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A Converse Lyapunov-Type Theorem for Control Systems with Regulated Cost

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Abstract
Given a nonlinear control system, a target set, a nonnegative integral cost, and a continuous function W, we say that the system is *globally asymptotically controllable to the target with W-regulated cost*, whenever, starting from any point z, among the strategies that achieve classical asymptotic controllability we can select one that also keeps the cost less than $W(z)$. In this paper, assuming mild regularity hypotheses on the data, we prove that a necessary and sufficient condition for global asymptotic controllability with regulated cost is the existence of a special, continuous Control Lyapunov Function, called a *Minimum Restraint Function*. The main novelty is the necessity implication, obtained here for the first time. Nevertheless, the sufficiency condition extends previous results based on semiconcavity of the Minimum Restraint Function, while we require mere continuity.

Keywords Converse Lyapunov-type theorem · Asymptotic controllability with regulated cost · Optimal control · Nonlinear theory · Viscosity solutions

Mathematics Subject Classification 93D30 · 93B05 · 49J15 · 93C10 · 49L25

1 Introduction

In this paper we investigate in an optimal control perspective the classical equivalence result between global asymptotic controllability to a closed set $\mathcal{C} \subseteq \mathbb{R}^n$ and

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the existence of a Control Lyapunov function, in the case where the control system is associated with a cost which has to be *regulated* (i.e., loosely speaking, kept bounded). Although controllability is a purely dynamic issue, in some applications, for instance to Lagrangian mechanics (see [13, 17]) it is very reasonable to associate a cost constraint with it. As might be expected, the presence of this constraint can drastically change the choice of control strategy (see [15, Example 1]).

Specifically, we consider a nonlinear control system of the form

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = z \in \mathbb{R}^n \setminus \mathcal{C}, \qquad u(t) \in U \subseteq \mathbb{R}^m, \tag{1}$$

and an integral cost

$$\int_0^{T_z(u)} l(x(t), u(t))\, dt, \tag{2}$$

where the running cost l is nonnegative and the time $T_z(u) \le +\infty$ satisfies[1]

$$x(t) \in \mathbb{R}^n \setminus \mathcal{C} \ \text{ for all } t \in [0, T_z(u)), \qquad \lim_{t \to T_z^-(u)} \mathbf{d}(x(t)) = 0 \tag{3}$$

(for any $y \in \mathbb{R}^n$, $\mathbf{d}(y)$ denotes the distance of y from $\mathcal{C}$). As customary, the control system (1) is called *globally asymptotically controllable to* $\mathcal{C}$ if there exists a $\mathcal{KL}$ function[2] β such that, for any initial point z, there is a measurable control u whose corresponding solution to (1) satisfies $\mathbf{d}(x(t)) \le \beta(\mathbf{d}(z), t)$ for all $t \ge 0$. If, in addition, there exists a continuous, proper, and positive definite function $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$, such that

$$\int_0^{T_z(u)} l(x(t), u(t))\, dt \le W(z),$$

we say that (1)–(2) is globally asymptotically controllable to $\mathcal{C}$ *with* W*-regulated* (or simply, *regulated*) *cost*. Slightly extending the original definition in [20], we define as *Minimum Restraint Function* any function $V : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to \mathbb{R}$, which is continuous, positive definite, proper, and satisfies the *decrease condition*

$$\min_{u \in U} \Big\{ \langle p,\ f(z,u) \rangle + p_0(V(z))\, l(z,u) \Big\} \le -\gamma(V(z)) \ \text{ for all } z \in \mathbb{R}^n \setminus \mathcal{C},\ \ p \in \partial_P V(z),$$

where $\partial_P V(z)$ is the proximal subdifferential, for some continuous increasing functions $p_0 : (0, +\infty) \to [0, 1]$ and $\gamma : (0, +\infty) \to (0, +\infty)$. Since p_0 and l are nonnegative, this decrease condition implies the classical relation which characterizes Control Lyapunov Functions, so that a Minimum Restraint Function is actually a special Control Lyapunov Function.

[1] We will consider assumptions under which, given z and u, the corresponding solution to (1) is uniquely determined.

[2] The definition of a $\mathcal{KL}$ function is recalled in Subsection 1.1.

The main result of the paper is a 'Converse Lyapunov-type theorem', which consists in proving that the existence of a continuous Minimum Restraint Function for some p_0 such that $1/p_0$ is integrable at 0^+, is not only sufficient but also necessary for global asymptotic controllability with regulated cost.

In the case without cost (i.e., for $l \equiv 0$), the equivalence between asymptotic controllability to a point or to a set of a nonlinear control system and the existence of a (possibly, non differentiable) Control Lyapunov Function, has been a central topic in control theory since the 1980s and nowadays it is established under very general assumptions (see e.g. the survey papers [3, 11], the references therein, and a recent extension to impulsive control systems [16]). The key idea of many converse Lyapunov theorems is to convert the control system into a differential inclusion (see, for instance, [5, 23], where $\mathcal{C} = \{0\}$, and [12], for a general target) and then to use the result on the existence of a Lyapunov function for the differential inclusion to get the promised Control Lyapunov Function. However, it seems difficult to adapt this approach to the case with a cost, which we need to estimate from above. Also note that the cost cannot be treated as an additional state variable, because it is increasing and cannot be associated with any target.

In this paper we are inspired instead by early work [25], in which a suitable cost is associated with the original control system, whose value function turns out to be a continuous Control Lyapunov Function. However, even following this approach, the generalization of the nonsmooth Lyapunov literature to allow for the presence of the cost (2) is far from trivial. In particular, we cannot, as one might think, simply add to the cost considered in [25] the current cost l. In fact, the presence of the cost (2) substantially changes the whole construction of both the function β, characterizing asymptotic controllability, and the cost used to obtain a Control Lyapunov Function.

The proof technique is also novel, in that it is based on a combination of the classical Lyapunov procedures adopted in [25] and a viscosity solutions approach. Thanks to this, differently from [25], in the proof of necessity we do not need to use relaxed controls and exhibit an explicit construction of the $\mathcal{KL}$ function β and of the bound W on the cost. Specifically, starting from the observation that a continuous, positive definite and proper function $V : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to \mathbb{R}$ solves the decrease condition if and only if it is a viscosity supersolution of the Hamilton–Jacobi–Bellman equation

$$\max_{u\in U} \Big\{ - Dv(z),\ f(z,u)\rangle - [p_0(v(z))\, l(z,u) + \gamma(V(z))]\Big\} = 0 \quad \text{for all } z \in \mathbb{R}^n \setminus \mathcal{C},$$

we derive both the facts that the value function built in the proof of the necessity implication satisfies the decrease condition and the sufficiency implication, from a viscosity super-optimality principle (Proposition 4.1 below). However, this principle is not included in the known theory, since p_0 is merely continuous and we do not assume the usual linear growth hypothesis on the dynamics function f, but x-local Lipschitz continuity only (see [1, Thm. 2.40] or [10, Thm. 3.3]). We prove it by introducing a slight generalization of a classical comparison principle for infinite horizon problems (Lemma 4.1 below), interesting in itself.

We leave for future investigation the issue of the existence of a semiconcave Minimum Restraint Function, which, as is well known, plays a key role in the feedback stabilizability of nonlinear systems, both with cost (see [13–15], and [8, 9] for a notion of degree-k Minimum Restraint Function) and without it (see e.g. [12, 16, 23, 24], the survey paper [3] and references therein, and [7, 21] for a notion of degree-k Control Lyapunov Function).

The paper is organized as follows. In the remaining part of this section we give some notations. In Sect. 2 we introduce precisely assumptions and definitions and state our Converse Lyapunov-type theorem. Section 3 is devoted to prove that global asymptotic controllability with W-regulated cost for some W, implies the existence of a continuous Minimum Restraint Function, while the converse implication is obtained in Sect. 4. In Sect. 5 we summarize the main outcomes of the work.

1.1 Notation

For $a, b \in \mathbb{R}$, we set $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$. Let $\Omega \subseteq \mathbb{R}^N$ for some integer $N \geq 1$ be a nonempty set. For every $r \geq 0$, we set $B_r(\Omega) := \{x \in \mathbb{R}^n \mid d(x, \Omega) \leq r\}$, where d is the usual Euclidean distance. We use $\overline{\Omega}$, $\partial\Omega$, and $\mathring{\Omega}$ to denote the closure, the boundary, and the interior of Ω, respectively. For any interval $I \subseteq \mathbb{R}$, $\mathcal{M}(I, \Omega)$, $AC(I, \Omega)$ are the sets of functions $x : I \to \Omega$, which are Lebesgue measurable or absolutely continuous, respectively, on I. When no confusion may arise, we simply write $\mathcal{M}(I)$, $AC(I)$. we use $\mathcal{KL}$ to denote the set of all continuous functions $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ such that: (1) $\beta(0, t) = 0$ and $\beta(\cdot, t)$ is strictly increasing and unbounded for each $t \geq 0$; (2) $\beta(r, \cdot)$ is strictly decreasing for each $r \geq 0$; (3) $\beta(r, t) \to 0$ as $t \to +\infty$ for each $r \geq 0$.

Given an open set $\Omega \subseteq \mathbb{R}^N$, a continuous function $W : \overline{\Omega} \to [0, +\infty)$ is said *positive definite* if $W(x) > 0$ $\forall x \in \Omega$ and $W(x) = 0$ $\forall x \in \partial\Omega$. It is called *proper* if the pre-image $W^{-1}(K)$ of any compact set $K \subset [0, +\infty)$ is compact. Let $x \in \Omega$. The set

$$D^- W(x) := \left\{ p \in \mathbb{R}^n \mid \liminf_{y \to x} \frac{W(y) - W(x) - p(y - x)}{|y - x|} \geq 0 \right\},$$

is the (possibly empty) *viscosity subdifferential of W at x*. We recall that $p \in D^- W(x)$ if and only if there exists $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = p$ and $W - \varphi$ has a local minimum at x (see e.g. [1]). We use $\partial_P W(x)$ to denote the *proximal subdifferential of W at x* (which may very well be empty). As it is known, p belongs to $\partial_P W(x)$ if and only if there exist σ and $\eta > 0$ such that

$$W(y) - W(x) + \sigma|y - x|^2 \geq \langle p,\ y - x\rangle \qquad \text{for all } y \in B_\eta(\{x\}).$$

The *limiting subdifferential* $\partial_L W(x)$ *of W at* $x \in \Omega$, is defined as

$$\partial_L W(x) := \left\{ p = \lim_{i \to +\infty} p_i \mid p_i \in \partial_P W(x_i),\ \lim_{i \to +\infty} x_i = x \right\}.$$

The set $\partial_L W(x)$ is always closed. If W is locally Lipschitz continuous on Ω, $\partial_L W(x)$ is compact, nonempty at every point, the set-valued map $x \rightsquigarrow \partial_L W(x)$ is upper semicontinuous, and the Clarke generalized gradient of W at x coincides with $\text{co}\,\partial_L W(x)$. As sources for nonsmooth analysis we refer e.g. to [2, 6, 28].

Let $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$ be a continuous, proper, and positive definite function. We can relate the level sets of W with the ones of the distance function $\mathbf{d}$ from $\mathcal{C}$, by introducing the functions $d_{W^+}, d_{W^-} : (0, +\infty) \to (0, +\infty)$, given by

$$d_{W^-}(r) := \sup\{\alpha > 0 \mid \ \{\tilde{z} \mid \ W(\tilde{z}) \le \alpha\} \subseteq \{\tilde{z} \mid \ \mathbf{d}(\tilde{z}) \le r\}\}, \tag{4}$$

$$d_{W^+}(r) := \inf\{\alpha > 0 \mid \ \{\tilde{z} \mid \ W(\tilde{z}) \le \alpha\} \supseteq \{\tilde{z} \mid \ \mathbf{d}(\tilde{z}) \le r\}\}. \tag{5}$$

By [14, Lemma 3.6], these functions are well-defined, increasing, and

$$\lim_{r\to 0^+} d_{W^+}(r) = \lim_{r\to 0^+} d_{W^-}(r) = 0, \quad \lim_{r\to +\infty} d_{W^+}(r) = \lim_{r\to +\infty} d_{W^-}(r) = +\infty. \tag{6}$$

Moreover, one has

$$d_{W^-}(\mathbf{d}(x)) \le W(x) \le d_{W^+}(\mathbf{d}(x)) \quad \text{for all } x \in \mathbb{R}^n \setminus \mathcal{C}. \tag{7}$$

Approximating d_{W^-} from below and d_{W^+} from above if necessary, we can thus assume the existence of continuous, strictly increasing functions, still denoted d_{W^-} and d_{W^+}, satisfying (6) and (7).

2 A Converse Theorem for Minimum Restraint Functions

Throughout the whole paper we assume that:

(i) *$U \subset \mathbb{R}^m$ is a nonempty compact set, $\mathcal{C} \subset \mathbb{R}^n$ is a nonempty, closed subset with compact boundary;*
(ii) *the functions $f : \mathbb{R}^n \times U \to \mathbb{R}^n$, $l : \mathbb{R}^n \times U \to [0, +\infty)$ are continuous on $\mathbb{R}^n \times U$, $x \mapsto f(x, u)$ and $x \mapsto l(x, u)$ are locally Lipschitz continuous, uniformly with respect to $u \in U$.*

Under these assumptions, given $z \in \mathbb{R}^n \setminus \mathcal{C}$ and $u \in \mathcal{M}([0, +\infty), U)$, there exist a maximal time $T^{max} \le +\infty$ and a unique solution $x \in AC([0, T^{\max}), \mathbb{R}^n)$ such that $x(0) = z$ and

$$\dot{x}(t) = f(x(t), u(t)), \qquad \text{a.e. } t \in [0, T^{\max}). \tag{8}$$

This solution (or trajectory) will be denoted $x(\cdot, u, z)$. If, in addition, some $c \ge 0$ is given, we define the corresponding cost $x^0(\cdot, u, c, z)$ as

$$x^0(t, u, c, z) := c + \int_0^t l(x(s, u, z), u(s))\, ds \qquad \text{for all } t \in [0, T^{\max}). \tag{9}$$

Let us preliminarily introduce some definitions.

Definition 2.1 (*Admissible controls, trajectories, and costs*) Given an initial condition $z \in \mathbb{R}^n \setminus \mathcal{C}$, a control $u \in \mathcal{M}([0,+\infty), U)$ is called *admissible from* z if there exists $0 < T_z(u) \le T^{\max} \le +\infty$ such that $x(t) := x(t,u,z) \notin \mathcal{C}$ for all $t \in [0, T_z(u))$ and $\lim_{t\to T_z^-(u)} \mathbf{d}(x(t)) = 0$ if $T_z(u) < +\infty$.

