the unique solution in $C^{1,\alpha}(\partial\omega)_0$ of

$$\frac{1}{2} \theta_{\bar{k}^a}^i(t) - W_\omega[\theta_{\bar{k}^a}^i](t) = \bar{k}^a! \left(u_{\#, \bar{k}^a}^a(t) - \int_{\partial\omega} u_{\#, \bar{k}^a}^a \rho_0^i d\sigma \right) \quad \forall t \in \partial\omega. \quad (24)$$

By (23), (24) and by Proposition 2.8, we deduce that

$$u_{m,k}^a = 0 \quad \forall k < \bar{k}^a, \quad u_{m, \bar{k}^a}^a = -\frac{1}{\bar{k}^a!} w^-[\partial\omega, \theta_{\bar{k}^a}^i]. \quad (25)$$

Classical potential theory implies that $u_{m, \bar{k}^a}^a$ is the unique solution in $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^2 \setminus \omega)$ of

$$\begin{cases} \Delta u_{m, \bar{k}^a}^a = 0 & \text{in } \mathbb{R}^2 \setminus \bar{\omega}, \\ u_{m, \bar{k}^a}^a(t) = u_{\#, \bar{k}^a}^a(t) - \int_{\partial\omega} u_{\#, \bar{k}^a}^a \rho_0^i d\sigma & \text{for all } t \in \partial\omega, \\ \sup_{t \in \mathbb{R}^2 \setminus \omega} |u_{m, \bar{k}^a}^a(t)| < +\infty. \end{cases}$$

Also, by assumption (22) and Proposition 2.8 we deduce that

$$g_k^a = 0 \quad \forall k < \bar{k}^a, \quad g_{\bar{k}^a}^a = \frac{1}{\bar{k}^a!} \sum_{h=0}^{\bar{k}^a} \binom{\bar{k}^a}{h} \int_{\partial\omega} s_1^h s_2^{\bar{k}^a-h} (\partial_1^h \partial_2^{\bar{k}^a-h} u^a)(0) \rho_0^i(s) d\sigma_s = \int_{\partial\omega} u_{\#, \bar{k}^a}^a \rho_0^i d\sigma. \quad (26)$$

Then by (22) and by Propostion 2.9 we verify that

$$u_{\#,k}^l = 0 \quad \forall k < \bar{k}^l, \quad l = a, b. \quad (27)$$

As a consequence, Proposition 2.9 and equations (25), (27) imply

$$\tilde{u}_k = 0 \quad \forall k < \bar{k}^a + \bar{k}^b, \quad \tilde{u}_{\bar{k}^a + \bar{k}^b} = \left(\frac{\partial u_{\#, \bar{k}^a}^a}{\partial \nu_\omega} \right) u_{\#, \bar{k}^b}^b|_{\partial\omega}. \quad (28)$$

Furthermore, by (26) and (27) we have

$$\tilde{g}_k = 0 \quad \forall k < \bar{k}^a + \bar{k}^b, \quad \tilde{g}_{\bar{k}^a + \bar{k}^b} = g_{\bar{k}^a}^a u_{\#, \bar{k}^b}^b|_{\partial\omega} = \left(\int_{\partial\omega} u_{\#, \bar{k}^a}^a \rho_0^i d\sigma \right) u_{\#, \bar{k}^b}^b|_{\partial\omega}. \quad (29)$$

To compute the coefficients of the expansion of the (u^a, u^b) -capacity $\text{Cap}_\Omega(\varepsilon\bar{\omega}, u^a, u^b)$, as an intermediate step, we consider the quantities $\tilde{a}_n, \tilde{\lambda}_{(n,l)}$ introduced in Theorem 2.10 for representing the behavior of $\nu_\omega(\cdot) \cdot \nabla(u_\varepsilon^a(\varepsilon \cdot))|_{\partial\omega} u^b(\varepsilon \cdot)$. A straightforward computation based on (28), (29) implies that

$$\tilde{a}_n = 0 \quad \forall n < \bar{k}^a + \bar{k}^b, \quad \tilde{a}_{\bar{k}^a + \bar{k}^b} = \tilde{g}_{\bar{k}^a + \bar{k}^b} \tilde{v}_0 = \tilde{v}_0 \left(\int_{\partial\omega} u_{\#, \bar{k}^a}^a \rho_0^i d\sigma \right) u_{\#, \bar{k}^b}^b|_{\partial\omega},$$

and accordingly

$$\tilde{\lambda}_{(n,0)} = 0 \quad \forall n < \bar{k}^a + \bar{k}^b, \quad \tilde{\lambda}_{\bar{k}^a + \bar{k}^b, 0} = \tilde{u}_{\bar{k}^a + \bar{k}^b} = \left(\frac{\partial u_{\#, \bar{k}^a}^a}{\partial \nu_\omega} \right) u_{\#, \bar{k}^b}^b|_{\partial\omega}, \quad (30)$$

$$\tilde{\lambda}_{(n,1)} = 0 \quad \forall n < \bar{k}^a + \bar{k}^b, \quad \tilde{\lambda}_{\bar{k}^a + \bar{k}^b, 1} = \tilde{a}_{\bar{k}^a + \bar{k}^b} = \tilde{v}_0 \left(\int_{\partial\omega} u_{\#, \bar{k}^a}^a \rho_0^i d\sigma \right) u_{\#, \bar{k}^b}^b|_{\partial\omega}, \quad (31)$$

and

$$\tilde{\lambda}_{(n,l)} = 0 \quad \forall (n,l) \text{ such that } n - l + 1 < \bar{k}^a + \bar{k}^b \text{ and that } 2 \leq l \leq n + 1. \quad (32)$$

In particular, $\tilde{\lambda}_{(n,l)} = 0$ for all (n,l) such that $n < \bar{k}^a + \bar{k}^b + 1$ and that $2 \leq l \leq n + 1$. Furthermore, by a simple computation we have that

$$\xi_n = 0 \quad \forall n < \bar{k}^a + \bar{k}^b, \quad \xi_{\bar{k}^a + \bar{k}^b} = \int_{\omega} \nabla u_{\#, \bar{k}^a}^a \cdot \nabla u_{\#, \bar{k}^b}^b dt.$$

Finally, by Theorem 2.13 and by integrating equalities (30)-(32), we obtain

$$\begin{aligned} c_{(n,0)} &= 0 \quad \forall n < \bar{k}^a + \bar{k}^b, \\ c_{\bar{k}^a + \bar{k}^b, 0} &= - \int_{\partial\omega} \tilde{u}_{\bar{k}^a + \bar{k}^b} d\sigma + \int_{\omega} \nabla u_{\#, \bar{k}^a}^a \cdot \nabla u_{\#, \bar{k}^b}^b dt \\ &= - \int_{\partial\omega} \left(\frac{\partial u_{\#, \bar{k}^a}^a}{\partial \nu_\omega} \right) u_{\#, \bar{k}^b}^b|_{\partial\omega} d\sigma + \int_{\omega} \nabla u_{\#, \bar{k}^a}^a \cdot \nabla u_{\#, \bar{k}^b}^b dt, \\ c_{(n,1)} &= 0 \quad \forall n < \bar{k}^a + \bar{k}^b, \quad c_{\bar{k}^a + \bar{k}^b, 1} = - \int_{\partial\omega} \tilde{a}_{\bar{k}^a + \bar{k}^b} d\sigma = - \int_{\partial\omega} u_{\#, \bar{k}^a}^a \rho_0^i d\sigma \int_{\partial\omega} \tilde{v}_0 u_{\#, \bar{k}^b}^b|_{\partial\omega} d\sigma, \end{aligned}$$

and

$$c_{(n,l)} = 0 \quad \forall (n,l) \text{ such that } n-l+1 < \bar{k}^a + \bar{k}^b \text{ and that } 2 \leq l \leq n+1.$$

In particular, $c_{(n,l)} = 0$ for all (n,l) such that $n < \bar{k}^a + \bar{k}^b + 1$ and that $2 \leq l \leq n+1$. We also note that, since $u_{m,\bar{k}^a}^a = -\frac{1}{\bar{k}^a} w^-[\partial\omega, \theta_{\bar{k}^a}^i]$, then $u_{m,\bar{k}^a}^a$ is harmonic at infinity (see (25)). As a consequence, by the decay properties of its radial derivative (see Folland [18, Prop. 2.75]) and by the Divergence Theorem one verifies that

$$\int_{\partial\omega} \frac{\partial u_{m,\bar{k}^a}^a}{\partial\nu_\omega} d\sigma = 0.$$

Then, for $l = a, b$, we denote by $u_{\bar{k}^l}^l$ the unique solution in $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^2 \setminus \omega)$ of

$$\begin{cases} \Delta u_{\bar{k}^l}^l = 0 & \text{in } \mathbb{R}^2 \setminus \bar{\omega}, \\ u_{\bar{k}^l}^l(t) = u_{\#,\bar{k}^l}^l(t) & \text{for all } t \in \partial\omega, \\ \sup_{t \in \mathbb{R}^2 \setminus \omega} |u_{\bar{k}^l}^l(t)| < +\infty. \end{cases} \quad (33)$$

