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Strichartz estimates for the 2D and 3D massless Dirac-Coulomb equations and applications



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ABSTRACT

In this paper we continue the analysis of the dispersive properties of the 2D and 3D massless Dirac-Coulomb equations that has been started in [7] and [8]. We prove a priori estimates of the solution of the mentioned systems, in particular Strichartz estimates with an additional angular regularity, exploiting the tools developed in the previous works. As an application, we show local well-posedness results for a Dirac-Coulomb equation perturbed with Hartree-type nonlinearities.

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1. Introduction

In this paper we consider the Cauchy problem associated with the massless Dirac equation with an electric *Coulomb potential* in 2 and 3 spatial dimensions. It reads as follows:

$$\begin{cases} i\partial_t u + \mathcal{D}_n u - \frac{\nu}{|x|} u = 0, \\ u(0, x) = u_0(x) \end{cases} \quad (1)$$

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where $u(t, x): \mathbb{R}_t \times \mathbb{R}_x^n \rightarrow \mathbb{C}^N$, $n \in \{2, 3\}$, $N = 2^{\lceil \frac{n}{2} \rceil}$, $\nu \in [-\frac{n-1}{2}, \frac{n-1}{2}]$ and $\mathcal{D}_n$ is the Dirac operator on $\mathbb{R}^n$. In order to define the operator we introduce two sets of matrices: the Pauli matrices $\sigma_j \in \mathcal{M}_{2 \times 2}(\mathbb{C})$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the Dirac matrices $\alpha_j \in \mathcal{M}_{4 \times 4}(\mathbb{C})$

$$\alpha_j = \begin{pmatrix} 0_{2 \times 2} & \sigma_j \\ \sigma_j & 0_{2 \times 2} \end{pmatrix}, \quad j = 1, 2, 3.$$

Then we can define the Dirac operator on $\mathbb{R}^2$ as

$$\mathcal{D}_2 = -i\sigma \cdot \nabla_{\mathbb{R}^2} = -i(\sigma_1 \partial_x + \sigma_2 \partial_y)$$

and on $\mathbb{R}^3$ as

$$\mathcal{D}_3 = -i\alpha \cdot \nabla_{\mathbb{R}^3} = -i \sum_{j=1}^3 \alpha_j \partial_j.$$

The massless Dirac equation is widely used to describe physical systems from Quantum Mechanics; the 3D equation is a model for the dynamics of massless fermions, such as the neutrinos. The 2D equation appears in the study of propagation of waves spectrally concentrated near some singular points on 2-dimensional honeycomb structures. We remark that among the materials which enjoy this structure one finds the *graphene*, a single-layer sheet of hexagonally-arranged carbon atoms, that is attracting a lot of interest in the recent years due to its countless technological applications (see [15] for a survey and the references therein). Notice that also with the Coulomb potential, describing interactions between particles, the system remains physically interesting (e.g., non-perfect graphene).

The restriction on the parameter ν comes from the fact that, according to Quantum Mechanics, we should work with self-adjoint operators on $L^2(\mathbb{R}^n; \mathbb{C}^N)$. We recall that $\mathcal{D}_n - \frac{\nu}{|x|}$, defined on $C_0^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^N)$, is essentially self-adjoint¹ with domain $H^1(\mathbb{R}^n; \mathbb{C}^N)$ if and only if $n = 2$ and $\nu = 0$ or $n = 3$ and $|\nu| < \frac{\sqrt{3}}{2}$. If $n = 3$ and $|\nu| = \frac{\sqrt{3}}{2}$ the Dirac-Coulomb operator is still essentially self-adjoint but with domain contained in $H^{\frac{1}{2}}(\mathbb{R}^3; \mathbb{C}^4)$. For any other values of ν there exist infinitely many different self-adjoint extensions. However, it has been shown that for ν in the range we consider it is possible to define a distinguished (i.e. “physically relevant”) self-adjoint extension. In particular, for $|\nu| < \frac{n-1}{2}$ one can choose the self-adjoint extension with domain contained in $H^{\frac{1}{2}}(\mathbb{R}^n; \mathbb{C}^N)$ (see the recent works [13], [21] and the references therein).

¹ That is, it admits only one self-adjoint extension, its closure.

The free massless Dirac equation can be listed within the class of *dispersive equations*. One way to see it is to look at the link between the Dirac equation and a system of decoupled wave equations; by conjugating $i\partial_t + \mathcal{D}_n$ one gets

$$(i\partial_t + \mathcal{D}_n)(i\partial_t - \mathcal{D}_n) = (-\partial_{tt}^2 + \Delta)\mathbb{1}_{N \times N}, \quad (2)$$

since the sets of matrices $\{\sigma_k\}_{k=1,2}$, $\{\alpha_j\}_{j=1,2,3}$ are chosen in order to satisfy the anti commutation relations

$$\begin{aligned} \sigma_k \sigma_j + \sigma_j \sigma_k &= 2\delta_{k,j} \mathbb{1}_{2 \times 2}, & k, j &= 1, 2 \\ \alpha_k \alpha_j + \alpha_j \alpha_k &= 2\delta_{k,j} \mathbb{1}_{4 \times 4}, & k, j &= 1, 2, 3. \end{aligned}$$

In the last years, many tools have been developed in order to quantify the dispersion of a system. Among these we find a priori estimates on the solutions, such as Strichartz estimates and their generalizations, which appear to be fundamental tools in the study, e.g., of local and global well-posedness of nonlinear systems (possibly with rough regularity data). We recall the *Strichartz estimates* for the free Dirac flow:

$$\|e^{it\mathcal{D}_n} u_0\|_{L_t^p L_x^q} \leq C \|u_0\|_{\dot{H}^{\frac{n}{2} - \frac{n}{q} - \frac{1}{p}}},$$

for any couple (p, q) satisfying the wave admissibility condition

$$p, q \in [2, +\infty], \quad \frac{1}{p} \leq \frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{q} \right), \quad (p, q) \neq \left(\frac{4}{n-1}, \infty \right), (\infty, \infty), \quad (3)$$

where we denote by $e^{it\mathcal{D}_n}$ the propagator for solutions of (1) with $\nu = 0$. Thanks to the relation (2), these estimates are deduced directly from the one that hold for the wave equation.

It has been observed (see [17] for a survey and the references therein) that one can enlarge the set of admissible couples, requiring some additional regularity in the angular variable. For the free Dirac equation one has the following *generalized Strichartz estimates*

$$\|e^{it\mathcal{D}_n} u_0\|_{L_t^p L_{|x|}^q L_\theta^2(\mathbb{R} \times \mathbb{R}^n)} \leq C \|u_0\|_{\dot{H}^{\frac{n}{2} - \frac{n}{q} - \frac{1}{p}}(\mathbb{R}^n)} \quad (4)$$

for (p, q) satisfying

$$p, q \in [2, +\infty], \quad \frac{n-1}{q} + \frac{1}{p} < \frac{n-1}{2} \text{ or } (p, q) = (\infty, 2), \quad (p, q) \neq (2, \infty), (\infty, \infty).$$

In presence of potential it is a natural question to ask whether the dispersion of the system, in the sense described above, is preserved. This analysis for the Coulomb case has been started in [7]. The authors showed the validity of the following local smoothing type estimates for solutions u of (1)

$$\left\| |x|^{-\alpha} \left| \mathcal{D}_n - \frac{\nu}{|x|} \right|^{\frac{1}{2}-\alpha} u \right\|_{L_t^2 L_x^2} \leq C \|u_0\|_{L^2} \quad (5)$$

where $\frac{1}{2} < \alpha < \sqrt{k_n^2 - \nu^2} + \frac{1}{2}$ and $k_n = \frac{n-1}{2}$. However, the case $\alpha = \frac{1}{2}$ is excluded, therefore it doesn't allow to deduce Strichartz estimates using the standard Duhamel formulation and the combination of it with the Strichartz estimates for the free flow. Then, in [8], the authors proved asymptotic estimates for the generalized eigenfunctions of the Dirac-Coulomb operator on $\mathbb{R}^3$ and, as an application, some generalized Strichartz estimates for the solutions of (1) on $\mathbb{R}^3$.