The set of admissible controls from z will be denoted $\mathcal{U}(z)$. Given $z \in \mathbb{R}^n \setminus \mathcal{C}$, $c \ge 0$, and a control $u \in \mathcal{U}(z)$, when $T_z(u) < +\infty$ we will extend the corresponding trajectory $x = x(\cdot,u,z)$ and cost $x^0 := x^0(\cdot,u,c,z)$ to $[0+\infty)$, by setting[3]

$$(x^0,x)(t) := \lim_{t\to T_z^-(u)} (x^0,x)(t) \qquad \text{for any } t \ge T_z(u).$$

We will call (x,u) and (x^0,x,u) (both defined on $[0,+\infty)$) an *admissible pair from* z and an *admissible triple from* (c,z), respectively.

We recall the notion of global asymptotic controllability with regulated cost, first introduced in [20]. In the following, we will refer to any function $\beta \in \mathcal{KL}$ as a *descent rate*.

Definition 2.2 (*GAC with regulated cost*) The control system (8) is *globally asymptotically controllable—in short, GAC—to* $\mathcal{C}$ if there exists a descent rate β such that, for any initial point $z \in \mathbb{R}^n \setminus \mathcal{C}$, there is an admissible pair (x,u) from z, satisfying

$$\mathbf{d}(x(t)) \le \beta(\mathbf{d}(z),t) \qquad \text{for all } t \ge 0. \tag{10}$$

If, in addition, there exists a continuous, proper, and positive definite function $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0,+\infty)$ such that, for any $z \in \mathbb{R}^n \setminus \mathcal{C}$, there is an admissible triple (x^0,x,u) from $(0,z)$ with (x,u) satisfying (10) and x^0 satisfying

$$x^0(t) = \int_0^t l(x(s),u(s))\,ds \le W(z) \qquad \text{for all } t \ge 0, \tag{11}$$

we say that (8) with the cost (9) is *globally asymptotically controllable to* $\mathcal{C}$ *with* W*-regulated cost*. In this case, we will often simply say that (8)–(9) is GAC (to $\mathcal{C}$) with regulated cost.

Given a descent rate β and a continuous, proper, and positive definite function $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0,+\infty)$, for any $z \in \mathbb{R}^n \setminus \mathcal{C}$ such that $\mathcal{U}(z) \ne \emptyset$, we set

$$\mathcal{U}_{\beta,W}(z) := \left\{ \begin{array}{r} u \in \mathcal{U}(z) \mid \text{the admissible triple } (x^0,x,u) \text{ from } (0,z) \text{ satisfies} \\ \mathbf{d}(x(t)) < \beta(\mathbf{d}(z),t) \text{ and } x^0(t) \le W(z) \text{ for all } t \ge 0 \end{array} \right\}.$$

Since in the definition of GAC it is clearly equivalent to replace the "$\le$" in (10) with "$<$", when (8)–(9) meet the properties in Definition 2.2 for some β, W, we can assume without loss of generality that $\mathcal{U}_{\beta,W}(z) \ne \emptyset$ for all $z \in \mathbb{R}^n \setminus \mathcal{C}$.

[3] The limit always exists, as $\partial\mathcal{C}$ is compact and (x^0,x) is Lipschitz continuous in any compact set $\overline{B_R(\mathcal{C}) \setminus \mathcal{C}}$, $R > 0$.

Let us now give a slightly extended version of the notion of Minimum Restraint function for (8)–(9), firstly introduced in [20]. To this end, we consider the Hamiltonian

$$H(x, p_0, p) := \min_{u \in U} \big\{ \langle p,\ f(x,u) \rangle + p_0\, l(x,u) \big\}. \tag{12}$$

Definition 2.3 (*MRF*) Let $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0,+\infty)$ be a continuous function, which is positive definite and proper. We say that W is a *Minimum Restraint Function—in short, a MRF*—for (8)–(9) if it satisfies the *decrease condition*:[4]

$$H(x, p_0(W(x)), \partial_P W(x)) \le -\gamma(W(x)) \quad \text{for all } x \in \mathbb{R}^n \setminus \mathcal{C}, \tag{13}$$

for some continuous, increasing function $p_0 : (0,+\infty) \to [0,1]$ and some continuous, strictly increasing function $\gamma : (0,+\infty) \to (0,+\infty)$.

As noted in the introduction, a MRF is a particular Control Lyapunov Function, in which the classical decrease condition

$$\min_{u \in U} \langle \partial_P W(x),\ f(x,u) \rangle \le -\gamma(W(x)) \quad \text{for all } x \in \mathbb{R}^n \setminus \mathcal{C},$$

is replaced by the stronger condition (13), also involving the current cost l.

Finally, we consider the following integrability condition.

Definition 2.4 Let $p_0 : (0,+\infty) \to [0,1]$ be an increasing, continuous function. We say that p_0 satisfies the *integrability condition* (IC), when $1/p_0$ is integrable at 0^+, namely, we can define the C^1, strictly increasing function $P : [0,+\infty) \to [0,+\infty)$, given by

$$P(v) := \int_0^v \frac{dv'}{p_0(v')} \qquad \text{for all } v \ge 0, \tag{14}$$

and, moreover, P satisfies $\lim_{v \to +\infty} P(v) = +\infty$.

Clearly, if p_0 is a positive constant, as in the original definition of MRF, it trivially satisfies condition (IC). We are now ready to state our main result.

Theorem 2.1 (Converse MRF Thm.) *The following properties are equivalent:*

(i) *System* (8) *with cost* (9) *is GAC to* $\mathcal{C}$ *with regulated cost;*
(ii) *There exists a continuous MRF for* (8)–(9)*, for some* p_0 *and* γ *such that* p_0 *satisfies the integrability condition* (IC)*.*

The rest of the paper is devoted to the proof of Theorem 2.1.

Remark 2.1 Starting with the assumption that the system is GAC with regulated cost, in the proof below we will explicitly build a MRF with $p_0 \equiv 1$, as unique continuous

[4] This means that $H(x, p_0(W(x)), p) \le -\gamma(W(x))$ for every $p \in \partial_P W(x)$, where $\partial_P W(x)$ is the proximal differential of W at x (see Sect. 1.1).

viscosity solution of the Hamilton–Jacobi–Bellman equation (in short, HJB) associated with an exit-time problem with vanishing lagrangian. As is well known, these HJB equations are highly degenerate and have in general multiple solutions, for which the continuity on the target does not propagate to the whole domain (see [22] and [18, 19, 27]). The proof technique thus consists in showing the continuity of the solution and establishing an ad hoc comparison principle. In addition to allowing us to extend the main result of [25] to the case with cost, this technique also provides an alternative approach to obtaining the classical result.

Remark 2.2 In this paper we continue the study, begun with [20], aimed at constructing a unified theory, which has as extreme situations asymptotic controllability (with cost $l \equiv 0$) on the one hand, and the minimum time problem (with cost $l \equiv 1$) on the other, and for which the $l \geq 0$ case represents, in a sense, the intermediate stage. In the original notion of MRF in [20], p_0 was a positive constant. Extending the definition by considering p_0 an increasing function, possibly vanishing at the origin but satisfying the integrability condition (IC), generalizes the cost bound obtained in [20], as it implies GAC with $\bar{W}$-regulated cost, where $\bar{W}(x) = 4\,P(W(x)/2)$ for P as in (14) (see estimate (43) below).[5] In the special case of the minimum time problem, this extension finally provides a result which is entirely consistent with the existing literature. Indeed, let $l \equiv 1$ and assume that the distance function $\mathbf{d}$ is a MRF for some functions p_0 and γ such that (IC) holds true. Then, from the decrease condition (13), it follows that $\mathbf{d}$ satisfies

$$\min_{u\in U}\langle \partial_P \mathbf{d}(x),\ f(x,u)\rangle \leq -\tilde{\gamma}(\mathbf{d}(z)) \qquad \text{for all } z \in \mathbb{R}^n \setminus \mathcal{C}, \tag{15}$$

where $\tilde{\gamma}(r) := p_0(r) + \gamma(r)$. As it is well-known, this condition combined with the integrability property (IC) for $\tilde{\gamma}$—sometimes called *weak Petrov condition*—guarantees the small time local controllability of system (8) to $\mathcal{C}$, and implies for the minimum time function T the estimate $T(z) \leq \int_0^{\mathbf{d}(z)} (1/\tilde{\gamma}(r))\,dr$, in line with our inequality (43) (see e.g. [2]). By the expression "weak", we mean that $\tilde{\gamma}$ can be 0 at 0, to distinguish it from the classical Petrov condition, in which $\tilde{\gamma}$ is replaced by a positive constant. Notice that, considering only $p_0 \equiv \bar{p}_0 > 0$ constant, we would have $\tilde{\gamma} \geq \bar{p}_0 > 0$, so our conditions would include just the ordinary, i.e. non-weak, Petrov condition, which implies local Lipschitz continuity of T (see e.g. [1]).

Furthermore, we think that considering p_0 not constant could play a role in regularizing a MRF, in the fashion of [23].

3 Proof of Implication (i) $\Longrightarrow$ (ii)

Suppose that (8)–(9) is GAC to $\mathcal{C}$ with W-regulated cost, for a cost bound W and a descent rate β. We split the proof into several lemmas. In particular, in Lemma 3.1, we build new functions $\bar{\beta}$ and $\bar{W}$ and an admissible triple $(\hat{x}^0, \hat{x}, \hat{u})$ from $(0, z)$ such that

[5] Actually, refining some estimates, we could likely get $\bar{W}(x) = P(W(x))$, as a consequence of the fact that $\bar{W}(x) \leq (1 + 2\varepsilon)(P(W(x)/(1 + \varepsilon)))$ for every $\varepsilon > 0$.

$\hat{u} \in \mathcal{U}_{\bar{\beta},\bar{W}}(z)$, for every $z \in \mathbb{R}^n \setminus \mathcal{C}$. These objects play a key role in the construction of a (larger) cost functional J, in Lemma 3.2. Furthermore, in Lemmas 3.3, 3.4 we show that the value function V associated with the control system (8) and the new cost J, is a continuous MRF. Lemmas 3.1, 3.2 are inspired by [25, Lemmas 3.8, 3.17], where, however, no cost is considered and relaxed controls are used. As already observed, differently from [25], the present results are formulated in terms of (explicitly built) new descent rate and cost bound, and are obtained by mixing nonsmooth analysis and viscosity methods, which incidentally allow us to disregard relaxed controls.

We begin by introducing some definitions. We consider a bilateral sequence $(r_i)_{i\in\mathbb{Z}}$, given by[6]

$$r_0 := 1, \qquad r_i := \min\left\{\beta^{-1}(r_{i-1}, 0), d_{W^+}^{-1}\left(\frac{1}{4}d_{W^-}(r_{i-1})\right)\right\} \quad \text{for all } i \in \mathbb{Z}. \tag{16}$$

Clearly, $(r_i)_{i\in\mathbb{Z}}$ is positive, strictly decreasing, so that $r_1 < 1$, and satisfies

$$\lim_{i\to-\infty} r_i = +\infty, \qquad \lim_{i\to+\infty} r_i = 0.$$

Hence, we have

$$\beta(r_i, 0) \le r_{i-1}, \quad d_{W^+}(r_i) \le \frac{1}{4}d_{W^-}(r_{i-1}) \quad \text{for all } i \in \mathbb{Z}$$

and, consequently, recalling that $d_{W^-} \le d_{W^+}$ (see (4) and (5)),

$$d_{W^+}(r_{i+N}) \le \frac{1}{4^N}d_{W^-}(r_i) \le \frac{1}{4^N}d_{W^+}(r_i) \quad \text{for all } i \in \mathbb{Z}, \ \ N \in \mathbb{N}, \ \ N \ge 1. \tag{17}$$

For any $i \in \mathbb{Z}$, let

$$\mathcal{B}_i := \{z \in \mathbb{R}^n \setminus \mathcal{C} \mid \ \mathbf{d}(z) \in [r_i, r_{i-1}]\}, \tag{18}$$

so that $\mathbb{R}^n \setminus \mathcal{C} = \cup_{i\in\mathbb{Z}}\mathcal{B}_i$. Finally, we define the *i-th (β, W)-strip* $\mathcal{A}_i$, as

$$\mathcal{A}_i := \left\{ \begin{array}{r} (x^0, x, u, z) \mid (x^0, x, u) \text{ admissible triple from } (0, z), \\ u \in \mathcal{U}_{\beta,W}(z), \ \ z \in \mathcal{B}_i \end{array} \right\}.$$

Lemma 3.1 *There exist a $\mathcal{KL}$ function $\bar{\beta} \ge \beta$, a continuous, unbounded, strictly increasing map $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$, and a function $T : (0, +\infty) \to [0, +\infty)$ with $T(R) = 0$ for all $R \le r_1$,*[7]

such that for any $z \in \mathbb{R}^n \setminus \mathcal{C}$ there exists an admissible triple $(\hat{x}^0, \hat{x}, \hat{u})$ from $(0, z)$ enjoying the following properties:

(i) $\mathbf{d}(\hat{x}(t)) \le \bar{\beta}(\mathbf{d}(z), t)$ *for all $t \ge 0$;*

6 As $\beta^{-1}(r, 0)$, we mean the inverse of the strictly increasing function $r' \mapsto r = \beta(r', 0)$.

7 The value r_1 is defined as in (16).

(ii) $\hat{x}^0(t) \leq \bar{W}(z) := \Phi(\mathbf{d}(z))$ *for all* $t \geq 0$*;*
(iii) $\mathbf{d}(\hat{x}(t)) \leq \bar{\beta}(1, t - T(\mathbf{d}(z)))$ *for all* $t \geq T(\mathbf{d}(z))$.