Then clearly

$$u_{\bar{k}^a}^a = u_{m,\bar{k}^a}^a + \int_{\partial\omega} u_{\#,\bar{k}^a}^a \rho_0^i d\sigma,$$

and thus

$$- \int_{\partial\omega} \left(\frac{\partial u_{m,\bar{k}^a}^a}{\partial\nu_\omega} \right) u_{\#,\bar{k}^b}^b|_{\partial\omega} d\sigma = - \int_{\partial\omega} \left(\frac{\partial u_{\bar{k}^a}^a}{\partial\nu_\omega} \right) u_{\bar{k}^b}^b|_{\partial\omega} d\sigma.$$

On the other hand, the harmonicity at infinity of $u_{\bar{k}^a}^a, u_{\bar{k}^b}^b$ and the Divergence Theorem imply that

$$\int_{\mathbb{R}^2 \setminus \bar{\omega}} \nabla u_{\bar{k}^a}^a \cdot \nabla u_{\bar{k}^b}^b dt = - \int_{\partial\omega} \left(\frac{\partial u_{\bar{k}^a}^a}{\partial\nu_\omega} \right) u_{\bar{k}^b}^b|_{\partial\omega} d\sigma$$

(see Folland [18, p. 118]). Hence,

$$- \int_{\partial\omega} \left(\frac{\partial u_{m,\bar{k}^a}^a}{\partial\nu_\omega} \right) u_{\#,\bar{k}^b}^b|_{\partial\omega} d\sigma = \int_{\mathbb{R}^2 \setminus \bar{\omega}} \nabla u_{\bar{k}^a}^a \cdot \nabla u_{\bar{k}^b}^b dt.$$

We now consider

$$- \int_{\partial\omega} u_{\#,\bar{k}^a}^a \rho_0^i d\sigma \int_{\partial\omega} \tilde{v}_0 u_{\#,\bar{k}^b}^b|_{\partial\omega} d\sigma.$$

First, we note that

$$\tilde{v}_0 = \nu_\omega \cdot \nabla v_{m,0}|_{\partial\omega} = \nu_\omega \cdot \nabla v^-[\partial\omega, \rho_0^i]|_{\partial\omega}.$$

On the other hand, Proposition 2.6 and the jump formula for the normal derivative of the single layer potential imply that

$$\nu_\omega \cdot \nabla v^-[\partial\omega, \rho_0^i]|_{\partial\omega} = \frac{1}{2} \rho_0^i + W_\omega^*[\rho_0^i] = \frac{1}{2} \rho_0^i + \frac{1}{2} \rho_0^i = \rho_0^i.$$

Accordingly,

$$\int_{\partial\omega} \tilde{v}_0 u_{\#,\bar{k}^b}^b|_{\partial\omega} d\sigma = \int_{\partial\omega} u_{\#,\bar{k}^b}^b \rho_0^i d\sigma.$$

By [16, Proof of Lem. 7.2], we have

$$\int_{\partial\omega} u_{\#, \bar{k}}^l \rho_0^i d\sigma = \lim_{t \rightarrow \infty} u_{\bar{k}}^l(t) \quad l = a, b,$$

which implies

$$-\int_{\partial\omega} u_{\#, \bar{k}}^a \rho_0^i d\sigma \int_{\partial\omega} \tilde{v}_0 u_{\#, \bar{k}}^b d\sigma = -\left(\lim_{t \rightarrow \infty} u_{\bar{k}}^a(t) \right) \left(\lim_{t \rightarrow \infty} u_{\bar{k}}^b(t) \right).$$

Under assumption (22), by Remark 2.14 and formula (21), we can now deduce the validity of the following.

Theorem 2.15. *Let assumption (22) hold. Then*

$$\begin{aligned} \text{Cap}_\Omega(\varepsilon\bar{\omega}, u^a, u^b) = & \varepsilon^{\bar{k}^a + \bar{k}^b} \left(\int_{\mathbb{R}^2 \setminus \bar{\omega}} \nabla u_{\bar{k}}^a \cdot \nabla u_{\bar{k}}^b dt + \int_{\omega} \nabla u_{\#, \bar{k}}^a \cdot \nabla u_{\#, \bar{k}}^b dt \right. \\ & \left. - \frac{\left(\lim_{t \rightarrow \infty} u_{\bar{k}}^a(t) \right) \left(\lim_{t \rightarrow \infty} u_{\bar{k}}^b(t) \right)}{(\lim_{t \rightarrow \infty} H_0^i(t) - H_0^o(0) + (2\pi)^{-1} \log |\varepsilon|)} \right) \\ & + \sum_{n=\bar{k}^a + \bar{k}^b + 1}^{\infty} \varepsilon^n \sum_{l=0}^{n-(\bar{k}^a + \bar{k}^b) + 1} \frac{c_{(n,l)}}{(\lim_{t \rightarrow \infty} H_0^i(t) - H_0^o(0) + (2\pi)^{-1} \log |\varepsilon|)^l}, \end{aligned} \quad (34)$$

for all $\varepsilon \in]-\varepsilon_c, \varepsilon_c[\setminus \{0\}$.

Remark 2.16. *Therefore, by (34) we have*

$$\text{Cap}_\Omega(\varepsilon\bar{\omega}, u^a, u^b) = \varepsilon^{\bar{k}^a + \bar{k}^b} \left(\int_{\mathbb{R}^2 \setminus \bar{\omega}} \nabla u_{\bar{k}}^a \cdot \nabla u_{\bar{k}}^b dt + \int_{\omega} \nabla u_{\#, \bar{k}}^a \cdot \nabla u_{\#, \bar{k}}^b dt \right) + o(\varepsilon^{\bar{k}^a + \bar{k}^b}) \quad \text{as } \varepsilon \rightarrow 0.$$

Moreover, we note that the terms $\int_{\mathbb{R}^2 \setminus \bar{\omega}} \nabla u_{\bar{k}}^a \cdot \nabla u_{\bar{k}}^b dt$ and $\int_{\omega} \nabla u_{\#, \bar{k}}^a \cdot \nabla u_{\#, \bar{k}}^b dt$ depend both on the geometrical properties of the set ω and on the behavior at 0 of the functions u^a and u^b (but not on Ω).

3 Perturbation of eigenvalues

In order to find an approximation of the perturbed eigenvalues $\{\lambda_j^\varepsilon\}$, we use a slight modification of a lemma from G. Courtois [12], itself based on the work of Y. Colin de Verdière [13]. For completeness, we give its proof in Appendix B.

Proposition 3.1. *Let $(\mathcal{H}, \|\cdot\|)$ be a Hilbert space and q be a quadratic form, semi-bounded from below (not necessarily positive), with domain $\mathcal{D}$ dense in $\mathcal{H}$ and with discrete spectrum $\{\nu_i\}_{i \geq 1}$. Let $\{g_i\}_{i \geq 1}$ be an orthonormal basis of eigenvectors of q . Let N and m be positive integers, F an m -dimensional subspace of $\mathcal{D}$ and $\{\xi_i^F\}_{i=1}^m$ the eigenvalues of the restriction of q to F .*

Assume that there exist positive constants γ and δ such that

$$(H1) \quad 0 < \delta < \gamma/\sqrt{2};$$

(H2) for all $i \in \{1, \dots, m\}$, $|\nu_{N+i-1}| \leq \gamma$, $\nu_{N+m} \geq \gamma$ and, if $N \geq 2$, $\nu_{N-1} \leq -\gamma$;

(H3) $|q(\varphi, g)| \leq \delta \|\varphi\| \|g\|$ for all $g \in \mathcal{D}$ and $\varphi \in F$.

Then we have

(i) $|\nu_{N+i-1} - \xi_i^F| \leq \frac{4}{\gamma} \delta^2$ for all $i = 1, \dots, m$;

(ii) $\|\Pi_N - \mathbb{I}\|_{\mathcal{L}(F, \mathcal{H})} \leq \sqrt{2} \delta / \gamma$, where Π_N is the projection onto the subspace of $\mathcal{D}$ spanned by $\{g_N, \dots, g_{N+m-1}\}$.

Our subsequent analysis is close to [12, Proof of Theorem 1.2], except that we replace the standard capacity with our generalized u -capacity (or (u, v) -capacity). Before proceeding, we recall the following crucial result from [2].

