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Some preconditioning techniques for a class of double saddle point problems

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Summary

In this paper, we describe and analyze the spectral properties of several exact block preconditioners for a class of double saddle point problems. Among all these, we consider an inexact version of a block triangular preconditioner providing extremely fast convergence of the (F)GMRES method. We develop a spectral analysis of the preconditioned matrix showing that the complex eigenvalues lie in a circle of center $(1, 0)$ and radius 1, while the real eigenvalues are described in terms of the roots of a third order polynomial with real coefficients. Numerical examples are reported to illustrate the efficiency of inexact versions of the proposed preconditioners, and to verify the theoretical bounds.

KEYWORDS

double saddle point problems, Krylov subspace methods, preconditioning

1 | INTRODUCTION

This paper is concerned with some block preconditioners for the numerical solution of a large and sparse linear system of equations of *double saddle point* type of the form

$$\mathcal{A}w \equiv \begin{pmatrix} A & B^T & 0 \\ B & 0 & C^T \\ 0 & C & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \equiv b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite (SPD) matrix, $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{l\times m}$ have full row rank, $f \in \mathbb{R}^n$, $g \in \mathbb{R}^m$ and $h \in \mathbb{R}^l$ are given vectors. Even if invertibility of $\mathcal{A}$ is guaranteed upon weaker conditions on matrices B and C (see e.g., Reference 1), however, the previous statement would make the spectral analysis more readable, especially in the inexact case.

Such linear systems arise in several scientific applications including constrained least squares problems,[2] constrained quadratic programming,[3] and magma-mantle dynamics,[4] to mention a few; see, for example, References 5–7. Similar block structures arise for example, in liquid crystal director modeling or in the coupled Stokes-Darcy problem, and the preconditioning of such linear systems has been considered in References 8–12.

Obviously, the coefficient matrix of system (1) is symmetric and can be considered as a 2×2 block matrix.[13] Due to the fact that these saddle point matrices are typically large and sparse, their iterative solution is recommended for example, by Krylov subspace iterative methods.[14] To improve the efficiency of iterative methods, some preconditioning techniques are employed, all combining in different ways the following Schur complement matrices:

$$S = BA^{-1}B^T, \ X = CS^{-1}C^T \tag{2}$$

and, in some cases, also $\hat{A}$, $\hat{S}$ and $\hat{X}$, which are defined as SPD approximations of A, S, and X, respectively.

To iteratively solve the linear system (1), different preconditioning methods have been investigated and studied in the literature. In Reference 15, Huang developed the block diagonal preconditioner $\mathcal{P}_D$ and its inexact version $\hat{\mathcal{P}}_D$ which are of the forms

$$\mathcal{P}_D = \begin{pmatrix} A & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & X \end{pmatrix}, \qquad \hat{\mathcal{P}}_D = \begin{pmatrix} \hat{A} & 0 & 0 \\ 0 & \hat{S} & 0 \\ 0 & 0 & \hat{X} \end{pmatrix}. \tag{3}$$

In addition, an inexact version of the diagonal preconditioner $\hat{\mathcal{P}}_D$ has been studied in Reference 16. Cao[13] considered the equivalent linear system

$$\mathcal{A}w \equiv \begin{pmatrix} A & B^T & 0 \\ -B & 0 & -C^T \\ 0 & C & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f \\ -g \\ h \end{pmatrix} \equiv b, \tag{4}$$

and proposed the shift-splitting iteration of the form

$$\frac{1}{2}(\alpha I + \mathcal{A})w^{(k+1)} = \frac{1}{2}(\alpha I - \mathcal{A})w^{(k)} + b, \tag{5}$$

which leads to the preconditioner

$$\mathcal{P}_{SS} = \frac{1}{2} \begin{pmatrix} \alpha I + A & B^T & 0 \\ -B & \alpha I & -C^T \\ 0 & C & \alpha I \end{pmatrix}, \tag{6}$$

where α is a positive constant and I is the identity matrix of appropriate size. In addition, a relaxed version of the shift-splitting preconditioner has been considered by dropping the shift parameter in the (1, 1) block of $\mathcal{P}_{SS}$.

In Reference 1, three exact block preconditioners for solving (1) have been introduced and analyzed which are defined as

$$\mathcal{P}_1 = \begin{pmatrix} A & 0 & 0 \\ B & -S & C^T \\ 0 & 0 & -X \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} A & 0 & 0 \\ B & -S & C^T \\ 0 & 0 & X \end{pmatrix}, \quad \mathcal{P}_3 = \begin{pmatrix} A & B^T & 0 \\ B & -S & 0 \\ 0 & 0 & -X \end{pmatrix}. \tag{7}$$

Moreover, it is shown that the preconditioned matrices corresponding to the above preconditioners only have at most three distinct eigenvalues. More recently, Wang and Li[17] have proposed an exact and inexact parameterized block symmetric positive definite preconditioner for solving the double saddle point problem (1). In Reference 18 also a block preconditioner depending on a parameter is analyzed in both its exact and inexact formulations. Finally an SPD preconditioner is proposed in Reference 19.

In this paper, we will describe several other block preconditioning approaches to be employed within a Krylov subspace method for the solution of a linear system of equations (1),

$$\mathcal{Q}_1 = \begin{pmatrix} A & B^T & 0 \\ 0 & -S & 0 \\ 0 & 0 & X \end{pmatrix}, \quad \mathcal{Q}_2 = \begin{pmatrix} A & B^T & 0 \\ 0 & S & C^T \\ 0 & 0 & -X \end{pmatrix}, \quad \mathcal{Q}_{3,\pm} = \begin{pmatrix} A & B^T & 0 \\ 0 & -S & C^T \\ 0 & 0 & \pm X \end{pmatrix}, \tag{8}$$

and the block preconditioners of the forms

$$Q_{4,\pm} = \begin{pmatrix} A & B^T & 0 \\ B & 0 & 0 \\ 0 & C & \pm X \end{pmatrix}, \quad Q_5 = \begin{pmatrix} A & B^T & 0 \\ B & 0 & 0 \\ 0 & 0 & X \end{pmatrix}. \tag{9}$$

We analyze the spectral distribution of the corresponding preconditioned matrices which in all cases have at most three distinct eigenvalues thus guaranteeing the finite termination of for example, the GMRES iterative method. In realistic problems, the proposed preconditioners can not be used exactly since their application requires

1. Solution of a system with A,
2. Explicit computation of $S = BA^{-1}B^T$,
3. Solution of a system with S,
4. Explicit computation of $X = CS^{-1}C^T$,
5. Solution of a system with X.

Particularly, steps 2 and 4 require inversion of the (possibly sparse) matrices A and S. The practical application of the described preconditioners is then subjected to the approximation of matrices A, S, and X with $\hat{A}$, $\hat{S}$, and $\hat{X}$, respectively.

To measure the effects of such approximation on the spectral properties of the preconditioned matrix we considered the block triangular preconditioner $Q_{3,+}$, and give bound on the complex and real eigenvalues in terms of the (real and positive) eigenvalues of $\hat{A}^{-1}A$, $\hat{S}^{-1}\tilde{S}$, $\hat{X}^{-1}\tilde{X}$, where $\tilde{S}$ and $\tilde{X}$ are further approximations of $\hat{S}$ and $\hat{X}$, respectively. Even though some realistic double saddle point linear systems have at least one nonzero $(2,2)$ and/or $(3,3)$ diagonal block, we are confident that the spectral analysis we will develop on this simpler case could be extended to more general double saddle point linear systems.

The outline of this work is described as follows. In section 2, we derive and analyze some exact block preconditioners for solving double saddle point problem (1). In section 3, we test the inexact versions of the proposed preconditioners on a test case in combination with the FGMRES iterative method. We then focus in section 4 on the spectral analysis of the block preconditioner which revealed the most efficient. Following an idea presented in References 20,21, bounds on truly complex as well as on real eigenvalues are developed for this preconditioner and compared with the actual spectral distribution. Some further results about the real eigenvalues of a simplified version of the triangular preconditioner are developed and tested in section 4.1. In section 5 we show the results of two of the previously described preconditioners in the GMRES solution of a realistic problem obtained by *enlarging* the discretized Stokes problem as in Reference 17. Finally, we state some conclusions in section 6.

2 | BLOCK PRECONDITIONERS AND EIGENVALUE ANALYSIS

Let us consider the preconditioners in (8) and (9). We observe that these preconditioners are nonsingular since A is symmetric positive definite and both B and C have full row rank. In the following, the eigenvalues of the preconditioned matrices corresponding to the proposed preconditioners are determined. The notations $\sigma(.)$ and $||.||$ denote the set of all eigenvalues of a matrix and the Euclidean norm of a vector, respectively. We use $\Re(\lambda)$ and $\Im(\lambda)$ to denote the real and imaginary parts of a complex eigenvalue λ.