Proof *Step 1 (Properties of the* (β, W)*-strips).* Fix $i \in \mathbb{Z}$ and let $(u^0, x, u, z) \in \mathcal{A}_i$. From the definitions of $(r_i)_{i\in\mathbb{Z}}$, d_{W^-}, and d_{W^+}, it follows that

$$\mathbf{d}(x(t)) < \beta(\mathbf{d}(z), t) \leq \beta(r_{i-1}, 0) < r_{i-2} \qquad \text{for all } t \geq 0, \tag{19}$$

and

$$W(z) \leq d_{W^+}(r_{i-1}) \leq \frac{1}{4} d_{W^-}(r_{i-2}). \tag{20}$$

Define

$$T_{i,z} := \inf \left\{ t \geq 0 \mid \mathbf{d}(x(t)) = \frac{r_i + r_{i+1}}{2} \right\}.$$

Clearly, $0 < T_{i,z} < T_z(u)$, where $T_z(u)$ is as in Definition 2.1. Set

$$\tilde{\varepsilon}_{i,z} := \inf \left\{ \frac{1}{2}\big(\beta(\mathbf{d}(z), t) - \mathbf{d}(x(t))\big) \mid t \in [0, T_{i,z}] \right\}$$

Note that by the continuity of β and of x, $\tilde{\varepsilon}_{i,z}$ is actually a minimum and $\tilde{\varepsilon}_{i,z} > 0$. Also define $\hat{\varepsilon}_i := \frac{1}{4} d_{W^+}(r_{i-1})$, $\bar{\varepsilon}_i := \frac{r_i - r_{i+1}}{4}$ and, finally, set

$$\varepsilon_{i,z} := \min\{\tilde{\varepsilon}_{i,z}, \hat{\varepsilon}_i, \bar{\varepsilon}_i\}.$$

Since by assumption $f(\cdot, u)$ and $l(\cdot, u)$ are locally Lipschitz continuous, uniformly with respect to $u \in U$, then there exists $\delta_{i,z} > 0$ such that, for all $\bar{z} \in \mathbb{R}^n \backslash \mathcal{C}$ satisfying $|\bar{z} - z| < \delta_{i,z}$, the cost $x^0(\cdot, u, 0, \bar{z})$ and the trajectory $x(\cdot, u, \bar{z})$ are defined on $[0, T_{i,z}]$ and satisfy

$$|x(t) - x(t, u, \bar{z})| \leq \varepsilon_{i,z}, \quad |x^0(t) - x^0(t, u, 0, \bar{z})| \leq \varepsilon_{i,z} \qquad \text{for all } t \in [0, T_{i,z}].$$

Hence, for all $t \in [0, T_{i,z}]$, using (20), we get

$$\begin{aligned} x^0(t, u, 0, \bar{z}) &\leq x^0(t) + \varepsilon_{i,z} \leq W(z) + \hat{\varepsilon}_i \leq W(z) + \frac{1}{4} d_{W^+}(r_{i-1}) \\ &\leq \frac{1}{4} d_{W^-}(r_{i-2}) + \frac{1}{4} d_{W^+}(r_{i-1}) \leq \frac{1}{2} d_{W^+}(r_{i-2}). \end{aligned}$$

Moreover, in view of the definition of $\varepsilon_{i,z}$, we also have

$$\begin{aligned} \mathbf{d}(x(t, u, \bar{z})) &\leq \mathbf{d}(x(t)) + \frac{1}{2}\big(\beta(\mathbf{d}(z), t) - \mathbf{d}(x(t))\big) = \frac{1}{2}\beta(\mathbf{d}(z), t) + \frac{1}{2}\mathbf{d}(x(t)) \\ &< \beta(\mathbf{d}(z), t) \leq \beta(r_{i-1}, 0) \leq r_{i-2}, \end{aligned}$$

whereas the definition of $T_{i,z}$ implies

$$\begin{aligned}\mathbf{d}(x(t,u,\bar{z})) &\geq \mathbf{d}(x(t)) - \bar{\varepsilon}_i \geq \mathbf{d}(x(T_{i,z})) - \bar{\varepsilon}_i \\ &\geq \frac{r_i + r_{i+1}}{2} - \frac{r_i - r_{i+1}}{4} = \frac{r_i}{4} + \frac{3}{4}r_{i+1} > r_{i+1},\end{aligned}$$

and

$$\begin{aligned}\mathbf{d}(x(T_{i,z},u,\bar{z})) &\leq \mathbf{d}(x(T_{i,z})) + \bar{\varepsilon}_i = \frac{r_i + r_{i+1}}{2} + \frac{r_i - r_{i+1}}{4} \\ &= \frac{3}{4}r_i + \frac{1}{4}r_{i+1} < r_i.\end{aligned}$$

Summarizing the above results, we can conclude that, for every $i \in \mathbb{Z}$ and $z \in \mathcal{B}_i$ ($\mathcal{B}_i$ as in (18)), with which we can consider an element $(x^0, x, u, z) \in \mathcal{A}_i$ to be associated, by the axiom of choice, there exists $\delta_{i,z} > 0$ such that, for all $\bar{z} \in \mathbb{R}^n \setminus \mathcal{C}$ with $|z - \bar{z}| < \delta_{i,z}$, one has

(a) $\mathbf{d}(x(t,u,\bar{z})) \in (r_{i+1}, r_{i-2})$ for all $t \in [0, T_{i,z}]$;
(b) $\mathbf{d}(x(T_{i,z},u,\bar{z})) \in (r_{i+1}, r_i)$, i.e. $x(T_{i,z},u,\bar{z}) \in \mathring{\mathcal{B}}_{i+1}$;
(c) $x^0(T_{i,z},u,0,\bar{z}) \leq \frac{1}{2}d_{W^+}(r_{i-2})$.

Step 2 (Construction of a suitable admissible triple) Preliminarily, observe that, since $\partial\mathcal{C}$ is compact, for every $i \in \mathbb{Z}$ the set $\mathcal{B}_i$ is compact. Therefore, the cover of $\mathcal{B}_i$ given by the open balls $\mathring{B}_{\delta_{i,z}}(\{z\})$, $z \in \mathcal{B}_i$, admits a finite subcover corresponding to the points $z \in Z_i$, for some finite subset Z_i of $\mathcal{B}_i$. Fix a positive bilateral sequence $(T_i)_{i\in\mathbb{Z}}$ such that

$$T_i \geq \max\{T_{i,z} \mid z \in Z_i\}, \qquad \sum_{j=0}^{\infty} T_{i+j} = +\infty, \quad \text{for every } i \in \mathbb{Z}. \tag{21}$$

Furthermore, thanks to the properties of the bilateral sequence $(r_i)_{i\in\mathbb{Z}}$, we can define the map $i : (0, +\infty) \to \mathbb{Z}$, given by

$$i(r) = i \qquad \text{if } r \in (r_i, r_{i-1}]. \tag{22}$$

Fix now $\bar{z} \in \mathbb{R}^n \setminus \mathcal{C}$ and let $i := i(\mathbf{d}(\bar{z}))$. Then, $\bar{z} \in \mathring{B}_{\delta_{i,z_0}}(\{z_0\})$ for some $z_0 \in Z_i \subset \mathcal{B}_i$. Let $(x_0^0, x_0, u_0, z_0) \in \mathcal{A}_i$ be the associated process from $(0, z_0)$. Since $|\bar{z} - z_0| < \delta_{i,z_0}$, from Step 1 it follows that $\hat{x}^0 := (\cdot, u_0, 0, \bar{z})$ and $\hat{x} := x(\cdot, u_0, \bar{z})$ are defined on the interval $[0, \hat{t}_0]$, $\hat{t}_0 := T_{i,z_0} \leq T_i$, and satisfy

(a.0) $\mathbf{d}(\hat{x}(t)) \in (r_{i+1}, r_{i-2})$ for all $t \in [0, \hat{t}_0]$;
(b.0) $\hat{x}(\hat{t}_0) \in \mathring{\mathcal{B}}_{i+1}$;
(c.0) $\hat{x}^0(\hat{t}_0) \leq \frac{1}{2}d_{W^+}(r_{i-2})$.

Repeating the above procedure with the initial conditions $\bar{z}_1 := \hat{x}(\hat{t}_0) \in \mathcal{B}_{i+1}$ and $\bar{c}_1 := \hat{x}_0(\hat{t}_0)$, we get the existence of an admissible control $u_1 \in \mathcal{U}_{\beta,W}(\bar{z}_1)$ and of a time $\hat{t}_1 \leq T_{i+1}$, such that extending $\hat{x}^0$ and $\hat{x}$ by setting $\hat{x}^0(t) = x^0(t - \hat{t}_0, u_1, \bar{c}_1, \bar{z}_1)$ and $\hat{x}(t) = x(t - \hat{t}_0, u_1, \bar{z}_1)$ for $t \in [\hat{t}_0, \hat{t}_0 + \hat{t}_1]$, respectively, one has

(a.1) $\mathbf{d}(\hat{x}(t)) \in (r_{i+2}, r_{i-1})$ for all $t \in [\hat{t}_0, \hat{t}_0 + \hat{t}_1]$;
(b.1) $\hat{x}(\hat{t}_0 + \hat{t}_1) \in \mathring{\mathcal{B}}_{i+2}$;
(c.1) $\hat{x}^0(\hat{t}_0 + \hat{t}_1) \leq \frac{1}{2} d_{W^+}(r_{i-2}) + \frac{1}{2} d_{W^+}(r_{i-1}) \leq \frac{1}{2} d_{W^+}(r_{i-2}) + \frac{1}{8} d_{W^+}(r_{i-2})$.

Of course, setting $\hat{u}(t) := u_0(t)\chi_{[0,\hat{t}_0]}(t) + u_1(t - \hat{t}_0)\chi_{(\hat{t}_0, \hat{t}_0+\hat{t}_1]}(t)$, we have $\hat{x}^0 = x(\cdot, \hat{u}, 0, \bar{z})$ and $\hat{x} = x(\cdot, \hat{u}, \bar{z})$. In this way, we can recursively construct sequences of controls $(u_N)_{N\in\mathbb{N}}$ and times $(\hat{t}_N)_{N\in\mathbb{N}}$ with $\hat{t}_N \leq T_{i+N}$ for all N, such that, setting

$$\hat{T}_{-1} := 0, \quad \hat{T}_N := \sum_{j=0}^{N} \hat{t}_j, \quad \hat{T}_\infty := \sum_{j=0}^{+\infty} \hat{t}_j$$

and

$$\hat{u}(t) := u_N(t - \hat{T}_{N-1}) \qquad \text{for all } t \in (\hat{T}_{N-1}, \hat{T}_N]$$

we have $u \in \mathcal{M}([0, \hat{T}_\infty), U)$ and the corresponding solution $(\hat{x}^0, \hat{x})$ from $(0, \bar{z})$ is defined on the whole interval $[0, \hat{T}_\infty)$ and satisfies

(a.N) $\mathbf{d}(\hat{x}(t)) \in (r_{i+N+1}, r_{i+N-2})$ for all $t \in [\hat{T}_{N-1}, \hat{T}_N]$;
(b.N) $\hat{x}(\hat{T}_N) \in \mathring{\mathcal{B}}_{i+N+1}$;
(c.N) $\hat{x}^0(\hat{T}_N) \leq \hat{x}_0(\hat{T}_{N-1}) + \frac{1}{2} d_{W^+}(r_{i+N-2}) \leq C d_{W^+}(r_{i-2})$, for $C := \frac{1}{2}\sum_{j=0}^{\infty} \frac{1}{4^j}$.

(The latter inequality can be easily proved by induction.) From these relations, it follows immediately that $\hat{T}_\infty = T_z(\hat{u})$, $\lim_{t\to\hat{T}_\infty^-} \mathbf{d}(\hat{x}(t)) = 0$, and $\hat{x}^0(\hat{T}_\infty) \leq C d_{W^+}(r_{i-2})$ (actually, even if $\hat{T}_\infty = +\infty$). Hence, if in case $\hat{T}_\infty < +\infty$ we extend the process $(\hat{x}^0, \hat{x}, \hat{u})$ to $[0, +\infty)$ by setting $\hat{u}(t) = w$ ($w \in U$ arbitrary), and $(\hat{x}^0, \hat{x})(t) = \lim_{t\to\hat{T}_\infty^-}(\hat{x}^0, \hat{x})(t)$ for any $t \geq \hat{T}_\infty$, we can conclude that $(\hat{x}^0, \hat{x}, \hat{u})$ is an admissible triple from $(0, \bar{z})$.
Step 3 (Construction of $\bar{\beta}$, Φ, and T). For all $i \in \mathbb{Z}$ and $N \in \mathbb{N}$, set

$$\bar{T}_{i,-1} := 0, \qquad \bar{T}_{i,N} := \sum_{j=0}^{N} T_{i+j},$$

where the times T_i are as in (21), so that for any $i \in \mathbb{Z}$, $\bar{T}_{i,N} \to +\infty$ as $N \to +\infty$. Then, we define the function $T : (0, +\infty) \to [0 + \infty)$, as

$$T(R) := \bar{T}_{i(R),-1\vee(-i(R)+1)} \qquad \text{for all } R > 0,$$

where $i(\cdot)$ is as in (22). Note that if $R \leq r_1 < 1$ then $i(R) \geq 2$ and this implies $T(R) = \bar{T}_{i(R),-1} = 0$. Since d_W^+ as well as $R \mapsto r_{i(R)}$ are increasing functions, the composition $R \mapsto d_W^+(r_{i(R)-2})$ is a positive, piecewise constant, increasing function, such that $d_W^+(r_{i(R)-2}) \to 0$ as $R \to 0^+$ and $d_W^+(r_{i(R)-2}) \to +\infty$ as $R \to +\infty$. There is thus a continuous, strictly increasing approximation from above $\Phi : [0, +\infty) \to [0, +\infty)$ of this composition times C (C as in (c.N) above), vanishing at zero and unbounded, namely

$$C\, d_W^+(r_{i(R)-2}) \leq \Phi(R) \qquad \text{for all } R > 0.$$

Finally, we introduce the function $b : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$, given by

$$\begin{aligned} b(R, t) &:= r_{i+N-2} \quad \text{if } (R, t) \in (r_i, r_{i-1}] \times [\bar{T}_{i,N-1}, \bar{T}_{i,N}), \quad i \in \mathbb{Z}, \; N \in \mathbb{N} \\ b(0, t) &:= 0 \quad \text{for all } t \geq 0. \end{aligned}$$