Lemma 3.2. ([2, Lemma A.1]) For any $f \in H_0^1(\Omega)$ we have

$$\int_{\Omega} |V_{K_\varepsilon, f}|^2 dx = o(\text{Cap}_\Omega(K_\varepsilon, f)) \quad \text{as } \varepsilon \rightarrow 0.$$

We also establish a series of preparatory lemmas. For $\varepsilon > 0$, we denote by Π_ε the linear mapping

$$\begin{aligned} \Pi_\varepsilon : H_0^1(\Omega) &\rightarrow H_0^1(\Omega \setminus K_\varepsilon) \\ u &\mapsto u - V_{K_\varepsilon, u}. \end{aligned}$$

Note that Π_ε is the q_Ω -orthogonal projection described in Remark 1.6. We recall the quantity χ_ε , defined in the statement of Theorem 1.9:

$$\chi_\varepsilon^2 \equiv \sup\{\text{Cap}_\Omega(K_\varepsilon, u) : u \in E(\lambda_N) \text{ and } \|u\| = 1\} \quad (35)$$

To simplify notation, we denote by V_u^ε the potential $V_{K_\varepsilon, u}$ for all $u \in H_0^1(\Omega)$.

Lemma 3.3. As $\varepsilon \rightarrow 0$, $\chi_\varepsilon \rightarrow 0$.

Proof. Let us pick $u \in E(\lambda_N)$ such that $\|u\| = 1$. We write $u = \sum_{i=1}^m c_i u_{N+i-1}$. Then

$$\begin{aligned} \text{Cap}_\Omega(K_\varepsilon, u) &= \left| \sum_{1 \leq i, j \leq m} c_i c_j \int_{\Omega} \nabla V_{u_{N+i-1}}^\varepsilon \cdot \nabla V_{u_{N+j-1}}^\varepsilon dx \right| \\ &\leq \sum_{1 \leq i, j \leq m} |c_i| |c_j| \left(\int_{\Omega} |\nabla V_{u_{N+i-1}}^\varepsilon|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla V_{u_{N+j-1}}^\varepsilon|^2 dx \right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^m |c_i| \left(\int_{\Omega} |\nabla V_{u_{N+i-1}}^\varepsilon|^2 dx \right)^{\frac{1}{2}} \right)^2 \\ &\leq m \left(\max_{1 \leq i \leq m} \int_{\Omega} |\nabla V_{u_{N+i-1}}^\varepsilon|^2 dx \right) \sum_{i=1}^m c_i^2 = m \max_{1 \leq i \leq m} \text{Cap}_\Omega(K_\varepsilon, u_{N+i-1}). \end{aligned}$$

Since $\text{Cap}_\Omega(K_\varepsilon, u_{N+i-1}) \rightarrow 0$ for all $1 \leq i \leq m$, the result follows. $\square$

Lemma 3.4. As $\varepsilon \rightarrow 0$, $M_\varepsilon \equiv \|\mathbb{I} - \Pi_\varepsilon\|_{\mathcal{L}(E(\lambda_N), L^2(\Omega))} = o(\chi_\varepsilon)$.

Proof. Let $u \in E(\lambda_N)$ such that $\|u\| = 1$. By definition, we have $(\mathbb{I} - \Pi_\varepsilon)u = V_u^\varepsilon$. We find

$$\begin{aligned} \|V_u^\varepsilon\| &\leq \sum_{i=1}^m |c_i| \|V_{N+i-1}^\varepsilon\| \leq \left(\sum_{i=1}^m c_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^m \|V_{N+i-1}^\varepsilon\|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^m \text{Cap}_\Omega(K_\varepsilon, u_{N+i-1}) \frac{\|V_{N+i-1}^\varepsilon\|^2}{\text{Cap}_\Omega(K_\varepsilon, u_{N+i-1})} \right)^{\frac{1}{2}} \\ &\leq \sqrt{m} \chi_\varepsilon \max_{1 \leq i \leq m} \frac{\|V_{N+i-1}^\varepsilon\|}{\text{Cap}_\Omega(K_\varepsilon, u_{N+i-1})^{(1/2)}}. \end{aligned}$$

The last term is $o(\chi_\varepsilon)$ according to Lemma 3.2. $\square$

Lemma 3.4 implies in particular that $M_\varepsilon < 1$, so that the restriction of Π_ε to E_N is injective, for ε small enough. We will always suppose this to be the case in the rest of this section.

3.1 Application of the abstract lemma

We are going to apply Proposition 3.1 in the following way. For $\varepsilon > 0$ small enough, we introduce the following set of definitions (36)–(39):

$$\mathcal{H}_\varepsilon \equiv L^2(\Omega \setminus K_\varepsilon); \quad (36)$$

$$\mathcal{D}_\varepsilon \equiv H_0^1(\Omega \setminus K_\varepsilon); \quad (37)$$

$$q_\varepsilon(u) \equiv \int_{\Omega \setminus K_\varepsilon} |\nabla u|^2 dx - \lambda_N \int_{\Omega \setminus K_\varepsilon} u^2 dx \text{ for all } u \in \mathcal{D}_\varepsilon; \quad (38)$$

$$F_\varepsilon \equiv \Pi_\varepsilon(E(\lambda_N)). \quad (39)$$

By construction, the eigenvalues of q_ε are $\{\lambda_i^\varepsilon - \lambda_N\}_{i \geq 1}$. We use the notation $\nu_i^\varepsilon \equiv \lambda_i^\varepsilon - \lambda_N$. Since $\lambda_i^\varepsilon \rightarrow \lambda_i$ for all $i \in \mathbb{N}^*$ and since λ_N is of multiplicity m , the assumption (H2) is fulfilled for $\varepsilon > 0$ small enough if we take, for instance,

$$\gamma \equiv \frac{1}{2} \min\{\lambda_N - \lambda_{N-1}, \lambda_{N+m} - \lambda_{N+m-1}\}$$

when $N \geq 2$ and, when $N = 1$ (in which case $m = 1$),

$$\gamma \equiv \frac{1}{2} (\lambda_2 - \lambda_1).$$

It remains to check whether condition (H3) is satisfied. Let us choose $v \in F_\varepsilon$ and $w \in \mathcal{D}_\varepsilon$. There exists a unique $u \in E(\lambda_N)$ such that $v = \Pi_\varepsilon u$. We have

$$\begin{aligned} q_\varepsilon(v, w) &= q_\Omega(v, w) - \lambda_N \langle v, w \rangle \\ &= q_\Omega(u, w) - \lambda_N \langle u, w \rangle - q_\Omega(V_u^\varepsilon, w) + \lambda_N \langle V_u^\varepsilon, w \rangle \\ &= \lambda_N \langle V_u^\varepsilon, w \rangle. \end{aligned}$$

We have used the facts that u is an eigenfunction of q_Ω and that V_u^ε is q_Ω -orthogonal to $\mathcal{D}_\varepsilon$. We obtain

$$|q_\varepsilon(v, w)| \leq \lambda_N \|V_u^\varepsilon\| \|w\| \leq \lambda_N M_\varepsilon \|u\| \|w\| \leq \lambda_N \frac{M_\varepsilon}{1 - M_\varepsilon} \|v\| \|w\|.$$

Lemma 3.4 then implies

$$|q_\varepsilon(v, w)| \leq o(\chi_\varepsilon) \|v\| \|w\|.$$

We can now apply Proposition 3.1, which tells us that for $1 \leq i \leq m$,

$$\lambda_{N+i-1}^\varepsilon = \lambda_N + \xi_i^\varepsilon + o(\chi_\varepsilon^2),$$

where $\{\xi_i^\varepsilon\}_{i=1}^m$ are the eigenvalues of the restriction of q_ε to F_ε .

3.2 Analysis of the restricted quadratic form. Proof of Theorem 1.9

In order to give a complete proof of Theorem 1.9, it remains to show that for $1 \leq i \leq m$,

$$|\mu_i^\varepsilon - \xi_i^\varepsilon| = o(\chi_\varepsilon^2).$$

Let us recall that $\{\mu_i^\varepsilon\}_{i=1}^m$ are the eigenvalues of the quadratic form r_ε defined on $E(\lambda_N)$ by

$$r_\varepsilon(u, v) \equiv q_\Omega(V_u^\varepsilon, V_v^\varepsilon) - \lambda_N \langle V_u^\varepsilon, V_v^\varepsilon \rangle.$$

Note that, from Lemma 3.2, $\mu_i^\varepsilon = O(\chi_\varepsilon^2)$ as $\varepsilon \rightarrow 0$.