Theorem 1. *Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{l\times m}$ are matrices with full row rank. Then the preconditioner Q_1 for $\mathcal{A}$ satisfies*

$$\sigma(\mathcal{A}Q_1^{-1}) = \left\{1, \frac{1}{2}(1 \pm \sqrt{3}i)\right\}.$$

Proof. First we must compute Q_1^{-1}. An easy calculation yields

$$\mathcal{A}Q_1^{-1} = \begin{pmatrix} I & 0 & 0 \\ BA^{-1} & I & C^TX^{-1} \\ 0 & -CS^{-1} & 0 \end{pmatrix}. \tag{10}$$

We now determine the eigenvalues of the preconditioned matrix $\mathcal{A}\mathcal{Q}_1^{-1}$ by using the Laplace expansion. Therefore, the characteristic polynomial of $\mathcal{A}\mathcal{Q}_1^{-1}$ is given by

$$q(\lambda) = \det(\lambda I - \mathcal{A}\mathcal{Q}_1^{-1}) = (\lambda - 1)^n \begin{vmatrix} (\lambda - 1)I & -C^T X^{-1} \\ CS^{-1} & \lambda I \end{vmatrix}. \tag{11}$$

Clearly, $\lambda = 1$ is an eigenvalue of $\mathcal{A}\mathcal{Q}_1^{-1}$ with algebraic multiplicity at least n. To determine the rest of eigenvalues, we seek $\lambda \neq 1$, x_2 and x_3 satisfying

$$(\lambda - 1)x_2 - C^T X^{-1} x_3 = 0, \tag{12}$$

$$CS^{-1} x_2 + \lambda x_3 = 0. \tag{13}$$

Computing $x_2 = \frac{1}{\lambda - 1} CS^{-1} x_3$ from Equation (12) and substituting the value x_2 into Equation (13), we get

$$\left(\lambda^2 I - \lambda I + I\right) x_3 = 0 \tag{14}$$

Note that the vector x_3 must be nonzero, otherwise if $x_3 = 0$, then $x_2 = 0$, and we saw that $x_1 = 0$ if $\lambda \neq 1$. Without loss of generality, we can assume that $x_3^* x_3 = 1$. Multiplying Equation (14) on the left by x_3^*, we obtain

$$\lambda^2 - \lambda + 1 = 0. \tag{15}$$

The roots of (15) are equal to $\lambda = \frac{1}{2}(1 \pm \sqrt{3}i)$, which completes the proof. ■

Remark 1. It is easy to verify that the preconditioned matrix $\mathcal{T} = \mathcal{A}\mathcal{Q}_1^{-1}$ satisfies the following polynomial equation

$$(\mathcal{T} - \mathcal{I})(\mathcal{T}^2 - \mathcal{T} + \mathcal{I}) = 0.$$

Since the above relation can be factorized into distinct linear factors, we conclude that $\mathcal{T}$ is diagonalizable and has at most three distinct eigenvalues $1, \frac{1}{2}(1 \pm \sqrt{3}i)$.

Theorem 2. *Suppose that $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite and $B \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{l \times m}$ are matrices with full row rank. Then the preconditioner $\mathcal{Q}_2$ for $\mathcal{A}$ satisfies*

$$\sigma(\mathcal{A}\mathcal{Q}_2^{-1}) = \{\pm 1, \pm i\}.$$

Proof. Straightforward calculations reveal that

$$\mathcal{Q}_2^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}B^T S^{-1} & -A^{-1}B^T S^{-1} C^T X^{-1} \\ 0 & S^{-1} & S^{-1} C^T X^{-1} \\ 0 & 0 & -X^{-1} \end{pmatrix}. \tag{16}$$

It follows from (16) that

$$\mathcal{A}\mathcal{Q}_2^{-1} = \begin{pmatrix} I & 0 & 0 \\ BA^{-1} & -I & -2C^T X^{-1} \\ 0 & CS^{-1} & I \end{pmatrix}. \tag{17}$$

To proceed we use the Laplace expansion to determine the eigenvalues of $\mathcal{A}\mathcal{Q}_2^{-1}$. Therefore, the characteristic polynomial of $\mathcal{A}\mathcal{Q}_2^{-1}$ is given by

$$\hat{q}(\lambda) = \det(\lambda I - \mathcal{A}\mathcal{Q}_2^{-1}) = (\lambda - 1)^n \begin{vmatrix} (\lambda + 1)I & 2C^T X^{-1} \\ -CS^{-1} & (\lambda - 1)I \end{vmatrix}. \tag{18}$$

It is clear that $\lambda = 1$ is an eigenvalue of $\mathcal{A}\mathcal{Q}_2^{-1}$ with algebraic multiplicity at least n. To find the remaining eigenvalues, we seek $\lambda \neq 1$, x_2 and x_3 satisfying

$$(\lambda + 1)x_2 + 2C^T X^{-1} x_3 = 0, \tag{19}$$

$$-CS^{-1}x_2 + (\lambda - 1)x_3 = 0. \tag{20}$$

Notice that $x_2 \neq 0$, otherwise if $x_2 = 0$, then $x_3 = 0$ from Equation (20), and we saw that $x_1 = 0$ if $\lambda \neq 1$. From Equation (20), we derive $x_3 = \frac{1}{\lambda - 1} CS^{-1}x_2$. If $x_2 \in \ker(CS^{-1})$, then $\lambda = -1$ is an eigenvalue of $\mathcal{A}\mathcal{Q}_2^{-1}$. Thus, it can be deduced that $(0; x_2; 0)$ is an eigenvector associated with $\lambda = -1$, where x_2 is an arbitrary vector. In the sequel we assume that $\lambda \neq -1$. Substituting the value x_3 into Equation (19), we get

$$\left((\lambda^2 - 1)I + 2C^T X^{-1} CS^{-1}\right) x_2 = 0. \tag{21}$$

We normalize x_2 such that $x_2^* x_2 = 1$, and multiply Equation (21) by x_2 on the left to obtain

$$\lambda^2 - 1 + 2x_2^* C^T X^{-1} CS^{-1} x_2 = 0. \tag{22}$$

From the above relation, the eigenvalue λ can be expressed

$$\lambda = \pm\sqrt{1 - 2x_2^* C^T X^{-1} CS^{-1} x_2}. \tag{23}$$

It is easy to verify that $C^T X^{-1} CS^{-1}$ is a projector onto $\mathcal{R}(C^T X^{-1}) = \mathcal{R}(C^T)$, where $\mathcal{R}$ denotes the range of a matrix. We rewrite the relation (19) as

$$x_2 = \left(\frac{-2}{\lambda + 1}\right) C^T X^{-1} x_3, \tag{24}$$

and hence we observe $x_2 \in \mathcal{R}(C^T X^{-1})$. Consequently, we have

$$x_2^* C^T X^{-1} CS^{-1} x_2 = 1. \tag{25}$$

It follows that $\mathcal{A}\mathcal{Q}_2^{-1}$ has eigenvalues $\lambda = \pm i$, and we have proved the theorem. ■

Remark 2. From Equations (19), (20) and the proof of Theorem 2, it can be seen that $\lambda = -1$ may or may not be an eigenvalue of $\mathcal{A}\mathcal{Q}_2^{-1}$. We observed that $\lambda = -1$ is an eigenvalue if $x_2 \in \ker(CS^{-1})$. Conversely, suppose that $\lambda = -1$ is an eigenvalue of $\mathcal{A}\mathcal{Q}_2^{-1}$. From Equations (19) and (20), we have $x_3 = 0$ and then $CS^{-1}x_2 = 0$ which means that $x_2 \in \ker(CS^{-1})$. This condition is necessary and sufficient for $\lambda = -1$ to be an eigenvalue of $\mathcal{A}\mathcal{Q}_2^{-1}$ with associated eigenvector of the form $(0; x_2; 0)$.

Remark 3. It is easy to check that the preconditioned matrix $\mathcal{F} = \mathcal{A}\mathcal{Q}_2^{-1}$ satisfies

$$(\mathcal{F} - \mathcal{I})(\mathcal{F} + \mathcal{I})(\mathcal{F}^2 + \mathcal{I}) = 0. \tag{26}$$

From the relation (26), we can conclude that $\mathcal{F}$ is diagonalizable and has at most four distinct eigenvalues $\pm 1, \pm i$.

Remark 4. According to the following partitioning

$$\left(\begin{array}{cc|c} * & * & 0 \\ * & * & * \\ \hline 0 & 0 & * \end{array}\right),$$

it is clear that the preconditioners $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_3$ and $\mathcal{Q}_{3,+}$ have the same structure and all are in the class of block triangular preconditioners. The eigenvalues of the preconditioned matrices $\mathcal{A}\mathcal{Q}_{3,\pm}^{-1}$ are also provided here for completeness.

Theorem 3. *Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{l\times m}$ have full row rank. Then the eigenvalues of the preconditioned matrices $\mathcal{A}\mathcal{Q}_{3,-}^{-1}$ and $\mathcal{A}\mathcal{Q}_{4,-}^{-1}$ are either 1 or -1 while those of $\mathcal{A}\mathcal{Q}_{3,+}^{-1}$ and $\mathcal{A}\mathcal{Q}_{4,+}^{-1}$ are all 1.*

Proof. After straightforward computations, we can obtain

$$\mathcal{A}\mathcal{Q}_{3,-}^{-1} = \begin{pmatrix} A & B^T & 0 \\ B & 0 & C^T \\ 0 & C & 0 \end{pmatrix}\begin{pmatrix} A^{-1} & A^{-1}B^TS^{-1} & A^{-1}B^TS^{-1}C^TX^{-1} \\ 0 & -S^{-1} & -S^{-1}C^TX^{-1} \\ 0 & 0 & -X^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ BA^{-1} & I & 0 \\ 0 & -CS^{-1} & -I \end{pmatrix},$$

and

$$\mathcal{Q}_{4,-}^{-1}\mathcal{A} = \begin{pmatrix} A^{-1} - A^{-1}B^TS^{-1}BA^{-1} & A^{-1}B^TS^{-1} & 0 \\ S^{-1}BA^{-1} & -S^{-1} & 0 \\ X^{-1}CS^{-1}BA^{-1} & -X^{-1}CS^{-1} & -X^{-1} \end{pmatrix}\begin{pmatrix} A & B^T & 0 \\ B & 0 & C^T \\ 0 & C & 0 \end{pmatrix} = \begin{pmatrix} I & 0 & A^{-1}B^TS^{-1}C^T \\ 0 & I & -S^{-1}C^T \\ 0 & 0 & -I \end{pmatrix},$$

An analogous derivation yields

$$\mathcal{A}\mathcal{Q}_{3,+}^{-1} = \begin{pmatrix} I & 0 & 0 \\ BA^{-1} & I & 0 \\ 0 & -CS^{-1} & I \end{pmatrix}, \quad \text{and} \quad \mathcal{Q}_{4,+}^{-1}\mathcal{A} = \begin{pmatrix} I & 0 & A^{-1}B^TS^{-1}C^T \\ 0 & I & -S^{-1}C^T \\ 0 & 0 & I \end{pmatrix}.$$

which provides the thesis of the Theorem. ∎

Remark 5. It is easy to check that the degree of the minimum polynomial of the preconditioned matrices $\mathcal{A}\mathcal{Q}_{3,\pm}^{-1}$ is 3 while that of $\mathcal{Q}_{4,\pm}^{-1}\mathcal{A}$ is 2, therefore, the GMRES method will reach the exact solution in at most three and two steps, respectively.