Note that $b(\cdot, t)$ is increasing and $b(R, t) \to +\infty$ as $R \to +\infty$, for all $t \geq 0$. Similarly, $b(R, \cdot)$ is positive, decreasing and $b(R, t) \to 0$ as $t \to +\infty$ for all $R > 0$. Using e.g. a linear interpolation procedure, it is not difficult to show that the discontinuous function b can be approximated from above by some $\mathcal{KL}$ function $\bar{\beta}$, which is $\geq \beta$ by construction.
Let $\bar{z} \in \mathbb{R}^n \setminus \mathcal{C}$ and set $i := i(\mathbf{d}(\bar{z}))$. Then, since $\hat{T}_N \leq \bar{T}_{i,N}$ for all $N \geq -1$, the admissible triple $(\hat{x}^0, \hat{x}, \hat{u})$ from $(0, \bar{z})$ built in Step 2, satisfies

$$\mathbf{d}(\hat{x}(t)) < r_{i+N-2} \cdot \chi_{[\bar{T}_{i,N-1}, \bar{T}_{i,N})}(t) \leq b(\mathbf{d}(\bar{z}), t) \leq \bar{\beta}(\mathbf{d}(\bar{z}), t) \;\; \text{for all } t \geq 0, \quad (23)$$

and

$$\hat{x}^0(t) \leq \bar{W}(\bar{z}) := \Phi(\mathbf{d}(\bar{z})) \qquad \text{for all } t \geq 0,$$

so that $\hat{u} \in \mathcal{U}_{\bar{\beta},\bar{W}}(\bar{z})$. Notice that $\bar{W} : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$ is a continuous, proper and positive definite function, in view of the properties of Φ and $\mathbf{d}$. This concludes the proof of statements (i) and (ii).
In order to prove (iii), let us first suppose $\mathbf{d}(\bar{z}) \leq r_1 < r_0 = 1$. In this case, $i(\mathbf{d}(\bar{z})) \geq 2$ and $T(\mathbf{d}(\bar{z})) = \bar{T}_{i(\mathbf{d}(z)),-1} = 0$. Hence (iii) follows from (23), because $\bar{\beta}(\cdot, t)$ is strictly increasing for every $t \geq 0$, so that

$$\mathbf{d}(\hat{x}(t)) < \bar{\beta}(\mathbf{d}(\bar{z}), t) \leq \bar{\beta}(1, t) = \bar{\beta}(1, t - T(\mathbf{d}(\bar{z}))) \quad \text{for all } t \geq T(\mathbf{d}(\bar{z}) = 0.$$

If instead $\mathbf{d}(\bar{z}) > r_1$, we have $i := i(\mathbf{d}(\bar{z})) \leq 1$. Set $N := -i(\mathbf{d}(z)) + 2 \;\; (\geq 1)$ and $\bar{z}_N := \hat{x}(\hat{T}_{N-1})$, so that, by property (b.N) of Step 2, $r_2 < \mathbf{d}(\bar{z}_N) < r_1 < r_0 = 1$. Note that, applying the above construction from the initial condition $(0, \bar{z}_N)$, the obtained admissible triple, say $(\hat{x}_N^0, \hat{x}_N, \hat{u}_N)$ from $(0, \bar{z}_N)$, satisfying

$$\mathbf{d}(\hat{x}_N(s)) < \bar{\beta}(\mathbf{d}(\bar{z}_N), s), \quad \hat{x}_N^0(s) \leq \Phi(\mathbf{d}(\bar{z}_N)) \quad \text{for all } s \geq 0,$$

is simply given by

$$\hat{u}_N(\cdot) = \hat{u}(\cdot + \hat{T}_{N-1}), \quad \hat{x}_N^0 = x^0(\cdot, \hat{u}_N, 0, \bar{z}_N), \quad \hat{x}_N = x(\cdot, \hat{u}_N, \bar{z}_N).$$

This fact is crucial for property (iii) to hold. Indeed, using the monotonicities of $\bar{\beta}$ and the inequality $\hat{T}_{N-1} \le \bar{T}_{i,N-1} = T(\mathbf{d}(\bar{z}))$, it implies that

$$\begin{aligned}\mathbf{d}(\hat{x}(t)) = \mathbf{d}(\hat{x}_N(t-\hat{T}_{N-1})) &< \beta(\mathbf{d}(\bar{z}_N), t-\hat{T}_{N-1}) \\ &< \bar{\beta}(1, t-\hat{T}_{N-1}) \le \bar{\beta}(1, t-T(\mathbf{d}(\bar{z}))) \qquad \text{for all } t \ge T(\mathbf{d}(\bar{z})). \qquad \square\end{aligned}$$

Lemma 3.2 *There exist two continuous, strictly increasing functions* $\ell, \Psi : [0,+\infty) \to [0,+\infty)$, *with* $\ell(0) = 0$, $\lim_{R\to+\infty} \ell(R) = +\infty$, *such that the functional* $J : (\mathbb{R}^n \setminus \mathcal{C}) \times \mathcal{M}([0,+\infty), U) \to [0,+\infty) \cup \{+\infty\}$, *defined as*

$$J(z,u) := \begin{cases} \displaystyle\int_0^{T_z(u)} [\ell(\mathbf{d}(x(t,z,u))) + l(x(t,z,u), u(t))]\, dt & \text{if } u \in \mathcal{U}(z), \\ +\infty & \text{if } u \in \mathcal{M}([0,+\infty), U) \setminus \mathcal{U}(z), \end{cases}$$

enjoys the properties (i)–(v) *below.*
For every $z \in \mathbb{R}^n \setminus \mathcal{C}$, *let* $(\hat{x}^0, \hat{x}, \hat{u})$ *be the admissible triple from* $(0, z)$ *built in Lemma* 3.1. *Using the notations of Lemma* 3.1, *we have*

(i) $J(z, \hat{u}) < +\infty$;
(ii) *if* $\mathbf{d}(z) < r_1$, *then* $J(z, \hat{u}) \le \bar{\beta}(\mathbf{d}(z), 0) + \Phi(\mathbf{d}(z))$;
(iii) *for any* $u \in \mathcal{M}([0,+\infty), U)$ *such that* $J(z,u) \le J(z,\hat{u})$, *(*$u \in \mathcal{U}(z)$ *and)* $\mathbf{d}(x(t,u,z)) \le \Psi(\mathbf{d}(z))$ *for all* $t \ge 0$;
(iv) *for all* $\alpha > 0$ *there exists* $\Theta > 0$ *such that for any* $u \in \mathcal{M}([0,+\infty), U)$ *with* $J(z,u) < \alpha$, *then* $\mathbf{d}(z) \le \Theta$;
(v) *for all* $\alpha > 0$ *there exists* $\delta > 0$ *such that, if* $\mathbf{d}(z) > \alpha$, *then* $J(z,u) > \delta$ *for all* $u \in \mathcal{M}([0,+\infty), U)$.

Proof Let $\bar{\beta}$ be a $\mathcal{KL}$ function as in Lemma 3.1. Extend the continuous strictly decreasing function $[0,+\infty) \ni t \mapsto \bar{\beta}(1,t)$ to a continuous strictly decreasing function defined on $\mathbb{R}$ and tending to $+\infty$ as $t \to -\infty$. Let $\tau : (0,+\infty) \to \mathbb{R}$ be the inverse of this map, so that, in particular, τ is continuous, strictly decreasing, $\tau(R) \to +\infty$ as $R \to 0^+$ and $\tau(R) \to -\infty$ as $R \to +\infty$.

Step 1. (Construction of a function ℓ_1*)* Define the continuous, strictly increasing, and unbounded function $\ell_1 : [0,+\infty) \to [0,+\infty)$, by setting

$$\ell_1(R) := Re^{-\tau(R)} \quad \text{for all } R > 0, \qquad \ell_1(0) := 0.$$

For any $0 < b < c$, we claim that there exists a function $\varkappa(b,c)$ such that, if $z_1, z_2 \in \mathbb{R}^n \setminus \mathcal{C}$, $u \in \mathcal{M}([0,+\infty), U)$, and $T > 0$ satisfy $z_2 = x(T,u,z_1)$ and $b \le \mathbf{d}(x(t,u,z_1)) \le c$ for all $t \in [0,T]$, then

$$\int_0^T \ell_1(\mathbf{d}(x(t,u,z_1)))dt \ge \varkappa(b,c)|z_1 - z_2| \ge \varkappa(b,c)|\mathbf{d}(z_1) - \mathbf{d}(z_2)|. \qquad (24)$$

Indeeed, if $z_1 = z_2$, (24) is trivially satisfied. If instead $z_1 \neq z_2$, then $\bar{M}(b, c) := \max\{|f(x, u)| \mid b \leq \mathbf{d}(x) \leq c,\ u \in U\} > 0$ and $|z_1 - z_2| \leq T\bar{M}(b, c)$. Setting $\varkappa(b, c) := \ell_1(b)\bar{M}(b, c)^{-1}$, since ℓ_1 is increasing, we have $\ell_1(\mathbf{d}(x(t, u, z_1))) \geq \ell_1(b)$ for all $t \in [0, T]$, and consequently

$$\int_0^T \ell_1(\mathbf{d}(x(t, u, z_1)))dt \geq T\ell_1(b) \geq \varkappa(b, c)|z_1 - z_2|.$$

Step 2 (Recursive definition of a sequence ℓ_j) Starting from ℓ_1 introduced in Step 1, for every $j \geq 1$ we will recursively define an increasing sequence of functions $\ell_j : [0, +\infty) \to \mathbb{R}$ with the following property:

$$0 \leq R \leq \bar{\beta}(j, 0) \quad \Longrightarrow \quad \ell_{j+1}(R) = \ell_j(R). \tag{25}$$

Then, we will prove that the pointwise limit $\ell := \lim_{j\to+\infty} \ell_j$ (finite, because of (25)) together with the map Ψ defined below, satisfies (i)–(v). Further, all the functions ℓ_j, (hence ℓ itself) will be continuous. We begin by assuming that, for some fixed $j \geq 2$, we already defined the functions ℓ_i for all $i \leq j$, satisfying (25) for all $i = 1, \ldots, j-1$. Given $z \in \mathbb{R}^n \setminus \mathcal{C}$ and any control $u \in \mathcal{U}(z)$, let us set

$$J_j(z, u) := \int_0^{T_z(u)} \ell_j(\mathbf{d}(x(t, u, z))) + l(x(t, u, z), u(t))dt.$$

Observe that, if we consider the admissible triple (x^0, x, u) from $(0, z)$ defined on $[0, +\infty)$ (even in case $T_z(u) < +\infty$, as in Definition 2.1), we can equivalently write

$$J_j(z, u) = \int_0^{+\infty} \ell_j(\mathbf{d}(x(t))) + \lim_{t\to T_z^-(u)} x^0(t),$$

as $\mathbf{d}(x(t)) = 0$ for all $t \geq T_z(u)$, whenever $T_z(u) < +\infty$. Assume $\mathbf{d}(z) \leq j$. Let $T(\cdot)$ and $(\hat{x}^0, \hat{x}, \hat{u})$ be the uniform time and the admissible triple from $(0, z)$ built in Lemma 3.1, respectively. Then, we have

$$\mathbf{d}(\hat{x}(t)) \leq \bar{\beta}(1, t - T(j)) \leq \bar{\beta}(1, 0) \qquad \text{for all } t \geq T(j).$$

In particular, (25) (together with the fact that $\bar{\beta}(\cdot, 0)$ is increasing) implies

$$\ell_j(\mathbf{d}(\hat{x}(t))) = \ell_1(\mathbf{d}(\hat{x}(t))) \qquad \text{for all } t \geq T(j).$$

If otherwise $t \leq T(j)$, we get

$$\mathbf{d}(\hat{x}(t)) < \bar{\beta}(\mathbf{d}(z), 0) \leq \bar{\beta}(j, 0).$$

Hence, we have

$$\begin{aligned}\int_0^{+\infty} \ell_j(\mathbf{d}(\hat{x}(t)))dt &= \int_0^{T(j)} \ell_j(\mathbf{d}((\hat{x}(t)))) + \int_{T(j)}^{+\infty} \ell_j(\mathbf{d}(\hat{x}(t)))dt \\ &\le \ell_j(\bar{\beta}(j,0))T(j) + \int_{T(j)}^{+\infty} \ell_1(\mathbf{d}(\hat{x}(t)))dt \\ &= \ell_j(\bar{\beta}(j,0))T(j) + \int_{T(j)}^{+\infty} \mathbf{d}(\hat{x}(t))e^{-t(\mathbf{d}(\hat{x}(t)))}dt \\ &\le \ell_j(\bar{\beta}(j,0))T(j) + \int_{T(j)}^{+\infty} \bar{\beta}(1,(t-T(j)))\, e^{-\tau(\bar{\beta}(1,(t-T(j)))}dt \\ &\le \ell_j(\bar{\beta}(j,0))T(j) + \bar{\beta}(1,0)\int_{T(j)}^{+\infty} e^{-t+T(j)}dt \\ &= \ell_j(\bar{\beta}(j,0))T(j) + \bar{\beta}(1,0) := L_j.\end{aligned}$$

Therefore, from Lemma 3.1, (ii) it follows that

$$J_j(z,\hat{u}) \le L_j + \Phi(j) \quad \text{for all } z \in \mathbb{R}^n \setminus \mathcal{C} \text{ such that } \mathbf{d}(z) \le j. \tag{26}$$

Now, set $\varkappa_j := \varkappa(\bar{\beta}(j,0)+1, \bar{\beta}(j,0)+2)$ and consider a continuous function $\rho_j : [0,+\infty) \to [0,+\infty)$ such that

$$\rho_j(R) = \begin{cases} 0 & \text{if } R \le \bar{\beta}(j,0) \text{ or } R \ge \bar{\beta}(j,0)+3, \\ \dfrac{L_j+\Phi(j)}{\varkappa_j} & \text{if } \bar{\beta}(j,0)+1 \le R \le \bar{\beta}(j,0)+2. \end{cases}$$

Finally, define

$$\ell_{j+1}(R) := (1+\rho_j(R))\ell_j(R) \qquad \text{for all } R \ge 0.$$

Clearly, ℓ_{j+1} satisfies (25).

Step 3. (Definition of ℓ and Ψ) As anticipated above, we define ℓ as the pointwise limit of the increasing sequence (ℓ_j). Incidentally notice that, by construction, ℓ is continuous and $\ell(R) \ge \ell_j(R)$ for all $R \in [0,+\infty)$ and all $j \ge 1$. Furthermore, let $\Psi : [0,+\infty) \to [0,+\infty)$ be any continuous strictly increasing function, satisfying

$$R \in [j-1, j] \quad \Rightarrow \quad \Psi(R) \ge \bar{\beta}(j,0)+2 \quad \text{for all } j \ge 1.$$

We show that ℓ and Ψ have the required properties. Let $z \in \mathbb{R}^n \setminus \mathcal{C}$ be given.