Lemma 3.5. *For all $u, v \in E(\lambda_N)$,*

$$q_\varepsilon(\Pi_\varepsilon u, \Pi_\varepsilon v) = r_\varepsilon(u, v).$$

Proof. We have

$$q_\varepsilon(\Pi_\varepsilon u, \Pi_\varepsilon v) = q_\Omega(u - V_u^\varepsilon, v - V_v^\varepsilon) - \lambda_N \langle u - V_u^\varepsilon, v - V_v^\varepsilon \rangle.$$

Since V_u^ε is q_Ω -orthogonal to $\mathcal{D}_\varepsilon$,

$$q_\varepsilon(\Pi_\varepsilon u, \Pi_\varepsilon v) = q_\Omega(u, v - V_v^\varepsilon) - \lambda_N \langle u, v - V_v^\varepsilon \rangle + \lambda_N \langle V_u^\varepsilon, v - V_v^\varepsilon \rangle$$

and since u is an eigenfunction of q_Ω associated with λ_N ,

$$q_\varepsilon(\Pi_\varepsilon u, \Pi_\varepsilon v) = \lambda_N \langle V_u^\varepsilon, v \rangle - \lambda_N \langle V_u^\varepsilon, V_v^\varepsilon \rangle.$$

We finally use the fact that v is an eigenfunction associated with λ_N and that V_v^ε is the projection of v on the q_Ω -orthogonal complement of $\mathcal{D}_\varepsilon$:

$$q_\varepsilon(\Pi_\varepsilon u, \Pi_\varepsilon v) = q_\Omega(V_u^\varepsilon, v) - \lambda_N \langle V_u^\varepsilon, V_v^\varepsilon \rangle = q_\Omega(V_u^\varepsilon, V_v^\varepsilon) - \lambda_N \langle V_u^\varepsilon, V_v^\varepsilon \rangle = r_\varepsilon(u, v). \quad \square$$

Let us now introduce the notation $v_i^\varepsilon \equiv \Pi_\varepsilon u_{N+i-1}$. Since $\{u_{N+i-1}\}_{1 \leq i \leq m}$ is an orthonormal basis of $E(\lambda_N)$ and $\Pi_\varepsilon : E(\lambda_N) \rightarrow F_\varepsilon$ is bijective, $\{v_i^\varepsilon\}_{i=1}^m$ is a basis of F_ε . According to Lemma 3.5, $q_\varepsilon(v_i^\varepsilon, v_j^\varepsilon) = r_\varepsilon(u_{N+i-1}, u_{N+j-1})$. Let us define the $m \times m$ matrix $A_\varepsilon \equiv [q_\varepsilon(v_i^\varepsilon, v_j^\varepsilon)]_{1 \leq i, j \leq m}$. We have just seen that A_ε is the matrix of the quadratic form r_ε in the orthonormal basis $\{u_{N+i-1}\}_{1 \leq i \leq m}$. Its eigenvalues are therefore $\{\mu_i^\varepsilon\}_{i=1}^m$.

On the other hand, A_ε is the matrix of the quadratic form q_ε , restricted to F_ε , in the basis $\{v_i^\varepsilon\}_{i=1}^m$. It follows from Lemma 3.2 that $v_i^\varepsilon \rightarrow u_{N+i-1}$ in $L^2(\Omega)$ as $\varepsilon \rightarrow 0$, and this means that the basis $\{v_i^\varepsilon\}_{i=1}^m$ is approximately orthonormal. More precisely, if we define the matrix $C_\varepsilon \equiv [\langle v_i^\varepsilon, v_j^\varepsilon \rangle]_{1 \leq i, j \leq m}$, we have that $C_\varepsilon = \mathbb{I} + E_\varepsilon$, with $\lim_{\varepsilon \rightarrow 0} E_\varepsilon = 0$. As detailed in Appendix C, this fact and the estimate $\mu_i^\varepsilon = O(\chi_\varepsilon^2)$ imply that

$$\xi_i^\varepsilon = \mu_i^\varepsilon + o(\chi_\varepsilon^2).$$

4 Connection with the order of vanishing

In this section, we again denote by

$$E(\lambda_N) = E_1 \oplus \cdots \oplus E_p$$

the order decomposition of the eigenspace $E(\lambda_N)$ (see Proposition 1.10), with

$$k_1 > \cdots > k_p \geq 0$$

the associated finite sequence of orders. Our goal is to prove Theorem 1.17. Let us note that our formulation of this theorem does not make any reference to a particular basis of $E(\lambda_N)$. Up to a change of basis, we can therefore assume, in the course of the proof, that the orthonormal basis $\{u_{N+i-1}\}_{i=1}^m$ has a form which is convenient for our computations. The final result will not depend on this choice of basis. More precisely, we can assume that the orthonormal basis $\{u_{N+i-1}\}_{i=1}^m$ agrees with the order decomposition and diagonalizes each of the quadratic forms $\mathcal{Q}_j$. Explicitly, this means that, for all $j \in \{1, \dots, p\}$,

$$E_j = \text{span}\{u_{N+m_1+\cdots+m_{j-1}}, \dots, u_{N+m_1+\cdots+m_{j-1}+m_j-1}\}$$

and, for all $1 \leq s < t \leq m_j$,

$$\mathcal{Q}_j(u_{N+m_1+\cdots+m_{j-1}+s-1}, u_{N+m_1+\cdots+m_{j-1}+t-1}) = 0.$$

It follows that, for all $1 \leq s \leq m_j$,

$$\mathcal{Q}_j(u_{N+m_1+\cdots+m_{j-1}+s-1}, u_{N+m_1+\cdots+m_{j-1}+s-1}) = \mu_{j,s}.$$

As a first step in the proof of Theorem 1.17, let us find the asymptotic behavior of those eigenvalues $\nu_i^\varepsilon \equiv \lambda_i^\varepsilon - \lambda_N$ that go most slowly to 0, that is to say the eigenvalues

$$\nu_{N+m_1+\cdots+m_{p-1}}^\varepsilon, \dots, \nu_{N+m-1}^\varepsilon.$$

According to Theorem 1.9, we have to find the m_p largest eigenvalues (as $\varepsilon \rightarrow 0$) of A_ε , the matrix of the quadratic form r_ε in the basis $\{u_{N+i-1}\}_{i=1}^m$. It follows from Lemma 3.2 and Corollary 1.15 that

$$A_\varepsilon = \begin{pmatrix} 0 & & 0 \\ & \mu_{p,1} \rho_{k_p}^\varepsilon & \\ 0 & & \ddots \\ & & & \mu_{p,m_p} \rho_{k_p}^\varepsilon \end{pmatrix} + o\left(\rho_{k_p}^\varepsilon\right),$$

with the functions $\{\rho_k^\varepsilon\}$ defined by Equation (9). Using the min-max characterization of eigenvalues, we conclude that, for $1 \leq i \leq m_p$,

$$\mu_{m_1+\cdots+m_{p-1}+i}^\varepsilon = \mu_{p,i} \rho_{k_p}^\varepsilon + o\left(\rho_{k_p}^\varepsilon\right)$$

and, for $1 \leq i \leq m_1 + \cdots + m_{p-1}$,

$$\mu_i^\varepsilon = o\left(\rho_{k_p}^\varepsilon\right).$$

Theorem 1.9, and the fact that χ_ε^2 and $\rho_{k_p}^\varepsilon$ are of the same order, tell us that the same estimates hold for the ν_i^ε 's: for $1 \leq i \leq m_p$,

$$\nu_{N-1+m_1+\dots+m_{p-1}+i}^\varepsilon = \mu_{p,i}\rho_{k_p}^\varepsilon + o\left(\rho_{k_p}^\varepsilon\right)$$

and, for $1 \leq i \leq m_1 + \dots + m_{p-1}$,

$$\nu_{N-1+i}^\varepsilon = o\left(\rho_{k_p}^\varepsilon\right).$$

The rest of the proof consists of a step-by-step procedure, in which we rescale the quadratic form q_ε and apply the same arguments in order to identify successive groups of eigenvalue converging to λ_N at the same rate. Let us sketch the next step. We set, for $u, v \in \mathcal{D}_\varepsilon$,

$$q_{p-1}^\varepsilon(u, v) \equiv \frac{1}{\rho_{k_p}^\varepsilon} q_\varepsilon(u, v),$$

and we define the subspace

$$F_{p-1}^\varepsilon = \Pi_\varepsilon(E_1 \oplus \dots \oplus E_{p-1}).$$

The eigenvalues of q_{p-1}^ε are $\left\{ \frac{1}{\rho_{k_p}^\varepsilon} \nu_i^\varepsilon \right\}_{i \geq 1}$. We know from the first step that, for $1 \leq i \leq m_p$,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\rho_{k_p}^\varepsilon} \nu_{N-1+m_1+\dots+m_{p-1}+i}^\varepsilon = \mu_{p,i} > 0.$$

It follows immediately that there exists $\gamma > 0$ such that, for $\varepsilon > 0$ small enough,

$$\left| \frac{1}{\rho_{k_p}^\varepsilon} \nu_{N-1+i}^\varepsilon \right| \leq \gamma \text{ for } 1 \leq i \leq m_1 + \dots + m_{p-1};$$

$$\frac{1}{\rho_{k_p}^\varepsilon} \nu_{N+m_1+\dots+m_{p-1}}^\varepsilon \geq 2\gamma;$$

and, in case $N \geq 2$,

$$\frac{1}{\rho_{k_p}^\varepsilon} \nu_{N-1}^\varepsilon \leq -2\gamma.$$

Repeating the arguments of Section 3.1, we can show that for all $v \in F_{p-1}^\varepsilon$ and $w \in \mathcal{D}_\varepsilon$,

$$|q_{p-1}^\varepsilon(v, w)| \leq o\left(\left(\frac{\rho_{k_{p-1}}^\varepsilon}{\rho_{k_p}^\varepsilon}\right)^{1/2}\right) \|v\| \|w\|.$$

