

Chapter VIII

Elementary fibrations and applications to Gauss-Manin

Introduction

This chapter develops a new approach to the study of direct images of connections (i.e., de Rham cohomology with coefficients, endowed with the Gauss-Manin connection) with respect to a smooth, not necessarily proper, morphism of smooth algebraic varieties in characteristic 0.

We present elementary and purely algebraic proofs of the *generic finiteness and base change* theorems, as well as of the fundamental *finiteness, regularity, monodromy and base change theorems for direct images of regular algebraic connections*.

Our proofs use neither resolution of singularities (beyond the classical case of embedded resolution of curves in surfaces), nor the theory of holonomy. In fact, in contrast to the now standard methods which consist of trying to extend all objects to a good compactification and studying ramification at infinity, our strategy relies upon a *dévissage* inspired by Artin's theory of elementary fibrations. This approach allows us to reduce problems to the simple case of an ordinary differential operator in one variable (but in a relative situation), which we handle directly.

We consider properties $\mathcal{P}$ of connections which are étale-local and strongly exact. Our main lemma of *dévissage*¹ asserts that *in order to establish that for any smooth morphism of smooth K -varieties $X \xrightarrow{f} S$, for any (quasi-coherent) connection $(\mathcal{E}, \nabla)$ on X satisfying $\mathcal{P}$, and for any $i \geq 0$, the restriction of $R_{\mathrm{DR}}^i f_*(\mathcal{E}, \nabla)$ to the Artin set $A(f) \subseteq S$ satisfies $\mathcal{P}$, it suffices to consider rational elementary*

¹The idea of this *dévissage* has already appeared in our work on G-functions [9]. It is interesting to note that Artin's technique of elementary fibrations was also used, independently, as a basic tool by A. de Jong in his now famous alteration theorem in [34], as we learned shortly after beginning this work.

fibrations f (in which case $A(f) = S$), and $i = 0, 1$. This essentially reduces the problem to the study of the kernel and cokernel of a connection in one variable.

25 Elementary fibrations and dévissage

25.1 Elementary fibrations

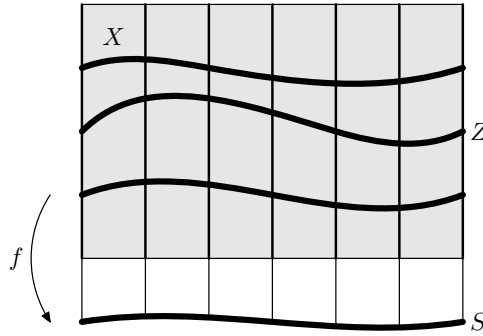
Let k be an algebraically closed field of characteristic 0. A k -variety will always be meant to be a reduced separated k -scheme of finite type, not necessarily irreducible.

Definition 25.1.1 (Elementary fibrations). A morphism $f : X \rightarrow S$ of smooth algebraic k -varieties is called an elementary fibration if it can be embedded in a commutative diagram

$$(25.1.2) \quad \begin{array}{ccccc} X & \xleftarrow{j} & \overline{X} & \xleftarrow{\quad} & Z \\ & \searrow f & \downarrow \bar{f} & \swarrow g & \\ & & S & & \end{array}$$

where

- (i) Z is a closed reduced subscheme of $\overline{X}$ and j is the complementary open immersion;
- (ii) $\bar{f}$ is projective and smooth with geometric fibers irreducible of dimension 1;
- (iii) g is an étale covering.



Lemma 25.1.3. An elementary fibration is a surjective affine morphism.

Proof. The question is local for the étale topology on S . Therefore, We may assume that Z is a disjoint union of divisors isomorphic to S via g . For each point $s \in S$, $Z_s = g^{-1}(s)$ is then a positive effective divisor on the projective curve $\overline{X}_s/\kappa(s)$,

hence it is ample [54] Chap. IV, Cor. 3.3]. By [47] 4.7.1], Z is then a relatively ample divisor in the relative projective curve $\bar{X}/S$. We conclude by [46] 5.5.7] that f is affine. $\square$

Definition 25.1.4 (Rational elementary fibrations). *An elementary fibration f as in [25.1.1] is rational if $\bar{X} = \mathbb{P}_S^1$, $\bar{f}$ is the projection on S and Z is a disjoint union of images of sections σ_i , $i = 1, \dots, r, \infty$, of $\bar{f}$, one of which is the section $\sigma_\infty : S \rightarrow Z_\infty = \{\infty\} \times S \subseteq \mathbb{P}_S^1$. We thus have the diagram*

$$(25.1.5) \quad \begin{array}{ccccc} X & \xrightarrow{j} & \mathbb{A}_S^1 & \longleftrightarrow & \coprod_{i=1}^r \sigma_i(S) \\ & \searrow f & \downarrow \text{pr}_S & & \swarrow g \\ & & S & & \end{array}$$

Remark 25.1.6. If $k = \mathbb{C}$, any rational elementary fibration is thus topologically locally trivial for the classical complex topology.

Definition 25.1.7 (Coordinatized elementary fibrations). *An elementary fibration f as in [25.1.1] is coordinatized if it can be embedded in a commutative diagram*

$$(25.1.8) \quad \begin{array}{ccccc} X & \xrightarrow{j} & \bar{X} & \longleftrightarrow & Z \\ \pi \downarrow & & \downarrow \bar{\pi} & & \downarrow \pi' \\ Y & \xrightarrow{j'} & \mathbb{P}_S^1 & \longleftrightarrow & Z' \\ & \searrow f' & \downarrow \text{pr}_S & & \swarrow g' \\ & & S & & \end{array}$$

where

- (i) $\bar{\pi}$ is finite and π is an étale covering;
- (ii) $f = f' \circ \pi$, $\bar{f} = \text{pr}_S \circ \bar{\pi}$, $g = g' \circ \pi'$;
- (iii) the lower part of the diagram is an elementary fibration with $j'(Y) \subseteq \mathbb{A}_S^1$.

Remark 25.1.9. We notice that the two squares in [25.1.8] are necessarily cartesian and that π' is an étale covering. A rational elementary fibration is *ipso facto* coordinatized. Conversely, given a coordinatized elementary fibration as in [25.1.8], there exists an étale covering $S' \rightarrow S$ such that the lower part of the pull-back of [25.1.8] over S' is a rational elementary fibration.

Remark 25.1.10. Let $f : X \rightarrow S$ be a smooth morphism of smooth k -varieties, of relative dimension d . Assume S is connected. It follows from M. Artin's theorem [2] Exp. XI, prop. 3.3] on “good neighborhoods” that there exist an étale dominant

morphism $\epsilon : S' \rightarrow S$ and a finite open (affine) cover $\{U_\alpha\}$ of $X' = X \times_S S'$, such that each $U_\alpha \rightarrow S'$ is a tower of elementary fibrations

$$(25.1.11) \quad U_\alpha = V_{\alpha,d} \longrightarrow V_{\alpha,d-1} \longrightarrow \cdots \longrightarrow V_{\alpha,0} = S'.$$

The integer $d \geq 0$ is called the *height* of the tower (a tower of height 0 is then an isomorphism $U_\alpha \xrightarrow{\cong} S'$). This result will play a fundamental role in the present chapter, in the following more precise form.

Proposition 25.1.12. *Let $f : X \rightarrow S$ be a smooth morphism of smooth k -varieties, with S connected. Then there exist an étale dominant morphism $\epsilon : S' \rightarrow S$ and a finite open (affine) cover $\{U_\alpha\}$ of $X' = X \times_S S'$ such that each $U_\alpha \rightarrow S'$ is a tower of coordinatized elementary fibrations.*

Proof. We may assume that f is of pure relative dimension $d \geq 0$. Standard thickening arguments allow us to replace S by its geometric generic point $\text{Spec } k(S)^{alg}$, and then replace the ground field k by $k(S)^{alg}$. The cases $d = 0, 1$ are then trivial.

By induction on d ($d \geq 2$), it suffices to find, for an arbitrary closed point $\xi \in X$, an open neighborhood U of ξ in X and a coordinatized elementary fibration $U \rightarrow T$.

Let us first recall, for convenience, Artin's construction of an elementary fibration $U \rightarrow T$, let alone the condition of being “coordinatized”. We may assume X affine and consider a normal projective closure $\bar{X}$. Let $\mathcal{L}$ be a very ample line bundle on $\bar{X}$ and let us embed $\bar{X}$ in a projective space $\mathbb{P}^N$ via $\mathcal{L}^{\otimes 2}$. Then a “general” linear space $L \subseteq \mathbb{P}^N$ of codimension $d-1$ passing through ξ cuts $\bar{X}$ and $\bar{X} \setminus X$ transversally and avoids the singular locus of $\bar{X}$. Let $H \subseteq \mathbb{P}^N$ be a “general” hyperplane (not passing through ξ and avoiding $L \cap (\bar{X} \setminus X)$), and let us consider the blow-up $\widetilde{\mathbb{P}^N}$ of $\mathbb{P}^N$ along $L \cap H$, and the morphism $p : \widetilde{\mathbb{P}^N} \rightarrow \mathbb{P}^{d-1}$ induced by the linear projection with center $L \cap H$. Let $\widetilde{\bar{X}}$ be the strict transform of $\bar{X}$ in $\widetilde{\mathbb{P}^N}$, and let $Z \subseteq \widetilde{\bar{X}}$ be the disjoint union of $\bar{X} \setminus X$ (considered as a subscheme of $\widetilde{\bar{X}}$ of codimension 1) and of the exceptional divisor $\simeq \mathbb{P}^{d-1} \times (X \cap L \cap H)$. Then $\widetilde{\bar{X}} \setminus Z$ identifies with an open subscheme of X containing ξ , and the pull-back of the diagram

$$(25.1.13) \quad \begin{array}{ccccc} \widetilde{\bar{X}} \setminus Z & \hookrightarrow & \widetilde{\bar{X}} & \hookleftarrow & Z \\ & \searrow & \downarrow p|_{\widetilde{\bar{X}}} & \swarrow & \\ & & \mathbb{P}^{d-1} & & \end{array}$$

over a suitable (affine) neighborhood T of $p(\xi)$ in $\mathbb{P}^{d-1}$ defines an elementary fibration. Moreover, the fiber of $p|_{\widetilde{\bar{X}}}$ above $p(\xi)$ identifies with $\bar{X} \cap L$. The proposition is then a consequence of the following lemma.

Lemma 25.1.14. *Given an elementary fibration as in (25.1.2)*

$$\begin{array}{ccccc} X & \xrightarrow{j} & \overline{X} & \xleftarrow{\quad} & Z \\ & \searrow f & \downarrow \overline{f} & \nearrow g & \\ & & S & & \end{array}$$

and a closed point $\xi \in X$, with image $\zeta = f(\xi)$, there is an open neighborhood U of ξ (resp. T of ζ), such that f induces a coordinatized elementary fibration $U \rightarrow T$.

Proof. We denote by $\{\xi_1, \dots, \xi_r\}$ the finite set $\{\xi\} \cup Z_\zeta$ of closed points of $\overline{X}_\zeta$. Let us choose a closed S -immersion $\overline{X} \rightarrow \mathbb{P}_S^M$. The fiber $\overline{X}_\zeta$ is a closed smooth curve in $\mathbb{P}^M$ to which the theory of Lefschetz pencils applies [3] exp. XVII]. There exists an open dense subset $\mathcal{U}$ of the Grassmannian of lines in the dual projective space $\mathbb{P}^M$ such that for any $D_\zeta \in \mathcal{U}$:

- (a) $_\zeta$ D_ζ is a Lefschetz pencil; in particular, the axis of D_ζ (which has codimension 2 in $\mathbb{P}^M$) does not cut $\overline{X}_\zeta$, so that D_ζ gives rise to a finite morphism

$$\overline{\pi}_\zeta : \overline{X}_\zeta \longrightarrow \mathbb{P}^1 \simeq D_\zeta;$$

- (b) $_\zeta$ if $i \neq j$, $\overline{\pi}_\zeta(\xi_i) \neq \overline{\pi}_\zeta(\xi_j)$;

- (c) $_\zeta$ for each i , $\overline{\pi}_\zeta$ is étale at each point of $\overline{\pi}_\zeta^{-1}\overline{\pi}_\zeta(\xi_i)$.

In order to justify (b) $_\zeta$ and (c) $_\zeta$, it suffices to show that the generic pencil satisfies these conditions; but this follows from the fact [3] exp. XVII, 3.2.8] that for any hyperplane $H_i \subseteq \mathbb{P}^M$ containing ξ_i , but not ξ_j for $j \neq i$, and cutting $\overline{X}_\zeta$ transversally, there exists a Lefschetz pencil of hyperplanes containing H_i .

Let us fix $D_\zeta \in \mathcal{U}$. Up to changing the coordinate in $\mathbb{P}^1 \simeq D_\zeta$, we may assume in addition that

- (d) $_\zeta$ the hyperplane H_∞ corresponding to $\infty \in D_\zeta$ cuts $\overline{X}_\zeta$ transversally, and does not meet the points ξ_i .

Let us now extend D_ζ to a relative line D/S in the dual projective space $\mathbb{P}_S^M$ (D is defined by an S -point in a suitable Grassmannian). After replacing S by a suitable Zariski neighborhood T of ζ , we may assume that the axis of D does not cut $\overline{X}$, so that D gives rise to a finite T -morphism

$$\overline{\pi} : \overline{X} \longrightarrow \mathbb{P}_T^1.$$

We may also assume that:

- (b) $\overline{\pi}(Z)$ is étale over T , and does not contain $\overline{\pi}(\xi)$;
- (c) $\overline{\pi}$ is étale at any point of $\overline{\pi}^{-1}\overline{\pi}(Z)$;

- (d) $\bar{\pi}^{-1}(\infty \times T)$ does not meet $\bar{\pi}^{-1}\bar{\pi}(Z)$ nor $\{\xi\}$, and the restriction of $\bar{\pi}$ to $\bar{\pi}^{-1}(\infty \times T)$ is étale.