Theorem 4. *Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{l\times m}$ are matrices with full row rank. Then the preconditioner $\mathcal{Q}_5$ for $\mathcal{A}$ satisfies*

$$\sigma(\mathcal{A}\mathcal{Q}_5^{-1}) \in \left\{1, \frac{1}{2}(1 \pm \sqrt{3}i)\right\}.$$

Proof. By simple calculations, we can obtain

$$\mathcal{Q}_5^{-1} = \begin{pmatrix} A^{-1} - A^{-1}B^TS^{-1}BA^{-1} & A^{-1}B^TS^{-1} & 0 \\ S^{-1}BA^{-1} & -S^{-1} & 0 \\ 0 & 0 & X^{-1} \end{pmatrix}.$$

We readily verify that

$$\mathcal{A}\mathcal{Q}_5^{-1} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & C^TX^{-1} \\ CS^{-1}BA^{-1} & -CS^{-1} & 0 \end{pmatrix}.$$

The rest of the proof is similar to that of Theorem 1; we omit the details here. It can also be shown that the preconditioned matrix $\mathcal{J} = \mathcal{A}\mathcal{Q}_5^{-1}$ satisfies

$$(\mathcal{J} - \mathcal{I})(\mathcal{J}^2 - \mathcal{J} + \mathcal{I}) = 0,$$

and it follows that $\mathcal{J}$ is diagonalizable and has three distinct eigenvalues $1, \frac{1}{2}(1 \pm \sqrt{3}i)$. ∎

Finally, we mention that although the preconditioners type $\mathcal{P}$ and $\mathcal{Q}$ are theoretically powerful, they are not practical in real-world problems. Solving linear systems involving S and X can be prohibitive. However, we may be able to devise effective inexact version of these preconditioners for $\mathcal{A}$ by approximating the Schur complement matrices. To apply the proposed type Q preconditioners within a Krylov subspace method, we require to solve the linear system of equations with the coefficient matrices A, $S = BA^{-1}B^T$ and $X = CS^{-1}C^T$. Instead of exactly performing solves with S and X, we approximate S and X by matrices $\hat{S}$ and $\hat{X}$, respectively, and then the linear systems involving these matrices can be iteratively solved.

3 | EXPERIMENTAL COMPARISON OF THE PRECONDITIONERS

In this section, we present the numerical solution of a test case solved by Flexible GMRES (FGMRES) with no restart, using the previously described preconditioners. Since the application of all the previously analyzed preconditioners is based on the inversion of matrix A and the explicit construction and inversion of Schur complements S and X, we selected to employ an *inexact* variant of all these matrices. This will be described below in this Section and sketched in Algorithms 1 and 2.

The comparisons will be carried on in terms of the number of FGMRES iterations (represented by ITS) and elapsed CPU time in seconds (represented by CPU). We also provide the norm of error vectors denoted as $\text{ERR} = \|w^{(k)} - w^*\|_2/\|w^*\|_2$. The initial guess is set to be the zero vector and the iterations will be stopped whenever

$$\|b - \mathcal{A}w^{(k)}\|_2/\|b\|_2 < \texttt{tol},$$

with the tolerance `tol` selected as will be explained below.

In all cases, the right-hand side vector b is computed after selecting the exact solution of (1) as

1. $w = e \equiv (1, 1, \ldots, 1)^T \in \mathbb{R}^{n+m+l}$.
2. w a random vector of the appropriate dimension.

The numerical experiments presented in this work have been carried out on a computer with an Intel Core i7-1185G7 CPU @ 3.00GHz processor and 16 GB RAM using MATLAB 2022a.

Example 1 ([1,15]). We consider the linear system of equations (1) for which

$$A = \text{diag}\left(2W^TW + D_1, D_2, D_3\right) \in \mathbb{R}^{n\times n},$$

is a block diagonal matrix;

$$B = [E, -I_{2p_1}, I_{2p_2}] \in \mathbb{R}^{m\times n}, \quad \text{and} \quad C = E^T \in \mathbb{R}^{l\times m},$$

are both full row rank matrices where $p_1 = p^2$, $p_2 = p(p+1)$; $W = (w_{ij}) \in \mathbb{R}^{p_2\times p_2}$ with $w_{ij} = e^{-2((i/3)^2+(j/3)^2)}$; $D_1 = I_{p_2}$ is the identity matrix; $D_i = \text{diag}(d_j^{(i)}) \in \mathbb{R}^{2p_1\times 2p_1}$, $(i = 2, 3)$ are diagonal matrices with

$$d_j^{(2)} = \begin{cases} 1, & \text{for} \quad 1 \le j \le p_1, \\ 10^{-5}(j - p_1^2), & \text{for} \quad p_1 + 1 \le j \le 2p_1, \end{cases}$$
$$d_j^{(3)} = 10^{-5}(j + p_1^2), \qquad \text{for} \quad 1 \le j \le 2p_1;$$

and

$$E = \begin{pmatrix} E_1 \otimes I_p \\ I_p \otimes E_1 \end{pmatrix}, \qquad \text{with} \qquad E_1 = \begin{pmatrix} 2 - 1 & & & \\ & 2 - 1 & & \\ & & \ddots \ddots & \\ & & & 2 - 1 \end{pmatrix} \in \mathbb{R}^{p\times(p+1)}.$$

Algorithm 1. Approximations of the blocks before the FGMRES solution.

1: $\widehat{A} = \text{diag}(A)$,
2: $\widehat{S} = \text{tridiag}(B\widehat{A}^{-1}B^T)$
3: L_S is the exact bidiagonal factor of $\widehat{S}$ such that $\widehat{S} = L_S L_S^T$.
4: Compute $X_0 = C\text{diag}(\widehat{S})^{-1}C^T$ and M its IC factor with drop tolerance 10^{-4}.

Algorithm 2. Computation of $w = Q_{3,+}^{-1} r$.

1: Split vector r into $r = \begin{bmatrix} r_1^T & r_2^T & r_3^T \end{bmatrix}^T$
2: Solve system $CL_S^{-T}L_S^{-1}C^T w_3 = r_3$ by PCG with preconditioner MM^T and $\texttt{tol}_{\texttt{PCG}} = 10^{-4}$
3: $v = C^T w_3 - r_2$
4: $w_2 = L_S^{-T}L_S^{-1}v$
5: $y = r_1 - B^T w_2$
6: Solve $Aw_1 = y$
7: Define $w = \begin{bmatrix} w_1^T & w_2^T & w_3^T \end{bmatrix}^T$

To summarize we have $n = 5p^2 + p$, $m = 2p^2$ and $l = p^2 + p$ and the size of the double saddle point matrix is $n + m + l = 8p^2 + 2p$.

As the block approximations, we take $\widehat{S} = \text{tridiag}(B\widehat{A}^{-1}B^T)$, the tridiagonal part of S, where $\widehat{A} = \text{diag}(A)$ and $\widehat{X} = CL_S^{-T}L_S^{-1}C^T$ with L_S the exact bidiagonal factor of $\widehat{S}$.

In addition to the previously analyzed preconditioners we also present the results of an inexact variant of the block preconditioner considered in Reference 22 (here denoted as Q_{ASB}), where the Schur complement matrix is simply approximate with the identity matrix ($\widehat{S} \equiv I$), showing outstanding results in the solution of this Example. However, these results (convergence in a small number of iterations) are somehow distorted by the choice of the right-hand-side ($b = Ae$) and of the initial (zero) vector, which promote the performance of the preconditioner. This will be more clear in the numerical results, in which we tried Q_{ASB} also for a random exact solution.

Figure 1 plots the eigenvalue distribution of the preconditioned matrices with the approximation matrices $\widehat{S}$ and $\widehat{X}$. We observe from this figure that the preconditioned matrix with $Q_{3,+}$ has more clustered eigenvalues than the other ones, which can considerably improve the convergence rate of the Krylov subspace iterative methods. Preconditioners Q_1 and Q_5 display quite similar eigenvalue distributions (the ideal versions have the same eigenvalues). We then remove Q_1 from the numerical results in view of its slightly higher application cost in comparison with Q_5.

Plots of eigenvalue distributions of $Q_{3,+}$ and $Q_{4,+}$ are even closer. A possible explanation for this behavior stems from the block structure of both preconditioned matrices in the inexact case. In fact:

$$Q_{4,+} = \begin{pmatrix} I & 0 & 0 \\ B\hat{A}^{-1} & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \hat{A} & B^T & 0 \\ 0 & -\hat{S} & 0 \\ 0 & C & X \end{pmatrix}, \qquad \text{while} \quad Q_{3,+} = \begin{pmatrix} \hat{A} & B^T & 0 \\ 0 & -\hat{S} & C^T \\ 0 & 0 & X \end{pmatrix}$$

We remove $Q_{4,+}$ from the numerical results in view of its slightly higher application cost in comparison with $Q_{3,+}$.

All other proposed preconditioners display also negative eigenvalues, predicting slow GMRES convergence. Finally, the spectral distribution with the Q_{ASB} preconditioner, does not seem to be favorable, as it spreads over a wide (real) interval.

The linear system with $\widehat{S}$ is solved exactly by solving two bidiagonal systems with L_S and L_S^T, while the system with $\widehat{X}$ is solved, without forming explicitly matrix $\widehat{X}$, by the PCG method accelerated by the incomplete Cholesky factorization of $C\text{diag}(\widehat{S})^{-1}C^T$ with a drop tolerance $\tau = 10^{-4}$. The work to be done before the beginning of the FGMRES process is described in Algorithm 1 while the application of the preconditioner at each FGMRES iteration is sketched in Algorithm 2, for the preconditioner $Q_{3,+}$. We finally notice that also in the application of preconditioner Q_5 the same approximation for the Schur complement S is employed, even if this matrix does not explicitly appear in the expression of Q_5.