In order to prove (i), it suffices to observe that there exists an integer $j \ge 1$ such that $j-1 < \mathbf{d}(z) \le j$. Then, (26) implies that

$$J(z,\hat{u}) = J_j(z,\hat{u}) \le L_j + \Phi(j) < +\infty. \tag{27}$$

Let us now prove (ii). Assume $\mathbf{d}(z) \le r_1 < 1$. Then, $T(\mathbf{d}(z)) = 0$, $\ell(\mathbf{d}(\hat{x}(t))) = \ell_1(\mathbf{d}(\hat{x}(t)))$ for all $t \ge 0$, and, arguing as in Step 2, we obtain

$$J(z, \hat{u}) = \int_0^{+\infty} \left[\ell_1(\mathbf{d}(\hat{x}(t)) + l(\hat{x}(t), \hat{u}(t))\right] dt \le \int_0^{+\infty} \bar{\beta}(\mathbf{d}(z), t)\, e^{-t}\, dt + \Phi(\mathbf{d}(z))$$
$$\le \bar{\beta}(\mathbf{d}(z), 0) + \Phi(\mathbf{d}(z)) \qquad \text{for all } t \ge 0.$$

To prove (iii), consider $u \in \mathcal{M}([0, +\infty), U)$ satisfying $J(z, u) \le J(z, \hat{u})$. Actually, $u \in \mathcal{U}(z)$, so let (x^0, x, u) be the corresponding admissible triple from $(0, z)$. As above, let j be the integer ≥ 1 such that $j - 1 < \mathbf{d}(z) \le j$. By the properties of $\bar{\beta}$, we have $\bar{\beta}(j, 0) > j$, so that $\mathbf{d}(z) < \bar{\beta}(j, 0)$. In contradiction to claim (iii), suppose that there exists $t > 0$ such that

$$\mathbf{d}(x(t)) > \Psi(\mathbf{d}(z)) > \bar{\beta}(j, 0) + 2.$$

Then, there exist $0 < t_1 < t_2 < T_z(u)$ such that

$$\begin{aligned} &\mathbf{d}(x(t_1)) = \bar{\beta}(j, 0) + 1, \qquad \mathbf{d}(x(t_2)) = \bar{\beta}(j, 0) + 2, \\ &\bar{\beta}(j, 0) + 1 \le \mathbf{d}(x(t)) \le \bar{\beta}(j, 0) + 2 \qquad \text{for all } t \in [t_1, t_2]. \end{aligned} \tag{28}$$

Therefore, in view of (27), (24), and the definition of $(\ell_j)_j$, we have

$$\begin{aligned} J(z, u) &= \int_0^{T_z(u)} [\ell(\mathbf{d}(x(t))) + l(x(t), u(t))]\, dt \ge \int_0^{T_z(u)} \ell(\mathbf{d}(x(t)))dt \\ &> \int_{t_1}^{t_2} \ell_{j+1}(\mathbf{d}(x(t)))dt \ge (1 + \rho_j(\bar{\beta}(j, 0) + 1)) \int_{t_1}^{t_2} \ell_1(\mathbf{d}(x(t)))dt \\ &\ge (1 + \rho_j(\bar{\beta}(j, 0) + 1))\varkappa_j = L_j + \Phi(j) \ge J_j(z, \hat{u}) = J(z, \hat{u}). \end{aligned}$$

This provides the required contradiction and the proof of (iii) is complete.

As in claim (iv), let us now suppose that $u \in \mathcal{U}(z)$ is a control satisfying $J(z, u) < \alpha$, for some $\alpha > 0$. Let (x^0, x, u) be the corresponding admissible triple from $(0, z)$, and let $j \ge 1$ be the smallest integer such that $L_j + \Phi(j) > \alpha$. We want to show that

$$\mathbf{d}(z) < \Theta := \bar{\beta}(j, 0) + 2.$$

Indeed, assume instead that $\mathbf{d}(z) \ge \bar{\beta}(j, 0) + 2$. If there exists a time $t \ge 0$ such that $\mathbf{d}(x(t)) < c := \bar{\beta}(j, 0) + 1$, then there are $0 < t_1 < t_2 < T_z(u)$ as in (28). Thus, arguing as in the proof of claim (iii) we can deduce the inequality $J(z, u) > L_j + \Phi(j) > \alpha$, in contradiction with the hypothesis $J(z, u) < \alpha$. If instead $\mathbf{d}(x(t, z, u)) \ge c$ for all $t \le T_z(u)$, then $T_z(u) = +\infty$ and we get the contradiction

$$J(z, u) \ge \int_0^{+\infty} \ell_1(\mathbf{d}(x(t)))dt \ge \int_0^{+\infty} \ell_1(c)dt = +\infty.$$

Let finally prove (v). Let $\alpha > 0$ and assume $\mathbf{d}(z) > \alpha$. For any $u \in \mathcal{M}([0, +\infty), U) \setminus \mathcal{U}(z)$, the cost $J(z, u) = +\infty$, so (v) is trivially true. If $u \in \mathcal{U}(z)$, by the last part of the proof of (iv) (for $c := \alpha/2$) there exists a positive, finite time $t_2 := \inf\{t \in [0, T_z(u)) \mid \mathbf{d}(x(t, u, z)) \leq \alpha/2\}$. Let $0 < t_1 < t_2$ be a time such that $\alpha/2 \leq \mathbf{d}(x(t, u, z) \leq \alpha$ for all $t \in [t_1, t_2]$. Then, in view of (24), we have

$$J(z, u) \geq \int_{t_1}^{t_2} \ell_1(\mathbf{d}(x(t, u, z)))dt \geq \frac{\alpha}{2}\varkappa(\alpha/2, \alpha) =: \delta > 0. \qquad \square$$

Let J be as in Lemma 3.2. For any $z \in \overline{\mathbb{R}^n \setminus \mathcal{C}}$, we define the value function

$$V(z) := \inf_{u \in \mathcal{M}([0,+\infty),U)} J(z, u) \quad \text{for all } z \in \mathbb{R}^n \setminus \mathcal{C}, \qquad V(z) := 0 \quad \text{for all } z \in \partial\mathcal{C}.$$

In the following two lemmas we show that V is a MRF for (8)–(9).

Lemma 3.3 *The function $V : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$ has the following properties:*

(i) *dom $V := \{z \in \overline{\mathbb{R}^n \setminus \mathcal{C}} \mid V(z) < +\infty\} = \overline{\mathbb{R}^n \setminus \mathcal{C}}$;*
(ii) *V is positive definite;*
(iii) *V is proper;*
(iv) *V is continuous.*

Proof Property (i) holds, because, if $z \in \partial\mathcal{C}$, then $V(z) = 0 < +\infty$, while, if $\mathbf{d}(z) > 0$, by Lemma 3.2, (i) we have $V(z) \leq J(z, \hat{u}) < +\infty$.

In order to prove (ii), observe that given $z \in \mathbb{R}^n \setminus \mathcal{C}$, then $\mathbf{d}(z) > \alpha$ for some $\alpha > 0$. Hence, by Lemma 3.2, (v) there exists $\delta > 0$, depending only on α, such that $J(z, u) > \delta$ for all $u \in \mathcal{M}([0, +\infty), U)$. As a consequence, $V(z) \geq \delta > 0$, that is V is positive outside the target.

The function V satisfies (iii) whenever, for all $\alpha > 0$, the sublevel set $E_\alpha := \{z \in \overline{\mathbb{R}^n \setminus \mathcal{C}} \mid V(z) < \alpha\}$ is bounded. If $V(z) < \alpha$, then by definition there exists $u \in \mathcal{U}(z)$ such that $J(z, u) < \alpha$, as well. Hence, by Lemma 3.2, (iv) we deduce that $\mathbf{d}(z) < \Theta$ for some $\Theta > 0$ (depending only on α) and, consequently, the set E_α is bounded, as $E_\alpha \subset \overline{B_\Theta(\mathcal{C}) \setminus \mathcal{C}}$.

Let us finally prove (iv), i.e. the continuity of V. Fix $\varepsilon > 0$ and let us first consider $\bar{z} \in \partial\mathcal{C}$. By virtue of Lemma 3.2, (ii), for any z such that $\mathbf{d}(z) < r_1$, we have $V(z) \leq J(z, \hat{u}) \leq \hat{\Phi}(\mathbf{d}(z)) := \bar{\beta}(\mathbf{d}(z), 0) + \Phi(\mathbf{d}(z))$, where, in particular, $\hat{\Phi}$ is continuous, strictly increasing and equal to 0 at 0. Hence, choosing

$$0 < \delta_\varepsilon < \frac{1}{2}\left(r_1 \wedge \hat{\Phi}^{-1}(\varepsilon)\right), \tag{29}$$

we get the continuity of V at $\bar{z}$, as

$$|V(z) - V(\bar{z})| = V(z) \leq J(z, \hat{u}) \leq \hat{\Phi}(\mathbf{d}(z)) < \varepsilon \quad \text{for all } z \in B_{2\delta_\varepsilon}(\bar{z}). \tag{30}$$

Assume now $\bar{z} \in \mathbb{R}^n \setminus \mathcal{C}$. Setting $\tilde{\delta}_{\bar{z},\varepsilon} := \delta_\varepsilon \wedge \frac{\mathbf{d}(\bar{z})}{4}$, we have $B_{2\tilde{\delta}_{\bar{z},\varepsilon}}(\bar{z}) \subset \mathbb{R}^n \setminus \mathcal{C}$. We claim that for any $z \in B_{\tilde{\delta}_{\bar{z},\varepsilon}/2}(\{\bar{z}\})$ there exists an admissible triple (x^0, x, u) from $(0, z)$ such that

$$\begin{aligned} &\mathbf{d}(x(t)) \le M_{\bar{z}} := \max\left\{\Psi(\mathbf{d}(z)) \mid z \in \overline{B_{r_1/4}(\{\bar{z}\})}\right\} \quad \text{for all } t \ge 0, \\ &J(z, u) \le V(z) + \varepsilon. \end{aligned} \tag{31}$$

Indeed, an ε-optimal triple (x^0, x, u) satisfying $J(z, u) \le V(z) + \varepsilon$ always exists, as $V(z) < +\infty$ by (i) above. Furthermore, we can clearly assume $J(z, u) \le J(z, \hat{u})$, where $(\hat{x}^0, \hat{x}, \hat{u})$ is the admissible triple from $(0, z)$ built in Lemma 3.2. But then the first of the inequalities above follows from Lemma 3.2, (iii).
Set

$$\tilde{T}_z := \inf\{t \ge 0 \mid \mathbf{d}(x(t)) < \tilde{\delta}_{\bar{z},\varepsilon}\}. \tag{32}$$

Note that $\tilde{T}_z > 0$, since $\mathbf{d}(z) > \mathbf{d}(\bar{z}) - \frac{\tilde{\delta}_{\bar{z},\varepsilon}}{2} > \frac{3}{2}\tilde{\delta}_{\bar{z},\varepsilon}$. Let $j = j_{\bar{z},\varepsilon}$ be a positive integer such that $j \ge \mathbf{d}(\bar{z}) + \frac{\tilde{\delta}_{\bar{z},\varepsilon}}{2} > \mathbf{d}(z)$. Then, using (27) (which is valid for every $j \ge \mathbf{d}(z)$, in view of (26)) and setting $m_{\bar{z},\varepsilon} := \min_{R \in [\tilde{\delta}_{\bar{z},\varepsilon}, M_{\bar{z}}]} \ell_1(R) > 0$, we get

$$L_j + \Phi(j) \ge J(z, \hat{u}) \ge J(z, u) \ge \int_0^{\tilde{T}_z} \ell_1(\mathbf{d}(x(t)))dt \ge \tilde{T}_z m_{\bar{z},\varepsilon}.$$

Thus, there exists an uniform upper bound for the times $\tilde{T}_z$. Precisely, we have

$$\tilde{T}_z \le T_{\bar{z},\varepsilon} := \frac{L_{j_{\bar{z},\varepsilon}} + \Phi(j_{\bar{z},\varepsilon})}{m_{\bar{z},\varepsilon}} \qquad \text{for all } z \in B_{\tilde{\delta}_{\bar{z},\varepsilon}/2}(\{\bar{z}\}).$$

To prove the continuity of V at $\bar{z}$, consider arbitrary points $z_1, z_2 \in B_{\tilde{\delta}_{\bar{z},\varepsilon}/2}(\{\bar{z}\})$ and suppose, for instance, $V(z_1) \le V(z_2)$. Let (x_1^0, x_1, u_1) be an admissible triple from $(0, z_1)$ satisfying (31) (for $z = z_1$). In particular, this implies that $x_1(t)$ lies in a compact set $\mathcal{K}$ depending only on $\bar{z}$ for all $t \ge 0$. Hence, if $L_{\bar{z}}$ denotes the Lipschitz constant in x of the dynamics function f on the compact set $\overline{B_1(\mathcal{K})}$, by a standard cut-off technique we can derive that the trajectory $x(\cdot, u_1, z_2)$ is defined for all $t \in [0, T_{\bar{z},\varepsilon}]$ and satisfies

$$\sup_{t \in [0, T_{\bar{z},\varepsilon}]} |x_1(t) - x(t, u_1, z_2)| \le |z_1 - z_2|\, e^{L_{\bar{z}}\, T_{\bar{z},\varepsilon}},$$

as soon as $|z_1 - z_2| < e^{-L_{\bar{z}}\, T_{\bar{z},\varepsilon}}$. Actually, from this inequality it also follows that, setting

$$\bar{\delta}_{\bar{z},\varepsilon} := \tilde{\delta}_{\bar{z},\varepsilon} \wedge \left(\frac{\delta_\varepsilon}{2} \wedge 1\right) e^{-L_{\bar{z}}\, T_{\bar{z},\varepsilon}}$$

(δ_ε as in (29)), and assuming $z_1, z_2 \in B_{\bar\delta_{\bar z,\varepsilon}/2}(\{\bar z\})$, we have that ($|z_1 - z_2| < \bar\delta_{\bar z,\varepsilon}$ and) $|x(\tilde T_{z_1}, u_1, z_2) - x_1(\tilde T_{z_1})| \le \frac{\delta_\varepsilon}{2}$ ($\tilde T_{z_1}$ is as in (32), for $x = x_1$). Since $\mathbf{d}(x_1(\tilde T_{z_1})) \le \delta_\varepsilon$, this implies that $\bar z_2 := x(\tilde T_{z_1}, u_1, z_2) \in B_{2\delta_\varepsilon}(\bar z)$. At this point, if $\hat u_2 \in \mathcal{U}(z_2)$ denotes an admissible control from $(0, \bar z_2)$ as in Lemma 3.2, the last part of (30) implies that $J(\bar z_2, \hat u_2) < \varepsilon$. Therefore, the control u_2 given by

$$u_2(t) := \begin{cases} u_1(t) & t \in [0, \tilde T_{z_1}] \\ \hat u_2(t - \tilde T_{z_1}) & t \in (\tilde T_{z_1}, +\infty) \end{cases}$$

belongs to $\mathcal{U}(z_2)$, $x_2(t) := x(t, u_2, z_2)$ belongs to $\overline{B_1(\mathcal{K})}$ for all $t \in [0, \tilde T_{z_1}]$, and denoting with $\omega_{\bar z}$ the modulus of continuity of $\overline{B_1(\mathcal{K})} \ni x \mapsto \ell(\mathbf{d}(x)) + l(x, u)$ (uniform w.r.t. the control, because of the assumptions on l), we finally obtain

$$\begin{aligned} 0 &\le V(z_2) - V(z_1) \le J(z_2, u_2) - J(z_1, u_1) + \varepsilon \\ &\le \int_0^{\tilde T_{z_1}} [|\ell(\mathbf{d}(x_2(t))) - \ell(\mathbf{d}(x_1(t)))|\, dt + |l(x_2(t), u_1(t)) - l(x_1(t), u_1(t))|]\, dt \\ &\quad + J(\bar z_2, \hat u_2) + \varepsilon \le T_{\bar z,\varepsilon}\, \omega_{\bar z}(\bar\delta_{\bar z,\varepsilon}) + 2\varepsilon \le 3\varepsilon, \end{aligned}$$

where the last inequality holds by replacing $\bar\delta_{\bar z,\varepsilon}$ with $\bar\delta_{\bar z,\varepsilon} \wedge \omega_{\bar z}^{-1}\left(\frac{\varepsilon}{T_{\bar z,\varepsilon}}\right)$. The continuity of V at $\bar z$ hence follows by the arbitrariness of ε. □

Lemma 3.4 *The value function V satisfies the decrease condition* (13).