According to [3] exp. XVII, 6.3], we may assume furthermore that:

- (a) for any closed point $\zeta' \in T$, $D_{\zeta'}$ is a Lefschetz pencil, and the branch locus $B \subseteq \mathbb{P}_T^1$ of $\bar{\pi}$ is étale over T .

Then there is at most one ramification point in each fiber of $\bar{\pi}^{-1}(B)$, so that $\bar{\pi}^{-1}(B)$ is étale over T , and does not meet $\bar{\pi}^{-1}\bar{\pi}(Z) \sqcup \bar{\pi}^{-1}(\infty \times T)$ nor $\{\xi\}$. It is now clear that

$$U = \bar{X} \setminus (\bar{\pi}^{-1}(B) \sqcup \bar{\pi}^{-1}\bar{\pi}(Z) \sqcup \bar{\pi}^{-1}(\infty \times T))$$

is a coordinatized elementary fibration (via $\bar{\pi}$), and $\xi \in U \subseteq X$.

This concludes the proof of lemma, hence of the proposition. $\square$

In the next chapter, we shall need the following slight generalization:

Lemma 25.1.15. *Given an elementary fibration (25.1.2)*

$$\begin{array}{ccccc} X & \xrightarrow{j} & \bar{X} & \longleftrightarrow & Z \\ & \searrow f & \downarrow \bar{f} & \swarrow g & \\ & & S & & \end{array}$$

and a finite set of closed points $F \subseteq X$, there is an open neighborhood U of F (resp. T of $f(F)$), such that f induces a coordinatized elementary fibration $U \rightarrow T$.

Proof. Let us choose a closed S -embedding $\bar{X} \rightarrow \mathbb{P}_S^M$. For each $\zeta \in f(F)$, we denote by $\{\xi_1^\zeta, \dots, \xi_r^\zeta\}$ the finite set $F \cup Z_\zeta$ of closed points of $\bar{X}_\zeta$. There exists an open dense subset $\mathcal{U}^\zeta$ of the Grassmannian of lines in the dual projective space $\mathbb{P}^M$ such that for any $D_\zeta \in \mathcal{U}^\zeta$:

- (a) $_\zeta$ D_ζ is a Lefschetz pencil; in particular, the axis of D_ζ (which has codimension 2 in $\mathbb{P}^M$) does not cut $\bar{X}_\zeta$, so that D_ζ gives rise to a finite morphism $\bar{\pi}_\zeta : \bar{X}_\zeta \rightarrow \mathbb{P}^1 \simeq D_\zeta$;
- (b) $_\zeta$ if $i \neq j$, then $\bar{\pi}_\zeta(\xi_i^\zeta) \neq \bar{\pi}_\zeta(\xi_j^\zeta)$;
- (c) $_\zeta$ for each i , $\bar{\pi}_\zeta$ is étale at each point of $\bar{\pi}_\zeta^{-1}\bar{\pi}_\zeta(\xi_i^\zeta)$.

There exists a relative line D/S in the dual projective space $\check{\mathbb{P}}_S^M$ such that $D_\zeta \in \mathcal{U}^\zeta$ for each ζ . Up to changing the coordinate in $\mathbb{P}^1 \simeq D_\zeta$, we may moreover assume that

- (d) $_\zeta$ the hyperplane H_∞ corresponding to $\infty \in D_\zeta$ cuts $\bar{X}_\zeta$ transversally, and does not meet the points ξ_i^ζ .

We then finish the proof as in Lemma 25.1.14. $\square$

Remark 25.1.16. In fact, Proposition 25.1.12 holds for a separably closed ground field k of any characteristic. The proof is the same, except that one must replace $k(S)^{\text{alg}}$ by the separable closure of $k(S)$, and that in Lemma 25.1.14 it may be necessary to replace the given embedding $\bar{X} \hookrightarrow \mathbb{P}_S^N$ by a Veronese multiple (in characteristic 2, one should also invoke [3, exp. XVII, 6.4], and replace $\mathcal{L}^{\otimes 2}$ by $\mathcal{L}^{\otimes 3}$ in (the proof of) 25.1.12).

25.2 Artin sets

Let $f : X \rightarrow S$ be a smooth morphism of k -varieties. We denote by d the maximum of the dimensions of the fibers. Let $\{U_\alpha\}_{\alpha=0,\dots,r}$ be a finite open cover of X . For $\underline{\alpha} = (\alpha_0, \dots, \alpha_p)$, $0 \leq \alpha_0 < \alpha_1 < \dots < \alpha_p \leq r$, $|\underline{\alpha}| = p+1$, we denote by $U_{\underline{\alpha}}$ the open subset $U_{\alpha_0} \cap \dots \cap U_{\alpha_p}$ of X , and by $f_{\underline{\alpha}}$ the restriction of f to $U_{\underline{\alpha}}$.

We introduce a decreasing chain of subsets of S , the Artin sets of f , which will be well-suited for the dévissage in Section 25.3 below.

Definition 25.2.1 (Artin sets). *The Artin set of f of level $i \geq 0$, denoted by $A_i(f)$, is the union of the images $\epsilon(S') \subseteq S$, for all étale morphisms $\epsilon : S' \rightarrow S$, finite over their image, such that:*

- (1) $X_{S'}$ admits a finite open cover $\{U_\alpha\}$ such that the restriction of $f_{S'}$ to each U_α is a tower of coordinatized elementary fibrations;
 - (2) each $U_{\underline{\alpha}}$, $|\underline{\alpha}| > 1$, admits a finite open cover $\{U_{\underline{\alpha},\beta}\}$ such that the restriction of $f_{S'}$ to each $U_{\underline{\alpha},\beta}$ is a tower of coordinatized elementary fibrations;
 - (3) each $U_{\underline{\alpha},\beta}$, $|\underline{\beta}| > 1$, admits a finite open cover $\{U_{\underline{\alpha},\beta,\gamma}\}$ such that the restriction of $f_{S'}$ to each $U_{\underline{\alpha},\beta,\gamma}$ is a tower of coordinatized elementary fibrations;
- ... and so on, for all $(\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \dots)$ such that

$$(|\underline{\alpha}| - 1) + (|\underline{\beta}| - 1) + (|\underline{\gamma}| - 1) + \dots \leq i.$$

Obviously, $A_i(f) \supseteq A_{i+1}(f)$ for all i , and, for any $U_{\underline{\alpha}}$ appearing in the definition, $A_{i-|\underline{\alpha}|+1}(f_{\underline{\alpha}}) \supseteq A_i(f)$.

We set

$$A(f) = A_{d+\dim X}(f)$$

(this definition is the most important and it will be justified later).

From Proposition 25.1.12 we deduce:

Corollary 25.2.2. *For any $i \geq 0$, $A_i(f)$ is a dense open subset of the image of the (open) morphism f in S .*

Remarks 25.2.3. (i) All Artin sets of a tower $X \rightarrow S$ of coordinatized elementary fibrations coincide with S itself.

(ii) If f is étale, they coincide with the maximal open subset of S on which f is an étale covering.

25.3 Dévissage

In the sequel, we consider properties of $\mathcal{O}_X$ -modules with integrable connection on arbitrary smooth k -varieties X . We write $\mathcal{P}((\mathcal{E}, \nabla))$ to indicate that $(\mathcal{E}, \nabla)$ satisfies property $\mathcal{P}$.

Definition 25.3.1. *We say that $\mathcal{P}$ is*

- (1) *local for the étale topology if for any X , any $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X)$ and any étale cover $\{V_i\}$ of X , one has*

$$\mathcal{P}((\mathcal{E}, \nabla)) \iff (\forall i, \mathcal{P}((\mathcal{E}, \nabla)|_{V_i}));$$

- (2) *stable under finite étale direct image if for any X , any $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X)$ and any finite étale $\pi : X \rightarrow X'$, one has*

$$\mathcal{P}((\mathcal{E}, \nabla)) \implies \mathcal{P}(\pi_*(\mathcal{E}, \nabla)).$$

Here we write simply $\pi_*(\mathcal{E}, \nabla)$ instead of $R_{\mathrm{DR}}^0 \pi_*(\mathcal{E}, \nabla)$, the underlying $\mathcal{O}_{X'}$ -module being $\pi_* \mathcal{E}$.

Definition 25.3.2. *We say in general that a property $\mathcal{P}$ of objects of an abelian category $\mathcal{A}$ is strongly exact if $\mathcal{P}(0)$ holds and for any exact sequence*

$$E_1 \longrightarrow E \longrightarrow E_2$$

in $\mathcal{A}$, one has

$$(\mathcal{P}(E_1) \text{ and } \mathcal{P}(E_2)) \implies \mathcal{P}(E).$$

Strong exactness is equivalent to the conjunction of exactness in the sense of [47] 3.1.1] and stability under taking subobjects.

Lemma 25.3.3. *Let $\mathcal{P}$ be a property of modules with integrable connection on smooth k -varieties which is strongly exact and local for the étale topology. Then $\mathcal{P}$ is stable under finite étale direct image.*

Proof. Let $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X)$ and $\pi : X \rightarrow Z$ be an étale covering. We may and shall assume that X is connected. Let $\tau : Y \rightarrow X$ be the Galois closure of X relative to π [1] Exp. V, §4 and §7], and let G be the associated Galois group of $\pi\tau$, so that $Z \cong Y/G$. Let $\sigma : Z' \rightarrow Z$ be an étale covering such that $Y_{Z'} \rightarrow Z'$ [1] Exp. V, Prop. 2.6, (ii bis)] is the trivial Galois covering $\mathrm{pr}_1 : Z' \times G \rightarrow Z'$. We have the commutative diagram with fibered squares

$$\begin{array}{ccccc} Y & \xrightarrow{\tau} & X & \xrightarrow{\pi} & Z \\ \uparrow \sigma'' & & \uparrow \sigma' & & \uparrow \sigma \\ Z' \times G = Y_{Z'} & \xrightarrow{\tau'} & X_{Z'} & \xrightarrow{\pi'} & Z' \end{array}$$

where all morphisms are finite étale and $\pi' \tau' = \text{pr}_1$.

Since $Z' \times G = \coprod_{\gamma \in G} Z' \times \{\gamma\}$, an object $(\mathcal{M}, \nabla)$ of $\mathbf{MIC}(Z' \times G)$, consists of local components $(\mathcal{M}, \nabla)_\gamma$ in $\mathbf{MIC}(Z' \times \{\gamma\}) = \mathbf{MIC}(Z')$, and $\mathcal{P}((\mathcal{M}, \nabla)) \Leftrightarrow (\mathcal{P}((\mathcal{M}, \nabla)_\gamma))$, $\forall \gamma \in G$. Since G operates on $Z' \times G$ by Z' -automorphisms, for any $\gamma \in G$

$$(\text{pr}_1^* \text{pr}_{1*}(\mathcal{M}, \nabla))_\gamma \cong \bigoplus_{\delta \in G} (\mathcal{M}, \nabla)_\delta$$

and

$$\mathcal{P}((\mathcal{M}, \nabla)) \iff \mathcal{P}(\text{pr}_1^* \text{pr}_{1*}(\mathcal{M}, \nabla)).$$

Applying this to $(\mathcal{M}, \nabla) = \tau'^* \sigma'^*(\mathcal{E}, \nabla)$, we derive $\mathcal{P}(\text{pr}_{1*} \tau'^* \sigma'^*(\mathcal{E}, \nabla))$. On the other hand,

$$\text{pr}_{1*} \tau'^* \sigma'^* = \sigma^* \pi_* \tau_* \tau^*,$$

whence $\mathcal{P}(\pi_* \tau_* \tau^*(\mathcal{E}, \nabla))$. Finally, since $(\mathcal{E}, \nabla)$ is a direct summand of $\tau_* \tau^*(\mathcal{E}, \nabla)$, we conclude $\mathcal{P}(\pi_*(\mathcal{E}, \nabla))$. $\square$

Lemma 25.3.4 (Lemma on dévissage, first form). *Let $\mathcal{P}$ be a property of modules with integrable connection on smooth k -varieties, which is strongly exact and local for the étale topology. Assume that*

- (*) *for any rational elementary fibration $f : X \rightarrow S$ with S affine, étale over some affine k -space, and for any quasi-coherent $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X)$,*

$$\mathcal{P}((\mathcal{E}, \nabla)) \Rightarrow \mathcal{P}(R_{\text{DR}}^j f_*(\mathcal{E}, \nabla)), \quad j = 0, 1.$$

Then for any $i \geq 0$, any smooth morphism $f : X \rightarrow S$ of smooth k -varieties, and any quasi-coherent $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X)$,

$$\mathcal{P}((\mathcal{E}, \nabla)) \Rightarrow \mathcal{P}((R_{\text{DR}}^i f_*(\mathcal{E}, \nabla))_{|A_i(f)}).$$

Proof. We proceed in three steps.

25.3.5. The case when f has relative dimension 0 is a simple consequence of the stability of $\mathcal{P}$ under finite étale direct image; we discard this trivial case in the sequel. Because $\mathcal{P}$ is local, we may assume that S is connected. On the other hand, if $f = f_1 \sqcup f_2 : X = X_1 \sqcup X_2 \rightarrow S$, then $R_{\text{DR}}^i f_*(\mathcal{E}, \nabla) = R_{\text{DR}}^i f_{1*}((\mathcal{E}, \nabla)|_{X_1}) \oplus R_{\text{DR}}^i f_{2*}((\mathcal{E}, \nabla)|_{X_2})$; because $\mathcal{P}$ is strongly exact, we may thus assume that X is connected so that f has pure relative dimension d . We may also assume that

$$0 \leq i \leq d + \dim X,$$

otherwise $R_{\text{DR}}^i f_*(\mathcal{E}, \nabla) = 0$, in which case the result is trivial.