The methods used to obtain these results are inspired by the ones developed in [3] and [19] to investigate dispersive properties of the wave and Schrödinger equations with inverse-square potential. The Coulomb and the inverse-square potentials leave, respectively, the massless Dirac and the wave/Schrödinger equations invariant under their natural scaling, that in the massless Dirac case is $u_\lambda(t, x) = u(\lambda^{-1}t, \lambda^{-1}x)$, $\lambda > 0$. This kind of invariance prevents, typically, the use of perturbative methods in the study of the effect of these potentials, requiring the development of new tools. Moreover, the Coulomb-type potentials seem to appear as a natural threshold for the validity of global-in-time Strichartz estimates, in their decays at infinity. Indeed, if we look at the literature concerning Strichartz estimates for the Dirac equation perturbed with electromagnetic potentials we find two types of results; in [11] the authors consider the equation $i\partial_t u + \mathcal{D}u + V(x)u = 0$ on $\mathbb{R}^3$, where $V(x) = V(x)^*$ is a 4×4 complex valued matrix, decaying (slightly) faster at infinity with respect to the Coulomb potential, precisely such that

$$|V(x)| \leq \frac{\delta}{|x|(1 + |\log|x||)^\sigma}, \quad \sigma > 1.$$

They showed that the dispersion of the system is preserved, i.e. the solution u enjoys the same Strichartz estimates, endpoint excluded, as the free one. The same result can be also extended in the 2-dimensional setting. Other results in this direction include e.g. [2], [12]. Instead, if one considers potential decaying slower at infinity, it is possible to construct potential such that the associated system is no more dispersive (in the sense described above). More precisely, in [1] the authors consider the magnetic Dirac equation $i\partial_t u + \mathcal{D}_A u = 0$, defining the vector field A as

$$A(x) = |x|^{-\delta} Mx, \quad 1 < \delta < 2, \quad M = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, the solutions u of the associated Cauchy problems do not satisfy the Strichartz estimates (4) for any admissible couple.

The aim of this work is two-folded; firstly, we continue the analysis of the dispersion of (1). We extend, compared to the result in [8], the set of admissible couples for the validity of generalized Strichartz estimates in 3D, we prove similar estimate for the 2D

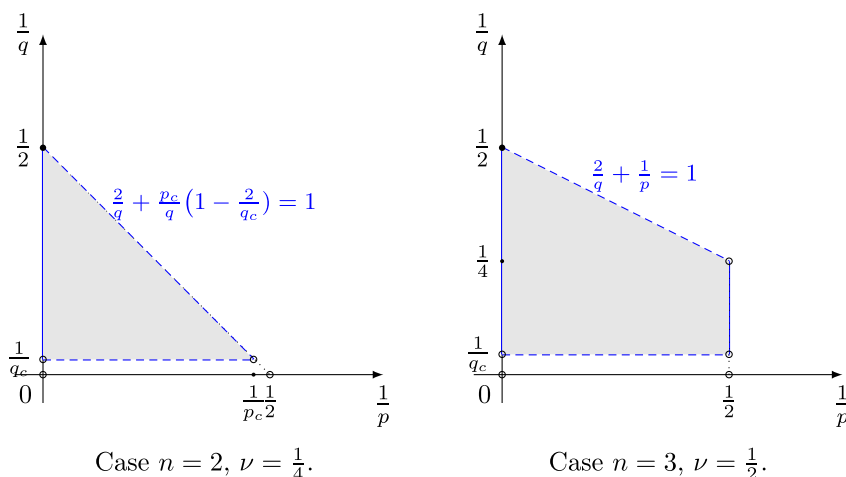


Fig. 1. Generalized Strichartz estimates. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

system and we also provide new local smoothing estimates. Then, we apply the obtained results to the study of local well-posedness of nonlinear systems. Before we state the results we introduce the following notations that will be used throughout the paper.

Notations. We denote with $\mathcal{D}_{n,\nu}$ the Dirac-Coulomb operator acting on $\mathbb{R}^n$, that is $\mathcal{D}_{n,\nu} := \mathcal{D}_n - \frac{\nu}{|x|}$ for $n = 2, 3$ and, with an abuse of notation, we will omit the index n when will be clear from the context.

We denote with $\dot{H}^s$, $s \in \mathbb{R}$, the standard homogeneous Sobolev spaces with the norm $\|u\|_{\dot{H}^s} = \| |D|^s u \|_{L^2}$ where $|D| = \sqrt{-\Delta}$. Instead we use $\dot{H}_{\mathcal{D}_\nu}^s$ to denote the homogeneous Sobolev spaces induced by the action of the Dirac-Coulomb operator, i.e., with norm $\|u\|_{\dot{H}_{\mathcal{D}_\nu}^s} = \| |\mathcal{D}_\nu|^s u \|_{L^2}$.

We say that $u \in L^2(\mathbb{R}^n; \mathbb{C}^N)$ is *Dirac-radial* if it coincides with his projection on the first partial wave subspaces. Then, we call $u \in L^2(\mathbb{R}^n; \mathbb{C}^N)$ *Dirac-non radial* if it is orthogonal to Dirac-radial functions. We postpone to Subsection 2.1 (Remark 4) the precise definitions.

With an abuse of notation, we use the term function to refer to both scalars and vector-valued functions. Their nature will be clear from the context.

Our first main result concerns Strichartz estimates with an additional angular regularity for solutions of (1), as explained above. As we will see we will have some technical conditions that will force us to slightly restrict the set of admissible ν . Moreover, respect to the free case, we obtain a smaller set of admissibility for the indexes p, q (as shown in Fig. 1). However we can recover the classical range (3) in two cases: if $\nu = 0$ ² or, for all

² except for the segment corresponding to $q = +\infty$.

$\nu \in [-\frac{n-1}{2}, \frac{n-1}{2}]$, if the initial datum is Dirac-non radial.³ We separate the cases $n = 2$ and $n = 3$ in order to lighten the notations. We have the following

Theorem 1 (Strichartz estimates 2D). *Let $|\nu| < \frac{\sqrt{2}}{3}$ and (p, q) such that*

$$p_c < p \leq +\infty, \quad 2 \leq q < q_c, \quad \frac{2}{q} + \frac{p_c}{p} \left(1 - \frac{2}{q_c}\right) < 1 \text{ or } (p, q) = (\infty, 2), \quad (6)$$

where $q_c = \frac{2}{\frac{1}{2} - \gamma_{\frac{1}{2}}}$, $p_c = \frac{\gamma_{\frac{1}{2}} + \frac{1}{2}}{\gamma_{\frac{1}{2}}}$ and $\gamma_{\frac{1}{2}} = \sqrt{\frac{1}{4} - \nu^2}$. Then, there exists a constant $C > 0$ such that for any $u_0 \in \dot{H}_{\mathcal{D}_\nu}^s(\mathbb{R}^2; \mathbb{C}^2)$ the following Strichartz estimates hold

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L_t^p L_{rdr}^q L_\theta^2} \leq C \|u_0\|_{\dot{H}_{\mathcal{D}_\nu}^s} \quad (7)$$

provided $s = 1 - \frac{1}{p} - \frac{2}{q}$.

Moreover, if u_0 is Dirac-non radial, then the Strichartz estimates hold for all $|\nu| \leq \frac{1}{2}$ and (p, q) satisfying the admissibility condition (6) with $(p_c, q_c) = (2, +\infty)$, $q = q_c$ included.

Theorem 2 (Strichartz estimates 3D). *Let $|\nu| < \frac{\sqrt{15}}{4}$ and (p, q) such that*

$$2 \leq p \leq +\infty, \quad 2 \leq q < q_c, \quad \frac{2}{q} + \frac{1}{p} < 1 \text{ or } (p, q) = (\infty, 2), \quad (8)$$

where $q_c = \frac{3}{1 - \sqrt{1 - \nu^2}}$. Then, there exists a constant $C > 0$ such that for any $u_0 \in \dot{H}_{\mathcal{D}_\nu}^s(\mathbb{R}^3; \mathbb{C}^4)$, the following Strichartz estimates hold

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L_t^p L_{r^2dr}^q L_\theta^2} \leq C \|u_0\|_{\dot{H}_{\mathcal{D}_\nu}^s} \quad (9)$$

provided $s = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}$.

Moreover, if u_0 is Dirac-non radial, then the Strichartz estimates hold for all $|\nu| \leq 1$ and (p, q) satisfying the admissibility condition (8) with $q = q_c = +\infty$, included.

We describe in Fig. 1 the admissible range (region in gray) in the cases $n = 2$, $\nu = \frac{1}{4}$ and $n = 3$, $\nu = \frac{1}{2}$. Notice that if $\nu = 0$ we can extend the region up to the segments $0\bar{P}$ where $P = (\frac{1}{2}, 0)$, reaching the classical range.

Remark 1. Observe that in the 2D case the norm in the R. H. S. is the one induced by the Dirac-Coulomb operator and it is not, in general, equivalent to the standard Sobolev norm. A more detailed discussion is to be found in Subsection 2.4.

³ The reason behind this additional restriction on the indexes will appear clearly in the proofs of Theorems 1 and 2; it is due to the behavior near the origin of the “first” generalized eigenfunctions.