Proof We divide the proof in three steps.

Step 1. Let us first show that, if V is a viscosity supersolution of the following Hamilton–Jacobi–Bellman equation

$$\max_{u \in U}\{-\langle DV(z), f(z, u)\rangle - l(z, u)\} = \ell(\mathbf{d}(z)) \qquad \text{for all } z \in \mathbb{R}^n \setminus \mathcal{C}, \tag{33}$$

then it satisfies the decrease condition (13), characterizing MRFs. Indeed, (33) implies that (see e.g. the survey paper [3])

$$\max_{u \in U}\{-\langle p, f(z, u)\rangle - l(z, u)\} \ge \ell(\mathbf{d}(z)) \qquad \text{for all } z \in \mathbb{R}^n \setminus \mathcal{C}, \quad \text{for all } p \in \partial_P V(z),$$

so that, for H defined as in (12) and $p_0 \equiv 1$, one has

$$H(z, 1, \partial_P V(z)) \le -\ell(\mathbf{d}(z)) \qquad \text{for all } z \in \mathbb{R}^n \setminus \mathcal{C}, \quad \text{for all } p \in \partial_P V(z),$$

Setting $\gamma := \ell \circ d_{V^+}^{-1}$ and recalling that L is increasing as well as d_{V^+}, by (7) we finally obtain that V satisfies condition (13) for $p_0 \equiv 1$ and such a γ, namely

$$H(z, 1, \partial_P V(x)) \le -\gamma(V(z)) \quad \text{for all } z \in \mathbb{R}^n \setminus \mathcal{C}.$$

Thus, the next two steps will be devoted to prove that V is a viscosity supersolution of (33). This proof is not completely standard, because we do not have the usual growth hypotheses on f, ℓ and l (see e.g. [1, Ch. III]). Actually, these assumptions can be avoided here thanks to the results in Lemma 3.2.

Step 2. Let us show that, for every $T > 0$ and every $z \in \mathbb{R}^n \setminus \mathcal{C}$, one has

$$V(z) \geq \inf_{u \in \hat{\mathcal{U}}(z)} \left\{ \int_0^{T_z(u) \wedge T} [\ell(\mathbf{d}(x(t))) + l(x(t), u(t))]dt + V(x(T_z(u) \wedge T)) \right\}, \tag{34}$$

where $x := x(\cdot, u, z)$, $\hat{\mathcal{U}}(z) := \{u \in \hat{\mathcal{U}}(z) \mid J(z, u) \leq J(z, \hat{u})\}$, and $\hat{u}$ is as in Lemma 3.2. In view of Lemma 3.2,(i), the set $\hat{\mathcal{U}}(z) \neq \emptyset$ and

$$V(z) = \inf_{u \in \hat{\mathcal{U}}(z)} \int_0^{T_z(u)} [\ell(\mathbf{d}(x(t))) + l(x(t), u(t))]dt < +\infty.$$

Let us refer to the right-hand side of (34) as $v_T(z)$. Given $u \in \hat{\mathcal{U}}(z)$, if $T_z(u) \leq T$ we have $V(x(T_z(u) \wedge T)) = 0$ and $J(x, u) \geq v_T(z)$. If instead $T_z(u) > T$, in view of the definition of v_T, we get

$$\begin{aligned}
J(z, u) &= \int_0^T [\ell(\mathbf{d}(x(t))) + l(x(t), u(t))]dt + \int_T^{T_z(u)} [\ell(\mathbf{d}(x(t))) + l(x(t), u(t))]dt \\
&= \int_0^T [\ell(\mathbf{d}(x(t))) + l(x(t), u(t))]dt + \int_0^{T_{x_T(0)}(u_T)} [\ell(\mathbf{d}(x_T(t))) + l(x_T(t), u_T(t))]dt \\
&\geq \int_0^T [\ell(\mathbf{d}(x(t))) + l(x(t), u(t))]dt + V(x(T)) \geq v_T(z),
\end{aligned}$$

where $x_T(t) := x(t+T)$, $u_T(t) := u(t+T)$, and $T_{x_T(0)}(u_T) = T_z(u) - T$. Therefore, $V(z) = \inf_{u \in \hat{\mathcal{U}}(z)} J(z, u) \geq v_T(z)$, that is, the relation (34) is proven.

Step 3. Let us now deduce from (34) that V is a viscosity supersolution of (33). To this aim, fixed $z \in \mathbb{R}^n \setminus \mathcal{C}$, we preliminarily observe that, in view of Lemma 3.2,(iii), every admissible trajectory-control pair (x, u) with $u \in \hat{\mathcal{U}}(z)$, satisfies

$$\mathbf{d}(x(t)) \leq \Psi(\mathbf{d}(z)) =: R_z \qquad \text{for all } t \geq 0.$$

Furthermore, on the compact set $\overline{B(\mathcal{C}, R_z) \setminus \mathcal{C}}$, depending only on z, the functions f and l are Lipschitz continuous in x, uniformly w.r.t. the control, and we can fix a modulus of continuity for ℓ and a bound $M_z > 0$ for $|f|$, ℓ, and l. Hence, choosing e.g. $\bar{T}_z = \frac{\mathbf{d}(z)}{2M_z}$, for any $u \in \hat{\mathcal{U}}(z)$ the trajectory $x(\cdot, u, z)$ is defined on $[0, \bar{T}_z]$ and satisfies $0 < \mathbf{d}(x(t, u, z)) \leq R_z$ for all $t \in [0, \bar{T}_z]$. For arbitrary $\varepsilon > 0$ and $0 < T < \bar{T}_z$, (34) implies that there exists some $\bar{u} = \bar{u}_{\varepsilon,T} \in \hat{\mathcal{U}}(z)$, such that

$$\int_0^T [\ell(\mathbf{d}(\bar{x}(t))) + l(\bar{x}(t), \bar{u}(t))]dt + V(\bar{x}(T)) \leq V(z) + \varepsilon T,$$

where $\bar{x} := x(\cdot, \bar{u}, z)$. In view of the above considerations, from now on, taken a test function $\varphi \in C^1(\mathbb{R}^n)$ such that $\varphi(z) = V(z)$ and $\varphi(\tilde{z}) \le V(\tilde{z})$ for all $\tilde{z} \in B(z, r)$, for some $r > 0$, the proof that V is a viscosity supersolution of (33) proceeds as usual, hence we omit it (see e.g. [1, Prop. III, 2.8]).

□

Surveying the results on V in Lemmas 3.3 and 3.4, we see that the proof of implication (i) $\Longrightarrow$ (ii) is concluded.

4 Proof of Implication (ii) $\Longrightarrow$ (i)

The proof that the existence of a MRF implies GAC to $\mathcal{C}$ with regulated cost relies on the following result, establishing a super-optimality principle satisfied by any MRF.

Proposition 4.1 *Let $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$ be a continuous MRF for (8)–(9) for some continuous and increasing function $p_0 : (0, +\infty) \to [0, 1]$ and some continuous and strictly increasing function $\gamma : (0, +\infty) \to (0, +\infty)$. Then, for any $z \in \mathbb{R}^n \setminus \mathcal{C}$, we have*

$$W(z) \ge \inf_{u \in \mathcal{U}(z)} \sup_{0 \le T < T_z(u)} \left\{ \begin{array}{r} \displaystyle\int_0^T \left[p_0(W(x(t))) l(x(t), u(t)) + \gamma(W(x(t))) \right] dt \\ +W(x(T)) \end{array} \right\}, \tag{35}$$

where $x(\cdot) := x(\cdot, u, z)$.

The proof of this proposition relies on the definite positiveness and properness of W coupled with an extension of results which are already known only under more restrictive assumptions than ours (see e.g. [1, Thm. 2.40] and [10, Thm. 3.3]), and it will be given at the end of the section.

Let $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \to [0, +\infty)$ be a continuous MRF for (8)–(9) for some functions $p_0 : (0, +\infty) \to [0, 1]$ and $\gamma : (0, +\infty) \to (0, +\infty)$ as in Definition 2.3. Assume that p_0 satisfies the integrability condition (IC). Fix $z \in \mathbb{R}^n \setminus \mathcal{C}$. From (35) it follows that, for $\varepsilon_1 := \frac{W(z)}{2} > 0$ there exists some admissible control $u_1 \in \mathcal{U}(z)$ such that, for any $0 \le T < T_z(u_1)$, we have

$$\int_0^T [p_0(W(x_1(t))) l(x_1(t), u_1(t)) + \gamma(W(x_1(t)))] dt + W(x_1(T)) \\ \le W(z) + \varepsilon_1 = W(z) + \frac{W(z)}{2}, \tag{36}$$

where $x_1(\cdot) := x(\cdot, u_1, z)$. We set

$$t_1 := \inf \left\{ t \in [0, T_z(u_1)) \;\middle|\; W(x_1(t)) \le \frac{W(z)}{2} \right\}, \qquad z_1 := x_1(t_1).$$

Clearly, t_1 is > 0 and is actually a minimum. Hence, (36) implies that

$$\begin{cases} \frac{1}{2}\, W(z) \leq W(x_1(t)) \leq \frac{3}{2}\, W(z) \qquad \text{for all } t \in [0, t_1], \\ W(z_1) = W(x_1(t_1)) = \frac{1}{2}\, W(z), \end{cases} \tag{37}$$

so that

$$p_0(W(z_1)) \int_0^{t_1} l(x_1(t), u_1(t))\, dt \leq \int_0^{t_1} p_0(W(x_1(t)))\, l(x_1(t), u_1(t))\, dt$$
$$\leq W(z) + \frac{W(z)}{2} - W(z_1) = W(z) = 2[W(z) - W(z_1)],$$

which yields the cost bound

$$\int_0^{t_1} l(x_1(t), u_1(t))\, dt \leq 2\frac{W(z) - W(z_1)}{p_0(W(z_1))}. \tag{38}$$

From (36), using the functions d_{W^-}, d_{W^+} introduced in (7), we also obtain

$$\gamma\left(\frac{1}{2}\, d_{W^-}(\mathbf{d}(z))\right) t_1 \leq \gamma(W(z_1))\, t_1 \leq \int_0^{t_1} \gamma(W(x_1(t)))\, dt \leq \frac{3}{2}\, W(z) \leq \frac{3}{2}\, d_{W^+}(\mathbf{d}(z)).$$

Define now the continuous function $\bar{T}_1 : (0, +\infty) \to (0, +\infty)$, given by

$$\bar{T}_1(R) := \frac{3 d_{W^+}(R)}{2\gamma\left(\frac{1}{2}\, d_{W^-}(R)\right)} \qquad \text{for all } R > 0.$$

Hence, the latter inequality yields the following uniform time bound

$$t_1 \leq \bar{T}_1(\mathbf{d}(z)). \tag{39}$$

Starting from z_1 and choosing $\varepsilon_2 := \frac{W(z_1)}{2} = \frac{W(z)}{4}$, arguing as above we can deduce from (36) the existence of a control $u_2 \in \mathcal{U}(z_1)$ and a time $t_2 > 0$, such that, denoting by x_2 the trajectory $x(\cdot, u_2, z_1)$ and setting $z_2 := x_2(t_2)$, we get relations (37)–(39) with z, z_1, u_1, x_1, and t_1 replaced by z_1, z_2, u_2, x_2, and t_2, respectively. Set $z_0 := z$. In a recursive way, for any integer $N \geq 1$, we can thus choose $\varepsilon_N := \frac{W(z)}{2^N}$ and construct z_N, u_N, x_N, and $t_N > 0$, such that $u_N \in \mathcal{U}(z_{N-1})$, and $x_N(\cdot) := x(\cdot, u_N, z_{N-1})$, $z_N := x_N(t_N)$, satisfy

$$\frac{1}{2}\, W(z_{N-1}) \leq W(x_N(t)) \leq \frac{3}{2}\, W(z_{N-1}) \qquad \text{for all } t \in [0, t_N],$$
$$W(z_N) = \frac{1}{2}\, W(z_{N-1}) = \frac{1}{2^N}\, W(z), \tag{40}$$

$$\int_0^{t_N} l(x_N(t), u_N(t))\, dt \leq 2\, \frac{W(z_{N-1}) - W(z_N)}{p_0(W(z_N))} = 4\, \frac{W(z_N) - W(z_{N+1})}{p_0(W(z_N))}, \tag{41}$$

and

$$t_N \leq \bar{T}_N(\mathbf{d}(z)), \tag{42}$$

where

$$\bar{T}_N(R) := \frac{3d_{W^+}(R)}{2^N \gamma\left(\frac{1}{2^N}\, d_{W^-}(R)\right)} \qquad \text{for all } R > 0.$$

Set now $T_0 := 0$, $T_N := \sum_{j=1}^{N} t_j$, and $T_\infty := \sum_{j=1}^{+\infty} t_j$ and, for every integer $N \geq 1$, define the control $u \in \mathcal{M}([0, +\infty), U)$ given by

$$\begin{aligned} &u(t) := u_N(t - T_{N-1}) \qquad \text{for all } t \in [T_{N-1},\ T_N), \\ &u(t) := w \qquad \text{for all } t \geq T_\infty, \quad \text{if } T_\infty < +\infty, \end{aligned}$$