25.3.6. We may replace at once S by $A_i(f)$, and then (because $\mathcal{P}$ is local for the étale topology and $R_{\text{DR}}^j f_*$ is compatible with étale localization for quasi-coherent $\mathcal{E}$, see [23.1](#)) by an affine connected étale neighborhood S' (étale over some affine

space if we wish) such that $X_{S'}$ admits a finite open cover $\{U_\alpha\}$ with the properties listed in definition [25.2.1](#). Let us consider the Zariski spectral sequence in **MIC**(S) [23.3.5](#)

$$E_1^{p,i-p} = \bigoplus_{\alpha_0 < \alpha_1 < \dots < \alpha_p} R_{\text{DR}}^{i-p} f_{\alpha*}(\mathcal{E}, \nabla)_{|U_{\underline{\alpha}}} \implies R_{\text{DR}}^i f_*(\mathcal{E}, \nabla).$$

By induction on i , since $A_{i-|\underline{\alpha}|+1}(f_{\underline{\alpha}}) = A_i(f) = S$, $\mathcal{P}(R_{\text{DR}}^{i-p} f_{\alpha*}(\mathcal{E}, \nabla)_{|U_{\underline{\alpha}}})$ for all $\underline{\alpha}$ such that $|\underline{\alpha}| = p+1 > 1$. Thus, by strong exactness of $\mathcal{P}$, this spectral sequence reduces us to prove $\mathcal{P}(R_{\text{DR}}^i f_{\alpha*}(\mathcal{E}, \nabla)_{|U_{\alpha}})$; hence, we may assume that f itself is a tower

$$X = X_d \longrightarrow X_{d-1} \longrightarrow \dots \longrightarrow X_0 = S$$

of coordinatized elementary fibrations over an affine connected base S (étale over some affine space if we wish). Such a tower gives rise to Leray spectral sequences in **MIC** (see [23.3](#)) and by strong exactness of $\mathcal{P}$ we are reduced to proving $\mathcal{P}$ for each level of the tower, i.e., we may assume that f itself is a coordinatized elementary fibration.

25.3.7. In case f is a coordinatized elementary fibration, [25.1.8](#)

$$\begin{array}{ccccc} X & \xrightarrow{j} & \bar{X} & \longleftrightarrow & Z \\ \pi \downarrow & & \downarrow \bar{\pi} & & \downarrow \\ Y & \xrightarrow{j'} & \mathbb{P}_S^1 & \longleftrightarrow & Z' \\ & \searrow f' & \downarrow & \swarrow & \\ & & S & & \end{array}$$

($f = f' \circ \pi$), we may assume that $i = 0, 1$: other cohomology sheaves are 0, since f is affine of relative dimension 1 (cf. [23.6](#)). Because $\mathcal{P}$ is étale-local, we may replace S by an étale covering so that f' becomes a rational elementary fibration. Since π is an étale covering, the Leray spectral sequence for $f = f' \circ \pi$ degenerates: $R_{\text{DR}}^i f_*(\mathcal{E}, \nabla) = R_{\text{DR}}^i f'_*(\pi_*(\mathcal{E}, \nabla))$. Then $\mathcal{P}((\mathcal{E}, \nabla)) \Rightarrow \mathcal{P}(\pi_*(\mathcal{E}, \nabla)) \Rightarrow \mathcal{P}(R_{\text{DR}}^i f'_*(\pi_*(\mathcal{E}, \nabla)))$, by (*). This proves the lemma. $\square$

Remarks 25.3.8. (i) In lemma [25.3.4](#) we may replace $A_i(f)$ by $A(f) = A_{d+\dim X}(f)$ since

$$R_{\text{DR}}^i f_*(\mathcal{E}, \nabla) = 0 \quad \text{for } j > d + \dim X.$$

(ii) If in lemma [25.3.4](#) assumption (*) holds only for $j = 0$, the conclusion keeps holding for $i = 0$.

Lemma 25.3.9 (Lemma on dévissage, second form). *Let $\mathcal{P}$ be a property of modules with integrable connection on smooth k -varieties that implies coherence, is strongly exact and local for the étale topology. Assume that*

- (*) for any rational elementary fibration $f : X \rightarrow S$ with S affine, étale over some affine k -space, and for any simple $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X)$, which is cyclic with respect to a generator of $T_{X/S}$, there exists an open dense subset $U \subseteq S$, such that

$$\mathcal{P}((\mathcal{E}, \nabla)) \implies \mathcal{P}((R_{\mathrm{DR}}^j f_*(\mathcal{E}, \nabla))|_U), \quad j = 0, 1.$$

Then for any smooth morphism $f : X \rightarrow S$ of smooth k -varieties, and any $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X)$, there exists a non-empty open subset $U \subseteq S$ (depending upon $(\mathcal{E}, \nabla)$) such that, for any $i \geq 0$,

$$\mathcal{P}((\mathcal{E}, \nabla)) \implies \mathcal{P}((R_{\mathrm{DR}}^i f_*(\mathcal{E}, \nabla))|_U).$$

Proof. We proceed in four steps.

25.3.10. We first show that (*) implies the apparently more stringent condition

- (*)' The conclusion of (*) holds true for every elementary fibration $f : X \rightarrow S$ and every $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X)$.

We show that $(*) \Rightarrow (*)'$. We first observe that, for any elementary fibration $f : X \rightarrow S$, if the conclusion of (*) holds true when $(\mathcal{E}, \nabla)$ is simple, then it holds true for any $(\mathcal{E}, \nabla)$. In fact, let $(\mathcal{E}, \nabla)$ be any object of $\mathbf{MIC}(X)$ satisfying $\mathcal{P}$. Since $\mathcal{P}$ implies coherence, $(\mathcal{E}, \nabla)$ has finite length. Applying the strong exactness of $\mathcal{P}$ to long exact cohomology sequences, we see that it is enough to establish (*) for the simple subquotients of $(\mathcal{E}, \nabla)$. Hence, we may assume that $(\mathcal{E}, \nabla)$ is simple.

We remark that we are free to replace S by some open dense subset; because $\mathcal{P}$ is étale-local, we may even replace S by an affine étale neighborhood. Hence we may even assume that the restriction of $(\mathcal{E}, \nabla)$ to the geometric generic fiber of f is simple.

It is clear that any elementary fibration may be coordinatized after replacing S by an open dense subset and then the argument in [25.3.7](#) further reduces the case of a coordinatized elementary fibration to the case of a rational one. So we may assume that $f : X \rightarrow S$ is indeed a rational elementary fibration.

On the other hand, according to Proposition [4.6.1](#) there exists a finite open affine cover $\{U_\alpha\}_{\alpha=0, \dots, r}$ of X , such that, for every α , $(\mathcal{E}, \nabla)|_{U_\alpha}$ admits a cyclic vector v_α with respect to any derivation $\partial_\alpha \in \mathcal{D}_{\mathrm{er}_{X/k}}(U_\alpha)$ which has no zero on U_α . Then after replacing S by some affine étale neighborhood, we may assume that for any α (resp. α, β) $f_{U_\alpha} : U_\alpha \rightarrow S$ (resp. $f_{U_\alpha \cap U_\beta} : U_\alpha \cap U_\beta \rightarrow S$) is a rational elementary fibration and $(\mathcal{E}, \nabla)|_{U_\alpha}$ (resp. $(\mathcal{E}, \nabla)|_{U_\alpha \cap U_\beta}$) admits a cyclic vector with respect to a generator of the relative tangent space $T_{X/S}$. The Čech spectral sequence for $\{U_\alpha\}$ then reduces us to the cyclic case, so that (*)' follows from (*).

We are left to prove that the conclusion of Lemma [25.3.9](#) holds if (*)' is verified.

25.3.11. As in [25.3.5](#) we reduce to the case when S is connected, f is equidimensional of relative dimension $d \geq 1$, and $0 \leq i \leq d + \dim X$. By Artin's theorem (i.e., Proposition [25.1.12](#) without insisting on “coordinatized” elementary fibrations), there exists an étale map $\epsilon : S' \rightarrow S$ such that $X_{S'}$ admits a finite open affine cover $\{U_\alpha\}$ such that the restriction of $f_{S'}$ to each U_α is a tower of elementary fibrations. We may then replace S by S' , and assume from the beginning that X admits a finite open affine cover $\{U_\alpha\}$ such that the restriction of f to each U_α is a tower of elementary fibrations.

Granting the previous reduction on $f : X \rightarrow S$, we further reduce the statement, by induction on i , to the case when f itself is a tower of elementary fibrations over a dense open subset of S . In fact, let us assume $\mathcal{P}((\mathcal{E}, \nabla))$, and let us consider the Zariski spectral sequence in $\mathbf{MIC}(S)$:

$$E_1^{p, i-p} = \bigoplus_{\alpha_0 < \alpha_1 < \dots < \alpha_p} R_{\mathrm{DR}}^{i-p} f_{\alpha*}(\mathcal{E}, \nabla)_{|U_{\underline{\alpha}}} \implies R_{\mathrm{DR}}^i f_*(\mathcal{E}, \nabla).$$

If $i = 0$, we are reduced to the case when f itself is a tower of d elementary fibrations and the result follows trivially from $(*)'$. By induction on i , we know that after further replacing S by a dense affine open subset, we may suppose that $R_{\mathrm{DR}}^{i-p} f_{\alpha*}((\mathcal{E}, \nabla)_{|U_{\underline{\alpha}}})$ satisfies $\mathcal{P}$, for all $\underline{\alpha}$ such that $|\underline{\alpha}| = p + 1 > 1$. Thus, by the strong exactness of $\mathcal{P}$, this spectral sequence reduces us to proving $\mathcal{P}(R_{\mathrm{DR}}^i f_{\alpha*}(\mathcal{E}, \nabla)_{|U_{\alpha}})$, after replacing S by a dense affine open subset. Hence we may assume that f is a tower of elementary fibrations over an affine connected base S .

25.3.12. We argue by induction on the height d of the tower. The induction hypothesis will be:

$(*)_d$ For any tower of elementary fibrations of height d ,

$$X_d \xrightarrow{f_{d-1}} X_{d-1} \longrightarrow \dots \xrightarrow{f_0} X_0,$$

with X_0 affine connected, and any $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X_d)$, there exists a dense affine open subset X'_0 of X_0 such that for any $(i_0, \dots, i_{d-1}) \in \{0, 1\}^d$, we have $\mathcal{P}((\mathcal{E}, \nabla)) \Rightarrow \mathcal{P}((R_{\mathrm{DR}}^{i_0} f_{0*} \dots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla))_{|X'_0})$.

Notice that $(*)_1 = (*)'$. We point out that via the Leray spectral sequences for $f = f_0 \circ \dots \circ f_{d-1}$, and taking into account the strong exactness of $\mathcal{P}$, it follows from $(*)_d$ that for any $i \geq 0$, $\mathcal{P}((\mathcal{E}, \nabla)) \Rightarrow \mathcal{P}((R_{\mathrm{DR}}^i f_*(\mathcal{E}, \nabla))_{|X'_0})$.

25.3.13. We now consider a tower of height $d \geq 2$ as in $(*)_d$, and $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X_d)$ satisfying $\mathcal{P}$ (in particular, $\mathcal{E}$ is torsion-free over $\mathcal{O}_{X_d}$). We apply the induction hypothesis $(*)_{d-1}$ to $X_d \longrightarrow \dots \longrightarrow X_1$, to the effect that there exists a dense affine open subset $X'_1 \subseteq X_1$ such that the $\mathcal{O}_{X'_1}$ -module with integrable connection $(R_{\mathrm{DR}}^{i_1} f_{1*} \dots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla))_{|X'_1}$ satisfies $\mathcal{P}$.

By shrinking X_0 , we may assume that the restriction of f_0 , say $f'_0 : X'_1 \rightarrow X_0$, is an elementary fibration and that X'_1 is the complement of a smooth divisor in X_1 . We then apply $(*)_1$ to

$$(R_{\mathrm{DR}}^{i_1} f_{1*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla))|_{X'_1},$$

with respect to f'_0 . Upon replacing X_0 again by a dense affine open subset, we obtain

$$\mathcal{P}(R_{\mathrm{DR}}^{i_0} f'_{0*}(R_{\mathrm{DR}}^{i_1} f_{1*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla))|_{X'_1}), \quad \text{for any } (i_0, \dots, i_{d-1}) \in \{0, 1\}^d.$$

We summarize the geometric situation, obtained at the price of shrinking X_0 , in the diagram

$$\begin{array}{ccccccc} X'_d & \xrightarrow{f'_{d-1}} & X'_{d-1} & \longrightarrow & \cdots & \longrightarrow & X'_1 \xrightarrow{f'_0} X_0 \\ j_d \downarrow & & j_{d-1} \downarrow & & & & j_1 \downarrow \\ X_d & \xrightarrow{f_{d-1}} & X_{d-1} & \longrightarrow & \cdots & \longrightarrow & X_1 \xrightarrow{f_0} X_0 \end{array}$$

in which the horizontal arrows are elementary fibrations, the vertical arrows are affine open embeddings, and all squares not involving the 0-subscript are fibered. Moreover, we may and will assume that, for any $k = 1, \dots, d$, $X_k \setminus X'_k$ is a smooth divisor in X_k .