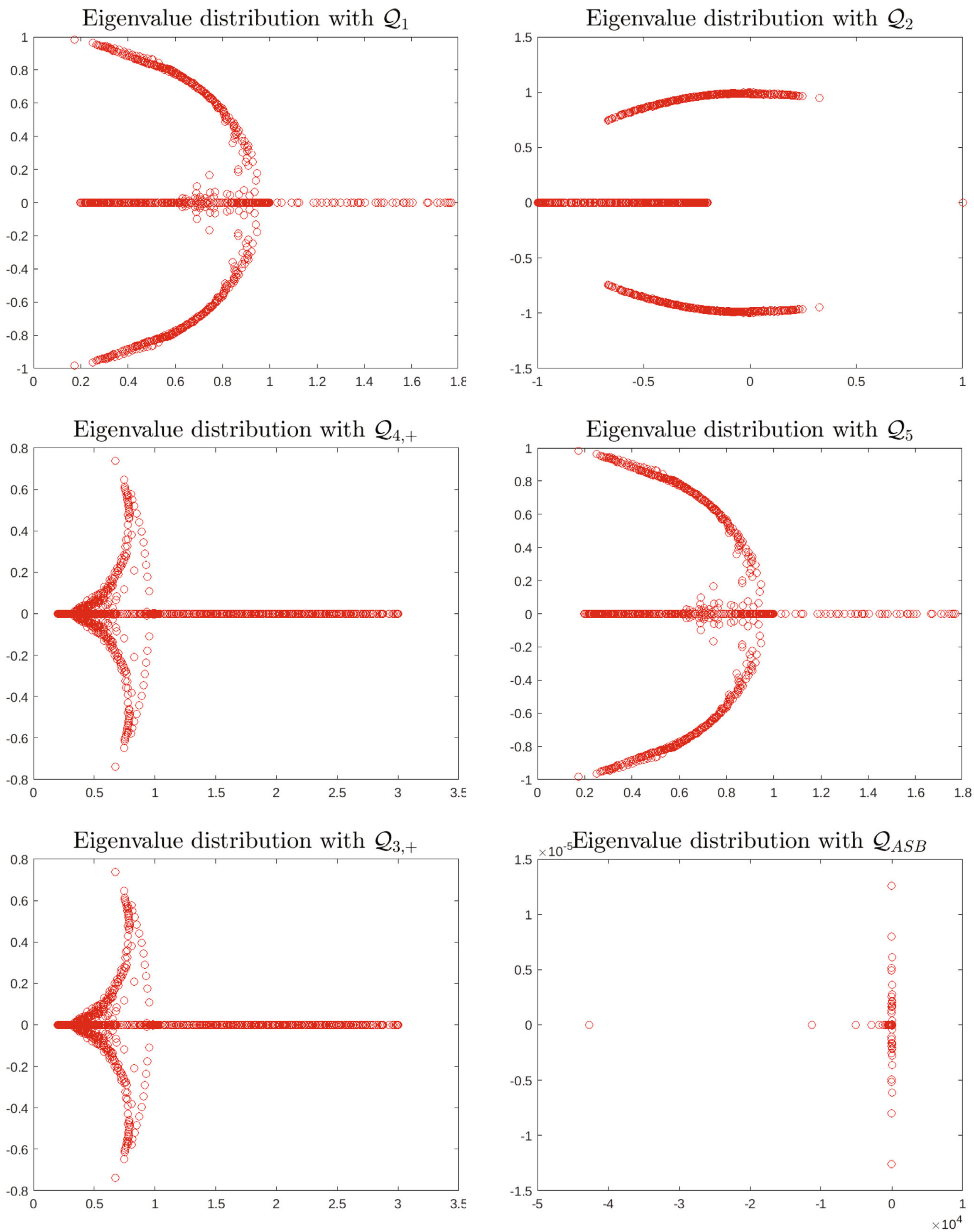

FIGURE 1 Eigenvalue distribution of the preconditioned matrix with $p = 16$ for different preconditioners.

TABLE 1 Numerical results for example 1 with unitary exact solution.

	Size	**2080**	**8256**	**32,896**	**131,328**	**524,800**	**2,098,176**	**8,390,656**
	$\boldsymbol{p}$	**16**	**32**	**64**	**128**	**256**	**512**	**1024**
$\mathcal{P}_D$	ITS	79	126	132	128	121	117	†
	CPU	0.15	0.83	3.28	13.25	49.90	165.06	†
	RES	0.21e−05	0.12e−06	0.88e−08	0.49e−09	0.35e−10	0.20e−11	†
	ERR	0.88e−05	0.64e−05	0.10e−04	0.11e−04	0.12e−04	0.11e−04	†
$\mathcal{P}_3$	ITS	51	83	87	84	80	77	†
	CPU	0.07	0.43	1.62	7.03	26.45	88.69	†
	RES	0.19e−05	0.12e−06	0.81e−08	0.53e−09	0.34e−10	0.19e−11	†
	ERR	0.68e−05	0.60e−05	0.10e−04	0.11e−04	0.12e−04	0.11e−04	†
$\mathcal{Q}_2$	ITS	66	98	100	99	95	91	†
	CPU	0.10	0.64	1.94	8.33	31.30	103.49	†
	RES	0.21e−05	0.13e−06	0.92e−08	0.57e−09	0.34e−10	0.22e−11	†
	ERR	0.66e−05	0.64e−05	0.97e−05	0.10e−04	0.10e−04	0.10e−04	†
$\mathcal{Q}_5$	ITS	38	57	59	57	54	52	52
	CPU	0.04	0.22	0.93	3.90	14.49	49.81	294.22
	RES	0.21e−05	0.14e−06	0.86e−08	0.53e−09	0.34e−10	0.19e−11	0.12e−12
	ERR	0.59e−05	0.68e−05	0.10e−04	0.11e−04	0.12e−04	0.10e−04	0.11e−04
$\mathcal{Q}_{3,+}$	ITS	30	44	46	45	43	41	39
	CPU	0.03	0.17	0.64	2.68	10.40	35.52	201.87
	RES	0.19e−05	0.14e−06	0.86e−08	0.47e−09	0.33e−10	0.20e−11	0.12e−12
	ERR	0.88e−05	0.69e−05	0.13e−04	0.12e−04	0.14e−04	0.13e−04	0.15e−04
$\mathcal{Q}_{ASB}$	ITS	19	17	16	15	13	12	45
	CPU	0.05	0.03	0.07	0.27	0.84	2.85	177.30
	RES	0.15e−05	0.12e−06	0.82e−08	0.38e−09	0.30e−10	0.16e−11	0.13e−12
	ERR	0.39e−05	0.31e−05	0.33e−05	0.25e−05	0.32e−05	0.26e−05	0.56e−05

† GMRES memory overflow.

The numerical results corresponding to the block preconditioned FGMRES for Example 1 are given in Table 1 (using the right hand side as $b = \mathcal{A}e$) and Table 2 where an exact random solution is employed.

We run FGMRES with all preconditioners for problems with $p = \{16, 32, 64, 128, 256, 512, 1024\}$ ending up with a problem with more than 8 million unknowns. To obtain a relative error of (roughly) the same order of magnitude we adjusted the tolerance as

$$\texttt{tol} = \frac{10}{N^2}, \qquad N = n + m + l.$$

From these tables, we see that the preconditioners $\mathcal{Q}_5$ and, particularly, $\mathcal{Q}_{3,+}$ outperform the other ones in terms of iteration number and CPU time, being all the proposed preconditioners more convenient than $\mathcal{P}_D$ and obtaining FGMRES convergence to the solution of (1) in a reasonable number of iterations and CPU time. For the largest problem and random right-hand-side, only preconditioners $\mathcal{Q}_{3,+}$ and $\mathcal{Q}_5$ could solve the given linear system within the memory of our laptop, due to the small number of iterations they required.

Regarding the $\mathcal{Q}_{ASB}$ preconditioner we observe that it is the best performing one when $w \equiv e$, whereas it does not reveal competitive with $\mathcal{Q}_{3,+}$ for a random exact solution.

TABLE 2 Numerical results for Example 1 with random exact solution.

	Size	2080	8256	32,896	131,328	524,800	2,098,176	8,390,656
	p	16	32	64	128	256	512	1024
$\mathcal{P}_D$	ITS	89	147	156	156	152	150	†
	CPU	0.13	0.90	4.31	16.58	61.67	257.74	†
	RES	0.20e−05	0.14e−06	0.85e−08	0.51e−09	0.35e−10	0.21e−11	†
	ERR	0.83e−05	0.71e−05	0.73e−05	0.73e−05	0.87e−05	0.83e−05	†
$\mathcal{P}_3$	ITS	58	97	103	101	100	97	†
	CPU	0.06	0.48	1.98	8.32	32.34	128.20	†
	RES	0.20e−05	0.11e−06	0.83e−08	0.57e−09	0.33e−10	0.22e−11	†
	ERR	0.87e−05	0.51e−05	0.72e−05	0.80e−05	0.78e−05	0.84e−05	†
$\mathcal{Q}_2$	ITS	74	114	118	117	114	112	†
	CPU	0.07	0.61	2.21	9.67	38.40	146.92	†
	RES	0.21e−05	0.13e−06	0.87e−08	0.52e−09	0.35e−10	0.22e−11	†
	ERR	0.59e−05	0.57e−05	0.71e−05	0.70e−05	0.81e−05	0.83e−05	†
$\mathcal{Q}_5$	ITS	43	66	69	68	66	65	60
	CPU	0.04	0.29	1.01	4.70	18.38	70.22	450.92
	RES	0.18e−05	0.12e−06	0.80e−08	0.50e−09	0.35e−10	0.22e−11	0.14e−12
	ERR	0.57e−05	0.54e−05	0.67e−05	0.69e−05	0.83e−05	0.86e−05	0.85e−05
$\mathcal{Q}_{3,+}$	ITS	33	51	54	53	52	52	51
	CPU	0.03	0.21	0.69	3.14	13.17	49.47	301.68
	RES	0.20e−05	0.13e−06	0.75e−08	0.49e−09	0.33e−10	0.18e−11	0.18e−12
	ERR	0.11e−04	0.57e−05	0.65e−05	0.73e−05	0.81e−05	0.71e−05	0.78e−05
$\mathcal{Q}_{ASB}$	ITS	66	98	113	117	117	116	†
	CPU	0.06	0.41	1.67	7.54	29.38	119.92	†
	RES	0.22e−05	0.14e−06	0.77e−08	0.56e−09	0.32e−10	0.21e−11	†
	ERR	0.65e−05	0.96e−05	0.16e−04	0.22e−04	0.21e−04	0.22e−04	†

†GMRES memory overflow.

