

# Robust Distributed Kalman [Filtering](#) with Event-Triggered Communication

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## Abstract

We consider the problem of distributed Kalman filtering for sensor networks in the case there are constraints in data transmission, necessary to deal with the limitation of the channel capacity, and there is model uncertainty. More precisely, we propose two distributed filtering strategies with event-triggered communication where the state estimators are computed according to the least favorable model. The latter belongs to a ball about the nominal model. We also show that both the methods are stable in the sense that the mean-square of the state estimation error is bounded in all the nodes.

*Keywords:* Distributed robust Kalman [Filtering](#), sensor networks (SNs), Event-triggered communication, model uncertainty.

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## 1. Introduction

Sensor networks are ubiquitous in many [fields](#), e.g. monitoring, security, data analysis and so on. The typical scenario is that the sensors collect measurements and from them, it is required to estimate some variables of interest, i.e. the state of a dynamical model. This task is performed in a distributed fashion and it can be accomplished in different ways, see [1, 2, 3, 4, 5, 6, 7]; for instance, each sensor can update its estimate and then share the latter with its neighbors.

Nowadays the devices at each node of the sensor network are usually low-cost and we have to deal with the limitation of the channel capacity.

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Therefore, it is fundamental to guarantee that the rate of data transmission across the network is below a certain threshold. Data communication is typically scheduled by means of event-triggered strategies (which can be data-driven or not) in distributed problems like state estimation [8, 9, 10, 11, 12, 13, 14, 15, 16, 17], observers design [18, 19] and control [20, 21]. In this paper we focus our attention to the distributed state estimation scheme proposed in [22] which guarantees mean-square boundedness of the state estimation error under minimal requirements of network connectivity and collective system observability: each node updates its estimate with the new measurement (if available); then, the latter is compared with the one just propagated in time through the state space model (i.e. the estimate that can be computed also by the neighbors of that node in the case there is no transmission). If the discrepancy between them is large, then the node sends out the updated estimate to its neighbors. Finally, each node performs a fusion of its estimate and the ones regarding its neighbors. The appealing property of this transmission rule is that it allows obtaining a distributed algorithm which enjoys nice stability properties (i.e., mean-square boundedness of the state estimation error in all the nodes) under minimal requirements.

In many situations the actual model is different from the nominal one (i.e. the one used in the estimation algorithm), [23, 24, 25]. Indeed, the fact that the devices placed at each node are low-cost can lead to parameter uncertainty. Moreover, the disturbances are modeled as Gaussian white noise processes, but the actual ones can be very different. In such a scenario, the performance of the estimator based on the nominal model will be poor. One possible way to address this issue is to consider the robust Kalman filter proposed in [26]: the idea is to consider a dynamic minimax game where one player is the estimator which minimizes the state prediction error, while the other one is the hostile player which selects at each time step the least favorable model in a set of plausible models called ambiguity set. The latter is a ball that is formed by placing an upper bound on the discrepancy between the nominal state space model and the models inside. The radius of this ball is called tolerance and defines the magnitude of the uncertainty in the nominal model.

Distributed strategies which are robust to model uncertainty have been already proposed in the literature, see for instance [27, 28, 29, 30, 31, 32, 33, 34], however, to the best of the authors' knowledge, only a few papers consider the case with event-triggered communication, see [35].

In this paper, we shall extend and formalize the ideas in the preliminary conference paper [36]. More precisely, we present two new distributed Kalman filters with event-triggered communication under model uncertainty. The difference between the two approaches is the way with which the tolerance at each node collecting measurements is designed. In the first approach, all nodes have the same tolerance. It is important to observe that there is one ambiguity set for the global model, containing the actual model, and additional local ambiguity sets for each node in the network. The centers of these sets are provided by the nominal global model and the nominal local models, respectively. The local tolerances for the node's ambiguity sets are set to be the same as the tolerance for the global ambiguity set. However, it is worth noting that the local ambiguity set of a node only represents a specific portion of the global model. Consequently, adopting a uniform approach by assigning identical local tolerances across the network, equal to the global tolerance, may not be the most optimal choice. Therefore, in the second approach, the tolerance depends on the local model associated with each individual node. These two approaches represent a robust generalization of the distributed strategy proposed in [22]. We also show that, under reasonable assumptions, both the methods are stable in the sense that the mean-square of the state estimation error is bounded in all the nodes. Finally, we also present a Monte Carlo study showing the effectiveness of the proposed approaches in the case of model uncertainty.

The outline of the paper is as follows. In Section 2 we formulate the distributed state estimation problem characterized by data transmission constraints and model uncertainty. In Section 3 we introduce the robust distributed approach with event-triggered communication and uniform tolerance across the network. In Section 4 we analyze the stability property of the proposed method. In Section 5 we propose the other robust distributed approach with [event-triggered](#) communication. In Section 6 we consider the Monte Carlo study showing the strength of our methods. Finally, in Section 7 we draw the conclusions.

*Notation.* Given a matrix  $A$ ,  $A^\top$  denotes its transpose matrix. Given a symmetric matrix  $Q$ , then  $Q > 0$  ( $Q \geq 0$ ) means that  $Q$  is positive (semi)definite;  $\sigma_{\min}(Q)$  and  $\sigma_{\max}(Q)$  denote the maximum and the minimum [eigenvalues](#) of  $Q$ , respectively. Given an index set  $\mathcal{N}$  and a set of matrices  $\{C^i, i \in \mathcal{N}\}$  having the same number of columns, then  $\text{col}(C^i, i \in \mathcal{N})$  is the matrix obtained by stacking  $C^i$ 's. Given a set of matrices  $\{R^i, i \in \mathcal{N}\}$ , then  $\text{diag}(R^i, i \in \mathcal{N})$  is the block diagonal matrix whose main blocks are

$R^i$ 's. Let  $x$  be a random vector, then  $\mathbb{E}[x]$  and  $\text{Var}[x]$  denote its expectation and variance, respectively. Finally,  $x \sim \mathcal{N}(m, R)$  means that  $x$  is Gaussian distributed with mean  $m$  and covariance matrix  $R$ .

## 2. Problem formulation

Consider a network of nodes described by the digraph  $(\mathcal{N}, \mathcal{A}, \mathcal{S})$  where:  $\mathcal{N} = \{1, \dots, N\}$  is the set of the nodes,  $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$  is the set of edges and  $\mathcal{S} \subseteq \mathcal{N}$  is the subset of sensors node. The latter are the only **ones** that have the capabilities to perform measurements, while the nodes in  $\mathcal{N} \setminus \mathcal{S}$  are used to increase the connectivity of the network. If  $(j, i) \in \mathcal{A}$ , then it means that node  $j$  can transmit data to node  $i$ ; moreover, all the possible self-loops belong to  $\mathcal{A}$ . For each node  $i \in \mathcal{N}$ , the subset  $\mathcal{N}_i := \{j : (j, i) \in \mathcal{A}, j \neq i\} \subseteq \mathcal{N}$  denotes the set of its in-neighbors, i.e. the set composed by the nodes that can send information to node  $i$ .

We attach to this network the following nominal state space model

$$x_{t+1} = Ax_t + Bw_t \quad (1)$$

$$y_t^i = C^i x_t + D^i v_t^i, \quad i \in \mathcal{S} \quad (2)$$

where  $x_t \in \mathbb{R}^n$  is the state,  $y_t^i \in \mathbb{R}^{p_i}$  is the output at the sensor node  $i$ . Furthermore, matrix  $B \in \mathbb{R}^{n \times n}$  and  $D^i \in \mathbb{R}^{p_i \times p_i}$  are full row rank matrices,  $w_t$  and  $v_t^i$  are zero-mean normalized Gaussian white noises. Notice that, the above model is reachable. The initial state  $x_0$  is with mean  $x_{0|-1}$  and with covariance matrix  $V_{0|-1}$ . Finally, we assume that  $w_t$ ,  $v_t^i$ 's and  $x_0$  are independent. Notice that, the model (1)-(2) can be written as

$$x_{t+1} = Ax_t + Bw_t \quad (3)$$

$$y_t = Cx_t + Dv_t \quad (4)$$

where  $C := \text{col}(C^i, i \in \mathcal{S})$ ,  $D := \text{diag}(D^i, i \in \mathcal{S})$ ,  $y_t := \text{col}(y_t^i, i \in \mathcal{S}) \in \mathbb{R}^p$  and  $v_t := \text{col}(v_t^i, i \in \mathcal{S}) \in \mathbb{R}^p$ .

The nominal model (1)-(2) over the time interval  $t = 1 \dots N$  can be equivalently described by the conditional probability densities  $\phi_t(z_t|x_t)$ , where  $z_t = [x_{t+1}^\top y_t^\top]^\top$ ,  $t = 1 \dots N$ , and  $f(x_0)$ . In many situations the actual model is very different from the nominal one. For instance, sensor's parameters can be characterized by uncertainty, or the noises in (1)-(2) can be very different from Gaussian white noise and known only approximately. Taking the robust framework proposed in [26, 37, 38], we assume that the actual model is

described by the conditional probability densities  $\tilde{\phi}_t(z_t|x_t)$ ,  $t = 1 \dots N$ , and  $f(x_0)$ . Moreover, we assume that  $\tilde{\phi}_t$  belongs to the ambiguity set

$$\mathcal{B}_t = \{ \tilde{\phi}_t \text{ s.t. } \tilde{\mathbb{E}}[\log(\tilde{\phi}_t/\phi_t)|Y_{t-1}] \leq b \},$$

$$\tilde{\mathbb{E}}[\log(\tilde{\phi}_t/\phi_t)|Y_{t-1}] := \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+p}} \tilde{\phi}_t(z_t|x_t) \check{f}_t(x_t|Y_{t-1}) \log \left( \frac{\tilde{\phi}_t(z_t|x_t)}{\phi_t(z_t|x_t)} \right) dz_t dx_t,$$

$Y_t := \{y_0 \dots y_t\}$ ,  $\check{f}_t$  is the actual probability density of  $x_t$  given  $Y_{t-1}$  and  $b > 0$  is called tolerance which accounts for model uncertainty. Notice that

$$\begin{aligned} & \tilde{\mathbb{E}}[\log(\tilde{\phi}_t/\phi_t)|Y_{t-1}] \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+p}} \tilde{\phi}_t(z_t|x_t) \check{f}_t(x_t|Y_{t-1}) \log \left( \frac{\tilde{\phi}_t(z_t|x_t) \check{f}_t(x_t|Y_{t-1})}{\phi_t(z_t|x_t) \check{f}_t(x_t|Y_{t-1})} \right) dz_t dx_t \geq 0 \end{aligned} \quad (5)$$

is the Kullback-Leibler divergence between the densities  $\tilde{\phi}_t(z_t|x_t) \check{f}_t(x_t|Y_{t-1})$  and  $\phi_t(z_t|x_t) \check{f}_t(x_t|Y_{t-1})$ . Hence,  $\tilde{\mathbb{E}}[\log(\tilde{\phi}_t/\phi_t)|Y_{t-1}] \geq 0$  and equality holds if and only if  $\tilde{\phi}_t = \phi_t$ . In plain words,  $\mathcal{B}_t$  is a ball, in the Kullback-Leibler topology, about the nominal model with with radius  $b$ . As pointed out in [39], this type of uncertainty arises when the nominal density  $\phi_t$  is estimated from data according to the maximum likelihood principle.

The corresponding robust state estimator is the one solving the following minimax game:

$$x_{t+1|t} = \arg \min_{g_t \in \mathcal{G}} \max_{\tilde{\phi}_t \in \mathcal{B}_t} \tilde{\mathbb{E}}[\|x_{t+1} - g_t(y_t)\|^2 | Y_{t-1}] \quad (6)$$

where  $x_{t+1|t}$  is the estimator of  $x_{t+1}$  given  $Y_t$ ;  $\mathcal{G}$  is the set of estimators having finite second order moments for any  $\tilde{\phi}_t \in \mathcal{B}_t$ ;

$$\begin{aligned} & \tilde{\mathbb{E}}[\|x_{t+1} - g_t(y_t)\|^2 | Y_{t-1}] := \\ & \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+p}} \|x_{t+1} - g_t(y_t)\|^2 \tilde{\phi}_t(z_t|x_t) \check{f}_t(x_t|Y_{t-1}) dz_t dx_t \end{aligned}$$

and it is assumed that  $\check{f}_t(x_t|Y_{t-1}) \sim \mathcal{N}(\hat{x}_{t|t-1}, V_{t|t-1})$ . The basic idea behind this paradigm is that when we are looking for an estimator that minimizes properly the selected loss function, a hostile player called “nature” conspires

to select the worst possible model in the ambiguity set  $\mathcal{B}_t$ . In [26] it has been shown that the (centralized) robust estimator solution to (6) admits a Kalman-like structure. It is not difficult to show that such a filter can be written in the information form as follows. Let  $P_{t|t-1}$  denote the pseudo-nominal covariance matrix of the prediction error at time  $t$ , i.e. the one obtained from  $\check{f}_t(x_t|Y_{t-1})$  and the nominal model (3)-(4); let  $V_{t|t-1}$  denote the least favorable covariance matrix of the prediction error at time  $t$ ; let  $x_{t|t}$  denote the estimator of  $x_t$  given  $Y_t$  and  $P_{t|t}$  denotes the covariance matrices of the corresponding estimation error at time  $t$ . We define the corresponding matrices in the information form as  $\Omega_{t|t-1} = P_{t|t-1}^{-1}$ ,  $\Psi_{t|t-1} = V_{t|t-1}^{-1}$ ,  $\Omega_{t|t} = P_{t|t}^{-1}$  and the information states as

$$q_{t|t-1} = \Psi_{t|t-1} x_{t|t-1}, \quad q_{t|t} = \Omega_{t|t} x_{t|t}.$$

Then, it is not difficult to see that the robust estimator obeys to

$$\begin{aligned} \text{Correction step: } & \begin{cases} \Omega_{t|t} = \Psi_{t|t-1} + C^\top R^{-1} C \\ q_{t|t} = q_{t|t-1} + C^\top R^{-1} y_t, \end{cases} \\ \text{Prediction step: } & \begin{cases} \Omega_{t+1|t} = \\ \quad Q^{-1} - Q^{-1} A (A^\top Q^{-1} A + \Omega_{t|t}) A^\top Q^{-1} \\ \text{Find } \theta_t > 0 \text{ s.t. } \gamma(\Omega_{t+1|t}, \theta_t) = b \\ \Psi_{t+1|t} = \Omega_{t+1|t} - \theta_t \mathbf{I}_n \\ q_{t+1|t} = \Psi_{t+1|t} A \Omega_{t|t}^{-1} q_{t|t}, \end{cases} \end{aligned}$$

where  $R := DD^\top$ ,  $Q := BB^\top$  are positive definite matrices and

$$\gamma(\Omega, \theta) := \frac{1}{2} \left\{ \text{tr}[(\mathbf{I}_n - \theta \Omega^{-1})^{-1} - \mathbf{I}_n] + \log \det(\mathbf{I}_n - \theta \Omega^{-1}) \right\}.$$

