Nonlinearity

PAPER • OPEN ACCESS

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To cite this article: Massimiliano Guzzo 2024 *Nonlinearity* **37** 105011

OPEN ACCESS
IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **37** (2024) 105011 (31pp) https://doi.org/10.1088/1361-6544/ad72c6

Parametric approximations of fast close encounters of the planar three-body problem as arcs of a focus-focus dynamics

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Received 30 November 2023; revised 27 June 2024
Accepted for publication 22 August 2024
Published 9 September 2024

Recommended by Dr Rafael de la Llave

Abstract

A gravitational close encounter of a small body with a planet may produce a substantial change of its orbital parameters which can be studied using the circular restricted three-body problem. In this paper we provide parametric representations of the fast close encounters with the secondary body of the planar CRTBP as arcs of non-linear focus-focus dynamics. The result is the consequence of a remarkable factorisation of the Birkhoff normal forms of the Hamiltonian of the problem represented with the Levi–Civita regularisation. The parameterisations are computed using two different sequences of Birkhoff normalisations of given order N. For each value of N, the Birkhoff normalisations and the parameters of the focus-focus dynamics are represented by polynomials whose coefficients can be computed iteratively with a computer algebra system; no quadratures, such as those needed to compute action-angle variables of resonant normal forms, are needed. We also provide

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some numerical demonstrations of the method for values of the mass parameter representative of the Sun–Earth and the Sun–Jupiter cases.

Keywords: three body problem, Birkhoff normalisation, Levi–Civita regularisation

Mathematics Subject Classification numbers: 70F07, 70F15, 70F16, 37N05, 70H09

1. Introduction

The problem of determining the effects of a gravitational close encounter had been considered few years after the publication of Newton's *Philosophiae Naturalis Principia Mathematica*, in an effort to better understand the apparitions of comets. Since comets are visible from Earth when they are close to the Sun, their apparitions at different epochs may be attributed to the same comet if they are linked by the same orbit. Using Newton's solutions of the 2-body problem Halley attributed the apparitions of 1531, 1607 and 1682 to the same comet by conjecturing a heliocentric elliptic orbit, and predicted its subsequent return [19]; this prediction had been later refined by Clairaut by considering also the effects of planetary perturbations [10]. A more puzzling situation manifested few years later, when the dramatic effects of close encounters with planet Jupiter had been indicated as possible explanation for the appearance of a new comet (Lexell's comet) and its subsequent disappearance[1]. The puzzling behaviour of Lexell's comet, which had an explanation within the theory of the circular restricted three-body problem, motivated the development of theories approximating the motion of Solar System bodies transiting close to a planet by Laplace, Le Verrier and Tisserand [25, 26, 38]. Since these pioneering papers the motion of a massless body P having close encounters with a planet is conveniently approximated by a heliocentric motion of the two-body problem defined by the primary body P_1 (the Sun), eventually modified by considering the effects of planetary perturbations, as long as P remains far from the secondary body P_2 (the planet). And during the short time of the close encounter, it is instead approximated by the motion of a planetocentric two-body problem defined by the secondary body P_2. The switch between the different heliocentric and planetocentric problems can produce a rapid and substantial alteration of the heliocentric orbital parameters (see figure 1 for basic examples; more details will be given in section 5). The analytic computation of the effects of close encounters remains relevant for the modern astronomical applications which are related to the dynamics of comets (a remarkable example is provided by the dynamics of the comets of the Jupiter family, such as for comet 67/P Churyumov–Gerasimenko, target of the recent mission Rosetta), for the study of asteroids whose orbit represents a risk for potential Earth impacts, and for the modern space mission design where close encounters are used to modify the orbital elements of a spacecraft. There is therefore the problem of computing the change of the orbital parameters and of representing parametrically the orbit during any individual close encounter.

There is a huge mathematical literature about collisions (for example the ejection-collision orbits) and near collision orbits in the restricted three-body problem and related topics (for example, for near collision orbits of Kolmogorov-Arnold-Moser (KAM) type moving on the

[1] Lexell's comet, discovered in 1770, despite having an elliptic orbit of period of about 5.6 years, was not seen before as well as in the next 10 years (not either afterwards). A possible explanation was that the comet had not been seen before because of a close encounter with Jupiter in 1767, and it would be never be seen again because of a subsequent close encounter.

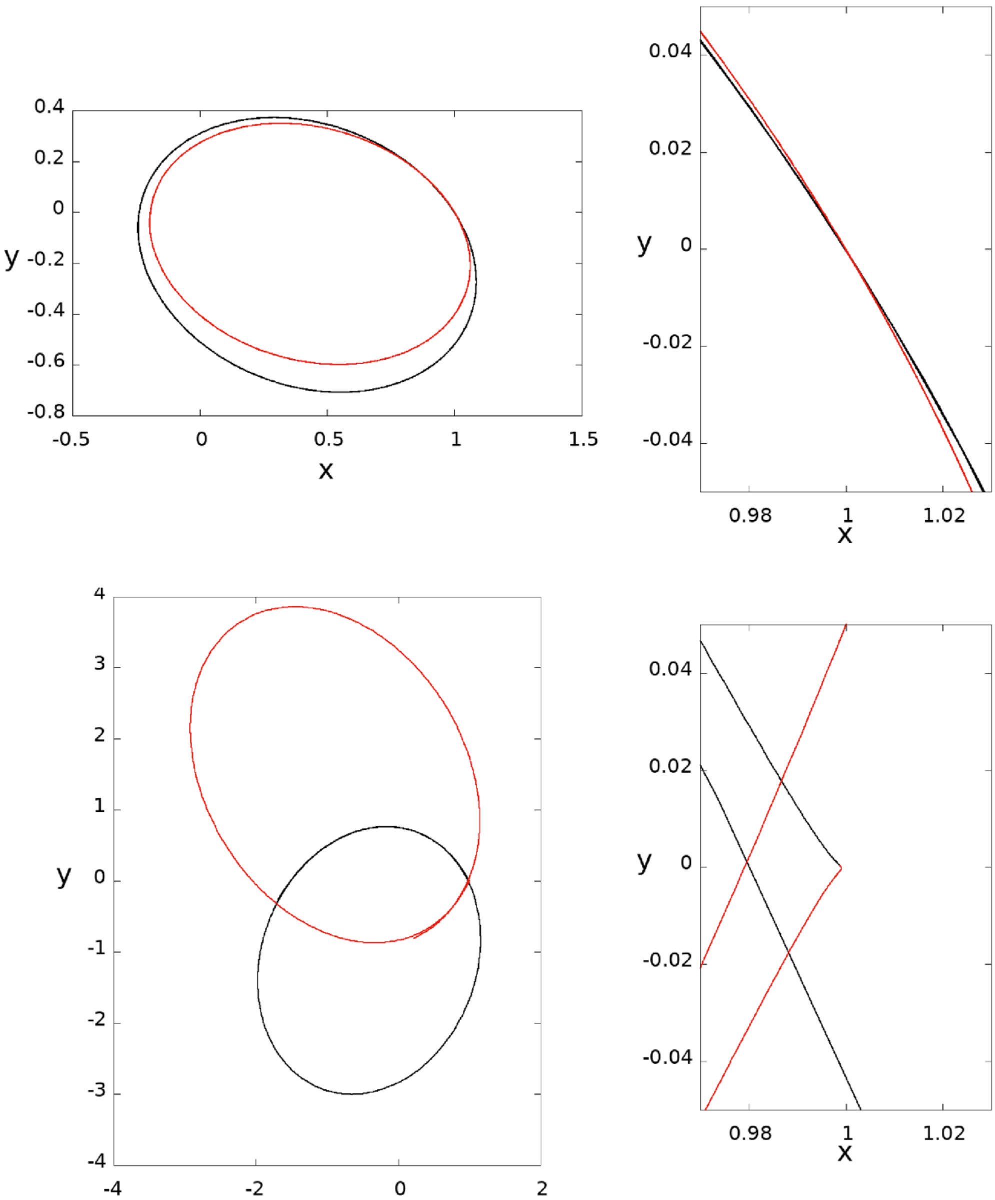

Figure 1. Representation on the inertial orbital plane Oxy of orbits with a close encounter at $t=0$: the black curves represent the orbits computed for positive times, the red curves represent the same orbits computed for negative times. On the top panels we represent an orbit computed for $\mu = 3\ 10^{-6}$ (close to the Sun–Earth mass ratio) for $E=-1.35$ and initial conditions $u_1(0) = -10^{-2}$, $u_2(0) = 10^{-2}$, $U_1(0) = -4\ 10^{-6}, U_2(0) = 0.016\,243\,781\,387\,232\,425$; the close encounter determines a change of the semi-major axis of $\Delta a \sim 0.043\,49$ and $\Delta e \sim -0.070\,84$. On the bottom panels we represent an orbit computed for $\mu = 10^{-3}$ (close to the Sun-Jupiter mass ratio) for $E=-1.35$ and initial conditions $u_1(0) = 10^{-2}$, $u_2(0) = 2\ 10^{-2}$, $U_1(0) = -2\ 10^{-5}, U_2(0) = 0.092\,703\,055\,510\,000\,729$; the close encounter determines a change of the semi-major axis of $\Delta a \sim -0.599$ and $\Delta e \sim -0.0708$. The right panels represent a zoom close to the secondary body P_2. Details about the numerical computation of these orbits and additional analysis are given in section 5.

so-called punctured tori, see [9, 11, 18, 39]; for near collision orbits arising from studies of Poincaré second species solutions see [1–3, 12, 13, 20, 21, 27, 29]; for computer assisted proofs see [4]; and references therein). This paper is about the representation of the arcs of solutions which intersect a small neighbourhood of P_2, obtained from computations of series.

Consider the planar circular restricted three-body problem defined by the motion of a body P of infinitesimally small mass in the gravitation field of two massive bodies P_1 and P_2, the primary and secondary body respectively, which rotate uniformly around their common center of mass. In a rotating frame, the Hamiltonian of the problem is:

$$h(x,y,p_x,p_y) = \frac{p_x^2+p_y^2}{2} + p_x y - p_y x - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \tag{1}$$

where $r_1 = \sqrt{(x+\mu)^2+y^2}$ and $r_2 = \sqrt{(x-1+\mu)^2+y^2}$ denote the distances of P from P_1, P_2 (as usual the units of mass, length and time have been chosen so that the masses of P_1 and P_2 are $1-\mu$ and μ $(0 \leqslant \mu \leqslant 1/2)$ respectively, their coordinates are $(x_1,0) = (-\mu,0)$, $(x_2,0) = (1-\mu,0)$ and their revolution period is 2π). Let σ be arbitrarily small; for any motion $(x(t),y(t))$ entering the ball centred at P_2 of radius σ at time t_0 and leaving it at time t_1, the problem is to provide a parametric representation of the arc $(x(t),y(t),p_x(t),p_y(t))$ for $t \in [t_0,t_1]$. For applications, a relevant point is to formulate parametric representations which are valid in neighbourhoods defined by σ as large as possible, for generic initial conditions (for example, without restricting the analysis to symmetric orbits), and using methods which possibly can be incrementally extended towards more realistic models of the Solar System. Different rigorous approaches have been developed in the literature, reducing the problem to the computation of series. Within the methods which exploit the regularisations at the secondary body P_2, we recall that a representation of close encounters for the planar problem had already been given in the paper by Levi–Civita [24] (generalised to the spatial problem in [6]), obtained from series representations of local solutions of the Hamilton–Jacobi equation of the regularised Hamiltonian. A different representation has been introduced by Henrard [21] who defined Birkhoff normalisations of the Hamiltonian regularised at P_2. For representations using the non-regularised equations of motion we quote the methods of [1] and [29], and subsequent developments.

Let us give more details about the method of [21], which applies to the class of fast close encounters, i.e. the encounters occurring for values E of the Hamilton function (1) satisfying[2]:

$$3 + 2E - 4\mu + \mu^2 := \alpha^2 > 0. \tag{2}$$

In [21], suitable resonant saddle-saddle Birkhoff normal forms of the Hamiltonian (1) represented with the Levi–Civita regularisation had been considered. At this regard, we introduce the regularisation of the Hamiltonian (1) following the method described in [24]: we first perform the phase-space translation:

$$X = x - x_2\ , \quad Y = y\ , \quad P_X = p_x\ , \quad P_Y = p_y - x_2\ \ , \tag{3}$$

[2] For comparison with Paper [21], we recall that the value of the parameter h appearing in equations (10) and (11) of [21] is related to E by $E = h + \mu/2 - \mu^2/2$. Also, the definition of the momenta U_1, U_2 conjugate to the Levi–Civita variables u_1, u_2 introduced in [21] is different from the definition given in Levi–Civita's paper [24], which we follow in this study. The two set of variables are related by a canonical transformation.

and then we introduce the Levi–Civita variables $u=(u_1,u_2)$, extended to the conjugate momenta $U=(U_1,U_2)$ and to the fictitious time τ,

$$X=u_1^2-u_2^2\,,Y=2u_1u_2 \tag{4}$$

$$P_X=\frac{U_1u_1-U_2u_2}{2|u|^2}\,,P_Y=\frac{U_1u_2+U_2u_1}{2|u|^2} \tag{5}$$

$$\mathrm{d}t=|u|^2\mathrm{d}\tau \tag{6}$$

where $|u|^2=u_1^2+u_2^2$. For any given value E of the Hamiltonian (1), the Levi–Civita Hamiltonian:

$$\begin{aligned}\mathcal{K}_E(u,U)=&\frac{1}{8}\left(U_1+2|u|^2u_2\right)^2+\frac{1}{8}\left(U_2-2|u|^2u_1\right)^2-\frac{1}{2}|u|^6-\mu\\&-|u|^2\left(E+\frac{(1-\mu)^2}{2}\right)-(1-\mu)|u|^2\left[\frac{1}{\sqrt{1+2\left(u_1^2-u_2^2\right)+|u|^4}}+u_1^2-u_2^2\right],\end{aligned} \tag{7}$$

is a regularisation of the planar circular restricted three-body problem at P_2, in the sense that the solutions $(u(\tau),U(\tau))$ of the Hamilton equations of $\mathcal{K}_E$ satisfying $\mathcal{K}_E(u(0),U(0))=0$ project, as long as $|u(\tau)|\neq 0$, on the solutions of the Hamilton equations of (1), up to the reparametrisation of the time $t=t(\tau)$.