In the next section, we will discuss in detail the eigenvalue distribution of the preconditioned matrix using the inexact version of $\mathcal{Q}_{3,+}$ and we leave other inexact versions of the proposed preconditioners as a topic for further research. Regarding the complex eigenvalues, we perform an analysis similar to the one in Reference 23 but generalized here for the 3×3 block triangular preconditioner.

4 | EIGENVALUE ANALYSIS OF THE INEXACT VARIANT OF $\mathcal{Q}_{3,+}$

We analyze in this section the eigenvalue distribution of the preconditioned matrix $\mathcal{A}\overline{\mathcal{Q}}^{-1}$, where, in the sequel,

$$\overline{\mathcal{Q}} \equiv \mathcal{Q}_{3,+} = \begin{pmatrix} \widehat{A} & B^T & 0 \\ 0 & -\widehat{S} & C^T \\ 0 & 0 & \widehat{X} \end{pmatrix}, \tag{27}$$

with $\widehat{A}$, $\widehat{S}$ and $\widehat{X}$ proper symmetric positive definite approximations (preconditioners) of A, S and X, respectively.

The relevant spectral properties of the preconditioned matrix $\mathcal{A}\overline{\mathcal{Q}}^{-1}$ will be given in terms of the eigenvalues of $\widehat{A}^{-1}A, \widehat{S}^{-1}\tilde{S}$ and $\widehat{X}^{-1}\tilde{X}$ where $\tilde{S} = B\widehat{A}^{-1}B^T$ and $\tilde{X} = C\widehat{S}^{-1}C^T$. To this aim, we define

$$\gamma_{\min}^A \equiv \lambda_{\min}(\widehat{A}^{-1}A), \qquad \gamma_{\max}^A \equiv \lambda_{\max}(\widehat{A}^{-1}A), \qquad \gamma_A \in [\gamma_{\min}^A, \gamma_{\max}^A], \tag{28}$$

$$\gamma_{\min}^S \equiv \lambda_{\min}(\widehat{S}^{-1}\tilde{S}), \qquad \gamma_{\max}^S \equiv \lambda_{\max}(\widehat{S}^{-1}\tilde{S}), \qquad \gamma_S \in [\gamma_{\min}^S, \gamma_{\max}^S], \tag{29}$$

$$\gamma_{\min}^X \equiv \lambda_{\min}(\widehat{X}^{-1}\tilde{X}), \qquad \gamma_{\max}^X \equiv \lambda_{\max}(\widehat{X}^{-1}\tilde{X}), \qquad \gamma_X \in [\gamma_{\min}^X, \gamma_{\max}^X]. \tag{30}$$

We will finally make the assumption that $1 \in [\gamma_{\min}^A, \gamma_{\max}^A]$. This assumption, very commonly satisfied in practice, will simplify some of the bounds mostly regarding real eigenvalues.

Let

$$\overline{D} = \begin{pmatrix} \widehat{A} & 0 & 0 \\ 0 & \widehat{S} & 0 \\ 0 & 0 & \widehat{X} \end{pmatrix}. \tag{31}$$

Then finding the eigenvalues of $\overline{\mathcal{Q}}^{-1}\mathcal{A}$ is equivalent to solving

$$\overline{D}^{-\frac{1}{2}}\mathcal{A}\overline{D}^{-\frac{1}{2}}\mathbf{w} = \lambda\overline{D}^{-\frac{1}{2}}\overline{\mathcal{Q}}\overline{D}^{-\frac{1}{2}}\mathbf{w},$$

or

$$\begin{pmatrix} \tilde{A} & R^T & 0 \\ R & 0 & K^T \\ 0 & K & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} I & R^T & 0 \\ 0 & -I & K^T \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{32}$$

where $\tilde{A} = \widehat{A}^{-\frac{1}{2}}A\widehat{A}^{-\frac{1}{2}}$, $R = \widehat{S}^{-\frac{1}{2}}B\widehat{A}^{-\frac{1}{2}}$ and $K = \widehat{X}^{-\frac{1}{2}}C\widehat{S}^{-\frac{1}{2}}$.

Theorem 5. *Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{l\times m}$ are matrices with full row rank. Let $\widehat{A}, \widehat{S}$ and $\widehat{X}$ be the symmetric positive approximations of A, S, and X, respectively. Assume that λ is an eigenvalue of the preconditioned matrix $\mathcal{A}\overline{\mathcal{Q}}^{-1}$ and $(x; y; z)$ is the corresponding eigenvector. If $\Im(\lambda) \neq 0$, then λ satisfies*

$$|\lambda - 1| < \sqrt{1 - \gamma_{\min}^A}, \qquad \text{if } Ky = 0,$$

$$|\lambda - 1| \leq \sqrt{1 - \gamma_{\min}^A\frac{\|x\|^2}{\|y\|^2}}, \qquad \text{if } Ky \neq 0.$$

Proof. Let λ be an eigenvalue of matrix $\mathcal{A}\overline{\mathcal{Q}}^{-1}$ and $(x; y; z)$ be the corresponding eigenvector such that $\|x\|^2 + \|y\|^2 + \|z\|^2 = 1$. It follows from (32) that

$$\tilde{A}x - \lambda x = (\lambda - 1)R^T y, \tag{33}$$

$$Rx - (\lambda - 1)K^T z = -\lambda y, \tag{34}$$

$$Ky = \lambda z. \tag{35}$$

Since $\widehat{A}, \widehat{S}$ and $\widehat{X}$ are SPD, and B and C are matrices with full row rank, then $\tilde{A}$ is SPD and K and R are matrices with full row rank. If $y = 0$, from (33) we can derive $\tilde{A}x = \lambda x$. Thus, it can be deduced that $\lambda \in [\gamma_{\min}^A, \gamma_{\max}^A]$

and then the eigenvalue λ is real. The associated eigenvector for this case is of the form $(x; 0; 0)$ where $x \neq 0$. Assume now that $\lambda \neq 1$ and $y \neq 0$. The rest of the proof is divided into two cases:

Case I. $Ky = 0$. From (35), we obtain $z = 0$. Multiplying (33) by x^* on the left and the transposed conjugate of (34) by y on the right, we get

$$x^* \tilde{A} x - \lambda \|x\|^2 = (\lambda - 1) x^* R^T y, \tag{36}$$

$$x^* R^T y = -\bar{\lambda} \|y\|^2. \tag{37}$$

Inserting Equations (37) into (36), we have

$$x^* \tilde{A} x - \lambda + (\lambda - \bar{\lambda}) \|y\|^2 + |\lambda|^2 \|y\|^2 = 0. \tag{38}$$

Let $\lambda = a + ib$, then taking the real and imaginary parts of (38) apart, we obtain

$$x^* \tilde{A} x - a + (a^2 + b^2) \|y\|^2 = 0, \tag{39}$$

$$b(2\|y\|^2 - 1) = 0. \tag{40}$$

From (40), we have $b = 0$ or $\|y\|^2 = \frac{1}{2}$. We assume that $b \neq 0$. From (38) and after some simple calculations, we have

$$2x^* \tilde{A} x - \lambda - \bar{\lambda} + |\lambda|^2 = 0. \tag{41}$$

Using identity $|\lambda|^2 - \lambda - \bar{\lambda} = |\lambda - 1|^2 - 1$, we obtain $|\lambda - 1|^2 = 1 - 2x^* \tilde{A} x$. If $1 - \gamma^A_{\min} \geq 0$, we deduce that

$$|\lambda - 1|^2 \leq 1 - \gamma^A_{\min},$$

which implies that

$$1 - \sqrt{1 - \gamma^A_{\min}} \leq \Re(\lambda) \leq 1 + \sqrt{1 - \gamma^A_{\min}}.$$

If $1 - \gamma^A_{\min} < 0$, therefore there exists no λ with nonzero imaginary part satisfying the equality in (41).