The parameter  $\theta_t > 0$  is called risk sensitivity parameter. It is worth noting that given  $\Omega > 0$  and  $b > 0$ , the equation  $\gamma(\Omega, \theta) = b$  always admits a unique solution  $\theta > 0$  such that  $\Omega - \theta \mathbf{I}_n > 0$ . Such property follows from the fact that  $\theta \mapsto \gamma(\Omega, \theta)$  is monotone increasing over  $(0, \sigma_{\min}(\Omega))$  and has for range  $\mathbb{R}_+ := \{x \in \mathbb{R} \text{ s.t. } x > 0\}$ , see [40] for the proof. Furthermore, in the special case where  $b = 0$ , i.e. there is no model uncertainty, we have that  $\gamma(\Omega, \theta) = 0$  implies that  $\theta_t = 0$  and thus the above equations degenerates in the usual Kalman equations in the information form.

In what follows, we face the problem of solving the minimax game in (6) where the minimizer works in a distributed way and under data transmission

constraints. More precisely, each node  $i \in \mathcal{N}$  must estimate the state  $x_t$ , with  $t \in \mathbb{Z}_+ = \{1, \dots, N\}$ , taking into account that: i) the actual model does not coincide with the nominal one (3)-(4); ii) each node  $i$  can selectively **transmit** only the most relevant data without compromising the stability properties.

### 3. Robust event-triggered strategy

Before to introduce our robust distributed estimation paradigm we consider the following quite simple scenario. We assume that  $\mathcal{N} = \mathcal{S}$ , i.e. all the nodes are sensor nodes, and  $\mathcal{A} = \{(j, j), j \in \mathcal{N}\}$ , i.e. the nodes do not communicate. In the presence of model uncertainty, at node  $i \in \mathcal{S}$  we can consider the robust Kalman filter in the information form based on the local model

$$\begin{aligned} x_{t+1} &= Ax_t + Bw_t \\ y_t^i &= C^i x_t + D^i v_t^i; \end{aligned}$$

thus, we obtain the following algorithm:

$$\text{Correction step: } \begin{cases} \Omega_{t|t}^i = \Psi_{t|t-1}^i + (C^i)^\top (R^i)^{-1} C^i \\ q_{t|t}^i = q_{t|t-1}^i + (C^i)^\top (R^i)^{-1} y_t^i \end{cases} \quad (7)$$

$$\text{Prediction step: } \begin{cases} \Omega_{t+1|t}^i = \\ Q^{-1} - Q^{-1} A (A^\top Q^{-1} A + \Omega_{t|t}^i) A^\top Q^{-1} \\ \text{Find } \theta_t^i > 0 \text{ s.t. } \gamma(\Omega_{t+1|t}^i, \theta_t^i) = b \\ \Psi_{t+1|t}^i = \Omega_{t+1|t}^i - \theta_t^i \mathbf{I}_n \\ q_{t+1|t}^i = \Psi_{t+1|t}^i A (\Omega_{t|t}^i)^{-1} q_{t|t}^i \end{cases} \quad (8)$$

where  $R^i := D^i (D^i)^\top$ . It is worth noting that each node has its own risk sensitivity parameter  $\theta_t^i$ .

Next, we consider the scenario in which: i)  $\mathcal{S} \subseteq \mathcal{N}$ ; ii) each node  $i$  can transmit its local estimate  $q_{t|t}^i$  and information matrix  $\Omega_{t|t}^i$  to all its out-neighbors  $\bar{\mathcal{N}}_i = \{j : (i, j) \in \mathcal{A}, i \neq j\}$  if necessary. In plain words, each node  $i$  can decide at any time step whether to transmit or not its data, i.e.  $(q_{t|t}^i, \Omega_{t|t}^i)$ , without compromising the stability properties of the algorithm. More precisely, we require that  $\mathbb{E}[\|x_t - x_{t|t}^i\|^2]$ , where the expectation operator is under the global least favorable model solution to (6), does not diverge for any  $i \in \mathcal{N}$  as  $t$  approaches infinity.

The estimation paradigm that we now present is a robust generalization of the distributed state estimation algorithm with event-triggered communication proposed in [22] and it is composed by four steps described below.

**Correction.** At time  $t$ , the predicted pair  $(q_{t|t-1}^i, \Psi_{t|t-1}^i)$  is available at node  $i \in \mathcal{N}$ . If  $i \in \mathcal{S}$ , i.e. it is a sensor node, then also the measurement  $y_t^i$  is available and thus the correction step coincides with (7). If  $i \notin \mathcal{S}$ , no measurement is available at the node, then we can only propagate the prediction couple. Therefore, the so called information pair is obtained as

$$(q_{t|t}^i, \Omega_{t|t}^i) = \begin{cases} \text{use equations (7),} & \text{if } i \in \mathcal{S} \\ (q_{t|t-1}^i, \Psi_{t|t-1}^i), & \text{if } i \in \mathcal{N} \setminus \mathcal{S}. \end{cases}$$

**Information exchange.** The communication from node  $i \in \mathcal{N}$  to all its out-neighbors is triggered according to the binary variable  $c_t^i$ :

- if  $c_t^i = 1$ , then node  $i$  transmits the information couple  $(q_{t|t}^i, \Omega_{t|t}^i)$  to all its out-neighbors at time  $t$ ;
- if  $c_t^i = 0$ , then node  $i$  does not transmit the information couple to all its out-neighbors at time  $t$ .

It remains to define the binary variable  $c_t^i$ . Let  $n_t^i \in \mathbb{N}$  be the number of time instants elapsed from the most recent transmission of node  $i$ , i.e. the most recently transmitted data is  $(q_{t-n_t^i|t-n_t^i}^i, \Omega_{t-n_t^i|t-n_t^i}^i)$ . Then, all the out-neighbors of node  $i$  propagate  $(q_{t-n_t^i|t-n_t^i}^i, \Omega_{t-n_t^i|t-n_t^i}^i)$  in time through a prediction step which takes into account the fact that the actual model does not coincide with the nominal one (see (14) in the prediction step below). Let  $(\bar{q}_t^i, \bar{\Psi}_t^i)$  denote this propagated pair at time  $t$ . Then, the transmission rule  $c_t^i$  computed at node  $i$  is defined similarly as in [22]:

$$c_t^i = \begin{cases} 0, & \text{if } \|x_{t|t}^i - \bar{x}_t^i\|_{\Omega_{t|t}^i}^2 \leq \alpha \\ & \text{and } \frac{1}{1+\beta}\Omega_{t|t}^i \leq \bar{\Psi}_t^i \leq (1+\delta)\Omega_{t|t}^i \\ 1, & \text{otherwise} \end{cases} \quad (9)$$

where  $\bar{x}_t^i = (\bar{\Psi}_t^i)^{-1}\bar{q}_t^i$  represents the robust state prediction based on the propagation of the most recent transmitted pair  $(q_{t-n_t^i|t-n_t^i}^i, \Omega_{t-n_t^i|t-n_t^i}^i)$ ;  $x_{t|t}^i = (\Omega_{t|t}^i)^{-1}q_{t|t}^i$  is the state estimate at node  $i$ . In plain words, the transmission rule in (9) checks the discrepancy between  $(q_{t|t}^i, \Omega_{t|t}^i)$  and  $(\bar{q}_t^i, \bar{\Psi}_t^i)$ . **If such**

discrepancy is large, then it means that the out-neighbors own a bad prediction corresponding to node  $i$  and thus the transmission from node  $i$  to all its out-neighbors is triggered. It is worth noting that the (9) is intrinsically different from the transmission rule in [22]: here the propagated prediction  $\bar{x}_t^i$  is constructed according to the least favorable statistics (see (14) below). In plain words, our transmission rule takes into account the presence of model uncertainty. The positive scalars  $\alpha$ ,  $\beta$  and  $\delta$  can be tuned by the user in order to reach a desired behavior in terms of transmission rate and performance. More precisely,  $\alpha$  tunes the bound on the discrepancy between  $x_{t|t}^i$  and  $\bar{x}_t^i$ , while  $\beta$  and  $\delta$  tunes the allowed mismatch between the covariance matrices  $\Omega_{t|t}^i$  and  $\bar{\Psi}_t^i$ . In [41] it has been shown that the transmission strategy in (9) guarantees the following upper bound. If we model the propagated and the information pairs as  $N(\bar{q}_t^i, \bar{\Psi}_t^i)$  and  $N(q_{t|t}^i, \Omega_{t|t}^i)$ , respectively, then Condition (9) guarantees that

$$D_{KL}(N(q_{t|t}^i, \Omega_{t|t}^i), N(\bar{q}_t^i, \bar{\Psi}_t^i)) \leq \frac{1}{2}[\alpha + \beta n + n \log(1 + \delta)]$$

where  $n$  is the state dimension and  $D_{KL}$  denotes the Kullback-Leibler divergence. Therefore, the relation above can be used in order to design these parameters. Moreover, from the definition of  $c_t^i$  in (9) we can see that when  $\alpha, \beta, \delta \rightarrow 0$  we have that  $c_t^i = 0$  only if  $\bar{q}_{t|t}^i \rightarrow q_{t|t}^i$  and  $\bar{\Psi}_{t|t}^i \rightarrow \Omega_{t|t}^i$ . Accordingly, the scheme tends to trigger the communication at any time step. Finally,  $c_t^i \rightarrow 0$  as  $\alpha, \beta, \delta \rightarrow \infty$  that is the scheme tends to suppress the communication at any time step as  $\alpha, \beta, \delta \rightarrow \infty$ .

**Information fusion.** In this step, any node merges its information with the ones corresponding to its in-neighbors. Let  $\Pi \in \mathbb{R}^{N \times N}$  denote the consensus matrix whose element in position  $(i, j)$  is defined as:

$$\pi_{i,j} = \begin{cases} (d_i + 1)^{-1}, & \text{if } (j, i) \in \mathcal{A} \\ 0, & \text{otherwise} \end{cases}$$

where  $d_i$  denotes the degree of node  $i$ ; in this way we have that  $\pi_{i,j}$  with  $j \in \mathcal{N}_i$  represents the coefficients of a convex combination. Then, the fusion step is performed through the following convex combination of the pairs:

$$q_{t|t}^{i,F} = \pi_{i,i} q_{t|t}^i + \sum_{j \in \mathcal{N}_i} \pi_{i,j} \left[ c_t^j q_{t|t}^j + (1 - c_t^j) \tilde{q}_t^j \right] \quad (10)$$

$$\Omega_{t|t}^{i,F} = \pi_{i,i} \Omega_{t|t}^i + \sum_{j \in \mathcal{N}_i} \pi_{i,j} \left[ c_t^j \Omega_{t|t}^j + (1 - c_t^j) \tilde{\Omega}_t^j \right] \quad (11)$$

where

$$\tilde{q}_t^j = \frac{1}{1+\delta} \bar{q}_t^j \quad , \quad \tilde{\Omega}_t^j = \frac{1}{1+\delta} \bar{\Psi}_t^j.$$

In view of (10)-(11), we can see that in the fusion step, we consider  $(q_{t|t}^j, \Omega_{t|t}^j)$ , if node  $j$  transmitted its information pair at time  $t$ . If node  $j$  does not transmit, then the aforementioned pair is not available at node  $i$ . To account for this lack, given that at each iteration the nodes can calculate the pair  $(\bar{q}_t^i, \bar{\Psi}_t^i)$ , which is certainly less informative than  $(q_{t|t}^i, \Omega_{t|t}^i)$ , we consider  $(\bar{q}_t^i, \bar{\Psi}_t^i)$  in (10)-(11) shrunk by the factor  $(1+\delta)^{-1}$  in order to decrease its importance in the fusion step, see [22] for more details.