The Levi–Civita Hamiltonian (7) has an equilibrium point at $(u,U)=(0,0,0,0)$. Although this point does not belong to the level set $\mathcal{K}_E(u,U)=0$ (and consequently the equilibrium does not correspond to a solution of the CR3BP), it is close to this set, which contains the circle $\{(u,U):u=0,\|U\|^2=8\mu\}$. Therefore, the equilibrium point can be used to defined Birkhoff normal forms which are meaningful to the CR3BP as long as their domain of definition intersects the level set $\mathcal{K}_E(u,U)=0$, and this is expected to happen for small values of μ (see also Remark (III) below).

In the paper [21] it was noticed that the equilibrium point $(u,U)=(0,0,0,0)$ is hyperbolic for values E satisfying (2); in particular the Jacobian of the Hamiltonian vector field of (7) computed at the equilibrium has eigenvalues $\pm\alpha/2$ of multiplicity 2. This property has been exploited in the paper [21] by considering resonant hyperbolic Birkhoff normal forms (Bnf hereafter) of the regularised Hamiltonian. In [21], the Bnf have been introduced to discuss the application of Hartman's theorem to the equilibrium point $(u,U)=(0,0,0,0)$; therefore no high order Bnf have been constructed, as would be required by a high precision representation of the close encounters.

We recall that Birkhoff normal forms offer a highly effective method to approximate the solutions which are close to the equilibrium points of Hamiltonian systems. The method is useful also when the equilibrium is hyperbolic, since in this case it allows to study the stable/unstable manifolds, as well as the transits close to the equilibrium; for example, the method has been extensively used in the last decades to represent the solutions transiting close to the Lagrangian points L_1,L_2 (see, for example, [7, 17, 23, 28, 30–35, 37]). Depending on the resonant properties of Hamilton's equations linearised at the equilibrium, a suitable normal form Hamiltonian of arbitrary order N is conjugate to the original one, having the property that non-resonant monomials appear only at degrees larger than N. For the case at hand, with a standard procedure (which is recalled with all the details in appendix A), for any $N\geqslant 4$ we first define a canonical transformation:

$$(u,U)\to(q,p)$$

in a neighbourhood of $(u_*, U_*) = (0,0,0,0)$ (which is a fixed point of the transformation) conjugating the Levi–Civita Hamiltonian (7) to

$$K_E^N(q,p) = -\mu + \frac{\alpha}{2}(q_1p_1 + q_2p_2) + \hat{K}_4(q,p) + \ldots + \hat{K}_N(q,p) + \mathcal{R}_{N+2}(q,p) \tag{8}$$

where the Taylor series of $\mathcal{R}_{N+2}$ in q,p starts with terms of degree at least $N+2$ and $\hat{K}_j$ are polynomials of degree j in q,p (only the polynomials with even j appear in the expansion) containing monomials:

$$c_{m,n}^{j,N} q_1^{m_1} q_2^{m_2} p_1^{n_1} p_2^{n_2} \tag{9}$$

with $m = (m_1, m_2), n = (n_1, n_2)$ satisfying:

$$n_1 + n_2 = m_1 + m_2. \tag{10}$$

Relation (10) is a direct consequence of the multiplicity of the characteristic multipliers, which qualifies the origin as 'resonant'. We will call 'non resonant' the monomials (9) which do not satisfy relation (10); monomials which satisfy relation (10) will be called 'integrable' if $n_1 = m_1$ and $n_2 = m_2$, 'resonant' otherwise. The Hamilton function which is obtained by neglecting in $K_E^N(q,p)$ the remainder $\mathcal{R}_{N+2}(q,p)$:

$$\hat{K}_E^N(q,p) = -\mu + \frac{\alpha}{2}(q_1p_1 + q_2p_2) + \hat{K}_4(q,p) + \ldots + \hat{K}^N(q,p) \tag{11}$$

will be called Bnf of order N. At variance with the full Hamiltonian (8), the Bnf (11) is Poisson commuting with the function $J(q,p) = q_1p_1 + q_2p_2$ (this is a direct consequence of property (10) of all the monomials of the normal form), and its Hamilton equations are integrable by quadratures. For example, by introducing the canonical variables I_j, θ_j (defined for $I_j > 0$; for $I_j < 0$ obvious modifications are required):

$$q_j = \sqrt{I_j} e^{\theta_j}, \;\; p_j = \sqrt{I_j} e^{-\theta_j}$$

the integrable monomials are conjugate to functions depending only on the actions I_j, while the resonant monomials are conjugate to functions depending on the actions I_j and on multiples of $\theta_1 - \theta_2$. As a consequence, the Hamiltonian flow is integrable by quadratures (see section 2 for more details). Nevertheless, working out the explicit solutions for any order N is cumbersome, because of the presence of the resonant terms (or equivalently, because of the dependence on $\theta_1 - \theta_2$). The computation by quadratures may be particularly tricky to perform explicitly because the resonant terms depend on different powers of the action variables I_1, I_2 and so the solution by quadratures requires to represent the roots of an algebraic polynomial in the action variables whose degree increases with the order N of approximation. This is a minor problem if one is interested only in some low order approximation of the flow, but it poses a serious challenge if one needs a parametric representation of the flow which is valid for arbitrary (or for suitably large) order N. This problem does not exist for the Bnf of Hamiltonian systems with an equilibrium point which, in the linear approximation, is not resonant (for example, this happens for equilibria which are linear elliptic with Diophantine frequencies). In fact, in this case the Birkhoff normal forms can be represented with a function depending only on action variables, whose flow is easily represented.

In this paper, we obtain a major improvement in the computation of the solutions of the flow of the Bnf (11). Precisely, after proving a remarkable factorisation property of Hamiltonians (11) for all N, we are able to give its Hamilton equations a different Hamiltonian representation, where the origin $(u, U) = (0,0,0,0)$ is a non-resonant focus-focus equilibrium point. As a consequence, we are able to perform an additional sequence of focus-focus Birkhoff normalisations conjugating the Hamiltonian to a normal form depending only on the action variables. Thus, we obtain a considerable simplification in the computation of close encounters, which is valid for small values of the mass parameter μ. The factorisation property that we prove for the Bnf is the following: for all N the Birkhoff normal forms (11) are divisible by the polynomial $J(q,p) = q_1p_1 + q_2p_2$, i.e. they are represented in the form:

$$\hat{K}_E^N(q,p) = -\mu + (q_1p_1 + q_2p_2)\left(\frac{\alpha}{2} + k_2(q,p) + \ldots + k_{N-2}(q,p)\right), \tag{12}$$

where $k_j(q,p)$ are polynomials of degree j. Since both functions $J(q,p)$ and $k(q,p) = \frac{\alpha}{2} + k_2(q,p) + \ldots + k_{N-2}(q,p)$ are first integrals for the Birkhoff normal form $\hat{K}_E^N$, the solutions of Hamilton's equations with initial data satisfying

$$k(q(0),p(0)) := \kappa, \quad J(q(0),p(0)) := \eta,$$

with $\kappa\eta = \mu$, satisfy the Hamilton equations of:

$$\mathcal{H}(q,p) = \kappa(q_1p_1 + q_2p_2) + \eta(k_2(q,p) + k_4(q,p) + \ldots + k_{N-2}(q,p)). \tag{13}$$

Since η is hereafter treated as a parameter, the terms $J(q,p)k_j(q,p)$ (which in the Hamiltonian (12) are polynomials of degree $j+2$), are transformed into polynomials $\eta k_j(q,p)$ which have degree j; in particular $\eta k_2(q,p)$ has degree 2 and modifies the linearisation of the Hamiltonian vector field at $(q,p) = (0,0,0,0)$. Therefore, we have the opportunity to construct a different Bnf for Hamiltonian (13) depending on this new linearisation.

For initial conditions satisfying $\hat{K}_E^N(q,p) = 0$ we have $\eta \neq 0$, we therefore proceed by considering the case $\eta \neq 0$. By representing explicitly the term $k_2(q,p)$, we notice that Hamiltonian (13) has the representation:

$$\mathcal{H}(q,p) = \sum_{j\geqslant 1}^{\frac{N-2}{2}} h_{2j}(q,p)$$

with quadratic part[3]:

$$h_2 = \kappa(q_1p_1 + q_2p_2) + \eta k_2(q,p) = \Lambda(q_1p_1 + q_2p_2) + \Omega(p_2q_1 - p_1q_2)$$

with $\Lambda = \kappa$ and $\Omega = \eta/(4\alpha)$, while for $j \geqslant 2$ we have: $h_{2j}(q,p) = \eta k_{2j}(q,p)$. The origin $(q,p) = (0,0,0,0)$ is an equilibrium point, with characteristic exponents:

$$\pm\Lambda \pm i\Omega,$$

so that the equilibrium is focus-focus with eigenvalues of multiplicity 1. Therefore, we can proceed by defining a second sequence of Birkhoff normalisations of $\mathcal{H}$ using the focus-focus character of $(q,p) = (0,0,0,0)$. It is important to remark that the functions generating these Birkhoff normalisations have small divisors proportional to η, which may be very small. However, this does not produce divergence of the norms since, by the factorisation property, the polynomial terms of order larger than 4 of the Bnf are proportional to η as well, so that the generating functions of Birkhoff normalisations are not singular at $\eta = 0$. Moreover, the generating functions depend in straightforward way on the parameters Λ, Ω (see section 4 for details), so that the two additional parameters do not introduce any substantial complexity in the symbolic computation of the normal forms.

[3] From the computation of the Bnf $\hat{K}_E^N$ of order $N \geqslant 4$ (see section 2, equation (24)) we have $k_2 = (q_1p_2 - p_2q_1)/(4\alpha)$.

We therefore show that Hamiltonian (13) is conjugate by a (complex) canonical transformation:

$$(q,p) \mapsto (Q,P)$$

defined in a neighbourhood of $(q,p) = (0,0,0,0)$ (which is a fixed point of the transformation) to the focus-focus Bnf:

$$H_E^N(Q,P) = (i\Omega - \Lambda)\,Q_1P_1 - (i\Omega + \Lambda)\,Q_2P_2 + \hat{h}\,(Q_1P_1, Q_2P_2) + \mathcal{R}_N(Q,P) \tag{14}$$

where $\hat{h}(I_1,I_2)$ is a polynomial in I_1, I_2 containing monomials of degree ranging between 2 and $(N-2)/2$, and the remainder $\mathcal{R}_N(Q,P)$ is a series which starts with terms of degree N. By neglecting the remainder we obtain the integrable Hamiltonian:

$$\hat{H}_E^N(Q,P) = (i\Omega - \Lambda)\,Q_1P_1 - (i\Omega + \Lambda)\,Q_2P_2 + \hat{h}\,(Q_1P_1, Q_2P_2) \tag{15}$$

whose flow is represented explicitly by the same formulas, independently on N, since the functions $I_1 = Q_1P_1, I_2 = Q_2P_2$ are first integrals:

$$Q_j(\tau) = Q_j(0)\,e^{\hat{\kappa}_j\tau},\;\; P_j(t) = P_j(0)\,e^{-\hat{\kappa}_j\tau},\;\; j = 1,2 \tag{16}$$

where:

$$\hat{\kappa}_j = \frac{\partial}{\partial I_j}\Big((i\Omega - \Lambda)\,I_1 - (i\Omega + \Lambda)\,I_2 + \hat{h}\,(I_1,I_2)\Big)_{|_{I_1=I_1(0),I_2=I_2(0)}}.$$

For each order of approximation, all the Birkhoff normalisations and the parameters of the focus-focus dynamics (16) are represented by polynomials whose coefficients can be computed iteratively with a computer algebra system (see appendix A and section 5).

Remarks. (I) Parametric representations of close encounters may be used to implement a numerical integrator of the close encounters by explicitly representing with a computer program the flow of the Bnf $K_E^N(q,p)$.

(II) The parametric representations of the solutions offered by the Bnf are approximate, because they are obtained by neglecting the remainders $\mathcal{R}_N(q,p)$, which are polynomials in the (q,p) variables of order N. Therefore, the distance of (q,p) from the equilibrium point $(0,0,0,0)$ is the small parameter of the problem. We recall that the normal form variables q,p are obtained from the composition of a linear transformation of the Levi–Civita variables (u,U) with the Birkhoff normalisations. During a transit in a ball of radius σ centred at P_2, we have $|u|^2 < \sigma$ (which becomes 0 at collision); but also $|U|$ remains small (with limit value proportional to $\sqrt{\mu}$ at collision).