Case II. $Ky \neq 0$. Multiplying (33) by x^* on the left, the transposed conjugate of (34) by y on the right and (35) by z^* on the left, we derive

$$x^* \tilde{A} x - \lambda \|x\|^2 = (\lambda - 1) x^* R^T y, \tag{42}$$

$$x^* R^T y - (\bar{\lambda} - 1) z^* K y = -\bar{\lambda} \|y\|^2, \tag{43}$$

$$z^* K y = \lambda \|z\|^2. \tag{44}$$

Inserting (44) and (43) into Equation (42) and after easy manipulations, we get

$$x^* \tilde{A} x - \lambda(\|x\|^2 + \|z\|^2) + (\lambda^2 + |\lambda|^2 - \lambda|\lambda|^2)\|z\|^2 + \bar{\lambda}(\lambda - 1)\|y\|^2 = 0. \tag{45}$$

Using identity $\|x\|^2 + \|z\|^2 = 1 - \|y\|^2$, the above expression becomes

$$x^* \tilde{A} x - \lambda + (\lambda - \bar{\lambda} + |\lambda|^2)\|y\|^2 + (\lambda^2 - \lambda|\lambda|^2 + |\lambda|^2)\|z\|^2 = 0, \tag{46}$$

and can be equivalently written as

$$x^* \tilde{A} x - \lambda + \left(\lambda - \bar{\lambda} + |\lambda|^2\right)\|y\|^2 - \lambda\left(|\lambda|^2 - (\lambda + \bar{\lambda})\right)\|z\|^2 = 0. \tag{47}$$

In case of complex eigenvalues, we will show that the real quantity

$$\rho = |\lambda|^2 - (\lambda + \overline{\lambda}) = |\lambda - 1|^2 - 1 = |\lambda|^2 - 2a, \tag{48}$$

is always negative, showing that the complex eigenvalues lie in an open circle with center $(1, 0)$ and prescribed radius. Let us write (46), exploiting the real and imaginary part,

$$x^*\tilde{A}x - a + |\lambda|^2\|y\|^2 - a\rho\|z\|^2 = 0, \tag{49}$$

$$b(-1 + 2\|y\|^2 - \rho\|z\|^2) = 0. \tag{50}$$

If λ is complex, then $b \neq 0$ and from (50) we obtain

$$\|z\|^2 = \frac{2\|y\|^2 - 1}{\rho}, \tag{51}$$

and substituting it in (49) we have

$$0 = x^*\tilde{A}x - a + |\lambda|^2\|y\|^2 - a(2\|y\|^2 - 1) = x^*\tilde{A}x + \|y\|^2(|\lambda|^2 - 2a) = x^*\tilde{A}x + \|y\|^2\rho,$$

from which

$$\rho = -\frac{x^*\tilde{A}x}{\|y\|^2}. \tag{52}$$

We can rewrite (52) as

$$\rho = -\gamma_A \frac{\|x\|^2}{\|y\|^2} \leq -\gamma_{\min}^A \frac{\|x\|^2}{\|y\|^2}, \qquad \gamma_A = \frac{x^*\tilde{A}x}{\|x\|^2}.$$

which together with (48) completes the proof of the theorem. ■

In the following, our aim is to characterize the real eigenvalues of the preconditioned matrix not lying in $[\gamma_{\min}^A, \gamma_{\max}^A]$. To this end, we premise two technical lemmas which will be useful for our analysis.

Lemma 1 ([20]). *Let $\lambda \notin [\gamma_{\min}^A, \gamma_{\max}^A]$. Then for arbitrary $z \neq 0$, there exists a vector $s \neq 0$ such that*

$$\frac{z^T(\tilde{A} - \lambda I)^{-1}z}{z^Tz} = \left(\frac{s^T\tilde{A}s}{s^Ts} - \lambda\right)^{-1} = (\gamma_A - \lambda)^{-1},$$

where $\gamma_A = \frac{s^T\tilde{A}s}{s^Ts}$.

Lemma 2. *Let $p(x)$ be the polynomial defined as*

$$p(x) = x^3 - a_1x^2 + a_2x - a_3, \qquad a_j > 0,\ j = 1, 2, 3,$$

and let $a = \min\{a_1, \frac{a_3}{a_2}\}$ and $b = \max\{a_1, \frac{a_3}{a_2}\}$. Then $p(x) < 0,\ \forall x \in (0, a)$ and $p(x) > 0,\ \forall x > b$.

Proof. The statement of the lemma comes from observing that $p(x)$ is the sum of the term $x^3 - a_1x^2$ which is negative in $(0, a_1)$ and positive for $x > a_1$ and of the term $a_2x - a_3$ which is increasing and changes sign once for $x = \frac{a_3}{a_2}$. ■

Let, as in the previous lemma, $\gamma_A = \frac{s^T\tilde{A}s}{s^Ts}$ and γ_S, γ_X defined in (30). We are now able to bound the real eigenvalues of the preconditioned matrix $\mathcal{A}\overline{\mathcal{Q}}^{-1}$. We split the main theorem considering two cases $Ky = 0$ and $Ky \neq 0$.

Theorem 6. *If* $Ky = 0$, *then the real eigenvalues of the preconditioned matrix not lying in* $[\gamma_{\min}^A, \gamma_{\max}^A]$ *satisfy*

$$\lambda^2 - (\gamma_A + \gamma_S)\lambda + \gamma_S = 0.$$

Moreover, the following synthetic bound holds:

$$\min\left\{\gamma_{\min}^A, \frac{\gamma_{\min}^S}{\gamma_{\max}^A + \gamma_{\min}^S}\right\} \leq \lambda \leq \gamma_{\max}^A + \gamma_{\max}^S. \tag{53}$$

Proof. From (33), we have

$$x = (\tilde{A} - \lambda I)^{-1}(\lambda - 1)R^T y. \tag{54}$$

Inserting x into the Equation (34) yields

$$R(\tilde{A} - \lambda I)^{-1}(1 - \lambda)R^T y = \lambda y. \tag{55}$$

Multiplying the above equation by $\frac{y^T}{y^T y}$ and using Lemma 1, we derive

$$\lambda^2 - (\gamma_A + \gamma_S)\lambda + \gamma_S = 0, \tag{56}$$

The solutions of Equation (56) are

$$\lambda_{1,2} = \frac{\gamma_A + \gamma_S \pm \sqrt{(\gamma_A + \gamma_S)^2 - 4\gamma_S}}{2}.$$

It is easy to see that

$$\lambda_1 = \frac{\gamma_A + \gamma_S + \sqrt{(\gamma_A + \gamma_S)^2 - 4\gamma_S}}{2} \leq \gamma_A + \gamma_S \leq \gamma_{\max}^A + \gamma_{\max}^S.$$

It is not hard to find that the smallest eigenvalue λ_2 is a decreasing function with respect to γ_A and it is increasing with respect to γ_S if $\gamma_A \geq 1$. Therefore, we have

$$\begin{aligned}\lambda_2 &= \frac{\gamma_A + \gamma_S - \sqrt{(\gamma_A + \gamma_S)^2 - 4\gamma_S}}{2} \\ &= \frac{2\gamma_S}{\gamma_A + \gamma_S + \sqrt{(\gamma_A + \gamma_S)^2 - 4\gamma_S}} \geq \frac{\gamma_{\min}^S}{\gamma_{\max}^A + \gamma_{\min}^S}.\end{aligned}$$

From the above discussion, we have proved that the real eigenvalues satisfy

$$\frac{\gamma_{\min}^S}{\gamma_{\max}^A + \gamma_{\min}^S} \leq \lambda \leq \gamma_{\max}^A + \gamma_{\max}^S. \tag{57}$$

■

Before developing bound on the real eigenvalues of the preconditioned matrix in the general case we state the following Lemma.

Lemma 3. *Let* $\zeta \in \mathbb{R}$ *be either* $0 < \zeta < \min\left\{\gamma_{\min}^A, \frac{\gamma_{\min}^S}{\gamma_{\max}^A + \gamma_{\min}^S}\right\}$ *or* $\zeta \geq \gamma_{\max}^A + \gamma_{\max}^S$. *Then the symmetric matrix*

$$Z(\zeta) = (1 - \zeta)R(\zeta I - \tilde{A})^{-1}R^T + \zeta I, \tag{58}$$

has either all positive or all negative eigenvalues.

Proof. Let w be a nonzero vector. Multiplying (58) by $\frac{w^T}{w^Tw}$ on the left and by w on the right and applying Lemma 1, since $\zeta I - \tilde{A}$ has all positive or all negative eigenvalues, yields

$$\begin{aligned}\frac{w^TZw}{w^Tw} &= (1-\zeta)\frac{w^TR(\zeta I-\tilde{A})^{-1}R^Tw}{w^Tw} + \zeta = \quad (\text{setting } z = R^Tw)\\ &= (1-\zeta)\frac{z^T(\zeta I-\tilde{A})^{-1}z}{z^Tz}\frac{w^TRR^Tw}{w^Tw} + \zeta = \\ &= \frac{1-\zeta}{\zeta-\gamma_A}\gamma_S + \zeta.\end{aligned} \tag{59}$$

The Rayleigh quotient associated to $Z(\zeta)$, namely the function $h(\zeta) = \frac{1-\zeta}{\zeta-\gamma_A}\gamma_S + \zeta$ can not be zero under the hypotheses on ζ. In fact,

$$h(\zeta) = 0 \Rightarrow \zeta^2 - (\gamma_A + \gamma_S)\zeta + \gamma_S = 0, \tag{60}$$

and applying (57) we obtain the desired result. ∎

The next theorem provides bounds on the real eigenvalues of the preconditioned matrix $\mathcal{A}\overline{\mathcal{Q}}^{-1}$ in the general case.