**Prediction.** Once each node  $i \in \mathcal{N}$  has computed the fused information couple  $(q_{t|t}^{i,F}, \Omega_{t|t}^{i,F})$ , the latter is propagated in time with the robust prediction step in (8) where  $(q_{t|t}^i, \Omega_{t|t}^i)$  is now replaced by  $(q_{t|t}^{i,F}, \Omega_{t|t}^{i,F})$ :

$$\begin{aligned} \Omega_{t+1|t}^i &= Q^{-1} - Q^{-1}A(A^\top Q^{-1}A + \Omega_{t|t}^{i,F})A^\top Q^{-1} \\ \text{Find } \theta_t^i &> 0 \text{ s.t. } \gamma(\Omega_{t+1|t}^i, \theta_t^i) = b \\ \Psi_{t+1|t}^i &= \Omega_{t+1|t}^i - \theta_t^i \mathbf{I}_n \\ q_{t+1|t}^i &= \Psi_{t+1|t}^i A (\Omega_{t|t}^{i,F})^{-1} q_{t|t}^{i,F}. \end{aligned} \tag{12}$$

In this step, we compute the propagated pair  $(\bar{q}_{t+1}^i, \bar{\Psi}_{t+1}^i)$ , i.e. the one used in the case node  $i$  does not transmit its information pair. Notice this operation is performed by both node  $i$  and its out-neighbors and it can be summarized as follows. At nodes  $i \cup \mathcal{N}_i$  we have the pair  $(\check{q}_t^i, \check{\Omega}_t^i)$  defined as

$$\begin{cases} \check{q}_t^i &= c_t^i q_{t|t}^i + (1 - c_t^i) \bar{q}_t^i \\ \check{\Omega}_t^i &= c_t^i \Omega_{t|t}^i + (1 - c_t^i) \bar{\Psi}_t^i; \end{cases} \tag{13}$$

then, it is propagated in time with the robust prediction step in (8) where  $(q_{t|t}^i, \Omega_{t|t}^i)$  is now replaced by  $(\check{q}_t^i, \check{\Omega}_t^i)$ :

$$\begin{aligned} \bar{\Omega}_{t+1}^i &= Q^{-1} - Q^{-1}A(A^\top Q^{-1}A + \check{\Omega}_t^i)A^\top Q^{-1} \\ \text{Find } \bar{\theta}_t^i &> 0 \text{ s.t. } \gamma(\bar{\Omega}_{t+1}^i, \bar{\theta}_t^i) = b \\ \bar{\Psi}_{t+1}^i &= \bar{\Omega}_{t+1}^i - \bar{\theta}_t^i \mathbf{I}_n \\ \bar{q}_{t+1}^i &= \bar{\Psi}_{t+1}^i A (\check{\Omega}_t^i)^{-1} \check{q}_t^i. \end{aligned} \tag{14}$$

The procedure is summarized in Algorithm 1. It is worth noting that each node  $i$  is characterized by two risk sensitivity parameters, i.e.  $\theta_t^i$  and

$\bar{\theta}_t^i$ . In the case that  $b = 0$ , i.e. there is no model uncertainty, in Algorithm 1 we have:  $\theta_t^i = 0$ ,  $\bar{\theta}_t^i = 0$  and thus  $\Psi_t^i = \Omega_t^i$ ,  $\bar{\Psi}_t^i = \bar{\Omega}_t^i$ , i.e. we recover the distributed Kalman algorithm with event-triggered communication proposed in [22].

**Comparison with existing algorithms.** Next, we analyze the differences between the proposed algorithm and the existing distributed algorithms with event-triggered communication. We start with the algorithm proposed in [35]. While our model uncertainty arises when the nominal model is inferred according to the maximum likelihood principle, the uncertainty considered in [35] acts only on the parameters of matrix  $A$  in (3); in particular, matrices  $C^i$  and  $D^i$ , with  $i \in \mathcal{S}$ , are not affected by model uncertainty. On the contrary, our ambiguity set also contains perturbed versions of the nominal model in which  $C^i$  and  $D^i$  are replaced by perturbed versions. Regarding the robust algorithm with event-triggered communication proposed in [42], the difference between it and our algorithm is the same as in the previous case.

**Computational complexity.** Next, we analyze the asymptotic computational complexity, understood as the number of floating point operations (flops) by using big O notation, of Algorithm 1 at any node  $i \in \mathcal{N}$ . The complexity of the correction step is  $O(np_i^2 + n^2p_i)$  if  $i \in \mathcal{S}$ , while there is no computational complexity if  $i \in \mathcal{N} \setminus \mathcal{S}$ . Then, there is no computational complexity in the information exchange step. The complexity of the information fusion step is  $O((|\mathcal{N}_i| + 1)n^2)$ . Finally, in the prediction step we have to find  $\theta \in (0, \sigma_{\min}(\Omega))$  such that  $\gamma(\Omega, \theta) = b$  for  $\Omega = \Omega_{t+1|t}^i$  and  $\Omega = \bar{\Omega}_{t+1}^i$ . This computation is accomplished by a bisection method, see Algorithm 2 in [43]. Since at each step we spend constant time to reduce the problem to an instance half its size, the complexity is  $O(n^3 \log_2(\sigma_{\min}(\Omega)/\varepsilon))$  where  $\varepsilon > 0$  is the selected accuracy, i.e. the solution found satisfies the condition  $|\gamma(\Omega, \theta) - b| \leq \varepsilon$ . Accordingly, the complexity of the prediction step is  $O(n^3 + n^3 \log_2(\max\{\sigma_{\min}(\Omega_{t+1|t}^i), \sigma_{\min}(\bar{\Omega}_{t+1}^i)\}/\varepsilon))$ .

#### 4. Stability analysis

Recall that the nominal (global) model (3)-(4) is different from the actual one. In this section, we analyze the stability properties of Algorithm 1 under the least favorable model which is given by the maximizer of the minimax problem in (6), that is, the centralized problem. It is worth noting that

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**Algorithm 1** RDKF with event-triggered communication

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**Initialization:** Set  $(q_{0|-1}^i, \Psi_{0|-1}^i)$  for any  $i \in \mathcal{N}$

For each  $t = 0, 1, \dots$

For each node  $i \in \mathcal{N}$

**Correction:**

$$(q_{t|t}^i, \Omega_{t|t}^i) = \begin{cases} \text{use (7),} & \text{if } i \in \mathcal{S} \\ (q_{t|t-1}^i, \Psi_{t|t-1}^i), & \text{if } i \in \mathcal{N} \setminus \mathcal{S} \end{cases}$$

**Information exchange:**

- if  $t = 0$  set  $c_t^i = 1$ , otherwise determine  $c_t^i$  according to (9)
- if  $c_t^i = 1$  transmit  $(q_{t|t}^i, \Omega_{t|t}^i)$  to the out-neighbors
- receive  $(q_{t|t}^j, \Omega_{t|t}^j)$  from all the in-neighbors  $j \in \mathcal{N}_i$  for which  $c_t^j = 1$

**Information fusion:**

$$\tilde{q}_t^j = \frac{1}{1 + \delta} \bar{q}_t^j, \quad \tilde{\Omega}_t^j = \frac{1}{1 + \delta} \bar{\Psi}_t^j \text{ with } j \in \mathcal{N}_i$$

$$q_{t|t}^{i,F} = \pi_{i,i} q_{t|t}^i + \sum_{j \in \mathcal{N}_i} \pi_{i,j} \left[ c_t^j q_{t|t}^j + (1 - c_t^j) \tilde{q}_t^j \right]$$

$$\Omega_{t|t}^{i,F} = \pi_{i,i} \Omega_{t|t}^i + \sum_{j \in \mathcal{N}_i} \pi_{i,j} \left[ c_t^j \Omega_{t|t}^j + (1 - c_t^j) \tilde{\Omega}_t^j \right]$$

**Prediction step:**

- Compute  $(q_{t+1|t}^i, \Psi_{t+1|t}^i)$  using (12)
  - Compute  $(\bar{q}_{t+1|t}^i, \bar{\Psi}_{t+1}^i)$  using (13) and (14)
-

the stability result in [22] cannot be applied in our setup for the following reasons:

- The least favorable model is characterized by noise processes which are colored, instead in [22] the nominal model is characterized by noise processes which are white and independent each other;
- Unlike [22], now the bounds on matrices  $\Omega_{t|t}^i$  and  $\Omega_{t|t}^{i,F}$  depends on the tolerance  $b$  characterizing the ambiguity sets.
- The stability analysis in [22] only involves  $\Omega_{t|t-1}^i$ , but in our setup we also have its least favorable counterpart  $\Psi_{t|t-1}^i$  whose influence on the stability is not clear yet.

In what follows we will adapt the stability result in [22] to our setup. In doing that, we need the following assumptions:

- A1.** The tolerance  $b$  defining the ambiguity set in (6) is taken sufficiently small;
- A2.** The transition matrix  $A$  is invertible;
- A3.** The system is collectively observable, i.e., the pair  $(A, C)$  is observable;
- A4.** The network is strongly connected, that is, there exists a directed path between any pair  $i, j \in \mathcal{N}$ .

It is worth noting that assumptions **A3-A4** are the same made in [22], in particular Assumption **A2** automatically holds in sampled-data systems where matrix  $A$  is obtained by discretizing the corresponding continuous-time matrix. Finally, Assumption **A1** is necessary to guarantee that the noise processes characterizing the global least favorable model have uniformly bounded variance as  $t \rightarrow \infty$ , see the next proposition.

**Proposition 1.** *Under assumptions **A1** and **A3**, the least favorable model solution to (6) takes the form*

$$x_{t+1} = Ax_t + B\tilde{w}_t \tag{15}$$

$$y_t = Cx_t + D\tilde{v}_t \tag{16}$$

where  $\tilde{w}_t$  and  $\tilde{v}_t := \text{col}(\tilde{v}_t^i, i \in \mathcal{S})$  are zero-mean Gaussian colored noises such that

$$\underline{\rho}_t I_{n+N} \leq \text{Var} \left( \begin{bmatrix} \tilde{w}_t \\ \tilde{v}_{t+1} \end{bmatrix} \right) \leq \bar{\rho}_t I_{n+N}$$

with  $\bar{\rho}_t$  and  $\underline{\rho}_t$  such that  $\underline{\rho}_t \rightarrow \underline{\rho}$  and  $\bar{\rho}_t \rightarrow \bar{\rho}$  as  $t \rightarrow \infty$ , with  $\bar{\rho} \geq \underline{\rho} \geq 0$ . Moreover,  $\tilde{w}_t$  and  $\tilde{v}_t$  are correlated.

PROOF. First notice that the least favorable model is the solution to the “global” problem (6) and thus we will prove the statement using results established in the non-distributed setup. The least favorable model (15)-(16) has been characterized in [26, Section V] where

$$\tilde{w}_t := H_{1,t}\epsilon_t + L_{1,t}\epsilon_t, \quad \tilde{v}_t := H_{2,t}\epsilon_t + L_{2,t}\epsilon_t;$$

$\epsilon_t$  is zero-mean normalized Gaussian white noise,

$$\epsilon_{t+1} = [A + BH_{1,t} - G_t(C + DH_{2,t})]\epsilon_t + \Gamma_t\epsilon_t;$$

$\Gamma_t := \mathbb{E}[\epsilon_{t+1}\epsilon_t^\top] = BL_{1,t} - G_tDL_{2,t}$ ; the characterization of the matrices  $H_{1,t}$ ,  $H_{2,t}$ ,  $L_{1,t}$ ,  $L_{2,t}$  and  $G_t$  can be done using reasonings similar to the ones in [26]. Let

$$H_t := \begin{bmatrix} H_{1,t} & 0 \\ 0 & H_{2,t+1} \end{bmatrix}, \quad E_t := \begin{bmatrix} \mathbb{E}[\epsilon_t\epsilon_t^\top] & \mathbb{E}[\epsilon_t\epsilon_{t+1}^\top]^\top \\ \mathbb{E}[\epsilon_{t+1}\epsilon_t^\top] & \mathbb{E}[\epsilon_{t+1}\epsilon_{t+1}^\top] \end{bmatrix},$$

$$L_t := \begin{bmatrix} L_{1,t} & 0 \\ 0 & L_{2,t+1} \end{bmatrix}, \quad M_t := \begin{bmatrix} 0 & 0 \\ \Gamma_t & 0 \end{bmatrix}.$$

Then,

$$\begin{aligned} \text{Var} \left( \begin{bmatrix} \tilde{w}_t \\ \tilde{v}_{t+1} \end{bmatrix} \right) &= \text{Var} \left( H_t \begin{bmatrix} \epsilon_t \\ \epsilon_{t+1} \end{bmatrix} + L_t \begin{bmatrix} \epsilon_t \\ \epsilon_{t+1} \end{bmatrix} \right) \\ &= H_t E_t H_t^\top + L_t L_t^\top + H_t \begin{bmatrix} \mathbb{E}[\epsilon_t\epsilon_t^\top] & \mathbb{E}[\epsilon_t\epsilon_{t+1}^\top]^\top \\ \mathbb{E}[\epsilon_{t+1}\epsilon_t^\top] & \mathbb{E}[\epsilon_{t+1}\epsilon_{t+1}^\top] \end{bmatrix} L_t^\top \\ &\quad + L_t \begin{bmatrix} \mathbb{E}[\epsilon_t\epsilon_t^\top] & \mathbb{E}[\epsilon_t\epsilon_{t+1}^\top]^\top \\ \mathbb{E}[\epsilon_{t+1}\epsilon_t^\top] & \mathbb{E}[\epsilon_{t+1}\epsilon_{t+1}^\top] \end{bmatrix} H_t^\top \\ &= H_t E_t H_t^\top + L_t L_t^\top + H_t M_t L_t^\top + L_t M_t^\top H_t^\top \end{aligned}$$

and thus

$$\begin{aligned}\underline{\rho}_t &= \sigma_{\min}(H_t E_t H_t^\top + L_t L_t^\top + H_t M_t L_t^\top + L_t M_t^\top H_t^\top), \\ \bar{\rho}_t &= \sigma_{\max}(H_t E_t H_t^\top + L_t L_t^\top + H_t M_t L_t^\top + L_t M_t^\top H_t^\top).\end{aligned}$$