(III) For given N (and $\alpha \geqslant \alpha_0 > 0$) one may formally prove the convergence of the canonical transformations providing the first and second Birkhoff normal forms of degree N, together with suitable estimates of the remainders and their derivatives, in a neighbourhood of radius R_0 of the equilibrium point $(0,0,0,0)$. This neighbourhood intersects the level set $\mathcal{K}_E(u,U) = 0$, supporting the solutions of the CR3BP, only if $\sqrt{8\mu} < R_0$. Therefore, initial conditions for the solutions of the CR3BP belong to the domain of definition of the Bnf for all $\mu < \mu_0 := R_0^8/8$, ensuring the asymptotic applicability of the method for small values of μ. Instead, the applicability of the method to specific values of μ would be ensured from the explicit computation of the threshold R_0, which could be obtained with computer assisted methods. In fact, a well known issue of the general methods of perturbation theory when applied to problems of Celestial Mechanics characterised by peculiar symmetries is related to the highly inefficient computation of the thresholds (such as R_0) of convergence of the composition of sequences of canonical transformations. Instead, realistic computations of the thresholds for the three–body problem have been obtained with computer assisted proofs (see for example [8, 15, 16]) including estimates of Birkhoff transformations at an equilibrium point (see [15, 16]; we refer to [5] for a fully rigorous computer-assisted procedure applying to the Birkhoff normal forms of any finite order and their remainders). In section 5 we use an indirect numerical method to check the convergence of the Bnf canonical transformations along specific solutions, by numerically computing the error ϵ_N on the conservation of the actions I_1, I_2 defined by the Bnf of different orders $N \in [2,\ldots,N_0]$, along the numerically computed solution. In fact, when a solution belongs to the domain of convergence the action variables I_1, I_2, which are constant in the approximated dynamics which is obtained by neglecting the remainders of the Bnf, are conserved within an error ϵ_N when we consider the flow of the complete Hamiltonian of the CR3BP. An exponential reduction of the error ϵ_N measured when the solution is at a distance ρ from the equilibrium point, for increasing values of $N \in [2,\ldots,N_0]$, provides a strong indication of the convergence of the method at the distance ρ up to the order N_0 (for details see section 5, figures 3 and 4).

(IV) As it is typical of normal form Hamiltonians, and also of numerical integrators of given order, the remainder $\mathcal{R}_N(q,p)$ may depend on N in a non-trivial case, so that the increment of N may produce an improvement of the method at a given distance ρ from the equilibrium point (i.e. a reduction of $\sup_{\|(q,p)\|\leqslant\rho} \|\mathcal{R}_N(q,p)\|$) only up to a finite large value N_0. For the numerical examples of section 5, obtained for the large order $N_0 = 30$, we see that we would still have the opportunity to reduce the error of the parametric representation by considering also larger values of $N > N_0$.

The paper is organised as follows: in section 2 we describe the properties of the first sequence of saddle-saddle Bnf; in section 3 we prove that all the Birkhoff normal forms presented in section 2 are divisible by $q_1p_1 + q_2p_2$; in section 4 we provide all the details needed to construct the second sequence of focus-focus Bnf, and the solution of its Hamilton equations; in section 5 we provide some numerical demonstrations of the method for values of the mass parameter representative of the Sun-Earth and the Sun-Jupiter cases; in the appendix A we present the technical details of the construction of the Birkhoff normal forms for 2-degrees of freedom Hamiltonian systems; in the appendix B we provide the generating functions which define the Bnf of order $N = 6$; in appendix C we provide an error analysis based on the Gronwall lemma; conclusions and perspectives are provided in section 6.

2. The resonant saddle-saddle Birkhoff normal form

We expand the Levi–Civita Hamiltonian (7) as a Taylor series of the variables u, U:

$$\mathcal{K}_E = -\mu + \mathcal{K}_2(u,U;E) + \mathcal{K}_4(u,U) + \sum_{j\geqslant 3} \mathcal{K}_{2j}(u)$$

where:

$$\mathcal{K}_2 = \frac{1}{8}\left(U_1^2+U_2^2\right) - \frac{1}{2}\left(3+2E-4\mu+\mu^2\right)\left(u_1^2+u_2^2\right) = \frac{1}{8}\left(U_1^2+U_2^2\right) - \frac{\alpha^2}{2}\left(u_1^2+u_2^2\right),$$

$$\mathcal{K}_4 = \frac{1}{2}\left(u_1^2+u_2^2\right)\left(U_1u_2 - U_2u_1\right),$$

and, for all $j \geqslant 3$,

$$\mathcal{K}_{2j} = -(1-\mu)\ T_{2j}\left[|u|^2\left(\frac{1}{\sqrt{1+2\left(u_1^2-u_2^2\right)+|u|^4}}\right)\right] \tag{17}$$

where $T_k f(u,U)$ denotes the Taylor term of degree k of a function $f(u,U)$. We remark that only terms of even degree appear in the expansion of $\mathcal{K}_E$, since the Hamiltonian (7) is the sum of a polynomial containing only terms of even degree and of a function which is even in u_1, u_2. For E satisfying

$$\alpha^2 = 3 + 2E - 4\mu + \mu^2 > 0$$

the origin is a hyperbolic equilibrium point with real eigenvalues $\pm\alpha/2$ of multiplicity 2. Therefore, we introduce hyperbolic variables by means of the canonical transformation:

$$(q_1, q_2, p_1, p_2) = C^{-1}(u_1, u_2, U_1, U_2) \tag{18}$$

where C is the real matrix:

$$C = \begin{pmatrix} \frac{1}{2\sqrt{\alpha}} & 0 & -\frac{1}{2\sqrt{\alpha}} & 0 \\ 0 & \frac{1}{2\sqrt{\alpha}} & 0 & -\frac{1}{2\sqrt{\alpha}} \\ \sqrt{\alpha} & 0 & \sqrt{\alpha} & 0 \\ 0 & \sqrt{\alpha} & 0 & \sqrt{\alpha} \end{pmatrix}.$$

The transformation (18) conjugates the Hamiltonian $\mathcal{K}_E$ to:

$$K_E(q,p) = -\mu + \sum_{j\geqslant 1} K_{2j}(q,p) \tag{19}$$

with:

$$K_2 = \frac{\alpha}{2}(q_1 p_1 + q_2 p_2)$$

and, for all j, we introduce the representation:

$$K_{2j} = \sum_{m,n\in\mathbb{N}^2:|m|+|n|=2j} c_{m,n} q^m p^n,$$

where $m = (m_1, m_2), n = (n_1, n_2)$ are multi-indices and, following standard notation for multi-indices, for any multi-index $\nu \in \mathbb{Z}^2$, we denote $|\nu| = \nu_1 + \nu_2$.

The Hamiltonian K_E is given the Birkhoff normal form by a canonical transformation $\mathcal{C}_N$, defined in a neighbourhood of $(q_*, p_*) = (0,0,0,0)$ (which is a fixed point of the transformation) conjugating K_E to:

$$K_E^N(q,p) = -\mu + \frac{\alpha}{2}(q_1 p_1 + q_2 p_2) + \hat{K}_4(q,p) + \ldots + \hat{K}_N(q,p) + \mathcal{R}_{N+2}(q,p) \tag{20}$$

where the Taylor series of $\mathcal{R}_{N+2}$ in q,p starts with terms of order at least $N+2$ and $\hat{K}_j$ are polynomials of order j in q,p (only the polynomials with j even number appear in the expansion) containing monomials:

$$c_{m,n}^{j,N} q_1^{m_1} q_2^{m_2} p_1^{n_1} p_2^{n_2} \tag{21}$$

with $m = (m_1, m_2), n = (n_1, n_2)$ satisfying:

$$n_1 + n_2 = m_1 + m_2. \tag{22}$$

The definition of the canonical transformation is defined following a standard procedure, which we describe in detail in the appendix A.

Remark. The standard procedure which is described in appendix A is applied by setting the parameters $\lambda_1, \lambda_2 = \alpha/2$, so that the divisors entering the definition of the generating functions (62):

$$\lambda_1 (n_1 - m_1) + \lambda_2 (n_2 - m_2) = (\alpha/2)(n_1 + n_2 - m_1 - m_2),$$

are proportional to the integer $n_1 + n_2 - m_1 - m_2$. As a consequence, the resonant relation (55) becomes the relation (22).

We consider the Hamiltonian which is obtained by neglecting the remainder $\mathcal{R}_{N+2}(q,p)$ in $K_E^N(q,p)$:

$$\hat{K}_E^N(q,p) = -\mu + \frac{\alpha}{2}(q_1 p_1 + q_2 p_2) + \hat{K}_4(q,p) + \ldots + \hat{K}^N(q,p), \tag{23}$$

which will be called Bnf of order N. For example, at order $N = 6$ we have:

$$\begin{aligned} \hat{K}_E^6(q,p) = &-\mu + \frac{\alpha}{2}(q_1 p_1 + q_2 p_2) - \frac{1}{4\alpha}(p_1 q_2 - p_2 q_1)(q_1 p_1 + q_2 p_2) \\ &- \frac{(p_1 q_1 + p_2 q_2)}{16\alpha^3}\Big[-5(1-\mu)\left(q_1^2 p_1^2 + q_2^2 p_2^2\right) + 2 p_1 p_2 q_1 q_2 (6 - 7\mu) \\ &+ \left(q_1^2 p_2^2 + q_2^2 p_1^2\right)(4 - 3\mu)\Big]. \end{aligned} \tag{24}$$

Higher order Bnf can be computed using modern computer algebra systems, with a symbolic representation of the coefficients of all their monomials as a function of the parameters μ, E.

The Bnf of any order N has the first integrals $\hat{K}_E^N(q,p)$ and $J(q,p) = q_1 p_1 + q_2 p_2$ (this is a direct consequence of relation (10) of all the monomials of the normal form), and is integrable by quadratures. In fact, by introducing the canonical variables I_j, θ_j (defined for $I_j > 0$; for $I_j < 0$ obvious modifications are required):

$$q_j = \sqrt{I_j} e^{\theta_j}, \;\; p_j = \sqrt{I_j} e^{-\theta_j}$$

the monomials of $\hat{K}_E^N$ are conjugate to:

$$\sqrt{I_1}^{m_1+n_1} \sqrt{I_2}^{m_2+n_2} e^{(m_1-n_1)\theta_1} e^{(m_2-n_2)\theta_2} = \sqrt{I_1}^{m_1+n_1} \sqrt{I_2}^{m_2+n_2} e^{(m_1-n_1)(\theta_1-\theta_2)},$$

where the last equality follows from relation (10). With a further change to the action-angle variables $\hat{I}_1, \hat{I}_2, \hat{\theta}_1, \hat{\theta}_2$:

$$\hat{I}_1 = I_1 \;\; , \;\; \hat{I}_2 = I_1 + I_2 \;\; , \;\; \hat{\theta}_1 = \theta_1 - \theta_2 \;\; , \;\; \hat{\theta}_2 = \theta_2$$

the Bnf is finally conjugate to a Hamiltonian $h_E^N(\hat{I}_1, \hat{I}_2, \hat{\theta}_1)$ which is independent on $\hat{\theta}_2$, and therefore for any value of the first integral $\hat{I}_2 = J(q,p)$ can be studied as a reduced 1-degree of freedom Hamiltonian system. Nevertheless, working out the explicit solutions for its Hamilton equations by quadratures, which in particular requires finding the solutions of the equation:

$$h_E^N\left(\hat{I}_1, \hat{I}_2, \hat{\theta}_1\right) = 0 \tag{25}$$

in the form $\hat{I}_1 := \hat{I}_1(\hat{I}_2, \hat{\theta}_1)$ for any N, is cumbersome.

3. A remarkable property of the Birkhoff normal forms

3.1. A permutation symmetry

We say that the polynomial $F(q,p)$ of the hyperbolic variables q,p:

$$F(q,p) := \sum_{\ell=1}^{L} \sum_{m,n:\ |m|+|n|=2\ell} c_{(m,n)} q^m p^n \tag{26}$$

has the permutation symmetry if for each multi-index $(m,n) = (m_1,m_2,n_1,n_2)$ we have:

$$c_{(m,n)} = (-1)^{\sigma_{(m,n)} + \frac{|m|+|n|-2}{2}} c_{\overline{(m,n)}} \ , \tag{27}$$

where $\overline{(m,n)} = (n_2,n_1,m_2,m_1)$ and $\sigma_{(m,n)} = 1$ is m_1+n_1 is odd, $\sigma_{(m,n)} = 0$ otherwise.

We denote by ΠF the expansion obtained by summing all the resonant terms of F:

$$\Pi F := \sum_{\ell=1}^{L} \sum_{|m|+|n|=2\ell, m_1+m_2=n_1+n_2} c_{(m,n)} q^m p^n.$$

We have the following:

Proposition 1. *For any polynomial $F(q,p)$ as in equation (26) satisfying the permutation symmetry, we have:*

$$\Pi F(q,p) = (q_1p_1 + q_2p_2)(c + f_2(q,p) + \ldots + f_{2L-2}(q,p))$$

where $f_j(q,p)$ are homogeneous polynomials of degree j in the variables q,p.

Let us prove proposition 1. First, from the property (27) we have:

(i) $c_{(m_1,m_2,m_2,m_1)} = 0$ for any m_1,m_2 with $m_1+m_2 \geqslant 1$. In fact, for $n_1 = m_2, n_2 = m_1$, equation (27) becomes:

$$c_{(m_1,m_2,m_2,m_1)} = (-1)^{\sigma_{(m,n)}} (-1)^{m_1+m_2-1} c_{(m_1,m_2,m_2,m_1)}.$$

If $m_1+n_1 = m_1+m_2$ is even, then $(-1)^{\sigma_{(m,n)}} = 1$ and $(-1)^{m_1+m_2} = 1$; if $m_1+n_1 = m_1+m_2$ is odd, then $(-1)^{\sigma_{(m,n)}} = -1$ and $(-1)^{m_1+m_2} = -1$. In both cases we have $(-1)^{\sigma_{(m,n)}+\frac{|m|+|n|-2}{2}} = (-1)^{\sigma_{(m,n)}}(-1)^{m_1+m_2-1} = -1$, from which it follows $c_{(m_1,m_2,m_2,m_1)} = 0$.