Remark 2. We notice that from the Strichartz estimates (7), (9) it is possible to deduce standard Strichartz estimates with an additional loss of angular derivatives on u_0 . The idea is to combine the estimates above with Sobolev embeddings on the sphere of dimension $n - 1$. We denote with Λ_θ^s the angular derivative operator, which is defined in terms of the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$

$$\Lambda_\theta^s = (1 - \Delta_{\mathbb{S}^{n-1}})^{\frac{s}{2}}.$$

This operator does not commute with the Dirac operator $\mathcal{D}_n$. However, it has been observed in [5] (see formula (2.45))⁴ that it is possible to define a modified operator $\tilde{\Lambda}_\theta^s$ commuting with $\mathcal{D}_n$ and such that

$$\|\tilde{\Lambda}_\theta^s f\|_{L_\theta^2} \simeq \|\Lambda_\theta^s f\|_{L_\theta^2}.$$

Then, by Sobolev embeddings $H_\theta^\sigma(\mathbb{S}^{n-1}) \hookrightarrow L_\theta^q(\mathbb{S}^{n-1})$, we get

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L_t^p L_x^q} \lesssim \|\Lambda_\theta^\sigma e^{it\mathcal{D}} u_0\|_{L_t^p L_{r^{n-1}dr}^q} \simeq \|e^{it\mathcal{D}} \tilde{\Lambda}_\theta^\sigma u_0\|_{L_t^p L_{r^{n-1}dr}^q} \lesssim \|\tilde{\Lambda}_\theta^\sigma u_0\|_{\dot{H}_s}$$

where $\sigma = \frac{n-1}{2} - \frac{n-1}{q}$, $0 < s < \frac{n-1}{2}$ and $\nu, (p, q)$ as in Theorems 1, 2.

The strategy developed in order to prove the above results is a refinement of the one in [8]; let us briefly describe it. Firstly, we exploit the radial structure of the Dirac-Coulomb operator, using the partial wave decomposition to reduce the Dirac-Coulomb operator to a differential operator acting only on the radial component; then the “relativistic” Hankel transform (see Section 2.2 for the definition) allows us to reduce the radial operator to a multiplicative and diagonal one. This gives us an explicit representation for the solutions of (1) (see (24)). Then, we use the pointwise estimates of the generalized eigenfunctions to estimate the $L_t^p(\mathbb{R})L_r^q([R, 2R])$ -norm of the solution, $R > 0$, on a fixed angular level for frequency localized initial data. Finally we conclude using the orthogonality of the partial waves, scaling argument and dyadic decompositions.

With the same tools, we can also show the validity of a new smoothing estimate for solutions of (1).

Theorem 3. *Let $u_0 \in L^2(\mathbb{R}^n; \mathbb{C}^N)$. Then the following estimate holds*

$$\sup_{R>0} R^{-\frac{1}{2}} \|e^{it\mathcal{D}_\nu} u_0\|_{L_t^2 L_{|x|\leq R}^2} \lesssim \|u_0\|_{L_x^2}. \quad (10)$$

Remark 3. We observe that the estimate in the Morrey spaces $L^{1,2}(\mathbb{R}^n)$ can be viewed as a limiting case of the local smoothing estimates (5) as $\alpha \rightarrow \frac{1}{2}$. We mention also that the

⁴ In [5] the authors work in the 3D setting. However, one can use the same definition to build a modified operator, with the same properties, also in the 2D setting.

same estimate has been proved in [2] for solutions of small magnetic Dirac equations with completely different techniques and in [6] for the Dirac equation in the Aharonov-Bohm magnetic field.

As mentioned above, Strichartz estimates can be used in the study of the dynamics of dispersive nonlinear systems. Therefore, as an application of Theorem 2 for Dirac-radial functions, we discuss a local well-posedness result for the following 3D nonlinear system

$$\begin{cases} i\partial_t u + \mathcal{D}_\nu u = N(u), \\ u(x, 0) = u_0(x), \end{cases} \quad (11)$$

where $|\nu| < \frac{\sqrt{3}}{2}$, u_0 is Dirac-radial and the nonlinearity is of the form

$$N(u) = (\omega * \langle \beta u, u \rangle) u, \quad \beta = \begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & -\mathbb{1}_2 \end{pmatrix} \quad (12)$$

with ω radially symmetric, i.e., $\omega(x) = \omega(|x|)$, $\omega \in L^p(\mathbb{R}^3)$ for some $p > 1$.

We have the following

Theorem 4. *Let ω be a radial function, $\omega \in L^p(\mathbb{R}^3)$, $p \in [\frac{3}{2}, +\infty]$. Let also $u_0 \in \dot{H}^s(\mathbb{R}^3)$, $s = \frac{3}{2p}$, be Dirac-radial. Then there exists a positive time $T = T(\|u_0\|_{\dot{H}^s}, \|\omega\|_{L^p})$ such that (11) has a unique solution $u \in C([0, T]; \dot{H}^s(\mathbb{R}^3))$.*

Theorem 5. *Let ω be a radial function, $\omega \in L^p(\mathbb{R}^3)$, $p \in (\frac{3}{1+2\sqrt{1-\nu^2}}, +\infty)$. Let also $u_0 \in \dot{H}^s(\mathbb{R}^3)$, $s = \frac{1}{p}$, be Dirac-radial. Then there exists a positive time $T = T(\|u_0\|_{\dot{H}^s}, \|\omega\|_{L^p})$ such that (11) has a unique solution $u \in C([0, T]; \dot{H}^s(\mathbb{R}^3)) \cap L^{2p}([0, T]; L^{2p'}(\mathbb{R}^3))$, where p' is the conjugate exponent of p .*

The choice of such initial datum is twofold motivated. First, we notice that (see Proposition 5) if u_0 is Dirac-radial then the Strichartz estimates hold in the classical way. Second, the Dirac-radiality of the solution is preserved by the nonlinearity (as observed in Remark 9). Then, the proofs of Theorems 4 and 5 will be based on standard fixed-point arguments on suitable complete metric spaces (see [10], [18] and references therein for related results). Observe that in the first one we do not use any kind of Strichartz estimates; they come into play if we want to require less integrability for ω . However, we shall remark that, in both cases, we strongly exploit the equivalence between $\dot{H}^s$ and $\dot{H}_{\mathcal{D}_\nu}^s$ norms, which holds for every $s \in [0, 1]$ if $|\nu| < \frac{\sqrt{3}}{2}$. The failing of this equivalence in the general 2D setting (see Subsection 2.4), prevents us from extending the result on $\mathbb{R}^2$. To conclude, notice that the conditions on ω are satisfied by the Yukawa potential $V_b(x) = c \frac{e^{-b|x|}}{|x|} \in L^p(\mathbb{R}^3)$ for all $p < 3$, $b > 0$ and by $\omega(x) = \langle x \rangle^{-\alpha} \in L^p(\mathbb{R}^3)$ for all $p > \frac{3}{\alpha}$, $\alpha > 0$. It would be interesting to consider the case $\omega = \delta_0$, in order to recover the standard cubic nonlinearity $N(u) = \langle \beta u, u \rangle u$. However, one expects to obtain local

well-posedness results in $H^s(\mathbb{R}^3)$ in the subcritical case $s > s_c$. The critical exponent s_c is given by the homogeneity of the Cauchy problem and it can be obtained by scaling arguments. In this case $s_c = 1$. However, for $s > 1$ we don't have anymore the equivalence between $\dot{H}^s$ and $\dot{H}_{\mathcal{D}_\nu}^s$ norms. This would prevent the use of standard tools developed in Sobolev spaces, e.g. Sobolev embeddings, and would require additional work to adapt them in the spaces obtained by the action of the Dirac-Coulomb operator. This would be the object of future works.

The paper is divided into 4 Sections. In Section 2 we introduce the necessary preliminaries: the partial wave decomposition, the “relativistic” Hankel transform and we recall the estimates for the generalized eigenfunction in 3D, deducing the ones in 2D. We also discuss the relationship between the standard Sobolev norms and the ones induced by the Dirac-Coulomb operator. Section 3 is devoted to the proofs of the a priori estimates, i.e., Theorems 1, 2 and 3. Lastly, in Section 4 we prove the nonlinear applications, i.e. Theorems 4 and 5.

2. Preliminaries

In this section we collect some tools that we will use in the proof of the results presented above.

2.1. Partial wave decomposition

A crucial aspect of the Dirac-Coulomb operator is that it can be seen as a radial operator with respect to some suitable decomposition, both in dimension 2 and 3. We recall these decompositions, referring to [23] for all the details.