Using the arguments in the proof of Theorem 1.9 and in the first step, we conclude that, for $1 \leq i \leq m_{p-1}$,

$$\frac{1}{\rho_{k_p}^\varepsilon} \nu_{N-1+m_1+\dots+m_{p-2}+i}^\varepsilon = \mu_{p-1,i} \frac{\rho_{k_{p-1}}^\varepsilon}{\rho_{k_p}^\varepsilon} + o\left(\frac{\rho_{k_{p-1}}^\varepsilon}{\rho_{k_p}^\varepsilon}\right)$$

and, for $1 \leq i \leq m_1 + \dots + m_{p-2}$,

$$\frac{1}{\rho_{k_p}^\varepsilon} \nu_{N-1+i}^\varepsilon = o\left(\frac{\rho_{k_{p-1}}^\varepsilon}{\rho_{k_p}^\varepsilon}\right).$$

This gives us finally, for $1 \leq i \leq m_{p-1}$,

$$\nu_{N-1+m_1+\dots+m_{p-2}+i}^\varepsilon = \mu_{p-1,i} \rho_{k_{p-1}}^\varepsilon + o\left(\rho_{k_{p-1}}^\varepsilon\right)$$

and, for $1 \leq i \leq m_1 + \dots + m_{p-2}$,

$$\nu_{N-1+i}^\varepsilon = o\left(\rho_{k_{p-1}}^\varepsilon\right).$$

Carrying on the procedure for j decreasing from $p-2$ to 1, we obtain Theorem 1.17.

5 Same vanishing order: the case of elliptic holes

In this section we provide further applications of the results established above. Let us consider the dimension to be $d = 2$ and the multiplicity of λ_N to be $m = 2$. We recall that Ω and ω are open bounded connected subsets of $\mathbb{R}^2$ satisfying assumption (11). Moreover,

$$\Omega_\varepsilon \equiv \Omega \setminus K_\varepsilon \quad \text{where} \quad K_\varepsilon \equiv \varepsilon \bar{\omega} \quad \forall \varepsilon \in]-\varepsilon_0, \varepsilon_0[.$$

Moreover, in this subsection we assume that

$$\text{all non-zero functions in } E(\lambda_N) \text{ have the same order of vanishing } k. \quad (40)$$

We will compute explicitly the 2×2 matrix of Corollary 1.18 in the case of elliptic holes and then analyze the special case of a circular hole to prove Corollary 1.19. To consider the case of elliptic holes, we refer to [1, Section 9]. Let $a > b > 0$. We consider the ellipse $\mathcal{E}_0$ defined as

$$\mathcal{E}_0(a, b) \equiv \left\{ (x, y) \in \mathbb{R}^2, \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{b^2 + c^2} + \frac{x_2^2}{b^2} < 1 \right\},$$

where c is the distance between the two foci, which satisfies $c^2 = a^2 - b^2$.

For the sake of simplicity and without loss of generality, we perform a change of variables by rotating the domain, in such a way that in the new domain, the major axis of the small elliptic hole is lying along the x_1 -axis, so that

$$\omega = \mathcal{E}_0(a, b).$$

Let us now choose $\{u_N, u_{N+1}\}$ an orthonormal basis of $E(\lambda_N)$. According to assumption (40), we have, for $l \in \{N, N+1\}$,

$$r^{-k} u_l(r \cos t, r \sin t) \rightarrow \beta_l \sin(kt + k\varphi_l) \quad \text{as } r \rightarrow 0, \quad (41)$$

uniformly in t for derivatives of all orders, where $\beta_l \in \mathbb{R} \setminus \{0\}$ and $\varphi_l \in]-\pi/2k, \pi/2k[$.

We apply Corollary 1.18. According to [1, Equation (9.12)], the diagonal entries of the matrix M representing $\mathcal{Q}_1$ in the basis $\{u_N, u_{N+1}\}$ are

$$M_{1,1} = \frac{-\pi \beta_{N+1}^2 c^{2k}}{2} C_k \cos(2k\varphi_{N+1}) + \pi \beta_{N+1}^2 Q_k(a, b) \quad (42)$$

$$M_{2,2} = \frac{-\pi \beta_N^2 c^{2k}}{2} C_k \cos(2k\varphi_N) + \pi \beta_N^2 Q_k(a, b), \quad (43)$$

C_k and $Q_k(a, b)$ being positive constants depending only on k and k, a, b , respectively (see [1, Section 9]). We recall that here u_N and u_{N+1} are not assumed to diagonalize $\mathcal{Q}_1$, so we need a similar formula for the mixed term, too. Following all steps in [1, Subsection 9.1], we can compute the first contribution. We introduce the function $F : (\xi, \eta) \mapsto (x_1, x_2)$ which changes the variables into elliptic coordinates by

$$\begin{cases} x_1 = c \cosh(\xi) \cos(\eta), \\ x_2 = c \sinh(\xi) \sin(\eta), \end{cases} \quad \xi \in [0, +\infty[, \eta \in [0, 2\pi[.$$

F is a C^∞ diffeomorphism from $D := [0, +\infty[\times [0, 2\pi[$ onto $\mathbb{R}^2$. It is actually a conformal map, as noted in [2, Subsection 3.2]. For any $l = N, N+1$ let $W_k^l = u_k^l \circ F$ and $a_{j,l}, b_{j,l}$ its Fourier coefficient defined as

$$\begin{aligned} a_{j,l}(\xi) &= \frac{1}{\pi} \int_0^{2\pi} W_k^l(\xi, \eta) \cos(j\eta) d\eta \quad \text{for } j \in \mathbb{N}, \\ b_{j,l}(\xi) &= \frac{1}{\pi} \int_0^{2\pi} W_k^l(\xi, \eta) \sin(j\eta) d\eta \quad \text{for } j \in \mathbb{N} \setminus \{0\}. \end{aligned}$$

Following the same computations as in [1, Subsection 9.1], we compute

$$\begin{aligned} \int_{\mathbb{R}^2 \setminus \bar{\omega}} \nabla u_k^{N+1} \cdot \nabla u_k^N dt &= \pi \sum_{j \geq 1} j (a_{j,N}(\bar{\xi}) a_{j,N+1}(\bar{\xi}) + b_{j,N}(\bar{\xi}) b_{j,N+1}(\bar{\xi})) \\ &= \sum_{\substack{1 \leq j \leq k \\ k+j \text{ even}}} \frac{\pi \beta_N \beta_{N+1} c^{2k}}{4^{k-1}} j \left(\frac{k}{\frac{k+j}{2}} \right)^2 (\sin(k\varphi_N) \sin(k\varphi_{N+1}) \cosh^2 j\bar{\xi} \\ &\quad + \cos(k\varphi_N) \cos(k\varphi_{N+1}) \sinh^2 j\bar{\xi}) \\ &= \sum_{\substack{1 \leq j \leq k \\ k+j \text{ even}}} \frac{\pi \beta_N \beta_{N+1} c^{2k}}{4^{k-1}} j \left(\frac{k}{\frac{k+j}{2}} \right)^2 (\sin(k\varphi_N) \sin(k\varphi_{N+1}) \\ &\quad + \sinh^2 j\bar{\xi} (\cos(k\varphi_N - k\varphi_{N+1}))) \\ &= \sum_{\substack{1 \leq j \leq k \\ k+j \text{ even}}} \frac{\pi \beta_N \beta_{N+1} c^{2k}}{4^{k-1}} j \left(\frac{k}{\frac{k+j}{2}} \right)^2 \left(\frac{1}{2} \cosh 2j\bar{\xi} \cos(k\varphi_N - k\varphi_{N+1}) \right. \\ &\quad \left. - \frac{1}{2} \cos(k\varphi_N + k\varphi_{N+1}) \right) \\ &= -\frac{\pi \beta^2 c^{2k}}{2} C_k \cos(k\varphi_N + k\varphi_N) + \pi \beta^2 c^{2k} D_k(\bar{\xi}) \cos(k\varphi_N - k\varphi_N) \end{aligned}$$

where the third equality follows from adding and subtracting the same quantity $\sin(k\varphi_N) \sin(k\varphi_{N+1}) \sinh^2 j\bar{\xi}$ and factorizing; the fourth equality follows from Werner's formula and identity

$$\sinh^2 j\bar{\xi} = \frac{1}{2} \cosh 2j\bar{\xi} - \frac{1}{2};$$

the last one follows from analogous steps as in [1, Subsection 9.1].

More easily, following [1, Subsection 9.2], we obtain

$$\int_{\omega} \nabla u_{\#,k}^{N+1} \cdot \nabla u_{\#,k}^N dt = \frac{\pi \beta_N \beta_{N+1} c^{2k}}{2} \left(\sum_{j=0}^k \binom{k}{j}^2 (k-2j) e^{2(k-2j)\bar{\xi}} \right) \cos(k\varphi_N - k\varphi_N)$$

in place of [1, Equation (9.10)].