(ii) Consider any m,n with $n_1,n_2 \neq m_2,m_1$ and $m_1+m_2 = n_1+n_2$. The two coefficients $c_{(m,n)}, c_{\overline{(m,n)}}$ satisfy:

$$c_{(m,n)}(-1)^{m_1} + c_{\overline{(m,n)}}(-1)^{n_2} = 0. \tag{28}$$

In fact, since $m_1+m_2 = n_1+n_2$ we rewrite equation (27) as:

$$c_{(m,n)} = -(-1)^{\sigma_{m,n}}(-1)^{m_1+m_2} c_{\overline{(m,n)}}.$$

If m_1+n_1, m_2+n_2 are both even, on the one hand we have $(-1)^{\sigma_{(m,n)}} = 1$ so that:

$$c_{(m,n)} = -(-1)^{m_1+m_2} c_{\overline{(m,n)}},$$

as well as:

$$c_{(m,n)}(-1)^{m_1}+c_{\overline{(m,n)}}(-1)^{m_2}=0.$$

On the other hand, since m_2+n_2 is even, we have $(-1)^{m_2}=(-1)^{-n_2}=(-1)^{n_2}$, and so we obtain (28). If instead m_1+n_1, m_2+n_2 are both odd, we have:

$$c_{(m,n)}=(-1)^{m_1+m_2}c_{\overline{(m,n)}},$$

as well as:

$$c_{(m,n)}(-1)^{m_1}-c_{\overline{(m,n)}}(-1)^{m_2}=0.$$

Since m_2+n_2 is odd, we have $(-1)^{m_2}=-(-1)^{-n_2}=-(-1)^{n_2}$, and so we obtain (28).

We now show that (i) and (ii) imply that we can represent ΠF in the form:

$$\Pi F(q,p)=(q_1p_1+q_2p_2)(c+f_2(q,p)+\ldots+f_{2L-2}(q,p))$$

where $f_j(q,p)$ are polynomials of degree j in the variables q,p.

In fact, in $\Pi F(q,p)$ we have all the monomials of F corresponding to m,n with $m_1+m_2=n_1+n_2$. If $m,n=\overline{m,n}$, from (i) we have $c_{(m,n)}=0$. If $m,n\neq\overline{m,n}$ we consider the sum of the symmetric terms:

$$S_{(m,n)}=c_{(m,n)}q_1^{m_1}q_2^{m_2}p_1^{n_1}p_2^{n_2}+c_{\overline{(m,n)}}q_1^{n_2}q_2^{n_1}p_1^{m_2}p_2^{m_1}, \tag{29}$$

and prove that $S_{(m,n)}$ is divisible by $q_1p_1+q_2p_2$. Necessary condition for a polynomial $S(q_1,q_2,p_1,p_2)$ to be divisible by $(q_1p_1+q_2p_2)$ is that the rational function $\mathcal{S}=S(-q_2p_2/p_1,q_2,p_1,p_2)$ is identically zero. Since the ring of polynomials of four variables has unique factorisation (see [22], p 153, 154), the condition is also sufficient. Let us therefore consider the rational function:

$$\begin{aligned}\mathcal{S}_{(m,n)}&=S_{(m,n)}(-q_2p_2/p_1,q_2,p_1,p_2)\\&=c_{(m,n)}(-1)^{m_1}q_2^{m_2+m_1}p_1^{n_1-m_1}p_2^{n_2+m_1}+c_{\overline{(m,n)}}(-1)^{n_2}q_2^{n_1+n_2}p_1^{m_2-n_2}p_2^{m_1+n_2}.\end{aligned}$$

Since the monomials of ΠF satisfy $m_1+m_2=n_1+n_2$, the powers of q_2,p_1,p_2 are the same for both terms appearing in $\mathcal{S}_{(m,n)}$, and therefore we have:

$$\mathcal{S}_{(m,n)}=\left(c_{(m,n)}(-1)^{m_1}+c_{\overline{(m,n)}}(-1)^{n_2}\right)q_2^{m_2+m_1}p_1^{n_1-m_1}p_2^{n_2+m_1}.$$

By property (ii) the rational function $\mathcal{S}_{(m,n)}$ vanishes identically, and therefore the polynomial $S_{(m,n)}$ in (29) divisible by $q_1p_1+q_2p_2$. Since ΠF is the sum of polynomials (29), it is divisible by $q_1p_1+q_2p_2$ as well, and proposition 1 is proved.

3.2. The Levi–Civita Hamiltonian has the permutation symmetry

In this subsection, we prove the following:

Proposition 2. *Any truncation $\sum_{j=1}^{L}K_{2j}(q,p)$ of the Taylor expansion of the Levi–Civita Hamiltonian $K(q,p)$ represented with the hyperbolic variables q,p has the permutation symmetry.*

From the definition of the Levi–Civita Hamiltonian and of the hyperbolic variables (q,p) introduced in section 2, one directly checks that the terms:

$$K_2 = \frac{\alpha}{2}\left(q_1p_1 + q_2p_2\right)$$
$$K_4 = \left(-\frac{q_1{p_1}^2p_2}{8\alpha} - \frac{q_1{q_2}^2p_2}{8\alpha}\right) + \left(-\frac{q_1q_2{p_1}^2}{4\alpha} - \frac{{q_2}^2p_1p_2}{4\alpha}\right) + \left(\frac{q_2{p_1}^3}{8\alpha} + \frac{{q_2}^3p_1}{8\alpha}\right)$$
$$+ \left(\frac{q_2p_1{p_2}^2}{8\alpha} + \frac{{q_1}^2q_2p_1}{8\alpha}\right) + \left(\frac{{q_1}^2p_1p_2}{4\alpha} + \frac{q_1q_2{p_2}^2}{4\alpha}\right) + \left(-\frac{q_1{p_2}^3}{8\alpha} - \frac{{q_1}^3p_2}{8\alpha}\right)$$

have the permutation symmetry (the symmetric terms have been grouped together; for all the terms of K_4 we have $\sigma_{(m,n)} = 1$ and $|m| + |n| = 4$, so that (27) becomes $c_{(m,n)} = c_{\overline{(m,n)}}$).

To prove the property for all the other terms K_{2j} with $j \geqslant 3$, we introduce the parametric function:

$$\mathcal{K}^{\geqslant 6}(u;\lambda) = -\sum_{\ell \geqslant 3} (1-\mu)\ T_{2\ell}\left[|u|^2\left(\frac{1}{\sqrt{1+2\lambda\left(u_1^2-u_2^2\right)+\lambda^2|u|^4}}\right)\right]$$

and the Taylor expansion:

$$\mathcal{K}^{\geqslant 6}(u;\lambda) := \sum_{j\geqslant 2} \lambda^j \mathcal{P}_j(u).$$

We notice that $\mathcal{P}_j(u)$ are polynomials in the variables u_1,u_2 of degree $2j+2$: in fact, the coefficient of λ^j of the Taylor expansion of $\frac{1}{\sqrt{1+2\lambda(u_1^2-u_2^2)+\lambda^2|u|^4}}$, which is a polymomial of degree $2j$ in u_1,u_2 (since in this function λ multiplies u_1^2,u_2^2 and λ^2 multiplies $|u|^4$), is further multiplied by $|u|^2$). For $\lambda = 1$ we have:

$$\mathcal{K}_E(u,U) = -\mu + \mathcal{K}_2(u,U) + \mathcal{K}_4(u,U) + \mathcal{K}^{\geqslant 6}(u;1).$$

The composition of the functions $\mathcal{K}^{\geqslant 6}$ and $\mathcal{P}_j$ with the canonical transformation:

$$(u,U) = C(q,p)$$

provides the functions $K^{\geqslant 6}, P_j$:

$$K^{\geqslant 6}(q,p;\lambda) = \sum_{j\geqslant 2}\lambda^j P_j(q,p)$$

with $P_j(q,p) = K_{2j+2}(q,p)$ for all $j \geqslant 2$.

For any (q,p), by denoting with $(u,U) := C(q,p)$, $(\tilde q_1,\tilde q_2,\tilde p_1,\tilde p_2) := (p_2,p_1,q_2,q_1)$ and $(\tilde u,\tilde U) := C(\tilde q,\tilde p)$, we have $(\tilde u,\tilde U) = (-u_2,-u_1,U_2,U_1)$ as well as:

$$\mathcal{K}^{\geqslant 6}(\tilde u;-\lambda) = \mathcal{K}^{\geqslant 6}(u;\lambda),$$

and consequently:

$$\sum_{j\geqslant 2}\lambda^j\mathcal{P}_j(u) = \sum_{j\geqslant 2}\lambda^j(-1)^j\mathcal{P}_j(\tilde u)\quad .$$

By identifying the coefficients of the two Taylor expansions in λ on both sides of the last equation, we obtain:

$$\mathcal{P}_j(u) = (-1)^j \mathcal{P}_j(\tilde{u}), \tag{30}$$

as well as:

$$P_j(q,p) = (-1)^j P_j(p_2,p_1,q_2,q_1). \tag{31}$$

From:

$$P_j = \sum_{m,n:|m|+|n|=2j+2} c_{(m,n)} q^m p^n,$$

by equating the coefficients of the same term $q^m p^n$ from both sides of equality (31) we obtain:

$$c_{(m,n)} = (-1)^j c_{\overline{(m,n)}} = (-1)^{\frac{|m|+|n|-2}{2}} c_{\overline{(m,n)}} \ . \tag{32}$$

Since the polynomials $\mathcal{P}_j$ depend on the u only through u_1^2, u_2^2, in the corresponding polynomial $P_j(q,p) = K_{2j+2}(q,p)$ there are only monomials $c_{(m,n)} q^m p^n$ where m_1+n_1, m_2+n_2 are even numbers, and so $\sigma_{(m,n)} = 0$. Therefore, equation (32) can be rewritten as:

$$c_{(m,n)} = (-1)^{\sigma_{(m,n)} + \frac{|m|+|n|-2}{2}} c_{\overline{(m,n)}}, \tag{33}$$

as it is required by (27). Therefore we have proved proposition 2.

3.3. The Birkhoff normal forms of the Levi–Civita Hamiltonian have the permutation symmetry

In this subsection, we prove the following:

Proposition 3. *The Bnf (23) of arbitrary order N have the permutation symmetry.*

from which, using also proposition 1, we get immediately the following:

Corollary 1. *The Bnf (23) of arbitrary order N are divisible by* $q_1p_1 + q_2p_2$.

Consider any couple of functions:

$$F = \sum_{j\geqslant 1} \sum_{|m|+|n|=2j} c_{(m,n)} q^m p^n$$

$$\tilde{F} = \sum_{j\geqslant 1} \sum_{|m|+|n|=2j} \tilde{c}_{(m,n)} q^m p^n$$

having the permutation symmetry, and the generating function:

$$\chi_{2i} = \sum_{|m|+|n|=2i, |m|\neq|n|} \frac{c_{(m,n)}}{\frac{\alpha}{2}(|m|-|n|)} q^m p^n$$

which is constructed from the coefficients of the monomials of degree $2i$ of the function F. We prove that $\{\tilde{F}, \chi_{2i}\}$ has the permutation symmetry. It is sufficient to prove that, for any $j \geqslant 1$, by considering:

$$\tilde{F}_{2j} = \sum_{|\tilde{m}|+|\tilde{n}|=2j} \tilde{c}_{(\tilde{m},\tilde{n})} q^{\tilde{m}} p^{\tilde{n}},$$

the function $\{\tilde F_{2j},\chi_{2i}\}$ has the permutation symmetry.

We have:

$$\begin{aligned}\{\tilde F_{2j},\chi_{2i}\} = & \sum_{|m|+|n|=2i,|m|\neq|n|}\ \sum_{|\tilde m|+|\tilde n|=2j} \frac{\tilde c_{(\tilde m,\tilde n)} c_{(m,n)}}{\frac{\alpha}{2}(|m|-|n|)} \\ & \times\Big[(\tilde m_1 n_1 - m_1\tilde n_1)\, q_1^{m_1+\tilde m_1-1} q_2^{m_2+\tilde m_2} p_1^{n_1+\tilde n_1-1} p_2^{n_2+\tilde n_2} \\ & + (\tilde m_2 n_2 - m_2\tilde n_2)\, q_1^{m_1+\tilde m_1} q_2^{m_2+\tilde m_2-1} p_1^{n_1+\tilde n_1} p_2^{n_2+\tilde n_2-1}\Big] \\ := & \sum_{|m|+|n|=2i,|m|\neq|n|}\ \sum_{|\tilde m|+|\tilde n|=2j} \left[\mathcal P_{(m,n),(\tilde m,\tilde n)} + \mathcal Q_{(m,n),(\tilde m,\tilde n)}\right] \end{aligned} \tag{34}$$

where:

$$\mathcal P_{(m,n),(\tilde m,\tilde n)} := \frac{\tilde c_{(\tilde m,\tilde n)} c_{(m,n)}}{\frac{\alpha}{2}(|m|-|n|)}(\tilde m_1 n_1 - m_1\tilde n_1)\, q_1^{m_1+\tilde m_1-1}\, q_2^{m_2+\tilde m_2} p_1^{n_1+\tilde n_1-1} p_2^{n_2+\tilde n_2}$$

$$\mathcal Q_{(m,n),(\tilde m,\tilde n)} := \frac{\tilde c_{(\tilde m,\tilde n)} c_{(m,n)}}{\frac{\alpha}{2}(|m|-|n|)}(\tilde m_2 n_2 - m_2\tilde n_2)\ q_1^{m_1+\tilde m_1} q_2^{m_2+\tilde m_2-1} p_1^{n_1+\tilde n_1} p_2^{n_2+\tilde n_2-1}.$$

The double sum in (34) can be rearranged as the sum of couples of symmetric monomials $\mathcal P_{(m,n),(\tilde m,\tilde n)} := \xi\, q_1^{M_1} q_2^{M_2} p_1^{N_1} p_2^{N_2}$ and $\mathcal Q_{\overline{(m,n)},\overline{(\tilde m,\tilde n)}} := \eta q_1^{N_2} q_2^{N_1} p_1^{M_2} p_2^{M_1}$ satisfying:

$$\xi = (-1)^{\sigma_{M,N}+\frac{|M|+|N|-2}{2}}\eta, \tag{35}$$

which implies that $\{\tilde F_j,\chi_{2i}\}$ has the permutation symmetry.