In [44] it was shown that, under Assumptions **A1** and **A3**,  $H_t \rightarrow H$ ,  $L_t \rightarrow L$  and  $G_t \rightarrow G$  as  $t \rightarrow \infty$ . Moreover,  $A + BH_{1,t} - G_t(C + DH_{2,t}) \rightarrow \tilde{A}$  and  $\tilde{A}$  is Schur stable. Accordingly  $E_t \rightarrow E$ , with  $E \geq 0$ , as  $t \rightarrow \infty$ . We conclude that

$$\begin{aligned}\underline{\rho} &= \sigma_{\min}(HEH^\top + LL^\top + HML^\top + LM^\top H^\top), \\ \bar{\rho} &= \sigma_{\max}(HEH^\top + LL^\top + HML^\top + LM^\top H^\top).\end{aligned}$$

□

Let  $e_t^i = x_t - x_{t|t}^i$  denote the estimation error at node  $i$  and  $e_t = \text{col}(e_t^i, i \in \mathcal{N})$  the collective estimation error. In order to prove the stability of the estimation error, we consider the Lyapunov function

$$\mathcal{V}_t(e_t) = \sum_{i \in \mathcal{N}} p_i \|e_t^i\|_{\Omega_{t|t}^i}^2 \quad (17)$$

where  $p_i$ 's, strictly positive, are the components of a vector  $p$  satisfying the condition  $p^\top = p^\top \Pi$ . Notice that, the existence of such a vector is guaranteed by the Perron-Frobenius theorem since, by Assumption **A4**,  $\Pi$  is a primitive matrix. The next proposition shows that  $\mathcal{V}_t(e_t)$  is a well-defined Lyapunov function because  $\Omega_{t|t}^i$ 's are uniformly bounded.

**Proposition 2.** *Assume that **A2-A4** hold. Consider the sequence  $\Omega_{t|t}^i$ ,  $i \in \mathcal{N}$ , generated by Algorithm 1 with  $\Psi_{0|-1}^i > 0$ . Then, there exist three positive constants  $\underline{\omega}$ ,  $\bar{\omega}$  and  $\underline{\omega}_F$  such that  $\underline{\omega}I \leq \Omega_{t|t}^i \leq \bar{\omega}I$  and  $\underline{\omega}I \leq \Omega_{t|t}^{i,F} \leq \bar{\omega}_F I$ .*

PROOF. We start by showing the upper bounds. By (8) we have that  $\Omega_{t+1|t}^i \leq Q^{-1}$  and thus  $\Psi_{t+1|t}^i \leq \Omega_{t+1|t}^i \leq Q^{-1}$  which is a uniform upper bound. By the correction step in Algorithm 1 we have that  $\Omega_{t|t}^i \leq Q^{-1} + (C^i)^\top (R^i)^{-1} C^i \leq \bar{\omega}I$  where

$$\bar{\omega} := \max_{i \in \mathcal{N}} \sigma_{\max}(Q^{-1} + (C^i)^\top (R^i)^{-1} C^i).$$

It is also worth noting that

$$\Omega_{t|t}^{i,F} \leq \bar{\omega}I \quad (18)$$

where we have exploited the definition in (11) and the facts that  $\Omega_{t|t}^i \leq \bar{\omega}I$  and  $\tilde{\Omega}_t^i \leq \bar{\omega}I$ , by the transmission rule (9). We now prove the uniform lower bound. First, notice that if  $\Psi_{0|-1}^i > 0 \forall i \in \mathcal{N}$  then  $\Omega_{t|t}^i > 0 \forall i \in \mathcal{N}$ . Indeed, by induction we have that if  $\Psi_{t|t-1}^i > 0$ , by the correction step in Algorithm 1 it follows that  $\Omega_{t|t}^i \geq \Psi_{t|t-1}^i > 0 \forall i \in \mathcal{N}$ ; thus,  $\Omega_{t|t}^{i,F} > 0$  because it is a convex combination of positive definite matrices; then, it follows  $\Omega_{t+1|t}^i = (A(\Omega_{t|t}^{i,F})^{-1}A^\top + Q)^{-1} > 0$  by (12); finally, the fact that  $\Psi_{t+1|t}^i > 0$  is guaranteed by the choice of  $\theta_t^i$  as the solution of  $\gamma(\Omega_{t+1|t}^i, \theta_t^i) = b$ , see [26, 45]. Let  $\mathbf{1}_{\mathcal{S}}(i)$  denote the indicator function taking value 1 if  $i \in \mathcal{S}$  and 0 otherwise. By Lemma 2 in Appendix, there exists a constant  $\mu > 0$  such that

$$\begin{aligned} \Omega_{t|t}^i &= \Psi_{t|t-1}^i + \mathbf{1}_{\mathcal{S}}(i)(C^i)^\top (R^i)^{-1}C^i \\ &= \Omega_{t|t-1}^i - \theta_{t-1}^i I + \mathbf{1}_{\mathcal{S}}(i)(C^i)^\top (R^i)^{-1}C^i \\ &\geq \mu \Omega_{t|t-1}^i + \mathbf{1}_{\mathcal{S}}(i)(C^i)^\top (R^i)^{-1}C^i. \end{aligned}$$

Moreover, in view of (18), by [4, Lemma 1 -fact (ii)] there exists a constant  $\nu > 0$  such that

$$\Omega_{t|t}^i \geq \mu \nu A^{-\top} \Omega_{t-1|t-1}^{i,F} A^{-1} + \mathbf{1}_{\mathcal{S}}(i)(C^i)^\top (R^i)^{-1}C^i. \quad (19)$$

In view of (9) and (11), it is not difficult to see that

$$\Omega_{t|t}^{i,F} \geq \frac{1}{(1+\beta)(1+\delta)} \sum_{j \in \mathcal{N}} \pi_{i,j} \Omega_{t|t}^j. \quad (20)$$

Taking into account (19), we obtain

$$\Omega_{t|t}^i \geq \nu A^{-\top} \sum_{j \in \mathcal{N}} \pi_{i,j} \Omega_{t|t}^j A^{-1} + \phi_i$$

where  $\nu := \mu \nu ((1+\beta)(1+\delta))^{-1} > 0$  and  $\phi_i := \mathbf{1}_{\mathcal{S}}(i)(C^i)^\top (R^i)^{-1}C^i$ . The last inequality is similar to the one obtained in the proof of Lemma 1 in [22]. Accordingly, using the same reasonings it is possible to conclude that there exists a constant  $\underline{\omega} > 0$  such that  $\Omega_{t|t}^i \geq \underline{\omega}I$ . It is also worth noting we have, by (20), that  $\Omega_{t|t}^{i,F} \geq \underline{\omega}_F I$  with  $\underline{\omega}_F := \underline{\omega}((1+\beta)(1+\delta))^{-1}$ .  $\square$

It is worth noting that, unlike [22], the constants  $\underline{\omega}$  and  $\underline{\omega}_F$  depend on the tolerance  $b$  through the constant  $\mu$ .

**Proposition 3.** *Assume that **A2-A4** hold. Let  $(q_{t|t}^i, \Omega_{t|t}^i)$  be the sequence generated according to Algorithm 1 with  $\Psi_{0|-1}^i > 0 \forall i \in \mathcal{N}$ . Then, under the least favorable model in (15)-(16), we have*

$$\|e_{t+1}^i\|_{\Omega_{t+1|t+1}^i}^2 \leq \gamma^2 \left( \pi_{i,i} \|e_t^i + \xi_t^i\|_{\Omega_{t|t}^i}^2 + \sum_{j \in \mathcal{N}_i} \pi_{i,j} \|e_t^j + \xi_t^j + \eta_t^j\|_{\Omega_{t|t}^j}^2 \right), \quad \forall i \in \mathcal{N}$$

where  $0 < \gamma < 1$  is a constant and

$$\begin{aligned} \xi_t^i &:= A^{-1}[B\tilde{w}_t - \mathbf{1}_S(i)(\Psi_{t+1|t}^i)^{-1}(C^i)^\top(R^i)^{-1}D^i\tilde{v}_{t+1}^i], \\ \eta_t^j &:= (1 - c_t^j)(\hat{x}_{t|t}^j - \bar{x}_t^j), \quad \bar{x}_t^j := (\bar{\Psi}_t^j)^{-1}\bar{q}_t^j. \end{aligned}$$

PROOF. Recall that  $q_{t+1|t}^i = \Psi_{t+1|t}^i x_{t+1|t}^i$  and  $q_{t|t}^i = \Omega_{t|t}^i x_{t|t}^i$ . Then, it is not difficult to see that

$$\begin{aligned} x_{t+1|t+1}^i &= (\Omega_{t+1|t+1}^i)^{-1} \times \\ &\quad (\Psi_{t+1|t}^i x_{t+1|t}^i + \mathbf{1}_S(i)(C^i)^\top(R^i)^{-1}C^i x_{t+1} + \mathbf{1}_S(i)(C^i)^\top(R^i)^{-1}D^i\tilde{v}_{t+1}^i) \\ x_{t+1} &= (\Omega_{t+1|t+1}^i)^{-1} (\Psi_{t+1|t}^i x_{t+1} \mathbf{1}_S(i)(C^i)^\top(R^i)^{-1}C^i x_{t+1}) \end{aligned}$$

which implies

$$e_{t+1}^i = x_{t+1} - x_{t+1|t+1}^i = (\Omega_{t+1|t+1}^i)^{-1} \Psi_{t+1|t}^i (x_{t+1} - x_{t+1|t}^i + \bar{v}_{t+1}^i)$$

where  $\bar{v}_{t+1}^i = -\mathbf{1}_S(i)(\Psi_{t+1|t}^i)^{-1}(C^i)^\top(R^i)^{-1}D^i\tilde{v}_{t+1}^i$ . Since  $\Psi_{t+1|t}^i \leq \Omega_{t+1|t+1}^i$ , we have

$$\begin{aligned} \|e_{t+1}^i\|_{\Omega_{t+1|t+1}^i}^2 &= \|x_{t+1} - x_{t+1|t}^i + \bar{v}_{t+1}^i\|_{\Psi_{t+1|t}^i (\Omega_{t+1|t+1}^i)^{-1} \Psi_{t+1|t}^i}^2 \\ &\leq \|x_{t+1} - x_{t+1|t}^i + \bar{v}_{t+1}^i\|_{\Psi_{t+1|t}^i}^2. \end{aligned} \tag{21}$$

By Lemma 1 - fact (iii) in [4] there exists a constant  $0 < \gamma \leq 1$  such that

$$\Psi_{t+1|t}^i = \Omega_{t+1|t}^i - \theta_t^i \leq \Omega_{t+1|t}^i \leq \gamma^2 A^{-T} \Omega_{t|t}^{i,F} A^{-1}$$

where we exploited the fact that  $\Omega_{t|t}^{i,F} \geq \underline{\omega}_F I$  by Proposition 2. Taking into account (21), it follows that

$$\begin{aligned} \|e_{t+1}^i\|_{\Omega_{t+1|t+1}^i}^2 &\leq \gamma^2 \|x_{t+1} - x_{t+1|t}^i + \bar{v}_{t+1}^i\|_{A^{-\top} \Omega_{t|t}^{i,F} A^{-1}}^2 \\ &= \gamma^2 \|A(x_t - x_{t|t}^{i,F}) + B\tilde{w}_t + \bar{v}_{t+1}^i\|_{A^{-\top} \Omega_{t|t}^{i,F} A^{-1}}^2 \\ &= \gamma^2 \|x_t - x_{t|t}^{i,F} + \xi_t^i\|_{\Omega_{t|t}^{i,F}}^2 \end{aligned}$$

where we exploited the fact that  $x_{t|t}^{i,F} = (\Omega_{t|t}^{i,F})^{-1} q_{t|t}^{i,F}$  and  $x_{t+1|t}^i = Ax_{t|t}^{i,F}$ . Then, using the same reasonings in the second part of the proof of Lemma 2 in [22] it is not difficult to see that

$$\begin{aligned} \|e_{t+1}^i\|_{\Omega_{t+1|t+1}^i}^2 &\leq \gamma^2 \left( \pi_{i,i} \|e_t^i + \xi_t^i\|_{\Omega_{t|t}^i}^2 \right. \\ &\quad \left. + \sum_{j \in \mathcal{N}_i} \pi_{i,j} \|e_t^j + \xi_t^j + \eta_t^j\|_{\Omega_{t|t}^j}^2 \right) \end{aligned}$$