Following [1, Subsection 9.3], we finally obtain

$$\begin{aligned} M_{1,2} &= \frac{-\pi \beta_N \beta_{N+1} c^{2k}}{2} C_k \cos(k\varphi_N + k\varphi_{N+1}) \\ &\quad + \pi \beta_N \beta_{N+1} Q_k(a, b) \cos(k\varphi_N - k\varphi_{N+1}). \end{aligned}$$

As already mentioned, the two eigenvalues of the symmetric matrix with entries $M_{1,1}$, $M_{1,2}$ and $M_{2,2}$ coincide if and only if $M_{1,1} = M_{2,2}$ and $M_{1,2} = 0$.

Taking advantage of the computations performed above, we are in position to prove Corollary 1.19.

Proof of Corollary 1.19. As mentioned in [1, Remark 9.3], we can recover the case of round holes letting $b \rightarrow a$. In this way we obtain

$$\begin{aligned} M_{1,1} &= 2\pi k a^{2k} \beta_{N+1}^2, & M_{2,2} &= 2\pi k a^{2k} \beta_N^2 \\ M_{1,2} &= 2\pi k a^{2k} \beta_N \beta_{N+1} \cos(k\varphi_N - k\varphi_{N+1}). \end{aligned}$$

If $\beta_N = \beta_{N+1}$, then $M_{1,1} = M_{2,2}$, but condition $M_{1,2} = 0$ implies

$$\varphi_N - \varphi_{N+1} = \pm \frac{\pi}{2k}.$$

□

We note that if $k = 1$, the condition in Corollary 1.19 means the two eigenfunctions do not have perpendicular nodal lines.

A Order decomposition of an eigenspace

Let us recall the setting. We consider $E \equiv E(\lambda_N)$, the eigenspace associated with eigenvalue $\lambda = \lambda_N(\Omega)$ of Problem (1) and we denote the multiplicity by m , i.e. $m = \dim(E)$. We study the behavior of eigenfunctions at a point $x_0 \in \Omega$ and we may assume $x_0 = 0$ without loss of generality. Our main goal is to establish the existence and uniqueness of the order decomposition (Proposition 1.10). Let us recall the corresponding result.

Proposition A.1. *There exists a decomposition of E into a sum of subspaces*

$$E = E_1 \oplus \cdots \oplus E_p,$$

orthogonal for the scalar product in $L^2(\Omega)$, and an associated finite decreasing sequence of integers

$$k_1 > \cdots > k_p \geq 0$$

such that, for all $1 \leq j \leq p$, a function in $E_j \setminus \{0\}$ has the order of vanishing k_j at 0. In addition, such a decomposition is unique.

Proof. We use the fact that all eigenfunctions are analytic, and thus any function in $E \setminus \{0\}$ has a finite order of vanishing. For any $k \in \mathbb{N}$, let us define the mapping $\Pi_k : E \rightarrow \mathbb{R}_k[X_1, \dots, X_d]$ that associates to a function its Taylor expansion at 0, truncated to order k (here $\mathbb{R}_k[X_1, \dots, X_d]$ is the space of polynomials in d variables of degree at most k). Explicitly, using the standard multi-index notation,

$$\Pi_k u = \sum_{\beta \in \mathbb{N}^d, |\beta| \leq k} \frac{1}{\beta!} D^\beta u(0) X^\beta.$$

The mapping Π_k is clearly linear, and we denote its kernel by N_k . We note that the sequence of subspaces $(N_k)_{k \geq 0}$ is non-increasing with respect to inclusion: $N_{k+1} \subseteq N_k$. We claim that $N_k = \{0\}$ for k large enough. Indeed, if it were not the case, the vector space

$$N_\infty \equiv \bigcap_{k \in \mathbb{N}} N_k$$

would not be trivial, while any function in N_∞ has an infinite order of vanishing, contradicting the remark at the beginning of the proof. It will be convenient in what follows to use the convention $N_{-1} \equiv E$.

We now define the finite sequence

$$k_1 > k_2 > \dots > k_p \geq 0$$

as the integers at which there is a jump in the non-increasing sequence $(\dim(N_k))_{k \geq -1}$. More explicitly, the sequence $(k_j)_{1 \leq j \leq p}$ consists of all the integers $k \geq 0$ for which $\dim(N_k) < \dim(N_{k-1})$, arranged in decreasing order. This means that

$$\{0\} = N_{k_1} \subsetneq N_{k_2} \subsetneq \dots \subsetneq N_{k_p} \subsetneq E$$

and moreover

- for all $1 \leq j \leq p-1$ and $k_{j+1} \leq k < k_j$, $N_k = N_{k_{j+1}}$;
- for all $k \geq k_1$, $N_k = \{0\}$.

Note that if E contains an eigenfunction which does not vanish at 0, $N_0 = \ker(\Pi_0) \neq E = N_{-1}$, and therefore $k_p = 0$. This motivates our convention $N_{-1} = E$. For convenience, we define the additional value $k_{p+1} \equiv -1$.

We now define, for $1 \leq j \leq p$,

$$E_j \equiv N_{k_{j+1}} \cap N_{k_j}^\perp,$$

so that we have the orthogonal decomposition

$$N_{k_{j+1}} = N_{k_j} \oplus E_j. \tag{44}$$

We obtain in this way the orthogonal decomposition

$$E = E_1 \oplus \dots \oplus E_p.$$

Let us now show that this decomposition satisfies the property: any $u \in E_j \setminus \{0\}$ has order of vanishing exactly k_j . We have $E_j \subseteq N_{k_{j+1}}$ and, by definition of the sequence $(k_j)_{1 \leq j \leq p+1}$,

$N_{k_{j+1}} = N_{k_j-1}$, so $E_j \subseteq N_{k_j-1}$. Therefore, any $u \in E_j$ has order of vanishing at least k_j . If the order is greater than k_j , u belongs to N_{k_j} , and by (44) this implies $u = 0$. This shows the required property. We have proved existence in the proposition.

To show uniqueness, let us consider a decomposition

$$E = E_1 \oplus \cdots \oplus E_p$$

and a sequence $(k_j)_{1 \leq j \leq p+1}$ satisfying the hypotheses of the theorem, with the additional term $k_{p+1} = -1$. Then, for all $1 \leq j \leq p+1$,

$$N_{k_j} = \bigoplus_{i=1}^{j-1} E_i.$$

Futhermore, we can check easily that $N_k = N_{k_j}$ for all $k_{j+1} \leq k < k_j$, with $1 \leq j \leq p$, and that $N_k = \{0\}$ for all $k \geq k_1$. The sequence $(N_k)_{k \geq -1}$ is thus determined by the decomposition, with the k_j 's corresponding to the jumps in dimension. The given decomposition therefore coincides with the one constructed in the first part of the proof. $\square$

Let us conclude by providing an upper bound for the dimension of the subspaces (Proposition 1.11).

Proposition A.2. *Let $E = E_1 \oplus \cdots \oplus E_p$, be the order decomposition of the previous Proposition. Then the dimension of E_j is at most the dimension of the space of spherical harmonics in d variables of degree k_j .*

Proof. For $1 \leq j \leq k$, we define the mapping $P_j : E_j \rightarrow \mathbb{R}_{k_j}[X_1, \dots, X_d]$, which associate to u its principal part, namely

$$P_j u \equiv u_{\#} = \sum_{\beta \in \mathbb{N}^d, |\beta|=k_j} \frac{1}{\beta!} D^{\beta} u(0) X^{\beta}.$$

According to the classical results on the local behavior of eigenfunctions, recalled in the introduction, $P_j u$ is a harmonic homogeneous polynomial in d variables of degree k_j . The mapping P_j is clearly linear. In order to prove that P_j is injective, let us consider $u, v \in E_j$ such that $P_j u = P_j v$. By linearity, $P_j(u - v) = 0$, which means $u - v \in N_{k_j}$. By (44), this implies that $u - v = 0$.

The dimension of E_j is therefore equal to that of its image by P_j , which is contained in the space of spherical harmonics in d variables of degree k_j . $\square$

B Lemma on small eigenvalues

We want to approximate the eigenvalues of a quadratic form which are close to 0. Let us recall the result.

Proposition B.1. *Let $(\mathcal{H}, \|\cdot\|)$ be a Hilbert space and q be a quadratic form, semi-bounded from below (not necessarily positive), with domain $\mathcal{D}$ dense in $\mathcal{H}$ and with discrete spectrum $\{\nu_i\}_{i \geq 1}$. Let $\{g_i\}_{i \geq 1}$ be an orthonormal basis of eigenvectors of q . Let N and m be positive integers, F an m -dimensional subspace of $\mathcal{D}$ and $\{\xi_i^F\}_{i=1}^m$ the eigenvalues of the restriction of q to F .*

Assume that there exist positive constants γ and δ such that

$$(H1) \quad 0 < \delta < \gamma/\sqrt{2};$$

(H2) for all $i \in \{1, \dots, m\}$, $|\nu_{N+i-1}| \leq \gamma$, $\nu_{N+m} \geq \gamma$ and, if $N \geq 2$, $\nu_{N-1} \leq -\gamma$;

(H3) $|q(\varphi, g)| \leq \delta \|\varphi\| \|g\|$ for all $g \in \mathcal{D}$ and $\varphi \in F$.