To see why, consider $m,n,\tilde m,\tilde n$ such that $(m,n)\neq\overline{(m,n)}$ and $(\tilde m,\tilde n)\neq\overline{(\tilde m,\tilde n)}$ (otherwise $\tilde c_{(\tilde m,\tilde n)} c_{(m,n)} = 0$ and there is nothing more to prove), and the term:

$$\begin{aligned}\mathcal P_{(m,n),(\tilde m,\tilde n)} := & \frac{\tilde c_{(\tilde m,\tilde n)} c_{(m,n)}}{\frac{\alpha}{2}(|m|-|n|)}(\tilde m_1 n_1 - m_1\tilde n_1)\, q_1^{m_1+\tilde m_1-1}\, q_2^{m_2+\tilde m_2} p_1^{n_1+\tilde n_1-1} p_2^{n_2+\tilde n_2} \\ = & \,\xi\, q_1^{M_1} q_2^{M_2} p_1^{N_1} p_2^{N_2}\end{aligned} \tag{36}$$

where we denote:

$$M_1 = m_1+\tilde m_1 - 1, M_2 = m_2+\tilde m_2, N_1 = n_1+\tilde n_1-1, N_2 = n_2+\tilde n_2. \tag{37}$$

and:

$$\xi = \frac{\tilde c_{(\tilde m,\tilde n)} c_{(m,n)}}{\frac{\alpha}{2}(|m|-|n|)}(\tilde m_1 n_1 - m_1\tilde n_1). \tag{38}$$

We consider the symmetric term:

$$\begin{aligned}\mathcal Q_{\overline{(m,n)},\overline{(\tilde m,\tilde n)}} := & \frac{\tilde c_{\overline{(\tilde m,\tilde n)}} c_{\overline{(m,n)}}}{\frac{\alpha}{2}(|n|-|m|)}(\tilde n_1 m_1 - n_1\tilde m_1)\ q_1^{n_2+\tilde n_2} q_2^{n_1+\tilde n_1-1} p_1^{m_2+\tilde m_2} p_2^{m_1+\tilde m_1-1} \\ = & \ \eta q_1^{N_2} q_2^{N_1} p_1^{M_2} p_2^{M_1}\end{aligned} \tag{39}$$

where M_1, M_2, N_1, N_2 have been defined in (37), and:

$$\eta = \frac{\tilde c_{\overline{(\tilde m,\tilde n)}} c_{\overline{(m,n)}}}{\frac{\alpha}{2}(|n|-|m|)}(\tilde n_1 m_1 - n_1\tilde m_1).$$

Since $F, \tilde{F}$ satisfy the permutation symmetry we have:

$$\xi = (-1)^{\sigma_{(m,n)}+\sigma_{(\tilde{m},\tilde{n})}+\frac{|m|+|n|+|\tilde{m}|+|\tilde{n}|-4}{2}} \frac{\tilde{c}_{\overline{(\tilde{m},\tilde{n})}} c_{\overline{(m,n)}}}{\frac{\alpha}{2}(|m|-|n|)} (\tilde{m}_1 n_1 - m_1 \tilde{n}_1)$$
$$= (-1)^{\sigma_{(m,n)}+\sigma_{(\tilde{m},\tilde{n})}+\frac{|M|+|N|-2}{2}} \eta.$$

We have the two possibilities:

- if m_1+n_1 and $\tilde{m}_1+\tilde{n}_1$ are both even or both odd we have $(-1)^{\sigma_{(m,n)}+\sigma_{(\tilde{m},\tilde{n})}} = 1$; $M_1+N_1 = m_1+\tilde{m}_1+n_1+\tilde{n}_1-2$ is even and therefore: $(-1)^{\sigma_{(m,n)}+\sigma_{(\tilde{m},\tilde{n})}} = (-1)^{\sigma_{(M,N)}}$.
- if m_1+n_1 is even and $\tilde{m}_1+\tilde{n}_1$ is odd (or m_1+n_1 is odd and $\tilde{m}_1+\tilde{n}_1$ is even), we have $(-1)^{\sigma_{(m,n)}+\sigma_{(\tilde{m},\tilde{n})}} = -1$; M_1+N_1 is odd therefore $(-1)^{\sigma_{(m,n)}+\sigma_{(\tilde{m},\tilde{n})}} = (-1)^{\sigma_{(M,N)}}$.

Therefore we have:

$$\xi = (-1)^{\sigma_{(M,N)}+\frac{|M|+|N|-2}{2}} \eta.$$

Since the Birkhoff normal forms are constructed with Lie series of functions having the permutation symmetry (and the generating functions are constructed from the coefficients of functions having the permutation symmetry) the Bnf of any order N has the permutation symmetry and proposition 3 is proved.

4. A different Hamiltonian representation of the Birkhoff normal forms

Since the Bnf of any order N has the permutation symmetry, it is divisible by $J(q,p) = q_1p_1 + q_2p_2$, i.e. we have:

$$\hat{K}_E^N(q,p) = -\mu + \frac{\alpha}{2}(q_1p_1+q_2p_2) + \hat{K}_4(q,p) + \ldots + \hat{K}_N(q,p)$$
$$= -\mu + (q_1p_1+q_2p_2)\,k(q,p)$$

where:

$$k(q,p) = \frac{\alpha}{2} + k_2(q,p) + \ldots + k_{N-2}(q,p)$$

and k_j are polynomials of order j in the variables q,p. We remark that all the monomials of $k(q,p)$ satisfy the resonant relation $m_1+m_2 = n_1+n_2$ and moreover we have: $\{k, \hat{K}_E^N\} = k\{k,J\} = 0$. Therefore, both functions $J(q,p), k(q,p)$ are first integrals of the Bnf.

Let us consider the Hamiltonian flow of the resonant Bnf of order N with initial conditions satisfying $\hat{K}_E^N(q(0),p(0)) = 0$. The Hamilton's equations of $\hat{K}_E^N$ have the form:

$$\dot{q}_i = q_i\,k(q,p) + J(q,p)\,\frac{\partial k}{\partial p_i}\ ,\ \ i = 1,2$$

$$\dot{p}_i = -p_i\,k(q,p) - J(q,p)\,\frac{\partial k}{\partial q_i}\ ,\ \ i = 1,2$$

and therefore the solutions with initial data satisfying

$$k(q(0),p(0)) := \Lambda, \quad J(q(0),p(0)) := \eta = 4\alpha\Omega,$$

satisfy also the Hamilton equations of:

$$\mathcal{H}(q,p) = \Lambda(q_1p_1 + q_2p_2) + 4\alpha\Omega(k_2(q,p) + k_4(q,p) + \ldots + k_{N-2}(q,p)). \tag{40}$$

The condition $\hat{K}_E^N(q(0),p(0)) = 0$, equivalent to $4\alpha\Lambda\Omega = \mu$, excludes in particular the cases[4] $\Lambda = 0$ and $\Omega = 0$. We therefore proceed by considering the case $\Omega \neq 0$ and we represent Hamiltonian (40) in the form:

$$\mathcal{H}(q,p) = h_2(q,p) + h_4(q,p) + \ldots + h_{N-2}(q,p)$$

with quadratic part (see equation (24)):

$$h_2 := \Lambda(q_1p_1 + q_2p_2) + \Omega(p_2q_1 - p_1q_2)$$

and $h_{2j}(q,p) = \eta k_{2j}(q,p) = 4\alpha\Omega k_{2j}(q,p)$ for $j \geqslant 2$.

The origin $(q,p) = (0,0,0,0)$ is an equilibrium point for the Hamiltonian flow of $\mathcal{H}$, with characteristic exponents:

$$\pm\Lambda \pm i\Omega$$

so that the equilibrium is of focus-focus type. We therefore proceed by defining a second Birkhoff normalisation of $\mathcal{H}$ using the focus-focus character of $(q,p) = (0,0,0,0)$.

4.1. The focus-focus Birkhoff normal forms

We first define the symplectic linear transformation:

$$\begin{aligned} q_1 &= \frac{P_1 - P_2}{\sqrt{2}} \ , \ p_1 = \frac{Q_2 - Q_1}{\sqrt{2}} \\ q_2 &= i\frac{P_1 + P_2}{\sqrt{2}} \ , \ p_2 = i\frac{Q_1 + Q_2}{\sqrt{2}}, \end{aligned} \tag{41}$$

which conjugates $\mathcal{H}$ to

$$\hat{\mathcal{H}} := \hat{h}_2(Q,P) + \hat{h}_4(Q,P) + \ldots + \hat{h}_{N-2}(Q,P) \tag{42}$$

where:

$$\hat{h}_2 = (i\Omega - \Lambda)Q_1P_1 - (i\Omega + \Lambda)Q_2P_2 \tag{43}$$

and, for $j \geqslant 4$, $\hat{h}_j$ is a polynomial of degree j containing only monomials

$$c_{(M,N)}Q_1^{M_1}Q_2^{M_2}P_1^{N_1}P_2^{N_2} \tag{44}$$

with $M_1 + M_2 = N_1 + N_2$. In fact, any monomial $q_1^{m_1}q_2^{m_2}p_1^{n_1}p_2^{n_2}$ is conjugate by (41) to the polynomial:

$$\left(\frac{P_1 - P_2}{\sqrt{2}}\right)^{m_1}\left(i\frac{P_1 + P_2}{\sqrt{2}}\right)^{m_2}\left(\frac{Q_2 - Q_1}{\sqrt{2}}\right)^{n_1}\left(i\frac{Q_1 + Q_2}{\sqrt{2}}\right)^{n_2},$$

which is a sum of terms proportional to the monomials:

$$P_1^{j_1}P_2^{m_1-j_1}P_1^{j_2}P_2^{m_2-j_2}Q_1^{i_1}Q_2^{n_1-i_1}Q_1^{i_2}Q_2^{n_2-i_2} = Q_1^{M_1}Q_2^{M_2}P_1^{N_1}P_2^{N_2}$$

[4] The case $\Lambda\Omega = 0$ is ruled out also if we consider the equation $K_E^N(q,p) = 0$, where $K_E^N(q,p)$ is the complete Hamiltonian (8) including also the remainder $\mathcal{R}_{N+2}$), by assuming a a mild condition on N (see appendix C for details).

with $M_1 = i_1 + i_2, M_2 = n_1 + n_2 - i_1 - i_2$, $N_1 = j_1 + j_2, N_2 = m_1 + m_2 - j_1 - j_2$; from $m_1 + m_2 = n_1 + n_2$ we have $M_1 + M_2 = N_1 + N_2$.

The Hamiltonian $\hat{\mathcal{H}}$ is in the suitable form to apply a sequence of Birkhoff normalizing canonical transformations following the procedure which is described in detail in the appendix A, by setting the parameters $\lambda_1 = i\Omega - \Lambda, \lambda_2 = -i\Omega - \Lambda$, so that the divisors entering the definition of the generating functions (62) are represented by:

$$\lambda_1 (N_1 - M_1) + \lambda_2 (N_2 - M_2) = i\Omega (N_1 - N_2 + M_2 - M_1) - \Lambda (N_1 + N_2 - M_1 - M_2). \quad (45)$$

For M, N satisfying $M_1 + M_2 = N_1 + N_2$, the resonant relation (55) becomes:

$$\lambda_1 (N_1 - M_1) + \lambda_2 (N_2 - M_2) = i\Omega (N_1 - N_2 + M_2 - M_1) = 0$$

which is satisfied if and only if $N_1 = M_1$ and $N_2 = M_2$.

Therefore, we have the opportunity to construct a non-resonant Birkhoff normalisation, i.e. a finite sequence of Birkhoff elementary canonical transformations conjugating Hamiltonian (42) to a Hamiltonian of the form:

$$H_E^N (Q,P) = (i\Omega - \Lambda) Q_1 P_1 - (i\Omega + \Lambda) Q_2 P_2 + \hat{h} (Q_1 P_1, Q_2 P_2) + \mathcal{R}_N (Q,P) \quad (46)$$

where $\hat{h}(I_1, I_2)$ is a polynomial in $I_1 = Q_1 P_1$, $I_2 = Q_2 P_2$ containing monomials of degree ranging between 2 and $(N-2)/2$, and $\mathcal{R}_N(Q,P)$ has Taylor series starting at order N.

Therefore, the Bnf which is obtained by neglecting in (46) the terms of order larger than $N-2$, is integrable since it contains only monomials satisfying $M_1 = N_1$ and $M_2 = N_2$, i.e. it depends on the variables Q, P only through the actions $I_1 = Q_1 P_1$, $I_2 = Q_2 P_2$.

Remark. Even if the small divisors (45) are proportional to Ω, which may be very small, the generating function (62) χ_4 is defined from the polynomial $\hat{h}_4$ which is the sum of monomials proportional to Ω as well. Therefore χ_4 is independent of Ω and the Lie series:

$$\sum_{j\geqslant 1} \frac{1}{j!} \mathcal{L}_\chi^j \hat{h}_2 = \sum_{j\geqslant 1} \frac{1}{j!} \mathcal{L}_{\chi_4}^{j-1} \left(\Pi_0 \hat{h}_4 - \hat{h}_4 \right).$$

$$\sum_{j\geqslant 0} \frac{1}{j!} \mathcal{L}_{\chi_4}^j \hat{h}_{2\ell}, \quad \ell \geqslant 2$$

are the sum of monomials proportional to Ω. The argument is repeated at all orders. Therefore, at any step of the sequence of Birkhoff normalisations we have generating functions which are independent on Ω, Λ, and the Lie series produced by the normalisation depend on these parameter in straightforward way. This means that we do not need to introduce Λ, Ω as additional parameters (beyond a straightforward multiplicative factor) to compute symbolically the Birkhoff normal forms of Hamiltonian $\hat{\mathcal{H}}$.