From these properties it follows that, for any $k \in \{1, \dots, d-1\}$ and any quasi-coherent $(\mathcal{N}, \nabla) \in \mathbf{MIC}(X_{k+1})$,

$$\begin{aligned} j_{k*} j_k^* R_{\mathrm{DR}}^{i_k} f_{k*}(\mathcal{N}, \nabla) &= j_{k*} R_{\mathrm{DR}}^{i_k} f'_{k*} j_{k+1}^*(\mathcal{N}, \nabla) \\ &= R_{\mathrm{DR}}^{i_k} (j_k \circ f'_k)_* j_{k+1}^*(\mathcal{N}, \nabla) \\ &= R_{\mathrm{DR}}^{i_k} f_{k*} (j_{k+1*} j_{k+1}^*(\mathcal{N}, \nabla)). \end{aligned}$$

By iteration, we obtain

$$(**)_k \quad j_{k*} j_k^* (R_{\mathrm{DR}}^{i_k} f_{k*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla)) = R_{\mathrm{DR}}^{i_k} f_{k*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*} (j_{d*} j_d^*(\mathcal{E}, \nabla)).$$

On the other hand, since $\mathcal{E}$ is torsion-free, there is an exact sequence

$$0 \longrightarrow (\mathcal{E}, \nabla) \longrightarrow j_{d*} j_d^*(\mathcal{E}, \nabla) \longrightarrow \mathrm{Coker} \longrightarrow 0.$$

By descending induction on $k \in \{0, \dots, d-1\}$, we shall show that the sequence of

natural morphisms

$$\begin{array}{c}
 (***)_k \\
 \downarrow \\
 R_{\text{DR}}^{i_k} f_{k*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla) \\
 \downarrow \\
 R_{\text{DR}}^{i_k} f_{k*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(j_{d*} j_d^*(\mathcal{E}, \nabla)) \\
 \downarrow \\
 R_{\text{DR}}^{i_k} f_{k*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*} \text{Coker} \\
 \downarrow \\
 0
 \end{array}$$

is a short *exact sequence* ($i_k \in \{0, 1\}$).

By descending induction on k , and via the long exact sequence attached to $R_{\text{DR}}^{\bullet} f_{k*}$, it is enough to show that

$$(***)_k' \quad R_{\text{DR}}^0 f_{k*} (R_{\text{DR}}^{i_{k+1}} f_{k+1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*} \text{Coker}) = 0.$$

By $(**)_{k+1}$ and $(***)_{>k}$,

$$\begin{aligned}
 & R_{\text{DR}}^{i_{k+1}} f_{k+1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*} \text{Coker} \\
 &= \text{Coker} \left(\begin{array}{c} R_{\text{DR}}^{i_{k+1}} f_{k+1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla) \\ \downarrow \\ j_{k+1*} j_{k+1}^* R_{\text{DR}}^{i_{k+1}} f_{k+1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla) \end{array} \right)
 \end{aligned}$$

which is torsion, supported by $X_{k+1} \setminus X'_{k+1}$. This implies $(***)_k'$. Indeed, let $m \neq 0$, if any, be a section of $R_{\text{DR}}^{i_{k+1}} f_{k+1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*} \text{Coker}$ supported on the component of $X_{k+1} \setminus X'_{k+1}$ of local equation t . Then $t^h m = 0$, for some $h \in \mathbb{Z}_{>0}$, which we assume to be minimal. So, for the $\mathcal{O}_{X_k}$ -linear derivation $\partial_t = \frac{d}{dt}$,

$$0 = \nabla_{\partial_t}(t^h m) = h t^{h-1} m + t^h \nabla_{\partial_t}(m).$$

This shows that $\nabla_{\partial_t}(m) \neq 0$, hence the assertion $(***)_k'$. We have now proved $(***)_0'$, which shows that $R_{\text{DR}}^{i_0} f_{0*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla)$ is a subobject of

$$\begin{aligned}
 & R_{\text{DR}}^{i_0} f_{0*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(j_{d*} j_d^*(\mathcal{E}, \nabla)) \\
 &= R_{\text{DR}}^{i_0} f_{0*} j_{1*} j_1^* R_{\text{DR}}^{i_1} f_{1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla) \\
 &= R_{\text{DR}}^{i_0} f_{0*} ((R_{\text{DR}}^{i_1} f_{1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla))|_{X_1'}).
 \end{aligned}$$

Since the latter satisfies $\mathcal{P}$, this completes the proof of $(*)_d$ and of the lemma. $\square$

Remark 25.3.14. The argument also shows that, for any $k = d - 1, \dots, 1, 0$, the natural map

(25.3.15)

$$R_{\mathrm{DR}}^{i_k} f_{k*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla) \longrightarrow R_{\mathrm{DR}}^{i_k} f_{k*} j_{k+1*} j_{k+1}^* \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla)$$

or, equivalently, the map

$$(25.3.16) \quad R_{\mathrm{DR}}^{i_k} f_{k*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla) \longrightarrow R_{\mathrm{DR}}^{i_k} f_{k*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(j_{d*} j_d^*(\mathcal{E}, \nabla))$$

is an isomorphism if $(i_k, i_{k+1}, \dots, i_{d-1}) \neq (1, 1, \dots, 1)$. Indeed, assume $h, k \leq h \leq d-1$, is maximal with the condition that $i_h = 0$. Then (25.3.15) is an isomorphism for $k = h$. But then, it is also an isomorphism for $k = h-1, \dots, 1, 0$, since it is obtained by applying the functor $R_{\mathrm{DR}}^{i_k} f_{k*} \cdots R_{\mathrm{DR}}^{i_{h-1}} f_{h-1*}$ to the isomorphism (25.3.16) for $k = h$.

Remark 25.3.17. One can give an *alternative proof* of Lemma 25.3.9 under the extra assumption that for any closed immersion $i : Y \rightarrow X$ of smooth k -varieties, $\mathcal{P}((\mathcal{E}, \nabla)) \Rightarrow \mathcal{P}(i^*(\mathcal{E}, \nabla))$. This second proof avoids the use of Proposition 25.1.12 (see [9] in a special case, using the language and results of [19]).

26 Main theorems on the Gauss-Manin connection

26.1 Generic finiteness of direct images

Theorem 26.1.1. *For any smooth morphism $f : X \rightarrow S$ of smooth k -varieties and any coherent $\mathcal{O}_X$ -module with integrable connection $(\mathcal{E}, \nabla)$, there exists a dense open subset $U \subseteq S$ such that for every $j \geq 0$, $R_{\mathrm{DR}}^j f_*(\mathcal{E}, \nabla)|_U$ is locally free of finite rank.*

For the proof, we may and shall assume that f is dominant.

26.1.2. We treat separately the case $j = 0$, for which one can in fact take $U = A_0(f)$, the Artin set of level 0 (see 25.2.2), independently of $(\mathcal{E}, \nabla)$.

Lemma 26.1.3. *Let $f : X \rightarrow S$ be a smooth morphism with $S = A_0(f)$. Then for any coherent $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X)$, $R_{\mathrm{DR}}^0 f_*(\mathcal{E}, \nabla)$ is locally free of finite rank as an $\mathcal{O}_S$ -module.*

Proof. (of the lemma) It suffices to check the coherence of $R_{\mathrm{DR}}^0 f_*(\mathcal{E}, \nabla)$, and this is an étale-local condition on S . By the definition of $A_0(f)$, we thus may assume that f is Zariski-locally on X a tower of elementary fibrations, hence (separating the connected components of X) that f has non-empty connected geometric fibers. The coherence of $R_{\mathrm{DR}}^0 f_*(\mathcal{E}, \nabla)$ then follows from Proposition 23.1.3 and faithfully flat descent [48, 2.5.2]. $\square$

26.1.4. For $j > 0$, we use the lemma on dévissage in its second form [25.3.9](#) with $\mathcal{P} = \text{coherence}$ (i.e., $(\mathcal{E}, \nabla)$ satisfies $\mathcal{P}$ if and only if $\mathcal{E}$ is $\mathcal{O}_X$ -coherent). Strong exactness and étale-localness are fulfilled. Therefore, we are left to show that for any *rational elementary fibration* $f : X \rightarrow S$ with S smooth affine, and for any $(\mathcal{E}, \nabla)$ *cyclic* with respect to a generator of $T_{X/S}$, there is a dense open subset $U \subseteq S$ such that $R_{\text{DR}}^1 f_*(\mathcal{E}, \nabla)|_U$ is $\mathcal{O}_U$ -coherent; in other words, such that the cokernel of the relative de Rham complex $\Gamma(U, \mathcal{E}) \rightarrow \Gamma(U, \mathcal{E})dx$ is finitely generated.

26.1.5. Let us write the diagram pertaining to a rational elementary fibration over a smooth affine k -variety S :

$$(26.1.6) \quad \begin{array}{ccc} X & \xrightarrow{j} & \mathbb{P}_S^1 \longleftrightarrow \sigma_\infty(S) \sqcup \coprod_{i=1}^r \sigma_i(S) \\ & \searrow f & \downarrow \text{pr}_S \\ & & S \end{array} \quad \begin{array}{c} \nearrow g \\ \nwarrow \end{array}$$

where $\sigma_\infty(S) = \infty \times S$, $\sigma_i : \zeta \mapsto (\theta_i(\zeta), \zeta)$, $\theta_i \in \mathcal{O}(S)$, $\theta_i - \theta_j \in \mathcal{O}(S)^\times$. As usual, x denotes the affine coordinate on $\mathbb{P}^1$ and ∂_x the corresponding generator of $T_{\mathbb{P}_S^1/S}$. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$ -module endowed with an integrable connection $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_{X/k}^1$. Assume that $\mathcal{E}$ is cyclic with respect to ∂_x with cyclic vector $v \in E := \Gamma(X, \mathcal{E})$, and let $(\mathcal{E}, \nabla_{\partial_x})$ be the sheaf of $(\mathcal{O}_X, \partial_x)$ -differential modules naturally obtained from $(\mathcal{E}, \nabla)$. Then $(\mathcal{E}, \nabla_{\partial_x})$ is also cyclic with cyclic vector v . The dual connection $(\mathcal{E}^\vee, \nabla^\vee)$ is also cyclic with respect to ∂_x and the construction $(\mathcal{E}^\vee, \nabla_{\partial_x}^\vee)$ gives the sheaf of $(\mathcal{O}_X, \partial_x)$ -differential modules dual to $(\mathcal{E}, \nabla_{\partial_x})$. We assume $v^\vee \in E^\vee := \Gamma(X, \mathcal{E}^\vee)$ is a ∂_x -cyclic vector, and let

$$\Lambda = \partial_x^\mu + \sum_{k=0}^{\mu-1} a_k \partial_x^k \in \mathcal{O}(X) \langle \partial_x \rangle$$

the associated differential operator (such that $\nabla_\Lambda^\vee v^\vee = 0$).

We recall (see [7.3.1](#)) that the *indicial polynomial* of Λ at θ_i , denoted by $\text{ind}_{\theta_i}(t)$ (resp. $\text{ind}_\infty(t)$), is the element of $\mathcal{O}(S)[t]$ defined by the condition that, for any $m \in \mathbb{Z}$,

$$(26.1.7) \quad \Lambda \frac{1}{(x - \theta_i)^m} = \text{ind}_{\theta_i}(-m) \frac{1}{(x - \theta_i)^{m+r_i}} + \text{lower order terms in } \frac{1}{x - \theta_i},$$

(resp.

$$(26.1.8) \quad \Lambda x^m = \text{ind}_\infty(-m) x^{m+r_\infty} + \text{l.o.t. in } x),$$

where $r_i = \max(k - \text{ord}_{\theta_i} a_k) \geq 0$ (resp. $r_\infty = \max(\deg_x a_k - k) \geq 0$).

Let $\gamma_i \in \mathcal{O}(S) \setminus \{0\}$ (resp. $\gamma_\infty \in \mathcal{O}(S) \setminus \{0\}$) be the *leading coefficient* of the indicial polynomial $\text{ind}_{\theta_i}(t)$ (resp. $\text{ind}_{\theta_\infty}(t)$), and let us set $\gamma = \gamma_\infty \prod_i \gamma_i$. Since ∇ is integrable, it follows from [8.2.3](#) that

$$(26.1.9) \quad \frac{\text{ind}_{\theta_i}(t)}{\gamma_i}, \frac{\text{ind}_{\theta_\infty}(t)}{\gamma_\infty} \in k[t].$$

Lemma 26.1.10. *In the above geometric situation and with the above notation, assume moreover that:*

- $(\mathcal{E}, \nabla_{\partial_x})$ is a cyclic $(\mathcal{O}_X, \partial_x)$ -differential module;
- γ is unit in $\mathcal{O}(S)$.

Then $\text{Coker}(\nabla_{\partial_x})$ is a locally free $\mathcal{O}_S$ -module of finite rank. In general, if $U \subseteq S$ is the open subset on which γ is invertible, and we restrict $(\mathcal{E}, \nabla)$ on $X_U = f^{-1}(U)$ and f to $X_U \rightarrow U$, then $\text{Coker}(\nabla_{\partial_x})$ is $\mathcal{O}_U$ -coherent.

Proof of the lemma. We use the dual situation of Lemma [3.2.14](#). Let

$$(v^\vee, \nabla_{\partial_x}^\vee v^\vee, \dots, (\nabla_{\partial_x}^\vee)^{\mu-1} v^\vee)$$

be the cyclic basis of $E^\vee = \Gamma(X, \mathcal{E}^\vee)$ corresponding to $v^\vee$ and let

$$(v_0, v_1, \dots, v_{\mu-1})$$

be the $\mathcal{O}(X)$ -basis of E , dual to it. The diagram [\(3.2.15\)](#) produces here a commutative diagram

$$(26.1.11) \quad \begin{array}{ccc} \mathcal{O}(X) & \xrightarrow{\Lambda} & \mathcal{O}(X) \\ \downarrow & & \downarrow \\ \Gamma(X, \mathcal{E}) = \bigoplus_{k=0}^{\mu-1} \mathcal{O}(X) v_k & \xrightarrow{\nabla_{\partial_x}} & \bigoplus_{k=0}^{\mu-1} \mathcal{O}(X) v_k = \Gamma(X, \mathcal{E}) \end{array}$$

where the vertical arrows are $\eta \mapsto (\eta, \dots, \partial_x^{\mu-1} \eta)$ and $\eta \mapsto (0, \dots, 0, \eta)$, respectively. Diagram [\(26.1.11\)](#) induces an isomorphism between the cokernel (resp. kernel) of Λ and the cokernel (resp. kernel) of ∇_{∂_x} . Therefore

$$\text{Coker}(\Gamma(X, \mathcal{E}) \xrightarrow{\nabla_{\partial_x}} \Gamma(X, \mathcal{E})) \simeq \mathcal{O}(X) / \Lambda \mathcal{O}(X).$$

Let

$$P(x) = \prod_{i=1}^r (x - \theta_i)^{s_i} \in \mathcal{O}(S)[x]$$

be such that

$$(26.1.12) \quad \Lambda' = P(x) \Lambda = \sum_{k=0}^{\mu} a'_k \partial_x^k \in \mathcal{O}(S)[x] \langle \partial_x \rangle$$

with $s_i \geq 0$ minimal. We also have

$$\text{Coker}(\Gamma(X, \mathcal{E}) \xrightarrow{\nabla_{\theta_3}} \Gamma(X, \mathcal{E})) \simeq \mathcal{O}(X)/\Lambda' \mathcal{O}(X).$$

26.1.13. The indicial polynomials of Λ and Λ' (denoted by ind'_{θ_i}) at θ_i differ by multiplication by a unit in $\mathcal{O}(S)$. Therefore, in the statement of the lemma we can replace γ by the corresponding element γ' pertaining to Λ' . We have

$$\Lambda' \frac{1}{(x - \theta_i)^m} = \text{ind}'_{\theta_i}(-m) \frac{1}{(x - \theta_i)^{m+r'_i}} + \text{l.o.t. in } \frac{1}{x - \theta_i},$$

(resp.

$$\Lambda' x^m = \text{ind}'_{\infty}(-m) x^{m+r'_\infty} + \text{l.o.t. in } x),$$

where $r'_i = \max(k - \text{ord}_{\theta_i} a'_k) \geq 0$ (resp. $r'_\infty = \max(\deg_x a'_k - k) \geq 0$), and $\frac{\text{ind}'_{\theta_i}(t)}{\gamma'_i(t)}$, $\frac{\text{ind}'_{\infty}(t)}{\gamma'_\infty(t)} \in k[t]$.