We use spherical coordinates to write

$$L^2(\mathbb{R}^n; \mathbb{C}^N) \cong L^2((0, \infty), r^{n-1} dr) \otimes L^2(\mathbb{S}^{n-1}; \mathbb{C}^N).$$

Then, we use the partial wave decompositions to define the isomorphisms

$$\begin{aligned} L^2(\mathbb{S}^1; \mathbb{C}^2) &\cong \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} \mathfrak{h}_k^2, \\ L^2(\mathbb{S}^2; \mathbb{C}^4) &\cong \bigoplus_{k \in \mathbb{Z}^*} \bigoplus_{m_k = -|k| + \frac{1}{2}}^{|k| - \frac{1}{2}} \mathfrak{h}_{k, m_k}^3 \end{aligned}$$

where each subspace $\mathfrak{h}^n$ is two-dimensional and it is left invariant under the action of the Dirac-Coulomb operator. The orthonormal basis of each $\mathfrak{h}_k^2$, $\{\Xi_k^+(\theta), \Xi_k^-(\theta)\}$ is expressed in terms of the “classic Fourier basis”, that is

$$\Xi_k^+(\theta) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} e^{i(k-1/2)\theta} \\ 0 \end{pmatrix}, \quad \Xi_k^-(\theta) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 0 \\ e^{i(k+1/2)\theta} \end{pmatrix}, \quad (13)$$

analogously, the orthonormal basis of each $\mathfrak{h}_{k,m_k}^3$, $\{\Xi_{k,m_k}^+(\theta_1, \theta_2), \Xi_{k,m_k}^-(\theta_1, \theta_2)\}$ is expressed in terms of the standard spherical harmonics $Y_l^m(\theta_1, \theta_2)$, that is

$$\Xi_{k,m_k}^+(\theta_1, \theta_2) = \begin{pmatrix} i\Omega_k^{m_k} \\ 0_2 \end{pmatrix}, \quad \Xi_{k,m_k}^- = \begin{pmatrix} 0_2 \\ \Omega_{-k}^{m_k} \end{pmatrix} \quad (14)$$

where

$$\Omega_{k,m_k} = \frac{1}{\sqrt{|2k+1|}} \begin{pmatrix} \sqrt{|k-m_k+\frac{1}{2}|} Y_{|k+\frac{1}{2}|-\frac{1}{2}}^{m_k-\frac{1}{2}} \\ \text{sgn}(-k) \sqrt{|k+m_k+\frac{1}{2}|} Y_{|k+\frac{1}{2}|-\frac{1}{2}}^{m_k+\frac{1}{2}} \end{pmatrix}.$$

We thus have the unitary isomorphisms

$$\begin{aligned} L^2(\mathbb{R}^2; \mathbb{C}^2) &\cong \bigoplus_k L^2((0, \infty), r dr) \otimes \mathfrak{h}_k^2 \\ L^2(\mathbb{R}^3; \mathbb{C}^4) &\cong \bigoplus_{k,m_k} L^2((0, \infty), r^2 dr) \otimes \mathfrak{h}_{k,m_k}^3 \end{aligned}$$

given, respectively, by the decompositions

$$\begin{aligned} \Phi^2(x) &= \sum_{k \in \mathbb{Z} + \frac{1}{2}} \phi_k^+(r) \Xi_k^+(\theta) + \phi_k^-(r) \Xi_k^-(\theta) \\ \Phi^3(x) &= \sum_{k \in \mathbb{Z}^*} \sum_{m=-|k|+\frac{1}{2}}^{|k|-\frac{1}{2}} \psi_{k,m_k}^+(r) \Xi_{k,m_k}^+(\theta_1, \theta_2) + \psi_{k,m_k}^-(r) \Xi_{k,m_k}^-(\theta_1, \theta_2). \end{aligned} \quad (15)$$

The action of $\mathcal{D}_{n,\nu}$ on each partial wave subspace $L^2((0, \infty), r^{n-1} dr)^2 \otimes \mathfrak{h}^n$ can be represented by the radial matrices

$$d_{\nu,k}^n = \begin{pmatrix} -\frac{\nu}{dr} + \frac{n-1}{2r} + \frac{k}{r} & -(\frac{d}{dr} + \frac{n-1}{2r}) + \frac{k}{r} \\ \frac{d}{dr} + \frac{n-1}{2r} + \frac{k}{r} & -\frac{\nu}{r} \end{pmatrix}, \quad (16)$$

which are well defined on $C_0^\infty((0, \infty), r^{n-1} dr)^2 \subset L^2((0, \infty), r^{n-1} dr)^2$.

Notice that formulas (16) only depend on k , then, in order to give a unified treatment of the two cases $n = 2, 3$, in what follows we will maintain only the dependence on the parameter k for $n = 3$. Moreover, we will omit the dependence on n of the angular part. Thus, if $\varphi \in L^2(\mathbb{R}^n; \mathbb{C}^4)$ we will decompose it as

$$\Phi(x) = \sum_{k \in \mathcal{A}_n} \varphi_k^+(r) \Xi_k^+(\theta) + \varphi_k^-(r) \Xi_k^-(\theta) = \sum_{k \in \mathcal{A}_n} \varphi_k(r) \cdot \Xi_k(\theta) \quad (17)$$

where if $n = 2$, $\mathcal{A}_2 = \mathbb{Z} + \frac{1}{2}$, $\theta \in \mathbb{S}^1$ and the functions Ξ_k are the ones defined in (13), instead if $n = 3$, $\mathcal{A}_3 = \mathbb{Z}^*$, $\theta \in \mathbb{S}^2$ and the functions Ξ_k are as in (14) (omitting

the dependence of m_k). With this decomposition, by Stone's theorem, the propagator is given by

$$e^{it\mathcal{D}_\nu}\Phi(x) = \sum_{k \in \mathcal{A}_n} \varphi_k^+(t, r) \Xi_k^+(\theta) + \varphi_k^-(t, r) \Xi_k^-(\theta)$$

where

$$\begin{pmatrix} \varphi_k^+(t, r) \\ \varphi_k^-(t, r) \end{pmatrix} = e^{itd_{\nu, k}} \begin{pmatrix} \varphi_k^+(r) \\ \varphi_k^-(r) \end{pmatrix}.$$

Remark 4. Observe that the radial functions (meaning a vector of four radial functions) are contained in the firsts eigenspaces (corresponding to $k = \pm \frac{1}{2}$ if $n = 2$ and to $k = \pm 1$ if $n = 3$) but they are not left invariant, in general, by the Dirac operator. In order to consider invariant sets of functions, we call $u \in L^2(\mathbb{R}^n; \mathbb{C}^N)$ *Dirac-radial* if in the decomposition given by (17), for $n = 2$ $u_k^\pm(r) = 0$ for all $|k| > \frac{1}{2}$ and for $n = 3$ $u_{k, m_k}^\pm(r) = 0$ for all $|k| > 1$. On the contrary, $u \in L^2(\mathbb{R}^n; \mathbb{C}^N)$ is *Dirac-non radial* if it is orthogonal to the first partial wave subspaces; more precisely if, in the decomposition given by (15), for $n = 2$, $u_k^\pm(r) = 0$ if $|k| = \frac{1}{2}$ and, for $n = 3$, $u_{k, m_k}^\pm(r) = 0$ if $|k| = 1$.

2.2. Relativistic Hankel transform

Once one has decomposed the Dirac-Coulomb operator in a sum of radial operators, the key idea is to looking for an isometry that transforms each radial differential operator into a multiplication operator. This is the rule of the “relativistic” Hankel transform, which is built with the generalized eigenfunctions $\psi_{k, \varepsilon}$ of $\mathcal{D}_{n, \nu}$. The idea of this construction was borrowed by [3] in which the author considered the Hankel transform. This is built with the Bessel functions that are the generalized eigenfunctions for the radial Schrödinger operator (see e.g. [3], Section 2.1). In this sense the transform we consider can be viewed as a relativistic counterpart of the standard one. For the sake of completeness we recall in this Subsection the definition and its properties, without proofs. We refer to [7] (Section 2.2) for a complete presentation.

Definition 1. Let $\Phi \in L^2((0, \infty), r^{n-1}dr) \otimes \mathfrak{h}_k^n$ for some fixed k and let $\varphi(r) = (\varphi_1(r), \varphi_2(r))$ be the vector of its radial coordinates in decomposition (17).