4.2. The Hamiltonian flow of the focus-focus Birkhoff normal form

By neglecting the remainder from Hamiltonian (46) we obtain the integrable Hamiltonian:

$$\hat{H}_E^N (Q,P) = (i\Omega - \Lambda) Q_1 P_1 - (i\Omega + \Lambda) Q_2 P_2 + \hat{h} (Q_1 P_1, Q_2 P_2) \quad (47)$$

which in particular is Poisson commuting with $I_1 = Q_1 P_1$, $I_2 = Q_2 P_2$. Since I_1, I_2 are first integrals, the Hamiltonian flow of (47) is represented explicitly by the formulas:

$$Q_j (\tau) = Q_j (0) e^{\hat{\kappa}_j \tau} \ , \ P_j (t) = P_j (0) e^{-\hat{\kappa}_j \tau} \ , \ j = 1,2 \quad (48)$$

where:

$$\hat{\kappa}_j = \frac{\partial}{\partial I_j}\Big((i\Omega - \Lambda) I_1 - (i\Omega + \Lambda) I_2 + \hat{h}(I_1, I_2)\Big)_{|_{I_1=I_1(0), I_2=I_2(0)}}.$$

The representation of the solutions is now complete. For example, for $N = 8$, we have:

$$\begin{aligned}\hat{H}^8_E(Q,P) &= (i\Omega - \Lambda) Q_1 P_1 - (i\Omega + \Lambda) Q_2 P_2 \\ &+ \frac{\eta}{32\alpha^3}\left(\left(P_1^2 Q_1^2 + P_2^2 Q_2^2\right)(3-\mu) + 4 Q_1 P_1 Q_2 P_2 (1-2\mu)\right) \\ &+ \frac{i\eta}{64\alpha^5}\left[(5-3\mu)\left(P_2^3 Q_2^3 - P_1^3 Q_1^3\right) - \frac{3}{4}\left(95 - 178\mu + 75\mu^2\right)\left(P_1^2 Q_1^2 P_2 Q_2 - P_1 Q_1 P_2^2 Q_2^2\right)\right].\end{aligned}$$

Remarks. (I) For each order of approximation, the Birkhoff normalisations and the parameters of the focus-focus dynamics (48) are represented by polynomials whose coefficients can be computed iteratively by a computer algebra system (see appendix B, for the generating functions of the Bnf of order $N = 6$).

(II) Solutions (48) of the Hamilton equations of the focus-focus normal form $\hat{H}^N_E(Q,P)$, when mapped back to original regularizing variables u, U, can be compared to solutions of the CR3BP using the methods of Hamiltonian perturbation theory. In fact, while all the transformations of variables from the original variables u, U to the focus-focus variables Q, P conjugate solutions of a Hamiltonian system to solutions of another Hamiltonian system as long as $\|(Q(\tau), P(\tau))\|$ remains smaller than a suitable positive R_0, two sources of errors have been introduced when we neglected the small remainders in the first and second Birkhoff normal forms (precisely, we neglected the small remainder $\mathcal{R}_N$ in Hamiltonian (46) and the small remainder $\mathcal{R}_{N+2}$ in Hamiltonian (20)). These errors can be controlled analytically using the Gronwall Lemma (see, for example, [36]), as explained in appendix C.

5. Numerical demonstrations

In this section we provide some numerical demonstrations of the application of the Birkhoff normalisations described in the previous sections. The goal is to show the effectiveness of the representations provided in this paper for some realistic choices of the parameters μ, E. A systematic investigation of the effectiveness of the method for a wider range of applications is left to a further paper.

The most important parameters of the problem are the mass parameter μ and the value E of the Hamiltonian characterizing the close encounter. We consider two numerical values for the mass parameter $\mu = 10^{-3}$ and $\mu = 3.\ 10^{-6}$ which are representative, up to a small correction, of the three-body problems which are defined by considering the couples Sun–Jupiter and Sun–Earth as primary bodies respectively. The method considered in this Paper applies to the fast close encounters, whose parameter E satisfies inequality (2). Since the parameter $\lambda = \alpha/2 > 0$ appears in the denominators of the coefficients of the Bnf (e.g. see equations (36) and (39)), the values of α close to zero (and in general $\alpha/2 \in (0,1)$) imply larger coefficients of the monomials of the Bnf as well as larger errors in the representation of the solutions transiting at a given distance from P_2. Therefore, we consider the value of $E = -1.35$ which is close to the limit value $\alpha = 0$ (precisely the small divisor in this case is $\alpha/2 \sim 0.27$); the results are expected to improve for larger values of E.

For each value of μ, we choose an initial condition $u(0), U(0)$ well inside the Hill's sphere of the secondary body and we numerically compute its orbit for positive and negative times; this is equivalent to consider orbits having a close encounter with P_2. The orbits corresponding to the selected initial conditions are computed by numerically integrating the Hamilton's equations of the Levi–Civita Hamiltonian using an explicit Runge–Kutta algorithm of order 6, with a very small integration time step of $\tau = 10^{-5}$, and extended floating point numerical precision (the absolute value of the regularised Hamiltonian remains smaller than 3.5 10^{-16}).

The numerically computed solutions will be used to check the conservation of the integrals of motions of the first and second Birkhoff normal forms, and compared with the solutions provided in parametric form by the equation (48). Two orbits, numerically computed for positive (black curve) and negative times (red curve), are represented in figure 1 for the case $\mu = 3\ 10^{-6}$ and $\mu = 10^{-3}$ respectively. The time variations of the semi-major axis a and eccentricity e are reported in figure 2: we appreciate that there is a sharp variation of a, e in both cases, associated to the close encounter; in the case of the smaller mass value $\mu = 3\ 10^{-6}$ the variation is almost stepwise.

These orbits are then analysed using the Birkhoff normalisations described in sections 2 and 4. The canonical transformations and the normal form Hamiltonian $\hat{H}_E^N$ are computed by implementing with a computer algebra system the algorithm described in appendix A. The implementation of the Birkhoff transformations requires:

- the computation of Taylor expansions of functions up to a given polynomial order N;
- the computation of the generating functions χ_J from the Taylor expansion of the Hamiltonian represented with the hyperbolic variables;
- the computation of the Lie series defined by χ_J up to the degree N.

The operations required by these three steps are performed by several available computer algebra systems. The implementation can be symbolic, i.e. the coefficients of all the monomials are given as functions of the parameters μ, α (and no floating point approximations are introduced), or numeric, i.e. for a given value of μ, α the coefficients of all the monomials are given as floating point numbers. The second choice allows to reach larger orders N. A symbolic implementation has been executed up to $N = 20$; a numerical implementation (where μ, E are treated as floating point numbers) has been executed up to $N = 30$, and this is the procedure whose results are reported below. In the numerical implementation the generating functions and the Birkhoff normal forms Hamiltonians are computed as polynomials with floating point coefficients. The canonical transformations, which are the flow at time $\tau = 1$ (or $\tau = -1$ for the inverse) of the generating functions, can be also represented by polynomials. The direct numerical evaluation of these polynomials along the flow is computationally expensive; a less expensive evaluation (which is used in the computations described below) is obtained by numerically computing the Hamiltonian flow of the generating functions using an explicit Runge-Kutta algorithm of order 6 with 100 steps (further details about the numerical implementation are provided in section 5.1).

For each orbit:

- We compute the function

$$J^N = q_1 p_1 + q_2 p_2$$

along the numerically computed orbit for different orders of the first Birkhoff normalisation ranging from $N = 2$ (no normalisation) up to $N = 30$. We also check the conservation of the actions Q_1P_1 and Q_2P_2 for different normalisation orders of the second Birkhoff normalisation (which are implemented from the first Bnf of order $N = 30$). Since the real and imaginary parts of Q_1P_1, Q_2P_2 are proportional to $q_1p_1 + q_2p_2$ (which is already conserved up to order $N = 30$ from the first Bnf) or $q_1p_2 - q_2p_1$, and in particular we have

$$Q_1P_1 = -\frac{q_1p_1 + q_2p_2}{2} + i\,\frac{p_1q_2 - p_2q_1}{2},$$

we just need to check the conservation of

$$W^{N,\tilde{N}} = \Im(Q_1P_1)$$

along the numerically computed orbits (N refers to the order of the first Bnf, $\tilde{N}$ to the order of the second Bnf). The relative variations of $J^N, W^{N,\tilde{N}}$ are represented versus the distance $r = |u|^2$ from the secondary body, providing us an estimate of the error which we have at some distance from P_2 using different normalisation orders. In fact, the approximation of the solutions of the three-body problem with the solutions of the Hamilton's equations of Hamiltonian (40) during the transit inside a sphere centred at P_2 of radius ρ is justified if the variations of the function $J^N, W^{N,\tilde{N}}$ remain extremely small during the transit, possibly smaller than the numerical precision.

– For the same initial conditions of the numerical integrations we also compute the solution provided by the parametric representation of equation (48) (mapped back to the original Levi–Civita variables u, U). The difference between the parametric and the numerical solution is represented versus the distance $|u|^2$ for different normalisation orders.

During the fast close encounters, the orbits have a sharp change of the semi-major axis and eccentricity (see figure 2). In the left panels of figures 3 and 4 we represent the relative variations:

$$\mathrm{DJ} = \frac{|J^N(q(t), p(t)) - J^N(q(0), q(0))|}{|J^N(q(0), q(0))|} \tag{49}$$

for different orders of the first Birkhoff normalisation ranging from $N = 2$ (no normalisation) up to $N = 30$, computed during the close encounter occurring from initial time until P reaches a distance $r = 0.1$ from P_2, for the orbit of the case $\mu = 3\,10^{-6}$ (figure 3) and $\mu = 10^{-3}$ (figure 4). The relative variations are indeed extremely small for $N = 30$ ($\sim 10^{-15}$) up to $r \sim 0.02$ for both orbits, with a possible improvement which still could be obtained by considering larger values of N. In the same panels we also report the relative variation of $W^{N,\tilde{N}}$, computed after the $N = 30$ normalisations of the first Bnf:

$$\mathrm{DW} = \frac{|W^{N,\tilde{N}}(q(t), p(t)) - W^{N,\tilde{N}}(q(0), q(0))|}{|W^{N,\tilde{N}}(q(0), q(0))|}, \tag{50}$$

which is very small up to $r \sim 0.02$ already for $\tilde{N} = 10$ (figure 3) and $\tilde{N} = 16$ (figure 4) respectively.

In the right panels of (figures 3 and 4) we represent the distance:

$$\mathrm{DIST} = \|\left(x^N - x^P, y^N - y^P\right)\| \tag{51}$$

in Cartesian variables (x, y) between the solutions (x^N, y^N) computed by numerically integrating the Hamilton's equations of the Levi Civita Hamiltonian and the solutions (x^P, y^P) provided by the parametric representation (48): the distances are smaller than 10^{-16} up to $r = 0.02$.

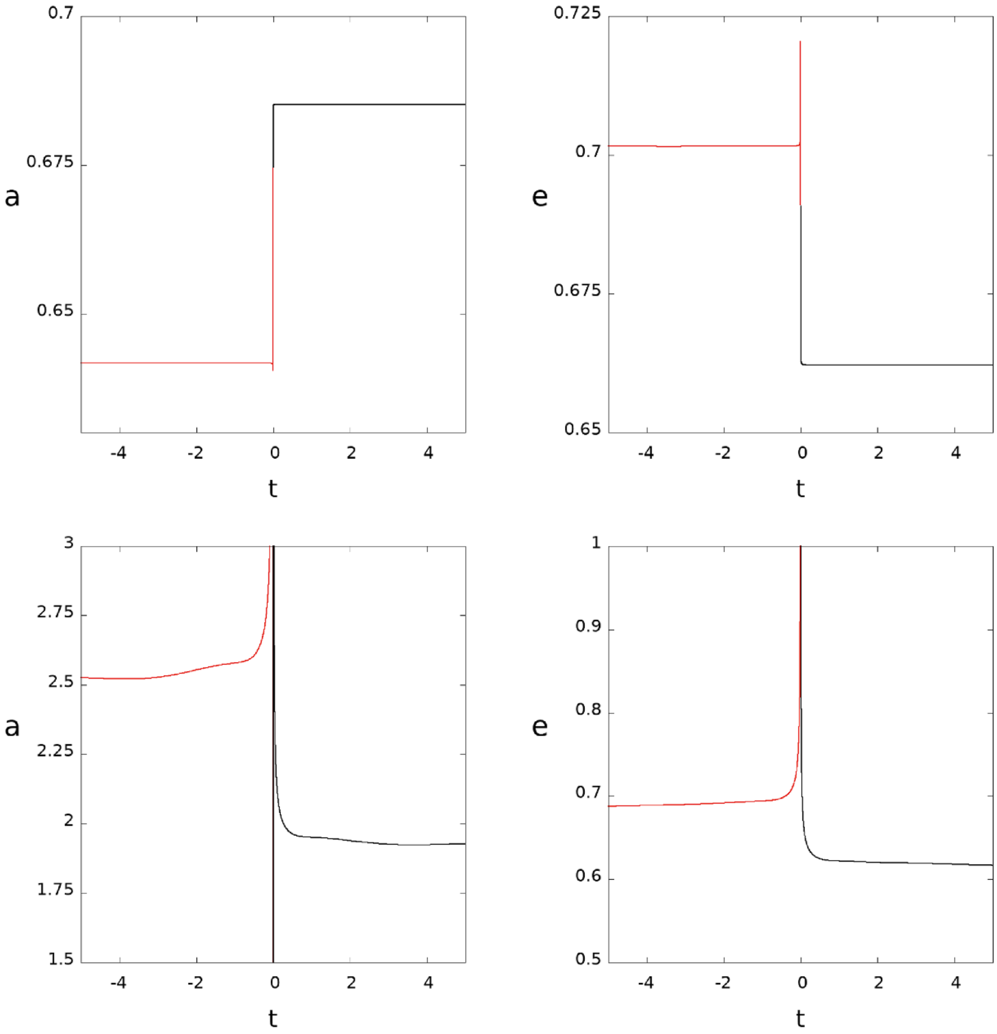

Figure 2. Representation of the time evolution of the semi-major axis a and eccentricity e for the orbits represented in figure 1: the top panels refer to the case $\mu = 3\ 10^{-6}$, the bottom panels to the case $\mu = 10^{-6}$.