Theorem 7. *Let $\lambda \in \mathbb{R}$ and $\lambda \notin \left[\min\left\{\gamma^A_{\min}, \frac{\gamma^S_{\min}}{\gamma^A_{\max}+\gamma^S_{\min}}\right\}, \gamma^A_{\max} + \gamma^S_{\max}\right]$. Then the remaining real eigenvalues of the preconditioned matrix $\mathcal{A}\overline{\mathcal{Q}}^{-1}$ satisfy*

$$\lambda^3 - (\gamma_A + \gamma_S + \gamma_X)\lambda^2 + (\gamma_S + \gamma_X + \gamma_A\gamma_X)\lambda - \gamma_A\gamma_X = 0. \tag{61}$$

Moreover, the following synthetic bound holds:

$$\min\left\{\frac{\gamma^S_{\min}}{\gamma^A_{\max}+\gamma^S_{\min}}, \frac{\gamma^A_{\min}\gamma^X_{\min}}{\gamma^X_{\min}+\gamma^S_{\max}+\gamma^A_{\min}\gamma^X_{\min}}\right\} \le \lambda \le \gamma^A_{\max} + \gamma^S_{\max} + \gamma^X_{\max}. \tag{62}$$

Proof. The Equation (33) can be written as

$$x = (1-\lambda)(\lambda I - \tilde{A})^{-1}R^Ty. \tag{63}$$

When we insert this into the second equation in (34), we obtain

$$Z(\lambda)y = (\lambda - 1)K^Tz, \tag{64}$$

where $Z(\lambda) = (1-\lambda)R(\lambda I - \tilde{A})^{-1}R^T + \lambda I$. The hypotheses on λ allow to use Lemma 3 which guarantee the matrix $Z(\lambda)$ is either SPD or symmetric negative definite. Hence, obtaining $y = (\lambda-1)Z(\lambda)^{-1}K^Tz$ from the previous equation and substituting in (35) yields

$$\left[K(\lambda-1)Z(\lambda)^{-1}K^T - \lambda I\right]z = 0. \tag{65}$$

Multiplying on the left by z^T and dividing by z^Tz yields

$$\frac{z^TK(\lambda-1)Z(\lambda)^{-1}K^Tz}{z^Tz} - \lambda = 0. \tag{66}$$

Setting $w = K^Tz$, we can obtain

$$(\lambda-1)\frac{w^TZ(\lambda)^{-1}w}{w^Tw}\frac{z^TKK^Tz}{z^Tz} - \lambda = 0. \tag{67}$$

Denoted the vector $u = Z(\lambda)^{-1/2}w$, the Equation (66) becomes

$$(\lambda - 1)\frac{u^T u}{u^T Z(\lambda) u}\frac{z^T K K^T z}{z^T z} - \lambda = 0. \tag{68}$$

Using now the relation (59) in Lemma 3, we get

$$\frac{\lambda - 1}{\frac{1-\lambda}{\lambda - \gamma_A}\gamma_S + \lambda}\gamma_X - \lambda = 0. \tag{69}$$

After simple algebra, we are left with the following polynomial cubic equation

$$q(\lambda) \equiv \lambda^3 - (\gamma_A + \gamma_S + \gamma_X)\lambda^2 + (\gamma_S + \gamma_X + \gamma_A\gamma_X)\lambda - \gamma_A\gamma_X = 0. \tag{70}$$

Applying Lemma 2 to this cubic polynomial we have

$$a = \frac{\gamma_A\gamma_X}{\gamma_X + \gamma_S + \gamma_A\gamma_X}, \qquad b = \gamma_A + \gamma_S + \gamma_X.$$

In this case, it is easily verified that $a < b$ from which we have that $a < \lambda < b$ and the statement of the theorem results by observing that the lower bound is an increasing function of both γ_A and γ_X and decreasing on γ_S. ■

Check of the bounds in Theorems 6 and 7. Figure 2 displays in depth the eigenvalue distribution of the preconditioned matrix $\overline{\mathcal{Q}}^{-1}\mathcal{A}$.

- The complex eigenvalues of the preconditioned matrix $\overline{\mathcal{Q}}^{-1}\mathcal{A}$ fall in the open circle with center (1,0) and radius 1;
- Regarding the real eigenvalues, the results are summarized in the following table:

$\min\{\lambda, \lambda \in \mathbb{R}\}$	$\max\{\lambda, \lambda \in \mathbb{R}\}$	Lower bound (62)	Upper bound (62)
0.1982	3.0019	0.1342	6.2110

In what follows, we will perform a more accurate eigenvalues analysis of the preconditioned matrix with the $\overline{\mathcal{Q}}$ preconditioner, under additional hypotheses.

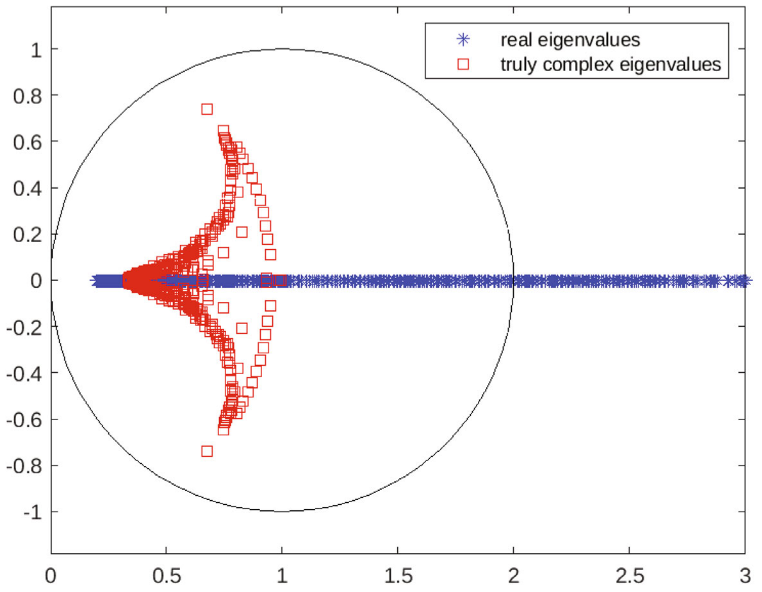

FIGURE 2 Eigenvalue distribution of preconditioned matrix $\overline{\mathcal{Q}}^{-1}\mathcal{A}$ for Example 1 with $p = 16$.

4.1 | Further characterization of the real eigenvalues

We will now consider a simplified preconditioner in which the only approximation is provided by $\widehat{A} \neq A$, whereas $\widehat{S} = B\widehat{A}^{-1}B^T \equiv \tilde{S}$ and $\widehat{X} = C\widehat{S}^{-1}C^T \equiv \tilde{X}$. Note that $RR^T = I_m$ and $KK^T = I_l$.

Theorem 8. *Let $\widehat{S} \equiv \tilde{S}$ and $\widehat{X} \equiv \tilde{X}$. Then any real eigenvalue λ of $\overline{\mathcal{Q}}^{-1}\mathcal{A}$ is bounded by*

$$\min\left\{\lambda^+(\gamma^A_{\min}), \gamma^A_{\min}, \frac{1}{\gamma^A_{\max}+1}\right\} \leq \lambda \leq \max\{\lambda^+(\gamma^A_{\max}), \gamma^A_{\max}+1\},$$

where $\lambda^+(.)$ is the (unique) positive root of the equation

$$\lambda^3 - (2+\gamma_A)\lambda^2 + (2+\gamma_A)\lambda - \gamma_A = 0.$$

Moreover, the following more synthetic bound holds:

$$\min\left\{\frac{1}{\gamma^A_{\max}+1}, \frac{\gamma^A_{\min}}{2}\right\} \leq \lambda \leq \gamma^A_{\max}+1. \tag{71}$$

Proof. For this simplified preconditioner we have $\gamma_S \equiv 1$ and $\gamma_X \equiv 1$. In this case the Equation (61) becomes

$$p(\lambda;\gamma_A) \equiv \lambda^3 - (2+\gamma_A)\lambda^2 + (2+\gamma_A)\lambda - \gamma_A = 0, \tag{72}$$

for all real

$$\lambda \notin \left[\min\left\{\gamma^A_{\min}, \frac{1}{1+\gamma^A_{\max}}\right\}, 1+\gamma^A_{\max}\right]. \tag{73}$$

The cubic polynomial Equation (72) can be written as

$$p(\lambda;\gamma_A) \equiv \lambda\left((\lambda-1)^2+1\right) - \gamma_A(\lambda^2-\lambda+1) = 0, \tag{74}$$

showing that the function $g(x) \equiv p(\lambda;x)$ is decreasing for each $x \geq 0$ and therefore the position of the largest positive root of (74) is increasing. Moreover, it is easy to show that for every $\gamma_A > 0$, there is a unique positive root to the equation $p(\lambda;\gamma_A) = 0$. In fact

$$p(0;\gamma_A) = -\gamma_A < 0, \quad p'(0;\gamma_A) = 2+\gamma_A, \qquad p'(\tilde{\lambda};\gamma_A) = 0, \quad \tilde{\lambda} = \frac{\gamma_A + 2 - \sqrt{\gamma_A^2+\gamma_A-2}}{3},$$

so that if $\gamma_A < 1$, the polynomial p is increasing for $\lambda > 0$ and it takes a local maximum in $\tilde{\lambda}$ if $\gamma_A > 1$ in which, however, $p(\tilde{\lambda};\gamma_A) < p(\tilde{\lambda};1) = 0$. Combining all these facts we finally have

$$\lambda^+(\gamma^A_{\min}) \leq \lambda \leq \lambda^+(\gamma^A_{\max}),$$

where $\lambda^+(\gamma_A)$ refers to the unique positive solution of $p(\lambda;\gamma_A) = 0$, and the thesis holds by observing that $p(\gamma^A_{\min};\gamma^A_{\min}) > 0 \Rightarrow \lambda^+(\gamma^A_{\min}) < \gamma^A_{\min}$ and $p(\gamma^A_{\max};\gamma^A_{\max}) < 0 \Rightarrow \lambda^+(\gamma^A_{\max}) > \gamma^A_{\max}$.