We define the following integral transform

$$\mathcal{P}_k \varphi(\varepsilon) = \begin{pmatrix} \mathcal{P}_k^+ \varphi(\varepsilon) \\ \mathcal{P}_k^- \varphi(\varepsilon) \end{pmatrix} = \int_0^{+\infty} H_k(\varepsilon r) \cdot \varphi(r) r^{n-1} dr$$

where

$$H_k(\varepsilon r) = \begin{pmatrix} F_{k, \varepsilon}(r) & G_{k, \varepsilon}(r) \\ F_{k, -\varepsilon}(r) & G_{k, -\varepsilon}(r) \end{pmatrix} = \begin{pmatrix} \psi_{k, \varepsilon}(r)^\top \\ \psi_{k, -\varepsilon}(r)^\top \end{pmatrix}.$$

Here

$$\psi_{k,\varepsilon}^n(r) = \begin{pmatrix} F_{k,\varepsilon}^n(r) \\ G_{k,\varepsilon}^n(r) \end{pmatrix}, \quad (18)$$

represents the vector of radial coordinates of the generalized eigenfunctions. That is, in the notation of (17), $\Psi^n(x) = \sum_{k \in \mathcal{A}_n} \psi_{k,\varepsilon}^n(r) \cdot \Xi_k(\theta)$ “solves”

$$\left(\mathcal{D}_n - \frac{\nu}{|x|} \right) \Psi^n = \varepsilon \Psi^n, \quad \varepsilon > 0.$$

For the sake of completeness we recall the formulas for $F_{k,\varepsilon}^n, G_{k,\varepsilon}^n$, that are

$$\begin{aligned} F_{k,\varepsilon}^n(r) &= \frac{\sqrt{2}|\Gamma(\gamma+1+i\nu)|}{\Gamma(2\gamma+1)} e^{\frac{\pi\nu}{2}} (2\varepsilon r)^{\gamma-\frac{n-1}{2}} \operatorname{Re}\{e^{i(\varepsilon r+\xi)} {}_1F_1(\gamma-i\nu, 2\gamma+1, -2i\varepsilon r)\}, \\ G_{k,\varepsilon}^n(r) &= \frac{i\sqrt{2}|\Gamma(\gamma+1+i\nu)|}{\Gamma(2\gamma+1)} e^{\frac{\pi\nu}{2}} (2\varepsilon r)^{\gamma-\frac{n-1}{2}} \operatorname{Im}\{e^{i(\varepsilon r+\xi)} {}_1F_1(\gamma-i\nu, 2\gamma+1, -2i\varepsilon r)\} \end{aligned} \quad (19)$$

where ${}_1F_1(a, b, z)$ are confluent hypergeometric functions, $\gamma = \sqrt{k^2 - \nu^2}$ and $e^{-2i\xi} = \frac{\gamma-i\nu}{k}$ is a phase shift.

Remark 5. We recall that the spectrum of $\mathcal{D}_n$ is purely absolutely continuous and it is the whole real line $\mathbb{R}$. The formulas in (19) give the generalized eigenstates of the continuous spectrum corresponding to the positive energies $\varepsilon > 0$. The ones corresponding to negative energies can be obtained using a charge conjugation argument. Then, one gets that

$$\psi_{k,-\varepsilon}^n(r) = \begin{pmatrix} F_{k,-\varepsilon}^n(r) \\ G_{k,-\varepsilon}^n(r) \end{pmatrix} = \begin{pmatrix} \tilde{F}_{-k,\varepsilon}^n(r) \\ \tilde{G}_{-k,\varepsilon}^n(r) \end{pmatrix}$$

where $\tilde{F}^n, \tilde{G}^n$ are the functions obtained by (19) by changing the sign of ν .

We also want to underlain that the homogeneity of the generalized eigenfunctions $\psi_{k,\varepsilon}^n(r)$ with respect to ε and r is the same, in particular we have that $\psi_{k,\varepsilon}^n(r) = \psi_{k,1}^n(\varepsilon r)$. The lack of this homogeneity in the massive case prevents us from extending the results to that case.

Proposition 1. *The following properties hold:*

- i) $\mathcal{P}_k$ is an L^2 -isometry,
- ii) $(\mathcal{P}_k d_{\nu,k}\varphi)(\varepsilon) = \sigma_3 \varepsilon (\mathcal{P}_k \varphi)(\varepsilon)$, $\varepsilon > 0$,

iii) The inverse transform of $\mathcal{P}_k$ is given by

$$\mathcal{P}_k^{-1}\varphi(r) = \int_0^{+\infty} H_k^*(\varepsilon r) \cdot \varphi(\varepsilon) \varepsilon^{n-1} d\varepsilon \quad (20)$$

$$\text{where } H_k^*(\varepsilon r) = \begin{pmatrix} F_{k,\varepsilon}(r) & F_{k,-\varepsilon}(r) \\ G_{k,\varepsilon}(r) & G_{k,-\varepsilon}(r) \end{pmatrix}.$$

2.3. Asymptotic estimates of the generalized eigenfunctions

Other fundamental tools that we will use are some asymptotic estimates of the generalized eigenfunctions ψ_k^n : these will allow us to obtain an estimate on the L^q norm of such functions on fixed interval, uniformly with respect to the parameter k . Notice that this is another substantial difference respect to the cases of the wave equation with an inverse square potential ([19]) and the Dirac equation with Aharonov-Bohm magnetic potential (see [6]). In these cases, indeed, the generalized eigenfunctions are basically Bessel functions for which asymptotic estimates are well known (see Remark 6 below).

Theorem 6. Given $\nu \in [-\frac{n-1}{2}, \frac{n-1}{2}]$ and $k \in \mathcal{A}_n$, let $\sqrt{k^2 - \nu^2}$ and consider the generalized eigenfunctions ψ_k^n of $\mathcal{D}_{\nu,n}$ with eigenvalue $\varepsilon = 1$ given by formulas (18) and (19). Then there exist positive constants C, D independent of k, ν such that the following pointwise estimate holds for all $\rho \in \mathbb{R} \setminus \{0\}$:

$$|\psi_k^n(\rho)| \leq C \begin{cases} (\min\{\frac{|\rho|}{2}, 1\})^{\gamma - \frac{n-1}{2}} e^{-D|k|}, & 0 < |\rho| \leq \max\{\frac{|k|}{2}, 2\}, \\ |k|^{-\frac{2n-3}{4}} (||k| - |\rho|| + |k|^{\frac{1}{3}})^{-\frac{1}{4}}, & \frac{|k|}{2} \leq |\rho| \leq 2|k|, \\ |\rho|^{-\frac{n-1}{2}}, & |\rho| \geq 2|k|. \end{cases} \quad (21)$$

Moreover, with a possibly larger C and a smaller $D > 0$, both independent of k, ν , the following estimate holds

$$|(\psi_k^n)'(\rho)| \leq C \begin{cases} (\min\{\frac{|\rho|}{2}, 1\})^{\gamma - \frac{n+1}{2}} e^{-D|k|}, & 0 < |\rho| \leq \max\{\frac{|k|}{2}, 2\}, \\ |k|^{-\frac{2n-3}{4}} (||k| - |\rho|| + |k|^{\frac{1}{3}})^{-\frac{1}{4}}, & \frac{|k|}{2} \leq |\rho| \leq 2|k|, \\ |\rho|^{-\frac{n-1}{2}}, & |\rho| \geq 2|k|, \end{cases} \quad (22)$$

Proof. The proof for the case $n = 3$ is given in [8], Thm. 1.1. For the case $n = 2$ we observe that, from formulas (19), we have a relation between the generalized eigenfunctions in dimensions 2 and 3; they differ from a factor $(2\varepsilon r)^{\frac{1}{2}}$ and from the range where the parameter k lives. Then, it is possible to adapt the proof in [8], with minor modifications, to deduce the estimates in dimension 2. $\square$

Remark 6. Notice that the obtained estimates are the same, for large value of k , that holds for the generalized eigenfunctions of the wave equation with inverse square potentials; in fact, in that case the functions are given by

$$\tilde{\psi}_\rho^n(r) = (\rho r)^{-\frac{n-2}{2}} J_{\nu(k)}(\rho r)$$

where J_ν is the Bessel function of order ν , $\nu(k) = \sqrt{\mu(k)^2 + a}$, $\mu(k) = \frac{n-2}{2} + k$, $k \in \mathbb{N}$ and $a > -\frac{(n-2)^2}{4}$. The Bessel functions J_ν , for $\nu \geq 2$, enjoy the pointwise bounds

$$|J_\nu(\rho)| \leq C \begin{cases} e^{-d\nu}, & 0 < \rho < \frac{\nu}{2}, \\ \nu^{-\frac{1}{4}}(|\rho - \nu| + \nu^{\frac{1}{3}})^{-\frac{1}{4}}, & \frac{\nu}{2} < \rho < 2\nu, \\ \rho^{-\frac{1}{2}}, & 2\nu < \rho < \infty, \end{cases}$$

for some C and d non depending on ν (see, e.g., [22]).

2.4. Equivalence of norms

As stated in the Introduction, in order to prove the local well-posedness of the non-linear system we exploit the equivalence between the Sobolev norms H^s and the ones induced by the action of the Dirac-Coulomb operator $\mathcal{D}_{n,\nu}$. In this Subsection we recall it and we also add some results in this direction about the 2D setting.

For what concerns the 3D case we have the following

Proposition 2. *Let $|\nu| < \frac{\sqrt{3}}{2}$ and $u \in \text{dom}(\mathcal{D}_{3,\nu})$. Then for every $s \in [0, 1]$ there exist two positive constants C_1, C_2 such that*

$$C_1 \|u\|_{\dot{H}^s(\mathbb{R}^3)} \leq \|u\|_{\dot{H}_{\mathcal{D}_\nu}^s(\mathbb{R}^3)} \leq C_2 \|u\|_{\dot{H}^s(\mathbb{R}^3)}.$$

Proof. We start observing that if $|\nu| < \frac{\sqrt{3}}{2}$ then the domain of the self-adjoint extension of $\mathcal{D}_{3,\nu}$ is $H^1(\mathbb{R}^3; \mathbb{C}^4)$ (see [23] Section 4.3.3). Moreover, for ν in this range, $\min\{1, \frac{1}{2} + \sqrt{1 - \nu^2}\} = 1$. Then, the statement comes directly from the following Lemma (it is Corollary 1.8 in [14]).