Let us now finish the proof of the lemma: we use the Mittag-Leffler decomposition

$$(26.1.14) \quad \mathcal{O}(X) = \mathcal{O}(S)[x] \oplus \bigoplus_{i=1}^r \frac{1}{x - \theta_i} \mathcal{O}(S) \left[\frac{1}{x - \theta_i} \right].$$

Let $M \geq \max r_i$ be such that $\text{ind}'_{\infty}(-n + r'_\infty) \prod_{j=1, \dots, r} \text{ind}'_{\theta_j}(-n + r'_j) \neq 0$ for $n > M$. Then

$$\frac{\gamma'_i}{(x - \theta_i)^n} \in \Lambda' \frac{\sigma_{i,n}}{(x - \theta_i)^{n-r'_i}} + \frac{1}{(x - \theta_i)^{n-1}} \mathcal{O}(S)[x], \quad \sigma_{i,n} \in \mathcal{O}(S)^\times,$$

and

$$\gamma'_\infty x^n \in \Lambda' (\sigma_{\infty,n} x^{n-r'_\infty}) + \bigoplus_{k=0}^{n-1} x^k \mathcal{O}(S), \quad \sigma_{\infty,n} \in \mathcal{O}(S)^\times.$$

Let F be the $\mathcal{O}(S)$ -submodule of $\mathcal{O}(X)/\Lambda' \mathcal{O}(X)$ generated by the images of

$$1, x, \dots, x^M, \frac{1}{(x - \theta_1)}, \dots, \frac{1}{(x - \theta_1)^M}, \dots, \frac{1}{(x - \theta_r)}, \dots, \frac{1}{(x - \theta_r)^M}.$$

By the Mittag-Leffler decomposition, for any element $[f] = f + \Lambda' \mathcal{O}(X)$ of $\mathcal{O}(X)/\Lambda' \mathcal{O}(X)$, there is a power $(\gamma')^m$, for some $m = 0, 1, \dots$, such that $(\gamma')^m [f] \in F$. Since $\gamma' \in \mathcal{O}(S)^\times$, we conclude that $F = \mathcal{O}(X)/\Lambda' \mathcal{O}(X)$ so that the latter is finitely generated. On the other hand, $\mathcal{O}(X)/\Lambda' \mathcal{O}(X)$ carries an integrable connection, so that it is projective of finite rank. This completes the proof of the lemma. $\square$

From the preceding lemma, one gets the following purity statement:

Corollary 26.1.15. *Let $f : X \rightarrow S$ be a rational elementary fibration, and let $(\mathcal{E}, \nabla)$ be a coherent $\mathcal{O}_X$ -module with integrable connection. Assume that there is a closed subset $T \subseteq S$ of codimension ≥ 2 such that $R_{\mathrm{DR}}^1 f_*(\mathcal{E}, \nabla)$ is locally free of finite rank over $S \setminus T$. Then $R_{\mathrm{DR}}^1 f_*(\mathcal{E}, \nabla)$ is locally free of finite rank over S .*

Remark 26.1.16. The proof does not use the full hypothesis that v is a cyclic vector on the whole of X (i.e., that one may take $P(x) = \prod_{i=1}^r (x - \theta_i)^{s_i}$). It would suffice to assume that v is a cyclic vector outside some divisor D whose Zariski closure in $\mathbb{P}_S^1$ is contained in X .

(Counter)example 26.1.17. Unlike the case of $R_{\mathrm{DR}}^0 f_*(\mathcal{E}, \nabla)$, it is not possible to choose an open subset $U \subseteq S$ of coherence for $R_{\mathrm{DR}}^1 f_*(\mathcal{E}, \nabla)$, which is independent of $(\mathcal{E}, \nabla)$. Indeed, let $X = \mathbb{A}_k^2 = \mathrm{Spec}(k[x, y]) \rightarrow S = \mathbb{A}_k^1 = \mathrm{Spec}(k[y])$ be the second projection, and consider, for $y_0 \in k$, the integrable $\mathcal{O}_X$ -connection of rank 1 $(\mathcal{E} = \mathcal{O}_X e, \nabla_{y_0})$, defined by

$$\begin{aligned} (\nabla_{y_0})_{\partial_x} e &= (y - y_0)e, \\ (\nabla_{y_0})_{\partial_y} e &= xe. \end{aligned}$$

The calculation of $\mathrm{Coker}((\nabla_{y_0})_{\partial_x})$ involves the ∂_x -adjoint of the operator $\partial_x - (y - y_0)$, that is, $\partial_x + (y - y_0)$. More precisely, the commutative square (26.1.11) takes the form

$$(26.1.18) \quad \begin{array}{ccc} k[x, y] & \xrightarrow{\partial_x + y - y_0} & k[x, y] \\ \downarrow & & \downarrow \\ k[x, y]e & \xrightarrow{\nabla_{\partial_x}} & k[x, y]e \end{array}$$

where both vertical maps are $f(x, y) \mapsto f(x, y)e$, and induces an isomorphism of $k[y]$ -modules

$$(26.1.19) \quad \mathrm{Coker}((\nabla_{y_0})_{\partial_x}) \cong k[x, y]/(\partial_x + (y - y_0))k[x, y].$$

The indicial polynomial $\mathrm{ind}_\infty(t)$ for the derivation ∂_x is the “constant” $y - y_0$. From

$$(\partial_x + (y - y_0))(x^j) = jx^{j-1} + (y - y_0)x^j, \quad \forall j = 0, 1, 2, \dots$$

we obtain in $k[x, y]/(\partial_x + (y - y_0))k[x, y]$ the congruences

$$x^j \equiv \frac{j!}{(y_0 - y)^j}, \quad \text{mod } (\partial_x + y - y_0)k[x, y].$$

We then have an isomorphism of $k[y]$ -modules

$$\begin{aligned} k[x, y]/(\partial_x + (y - y_0))k[x, y] &\longrightarrow k\left[y, \frac{1}{y - y_0}\right] \\ x^j &\longmapsto \frac{j!}{(y_0 - y)^j}, \end{aligned}$$

so that the maximal open subset of coherence of $\mathcal{E}/(\nabla_{y_0})_{\partial_x}\mathcal{E}$ is $S \setminus \{y_0\}$, which depends upon y_0 .

In order to determine the structure of left $k[y]\langle\partial_y\rangle$ -module of $\text{Coker}((\nabla_{y_0})_{\partial_x})$, that is, the action $\square_{\partial_y}$ of ∂_y on it, we need to fully replace $(\mathcal{E} = \mathcal{O}_X e, \nabla_{y_0})$ by the dual connection $(\mathcal{E}^\vee = \mathcal{O}_X e^\vee, \nabla_{y_0}^\vee)$ defined by

$$\begin{aligned} (\nabla_{y_0}^\vee)_{\partial_x} e^\vee &= (y_0 - y)e, \\ (\nabla_{y_0}^\vee)_{\partial_y} e^\vee &= -x e^\vee. \end{aligned}$$

Then, we must equip $k[x, y]$ in the upper line of the diagram (26.1.18) with the left $k[x, y]\langle\partial_y\rangle$ -module in which ∂_y acts as $\partial_y + x$. (Notice that $\partial_y + x$ and $\partial_x + y - y_0$ commute.) Finally, the action $\square_{\partial_y}$ of ∂_y on $\Gamma(S, \mathcal{E}/(\nabla_{y_0})_{\partial_x}\mathcal{E})$ is induced by $\partial_y + x$ the isomorphism (26.1.19). In the end, since the operators $\partial_y + x$ and $\partial_y - \frac{1}{y-y_0}$ coincide in $k[x, y]/(\partial_x + (y - y_0))k[x, y]$, the isomorphism of $k[y]$ -modules

$$k[y, \frac{1}{y-y_0}] \longrightarrow \text{Coker}((\nabla_{y_0})_{\partial_x}) \quad , \quad f(y) \longmapsto f(y)[e]$$

extends to an isomorphism of differential modules over $(k[y], \partial_y)$

$$(k[y, \frac{1}{y-y_0}], \partial_y - \frac{1}{y-y_0}) \xrightarrow{\sim} (\Gamma(S, \mathcal{E}/(\nabla_{y_0})_{\partial_x}\mathcal{E}), \square_{\partial_y}).$$

26.2 Generic base change for direct images

Theorem 26.2.1. *Let $f : X \rightarrow S$ be a smooth morphism of smooth k -varieties, and let $(\mathcal{E}, \nabla)$ be a coherent $\mathcal{O}_X$ -module with integrable connection. There is a dense open subset $U \subseteq S$ with the following property. For any smooth k -variety $S^\sharp$ and any morphism $u : S^\sharp \rightarrow S$, construct the fibered diagram*

$$\begin{array}{ccccc} X^\sharp & \xrightarrow{f^\sharp} & S^\sharp & \longleftrightarrow & U^\sharp \\ u^\sharp \downarrow & & \downarrow u & & \downarrow \\ X & \xrightarrow{f} & S & \longleftrightarrow & U \end{array}$$

Then, for any $i \geq 0$, the restriction to $U^\sharp$ of the base change morphism

$$\varphi^i : u^* R_{\text{DR}}^i f_* (\mathcal{E}, \nabla) \longrightarrow R_{\text{DR}}^i f_*^\sharp u^{\sharp*} (\mathcal{E}, \nabla)$$

is an isomorphism in $\mathbf{MIC}(U^\sharp)$.

26.2.2 (Notation). In the sequel, we set $(\mathcal{E}^\sharp, \nabla^\sharp) = u^{\sharp*}(\mathcal{E}, \nabla)$.

Remarks 26.2.3. (i) $U^\sharp$ is an open subset of $S^\sharp$, possibly empty (see 26.2.9 below).

The following assertions are straightforward consequences of the compatibility of the formation of $R_{\mathrm{DR}}^i f_*$ with flat base change (23.5).

(ii) It would suffice to consider the case of a closed immersion u (write u as a closed immersion given by its graph, followed by a projection).

(iii) The statement is equivalent to the following. For any $u : S^\# \rightarrow S$ such that $\mathrm{Im}(u) \subseteq U$, φ^i is an isomorphism.

(iv) In order to prove Theorem (26.2.1) one may replace S by any S' étale and dominant over S .

26.2.4. Replacing S by affine étale neighborhoods of its connected components, we may assume that X admits a finite open affine cover $\{U_\alpha\}$ such that the restriction f_α of f to U_α is a tower of elementary fibrations. Considering an affine cover of $S^\#$, or using (ii) of the preceding remark, we may assume that $u^{\#-1}(U_\alpha)$ are affine. With the notation of (25.2) there is a natural morphism of Čech spectral sequences in $\mathbf{MIC}(S^\#)$

$$\begin{array}{ccc} \bigoplus_{\alpha_0 < \dots < \alpha_p} u^* R_{\mathrm{DR}}^{i-p} f_{\alpha*}(\mathcal{E}, \nabla)|_{U_{\alpha}} & \Longrightarrow & u^* R_{\mathrm{DR}}^i f_*(\mathcal{E}, \nabla) \\ \downarrow \varphi_{i-p} & & \downarrow \varphi_i \\ \bigoplus_{\alpha_0 < \dots < \alpha_p} R_{\mathrm{DR}}^{i-p} f_{\alpha*}^\#(\mathcal{E}^\#, \nabla^\#)|_{u^{\#-1}U_{\alpha}} & \Longrightarrow & R_{\mathrm{DR}}^i f_*^\#(\mathcal{E}^\#, \nabla^\#) \end{array}$$

Arguing by induction on i , we may assume that f is a tower of elementary fibrations.

26.2.5. We argue by induction on the height d of the tower, as in the proof of (25.3.9). The induction hypothesis will be:

(*)_d For any tower of elementary fibrations of height d

$$X_d \xrightarrow{f_{d-1}} X_{d-1} \longrightarrow \dots \xrightarrow{f_0} X_0,$$

with X_0 affine connected, and any coherent $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X_d)$, there exists a dense affine open subset X'_0 of X_0 with the following property.