Also the second part of the theorem holds since $p\left(\frac{\gamma^A_{\min}}{2};\gamma^A_{\min}\right) = -\frac{(\gamma^A_{\min})^3}{8} < 0$, then $\lambda^+(\gamma^A_{\min}) > \frac{\gamma^A_{\min}}{2}$. Moreover, from $p\left(\gamma^A_{\max}+1;\gamma^A_{\max}\right) = 1 > 0$, we have that $\lambda^+(\gamma^A_{\max}) < \gamma^A_{\max}+1$. Combining this with (73) we conclude the proof. ∎

Remark 6. Note that we could have applied directly Lemma 2 to Equation (72), obtaining the following bounds

$$\frac{\gamma^A_{\min}}{2+\gamma^A_{\min}} \leq \lambda \leq \gamma^A_{\max}+2,$$

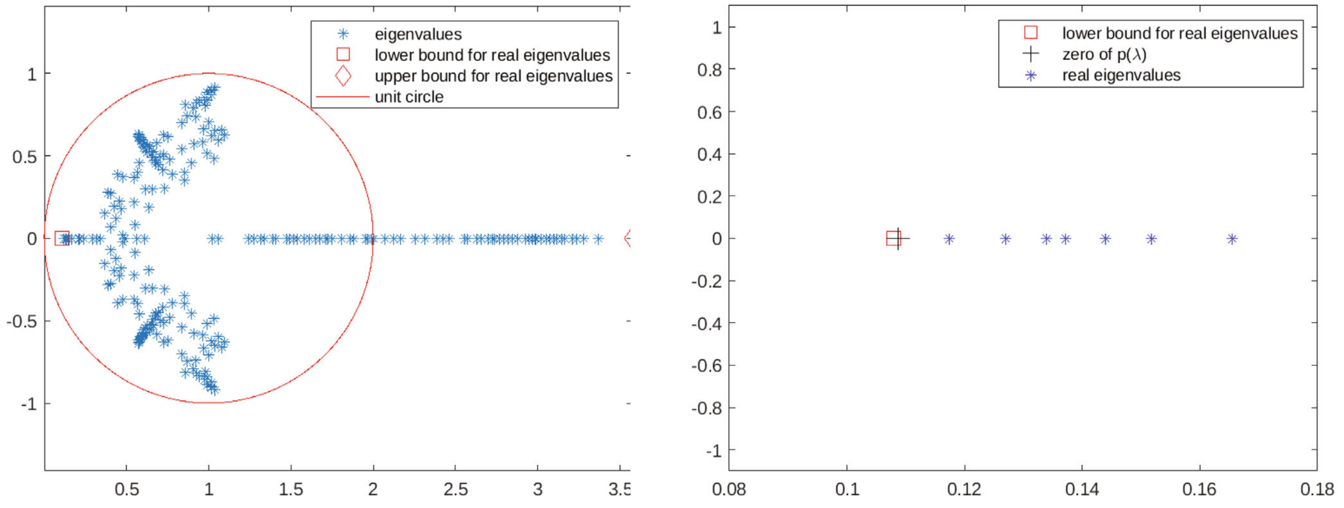

FIGURE 3 Eigenvalue distribution of the preconditioned matrix $\overline{\mathcal{Q}}^{-1}\mathcal{A}$ for Example 2. Left panel: real eigenvalues—blue asterisks, complex eigenvalues—red circles, upper bound by (71)—red diamond. Right panel: smallest eigenvalues compared with the two lower bounds of Theorem 8.

which are looser than those proved in Theorem 8.

Check of the bounds in Theorem 8. The following example is given to assess the theoretical results developed in Theorem 8.

Example 2. Consider the linear system (1) with the block matrices randomly generated by the following MATLAB code:

```
n = 100; m = 80; l = 60;
z = 1+10*rand(1); w = z*rand(n,1);  w = 0.1+sort(w);  w(1:10) = w(1);
A = diag(w); B = rand(m,n); C = rand(l,m);
```

In this example, matrix A is diagonal with a random eigenvalue distribution in $[0.1, 11]$ and $\widehat{A} = I$.

In Figure 3 (left) we show the whole spectrum of the preconditioned matrix $\overline{\mathcal{Q}}^{-1}\mathcal{A}$ together with the bounds for the real eigenvalues. In Figure 3 (right) a zoom of the smallest (real) eigenvalues is provided showing that both the lower bounds, namely $\gamma_{\min}^{A}$ (red box) and λ^{+} (black plus) are smaller, yet very close, than the smallest real eigenvalue of the preconditioned matrix. The results of this experiment as well as the observation of the figures point out that:

- The complex eigenvalues of the related preconditioned matrix are located in a circle centered at $(1, 0)$ with radius 1;
- The real eigenvalues lie in the real interval $[0.1024, 3.373]$;
- Here $\min\left\{\frac{1}{\gamma_{\max}^{A}+1}, \frac{\gamma_{\min}^{A}}{2}\right\} = \frac{\gamma_{\min}^{A}}{2} = 0.1003$, $\lambda^{+} = 0.1008$ and $\gamma_{\max}^{A} + 1 = 3.552$

We can appreciate the closeness of the bounds to the endpoints of the real eigenvalue interval.

5 | NUMERICAL RESULTS ONTO A REALISTIC TEST CASE

We analyze here the behavior of the preconditioners $\mathcal{Q}_{3,+}$ and $\mathcal{Q}_{5}$ in solving a realistic test case taken from Reference 17. Consider the block three-by-three saddle point problem (1), where the block matrices A and B arise from the two-dimensional Stokes problem, namely

$$-\Delta u + \nabla p = 0 \tag{75}$$

$$\nabla \cdot u = 0 \tag{76}$$

Algorithm 3. Approximations of the blocks before the GMRES solution.

1: $\widehat{S} = \text{diag}(Q_p)$
2: $\widehat{X} = \text{tridiag}\left(C\widehat{S}^{-1}C^T\right)$

Algorithm 4. Computation of $w = Q_{3,+}^{-1}r$.

1: Split vector r into $r = \begin{bmatrix} r_1^T & r_2^T & r_3^T \end{bmatrix}^T$
2: Solve exactly the tridiagonal system $\widehat{X}w_3 = r_3$
3: $v = C^T w_3 - r_2$
4: Solve exactly the diagonal system $\widehat{S}w_2 = v$
5: $y = r_1 - B^T w_2$
6: Compute $w_1 = \text{AMG}(y)$
7: Define $w = \begin{bmatrix} w_1^T & w_2^T & w_3^T \end{bmatrix}^T$

Algorithm 5. Computation of $w = Q_5^{-1}r$

1: Split vector r into $r = \begin{bmatrix} r_1^T & r_2^T & r_3^T \end{bmatrix}^T$
2: Compute $y = \text{AMG}(r_1)$
3: $v = By - r_2$
4: Solve exactly the system $\widehat{S}w_2 = v$
5: Solve exactly the system $\widehat{X}w_3 = r_3$
6: $y = r_1 - B^T w_2$
7: Compute $w_1 = \text{AMG}(y)$
8: Define $w = \begin{bmatrix} w_1^T & w_2^T & w_3^T \end{bmatrix}^T$

in the square $(-1, 1) \times (-1, 1)$, with suitable boundary conditions. To obtain A and B we use the $Q_2 - Q_1$ finite element method (FEM) on uniform grids with parameters

$$h = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128} \quad \text{and} \quad \frac{1}{256}.$$

The third dimension is set as $l = m - 2$ and matrix C is defined

$$C = \begin{bmatrix} \text{diag}(1, 3, 5, \dots, 2l - 1) & \text{randn}(l, 2) \end{bmatrix}$$

Differently from Reference 17 we choose optimal approximations for matrix A^{-1} and the Schur complement $S = BA^{-1}B^T$, which are known to be: multigrid for approximating the inverse of the $(1, 1)$ block and the diagonal of the **pressure mass matrix** Q_p as our $\hat{S}$ (see Reference 24), respectively. With these choices, we gain (a) optimality of the two approximations and (b) enormous savings in computing time since $S = B\widehat{A}^{-1}B^T$ does not need to be formed. Finally, our approximation for the second-level Schur complement is selected as in Reference 17. For this particular test case, Algorithms 1 and 2 change accordingly as follows (see Algorithm 3):

By comparing Algorithms 4 and 5 it turns out that the main computational cost of $Q_{3,+}$ is the solution of three linear systems with $\hat{A}$ (by AMG in this case), $\hat{S}$ and $\hat{X}$, respectively, while solving with Q_5 requires one more application of the multigrid on the $(1, 1)$ block. This implies that the cost of applying Q_5 at each iteration of the solver is roughly twice the cost of applying $Q_{3,+}$.

We report in Table 3 the numerical results of GMRES, accelerated with $Q_{3,+}$ and Q_5 with, as in Reference 17, exact solution as the vector of all ones, and tolerance on the relative residual of 10^{-9}. In the table, we provide the spatial discretization parameter, the total number of degrees of freedom, the number of iterations, and the total CPU time.

TABLE 3 Iterations and CPU time for GMRES preconditioned with $Q_{3,+}$ and Q_5, for the enlarged Stokes problem.

$\frac{1}{h}$	N	prec. $Q_{3,+}$		prec. Q_5	
		iter	CPU	iter	CPU
16	2498	47	0.158	60	0.234
32	10,114	47	0.532	63	2.057
64	40,706	47	2.011	63	7.518
128	163,330	39	6.519	50	21.163
256	654,338	38	32.960	48	107.408

The results show

1. the perfect scalability of both the proposed preconditioners (no increase of the number of iterations, by reducing h) and more than one order of magnitude lower CPU times than those reported in Reference 17. This is due to the suitable choice of the three approximations for A, S and X.
2. that the $Q_{3,+}$ preconditioner reveals the clear winner in terms of both number of iterations and CPU time with respect to Q_5.

6 | CONCLUSIONS

In this work, we have considered several *exact* block preconditioners, developing the spectral distribution of the corresponding preconditioned matrices, for a class of double saddle point problems. Some numerical experiments are performed, which show the good behavior of the preconditioned (F)GMRES method using an inexact counterpart of these preconditioners, in comparison with other preconditioners from the literature.

We have then concentrated on the inexact variants of a specific block triangular preconditioner, performing a complete spectral analysis and relating the eigenvalue distribution of the preconditioned matrix with the extremal eigenvalues of the (symmetric and positive definite) preconditioned $(1, 1)$ block and the Schur complement matrices. Numerical tests are reported, which confirm the validity of the developed theoretical bounds, also showing the good performance of the block triangular preconditioner in the solution of a more realistic test case.

Future work is aimed at generalizing this work to provide the eigenvalue distribution of more general double saddle point matrices, in particular, those with nonzero $(2, 2)$ and $(3, 3)$ blocks, and to test them on a wide number of realistic applications, such as, for example, coupled poromechanical models,[25] and the coupled Stokes-Darcy equation.[12]

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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