Lemma 1. *Let $|\nu| \leq 1$. For any $f \in C_c^\infty(\mathbb{R}^3; \mathbb{C}^4)$ we have*

i) *if $s \in [0, \min\{1, \frac{1}{2} + \sqrt{1 - \nu^2}\}]$, then*

$$\| |-\Delta|^{\frac{s}{2}} f \|_{L^2} \lesssim_{\nu,s} \| |\mathcal{D}_\nu|^s f \|_{L^2};$$

ii) *if $s \in [0, 1]$, then*

$$\| |\mathcal{D}_\nu|^s f \|_{L^2} \lesssim_{\nu,s} \| |-\Delta|^{\frac{s}{2}} f \|_{L^2} \quad \square$$

In the 2D case we cannot hope to obtain the same results; firstly, we recall that the domain of the distinguished self-adjoint extension of $\mathcal{D}_{2,\nu}$ is not contained in $H^1(\mathbb{R}^2; \mathbb{C}^2)$ (see [20], Corollary 16), for any $\nu \neq 0$. Secondly, the property *ii*) is proven using the Hardy's inequality, which we know to fail on $\mathbb{R}^2$.

However, the distinguished self-adjoint extension is chosen such that its domain is contained in $H^{\frac{1}{2}}(\mathbb{R}^2; \mathbb{C}^4)$, if $|\nu| < \frac{1}{2}$. This suggests that at least inequality *i*) in Lemma 1 could hold if we take $s \in [0, \frac{1}{2}]$. In fact, we have the following

Proposition 3. *Let $|\nu| < \frac{1}{2}$. For any $u \in \text{dom}(\mathcal{D}_{2,\nu})$ and $s \in [0, \frac{1}{2}]$, we have*

$$\|u\|_{\dot{H}^s} \lesssim_{\nu,s} \| |\mathcal{D}_\nu|^s u \|_{L^2}.$$

Proof. From [20] [Thm 1] we have that for every $\nu \in (-\frac{1}{2}, \frac{1}{2})$ there exists $C_\nu > 0$ such that

$$|\mathcal{D}_\nu| \geq C_\nu \sqrt{-\Delta} \otimes \mathbb{1}_2.$$

Then, by operator monotonicity of the map $f(t) = t^p$, $p \in [0, 1]$ (see [9] [Thm 2.6]), we conclude that

$$|\mathcal{D}_\nu|^p \geq C_\nu^p \sqrt{-\Delta}^p \otimes \mathbb{1}_2,$$

that is the estimate with $2s = p$. $\square$

For what concerns the reverse inequality, we observe that even if the Hardy's inequality fails in 2D, it still holds for non-radial functions (see [3], Prop. 1 p. 8). Then we have the following

Proposition 4. *Let $|\nu| < \frac{1}{2}$. For any $u \in \text{dom}(\mathcal{D}_{2,\nu})$, u Dirac-non radial and $s \in [0, \frac{1}{2}]$ we have*

$$\| |\mathcal{D}_\nu|^s u \|_{L^2} \lesssim_{\nu,s} \| |-\Delta|^{\frac{s}{2}} u \|_{L^2}$$

Proof. We first prove that, if $f \in C_c^\infty(\mathbb{R}^2; \mathbb{C}^2)$ and Dirac-non radial, then the Hardy's inequality holds, i.e.

$$\| |x|^{-1} f \|_{L^2} \leq \| \nabla f \|_{L^2}. \quad (23)$$

We can decompose f as:

$$f(x) = \sum_{k \in \mathbb{Z} + \frac{1}{2}, k \neq \frac{1}{2}} f_k^+(r) \Xi_k^+(\theta) + f_k^-(r) \Xi_k^-(\theta) =: f^+(r, \theta) + f^-(r, \theta),$$

and we recall that $\partial_\theta \Xi_k^\pm(\theta) = (k \mp \frac{1}{2}) \Xi_k^\pm(\theta)$. Then,

$$\int_0^{2\pi} |f^\pm|^2 d\theta = \sum_{k \in \mathbb{Z} + \frac{1}{2}, k \neq \frac{1}{2}} |f_k^\pm|^2 \leq \sum_{k \in \mathbb{Z} + \frac{1}{2}, k \neq \frac{1}{2}} (k \mp \frac{1}{2})^2 |f_k^\pm|^2 = \int_0^{2\pi} |\partial_\theta f^\pm|^2 d\theta$$

and thus

$$\| |x|^{-1} f \|_{L^2}^2 \leq \int_0^\infty \int_0^{2\pi} \frac{1}{r^2} [|\partial_\theta f^+|^2 + |\partial_\theta f^-|^2] d\theta r dr \leq \|\nabla f\|_{L^2}^2.$$

Hence, by the Cauchy-Schwarz inequality and by (23), we have

$$(\mathcal{D}_\nu)^2 \lesssim (-\Delta) \otimes \mathbb{1}_2.$$

Then, by monotonicity,⁵ we get that for any $f \in C_c^\infty(\mathbb{R}^2; \mathbb{C}^2)$, f Dirac-non radial and for any $s \in [0, 1]$

$$\|f\|_{\dot{H}_{\mathcal{D}_\nu}^s(\mathbb{R}^2)} \lesssim_{\nu, s} \|f\|_{\dot{H}^s(\mathbb{R}^2)}.$$

We conclude recalling that if $|\nu| < \frac{1}{2}$ then $\text{dom}(\mathcal{D}_{2, \nu}) \subset H^{\frac{1}{2}}(\mathbb{R}^2; \mathbb{C}^2)$, but not in $H^1(\mathbb{R}^2; \mathbb{C}^2)$ (see [21]). $\square$

3. Proofs of the results

Before going into the proofs of the above mentioned results, we exploit the tools described in Subsections 2.1 and 2.2 in order to work with a “nice” representation of the solution. Let $u_0 \in L^2(\mathbb{R}^n; \mathbb{C}^N)$, by (17) and property *ii*) in Proposition 1, we have

$$\begin{aligned} u_0(x) &= \sum_{k \in \mathcal{A}_n} u_{0,k}(r) \cdot \Xi_k(\theta), \\ e^{it\mathcal{D}_\nu} u_0(x) &= \sum_{k \in \mathcal{A}_n} \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho)](r) \cdot \Xi_k(\theta) \end{aligned} \quad (24)$$

where $x = (r, \theta) \in \mathbb{R}^n$. Moreover, thanks to the L^2 -orthogonal decomposition,

$$\|u_0\|_{L_x^2} = \left(\sum_{k \in \mathcal{A}_n} \|u_{0,k}(r)\|_{L_{r^{n-1}dr}^2}^2 \right)^{\frac{1}{2}} \quad (25)$$

$$\begin{aligned} \|e^{it\mathcal{D}_\nu} u_0\|_{L_t^2 L_x^2}^2 &= \left\| \sum_{k \in \mathcal{A}_n} \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho)](r) \cdot \Xi_k(\theta) \right\|_{L_t^2 L_{r^{n-1}dr}^2 L_\theta^2}^2 \\ &= \sum_{k \in \mathcal{A}_n} \left\| \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho)](r) \right\|_{L_t^2 L_{r^{n-1}dr}^2}^2. \end{aligned} \quad (26)$$

⁵ as in the proof of Proposition 3.

More generally, for $p, q \geq 2$, by Minkowski's inequality $\|\cdot\|_{L^q l^2} \leq \|\cdot\|_{l^2 L^q} \forall q \geq 2$, we get

$$\begin{aligned} \|e^{it\mathcal{D}_\nu} u_0\|_{L_t^p L_{r^{n-1}dr}^q L_\theta^2} &= \left\| \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho)](r) \right\|_{L_t^p L_{r^{n-1}dr}^q L_k^2(\mathcal{A}_n)} \\ &\leq \left(\sum_{k \in \mathcal{A}_n} \left\| \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho)](r) \right\|_{L_t^p L_{r^{n-1}dr}^q}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (27)$$

3.1. Proof of Theorem 3

Thanks to (25) and (26), it suffices to show that, for any fixed $k \in \mathcal{A}_n$,

$$R^{-1} \left\| \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho)](r) \right\|_{L_t^2 L_{r^{n-1}dr}^2([0,R])}^2 \leq C \|u_{0,k}(r)\|_{L_{r^{n-1}dr}^2}^2$$

where C is a positive constant independent of k and R .