where

$$\begin{aligned} \eta_t^j &:= x_{t|t}^j - \check{x}_{t|t}^j, \quad \check{x}_{t|t}^j = (\check{\Omega}_{t|t}^j)^{-1} \check{q}_{t|t}^j \\ \check{q}_{t|t}^j &:= c_t^j q_{t|t}^j + (1 - c_t^j) \tilde{q}_t^j, \quad \check{\Omega}_{t|t}^j := c_t^j \Omega_{t|t}^j + (1 - c_t^j) \tilde{\Omega}_t^j \end{aligned}$$

which concludes the proof.  $\square$

In view of Proposition 1 and Proposition 2, it follows that  $\xi_t^i$ , with  $i \in \mathcal{N}$ , is uniformly bounded in mean-square as  $t \rightarrow \infty$  under the assumptions **A1-A4**. Indeed, if  $i \notin \mathcal{S}$ , then

$$\begin{aligned} \xi_t^i &= A^{-1} B \tilde{w}_t = A^{-1} \begin{bmatrix} B & 0 \end{bmatrix} \begin{bmatrix} \tilde{w}_t \\ \tilde{v}_{t+1} \end{bmatrix} \\ \text{Var}(\xi_t^i) &= A^{-1} \begin{bmatrix} B & 0 \end{bmatrix} \text{Var} \left( \begin{bmatrix} \tilde{w}_t \\ \tilde{v}_{t+1} \end{bmatrix} \right) \begin{bmatrix} B^\top \\ 0 \end{bmatrix} A^{-\top} \\ &\leq \bar{\rho}_t^\top A^{-1} B B^\top A^{-\top} \leq \bar{\rho}_t \sigma_{\max}(A^{-1} B B^\top A^{-\top}) I; \end{aligned}$$

if  $i \in \mathcal{S}$ , then

$$\xi_t^i = A^{-1} \begin{bmatrix} B & H_i \end{bmatrix} \begin{bmatrix} \tilde{w}_t \\ \tilde{v}_{t+1} \end{bmatrix}$$

where

$$J_{i,t} := [0 \ \dots \ \underbrace{-(\Psi_{t+1|t}^i)^{-1}(C^i)^\top(R^i)^{-1}D^i}_{i+1\text{-th block}} \ \dots \ 0]$$

and thus

$$\begin{aligned} \text{Var}(\xi_t^i) &= A^{-1} \begin{bmatrix} B & J_{i,t} \end{bmatrix} \text{Var} \left( \begin{bmatrix} \tilde{w}_t \\ \tilde{v}_{t+1} \end{bmatrix} \right) \begin{bmatrix} B^\top \\ J_{i,t}^\top \end{bmatrix} A^{-\top} \\ &\leq \bar{\rho}_t A^{-1} (BB^\top + J_{i,t} J_{i,t}^\top) A^{-\top} \leq \bar{\rho}_t A^{-1} (BB^\top + v_i (\Psi_{t+1|t}^i)^{-2}) A^{-\top} \end{aligned}$$

where  $v_i = \sigma_{\max}((C^i)^\top(R^i)^{-1}D^i(D^i)^\top(R^i)^{-1}C^i)$ . By Lemma 2 there exists a constant  $\mu > 0$  such that

$$\Psi_{t+1|t}^i \geq \mu \Omega_{t+1|t}^i = \mu (A(\Omega_{t|t}^{i,F})^{-1}A^\top + Q)^{-1} \geq \mu (\underline{\omega}^{-1}AA^\top + Q)^{-1}$$

where we also exploited  $\Omega_{t|t}^{i,F} \geq \underline{\omega}I$  by Proposition 2. Thus,

$$\text{Var}(\xi_t^i) \leq \bar{\rho}_t \sigma_{\max}(A^{-1}(BB^\top + v_i \mu^2 (\underline{\omega}^{-1}AA^\top + Q)^{-2})A^{-\top})I.$$

We conclude that  $\text{Var}[\xi_t^i] \leq \rho_{\xi,t}^2$  where

$$\rho_{\xi,t}^2 := \bar{\rho}_t \max_{i \in \mathcal{N}} \sigma_{\max}(A^{-1}(BB^\top + v_i \mu^2 (\underline{\omega}^{-1}AA^\top + Q)^{-2})A^{-\top})$$

and  $\rho_{\xi,t}^2 \rightarrow \rho_\xi^2$ , as  $t \rightarrow \infty$ , with

$$\rho_\xi^2 := \bar{\rho} \max_{i \in \mathcal{N}} \sigma_{\max}(A^{-1}(BB^\top + v_i \mu^2 (\underline{\omega}^{-1}AA^\top + Q)^{-2})A^{-\top})$$

where we have exploited Proposition 1. Therefore,

$$\lim_{t \rightarrow \infty} \text{Var}[\xi_t^i] \leq \rho_\xi^2. \quad (22)$$

Finally, it is worth noting that  $\|\eta_t^j\|_{\Omega_{t|t}^j}^2 \leq \alpha$ , indeed if  $c_t^j = 0$ , i.e. there is not transmission from node  $j$ , then  $\|\eta_t^j\|_{\Omega_{t|t}^j}^2 = \|x_{t|t}^j - \bar{x}_t^j\|_{\Omega_{t|t}^j}^2 \leq \alpha$  by the transmission rule in (9); otherwise, if  $c_t^j = 1$  then  $\|\eta_t^j\|_{\Omega_{t|t}^j}^2 = 0 \leq \alpha$ .

**Theorem 1.** Assume that the hypotheses **A1-A4** hold. Then, the estimation error  $e_t^i$ , with  $i \in \mathcal{N}$ , is uniformly bounded in mean-square as  $t \rightarrow \infty$ , i.e.

$$\lim_{t \rightarrow \infty} \mathbb{E}[\|e_t^i\|^2] \leq \left( \frac{\sqrt{2}\gamma \sum_{i \in \mathcal{N}} \sqrt{p_i} (\sqrt{\bar{\omega}} \rho_\xi + \sqrt{\alpha})}{1 - \gamma \sqrt{\bar{\omega}} \min_{i \in \mathcal{N}} \sqrt{p_i}} \right)^2$$

where  $\underline{\omega}, \bar{\omega}, \alpha, \gamma$  are the constants given in Proposition 2 and Proposition 3.

PROOF. Using the same reasonings in the first part of the proof of Theorem 1 in [22], it is not difficult to prove that

$$\sqrt{\mathbb{E}[\mathcal{V}_{t+1}(e_{t+1})]} \leq \gamma \left( \sqrt{\mathbb{E}[\mathcal{V}_t(e_t)]} + \sqrt{\mathbb{E}[\mathcal{V}_t(\omega_t)]} \right) \quad (23)$$

$$\sqrt{\mathcal{V}_t(\omega_t)} \leq \sum_{i \in \mathcal{N}} \sqrt{p_i} \sqrt{\mathbb{E}[\|\omega_t^i\|_{\Omega_{t|t}^i}^2]} \quad (24)$$

where  $0 < \gamma < 1$  is the constant given by Proposition 3,  $\omega_t := \text{col}(\omega_t^i; i \in \mathcal{N})$ ,

$$\omega_t^i := \begin{cases} \bar{\omega}_t^i, & \text{if } e_t^i = 0 \\ \|\bar{\omega}_t^i\|_{\Omega_{t|t}^i}^2 / \|e_t^i\|_{\Omega_{t|t}^i}^2 e_t^i, & \text{otherwise} \end{cases}$$

and  $\bar{\omega}_t^i$  is the vector in the set  $\{\xi_t^i\} \cup \{\xi_t^j + \eta_t^i, j \in \bar{\mathcal{N}}_i\}$  maximizing the weighted norm  $\|\cdot\|_{\Omega_{t|t}^i}$ . Notice that

$$0 \leq \mathbb{E}[\|\xi_t^j - \eta_t^i\|_{\Omega_{t|t}^i}^2] = \mathbb{E}[\|\xi_t^j\|_{\Omega_{t|t}^i}^2 + \|\eta_t^i\|_{\Omega_{t|t}^i}^2 - 2(\xi_t^j)^\top \Omega_{t|t}^i \eta_t^i]$$

and thus

$$\mathbb{E}[2(\xi_t^j)^\top \Omega_{t|t}^i \eta_t^i] \leq \mathbb{E}[\|\xi_t^j\|_{\Omega_{t|t}^i}^2 + \|\eta_t^i\|_{\Omega_{t|t}^i}^2].$$

It is worth noting that the term  $\mathbb{E}[2(\xi_t^j)^\top \Omega_{t|t}^i \eta_t^i]$ , unlike [22], is different from zero in general because  $\tilde{w}_t$  and  $\tilde{v}_t$  are colored noises. Moreover,

$$\begin{aligned} \mathbb{E}[\|\omega_t^i\|_{\Omega_{t|t}^i}^2] &= \mathbb{E}[\|\bar{\omega}_t^i\|_{\Omega_{t|t}^i}^2] \\ &\leq \max\{\mathbb{E}[\|\xi_t^i\|_{\Omega_{t|t}^i}^2], \max_{j \in \mathcal{N}} \mathbb{E}[\|\xi_t^j\|_{\Omega_{t|t}^i}^2 + \|\eta_t^i\|_{\Omega_{t|t}^i}^2 + 2(\xi_t^j)^\top \Omega_{t|t}^i \eta_t^i]\} \\ &\leq \max\{\mathbb{E}[\|\xi_t^i\|_{\Omega_{t|t}^i}^2], \max_{j \in \mathcal{N}} 2 \mathbb{E}[\|\xi_t^j\|_{\Omega_{t|t}^i}^2 + \|\eta_t^i\|_{\Omega_{t|t}^i}^2]\} \\ &\leq 2 \max_{j \in \mathcal{N}} \mathbb{E}[\|\xi_t^j\|_{\Omega_{t|t}^i}^2 + \|\eta_t^i\|_{\Omega_{t|t}^i}^2]. \end{aligned}$$

Taking into account (24), we obtain

$$\begin{aligned}
\sqrt{\mathcal{V}_t(\omega_t)} &\leq \sqrt{2} \sum_{i \in \mathcal{N}} \sqrt{p_i} \max_{j \in \mathcal{N}} \sqrt{\mathbb{E}[\|\xi_t^j\|_{\Omega_{t|t}^i}^2 + \|\eta_t^i\|_{\Omega_{t|t}^i}^2]} \\
&\leq \sqrt{2} \sum_{i \in \mathcal{N}} \sqrt{p_i} \left( \max_{j \in \mathcal{N}} \sqrt{\mathbb{E}[\|\xi_t^j\|_{\Omega_{t|t}^i}^2]} + \sqrt{\mathbb{E}[\|\eta_t^i\|_{\Omega_{t|t}^i}^2]} \right) \\
&\leq \sqrt{2} \sum_{i \in \mathcal{N}} \sqrt{p_i} \left( \sqrt{\bar{\omega}} \rho_{\xi,t} + \sqrt{\alpha} \right)
\end{aligned}$$

where we exploited the upper bounds for  $\|\xi_t^j\|_{\Omega_{t|t}^i}^2$  and  $\|\eta_t^i\|_{\Omega_{t|t}^i}^2$ . Taking into account (23) we obtain

$$\sqrt{\mathbb{E}[\mathcal{V}_{t+1}(e_{t+1})]} \leq \gamma \sqrt{\mathbb{E}[\mathcal{V}_t(e_t)]} + \sqrt{2}\gamma \sum_{i \in \mathcal{N}} \sqrt{p_i} \left( \sqrt{\bar{\omega}} \rho_{\xi,t} + \sqrt{\alpha} \right)$$

which implies

$$\begin{aligned}
\limsup_{t \rightarrow \infty} \sqrt{\mathbb{E}[\mathcal{V}_t(e_t)]} &\leq \lim_{t \rightarrow \infty} \frac{\sqrt{2}\gamma}{1-\gamma} \sum_{i \in \mathcal{N}} \sqrt{p_i} \left( \sqrt{\bar{\omega}} \rho_{\xi,t} + \sqrt{\alpha} \right) \\
&= \frac{\sqrt{2}\gamma}{1-\gamma} \sum_{i \in \mathcal{N}} \sqrt{p_i} \left( \sqrt{\bar{\omega}} \rho_{\xi} + \sqrt{\alpha} \right).
\end{aligned}$$

Finally, recalling that  $\Omega_{t|t}^i \geq \underline{\omega}I$  by Proposition 2, it is sufficient to note that

$$\mathbb{E}[\|e_t\|^2] \leq \frac{\mathbb{E}[\mathcal{V}_t(e_t)]}{\underline{\omega} \min_{i \in \mathcal{N}} p_i}$$

to obtain the claim.  $\square$

It is worth noting that the upper bound found in Theorem 1 depends on the network topology (e.g. through  $p_i$ ), the parameters of the trigger rule (e.g. through  $\alpha$ ) and the least favorable model (e.g. through  $\rho_{\xi}$ ). For instance, if we choose  $\alpha$  very large (i.e. the communication is triggered only when the information at the node is very different from the one owned by its out-neighbors nodes), then such upper bound will be very large.