(for $w \in U$ arbitrary). Recalling that W is proper and positive definite, from (40) it is easy to deduce that $\lim_{t\to T_\infty^-} \mathbf{d}(x(t, u, z)) = 0$, so $u \in \mathcal{U}(z)$. Let (x^0, x, u) be the corresponding admissible triple from $(0, z)$ (defined on $[0, +\infty)$). In view of (41) and recalling that $1/p_0$ is decreasing, the cost x^0 satisfies

$$\begin{aligned} x^0(t) \leq \int_0^{T_\infty} l(x(t), u(t))\, dt &\leq \sum_{N=1}^{+\infty} 4\, \frac{W(z_N) - W(z_{N+1})}{p_0(W(z_N))} \\ &\leq 4 \int_0^{W(z)/2} \frac{dv}{p_0(v)} = \bar{W}(z) \qquad \text{for every } t \geq 0, \end{aligned} \tag{43}$$

as soon as we set $\bar{W}(z) := 4P(W(z)/2)$ for all $z \in \overline{\mathbb{R}^n \setminus \mathcal{C}}$ (P as in (14)). Notice that this function $\bar{W}$ is continuous, proper and positive definite, by the integrability assumption (IC). So, the triple (x^0, x, u) satisfies the cost bound condition (11) with regulation function $\bar{W}$. To conclude the proof that control system (8) with cost (9) is GAC to $\mathcal{C}$ with regulated cost, it remains only to prove the existence of a descent rate β, such that

$$\mathbf{d}(x(t)) \leq \beta(\mathbf{d}(z), t) \qquad \text{for every } t \geq 0. \tag{44}$$

First of all, we claim that there exist a strictly increasing, unbounded, continuous function $\Gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\Gamma(0) = 0$, and a function $\mathbf{T} : \mathbb{R}^2_{>0} \to \mathbb{R}_{>0}$, such that, for any $0 < r < R$, for every $z \in \mathbb{R}^n \setminus \mathcal{C}$ with $\mathbf{d}(z) \leq R$, the trajectory x from z considered above satisfies the following conditions:

$$\begin{aligned} &\text{(a) } \mathbf{d}(x(t)) \leq \boldsymbol{\Gamma}(R) && \text{for all } t \geq 0, \\ &\text{(b) } \mathbf{d}(x(t)) \leq r && \text{for all } t \geq \mathbf{T}(R, r). \end{aligned}$$

Condition (a) follows from (40), because by (7) we have

$$\mathbf{d}(x(t)) \leq d_{W^-}^{-1}(W(x(t))) \leq d_{W^-}^{-1}\left(\frac{3}{2}\, d_{W^+}(\mathbf{d}(z))\right) \leq \varGamma(R) \quad \text{for all } t \geq 0,$$

as soon as we choose $\varGamma(R) := d_{W^-}^{-1}\left(\frac{3}{2}\, d_{W^+}(R)\right)$, $R \geq 0$. This $\varGamma$ has all the required properties in view of the properties of d_{W^-} and d_{W^+}. In order to derive (b), we observe that (40) implies

$$\mathbf{d}(x(t)) \leq d_{W^-}^{-1}(W(x(t))) \leq d_{W^-}^{-1}\left(\frac{3}{2^N}\, d_{W^+}(R)\right) \quad \text{for all } t \geq T_N,$$

so, if $N(R,r)$ is the smallest integer $\geq \log_2\left(3\frac{d_{W^+}(R)}{d_{W^-}(r)}\right)$, we get

$$\mathbf{d}(x(t)) \leq r \qquad \text{for all } t \geq T_{N(R,r)}.$$

The time $T_{N(R,r)}$ depends on z, but, by (42), the value

$$\mathbf{T}(R,r) := \sum_{j=1}^{N(R,r)} \bar{T}_j(R)$$

is a uniform upper bound for $T_{N(R,r)}$. Hence, also condition (b) is valid. Now, arguing as in [26, Sec. 5], for any $R > 0$ let us introduce a strictly increasing, diverging sequence of positive times $(t_j)_{j\geq 1}$ depending on R, such that

$$t_j \geq \mathbf{T}\left(R, \frac{R}{j+1}\right)$$

and define the function $b : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, given by

$$b(R,t) := \begin{cases} \varGamma(R) & \text{for all } t \in [0, t_1), \\ \dfrac{R}{j+1} & \text{for all } t \in [t_j, t_{j+1}),\ j \geq 1. \end{cases}$$

From (a) and (b) it follows that

$$\mathbf{d}(x(t)) \leq b(\mathbf{d}(z), t) \qquad \text{for all } t \geq 0.$$

As already noticed in the proof of Lemma 3.1, Step 3, it is actually a routine exercise to find a $\mathcal{KL}$ function $\beta \geq b$. Therefore, the proof of implication (ii) $\Longrightarrow$ (i) is thus complete. □

4.1 A Comparison Principle and the Proof of Proposition 4.1

In the proof of Proposition 4.1, we will use the slightly modified version below of the classical Comparison Principle for the infinite horizon problem. In particular, in the known results it is basically necessary to assume the unilateral Lipschitz continuity hypothesis (i) below[8] (see e.g. the comments after [1, III. Thm. 2.12]). In the following lemma we show that, when the state has two components and the dynamics F depends on only one of them, x-Lipschitz continuity of the F-component that corresponds to the missing variable is not necessary.

Lemma 4.1 (Comparison Principle) *Let $U \subset \mathbb{R}^m$, $A \subset \mathbb{R}^{m'}$ be compact control sets (not both empty), let $L : \mathbb{R}^n \times \mathbb{R}^{n'} \times U \times A \to \mathbb{R}$ be a continuous running cost, and let $F_1 : \mathbb{R}^n \times U \times A \to \mathbb{R}^n$, $F_2 : \mathbb{R}^n \times U \times A \to \mathbb{R}^{n'}$ be continuous dynamics components, such that*

(i) *for some $C > 0$, $\langle F_1(x, u, a) - F_1(x', u, a), x - x'\rangle \le C|x - x'|^2$ for all x, $x' \in \mathbb{R}^n$ and $u \in U$, $a \in A$;*
(ii) *for some $\bar{K} > 0$, $|F_2(x, u, a)| \le \bar{K}(1 + |x|)$ for all $(x, u, a) \in \mathbb{R}^n \times U \times A$ and $x \mapsto F_2(x, u, a)$ is uniformly continuous, uniformly w.r.t. the controls;*
(iii) *L is bounded and $(x, z) \mapsto L(x, z, u, a)$ is uniformly continuous, uniformly w.r.t. the controls.*

Let $H : \mathbb{R}^n \times \mathbb{R}^{n'} \times \mathbb{R}^n \times \mathbb{R}^{n'} \to \mathbb{R}$ be the Hamiltonian defined as

$$H(x, z, p_1, p_2) := \min_{a\in A} \max_{u\in U} \Big\{ - \langle (p_1, p_2), (F_1(x, u, a), F_2(x, u, a))\rangle - L(x, z, u, a)\Big\}.$$

Given $\sigma > 0$, if $v_1, v_2 : \mathbb{R}^n \times \mathbb{R}^{n'} \to \mathbb{R}$, bounded and continuous, are, respectively, a viscosity sub- and supersolution of

$$\sigma\, v(x, z) + H(x, z, Dv(x, z)) = 0 \qquad \text{for all } (x, z) \in \mathbb{R}^n \times \mathbb{R}^{n'},$$

then $v_1(x, z) \le v_2(x, z)$ for all $(x, z) \in \mathbb{R}^n \times \mathbb{R}^{n'}$.

Proof The proof is a careful adaptation of the proof of [1, III. Thm. 2.12], which we give in detail for the sakes of clarity and selfconsistency. If we divide the equation by $\sigma > 0$, the functions F_1/σ, F_2/σ and L/σ satisfy the same structural hypotheses as above. Hence, we can assume $\sigma = 1$ without loss of generality. In view of [1, III. Rem. 2.13] we can also assume v_1 and v_2 uniformly continuous.

Set $\langle x\rangle := (1 + |x|^2)^{\frac{1}{2}}$. Take the map $\Phi : \mathbb{R}^n \times \mathbb{R}^{n'} \times \mathbb{R}^n \times \mathbb{R}^{n'} \to \mathbb{R}$, given by

$$\Phi(x, z, y, w) := v_1(x, z) - v_2(y, w) - \frac{|x - y|^2}{2\varepsilon} - \frac{|z - w|^2}{2\rho} - \beta(\langle x\rangle^N + \langle z\rangle^N + \langle y\rangle^N + \langle w\rangle^N)$$

where ε, ρ, β, N are positive parameters to be chosen conveniently.

[8] An alternative hypothesis to (i) is local Lipschitz continuity and at most linear growth in the state, uniformly w.r.t. the control.

Suppose by contradiction that there exist $\delta > 0$ and $(\tilde{x}, \tilde{z}) \in \mathbb{R}^n \times \mathbb{R}^{n'}$ such that $v_1(\tilde{x}, \tilde{z}) - v_2(\tilde{x}, \tilde{z}) = \delta$. Choose $\beta > 0$ such that $\beta\langle\tilde{x}\rangle \le \delta/8$ and $\beta\langle\tilde{z}\rangle \le \delta/8$, so that for all $0 < N \le 1$ we have $2\beta(\langle\tilde{x}\rangle^N + \langle\tilde{z}\rangle^N) \le 2\left(\frac{\delta}{8} + \frac{\delta}{8}\right) = \frac{\delta}{2}$, and

$$\frac{\delta}{2} \le \delta - \frac{\delta}{2} \le v_1(\tilde{x}, \tilde{z}) - v_2(\tilde{x}, \tilde{z}) - 2\beta(\langle\tilde{x}\rangle^N + \langle\tilde{z}\rangle^N) = \Phi(\tilde{x}, \tilde{z}, \tilde{x}, \tilde{z}) \le \sup \Phi.$$

Since Φ is continuous and tends to $-\infty$ as $|x| + |z| + |y| + |w| \to +\infty$, there exists a maximum point $(\bar{x}, \bar{z}, \bar{y}, \bar{w})$, for which, in particular, we get

$$0 < \frac{\delta}{2} \le \Phi(\bar{x}, \bar{z}, \bar{y}, \bar{w}) = \sup \Phi. \tag{45}$$

The obvious inequality $\Phi(\bar{x}, \bar{z}, \bar{x}, \bar{z}) + \Phi(\bar{y}, \bar{w}, \bar{y}, \bar{w}) \le 2\Phi(\bar{x}, \bar{z}, \bar{y}, \bar{w})$, yields

$$\frac{|\bar{x} - \bar{y}|^2}{\varepsilon} + \frac{|\bar{z} - \bar{w}|^2}{\rho} \le v_1(\bar{x}, \bar{z}) - v_1(\bar{y}, \bar{w}) + v_2(\bar{x}, \bar{z}) - v_2(\bar{y}, \bar{w}),$$

so the boundedness of v_1 and v_2 implies that

$$|\bar{x} - \bar{y}| \le c\sqrt{\varepsilon}, \qquad |\bar{z} - \bar{w}| \le c\sqrt{\rho}, \tag{46}$$

for some $c > 0$. From the inequality $\Phi(\bar{x}, \bar{z}, \bar{x}, \bar{w}) + \Phi(\bar{y}, \bar{z}, \bar{y}, \bar{w}) \le 2\Phi(\bar{x}, \bar{z}, \bar{y}, \bar{w})$ and thanks to the uniform continuity of v_1, v_2, we can thus derive that

$$\frac{|\bar{x} - \bar{y}|^2}{\varepsilon} \le v_1(\bar{x}, \bar{z}) - v_1(\bar{y}, \bar{z}) + v_2(\bar{x}, \bar{w}) - v_2(\bar{y}, \bar{w}) \le \omega(|\bar{x} - \bar{y}|) \le \omega(c\sqrt{\varepsilon}), \tag{47}$$

for some modulus of continuity ω.

Consider now the C^1 test functions

$$\varphi(x, z) := v_2(\bar{y}, \bar{w}) + \frac{|x - \bar{y}|^2}{2\varepsilon} + \frac{|z - \bar{w}|^2}{2\rho} + \beta(\langle x\rangle^N + \langle z\rangle^N + \langle\bar{y}\rangle^N + \langle\bar{w}\rangle^N),$$
$$\psi(y, w) := v_1(\bar{x}, \bar{z}) - \frac{|\bar{x} - y|^2}{2\varepsilon} - \frac{|\bar{z} - w|^2}{2\rho} - \beta(\langle\bar{x}\rangle^N + \langle\bar{z}\rangle^N + \langle y\rangle^N + \langle w\rangle^N).$$

By definition of $(\bar{x}, \bar{z}, \bar{y}, \bar{w})$, the function $v_1(x, z) - \varphi(x, z) = \Phi(x, z, \bar{y}, \bar{w})$ obtains its maximum at $(\bar{x}, \bar{z})$, while $v_2(y, w) - \psi(y, w) = -\Phi(\bar{x}, \bar{z}, y, w)$ obtains its minimum at $(\bar{y}, \bar{w})$. As it is easy to see, we have

$$D\varphi(\bar{x}, \bar{z}) = (D_x\varphi, D_z\varphi)(\bar{x}, \bar{z}) = \left(\frac{\bar{x} - \bar{y}}{\varepsilon} + \gamma_1\bar{x}, \frac{\bar{z} - \bar{w}}{\rho} + \gamma_2\bar{z}\right),$$
$$D\psi(\bar{y}, \bar{w}) = (D_x\psi, D_z\psi)(\bar{y}, \bar{w}) = \left(\frac{\bar{x} - \bar{y}}{\varepsilon} + \tau_1\bar{y}, \frac{\bar{z} - \bar{w}}{\rho} + \tau_2\bar{w}\right),$$

if $\gamma_1 := \beta N\langle\bar{x}\rangle^{N-2}$, $\gamma_2 := \beta N\langle\bar{z}\rangle^{N-2}$, $\tau_1 := \beta N\langle\bar{y}\rangle^{N-2}$, $\tau_2 := \beta N\langle\bar{w}\rangle^{N-2}$. The definition of viscosity sub- and supersolution yields

$$v_1(\bar{x},\bar{z}) + H\left(\bar{x},\bar{z},D\varphi(\bar{x},\bar{z})\right) \le 0 \le v_2(\bar{y},\bar{w}) + H\left(\bar{y},\bar{w},D\psi(\bar{y},\bar{w})\right),$$

so that, for some $u \in U$ and $a \in A$, we have

$$\begin{aligned} v_1(\bar{x},\bar{z}) - v_2(\bar{y},\bar{w}) \le\ & H\left(\bar{y},\bar{w},\frac{\bar{x}-\bar{y}}{\varepsilon}+\tau_1\bar{y},\frac{\bar{z}-\bar{w}}{\rho}+\tau_2\bar{w}\right) \\ & - H\left(\bar{x},\bar{z},\frac{\bar{x}-\bar{y}}{\varepsilon}+\gamma_1\bar{x},\frac{\bar{z}-\bar{w}}{\rho}+\gamma_2\bar{z}\right) \\ \le\ & \left\langle\frac{\bar{x}-\bar{y}}{\varepsilon}+\gamma_1\bar{x},F_1(\bar{x},u,a)\right\rangle + \left\langle\frac{\bar{z}-\bar{w}}{\rho}+\gamma_2\bar{z},F_2(\bar{x},u,a)\right\rangle \\ & + L(\bar{x},\bar{z},u,a) \\ & - \left\langle\frac{\bar{x}-\bar{y}}{\varepsilon}+\tau_1\bar{y},F_1(\bar{y},u,a)\right\rangle + \left\langle\frac{\bar{z}-\bar{w}}{\rho}+\tau_2\bar{w},F_2(\bar{y},u,a)\right\rangle \\ & - L(\bar{y},\bar{w},u,a). \end{aligned}$$