For any smooth k -variety $X_0^\#$ and any morphism $u_0 : X_0^\# \rightarrow X_0$, construct the fibered diagram

$$\begin{array}{ccccccc} X_d^\# & \xrightarrow{f_{d-1}^\#} & X_{d-1}^\# & \longrightarrow & \dots & \xrightarrow{f_0^\#} & X_0^\# \\ u_d \downarrow & & u_{d-1} \downarrow & & & & u_0 \downarrow \\ X_d & \xrightarrow{f_{d-1}} & X_{d-1} & \longrightarrow & \dots & \xrightarrow{f_0} & X_0 \end{array}$$

Then for any $(i_0, \dots, i_{d-1}) \in \{0, 1\}^d$,

$$(R_{\mathrm{DR}}^{i_0} f_{0*} R_{\mathrm{DR}}^{i_1} f_{1*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla))|_{X'_0}$$

is coherent, and the restriction to $u_0^{-1}(X'_0)$ of the base change morphism

$$\begin{aligned} & u_0^*(R_{\mathrm{DR}}^{i_0} f_{0*} R_{\mathrm{DR}}^{i_1} f_{1*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*} \mathcal{E}, \nabla) \\ & \xrightarrow{\psi} R_{\mathrm{DR}}^{i_0} f_{0*}^\# R_{\mathrm{DR}}^{i_1} f_{1*}^\# \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}^\# u_d^*(\mathcal{E}, \nabla) \end{aligned}$$

is an isomorphism in $\mathbf{MIC}(u_0^{-1}(X'_0))$.

We notice that via base change for the Leray spectral sequences of the composed morphism $f = f_0 \circ \cdots \circ f_{d-1}$, the statement [26.2.1](#) in the case of towers of elementary fibrations of height $\leq d$ follows from $(*)_{\leq d}$.

Remark 26.2.6. If $i_0 = i_1 = \cdots = i_{d-1} = 1$, then the base change morphism ψ is an isomorphism; this follows from the fact that for f affine smooth of relative dimension one the formation of $R_{\mathrm{DR}}^1 f_*$ commutes with arbitrary base change, being a cokernel.

26.2.7. Let us consider first $(*)_d$ in the case $d = 1$, i.e., f is an elementary fibration $X_1 = X \rightarrow S = X_0$. In this case, it turns out that *one can choose for $X'_0 = U$ any dense open subset such that $R_{\mathrm{DR}}^1 f_*(\mathcal{E}, \nabla)|_U$ is coherent.*

For simplicity of notation, assume that $S = U$. We note that only $i = 0$ or 1 occur. Since f is affine and flat, we have the exact sequence (cf. [23.6](#))

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_{\mathrm{DR}}^0 f_*(\mathcal{E}, \nabla) & \longrightarrow & f_*(\nabla) \xrightarrow{f_*(\nabla)} & f_*(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^1) & \longrightarrow & R_{\mathrm{DR}}^1 f_*(\mathcal{E}, \nabla) & \longrightarrow & 0 \\ & & \parallel & & & & & \parallel & & \\ & & \mathrm{Ker} f_*(\nabla) & & & & & \mathrm{Coker} f_*(\nabla) & & \end{array}$$

in which all terms are flat over S , which we split into two short exact sequences of flat $\mathcal{O}_S$ -modules:

$$\begin{aligned} 0 & \longrightarrow \mathrm{Im} f_*(\nabla) \longrightarrow f_*(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^1) \longrightarrow \mathrm{Coker} f_*(\nabla) \longrightarrow 0, \\ 0 & \longrightarrow \mathrm{Ker} f_*(\nabla) \longrightarrow f_* \mathcal{E} \longrightarrow \mathrm{Im} f_*(\nabla) \longrightarrow 0. \end{aligned}$$

Let us compare the exact sequences obtained by taking u^* with the analogous exact sequences built in terms of $f^\#$ and $\mathcal{E}^\#$: since $u^* f_* \mathcal{E} = f_*^\# \mathcal{E}^\#$, and $u^* f_*(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^1) = f_*^\#(\mathcal{E}^\# \otimes_{\mathcal{O}_{X^\#}} \Omega_{X^\#/S^\#}^1)$ [\[50\] 9.3.3](#), we see that these respective exact sequences coincide, so that

$$\begin{aligned} u^* \mathrm{Im} f_*(\nabla) &= \mathrm{Im} f_*^\#(\nabla^\#), \\ u^* \mathrm{Coker} f_*(\nabla) &= \mathrm{Coker} f_*^\#(\nabla^\#) \cong R_{\mathrm{DR}}^1 f_*^\#(\mathcal{E}^\#, \nabla^\#), \\ u^* \mathrm{Ker} f_*(\nabla) &= \mathrm{Ker} f_*^\#(\nabla^\#) \cong R_{\mathrm{DR}}^0 f_*^\#(\mathcal{E}^\#, \nabla^\#), \end{aligned}$$

hence φ^0 and φ^1 are isomorphisms in $\mathbf{MIC}(S^\#)$, which establishes $(*)_1$.

26.2.8. We now consider a tower of height $d \geq 2$ as in $(*)_d$. We apply the induction hypothesis $(*)_{d-1}$ to $X_d \rightarrow \cdots \rightarrow X_1$, to the effect that there exists a dense affine open subset $j_1 : X'_1 \hookrightarrow X_1$ such that $(R_{\text{DR}}^{i_1} f_{1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla))|_{X'_1}$ is coherent and the natural morphism

$$\begin{aligned} & (u_1^*(R_{\text{DR}}^{i_1} f_{1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla)))|_{u_1^{-1}(X'_1)} \\ & \xrightarrow{\varphi_{(1)}} (R_{\text{DR}}^{i_1} f_{1*}^\# \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}^\# u_d^*(\mathcal{E}, \nabla))|_{u_1^{-1}(X'_1)} \end{aligned}$$

is an isomorphism in $\mathbf{MIC}(u_1^{-1}(X'_1))$.

Let us consider the fibered prism

$$\begin{array}{ccccc} X_1^\# & \xrightarrow{j_1^\#} & X_1^\# & & \\ & \searrow f_0^\# & \swarrow f_0^\# & & \\ & & X_0^\# & & \\ & \swarrow u_1' & \downarrow u_0 & \searrow u_1 & \\ X_1' & \xrightarrow{j_1} & X_1 & & \\ & \searrow f_0' & \swarrow f_0 & & \\ & & X_0 & & \end{array}$$

By shrinking X_0 , we may assume that f_0' is an elementary fibration, and (by applying $(*)_1$ to f_0') that

(1) $R_{\text{DR}}^{i_0} f_{0*}'((R_{\text{DR}}^{i_1} f_{1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla))|_{X_1'})$ is coherent over X_0 .

(2) The natural morphism

$$\begin{aligned} & u_0^* R_{\text{DR}}^{i_0} f_{0*}'(R_{\text{DR}}^{i_1} f_{1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla))|_{X_1'} \\ & \downarrow \\ & R_{\text{DR}}^{i_0} f_{0*}' u_1'^*(R_{\text{DR}}^{i_1} f_{1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla))|_{X_1'} \\ & \parallel \wr \\ & R_{\text{DR}}^{i_0} f_{0*}' j_1'^* u_1^* R_{\text{DR}}^{i_1} f_{1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla) \end{aligned}$$

is an isomorphism in $\mathbf{MIC}(X_0^\#)$.

On combining with $\varphi_{(1)}$, we get a natural isomorphism φ in $\mathbf{MIC}(X_0^\#)$

$$\begin{aligned} (3) \quad & u_0^* R_{\text{DR}}^{i_0} f_{0*}' j_1'^* j_1^* R_{\text{DR}}^{i_1} f_{1*} \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla) \\ & \downarrow \varphi \cong \\ & R_{\text{DR}}^{i_0} f_{0*}' j_1'^* j_1^* R_{\text{DR}}^{i_1} f_{1*}^\# \cdots R_{\text{DR}}^{i_{d-1}} f_{d-1*}^\# u_d^*(\mathcal{E}, \nabla) . \end{aligned}$$

By Remark 26.2.6 we may assume that one of the i_j is 0; by Remark 25.3.14 one then has that the natural maps

$$\begin{aligned} R_{\mathrm{DR}}^{i_0} f_{0*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla) &\longrightarrow R_{\mathrm{DR}}^{i_0} f_{0*} j_{1*} j_1^* R_{\mathrm{DR}}^{i_1} f_{1*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla), \\ R_{\mathrm{DR}}^{i_0} f_{0*}^\# \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}^\# u_d^*(\mathcal{E}, \nabla) &\longrightarrow R_{\mathrm{DR}}^{i_0} f_{0*}^\# j_{1*}^\# j_1^* R_{\mathrm{DR}}^{i_1} f_{1*}^\# \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}^\# u_d^*(\mathcal{E}, \nabla) \end{aligned}$$

are isomorphisms. In particular, $R_{\mathrm{DR}}^{i_0} f_{0*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla)$ is coherent by (1). Moreover, these isomorphisms fit in a commutative square

$$\begin{array}{ccc} u_0^* R_{\mathrm{DR}}^{i_0} f_{0*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla) & \xrightarrow{\sim} & u_0^* R_{\mathrm{DR}}^{i_0} f_{0*} j_{1*} j_1^* R_{\mathrm{DR}}^{i_1} f_{1*} \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}(\mathcal{E}, \nabla) \\ \psi \downarrow & & \downarrow \varphi \cong (3) \\ R_{\mathrm{DR}}^{i_0} f_{0*}^\# \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}^\# u_d^*(\mathcal{E}, \nabla) & \xrightarrow{\sim} & R_{\mathrm{DR}}^{i_0} f_{0*}^\# j_{1*}^\# j_1^* R_{\mathrm{DR}}^{i_1} f_{1*}^\# \cdots R_{\mathrm{DR}}^{i_{d-1}} f_{d-1*}^\# u_d^*(\mathcal{E}, \nabla) \end{array}$$

We conclude that ψ (the base change map appearing in $(*)_d$) is an isomorphism, and this achieves the inductive proof of 26.2.1 $\square$

(Counter)example 26.2.9. Consider the fibered square

$$\begin{array}{ccc} \mathrm{Spec} k[x] & \xrightarrow{u^\#} & \mathrm{Spec} k[x, y] = X \\ f^\# \downarrow & & \downarrow f \\ \{y_0\} = \mathrm{Spec} k & \xrightarrow{u} & \mathrm{Spec} k[y] = S \end{array}$$

in which f is the projection and u is the embedding of the point defined by $y = y_0$. Next, consider the integrable connection on X of rank 1 $(\mathcal{E}, \nabla_{y_0})$ defined in 26.1.17. It is clear that $R_{\mathrm{DR}}^0 f_* (\mathcal{E}, \nabla_{y_0}) = 0$, while $R_{\mathrm{DR}}^0 f_*^\# u^\# (\mathcal{E}, \nabla_{y_0}) \cong k$, because $u^\# (\mathcal{E}, \nabla_{y_0}) \cong (k[x], d)$.

This shows the necessity of localizing in 26.2.1 when u is not flat. This example is also interesting in connection with the base change theorem of [19] VI, 8.4].

27 Gauss-Manin connection in the regular case

27.1 Main theorems (in the regular case)

Let i be a non-negative integer. Let $f : X \rightarrow S$ be a smooth morphism of smooth k -varieties, such that the Artin subset $A_i(f)$ is S itself (cf. 25.2.1). Let $\mathcal{E}$ be a coherent $\mathcal{O}_X$ -module endowed with an integrable regular connection ∇ .

Theorem 27.1.1 (Finiteness theorem). $R_{\mathrm{DR}}^i f_*(\mathcal{E}, \nabla)$ is a coherent $\mathcal{O}_S$ -module.

Theorem 27.1.2 (Regularity theorem). The Gauss-Manin connection $\aleph$ on $R_{\mathrm{DR}}^i f_*(\mathcal{E}, \nabla)$ is regular.

Theorem 27.1.3 (Monodromy theorem). *If the exponents of ∇ all belong to some $\mathbb{Q}$ -subspace Δ of k ($\Delta \supseteq \mathbb{Q}$), so do the exponents of $\mathfrak{K}$.*

Theorem 27.1.4 (Base change theorem). *For any smooth k -variety $S^\sharp$ and any morphism $u : S^\sharp \rightarrow S$, the base change morphism*

$$u^* R_{\mathrm{DR}}^i f_* (\mathcal{E}, \nabla) \longrightarrow R_{\mathrm{DR}}^i f_* u^* (\mathcal{E}, \nabla)$$

attached to the fibered square

$$\begin{array}{ccc} X^\sharp & \xrightarrow{u^\sharp} & X \\ f^\sharp \downarrow & & \downarrow f \\ S^\sharp & \xrightarrow{u} & S \end{array}$$

is an isomorphism.

Proofs. Let us say that $(\mathcal{E}, \nabla)$ satisfies property $\mathcal{P}_\Delta$ if and only if $\mathcal{E}$ is coherent, ∇ is regular, and all exponents of ∇ belong to Δ . Then $\mathcal{P}_\Delta$ is a strongly exact property (cf. Proposition 10.3.2 and Section 13), local for the étale topology (cf. Section 13). By the lemma on dévissage 25.3.4 in order to prove Theorems 27.1.1 27.1.2 27.1.3 it suffices to consider the case when f is a rational elementary fibration and $i = 0$ or 1, which will be settled in Sections 27.2 and 27.3.

As for the base change theorem, one reduces, as in Section 26.2 to the case when f is a tower of coordinatized elementary fibrations. The associated Leray spectral sequences allow us to further reduce to the case when f is a single coordinatized elementary fibration, which will be settled in 27.2.20. $\square$

27.2 Coherence of the cokernel of a regular connection

27.2.1. In this section, we establish that for any rational elementary fibration $f : X \rightarrow S$ as in 26.1.6 and any locally free $\mathcal{O}_X$ -module $(\mathcal{E}, \nabla)$ of finite rank μ with regular integrable connection, $R_{\mathrm{DR}}^1 f_* (\mathcal{E}, \nabla)$ is locally free of finite rank (over the whole of S). This is the key point in the proof of 27.1.1 (and 27.1.4).