## 5. A robust strategy with local tolerances

In Section 3 we have proposed a distributed strategy with event-triggered communication which provides a solution to the minimax problem in (6) which is suboptimal. Indeed, at each sensor node the state prediction in (12) is the solution to the (local) minimax problem

$$x_{t+1|t}^i = \arg \min_{g_t \in \mathcal{G}_i} \max_{\tilde{\phi}_t^i \in \mathcal{B}_t^i} \tilde{\mathbb{E}}_i [\|x_{t+1} - g_t(y_t^i)\|^2 | Y_{t-1}] \quad (25)$$

where

$$\begin{aligned} \tilde{\mathbb{E}}_i [\|x_{t+1} - g_t(y_t^i)\|^2 | Y_{t-1}] &:= \\ &\int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+p_i}} \|x_{t+1} - g_t(y_t)\|^2 \tilde{\phi}_t^i(z_t^i | x_t) \check{f}_t(x_t | Y_{t-1}) dz_t^i dx_t \end{aligned}$$

with  $z_t^i := [x_{t+1}^\top (y_t^i)^\top]^\top$ ;  $\mathcal{B}_t^i$  is the ambiguity set about the local and nominal model in (1)-(2), whose nominal density is denoted by  $\phi_t^i$ , with tolerance  $b$ ;  $\mathcal{G}_i$  is the set of estimators having finite second order moments for any  $\tilde{\phi}_t^i \in \mathcal{B}_t^i$ . Thus, the local least favorable transition probability density  $\tilde{\phi}_t^i$  solution to (25) does not necessarily agree with the global least favorable density  $\tilde{\phi}_t$  obtained by solving the centralized problem in (6).

Let

$$\begin{aligned} \bar{f}_t(z_t | Y_{t-1}) &:= \int_{\mathbb{R}^n} \phi_t(z_t | x_t) \check{f}_t(x_t | Y_{t-1}) dx_t \\ \tilde{f}_t(z_t | Y_{t-1}) &:= \int_{\mathbb{R}^n} \tilde{\phi}_t(z_t | x_t) \check{f}_t(x_t | Y_{t-1}) dx_t \end{aligned}$$

be the pseudo-nominal and the least favorable conditional probability densities of  $z_t$  given  $Y_{t-1}$ , respectively. In a similar way, we define the pseudo-nominal and the least favorable conditional probability densities of  $z_t^i$  given  $Y_{t-1}$ , respectively, as

$$\begin{aligned} \bar{f}_t^i(z_t | Y_{t-1}) &:= \int_{\mathbb{R}^n} \phi_t^i(z_t | x_t) \check{f}_t(x_t | Y_{t-1}) dx_t \\ \tilde{f}_t^i(z_t | Y_{t-1}) &:= \int_{\mathbb{R}^n} \tilde{\phi}_t^i(z_t | x_t) \check{f}_t(x_t | Y_{t-1}) dx_t. \end{aligned}$$

In [26] it has been shown that the least favorable density  $\tilde{\phi}_t^i$  solution to (25) is such that the Kullback-Leibler divergence between  $\tilde{f}_t^i$  and  $\bar{f}_t^i$ , i.e.

$$D_{KL}(\tilde{f}_t^i, \bar{f}_t^i) := \int_{\mathbb{R}^{n+p_i}} \tilde{f}_t^i(z_t^i | Y_{t-1}) \log \left( \frac{\tilde{f}_t^i(z_t^i | Y_{t-1})}{\bar{f}_t^i(z_t^i | Y_{t-1})} \right) dz_t^i$$

is equal to the tolerance of the ambiguity set  $\mathcal{B}_t^i$  used in (25).

Drawing inspiration from [30], it is possible to mitigate the fact that the local least favorable density  $\tilde{\phi}_t^i$  is different from the one obtained from the global least favorable density  $\tilde{\phi}_t$  by considering the ambiguity set  $\mathcal{B}_t^i$  about the nominal density  $\phi_t^i$  and with “local” tolerance

$$b_t^i = D_{KL}(\tilde{f}_t^{i,g}, \bar{f}_t^i) := \int_{\mathbb{R}^{n+p_i}} \tilde{f}_t^{i,g}(z_t^i | Y_{t-1}) \log \left( \frac{\tilde{f}_t^{i,g}(z_t^i | Y_{t-1})}{\bar{f}_t^i(z_t^i | Y_{t-1})} \right) dz_t^i \quad (26)$$

where  $\tilde{f}_t^{i,g}$  is the least favorable density at the sensor node  $i$  obtained by marginalizing  $\tilde{f}_t$  with respect to  $y_t^l$  with  $l \neq i$ . Although such a choice does not guarantee that  $\tilde{\phi}_t^i$ , i.e. the least favorable density solution to (25) with tolerance (26), leads to a density  $\tilde{f}_t^i$  which coincides with  $\tilde{f}_t^{i,g}$ , at least it is ensured that

$$D_{KL}(\tilde{f}_t^i, \bar{f}_t^i) = D_{KL}(\tilde{f}_t^{i,g}, \bar{f}_t^i).$$

In [44] it has been shown that if  $b > 0$  in (6) is taken sufficiently small, then the least favorable model  $\tilde{\phi}_t$  solution to (6) converges to a stationary Gaussian model as  $t \rightarrow \infty$ . Moreover, the conditional mean of  $z_t$  given  $Y_{t-1}$  under  $\tilde{f}_t$  is the same of the one under  $\bar{f}_t$ , see [30, Theorem 1]. Let  $K$  and  $\tilde{K}$  denote the asymptotic covariance matrices of  $z_t$  given  $Y_{t-1}$  under the Gaussian densities  $\bar{f}_t$  and  $\tilde{f}_t$ , respectively. Then,  $b_t^i \rightarrow b^i$  as  $t \rightarrow \infty$  where

$$b^i = \frac{1}{2} \left[ \log \det(K_i \tilde{K}_i^{-1}) + \text{tr}(\tilde{K}_i K_i^{-1}) - (n + p_i) \right];$$

$K_i$  is the matrix obtained from  $K$  by deleting the rows and the columns corresponding to  $y_t^l$  with  $l \neq i$ ;  $\tilde{K}_i$  is obtained from  $\tilde{K}$  likewise.

In conclusion, we propose the following distributed filtering strategy with event-triggered communication. The central unit can compute offline the asymptotic tolerance  $b^i$  for each sensor node  $i \in \mathcal{S}$  from the global least

favorable density  $\tilde{\phi}_t$  and send it to the corresponding sensor node. After this offline step, the remaining part of the algorithm is as Algorithm 1 with the exception of (12) and (14) which become, respectively,

$$\begin{aligned}
\Omega_{t+1|t}^i &= Q^{-1} - Q^{-1}A(A^\top Q^{-1}A + \Omega_{t|t}^{i,F})A^\top Q^{-1} \\
\text{Find } \theta_t^i &> 0 \text{ s.t. } \gamma(\Omega_{t+1|t}^i, \theta_t^i) = b^i \\
\Psi_{t+1|t}^i &= \Omega_{t+1|t}^i - \theta_t^i \mathbf{I}_n \\
\bar{q}_{t+1|t}^i &= \Psi_{t+1|t}^i A(\Omega_{t|t}^{i,F})^{-1} q_{t|t}^{i,F}
\end{aligned} \tag{27}$$

$$\begin{aligned}
\bar{\Omega}_{t+1}^i &= Q^{-1} - Q^{-1}A(A^\top Q^{-1}A + \check{\Omega}_t^i)A^\top Q^{-1} \\
\text{Find } \bar{\theta}_t^i &> 0 \text{ s.t. } \gamma(\bar{\Omega}_{t+1}^i, \bar{\theta}_t^i) = b^i \\
\bar{\Psi}_{t+1}^i &= \bar{\Omega}_{t+1}^i - \bar{\theta}_t^i \mathbf{I}_n \\
\bar{q}_{t+1}^i &= \bar{\Psi}_{t+1}^i A(\check{\Omega}_t^i)^{-1} \check{q}_t^i.
\end{aligned} \tag{28}$$

It is worth noting this distributed algorithm characterizes the local ambiguity set of each node in a more accurate way than the algorithm in Section 3. The algorithm procedure is outlined in Algorithm 2.

In regard to the stability properties of this algorithm with local tolerances, it is not difficult to see that under Assumptions **A1-A4**

$$\lim_{t \rightarrow \infty} \mathbb{E}[\|e_t^i\|^2] \leq \left( \frac{\sqrt{2}\gamma \sum_{i \in \mathcal{N}} \sqrt{p_i} (\sqrt{\underline{\omega}} \rho_\xi + \sqrt{\underline{\alpha}})}{1 - \gamma \sqrt{\underline{\omega}} \min_{i \in \mathcal{N}} \sqrt{p_i}} \right)^2$$

for any  $i \in \mathcal{N}$ . The proof follows the same ideas exploited in Section 4. The unique difference is the derivation of the constant  $\underline{\omega}$ . More precisely, in the proof of Proposition 2 the constant  $\mu$  is derived as follows. Since each sensor node  $i$  has its own tolerance  $b^i$ , then we have that  $\gamma(\Omega_{t+1|t}^i, \theta_t^i) = b^i$ . By Lemma 2 in Appendix, it follows that there exists  $\mu_i > 0$  such that  $\Omega_{t|t}^i \geq \mu_i \Omega_{t|t-1}^i \forall i \in \mathcal{S}$ . Then,  $\Omega_{t|t}^i \geq \mu \Omega_{t|t-1}^i \forall i \in \mathcal{S}$  with  $\mu := \min_{i \in \mathcal{S}} \mu_i > 0$ .

## 6. Simulations

In this section, we evaluate the performance of the proposed robust distributed Kalman algorithms with event-triggered communication. We consider the problem of tracking the position of a target by using noisy position measurements obtained by a network of dimension  $N = 100$  where there are

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**Algorithm 2** RDKF with local tolerances

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**Initialization:** Set  $(q_{0|-1}^i, \Psi_{0|-1}^i)$  for any  $i \in \mathcal{N}$

For each  $t = 0, 1, \dots$

For each node  $i \in \mathcal{N}$

**Correction:**

$$(q_{t|t}^i, \Omega_{t|t}^i) = \begin{cases} \text{use (7),} & \text{if } i \in \mathcal{S} \\ (q_{t|t-1}^i, \Psi_{t|t-1}^i), & \text{if } i \in \mathcal{N} \setminus \mathcal{S} \end{cases}$$

**Information exchange:**

- if  $t = 0$  set  $c_t^i = 1$ , otherwise determine  $c_t^i$  according to (9)
- if  $c_t^i = 1$  transmit  $(q_{t|t}^i, \Omega_{t|t}^i)$  to the out-neighbors
- receive  $(q_{t|t}^j, \Omega_{t|t}^j)$  from all the in-neighbors  $j \in \mathcal{N}_i$  for which  $c_t^j = 1$

**Information fusion:**

$$\tilde{q}_t^j = \frac{1}{1 + \delta} \bar{q}_t^j, \quad \tilde{\Omega}_t^j = \frac{1}{1 + \delta} \bar{\Psi}_t^j \text{ with } j \in \mathcal{N}_i$$

$$q_{t|t}^{i,F} = \pi_{i,i} q_{t|t}^i + \sum_{j \in \mathcal{N}_i} \pi_{i,j} \left[ c_t^j q_{t|t}^j + (1 - c_t^j) \tilde{q}_t^j \right]$$

$$\Omega_{t|t}^{i,F} = \pi_{i,i} \Omega_{t|t}^i + \sum_{j \in \mathcal{N}_i} \pi_{i,j} \left[ c_t^j \Omega_{t|t}^j + (1 - c_t^j) \tilde{\Omega}_t^j \right]$$

**Prediction step:**

- Compute  $(q_{t+1|t}^i, \Psi_{t+1|t}^i)$  using (27)
  - Compute  $(\bar{q}_{t+1|t}^i, \bar{\Psi}_{t+1}^i)$  using (13) and (28)
-

80 sensors and 20 sensor nodes. The possible connections among the nodes have been randomly generated in such a way the network is strongly connected. Moreover, the percentage of node connections is equal to 4%. The model for the motion of the target is

$$\dot{x}_s^c = \Phi x_s^c + \dot{w}_s^c, \quad s \in \mathbb{R} \quad (29)$$

where

$$\Phi = \begin{bmatrix} 0 & 0 \\ \mathbf{I}_3 & 0 \end{bmatrix},$$

$x_s^c = [v_{x,s} \ v_{y,s} \ v_{z,s} \ p_{x,s} \ p_{y,s} \ p_{z,s}]^\top$  with  $v$  denoting the velocity,  $p$  the position and the subscripts  $x, y, z$  denoting the three spatial dimensions;  $w_s^c$  is a Wiener process with zero mean and rate of variance equal to 0.1. We discretize (29) with a sampling time equal to 0.1. The corresponding discrete time model is  $x_{t+1} = Ax_t + Bw_t$  where  $x_t$  is the sampled version of  $x_s^c$ ,  $A = \mathbf{I}_6 + 0.1\Phi$ ,  $B = \sqrt{0.001}I$ ,  $w_t$  is a zero-mean normalized Gaussian white noise and thus  $Bw_t$  is the sampled version of  $w_s^c$ . We assume that every sensor measures the position of the target in either two horizontal dimensions or a combination of one horizontal dimension and the vertical dimension; in plain words, one sensor does not have measurements in all the three dimensions. Therefore, we obtain the nominal discrete state-space model (1)-(2) where  $C^i = [0 \ 0 \ 0 \ \text{diag}(1, 1, 0)]$ , in the case that the sensor measures only the horizontal positions, or  $C^i = [0 \ 0 \ 0 \ \text{diag}(1, 0, 1)]$ ,  $C^i = [0 \ 0 \ 0 \ \text{diag}(0, 1, 1)]$ , in the case that the sensor measures one horizontal position and the vertical position. Moreover,  $R^i = D^i(D^i)^\top = \sqrt{k}PR_0P^\top$  where  $R_0 = 0.5 \cdot \text{diag}(1, 4, 7)$  and  $P$  is a permutation matrix randomly chosen for every node. Finally, the initial state  $x_0$  is a Gaussian random vector with zero-mean and covariance matrix  $V_{0|-1} = I$ . Since the previous model is just an idealization of the underlying physical system, we assume that the actual state-space model belongs to the ambiguity set  $\mathcal{B}_t$  about the aforementioned nominal model and with tolerance  $b = 0.05$ . More precisely, we assume that the actual model is the least favorable model solution to (i.e. the maximizer of) the centralized problem in (6).