Hence, by standard calculations (see also [1, Lemma 2.11]), using the definitions of γ_1, γ_2, τ_1, and τ_2, (46) and (47), we get

$$\begin{aligned} v_1(\bar{x},\ \bar{z}) - v_2(\bar{y},\bar{w}) \le\ & C\frac{|\bar{x}-\bar{y}|^2}{\varepsilon} + \frac{|\bar{z}-\bar{w}|}{\rho}\,\omega_2(|\bar{x}-\bar{y}|) + \omega_L(|\bar{x}-\bar{y}|+|\bar{z}-\bar{w}|) \\ & + \gamma_1 K(1+|\bar{x}|^2) + \tau_1 K(1+|\bar{y}|^2) + \gamma_2 K(1+|\bar{z}|^2) \\ & + \tau_2 K(1+|\bar{w}|^2) \le C\omega(c\sqrt{\varepsilon}) + \frac{c\,\omega_2(c\sqrt{\varepsilon})}{\sqrt{\rho}} + \omega_L(c\sqrt{\varepsilon}+c\sqrt{\rho}) \\ & + K\beta N(\langle\bar{x}\rangle^N + \langle\bar{z}\rangle^N + \langle\bar{y}\rangle^N + \langle\bar{w}\rangle^N), \end{aligned}$$

where C is as in (i) above, K is a suitable positive constant (depending on C and $\bar{K}$, $\bar{K}$ as in (ii)), and ω_2, ω_L are the moduli of continuity of F_2 and L, respectively. At this point, by choosing $N := 1 \wedge \frac{1}{K}$, $\rho := \omega_2(c\sqrt{\varepsilon})$, and using (45), we obtain

$$\begin{aligned} \frac{\delta}{2} &\le \Phi(\bar{x},\bar{z},\bar{y},\bar{w}) \le v_1(\bar{x},\bar{z}) - v_2(\bar{y},\bar{w}) - \beta(\langle\bar{x}\rangle^N + \langle\bar{z}\rangle^N + \langle\bar{y}\rangle^N + \langle\bar{w}\rangle^N) \\ &\le C\omega(c\sqrt{\varepsilon}) + c\,\omega_2^{1/2}(c\sqrt{\varepsilon}) + \omega_L(c\sqrt{\varepsilon} + c\,\omega_2(c\sqrt{\varepsilon})), \end{aligned}$$

which leads to a contradiction as soon as we make $\varepsilon > 0$ small enough. □

This comparison principle is interesting in itself. For instance, it implies that the lipschitzianity conditions on the current cost under which the optimality principles in [10, 19, 22, 27] were obtained, can be replaced by mere continuity plus growth assumptions.

Proof of Proposition 4.1 From [4, Thm. 8.1] it follows immediately that, given a MRF W for some p_0 and γ, the decrease condition (13) is equivalent to the viscosity supersolution condition

$$\max_{u\in U}\left\{-\langle D^-W(z),\, f(z,u)\rangle - p_0(W(z))l(z,u) - \gamma(W(z))\right\} \geq 0 \tag{48}$$

for all $z \in \mathbb{R}^n \setminus \mathcal{C}$. For any $0 < b < c$, we define the sets

$$\mathcal{S}_b := \{z \in \overline{\mathbb{R}^n \setminus \mathcal{C}} \mid \ W(z) < b\}, \quad \mathcal{S}_{(b,c)} := \{z \in \overline{\mathbb{R}^n \setminus \mathcal{C}} \mid \ b < W(z) < c\}.$$

Since W is continuous, proper, and positive definite and $\partial\mathcal{C}$ is compact, these sets are open, bounded, and nonempty.

Step 1. Fix $M > 0$. Then, the set $\mathcal{S} := \overline{\mathcal{S}}_{M+1}$ is contained in the interior of $\mathcal{S}' := \overline{\mathcal{S}}_{M+2}$ and we can define a C^1 function $\tilde{\eta} : \mathbb{R}^n \to [0, 1]$ such that

$$\tilde{\eta}(z) := \begin{cases} 1 & \text{if } z \in \mathcal{S}, \\ 0 & \text{if } z \in \mathbb{R}^n \setminus \mathcal{S}'. \end{cases}$$

For all $(z, u) \in \mathbb{R}^n \times U$, we set

$$\tilde{f}(z,u) := \tilde{\eta}(z) f(z,u), \quad \tilde{\ell}(z,u) := \tilde{\eta}(z)\left[p_0(W(z))l(z,u) + \gamma(W(z))\right] \ (\geq 0).$$

The functions $\tilde{f}$, $\tilde{\ell}$ are continuous, bounded, $x \mapsto \tilde{f}(x,u)$ is (globally) Lipschitz continuous and $x \mapsto \tilde{\ell}(x,u)$ is uniformly continuous, uniformly w.r.t. the control. Since $\tilde{\eta} \geq 0$, from (48) it follows that W also satisfies

$$\max_{u\in U}\left\{-\langle D^-W(z),\, \tilde{f}(z,u)\rangle - \tilde{\ell}(z,u)\right\} \geq 0 \quad \text{for all } z \in \mathbb{R}^n \setminus \mathcal{C}. \tag{49}$$

Step 2. Fix $\varepsilon \in (0, 1)$ and let $\eta_\varepsilon : \mathbb{R}^n \to [0, 1]$ be a C^1 function such that

$$\eta_\varepsilon(z) := \begin{cases} 1 & \text{if } z \in \mathcal{S}_{\left(\varepsilon, \frac{1}{\varepsilon}\right)}, \\ 0 & \text{if } z \in \mathbb{R}^n \setminus \mathcal{S}_{\left(\frac{\varepsilon}{2}, \frac{2}{\varepsilon}\right)}. \end{cases}$$

Setting, for all $(z, u) \in \mathbb{R}^n \times U$,

$$\tilde{f}_\varepsilon(z,u) := \eta_\varepsilon(z)\tilde{f}(z,u), \quad \tilde{\ell}_\varepsilon(z,u) := \eta_\varepsilon(z)\tilde{\ell}(z,u),$$

we finally obtain that W satisfies

$$\max_{u\in U}\left\{-\langle D^-W(z),\, \tilde{f}_\varepsilon(z,u)\rangle - \tilde{\ell}_\varepsilon(z,u)\right\} \geq 0 \quad \text{for all } z \in \mathbb{R}^n. \tag{50}$$

Let $\lambda : \mathbb{R} \to (0, 1)$ be a C^1 function such that $0 < \dot{\lambda} \le \bar{C}$ for some $\bar{C} > 0$, $\lambda(s) \to 0$ as $s \to -\infty$ and $\lambda(s) \to 1$ as $s \to +\infty$, as, for instance, $\lambda(s) = \frac{1}{\pi}\left(\arctan(s) + \frac{\pi}{2}\right)$. Hence, by [1, Prop. 2.5], the function

$$V(z, r) := \lambda(W(z) + r)$$

turns out to be a bounded, continuous, and nonnegative viscosity supersolution of the Hamilton–Jacobi–Bellman equation

$$\max_{u\in U}\left\{-\langle Dv(z, r),\ \tilde{F}_\varepsilon(z, u)\rangle\right\} = 0 \quad \text{for all } (z, r) \in \mathbb{R}^{n+1},$$

where $\tilde{F}_\varepsilon(z, u) := (\tilde{f}_\varepsilon(z, u), \tilde{\ell}_\varepsilon(z, u))$. Set $\Psi(z, r) := V(z, r)$. Since $V \ge 0$, given $\sigma > 0$, V is also a viscosity supersolution of the obstacle equation

$$\min\left\{\sigma\, v(z, r) + \max_{u\in U}\left\{-\langle Dv(z, r),\ \tilde{F}_\varepsilon(z, u)\rangle\right\}, v(z, r) - \Psi(z, r)\right\} = 0 \text{ on } \mathbb{R}^{n+1}. \tag{51}$$

By introducing a new control $a \in \{0, 1\}$ and defining $\hat{F}_\varepsilon(z, u, a) := a\,\tilde{F}_\varepsilon(z, u)$, $\hat{L}(z, r, u, a) := (1 - a)\sigma\Psi(z, r)$, we can reformulate (51) as

$$\sigma\, v(z, r) + \min_{a\in\{0,1\}}\max_{u\in U}\left\{-\langle Dv(z, r),\ \hat{F}_\varepsilon(z, u, a)\rangle - \hat{L}(z, r, u, a)\right\} = 0 \text{ on } \mathbb{R}^{n+1}, \tag{52}$$

where all the assumptions of Lemma 4.1 are met. Thus, (52) (equivalently, (51)) satisfies the comparison principle for bounded viscosity sub- and supersolutions. In particular, by a standard dynamic programming procedure, the unique bounded, continuous solution of (51) is the value function

$$V_\sigma^\varepsilon(z, r) := \inf_{u\in\mathcal{M}([0,+\infty),U)}\ \sup_{T\ge 0} e^{-\sigma T}\Psi(\tilde{x}_\varepsilon(T), \tilde{x}_\varepsilon^0(T)) \qquad \text{for all } (z, r) \in \mathbb{R}^{n+1},$$

where $(\tilde{x}_\varepsilon, \tilde{x}_\varepsilon^0)$ is the unique solution of the control system $(\dot{x}, \dot{x}^0) = \tilde{F}_\varepsilon(x, u)$ with initial condition (z, r). From the comparison principle,

$$V(z, r) \ge V_\sigma^\varepsilon(z, r) \qquad \text{for all } (z, r) \in \mathbb{R}^{n+1}. \tag{53}$$

Given $z \in \mathcal{S}_{\left(\varepsilon, \frac{1}{\varepsilon}\right)}$ and for every $u \in \mathcal{M}([0, +\infty), U)$, set

$$\tilde{T}_z^\varepsilon(u) := \inf\left\{t \ge 0 \mid \quad W(\tilde{x}_\varepsilon(t)) \ge \frac{1}{\varepsilon} \text{ or } W(\tilde{x}_\varepsilon(t)) \le \varepsilon\right\} \le +\infty.$$

Clearly, $\tilde{T}_z^\varepsilon(u) > 0$, as $\varepsilon < W(z) < \frac{1}{\varepsilon}$. Furthermore, for all $t \in [0, \tilde{T}_z^\varepsilon(u))$, the solution $(\tilde{x}_\varepsilon, \tilde{x}_\varepsilon^0)$ corresponding to u coincides with the solution, say $(\tilde{x}, \tilde{x}^0)$, of the control system $(\dot{x}, \dot{x}^0) = (\tilde{f}, \tilde{\ell})(x, u)$ with $(\tilde{x}, \tilde{x}^0)(0) = (z, r)$. Hence, letting σ tend to zero in (53), for every $(z, r) \in \mathcal{S}_{\left(\varepsilon, \frac{1}{\varepsilon}\right)} \times \mathbb{R}$, we get the inequality

$$V(z, r) \geq \inf_{u \in \mathcal{M}([0,+\infty),U)} \sup_{T \in \left[0, \tilde{T}_z^{\varepsilon}(u)\right)} V(\tilde{x}(T), \tilde{x}^0(T)),$$

that, by a recursive procedure as in [27], finally implies

$$V(z, r) \geq \inf_{u \in \mathcal{M}([0,+\infty),U)} \sup_{T \in \left[0, \tilde{T}_z(u)\right)} V(\tilde{x}(T), \tilde{x}^0(T)) \quad \text{for all } (z, r) \in \mathbb{R}^{n+1}, \tag{54}$$

if $\tilde{T}_z(u)$ is the first time at which $\tilde{x}$ reaches the target $\mathcal{C}$ (possibly equal to $+\infty$).

Step 3. Let $z \in \overline{\mathcal{S}}_M \setminus \mathcal{C}$, namely $0 < W(z) \leq M$, and fix $0 < \rho \leq -\lambda(M) + \lambda(M + 1)$. From (54) with initial point $(z, 0)$, it follows that there exists a control $u \in \mathcal{M}([0, +\infty), U)$ such that, for any $T \in [0, \tilde{T}_z(u))$, we get

$$V(\tilde{x}(T), \tilde{x}^0(T)) = \lambda \left(W(\tilde{x}(T)) + \int_0^T \tilde{\ell}(\tilde{x}(t), u(t))\, dt \right) \leq V(z, 0) + \rho \leq \lambda(M + 1),$$

namely,

$$W(\tilde{x}(T)) + \int_0^T \tilde{\ell}(\tilde{x}(t), u(t))\, dt \leq M + 1.$$

Therefore, $\tilde{x}(t)$ belongs to $\mathcal{S}$ for all $t \in [0, \tilde{T}_z(u))$, so tha44441t $\tilde{\eta}$ as in Step 1 is identically 1. As a consequence, we have $\tilde{f}(x, u) \equiv f(x, u)$, $\tilde{\ell}(x, u) \equiv p_0(W(x))l(x, u) + \gamma(W(x))$, $\tilde{x}(\cdot) \equiv x(\cdot\,, u, z)$, and $\tilde{T}_z(u) \equiv T_z(u)$ as in Definition 2.1, so that, in particular $u \in \mathcal{U}(z)$. By the arbitrariness of $M > 0$, the above considerations imply that W satisfies (35) for all $z \in \mathbb{R}^n \setminus \mathcal{C}$. The proof of Proposition 4.1 is thus complete.

5 Conclusions

In this paper, we extend the well-known result that a nonlinear control system is globally asymptotically controllable to a target if and only if a Control Lyapunov Function is associated with it. Our extension considers the case where a cost is associated with the control system –in the form of the integral of a non-negative Lagrangian depending both on state and control–, which must be "regulated", i.e. bounded above by a continuous function of the initial state. Specifically, we show that a necessary and sufficient condition for a strategy to exist for each initial state, for which the system is GAC to $\mathcal{C}$ with corresponding regulated cost, is the existence of a continuous Minimum Restraint Function. A Minimum Restraint Function is a special Control Lyapunov Function, that satisfies a dissipative inequality including the Lagrangian in the cost.

The main novelty is the necessity implication, whose proof relies on a combination of classical nonsmooth analysis techniques adopted in converse Lyapunov theorems and viscosity solutions arguments.

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