Let $g_k(\rho) = \mathcal{P}_k u_{0,k}(\rho)$. Then, from (20) and Plancherel's theorem in t (on each component), we get

$$\begin{aligned} &\left\| \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (g_k(\rho))](r) \right\|_{L_t^2 L_{r^{n-1}dr}^2([0,R])}^2 \\ &= \left\| \mathcal{F}_{\rho\sigma_3 \rightarrow t}^{-1} \{H_k^*(r\rho) \cdot g_k(\rho) \rho^{n-1} \chi_{\mathbb{R}^+}(\rho)\} \right\|_{L_{rdr}^2([0,R]) L_t^2}^2 \\ &\leq \int_0^{+\infty} \int_0^R |H_k^*(r\rho) g_k(\rho) \rho^{n-1}|^2 r^{n-1} dr d\rho \\ &\lesssim \int_0^{+\infty} \int_0^R (|\psi_k^n(r\rho)|^2 + |\psi_k^n(-r\rho)|^2) |g_k(\rho)|^2 \rho^{2(n-1)} r^{n-1} dr d\rho. \end{aligned}$$

We thus need to estimate

$$\int_0^{+\infty} \left(\int_0^R |\psi_k^n(r\rho)|^2 r^{n-1} dr \right) |g_k(\rho)|^2 \rho^{2(n-1)} d\rho. \quad (28)$$

Lemma 2. Let ψ_k^n be the generalized eigenfunction of $\mathcal{D}_{n,\nu}$ with eigenvalue 1. Then, there exists a positive constant C depending on n but not on k and ν , such that

$$\frac{1}{R} \int_0^R |\psi_k^n(r)|^2 r^{n-1} dr \leq C. \quad (29)$$

Remark 7. We observe that the same estimate holds for $\psi_k^n(-r)$, that is the generalized eigenfunction with eigenvalue -1 . In fact, as stated in Remark 5, such eigenfunctions

are obtained by ψ_k^n by changing the sign of k and ν . However, all the estimates in the proof will be independent by the sign of ν and k .

Proof. Let $R > 0$. We use (21) to estimate the integral in (29); we need to consider three different cases:

i) if $0 < R < \frac{|k|}{2}$, then we have

$$\begin{aligned} \frac{1}{R} \int_0^R |\psi_k^n(r)|^2 r^{n-1} dr &\lesssim \frac{1}{R} \int_0^R (\min\{\frac{r}{2}, 1\})^{2\gamma-(n-1)} e^{-2D|k|} r^{n-1} dr \\ &\leq \frac{1}{R} \int_0^R (\min\{\frac{r}{2}, 1\})^{-(n-1)} e^{-2D|k|} r^{n-1} dr, \end{aligned} \quad (30)$$

if $\min\{\frac{r}{2}, 1\} = 1$ then

$$(30) \lesssim \frac{1}{R} R^n e^{-2DR} \leq C;$$

otherwise, if $\frac{r}{2} \leq 1$,

$$(30) \leq \frac{1}{R} e^{-2D|k|} 2^{n-1} R \leq C;$$

ii) if $\frac{|k|}{2} < R < 2|k|$, then we split the interval $[0, R] = [0, |k|/2] + [|k|/2, R] = I_1 + I_2$:

$$\begin{aligned} \frac{1}{R} \int_{I_1} |\psi_k^n(r)|^2 r^{n-1} dr &\lesssim \frac{1}{R} e^{-2D|k|} \frac{|k|^n}{2} \leq C, \\ \frac{1}{R} \int_{I_2} |\psi_k^n(r)|^2 r^{n-1} dr &\lesssim R^{-1} \int_{I_2} |k|^{-\frac{2n-3}{2}} (||k| - r| + |k|^{\frac{1}{3}})^{-\frac{1}{2}} r^{n-1} dr \\ &\lesssim |k|^{-\frac{1}{2}} \int_{\frac{|k|}{2}}^{2|k|} ||k| - r|^{-\frac{1}{2}} dr \leq C; \end{aligned}$$

iii) if $R > 2|k|$, then $[0, R] = [0, |k|/2] + [|k|/2, 2|k|] + [2|k|, R] = I_1 + I_2 + I_3$: estimates over I_1, I_2 as before,

$$R^{-1} \int_{I_3} |\psi_k^n(r)|^2 r^{n-1} dr \lesssim R^{-1} \int_{2|k|}^R dr \leq C. \quad \square$$

Then, coming back to (28), we have, after a change of variable $\xi = r\rho$,

$$\begin{aligned} (28) &= \int_0^{+\infty} \left(\int_0^{R\rho} |\psi_k(\xi)|^2 \xi d\xi \right) |g_k(\rho)|^2 \rho^{n-2} d\rho \\ &\lesssim R \int_0^{+\infty} |g_k(\rho)|^2 \rho^{n-1} d\rho = CR \|\mathcal{P}_k u_{0,k}\|_{L^2_{\rho d\rho}}^2, \end{aligned}$$

then the claim since $\mathcal{P}_k$ is an L^2 -isometry.

3.2. Proofs of Strichartz estimates

In order to lighten the notation, in the following we will treat separately the cases $n = 2, 3$ and we will omit the dependence on n of the generalized eigenfunctions. We start with the 2-dimensional case.

Proof of Theorem 1. The proof is divided in four steps. The key step is the second one in which we prove the following Lemma. This provides us Strichartz estimates on a fixed radial interval for a frequency localized solution on a fixed angular level. Then the linear estimates in Theorem 1 will follow by scaling arguments and interpolation with the standard $L_t^\infty L_x^2$ -estimate.

Lemma 3. *Let $(p, q) \in (2, \infty] \times [2, +\infty)$ and $k \in \mathbb{Z} + \frac{1}{2}$. Let $I = [\frac{1}{2}, 1]$ and $g_k \in L^2_{\rho d\rho}((0, +\infty))$ such that $\text{supp}(g_k) \subset I$. Then*

$$\left\| \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} g_k(\rho)](r) \right\|_{L_t^p L_{rdr}^q([R, 2R])} \leq C \|g_k(\rho)\|_{L^2_{\rho d\rho}(I)} \times \begin{cases} R^{\gamma - \frac{1}{2} + \frac{2}{q}}, & R \leq 1 \\ R^{\frac{1}{q} + \delta(p)(1 - \frac{2}{q})}, & R \geq 2, \end{cases}$$

where

$$\delta(p) = \begin{cases} \frac{1}{p} - \frac{1}{2} & \text{if } p < 4, \\ \frac{1}{4\varepsilon} - \frac{1}{2} & \text{if } p \geq 4 \end{cases}$$

and the constant C is independent of k, ν , but eventually depends on p and q . Moreover, if $q = +\infty$, for all $p > 2$ we have

$$\left\| \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} g_k(\rho)](r) \right\|_{L_t^p L_{dr}^\infty([R, 2R])} \leq C \|g_k(\rho)\|_{L^2_{\rho d\rho}(I)} \times \begin{cases} R^{\gamma-1}, & R \leq 1 \\ R^{\delta(p)}, & R \geq 2, \end{cases}$$

Step 1: By relying on the pointwise estimates of Theorem 6, we estimate the L^q norms of the generalized eigenfunctions on a fixed interval of length R ;

Lemma 4. Let $k \in \mathbb{Z} + \frac{1}{2}$, $\gamma = \sqrt{k^2 - \nu^2}$, $|\nu| \leq \frac{1}{2}$ and $q \in [2, +\infty]$. Then, the following estimates hold

$$\|\psi_k\|_{L^q([R, 2R])} \leq C \times \begin{cases} R^{\gamma + \frac{1}{q} - \frac{1}{2}}, & \text{if } R \leq 1, \\ R^{\beta(q)}, & \text{if } R \geq 1, \end{cases}$$

and

$$\|\psi'_k\|_{L^q([R, 2R])} \leq C \times \begin{cases} R^{\gamma + \frac{1}{q} - \frac{3}{2}}, & \text{if } R \leq 1, \\ R^{\beta(q)}, & \text{if } R \geq 1, \end{cases}$$

where

$$\beta(q) = \begin{cases} \frac{1}{q} - \frac{1}{2} & \text{if } q \in [2, 4), \\ \frac{1}{q} - \frac{1}{3} & \text{if } q \in [4, +\infty] \end{cases}$$

and all the constants are independent of γ, k , but eventually dependent on q .