In the following, we consider the distributed algorithms:

- RDKF – the distributed robust Kalman filter with event-triggered communication in Algorithm 1 with  $b = 0.05$ . Here, the transmission rule (9) is with  $\alpha = 10$ ,  $\beta = 0.2$  and  $\delta = 0.5$ ;

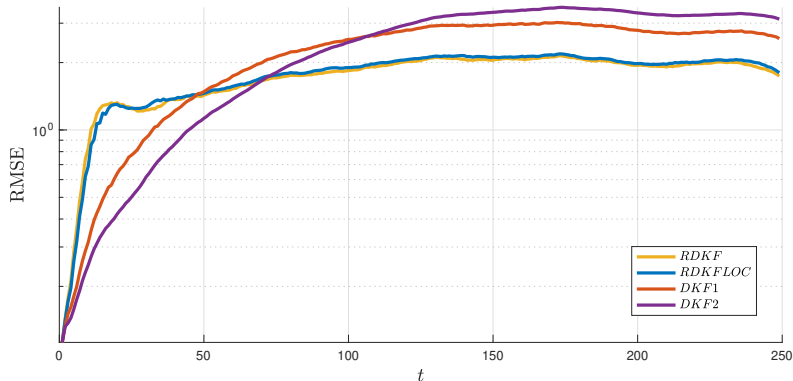


Figure 1: Average RMSE (in logarithmic scale) across the network.

- RDKFLOC – the distributed robust Kalman filter with event-triggered communication of Section 5 where each sensor nodes  $i$  has its own tolerance  $b^i$  computed from the global least favorable model; the transmission rule (9) is with  $\alpha = 10$ ,  $\beta = 0.2$  and  $\delta = 0.5$ ;
- DKF1 – the distributed Kalman filter with event-triggered communication proposed in [22] and the transmission rule is with  $\alpha = 10$ ,  $\beta = 0.2$  and  $\delta = 0.5$ ;
- DKF2 – is the same as DKF1 but the transmission rule is with  $\alpha = 0.01$ ,  $\beta = 0.2$  and  $\delta = 0.5$ .

It is worth noting that RDKF, RDKFLOC and DKF1 have the same parameters for the transmission rule. As we will see later, taking the parameter  $\alpha$  in (9) the same for RDKF, RDKFLOC and DKF1 provides a transmission rate for DKF1 which is smaller than the robust versions. Such difference is due, as observed in Section 3, by the fact that the transmission rule (9) is intrinsically different from the one in [22]. For this reason, we also consider DKF2 where the parameter  $\alpha$  has been decreased **in order to** increase the transmission rate. We also tried to increase the transmission rate by keeping fixed  $\alpha = 10$  and changing  $\beta, \delta$ ; however, we did not notice a significant growth in terms of transmission rate.

We consider a Monte Carlo study which is composed by 500 independent runs (which correspond to different target paths generated by the least favor-

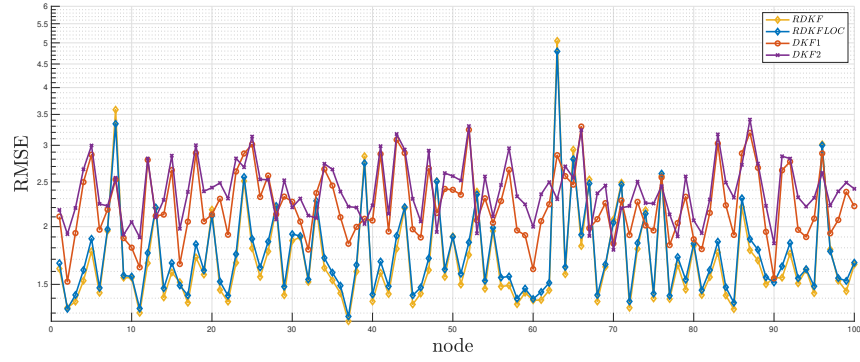


Figure 2: Average RMSE (in logarithmic scale) at each node over the time horizon  $[1, 250]$ .

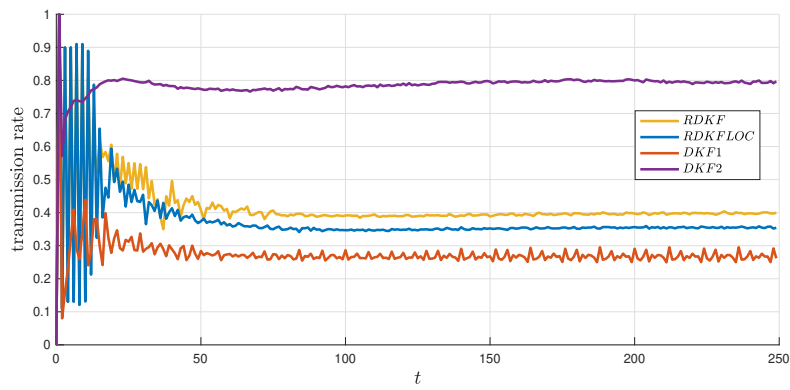


Figure 3: Average transmission rate across the network.

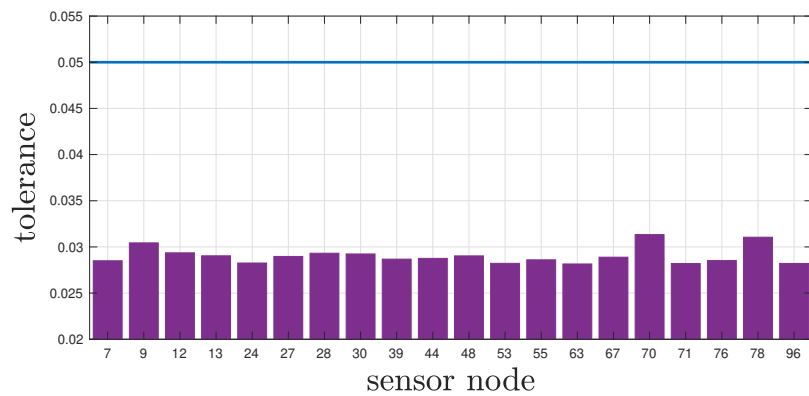


Figure 4: Tolerance used in RDKF (blue line) versus the local tolerances at each sensors node used in RDKFLOC (purple bars).

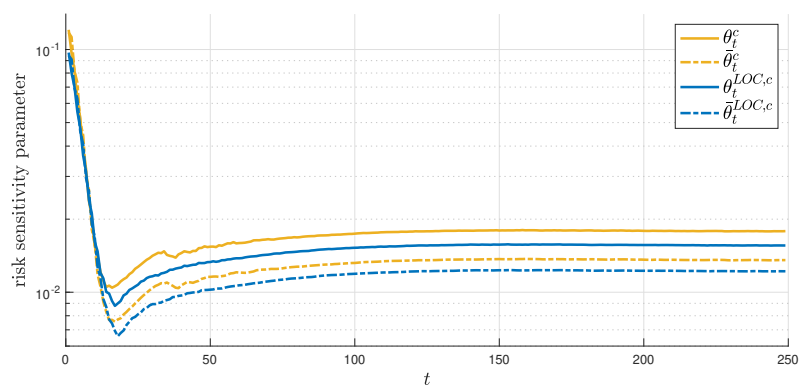


Figure 5: Average risk sensitivity parameters across the communication nodes of the network (in logarithmic scale).

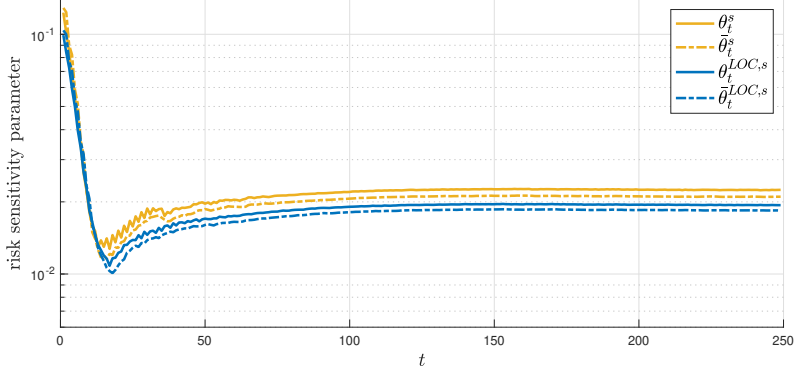


Figure 6: Average risk sensitivity parameter across the sensor nodes of the network (in logarithmic scale).

able model <sup>1)</sup> over a time horizon of 250 seconds. For each run, we estimate the state using the aforementioned distributed algorithms and for each of them we consider the following performance indexes:

- The average root mean square error across the network at time  $t$ :

$$\text{RMSE}_t = \frac{1}{N} \sum_{i=1}^N \|x_t - x_{t|t}^i\|^2.$$

- The average root mean square error at node  $i$  over the time horizon:

$$\text{RMSE}_i = \frac{1}{250} \sum_{t=1}^{250} \|x_t - x_{t|t}^i\|^2.$$

- The transmission rate across the network at time  $t$ , i.e. the fraction of nodes that transmit their data at time  $t$ .

Figure 1 and Figure 2 show the two aforementioned root mean square errors averaged over the runs, while Figure 3 shows the corresponding averaged

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<sup>1</sup>For more details on how to generate a realization from the least favorable model see [26, Section V].

transmission rate. As we can see the robust filters outperform DKF1 and DKF2. In particular, even in the case we increase the transmission rate, i.e. as in DKF2, RDKF and RDKFLOC outperform the standard algorithm. Such result highlights the fact that the increase of the communication rate does not substantially improve the performance of the distributed algorithm in the presence of model uncertainty. In plain words, model uncertainty cannot be cured simply by increasing the transmission rate. Clearly, DKF2 outperforms DKF1 in the steady state because the latter is penalized by the low transmission rate across the network. Finally, RDKFLOC is slightly better than RDKF: it requires a lower transmission rate and it exhibits a better RMSE in the nodes characterized by a large estimation error, see Figure 2. This is due to the fact that the local filters of the sensor nodes in RDKF are too conservative, indeed their tolerances are larger than the ones in RDKFLOC, see Figure 4.

Let the average risk sensitivity parameters across the communication nodes of RDKF be defined as:

$$\theta_t^c := \frac{1}{|\mathcal{C}|} \sum_{i \in \mathcal{C}} \theta_t^i, \quad \bar{\theta}_t^c := \frac{1}{|\mathcal{C}|} \sum_{i \in \mathcal{C}} \bar{\theta}_t^i$$

where  $\mathcal{C} = \mathcal{N} \setminus \mathcal{S}$  and  $|\mathcal{C}|$  denotes the cardinality of set  $\mathcal{C}$ . The average risk sensitivity parameters across the communication nodes in RDKFLOC, denoted by  $\theta_t^{LOC,c}$  and  $\bar{\theta}_t^{LOC,c}$ , are defined likewise. Figure 5 shows the aforementioned quantities averaged over the Monte Carlo runs. We can notice that  $\theta_t^c \geq \bar{\theta}_t^c$  and  $\theta_t^{LOC,c} \geq \bar{\theta}_t^{LOC,c}$ , moreover we have checked that  $\theta_t^i \geq \bar{\theta}_t^i$  for many communication nodes both in RDKF and RDKFLOC. Since the mapping  $\theta \mapsto \gamma(\Omega, \theta)$  is monotone increasing, the mapping  $\Omega \mapsto \gamma(\Omega, \theta)$  is monotone decreasing according to the partial order of positive definite matrices, see [46, 47], and in view of the fact that

$$\gamma(\Omega_{t+1|t}^i, \theta_t^i) = \gamma(\bar{\Omega}_{t+1}^i, \bar{\theta}_t^i),$$

it follows that  $\Omega_{t+1|t}^i \geq \bar{\Omega}_{t+1}^i \geq \bar{\Psi}_{t+1}^i$  for many nodes both in RDKF and RDKFLOC. The latter inequality means that the transmission from node  $i$  typically produces an increase of information in the corresponding **out-neighbor** nodes, which is the expected scenario. Notice that  $\theta_t^c \geq \theta_t^{LOC,c}$  and  $\bar{\theta}_t^c \geq \bar{\theta}_t^{LOC,c}$  which is just a consequence of the fact that RDKF is more conservative than RDKFLOC; indeed, recall that  $b^i \leq b \forall i \in \mathcal{S}$  see Figure 4.