Then we have

(i) $|\nu_{N+i-1} - \xi_i^F| \leq \frac{4}{\gamma} \delta^2$ for all $i = 1, \dots, m$;

(ii) $\|\Pi_N - \mathbb{I}\|_{\mathcal{L}(F, \mathcal{H})} \leq \sqrt{2} \delta / \gamma$, where Π_N is the projection onto the subspace of $\mathcal{D}$ spanned by $\{g_N, \dots, g_{N+m-1}\}$.

Proof. Let us prove only the case $N \geq 2$. The case $N = 1$ is analogous but simpler. We write $E_N \equiv \text{span}\{g_N, \dots, g_{N+m-1}\}$. For all $\varphi \in F$, we set $g \equiv \Pi_N \varphi$ and $h \equiv \varphi - g$. Furthermore, we write $h = h_- + h_+$, with

$$h_- = \sum_{i=1}^{N-1} c_i g_i;$$

$$h_+ = \sum_{i=N+m}^{\infty} c_i g_i.$$

Note that in the case $N = 1$ there is no h_- in the decomposition of h ; the proof follows accordingly.

The vectors φ and h are orthogonal both for the scalar product in $\mathcal{H}$ and for the quadratic form q , so that

$$\|\varphi\|^2 = \|g\|^2 + \|h\|^2; \quad (45)$$

$$q(\varphi) = q(g) + q(h). \quad (46)$$

In addition, h_+ and h_- are orthogonal both for the scalar product and for q , so that

$$\|h\|^2 = \|h_-\|^2 + \|h_+\|^2; \quad (47)$$

$$q(h) = q(h_-) + q(h_+). \quad (48)$$

Note that

$$\|h_-\|^2 = \sum_{i=1}^{N-1} |c_i|^2;$$

$$\|h_+\|^2 = \sum_{i=N+m}^{\infty} |c_i|^2;$$

$$q(h_-) = \sum_{i=1}^{N-1} \nu_i |c_i|^2 \leq 0;$$

$$q(h_+) = \sum_{i=N+m}^{\infty} \nu_i |c_i|^2 \geq 0.$$

Hypothesis (H2) implies that

$$\gamma \|h_{\pm}\|^2 \leq |q(h_{\pm})|. \quad (49)$$

Recalling that $\varphi = h + g$, Assumption (H3) implies

$$|q(h_{\pm})| = |q(\varphi, h_{\pm})| \leq \delta \|\varphi\| \|h_{\pm}\| \leq \frac{1}{\sqrt{\gamma}} \delta \|\varphi\| \sqrt{|q(h_{\pm})|},$$

from which

$$|q(h_{\pm})| \leq \frac{1}{\gamma} \delta^2 \|\varphi\|^2.$$

Equation (49) thus implies

$$\|h_{\pm}\| \leq \frac{1}{\gamma} \delta \|\varphi\|,$$

and finally

$$\|h\| \leq \frac{\sqrt{2}}{\gamma} \delta \|\varphi\|.$$

We have proved (ii).

To prove (i), let us first remark that (H1), together with (ii), implies that $\Pi_N : F \rightarrow E_N$ is injective, and therefore bijective since $\dim(F) = m = \dim(E_N)$. We now assume that φ (and therefore g) is non-zero. Then, using Identities (45)-(48),

$$\begin{aligned} \left| \frac{|q(\varphi)|}{\|\varphi\|^2} - \frac{|q(g)|}{\|g\|^2} \right| &= \left| \frac{(q(g) + q(h))\|g\|^2 - q(g)(\|g\|^2 + \|h\|^2)}{\|\varphi\|^2 \|g\|^2} \right| = \left| \frac{q(h)}{\|\varphi\|^2} - \frac{q(g)}{\|g\|^2} \frac{\|h\|^2}{\|\varphi\|^2} \right| \\ &\leq \frac{|q(h)|}{\|\varphi\|^2} + \frac{|q(g)|}{\|g\|^2} \frac{\|h\|^2}{\|\varphi\|^2} \leq \frac{|q(h_-)| + |q(h_+)|}{\|\varphi\|^2} + \frac{|q(g)|}{\|g\|^2} \frac{\|h_- \|^2 + \|h_+ \|^2}{\|\varphi\|^2}. \end{aligned}$$

Hypothesis (H2) implies that $|q(g)|/\|g\|^2 \leq \gamma$. We finally find

$$\left| \frac{|q(\varphi)|}{\|\varphi\|^2} - \frac{|q(g)|}{\|g\|^2} \right| \leq \frac{4\delta^2}{\gamma}. \quad (50)$$

We recall the min-max characterization of the eigenvalues $\{\xi_i^F\}_{i=1}^m$:

$$\xi_i^F = \min_{W \in \mathcal{F}_i} \max_{\varphi \in W \setminus \{0\}} \frac{q(\varphi)}{\|\varphi\|^2},$$

where $\mathcal{F}_i$ is the set of i -dimensional subspaces of F . By construction of E_N , the eigenvalues of the restriction of q to E_N are $\{\nu_i\}_{i=N}^{N+m-1}$. They can also be computed by the min-max characterization:

$$\nu_{N+i-1} = \min_{V \in \mathcal{E}_i} \max_{g \in V \setminus \{0\}} \frac{q(g)}{\|g\|^2},$$

where $\mathcal{E}_i$ is the set of i -dimensional subspaces of E_N . If we combine these characterizations with Inequality (50) and use the fact that Π_N maps $\mathcal{F}_i$ to $\mathcal{E}_i$ bijectively, we obtain (i). $\square$

C Approximation of eigenvalues

Let us recall the situation we are considering. We are studying the restriction of the quadratic form q_{ε} to the m -dimensional subspace $F_{\varepsilon} \subseteq \mathcal{D}_{\varepsilon}$. We have found a basis $\{v_i^{\varepsilon}\}_{i=1}^m$ of F_{ε} such that

1. the matrix $A_\varepsilon \equiv [q_\varepsilon(v_i^\varepsilon, v_j^\varepsilon)]$ of q_ε in the basis $\{v_i^\varepsilon\}$ has eigenvalues $\{\mu_i^\varepsilon\}$, with $\mu_i^\varepsilon = O(\chi_\varepsilon^2)$;
2. the Gram matrix $C_\varepsilon \equiv [\langle v_i^\varepsilon, v_j^\varepsilon \rangle]$ is of the form $C_\varepsilon = \mathbb{I} + o(1)$.

We want to show that the eigenvalues $\{\xi_i^\varepsilon\}$ of $q|_{F^\varepsilon}$ satisfy

$$\xi_i^\varepsilon = \mu_i^\varepsilon + o(\chi_\varepsilon^2). \quad (51)$$

Let us denote by $\{w_i^\varepsilon\}_{i=1}^m$ the basis of F_ε obtained from $\{v_i^\varepsilon\}_{i=1}^m$ by the Gram-Schmidt orthogonalization procedure. It follows from the form of C_ε that the change-of-basis matrix P_ε satisfies $P_\varepsilon = \mathbb{I} + o(1)$ (this can easily be checked by writing down the expression of the w_i^ε 's in terms of the v_i^ε 's). Let us denote by B_ε the matrix of $q|_{F^\varepsilon}$ in the orthogonal basis $\{w_i^\varepsilon\}_{i=1}^m$. Then B_ε has eigenvalues $\{\xi_i^\varepsilon\}$. We have

$$B_\varepsilon = P_\varepsilon^T A_\varepsilon P_\varepsilon = (\mathbb{I} + o(1))^T A_\varepsilon (\mathbb{I} + o(1)) = A_\varepsilon + o(\chi_\varepsilon^2). \quad (52)$$

The expansions (51) follow directly from (52) and the min-max characterization of eigenvalues.

Acknowledgements

L. Abatangelo and P. Musolino are members of the “Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni” (GNAMPA) of the “Istituto Nazionale di Alta Matematica” (INdAM). They are partially supported by the GNAMPA-INdAM 2020 project “Analisi e ottimizzazione asintotica per autovalori in domini con piccoli buchi”.

C. Léna acknowledges the support of COST (European Cooperation in Science and Technology) through the COST Action CA18232 MAT-DYN-NET.

P. Musolino acknowledges the support of the Project BIRD191739/19 “Sensitivity analysis of partial differential equations in the mathematical theory of electromagnetism” of the University of Padova and of the grant “Challenges in Asymptotic and Shape Analysis - CASA” of the Ca’ Foscari University of Venice.