5.1. Numerical tools

The sequence of functions χ_J generating the canonical normal forms which conjugate the Levi–Civita Hamiltonian to the saddle-saddle Birkhoff normal forms (see appendix A) and then to the focus-focus normal forms, as well as the normal form Hamiltonians, are computed using the algebra system provided by the software package Mathematica. Within the program, all the functions are represented as polynomials in the variables q, p (or Q, P), whose coefficients can be expressed either by explicit functions of the parameters μ and α (symbolic representation) or by floating point numbers, for given values of μ and α (numerical representation).

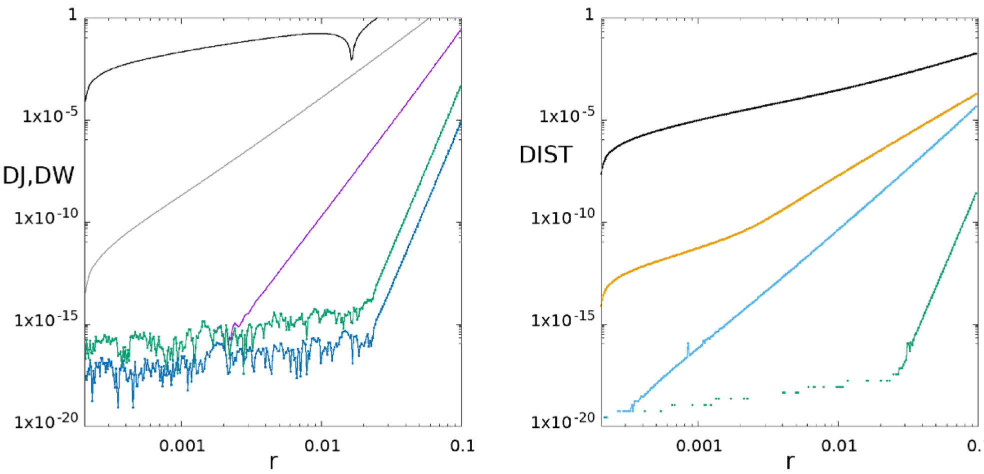

Figure 3. Left panel: relative variation DJ (defined in equation (49)) computed for the same orbit of figure 1, top panels (case $\mu = 3\ 10^{-6}$) represented versus the distance $r := |u|^2$ from P_2 for $N = 2$ (black) and the improvements provided by the Birkhoff normalisations with $N = 8$ (gray), $N = 16$, (violet), $N = 30$ (green). The blue line represents instead the relative variation DW (defined in equation (50)) computed for $N = 30, \tilde{N} = 10$. Right panel: variation of DIST (defined in equation (51)) represented versus the distance $r := |u|^2$ for $N, \tilde{N} = 2$ (black) and the improvements provided by the Birkhoff normalisations with $N, \tilde{N} = 4$ (gold), $N, \tilde{N} = 10$ (light blue), $N = 30, \tilde{N} = 14$ (green).

Apart from the elementary algebra, the normal form computations required: the symbolic computation of the derivatives of any function (which are needed to define the Taylor expansions as well as the Poisson brackets and Lie derivatives) and the Replace command (which is used to compute the compositions of functions; each composition is then replaced by its Taylor expansion computed up to order $N+2$). The factor $k(q,p)$ of the resonant normal form $\hat{K}_E^N$ is computed by processing the polynomial representation of the normal form with the Mathematica command Factor. Then, the product of functions $J(q,p)k(q,p)$ is processed with the Mathematica command Expand to check the correspondence of $-\mu + J(q,p)k(q,p)$ with the polynomial representation of $\hat{K}_E^N$. The coefficients of the generating functions χ_J, of the normal form Hamiltonians and their derivatives have been then exported to external ASCII files using the Mathematica commands FortranForm and Export, so they could be used by programs written in Fortran.

The other numerical results were obtained using numerical computations performed with the GNU Fortran f95 compiler.

6. Conclusions and perspectives

The method presented in this paper allows to compute the close encounters with the secondary body of the planar circular restricted three–body problem using integrable dynamics approximating the regularised Hamiltonian at any order N of its Birkhoff approximations. The numerical implementations of the method show that it indeed can be applied to problems whose parameters are compatible with important Solar System three-body problems, with errors on

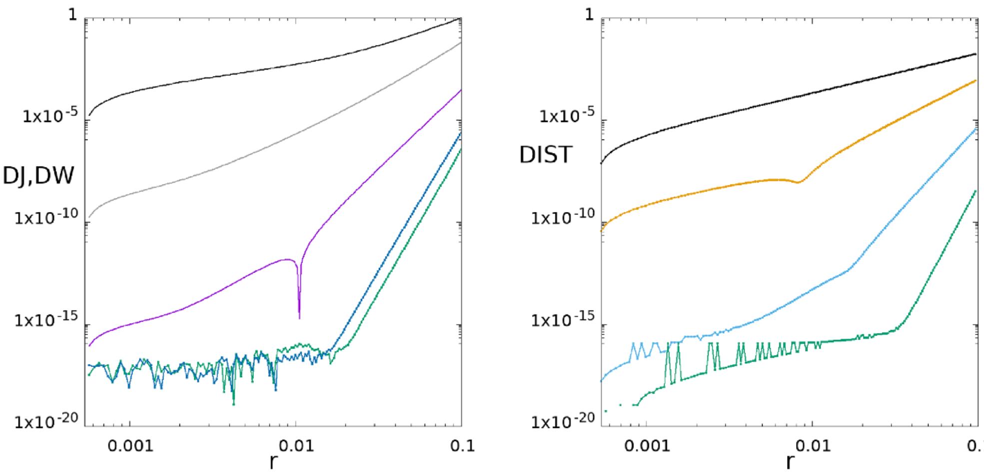

Figure 4. Left panel: relative variation DJ (defined in equation (49)) computed for the same orbit of figure 1, bottom panels (case $\mu = 10^{-3}$) represented versus the distance $r := |u|^2$ for $N = 2$ (black) and the improvements provided by the Birkhoff normalisations with $N = 8$ (gray), $N = 16$, (violet), $N = 30$ (green). The blue line represents instead the relative variation DW (defined in equation (50)) computed for $N = 30, \tilde{N} = 16$. Right panel: variation of DIST (defined in equation (51)) represented versus the distance $r := |u|^2$ for $N = \tilde{N} = 2$ (black), and the improvements provided by the Birkhoff normalisations with $N, \tilde{N} = 4$ (gold), $N, \tilde{N} = 14$ (light blue), $N = 30, \tilde{N} = 14$ (green).

the predicted orbits which can be reduced below the round-off approximation. Therefore, it provides a Hamiltonian integrator of the close encounters as a single step.

We have therefore the opportunity to compare the output of numerical integrations with these Hamiltonian dynamics to confirm their correctness when they agree below a precision threshold. In fact, there is no way to state the correctness of the numerical integration of a close encounter, apart from the necessary conservation of the Hamiltonian (which implies the approximate conservation of the Tisserand parameter before and after the close encounter).

Let us discuss the directions of future investigations aimed at establishing the boundaries of the method's applicability. The numerical tests presented in this paper (which are concerned with two values of the mass ratios, representative of the Sun–Earth and Sun–Jupiter cases, the value of $E := -1.35$, and $N = 30$) already provide an indication of the applicability of the method to study the planetary close encounters (since all the planets of the Solar System have mass ratio smaller than Jupiter) for values of $E \geqslant -1.35$; possible improvements of the precision can be obtained by further increasing the order N of normalisations. In future research, applications of the method for a full range of parameters μ, E which are relevant for Solar System applications will be investigated: on the one hand, it would be interesting to know which is the lower value of the energy E for which the method applies; on the other hand it would be interesting to know if the method applies also for larger values of the mass ratios, possibly including the Earth-Moon case. From preliminary results obtained for the Earth–Moon case (and $E := -1.35$) we still appreciate a reduction of the errors for increasing values of N, but the error obtained with the best normal form currently available (i.e $N = 30$) is above the required precision (for $N = 30$ we obtain an error in the conservation of $J(q,p)$ of about

$5.\ 10^{-7}$ at $\|P-P_2\| = 10^{-2}$). Therefore, current results are compatible with a possible applicability of the method also to the Earth–Moon case, which however needs an increment of the implementation to significantly larger orders N to receive confirmation.

Also, in view of applications, future research will be devoted to consider the implications of the present results for the dynamics of models which are more representative of the dynamics of a realistic model of the Solar System.

Data availability statement

There are no external data used for this research. The data that support the findings of this study are available upon reasonable request from the authors.

Acknowledgments

The author thanks Profs. E Detomi and A Lucchini for presenting to me the sufficient condition for the factorisation of polynomials of four variables. He also acknowledges the project MIUR-PRIN 20178CJA2B 'New frontiers of Celestial Mechanics: theory and applications'.

Appendix A. Definition of the Birkhoff normal forms for 2-degrees of freedom Hamiltonians

Consider the two-degrees of freedom Hamiltonian system of the form:

$$H(q,p) = h_2(q,p) + h_4(q,p) + h_6(q,p)\dots \tag{52}$$

where:

$$h_2(q,p) = \lambda_1 q_1 p_1 + \lambda_2 q_2 p_2,$$

and the functions $h_{2j}(q,p)$ are polynomials of degree $2j$.

For any even number $N \geqslant 4$, we apply iteratively a finite sequence of canonical transformations, called Birkhoff normalisations, whose composition is a close to the identity canonical transformation defined in a neighbourhood of $(q,p) = (0,0,0,0)$ (which is a fixed point of the transformation), conjugating H to

$$H^N(q,p) = h_2(q,p) + \hat{h}_4(q,p) + \ldots + \hat{h}_N(q,p) + \mathcal{R}_{N+2}(q,p) \tag{53}$$

where $\mathcal{R}_{N+2}$ is a series in q,p starting with terms of degree at least $N+2$ and the $\hat{h}_j$ are polynomials of order j in q,p (only the polynomials with j even number appear in the expansion) containing monomials:

$$c^{j,N}_{m,n} q_1^{m_1} q_2^{m_2} p_1^{n_1} p_2^{n_2} \tag{54}$$

with $m = (m_1,m_2), n = (n_1,n_2)$ satisfying:

$$\lambda_1(n_1 - m_1) + \lambda_2(n_2 - m_2) = 0. \tag{55}$$

The Hamiltonian which is obtained by neglecting the remainder $\mathcal{R}_{N+2}(q,p)$ in $H^N(q,p)$:

$$\hat{H}^N(q,p) = h_2(q,p) + \hat{h}_4(q,p) + \ldots + \hat{h}_N(q,p),$$

will be called Birkhoff normal form of H of order N. The Birkhoff transformations and the transformed Hamiltonians can be computed explicitly with a computer algebra system, by representing them with the Lie series method (for a modern presentation of the Lie series method we refer to [14]), following the procedure described below.