Let the average risk sensitivity parameters across the sensor nodes of RDKF be defined as:

$$\theta_t^s := \frac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} \theta_t^i, \quad \bar{\theta}_t^s := \frac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} \bar{\theta}_t^i.$$

The average risk sensitivity parameters across the sensor nodes in RDKFLOC, denoted by  $\theta_t^{LOC,s}$  and  $\bar{\theta}_t^{LOC,s}$ , are defined likewise. Figure 6 shows the aforementioned quantities averaged over the Monte Carlo runs. The observations done before nodes hold also in this case.

## 7. Conclusion

In this paper, we have considered the problem of [estimating](#) the state of a sensor network under model uncertainty and communication constraints. We have proposed two robust distributed strategies with event-triggered communication. More precisely, the sensor nodes compute their state estimate by solving a minimax game: one player (i.e. the estimator) aims to minimize the estimation error, while the other player selects the model in the ambiguity set which maximizes such error. The communication among nodes is governed by a data-driven rule which essentially allows the data transmission only in the case the latter provides a substantial increase of information in the nodes receiving it. The difference between the two distributed strategies is the way the ambiguity sets are formed at each sensor node. A stability analysis of the algorithms has been carried out showing that it is guaranteed mean-square boundedness of the state estimation error in all the nodes, under the global least favorable model, provided that the network is strongly connected, the system collectively observable and the tolerance of the ambiguity set corresponding to the global model is sufficiently small. Finally, a numerical experiment showed that the proposed strategies are effective in the case there is model uncertainty.

**Lemma 2.** *Let  $\Omega > 0$  and  $b, \theta > 0$  such that  $\gamma(\Omega, \theta) = b$ . Then, there exists a constant  $\mu > 0$  such that*

$$\Omega - \theta I \geq \mu \Omega.$$

PROOF. The constraint  $\gamma(\Omega, \theta) = b$  can be written as

$$\text{tr}(X) - \log \det(X) - n = 2b \tag{.1}$$

where  $X := (I - \theta\Omega^{-1})^{-1} > 0$ . Let  $\lambda_k > 0$  denote the  $k$ -th eigenvalue of  $X$ , then (.1) can be written as

$$\sum_{k=1}^n \lambda_k - \log \lambda_k - 1 = 2b \quad (.2)$$

and the terms  $\lambda_k - \log \lambda_k - 1$  in the summation are nonnegative. Hence, condition (.2) implies that

$$f(\lambda_k) := \lambda_k - \log \lambda_k - 1 \leq 2b, \quad k = 1 \dots n. \quad (.3)$$

Notice that  $f$  is continuous for  $\lambda > 0$  and it is not difficult to see that

$$\lim_{\lambda \rightarrow 0^+} f(\lambda) = \infty, \quad \lim_{\lambda \rightarrow \infty} f(\lambda) = \infty, \quad \operatorname{argmin}_{\lambda > 0} f(\lambda) = 1.$$

Accordingly, there exists  $\bar{\lambda} > 1$ , which only depends on  $b$ , such that condition (.3) is satisfied for  $1 \leq \lambda_k \leq \bar{\lambda}$ , see Figure .7. Accordingly, the constraint in

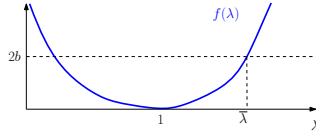


Figure .7: Pictorial description of  $\bar{\lambda}$ .

(.1) implies that

$$\begin{aligned} (I - \theta\Omega^{-1})^{-1} &= X \leq \bar{\lambda}I \\ I - \theta\Omega^{-1} &\geq \bar{\lambda}^{-1}I \\ \Omega - \theta I &\geq \bar{\lambda}^{-1}\Omega \end{aligned}$$

and thus  $\mu = \bar{\lambda}^{-1}$ . □

## References

- [1] F. S. Cattivelli, A. H. Sayed, Diffusion strategies for distributed Kalman filtering and smoothing, *IEEE Trans. Automatic Control* 55 (9) (2010) 2069–2084.

- [2] D. P. Spanos, R. Olfati-Saber, R. M. Murray, Approximate distributed Kalman filtering in sensor networks with quantifiable performance, in: Fourth International Symposium on Information Processing in Sensor Networks, 2005, pp. 133–139.
- [3] W. Li, Y. Jia, J. Du, Distributed Kalman consensus filter with intermittent observations, *Journal of the Franklin Institute* 352 (9) (2015) 3764–3781.
- [4] G. Battistelli, L. Chisci, Kullback-Leibler average, consensus on probability densities, and distributed state estimation with guaranteed stability, *Automatica* 50 (3) (2014) 707–718.
- [5] A. Kamal, J. Farrell, A. Roy-Chowdhury, Information weighted consensus filters and their application in distributed camera networks, *IEEE Trans. Automatic Control* 58 (12) (2013) 3112–3125.
- [6] T. Ding, Q. Yang, Y. Wen, Y. Ning, Y. Yang, F. Blaabjerg, Zonally robust decentralized optimization for global energy interconnection: Case study on northeast asian countries, *IEEE Trans. Automation Science and Engineering* 17 (4) (2020) 2120–2129.
- [7] W. Li, Y. Jia, J. Du, Distributed extended Kalman filter with nonlinear consensus estimate, *Journal of the Franklin Institute* 354 (17) (2017) 7983–7995.
- [8] G. Battistelli, A. Benavoli, L. Chisci, Data-driven communication for state estimation with sensor networks, *Automatica* 48 (5) (2012) 926–935.
- [9] D. Han, Y. Mo, J. Wu, S. Weerakkody, B. Sinopoli, L. Shi, Stochastic event-triggered sensor schedule for remote state estimation, *IEEE Trans. Automatic Control* 60 (10) (2015) 2661–2675.
- [10] D. Shi, L. Shi, T. Chen, Event-based state estimation A Stochastic Perspective, Springer, Switzerland, 2016.
- [11] Q. Liu, Z. Wang, X. He, D. Zhou, Event-based recursive distributed filtering over wireless sensor networks, *IEEE Trans. Automatic Control* 60 (9) (2015) 2470–2475.

- [12] W. Li, Y. Jia, J. Du, Event-triggered Kalman consensus filter over sensor networks, *IET Control Theory & Applications* 10 (1) (2015) 103–110.
- [13] L. Yan, X. Zhang, Z. Zhang, Y. Yang, Distributed state estimation in sensor networks with event-triggered communication, *Nonlinear Dynamics* 76 (1) (2014) 169–181.
- [14] L. Shi, K. Johansson, L. Qiu, Time and event-based sensor scheduling for networks with limited communication resources, *IFAC Proceedings Volumes* 44 (1) (2011) 13263–13268.
- [15] J. Hu, Z. Wang, G. Liu, C. Jia, J. Williams, Event-triggered recursive state estimation for dynamical networks under randomly switching topologies and multiple missing measurements, *Automatica* 115 (2020) 108908.
- [16] X. Ge, Q. Han, X. Zhang, L. Ding, F. Yang, Distributed event-triggered estimation over sensor networks: A survey, *IEEE Trans. Cybernetics* 50 (3) (2019) 1306–1320.
- [17] H. Cong, W. Ding, R. Gao, P. Mei, H. Karimi, Distributed state-of-charge estimation for lithium-ion batteries with random sensor failure under dynamic event-triggering protocol, *Information Fusion* 95 (2) (2023) 293–305.
- [18] C. Deng, C. Wen, J. Huang, X. M. Zhang, Y. Zou, Distributed observer-based cooperative control approach for uncertain nonlinear MASs under event-triggered communication, *IEEE Trans. Automatic Control* 67 (5) (2021) 2669–2676.
- [19] W. Xu, W. He, D. Ho, J. Kurths, Fully distributed observer-based consensus protocol: Adaptive dynamic event-triggered schemes, *Automatica* 139 (2022) 110188.
- [20] C. Peng, H. Sun, Switching-like event-triggered control for networked control systems under malicious denial of service attacks, *IEEE Trans. Automatic Control* 65 (9) (2020) 3943–3949.
- [21] L. Cao, H. Li, G. Dong, R. Lu, Event-triggered control for multiagent systems with sensor faults and input saturation, *IEEE Trans. Systems, Man, and Cybernetics: Systems* 51 (6) (2019) 3855–3866.

- [22] G. Battistelli, L. Chisci, D. Selvi, A distributed Kalman filter with event-triggered communication and guaranteed stability, *Automatica* 93 (7) (2018) 75–82.
- [23] S. Yi, M. Zorzi, Robust Kalman filtering under model uncertainty: The case of degenerate densities, *IEEE Trans. Automatic Control* 67 (7) (2022) 3458–3471.
- [24] Y. Huang, Y. Zhang, Y. Zhao, L. Mihaylova, J. Chambers, Robust Rauch–Tung–Striebel smoothing framework for heavy-tailed and/or skew noises, *IEEE Trans. Aerospace and Electronic Systems* 56 (1) (2019) 415–441.
- [25] Y. Huang, F. Zhu, G. Jia, Y. Zhang, A slide window variational adaptive Kalman filter, *IEEE Trans. Circuits and Systems II: Express Briefs* 67 (12) (2020) 3552–3556.
- [26] B. Levy, R. Nikoukhah, Robust state-space filtering under incremental model perturbations subject to a relative entropy tolerance, *IEEE Trans. Automatic Control* 58 (3) (2013) 682–695.
- [27] B. Shen, Z. Wang, Y. Hung, Distributed  $H_\infty$ -consensus filtering in sensor networks with multiple missing measurements: The finite-horizon case, *Automatica* 46 (10) (2010) 1682–1688.
- [28] Y. Luo, Y. Zhu, D. Luo, J. Zhou, E. Song, D. Wang, Globally optimal multisensor distributed random parameter matrices Kalman filtering fusion with applications, *Sensors* 8 (12) (2008) 8086–8103.
- [29] M. Zorzi, Distributed Kalman filtering under model uncertainty, *IEEE Trans. Control of Network Systems* 7 (2) (2020) 990–1001.
- [30] A. Emanuele, F. Gasparotto, G. Guerra, M. Zorzi, Robust distributed Kalman filtering: On the choice of the local tolerance, *Sensors* 20 (11) (2020) 3244.
- [31] X. Yu, J. Li, Distributed robust Kalman filters under model uncertainty and multiplicative disturbance, *IEEE Trans. Aerospace and Electronic Systems* 59 (2) (2023) 973–988.

- [32] J. Wang, P. Dong, K. Shen, X. Song, X. Wang, Distributed consensus Student-t filter for sensor networks with heavy-tailed process and measurement noises, *IEEE Access* 8 (2020) 167865–167874.
- [33] R. Caballero-Águila, A. Hermoso-Carazo, J. Linares-Pérez, A two-phase distributed filtering algorithm for networked uncertain systems with fading measurements under deception attacks, *Sensors* 20 (22) (2020) 6445.
- [34] J. Huang, D. Shi, T. Chen, Distributed robust state estimation for sensor networks: A risk-sensitive approach, in: *IEEE Conference on Decision and Control (CDC)*, 2018, pp. 6378–6383.
- [35] H. Dong, Z. Wang, F. Alsaadi, B. Ahmad, Event-triggered robust distributed state estimation for sensor networks with state-dependent noises, *International Journal of General Systems* 44 (2) (2015) 254–266.
- [36] D. Ghion, M. Zorzi, Distributed Kalman filtering with event-triggered communication: a robust approach, in: *Mediterranean Conference on Control and Automation (MED)*, 2022, pp. 785–790.
- [37] M. Zorzi, Robust Kalman filtering under model perturbations, *IEEE Trans. Automatic Control* 62 (6) (2017) 2902–2907.
- [38] M. Zorzi, On the robustness of the Bayes and Wiener estimators under model uncertainty, *Automatica* 83 (9) (2017) 133–140.
- [39] L. Hansen, T. Sargent, *Robustness*, Princeton University Press, Princeton, 2008.
- [40] B. Levy, R. Nikoukhah, Robust least-squares estimation with a relative entropy constraint, *IEEE Trans. Information Theory* 50 (1) (2004) 89–104.
- [41] G. Battistelli, L. Chisci, D. Selvi, Distributed Kalman filtering with data-driven communication, in: *International Conference on Information Fusion (FUSION)*, 2016, pp. 1042–1048.
- [42] J. Hu, Z. Wang, J. Liang, H. Dong, Event-triggered distributed state estimation with randomly occurring uncertainties and nonlinearities over sensor networks: a delay-fractioning approach, *Journal of the Franklin Institute* 352 (9) (2015) 3750–3763.

- [43] A. Zenere, M. Zorzi, On the coupling of model predictive control and robust Kalman filtering, *IET Control Theory Applications* 12 (13) (2018) 1873–1881.
- [44] M. Zorzi, B. Levy, Robust Kalman filtering: Asymptotic analysis of the least favorable model, in: *IEEE Conference on Decision and Control (CDC)*, 2018, pp. 7124–7129.
- [45] M. Zorzi, B. Levy, On the convergence of a risk sensitive like filter, in: *IEEE Conference on Decision and Control (CDC)*, 2015, pp. 4990–4995.
- [46] M. Zorzi, Convergence analysis of a family of robust Kalman filters based on the contraction principle, *SIAM J. Control and Optimization* 55 (5) (2017) 3116–3131.
- [47] B. Levy, M. Zorzi, A contraction analysis of the convergence of risk-sensitive filters, *SIAM J. Control and Optimization* 54 (4) (2016) 2154–2173.