We may assume that S is affine and connected and of dimension ≥ 1 . The statement amounts to the finite generation of $\Gamma(X, \mathcal{E})/\nabla_{\partial_x} \Gamma(X, \mathcal{E})$ over $\mathcal{O}(S)$.

27.2.2. If $(\mathcal{E}, \nabla)$ is cyclic and regular, the leading coefficients of the indicial polynomials are units (cf. Proposition 12.1.3), hence the result follows from 26.1.10.

However, unlike the situation of 25.3.10 the reduction to the cyclic case is problematic: it is not clear that one can cover S by étale neighborhoods S' with the property that $X_{S'}$ admits a Zariski open cover $\{U_\alpha\}$ such that $U_\alpha \rightarrow S'$ and $U_\alpha \cap U_\beta \rightarrow S'$ are rational elementary fibrations and such that $(\mathcal{E}, \nabla)|_{U_\alpha}$ is cyclic outside some divisor whose Zariski closure in $\mathbb{P}_{S'}^1$ is contained in U_α , see Section 18.1. We shall take another approach, via τ -extensions.

27.2.3. For notational convenience, we shall assume, by translating the “vertical” variable x , that one of the $\theta_1, \dots, \theta_r$, say θ_1 , is 0 (if $r \neq 0$).

Let us fix a set-theoretic section τ of the projection $k \rightarrow k/\mathbb{Z}$ with $\tau(0) = 0$.

Let us consider the τ -extension $(\tilde{\mathcal{E}}, \tilde{\nabla})$ of $(\mathcal{E}, \nabla)$ on $\mathbb{P}_S^1$, as constructed in [10.2.5](#) [11.2.2](#): $\tilde{\mathcal{E}}$ is a locally free $\mathcal{O}_{\mathbb{P}_S^1}$ -module, endowed with an integrable connection $\tilde{\nabla} : \text{Der}_Z \mathbb{P}_S^1 \rightarrow \text{End}_{\mathcal{O}_S} \tilde{\mathcal{E}}$ with logarithmic poles along $\mathbb{P}_S^1 \setminus X = Z = \prod_{\nu=1, \dots, r, \infty} Z_\nu$, such that the eigenvalues of the residue endomorphisms $\text{Res}_{\theta_i} \tilde{\nabla}$ belong to the image of τ .

Lemma 27.2.4. *Let $\mathcal{F}$ be a locally free $\mathcal{O}_{\mathbb{P}_S^1}$ -module of finite rank. Then there exists a locally free $\mathcal{O}_S$ -module $\mathcal{M}$ such that $\mathcal{F}|_{\mathbb{A}_S^1} \cong \text{pr}^* \mathcal{M}$, where pr stands for the projection $\mathbb{A}_S^1 \rightarrow S$.*

Proof. (of the lemma) The graded $\mathcal{O}(S)[x_0, x_1]$ -module F attached to $\mathcal{F}$ is projective of finite rank (since so are its localizations with respect to the open cover $\mathcal{U} = \{U_0, U_1\}$ of $\mathbb{P}_S^1$ with $U_0 := \mathbb{A}_S^1$, $U_1 := \mathbb{P}_S^1 \setminus (\{0\} \times S)$) and

$$F_{(x_i)} \cong \Gamma(U_i, \mathcal{F}),$$

for $i = 0, 1$. According to [\[26\]](#) X, p.144] there is a projective $\mathcal{O}(S)$ -module M such that $F \cong M \otimes_{\mathcal{O}(S)} \mathcal{O}(S)[x_0, x_1]$. One may then take for $\mathcal{M}$ the locally free $\mathcal{O}_S$ -module attached to M . $\square$

Remark 27.2.5. Lemma [27.2.4](#) is a special, elementary case of a result of H. Lindel [\[75\]](#) extending the work of Quillen and Suslin on the Serre conjecture, and which says that in fact, for affine smooth S/k , any locally free $\mathcal{O}_{\mathbb{A}_S^1}$ -module of finite rank has the form $\text{pr}^* \mathcal{M}$, for some locally free $\mathcal{O}_S$ -module $\mathcal{M}$.

27.2.6. We apply [27.2.4](#) to $\mathcal{F} = \tilde{\mathcal{E}}$, to the effect that there are projective $\mathcal{O}(S)$ -modules M and M' of rank μ , such that

$$\Gamma(\mathbb{A}_S^1, \tilde{\mathcal{E}}) = M[x], \quad \Gamma(\mathbb{P}_S^1 \setminus (\{0\} \times S), \tilde{\mathcal{E}}) = M' \left[\frac{1}{x} \right],$$

and $M[x, \frac{1}{x}] = M'[x, \frac{1}{x}]$. Since M and M' are finitely generated, there is a positive integer N such that

$$x^N M'[x] \subseteq M[x] \subseteq x^{-N} M'[x].$$

27.2.7. Set

$$(27.2.8) \quad \Theta(x) = \prod_{i=1}^r (x - \theta_i).$$

We have $\tilde{\nabla}_{\Theta(x)\partial_x} M[x] \subseteq M[x]$, and, for any $i = 1, \dots, r$, we are given an endomorphism $\text{Res}_{\theta_i} \tilde{\nabla}$ of M which enjoys the following properties (cf. Section [11.1](#)):

(1) the characteristic polynomial of $\text{Res}_{\theta_i} \tilde{\nabla}$ has coefficients in k and roots in the image of τ (in particular, none of the roots is a non-zero integer);

(2) for any $m \in M$, identified with $m(x - \theta_i)^0 \in M[x - \theta_i] = M[x]$, we have

$$\tilde{\nabla}_{(x-\theta_i)\partial_x} m - (\text{Res}_{\theta_i} \tilde{\nabla})m \in (x - \theta_i)M \left[x, \frac{1}{\prod_{k \neq i} (x - \theta_k)} \right].$$

Set $\gamma_i = \prod_{k \neq i} (\theta_i - \theta_k)$, which is a unit in $\mathcal{O}(S)$. It follows from (2) that for any $m \in M = M \cdot 1 \subseteq M[x, \frac{1}{x - \theta_i}]$,

$$(3) \quad \tilde{\nabla}_{\Theta(x)\partial_x} (m(x - \theta_i)^{-n}) - \gamma_i (\text{Res}_{\theta_i} \tilde{\nabla} - n1_M)(m)(x - \theta_i)^{-n} \in (x - \theta_i)^{-n+1} M[x].$$

On the other hand, (1) implies that $\text{Res}_{\theta_i} \tilde{\nabla} - n1_M$ is an invertible endomorphism of M for any $n > 0$ (by Cayley-Hamilton, the endomorphism $(\text{Res}_{\theta_i} \tilde{\nabla} - n1_M)^{-1}$ of $M \otimes_{\mathcal{O}(S)} \kappa(S)$ is a polynomial in $\text{Res}_{\theta_i} \tilde{\nabla} - n1_M$ with coefficients in k , hence induces an endomorphism of M). This shows that

$$m(x - \theta_i)^{-n} \in \tilde{\nabla}_{\Theta(x)\partial_x} (M(x - \theta_i)^{-n}) + (x - \theta_i)^{-n+1} M[x].$$

In virtue of the Mittag-Leffler decomposition

$$M \left[x, \frac{1}{\Theta(x)} \right] = M \otimes \mathcal{O}(S) \left[x, \frac{1}{\Theta(x)} \right] = M[x] \oplus \bigoplus_{i=1}^r \frac{1}{x - \theta_i} M \left[\frac{1}{x - \theta_i} \right],$$

one concludes that

$$(4) \quad M \left[x, \frac{1}{\Theta(x)} \right] = \tilde{\nabla}_{\Theta(x)\partial_x} M \left[x, \frac{1}{\Theta(x)} \right] + M[x].$$

At this point the proof splits into two alternative arguments 27.2.9 and 27.2.13.

27.2.9 (First conclusion of the proof). From

$$\tilde{\nabla}_{\frac{1}{x} \left(\prod_{i=2}^r \left(\frac{1}{x} - \frac{1}{\theta_i} \right) \right) \partial_{\frac{1}{x}}} M' \left[\frac{1}{x} \right] \subseteq M' \left[\frac{1}{x} \right],$$

we deduce that

$$\tilde{\nabla}_{x^{1-r}\Theta(x)\partial_x} M' \subseteq \bigoplus_{j=0}^l \frac{1}{x^j} M',$$

for sufficiently large $l \geq 1$. We consider the endomorphism $\text{Res}_{\infty} \tilde{\nabla}$ of M' , which satisfies 27.2.7(1) above, and

$$(27.2.10) \quad \tilde{\nabla}_{x\partial_x} m + (\text{Res}_{\infty} \tilde{\nabla})m \in \frac{1}{x} M' \left[\left[\frac{1}{x} \right] \right],$$

or, equivalently,

$$(27.2.11) \quad \tilde{\nabla}_{x^{1-r}\Theta(x)\partial_x} m + (\text{Res}_{\infty} \tilde{\nabla})m \in \bigoplus_{j=1}^l \frac{1}{x^j} M',$$

for any $m \in M'$. So, for any $n \geq 0$ and $m \in M'$,

$$(27.2.12) \quad \tilde{\nabla}_{x^{1-r}\Theta(x)\partial_x}(x^{n+l}m) + x^{n+l}(\text{Res}_\infty \tilde{\nabla} - (n+l)1_{M'})m \in \bigoplus_{j=0}^{l-1} x^j M'.$$

The invertibility of $\text{Res}_\infty \tilde{\nabla} - (n+l)1_{M'}$ implies that

$$M'[x]/\tilde{\nabla}_{x^{1-r}\Theta(x)\partial_x}x^l M'[x]$$

is generated by $\bigoplus_{j=0}^{l-1} x^j M'$. So, for any $h \geq 0$,

$$x^{r-h-1}M'[x]/\tilde{\nabla}_{\Theta(x)\partial_x}x^l M'[x]$$

is generated by $\bigoplus_{j=r-h-1}^{r+l-2} x^j M'$. We now pick l, h sufficiently large so that

$$x^l M'[x] \subseteq M[x] \subseteq x^{r-h-1}M'[x].$$

This proves that $M[x]/\tilde{\nabla}_{\Theta(x)\partial_x}M[x]$ is finitely generated over $\mathcal{O}(S)$.

A fortiori, $\Gamma(X, \mathcal{E})/\nabla_{\partial_x}\Gamma(X, \mathcal{E})$ is finitely generated over $\mathcal{O}(S)$. $\square_1$

27.2.13 (Alternative conclusion of proof). We come back to 27.2.7 and derive the coherence of $\text{Coker}_{\Gamma(X, \mathcal{E})}\nabla_{\partial_x}$ using another argument which will be of importance in the next section.

Let us first remark that

$$(27.2.14) \quad M[x] \cap \tilde{\nabla}_{\Theta(x)\partial_x}M[x, \frac{1}{\Theta(x)}] = \tilde{\nabla}_{\Theta(x)\partial_x}M[x].$$

Indeed, if an element of $M[x, \frac{1}{\Theta(x)}]$ has a pole of order $n \geq 0$ at θ_i , with Mittag-Leffler component $m(x - \theta_i)^{-n}$, and is sent to $M[x]$ by $\tilde{\nabla}_{\Theta(x)\partial_x}$, then $m \in \text{Ker}(\text{Res}_{\theta_i} \tilde{\nabla} - n1_M)$, in view of 27.2.7 (3); hence, $n \in \text{Im} \tau \cap \mathbb{Z} = \{0\}$. This argument also shows that

$$(27.2.15) \quad \text{Ker}_{M[x]}\tilde{\nabla}_{\Theta(x)\partial_x} = \text{Ker}_{M[x, \frac{1}{\Theta(x)}]}\nabla_{\partial_x}.$$

Putting 27.2.7 (4) and 27.2.14 together, we see that multiplication by $\Theta(x)^{-1}$ induces an isomorphism

$$(27.2.16) \quad M[x]/\tilde{\nabla}_{\Theta(x)\partial_x}M[x] \cong M[x, \frac{1}{\Theta(x)}]/\nabla_{\partial_x}M[x, \frac{1}{\Theta(x)}].$$

By the symmetry $x \mapsto \frac{1}{x}$, we also have

$$(27.2.15)' \quad \text{Ker}_{M'[\frac{1}{x}]}\tilde{\nabla}_{\frac{1}{x}}(\prod_{i=2}^r(\frac{1}{x} - \frac{1}{\theta_i}))\partial_{\frac{1}{x}} = \text{Ker}_{M'[\frac{1}{x, \Theta(x)}]}\nabla_{\partial_x}$$

and

$$(27.2.16)' \quad M'[\frac{1}{x}]/\tilde{\nabla}_{\frac{1}{x}}(\prod_{i=2}^r(\frac{1}{x}-\frac{1}{\theta_i}))_{\partial_{\frac{1}{x}}} M'[\frac{1}{x}] \cong M[x, \frac{1}{\Theta(x)}]/\nabla_{\partial_x} M[x, \frac{1}{\Theta(x)}].$$

This can be summarized in the following diagram of isomorphisms:

$$(27.2.17) \quad \begin{array}{ccc} (\text{Co})\text{Ker}_{M[x]} \tilde{\nabla}_{\Theta(x)\partial_x} & & \\ \downarrow & & \\ (\text{Co})\text{Ker}_{M[x, \frac{1}{x}]=M'[\frac{1}{x}]} \tilde{\nabla}_{\Theta(x)\partial_x} & \xrightarrow{\cdot \Theta(x)^{-1}} & (\text{Co})\text{Ker}_{M[x, \frac{1}{\Theta(x)}]} \nabla_{\partial_x} \\ \uparrow \cdot (-1)^r x^{r-1} \prod_{i=2}^r \theta_i & & \\ (\text{Co})\text{Ker}_{M'[\frac{1}{x}]} \tilde{\nabla}_{\frac{1}{x}}(\prod_{i=2}^r(\frac{1}{x}-\frac{1}{\theta_i}))_{\partial_{\frac{1}{x}}} & & \end{array}$$

In terms of the open cover $\mathcal{U} = \{U_0, U_1\}$ of $\mathbb{P}_S^1$ with $U_0 := \mathbb{A}_S^1$, $U_1 := \mathbb{P}_S^1 \setminus (\{0\} \times S)$, ($X \subseteq U_{01} := \mathbb{A}_S^1 \setminus (\{0\} \times S)$), this diagram may be rewritten as follows. For $i = 0, 1$, the maps

$$(27.2.18) \quad \begin{array}{ccc} \underline{h}^i \bar{f}_{|U_0*}(\Omega_{U_0/S}^\bullet(\log(U_0 \setminus X)) \otimes \tilde{\mathcal{E}}_{|U_0}) & & \\ \downarrow & & \\ \underline{h}^i \bar{f}_{|U_{01}*}(\Omega_{U_{01}/S}^\bullet(\log(U_{01} \setminus X)) \otimes \tilde{\mathcal{E}}_{|U_{01}}) & \longrightarrow & \underline{h}^i f_*(\Omega_{X/S}^\bullet \otimes \mathcal{E}) \xrightarrow{\cong} R_{\text{DR}}^i f_*(\mathcal{E}, \nabla) \\ \uparrow & & \\ \underline{h}^i \bar{f}_{|U_1*}(\Omega_{U_1/S}^\bullet(\log(U_1 \setminus X)) \otimes \tilde{\mathcal{E}}_{|U_1}) & & \end{array}$$

are isomorphisms of $\mathcal{O}_S$ -modules (in fact, isomorphisms in $\mathbf{MIC}(S)$).