The Birkhoff normalisation of Hamiltonian (52) is obtained, for any $N \geqslant 4$, with a canonical transformation $\mathcal{C}_N$ defined as the composition of a sequence of canonical Birkhoff transformations:

$$(q,p) := \left(q^{(2)},p^{(2)}\right) \mapsto \left(q^{(4)},p^{(4)}\right) \mapsto \ldots \mapsto \left(q^{(N)},p^{(N)}\right) \tag{56}$$

where each elementary transformation:

$$(q^{(J-2)},p^{(J-2)}) = \Phi^1_{\chi_J}(q^{(J)},p^{(J)})) \tag{57}$$

is the Hamiltonian flow $\Phi^1_{\chi_J}$ at time 1 of a Hamiltonian χ_J, which is polynomial of degree J. The elementary transformations (57) can be computed explicitly as the Lie series:

$$\zeta = \left[e^{L_{\chi_J}}\zeta\right]\left(q^{(J)},p^{(J)}\right) := \zeta' + \{\zeta,\chi_J\}\left(q^{(J)},p^{(J)}\right) + \frac{1}{2}\{\{\zeta,\chi_J\},\chi_J\}\left(q^{(J)},p^{(J)}\right) + \ldots, \tag{58}$$

where $L_{\chi_J} := \{\cdot,\chi_J\}$, and ζ,ζ' denote any couple of variables $q^{(J-2)},q^{(J)}$ or $p^{(J-2)},p^{(J)}$. The Hamiltonian $H^{(2)} := H$ is conjugate to the sequence of Hamiltonians $H^{(4)},\ldots,H^{(J)}$ which can be computed as Lie series as well:

$$H^{(J)} = e^{L_{\chi_J}} H^{(J-2)}, \tag{59}$$

and the iteration ends for $J = N$. The aim is to define the generating functions χ_J such that, at any step J, the intermediate normalised Hamiltonians:

$$H^{(J)} := h_2 + \sum_{j=2}^{\frac{J}{2}} H^{(J)}_{2j} + \sum_{j \geqslant (J+2)/2} \tilde{H}^{(J)}_{2j} \tag{60}$$

have the property that $H^{(J)}_{2j}$ and $\tilde{H}^{(J)}_{2j}$ are polynomials of degree $2j$, and moreover $H^{(J)}_{2j}$ contain only monomials satisfying:

$$\lambda_1(n_1 - m_1) + \lambda_2(n_2 - m_2) = 0. \tag{61}$$

We suppose that the Hamiltonian has been normalised up to given $J \geqslant 2$, and we define the subsequent normalisation with a generating function χ_{J+2} which is polynomial of degree $J+2$. Therefore, from equations (58) and (59), and the representation (60), we have:

$$H^{(J+2)} = h_2 + \sum_{j=2}^{\frac{J}{2}} H^{(J)}_{2j} + \left(\{h_2,\chi_{J+2}\} + \tilde{H}^{(J)}_{J+2}\right) + \mathcal{O}(J+4),$$

whose term of degree $J+2$ is $\{h_2,\chi_{J+2}\} + \tilde{H}^{(J)}_{J+2}$. By denoting with $a^{(J)}_{m_1,m_2,n_1,n_2} q_1^{m_1} q_2^{m_2} p_1^{n_1} p_2^{n_2}$ the monomials of $\tilde{H}^{(J)}_{J+2}$, and defining the generating function χ_{J+2} by:

$$\chi_{J+2} = \sum_{(m_1,m_2,n_1,n_2) \in \mathcal{L}_{J+2}} \frac{a^{(J)}_{m_1,m_2,n_1,n_2}}{[\lambda_1(m_1 - n_1) + \lambda_2(m_2 - n_2)]} q_1^{m_1} q_2^{m_2} p_1^{n_1} p_2^{n_2} \tag{62}$$

where:

$$\mathcal{L}_J = \left\{ (m_1, m_2, n_1, n_2) \in \mathbb{N}^4 : \sum_{j=1}^{2} (n_j + m_j) = J, \text{ and } \lambda_1 (m_1 - n_1) + \lambda_2 (m_2 - n_2) \neq 0 \right\},$$

we have

$$\{h_2, \chi_{J+2}\} + \tilde{H}^{(J)}_{J+2} = \Pi \tilde{H}^{(J)}_{J+2} := \sum_{m,n:|m|+|n|=J+2, \lambda_1(n_1-m_1)+\lambda_2(n_2-m_2)=0} a^{(J)}_{m,n} q^m p^n. \tag{63}$$

Therefore, $H^{(J+2)}$ is normalised up to degree $J+2$. Since the generating function χ_J is polynomial of order larger than 4, its flow at time 1 is close to the identity and has the origin $(q,p) = (0,0,0,0)$ as a fixed point.

Appendix B. The Birkhoff normalisation for $N = 6$

The saddle-saddle Birkhoff normalisation of order $N = 6$ is obtained using the generating functions:

$$\chi_4 = \frac{p_1{}^2 p_2 q_1}{8\alpha^2} - \frac{p_1{}^3 q_2}{8\alpha^2} - \frac{p_1 p_2{}^2 q_2}{8\alpha^2} + \frac{p_1 q_1{}^2 q_2}{8\alpha^2} + \frac{p_1 q_2{}^3}{8\alpha^2} + \frac{p_2{}^3 q_1}{8\alpha^2} - \frac{p_2 q_1{}^3}{8\alpha^2} - \frac{p_2 q_1 q_2{}^2}{8\alpha^2}$$

$$\begin{aligned}
\chi_6 = {} & \frac{p_1{}^2 p_2{}^2 q_1{}^2 (9\mu - 11)}{32\alpha^4} - \frac{3p_1{}^3 p_2{}^2 q_1 (\mu - 1)}{32\alpha^4} + \frac{p_1{}^2 p_2{}^2 q_2{}^2 (9\mu - 11)}{32\alpha^4} - \frac{3p_1{}^2 p_2{}^3 q_2 (\mu - 1)}{32\alpha^4} \\
& + \frac{p_1{}^4 p_2{}^2 (\mu - 1)}{64\alpha^4} + \frac{p_1{}^2 p_2{}^4 (\mu - 1)}{64\alpha^4} + \frac{p_1{}^3 p_2 q_1 q_2 (3\mu - 2)}{8\alpha^4} - \frac{3p_1{}^4 p_2 q_2 (\mu - 1)}{64\alpha^4} \\
& + \frac{p_1{}^2 q_1{}^2 q_2{}^2 (11 - 9\mu)}{32\alpha^4} - \frac{15p_1{}^4 q_1{}^2 (\mu - 1)}{64\alpha^4} + \frac{15p_1{}^2 q_1{}^4 (\mu - 1)}{64\alpha^4} + \frac{3p_1{}^5 q_1 (\mu - 1)}{64\alpha^4} \\
& + \frac{p_1{}^4 q_2{}^2 (3\mu - 7)}{64\alpha^4} + \frac{p_1{}^2 q_2{}^4 (7 - 3\mu)}{64\alpha^4} - \frac{p_1{}^6 (\mu - 1)}{192\alpha^4} + \frac{p_1 p_2{}^3 q_1 q_2 (3\mu - 2)}{8\alpha^4} \\
& - \frac{3p_1 p_2{}^4 q_1 (\mu - 1)}{64\alpha^4} + \frac{p_1 p_2 q_1{}^3 q_2 (2 - 3\mu)}{8\alpha^4} + \frac{p_1 p_2 q_1 q_2{}^3 (2 - 3\mu)}{8\alpha^4} + \frac{3p_1 q_1{}^3 q_2{}^2 (\mu - 1)}{32\alpha^4} \\
& - \frac{3p_1 q_1{}^5 (\mu - 1)}{64\alpha^4} + \frac{3p_1 q_1 q_2{}^4 (\mu - 1)}{64\alpha^4} + \frac{p_2{}^2 q_1{}^2 q_2{}^2 (11 - 9\mu)}{32\alpha^4} + \frac{p_2{}^2 q_1{}^4 (7 - 3\mu)}{64\alpha^4} \\
& + \frac{p_2{}^4 q_1{}^2 (3\mu - 7)}{64\alpha^4} + \frac{15p_2{}^2 q_2{}^4 (\mu - 1)}{64\alpha^4} - \frac{15p_2{}^4 q_2{}^2 (\mu - 1)}{64\alpha^4} + \frac{3p_2{}^5 q_2 (\mu - 1)}{64\alpha^4} \\
& - \frac{p_2{}^6 (\mu - 1)}{192\alpha^4} + \frac{3p_2 q_1{}^2 q_2{}^3 (\mu - 1)}{32\alpha^4} + \frac{3p_2 q_1{}^4 q_2 (\mu - 1)}{64\alpha^4} - \frac{3p_2 q_2{}^5 (\mu - 1)}{64\alpha^4} \\
& - \frac{q_1{}^2 q_2{}^4 (\mu - 1)}{64\alpha^4} - \frac{q_1{}^4 q_2{}^2 (\mu - 1)}{64\alpha^4} + \frac{q_1{}^6 (\mu - 1)}{192\alpha^4} + \frac{q_2{}^6 (\mu - 1)}{192\alpha^4}
\end{aligned}$$

The focus-focus Birkhoff normalisation of order $N = 6$ is obtained using the generating functions:

$$\chi_4 = \frac{15iP_2{}^2Q_1{}^2(\mu-1)}{32\alpha^2} - \frac{15iP_1{}^2Q_2{}^2(\mu-1)}{32\alpha^2}$$

$$\begin{aligned}\chi_6 = &\frac{15iP_1{}^2P_2Q_1{}^3(\mu-1)}{128\alpha^3} - \frac{75iP_1{}^2P_2Q_1Q_2{}^2(\mu-1)}{128\alpha^3} - \frac{45P_1{}^2P_2Q_2{}^3\left(\mu^2-6\mu+5\right)}{256\alpha^4}\\ &- \frac{15iP_1{}^3Q_1{}^2Q_2(\mu-1)}{128\alpha^3} + \frac{45P_1{}^3Q_1Q_2{}^2\left(\mu^2-6\mu+5\right)}{256\alpha^4} - \frac{175iP_1{}^3Q_2{}^3(\mu-1)}{384\alpha^3}\\ &+ \frac{75iP_1P_2{}^2Q_1{}^2Q_2(\mu-1)}{128\alpha^3} - \frac{45P_1P_2{}^2Q_1{}^3\left(\mu^2-6\mu+5\right)}{256\alpha^4} - \frac{15iP_1P_2{}^2Q_2{}^3(\mu-1)}{128\alpha^3}\\ &+ \frac{45P_2{}^3Q_1{}^2Q_2\left(\mu^2-6\mu+5\right)}{256\alpha^4} + \frac{175iP_2{}^3Q_1{}^3(\mu-1)}{384\alpha^3} + \frac{15iP_2{}^3Q_1Q_2{}^2(\mu-1)}{128\alpha^3}\end{aligned}$$

Appendix C. Error estimates through the Gronwall lemma

In this appendix we describe how a standard application of the Gronwall lemma (see, for example, [36]) together with an estimate of the time intervals characterizing the close encounters allow to bound the errors which are introduced when we neglect the small remainders of the Birkhoff normal forms (20) and (46).

Consider any Hamiltonian of the form:

$$k(q,p) = k_2(q,p) + k_4(q,p) + \ldots + k_J(q,p) + \mathcal{R}_{J+2}(q,p)$$

where k_j are homogeneous polynomials of degree j and $C, R > 0$ are constants such that for any $\rho \in [0,R]$ we have[5]:

$$\sup_{\|(q,p)\|\leqslant\rho} \max\{|\mathcal{R}_{J+2}(q,p)|, \rho\|X_{J+2}(q,p)\|\} < C\rho^{J+2} \tag{64}$$

where X_{J+2} denotes the Hamiltonian vector field X_{J+2} of $\mathcal{R}_{J+2}$. Let $(q(\tau),p(\tau))$ be a solution of the Hamilton equations of $k(q,p)$ satisfying $\|(q(\tau),p(\tau))\| \leqslant \rho$ for all $\tau \in [0,T]$, and $(\tilde{q}(\tau),\tilde{p}(\tau))$ a solution of the Hamilton equations of

$$\hat{k}(q,p) = k_2(q,p) + k_4(q,p) + \ldots + k_J(q,p)$$

with $(\tilde{q}(0),\tilde{p}(0)) = (q(0),p(0))$ satisfying $\|(\tilde{q}(\tau),\tilde{p}(\tau))\| \leqslant \rho$ for all $\tau \in [0,T]$. As a standard application of the Gronwall lemma we obtain:

$$\|(q(\tau),p(\tau)) - (\tilde{q}(\tau),\tilde{p}(\tau))\| \leqslant \frac{C}{\lambda_{\hat{k}}}\rho^{J+1}\left[e^{\lambda_{\hat{k}}\tau} - 1\right] \tag{65}$$

[5] In the analytic setting, when $\mathcal{R}_{J+2}$ is a Taylor series in the variables (q,p) starting at degree $J+2$, inequalities (64) follow from standard Cauchy estimates.

where $\lambda_{\hat{k}}$ is a Lipschitz constant for the Hamiltonian vector field of $\hat{k}$ in the ball of radius R. Estimate (65) allows to compare the solutions of the Hamilton equations of Hamiltonians (11) and (20), as well as the solutions of the Hamilton equations of Hamiltonians (47) and (46) expressed using the real variables (q,p) (see equation (41)). Let us discuss, for example, the comparison of the solutions of the Hamilton equations of Hamiltonians (47) and (46) for a given value of α and N (the other comparison is indeed very similar).

Using equation (48), one first estimates the time τ characterizing the close-encounter as follows. Each solution $(q(\tau),p(\tau))$ of the Hamilton equations of (47) transits through a set $\|q\|,\|p\| \leqslant \rho$ in the time:

$$\tau(I_1) = \frac{1}{\tilde{\Lambda}} \ln \frac{\rho^2}{2|I_1|} \tag{66}$$

where $\tilde{\Lambda} = |\Re \hat{\kappa}_1|$ and $I_1 = Q_1 P_1$ is the complex action of the focus-focus dynamics, which in particular satisfies $|I_1| \geqslant |J(q(0),p(0))|/2 = 2\alpha\Omega$. Since the initial condition is on the zero level set of the complete Hamiltonian (20), we have:

$$|\mu - 4\alpha\Omega\Lambda| \leqslant \|\mathcal{R}_{N+2}\|. \tag{67}$$

By assuming the mild condition on ρ, N ensuring $\|\mathcal{R}_{N+2}\| \leqslant \mu/2$, on the one hand we rule out the possibility of $\Lambda\Omega = 0$, on the other hand we obtain:

$$|I_1| \geqslant \frac{\mu}{4\Lambda}.$$

Therefore, the right-hand side of inequality (65) is bounded by:

$$\frac{C}{\lambda_{\hat{k}}} \rho^{N-1} \left[e^{\frac{\lambda_{\hat{k}}}{\tilde{\Lambda}} \ln \frac{2\Lambda\rho^2}{\mu}} - 1 \right] \leqslant \frac{C}{\lambda_{\hat{k}}} \rho^{N-1} \left(\frac{2\Lambda\rho^2}{\mu} \right)^{\frac{\lambda_{\hat{k}}}{\tilde{\Lambda}}}. \tag{68}$$

In the analytic setting, using standard estimates and inequality (67), one proves that for suitably small ρ the constants $\lambda_{\hat{k}}, \tilde{\Lambda}, \Lambda$ are close to $\alpha/2$, making the dependence on ρ and μ of the left-hand side of inequality (68) completely transparent.

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