References

- [1] L. Abatangelo, V. Bonnaillie-Noël, C. Léna, and P. Musolino, Asymptotic behavior of u -capacities and singular perturbations for the Dirichlet-Laplacian, *ESAIM Control Optim. Calc. Var.* **27** (2021), suppl., Paper No. S25, 43 pp.
- [2] L. Abatangelo, V. Felli, L. Hillaire, and C. Léna, Spectral stability under removal of small capacity sets and applications to Aharonov-Bohm operators, *J. Spectr. Theory* **9** (2019), no. 2, 379–427.
- [3] H. Ammari, H. Kang, and H. Lee, *Layer potential techniques in spectral analysis*, Mathematical Surveys and Monographs **153**, American Mathematical Society, Providence, RI, 2009.
- [4] M. Berger, P. Gauduchon, and E. Mazet, *Le spectre d’une variété riemannienne*. Lecture notes in Mathematics, 194, Springer-Verlag, Berlin-New York, 1971.

- [5] J. Bertrand and B. Colbois, Capacité et inégalité de Faber-Krahn dans $\mathbb{R}^n$. (French) [Capacity and Faber-Krahn inequality in $\mathbb{R}^n$], *J. Funct. Anal.* **232** (2006), no. 1, 1–28.
- [6] G. Besson, Comportement asymptotique des valeurs propres du laplacien dans un domaine avec un trou. (French) [Asymptotic behavior of the eigenvalues of the Laplacian in a domain with a hole], *Bull. Soc. Math. France* **113** (1985), no. 2, 211–230.
- [7] R. Böhme and F. Tomi, Zur Struktur der Lösungsmenge des Plateauproblems, *Math. Z.* **133** (1973), 1–29.
- [8] O.P. Bruno and F. Reitich, Boundary-variation solution of eigenvalue problems for elliptic operators, *J. Fourier Anal. Appl.* **7** (2001), no. 2, 169–187.
- [9] I. Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics, 115. Academic Press, Inc., Orlando, FL, 1984.
- [10] I. Chavel and E.A. Feldman, Spectra of manifolds less a small domain, *Duke Math. J.* **56** (1988), no. 2, 399–414.
- [11] B. Colbois and G. Courtois, Convergence de variétés and convergence du spectre du Laplacien. (French) [Convergence of manifolds and convergence of the spectrum of the Laplacian], *Ann. Sci. École Norm. Sup. (4)* **24** (1991), no. 4, 507–518.
- [12] G. Courtois, Spectrum of manifolds with holes, *J. Funct. Anal.* **134** (1995), no. 1, 194–221.
- [13] Y. Colin de Verdière, Sur la multiplicité de la première valeur propre non nulle du laplacien, *Comment. Math. Helv.*, **61** (1986), no. 2: 254–270.
- [14] M. Dalla Riva, M. Lanza de Cristoforis, and P. Musolino, *Singularly Perturbed Boundary Value Problems. A Functional Analytic Approach*, Springer, Cham, 2021.
- [15] M. Dalla Riva and P. Musolino, Real analytic families of harmonic functions in a planar domain with a small hole, *J. Math. Anal. Appl.* **422** (2015), 37–55.
- [16] M. Dalla Riva, P. Musolino, and S.V. Rogosin, Series expansions for the solution of the Dirichlet problem in a planar domain with a small hole, *Asymptot. Anal.* **92** (2015), 339–361.
- [17] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985.
- [18] G.B. Folland, *Introduction to partial differential equations*, Princeton University Press, Princeton NJ, 1995.
- [19] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
- [20] D. Henry, *Topics in nonlinear analysis*, Trabalho de Matemática, 192, Brasília, 1982.
- [21] L. Hörmander, *Linear partial differential operators. 3rd rev. print.*, Grundlehren Math. Wiss., Springer-Verlag, Berlin, Heidelberg, New York, 1969.
- [22] T. Kato, *Perturbation theory for linear operators*. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.

- [23] R. Kress, *Linear integral equations*, Third edition. Applied Mathematical Sciences, **82**, Springer-Verlag, New York, 2014.
- [24] P.D. Lamberti and M. Lanza de Cristoforis, An analyticity result for the dependence of multiple eigenvalues and eigenspaces of the Laplace operator upon perturbation of the domain, *Glasg. Math. J.* **44** (2002), no. 1, 29–43.
- [25] P.D. Lamberti and M. Lanza de Cristoforis, A real analyticity result for symmetric functions of the eigenvalues of a domain dependent Dirichlet problem for the Laplace operator, *J. Nonlinear Convex Anal.* **5** (2004), no. 1, 19–42.
- [26] P.D. Lamberti and M. Lanza de Cristoforis, A real analyticity result for symmetric functions of the eigenvalues of a domain-dependent Neumann problem for the Laplace operator, *Mediterr. J. Math.* **4** (2007), no. 4, 435–449.
- [27] M. Lanza de Cristoforis, Asymptotic behaviour of the conformal representation of a Jordan domain with a small hole in Schauder spaces, *Comput. Methods Funct. Theory* **2** (2002), 1–27.
- [28] M. Lanza de Cristoforis, Asymptotic behaviour of the conformal representation of a Jordan domain with a small hole, and relative capacity, in *Complex analysis and dynamical systems*, vol. **364** of Contemp. Math., Amer. Math. Soc., Providence, RI, 2004, 155–167.
- [29] M. Lanza de Cristoforis, Asymptotic behaviour of the solutions of the Dirichlet problem for the Laplace operator in a domain with a small hole. A functional analytic approach, *Analysis (Munich)* **28** (2008), 63–93.
- [30] M. Lanza de Cristoforis, Simple Neumann eigenvalues for the Laplace operator in a domain with a small hole. A functional analytic approach, *Rev. Mat. Complut.* **25** (2012), no. 2, 369–412.
- [31] A. Laurain, S. Nazarov, and J. Sokolowski, Singular perturbations of curved boundaries in three dimensions. The spectrum of the Neumann Laplacian, *Z. Anal. Anwend.* **30** (2011), no. 2, 145–180.
- [32] V.G. Maz'ya, S.A. Nazarov, and B.A. Plamenevskiĭ, Asymptotic expansions of eigenvalues of boundary value problems for the Laplace operator in domains with small openings, (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), no. 2, 347–371. English translation: *Math. USSR-Izv.* **24** (1985), no. 2, 321–346.
- [33] V.G. Maz'ya, S.A. Nazarov, and B.A. Plamenevskij, *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. Vol. I*, volume 111 of *Operator Theory: Advances and Applications*, Birkhäuser Verlag, Basel, 2000.
- [34] M.T. Nguyen, A bifurcation of multiple eigenvalues and eigenfunctions for boundary value problems in a domain with a small hole, *J. Math. Sci. Univ. Tokyo* **1** (1994), no. 3, 567–587.
- [35] S.A. Nazarov and J. Sokolowski, Shape sensitivity analysis of eigenvalues revisited, *Control Cybernet.* **37** (2008), no. 4, 999–1012.
- [36] A.A. Novotny and J. Sokolowski, *Topological derivatives in shape optimization*. Interaction of Mechanics and Mathematics. Springer, Heidelberg, 2013.

- [37] S. Ozawa, Singular Hadamard's variation of domains and eigenvalues of the Laplacian, *Proc. Japan Acad. Ser. A Math. Sci.* **56** (1980), no. 7, 306–310.
- [38] S. Ozawa, Singular Hadamard's variation of domains and eigenvalues of the Laplacian. II, *Proc. Japan Acad. Ser. A Math. Sci.* **57** (1981), no. 5, 242–246.
- [39] S. Ozawa, Singular variation of domains and eigenvalues of the Laplacian, *Duke Math. J.* **48** (1981), no. 4, 767–778.
- [40] S. Ozawa, An asymptotic formula for the eigenvalues of the Laplacian in a domain with a small hole, *Proc. Japan Acad. Ser. A Math. Sci.* **58** (1982), no. 1, 5–8.
- [41] S. Ozawa, Potential theory and eigenvalues of the Laplacian, *Proc. Japan Acad. Ser. A Math. Sci.* **58** (1982), no. 4, 134–136.
- [42] J. Rauch and M. Taylor, Potential and scattering theory on wildly perturbed domains, *J. Funct. Anal.*, **18** (1975), 27–59.
- [43] A. Samarskiĭ, On the influence of constraints on the characteristic frequencies of closed volumes, (Russian) *Doklady Akad. Nauk SSSR (N.S.)* **63** (1948), 631–634.
- [44] J. Schauder, Potentialtheoretische Untersuchungen, *Math. Z.* **33** (1931), 602–640.
- [45] J. Schauder, Bemerkung zu meiner Arbeit “Potentialtheoretische Untersuchungen I (Anhang)”, *Math. Z.* **35** (1932), 536–538.
- [46] T. Valent, *Boundary value problems of finite elasticity. Local theorems on existence, uniqueness and analytic dependence on data*, Springer-Verlag, New York, 1988.
- [47] W. Wendland, Die Fredholmsche Alternative für Operatoren, die bezüglich eines bilinearen Funktional adjungiert sind, *Math. Z.* **101** (1967), 61–64.
- [48] W. Wendland, Bemerkungen über die Fredholmschen Sätze. *Methoden Verfahren Math. Phys.* **3**, B.I.-Hochschulschriften **722/722a** (1970), 141–176.