Let us now compute $\mathbf{R}^i \bar{f}_*(\Omega_{\mathbb{P}_S^1/S}^\bullet(\log Z) \otimes \tilde{\mathcal{E}})$, using the (2 by 2) Čech bi-complex of $\mathcal{O}_S$ -modules $C^\bullet(\mathcal{U}, \Omega_{\mathbb{P}_S^1/S}^\bullet(\log Z) \otimes \tilde{\mathcal{E}})$. The differentials $d_2^{0,0}$ and $d_2^{1,0}$ are given by $\tilde{\nabla}$, while $d_1^{0,0}$ (resp. $d_1^{0,1}$) is induced by $(j_0^*, -j_1^*)$ (resp. $(-j_0^*, j_1^*)$), where j_k stands for the inclusion $U_{01} \subseteq U_k$. For $k = 0, 1$, (27.2.17) identifies $\text{Ker } d_2^{k,0}$ (resp. $\text{Coker } d_2^{k,0}$) with $2-k$ copies of $R_{\text{DR}}^0 f_*(\mathcal{E}, \nabla)$ (resp. $R_{\text{DR}}^1 f_*(\mathcal{E}, \nabla)$), and for the first spectral sequence of the bicomplex, we find $\mathbf{I}_2^{0,0} \cong R_{\text{DR}}^0 f_*(\mathcal{E}, \nabla)$, $\mathbf{I}_2^{0,1} \cong R_{\text{DR}}^1 f_*(\mathcal{E}, \nabla)$, $\mathbf{I}_2^{1,0} = \mathbf{I}_2^{1,1} = 0$. Applying [29, chap. XV, sect. 6, case 4] we obtain:

Proposition. *The morphism of $\mathcal{O}_S$ -modules*

$$(27.2.19) \quad \mathbf{R}^i \bar{f}_*(\Omega_{\mathbb{P}_S^1/S}^\bullet(\log Z) \otimes \tilde{\mathcal{E}}) \longrightarrow \mathbf{R}^i \bar{f}_*(j_*(\Omega_{X/S}^\bullet \otimes \mathcal{E})) \cong R_{\text{DR}}^i f_*(\mathcal{E}, \nabla),$$

induced by the open S -embedding $j : X \hookrightarrow \mathbb{P}_S^1$, is an isomorphism.

Since $\mathbf{R}^i \bar{f}_*(\Omega_{\mathbb{P}_S^1/S}^*(\log Z) \otimes \tilde{\mathcal{E}})$ is the abutment of a spectral sequence starting with $E_1^{pq} = R^q \bar{f}_*(\Omega_{\mathbb{P}_S^1/S}^p(\log Z) \otimes \tilde{\mathcal{E}})$, the coherence of $R_{\text{DR}}^i f_*(\mathcal{E}, \nabla)$ now follows from the stability of coherence under direct image by the projective morphism $\bar{f}$. $\square_2$

27.2.20 (Base change). Combining [27.2.1](#) and [26.2.7](#) we conclude that for any fibered square of smooth k -varieties

$$\begin{array}{ccc} X^\# & \xrightarrow{u^\#} & X \\ f^\# \downarrow & & \downarrow f \\ S^\# & \xrightarrow{u} & S \end{array}$$

with f a rational elementary fibration, and for any regular object $(\mathcal{E}, \nabla)$ of $\text{MIC}(X)$, the base change morphism

$$u^* R_{\text{DR}}^i f_*(\mathcal{E}, \nabla) \longrightarrow R_{\text{DR}}^i f_*^\# u^{\#*}(\mathcal{E}, \nabla)$$

is an isomorphism for $i = 0, 1$.

In fact it is immediate to generalize [27.2.1](#) (coherence) and this base change property to the case where f is a coordinatized elementary fibration.

27.3 Regularity and exponents of the cokernel of a regular connection

27.3.1. In this section, we establish that for any rational elementary fibration

$$(27.3.2) \quad \begin{array}{ccc} X & \xleftarrow{j} & \bar{X} \\ & \searrow f & \swarrow \bar{f} \\ & S & \end{array}$$

and for any locally free finite-rank $\mathcal{O}_X$ -module $(\mathcal{E}, \nabla)$ with regular integrable connection, the Gauss-Manin connection on $R_{\text{DR}}^i f_*(\mathcal{E}, \nabla)$ is regular for $i = 0, 1$; moreover, if the exponents of $(\mathcal{E}, \nabla)$ belong to a certain $\mathbb{Q}$ -space Δ ($\mathbb{Q} \subseteq \Delta \subseteq k$), so do the exponents of $R_{\text{DR}}^i f_*(\mathcal{E}, \nabla)$. This is the key point in the proof of [27.1.2](#) (and [27.1.3](#)).

We notice that the case $i = 0$ is an easy consequence of [23.1.3](#).

27.3.3. By Theorem [13.1.5](#) and by base change [27.2.20](#) we may assume that S is the complement of a finite set of points Σ in a smooth projective curve $\bar{S}$. By the classical theory of embedded resolution of curves (Zariski), there exist a smooth

projective surface $\overline{\overline{X}}$ containing $\overline{X}$, such that $\overline{Z} := \overline{\overline{X}} \setminus X$ is a union $\bigcup_{i=1}^s \overline{Z}_i$ of smooth connected curves with normal crossings, and a projective morphism $\overline{\overline{f}} : \overline{\overline{X}} \rightarrow \overline{S}$ extending $\overline{f}$. By rearranging the $\overline{Z}_j$, we may assume that $\overline{Z}_j \cap \overline{X} \cong Z_j$, for $i \leq r$, while $\overline{\overline{f}}^{-1}(\Sigma) = \sum_{j=r+1}^s e_j \overline{Z}_j$, for some positive integers e_j . Following Katz [62], we shall construct a locally free extension of $R_{\text{DR}}^i f_*(\mathcal{E}, \nabla)$ on $\overline{S}$ and an extension of the Gauss-Manin connection with logarithmic poles along Σ .

27.3.4. Let us set

$$\Omega_{\overline{X}/\overline{S}}^p(\log \overline{Z}/\Sigma) = \bigwedge_{\mathcal{O}_{\overline{X}}}^p (\Omega_{\overline{X}}^1(\log \overline{Z})) / \overline{\overline{f}}^* \Omega_{\overline{S}}^1(\log \Sigma).$$

Let $(\tilde{\mathcal{E}}, \tilde{\nabla})$ be a locally free extension of $(\mathcal{E}, \nabla)$ on $\overline{\overline{X}}$ with logarithmic poles along $\overline{Z}$ (see Section 11.1). Let us filter the logarithmic de Rham complex $\Omega_{\overline{X}}^\bullet(\log \overline{Z}) \otimes_{\mathcal{O}_{\overline{X}}} \tilde{\mathcal{E}}$ by

$$F^p = \text{image of } \overline{\overline{f}}^* \Omega_{\overline{S}}^p(\log \Sigma) \otimes_{\mathcal{O}_{\overline{X}}} \Omega_{\overline{X}/\overline{S}}^{\bullet-p}(\log \overline{Z}) \otimes_{\mathcal{O}_{\overline{X}}} \tilde{\mathcal{E}},$$

so that

$$gr^p = F^p / F^{p+1} = \overline{\overline{f}}^* \Omega_{\overline{S}}^p(\log \Sigma) \otimes_{\mathcal{O}_{\overline{X}}} \Omega_{\overline{X}/\overline{S}}^{\bullet-p}(\log \overline{Z}/\Sigma) \otimes_{\mathcal{O}_{\overline{X}}} \tilde{\mathcal{E}}.$$

We define

$$R_{\text{DR}}^i \overline{\overline{f}}_*(\tilde{\mathcal{E}}, \tilde{\nabla}) = \mathbf{R}^i \overline{\overline{f}}_*(\Omega_{\overline{X}/\overline{S}}^\bullet(\log \overline{Z}/\Sigma) \otimes_{\mathcal{O}_{\overline{X}}} \tilde{\mathcal{E}}) / \{\mathcal{O}_{\overline{S}} - \text{torsion}\},$$

which is a locally free $\mathcal{O}_{\overline{S}}$ -module of finite rank because $\overline{\overline{f}}$ is projective and $\overline{S}$ is a curve. Moreover,

$$\mathbf{R}^i \overline{\overline{f}}_*(\Omega_{\overline{X}/\overline{S}}^\bullet(\log \overline{Z}/\Sigma) \otimes_{\mathcal{O}_{\overline{X}}} \tilde{\mathcal{E}})|_S \cong \mathbf{R}^i \overline{\overline{f}}_*(\Omega_{\overline{X}/\overline{S}}^\bullet(\log Z) \otimes_{\mathcal{O}_{\overline{X}}} \tilde{\mathcal{E}}|_{\overline{X}}) \cong R_{\text{DR}}^i f_*(\mathcal{E}, \nabla),$$

according to [27.2.19], so that $R_{\text{DR}}^i \overline{\overline{f}}_*(\tilde{\mathcal{E}}, \tilde{\nabla})$ is a locally free extension of $R_{\text{DR}}^i f_*(\mathcal{E}, \nabla)$. On the other hand, an extension of the Gauss-Manin connection

$$\mathbf{R}^i \overline{\overline{f}}_*(\Omega_{\overline{X}/\overline{S}}^\bullet(\log \overline{Z}/\Sigma) \otimes_{\mathcal{O}_{\overline{X}}} \tilde{\mathcal{E}}) \longrightarrow \Omega_{\overline{S}}^1(\log \Sigma) \otimes_{\mathcal{O}_{\overline{S}}} \mathbf{R}^i \overline{\overline{f}}_*(\Omega_{\overline{X}/\overline{S}}^\bullet(\log \overline{Z}/\Sigma) \otimes_{\mathcal{O}_{\overline{X}}} \tilde{\mathcal{E}})$$

is provided by the coboundary map of the long exact sequence

$$0 \longrightarrow gr^1 \longrightarrow F^0 / F^2 \longrightarrow gr^0 \longrightarrow 0$$

(since $\Omega_{\overline{S}}^1(\log \Sigma) \otimes_{\mathcal{O}_{\overline{S}}} \mathbf{R}^i \overline{\overline{f}}_*(\Omega_{\overline{X}/\overline{S}}^\bullet(\log \overline{Z}/\Sigma) \otimes_{\mathcal{O}_{\overline{X}}} \tilde{\mathcal{E}})$ may be identified with $\mathbf{R}^{i+1} \overline{\overline{f}}_*(gr^1)$). That map factors through torsion, and induces an extension of the Gauss-Manin connection on $R_{\text{DR}}^i \overline{\overline{f}}_*(\tilde{\mathcal{E}}, \tilde{\nabla})$, with logarithmic poles along Σ .

27.3.5. Katz's argument [62] 7] applies without change to our situation and shows that the indicial polynomial of $R_{\text{DR}}^i \bar{f}_*^{\approx}(\mathcal{E}, \nabla)$ at $s \in \Sigma$ divides

$$(27.3.6) \quad \prod_{j \in J} \prod_{f_j=0}^{e_j-1} \text{ind}_j(e_j X - f_j),$$

where ind_j is the indicial polynomial of $(\mathcal{E}, \nabla)$ at $\bar{Z}_j$, $\bar{f}^{-1}(s) = \sum_{j \in J} e_j \bar{Z}_j$ (so, $J \subseteq \{r+1, \dots, s\}$). This completes the proof of the statement in [27.3].

Remarks 27.3.7. (i) A closer examination of Katz's argument allows to replace [27.3.6] by the least common multiple of the indicial polynomials $\text{ind}_j(e_j X - f_j)$, for $j \in J$ and $f_j \in \{0, 1, \dots, e_j - 1\}$.

(ii) If $k \subseteq \mathbb{C}$ and τ is the canonical section $k/\mathbb{Z} \rightarrow k$, defined by $\text{Re}(\tau(z)) \in [0, 1[$, and if $(\mathcal{E}, \nabla)$ is the τ -extension of $(\mathcal{E}, \nabla)$ on $\bar{X}$, then [27.3.5] shows that $R_{\text{DR}}^i \bar{f}_*^{\approx}(\mathcal{E}, \nabla)$ is the τ -extension of $R_{\text{DR}}^i f_*(\mathcal{E}, \nabla)$ on $\bar{S}$; indeed $\text{Re}(z) \in [0, 1[$ implies that $\text{Re}(\frac{z+f_j}{e_j}) \in [0, 1[$.

(iii) The arguments [27.3.3] [27.3.4] [27.3.5] work as well for any elementary fibration, not necessarily rational, over a curve.