Proof. We split the proof in two cases:

i) $R \leq 1$:

Let $q \in [2, +\infty)$; observing that $R \leq r \leq 2R$ implies $r \leq 2$, we have

$$\int_R^{2R} |\psi_k(r)|^q dr \lesssim \int_R^{2R} (\min\{\frac{r}{2}, 1\})^{q\gamma - \frac{q}{2}} e^{-Dq|k|} dr \lesssim \int_R^{2R} r^{q\gamma - \frac{q}{2}} dr \lesssim R^{q\gamma - \frac{q}{2} + 1},$$

and

$$\int_R^{2R} |\psi'_k(r)|^q dr \lesssim \int_R^{2R} (\min\{\frac{r}{2}, 1\})^{q\gamma - \frac{3}{2}q} e^{-Dq|k|} dr \lesssim \int_R^{2R} r^{q\gamma - \frac{3}{2}q} dr \lesssim R^{q\gamma - \frac{3}{2}q + 1}.$$

Now let $q = +\infty$. Then, as before,

$$\sup_{r \in [R, 2R]} |\psi_k(r)| \lesssim \sup_{r \in [R, 2R]} r^{\gamma - \frac{1}{2}} 2^{-\gamma + \frac{1}{2}} \leq 2^{\frac{1}{2}} R^{\gamma - \frac{1}{2}}.$$

In the same way we estimate $\|\psi'\|_{L^\infty}$.

ii) $R \geq 1$:

We write the interval of integration as $[R, 2R] = I_1 + I_2 + I_3$, where

$$I_1 = [R, 2R] \cap [0, \frac{|k|}{2}], \quad I_2 = [R, 2R] \cap [\frac{|k|}{2}, 2|k|], \quad I_3 = [R, 2R] \cap [2|k|, +\infty)$$

and we estimate each interval separately.

For I_1 we can assume $2R \leq |k|$, otherwise $I_1 = \emptyset$. Let $q \in [2, +\infty)$, then

$$\int_{I_1} |\psi_k(r)|^q dr \lesssim \int_{I_1} (\min\{\frac{r}{2}, 1\})^{q\gamma - \frac{q}{2}} e^{-Dq|k|} dr \leq C_\alpha R^{-\alpha}, \quad \forall \alpha > 0;$$

in fact, if $\min\{\frac{r}{2}, 1\} = 1$, then

$$\int_{I_1} |\psi_k(r)|^q dr \lesssim e^{-Dq|k|} |k|^\alpha |k|^{-\alpha} \leq C_\alpha R^{-\alpha},$$

otherwise $r \leq 2$ and, again

$$\int_{I_1} |\psi_k(r)|^q dr \lesssim \int_R^{2R} r^{-\frac{q}{2}} e^{-Dqr} dr \leq C_\alpha R^{-\alpha};$$

Let now $q = +\infty$, then

$$|\psi_k(r)| \lesssim \begin{cases} e^{-DR} & \text{if } r \geq 2, \\ r^{-\frac{1}{2}} 2^{\frac{1}{2}} e^{-DR} & \text{if } r \leq 2 \end{cases}$$

and we get the claim.

For I_2 we can assume $\frac{|k|}{4} \leq R \leq 2|k|$, otherwise $I_2 = \emptyset$. Let $q \in [2, 4)$, we compute the integral and obtain

$$\begin{aligned} \int_{I_2} |\psi_k(r)|^q dr &\lesssim \int_{\frac{|k|}{2}}^{2|k|} |k|^{-\frac{q}{4}} (|k| - r + |k|^{\frac{1}{3}})^{-\frac{q}{4}} dr \\ &= |k|^{1-\frac{q}{4}} \int_{\frac{1}{2}}^2 (|k|^{\frac{1}{3}} (|k|^{\frac{2}{3}} |1 - \tau| + 1))^{-\frac{q}{4}} d\tau \\ &= |k|^{1-\frac{q}{4}-\frac{q}{12}} \int_{-\frac{1}{2}}^2 (|k|^{\frac{2}{3}} |y| + 1)^{-\frac{q}{4}} dy = |k|^{1-\frac{q}{4}-\frac{q}{12}-\frac{2}{3}} \int_{-\frac{|k|^{\frac{2}{3}}}{2}}^{\frac{|k|^{\frac{2}{3}}}{2}} (1 + |x|)^{-\frac{q}{4}} dx \\ &= |k|^{\frac{1}{3}-\frac{q}{3}} \frac{4}{4-q} \left[\left(1 + \frac{|k|^{\frac{2}{3}}}{2}\right)^{1-\frac{q}{4}} + \left(1 + |k|^{\frac{2}{3}}\right)^{1-\frac{q}{4}} - 2 \right] \\ &\lesssim |k|^{\frac{1}{3}-\frac{q}{3}+\frac{2}{3}-\frac{q}{6}} = |k|^{1-\frac{q}{2}} \simeq R^{1-\frac{q}{2}}. \end{aligned}$$

For $q \in [4, +\infty)$ we estimate the norm as

$$\int_{I_2} |\psi_k(r)|^q dr \lesssim \int_{I_2} |k|^{-\frac{3}{4}q} (|k| - r + |k|^{\frac{1}{3}})^{-\frac{q}{4}} r^{\frac{q}{2}} dr \lesssim |k|^{-\frac{5}{6}q} \int_R^{2R} r^{\frac{q}{2}} \lesssim R^{1-\frac{1}{3}q}. \quad (31)$$

It remains to estimate the L^∞ norm, for which is enough to observe that, if $r \in I_2$

$$|\psi_k(r)| \lesssim |k|^{-\frac{1}{4}-\frac{1}{12}} \simeq R^{-\frac{1}{3}}.$$

Lastly, for I_3 we get

$$\int_{I_3} |\psi_k(r)|^q dr \lesssim \int_R^{2R} r^{-\frac{q}{2}} dr \lesssim R^{1-\frac{q}{2}}$$

and

$$\sup_{r \in [R, 2R]} |\psi_k(r)| \lesssim R^{-\frac{1}{2}}.$$

Similar computations bring to the estimates of $\|\psi'_k\|_{L^q([R, 2R])}$ for $R \geq 1$. $\square$

Step 2: We can now provide an estimate for the localized solution on a fixed angular level k and with the radius r varying on a fixed interval of length R .

In order to lighten the notation, in the following we will write $e^{it\rho}$ instead of $e^{it\sigma_3\rho}$ since all the estimates are to be understood on every component of the functions with values on $\mathbb{C}^N$. Moreover, we observe that all the estimates hold also for the generalized eigenfunctions corresponding to negative energies.

Proof of Lemma 3. i) $R \leq 1$:

Let $p \in [2, +\infty]$ and Ω be an interval. Then by the embedding $\dot{H}^{\frac{1}{2}-\frac{1}{q}}(\Omega) \hookrightarrow L^q(\Omega)$, that holds for every $q \in [2, +\infty)$, and interpolation we have

$$\begin{aligned} & \left\| \int_0^\infty e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{dr}^q([R, 2R])} \\ & \lesssim \left\| \int_0^\infty e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p \dot{H}^{\frac{1}{2}-\frac{1}{q}}([R, 2R])} \\ & \lesssim \left\| \int_0^\infty e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{dr}^2([R, 2R])}^{\frac{1}{2}+\frac{1}{q}} \times \left\| \int_0^\infty e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p \dot{H}^1([R, 2R])}^{\frac{1}{2}-\frac{1}{q}} \\ & \lesssim R^{\gamma-\frac{1}{2}+\frac{1}{q}} \|g_k(\rho)\|_{L_{\rho d\rho}^{p'}(I)}. \end{aligned}$$

Where the last inequality comes from Minkowski and Hausdorff-Young inequalities and Lemma 4⁶:

$$\begin{aligned} \left\| \int_0^\infty e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{dr}^2([R, 2R])} &\lesssim \|\psi_k(r\rho) g_k(\rho)\|_{L_{d\rho}^{p'}(I) L_{dr}^2([R, 2R])} \\ &\lesssim \left\| |g_k(\rho)| |\psi_k| \right\|_{L_{dr}^2([R\rho, 2R\rho])} \Big\|_{L_{d\rho}^{p'}(I)} \\ &\lesssim R^\gamma \|g_k(\rho)\|_{L_{\rho d\rho}^{p'}(I)}, \end{aligned}$$

and

$$\begin{aligned} \left\| \int_0^\infty e^{it\rho} \psi'_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{dr}^2([R, 2R])} &\lesssim \|\psi'_k(r\rho) g_k(\rho)\|_{L_{\rho d\rho}^{p'}(I) L_{dr}^2([R, 2R])} \\ &\lesssim \left\| |g_k(\rho)| |\psi'_k| \right\|_{L_{dr}^2([R\rho, 2R\rho])} \Big\|_{L_{\rho d\rho}^{p'}(I)} \\ &\lesssim R^{\gamma-1} \|g_k(\rho)\|_{L_{\rho d\rho}^{p'}(I)}. \end{aligned}$$

Therefore we obtain

$$\left\| \int_0^\infty e^{it\rho} \psi'_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{rdr}^q([R, 2R])} \lesssim R^{\gamma-\frac{1}{2}+\frac{2}{q}} \|g_k(\rho)\|_{L_{\rho d\rho}^2(I)}.$$

ii) $R \geq 2$:

Let $p \in [2, +\infty]$. We prove the following estimates

$$\left\| \int_0^\infty e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{rdr}^\infty([R, 2R])} \lesssim R^{\delta(p)} \|g_k(\rho)\|_{L_{\rho d\rho}^2(I)} \quad (32)$$

and

$$\left\| \int_0^\infty e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{rdr}^2([R, 2R])} \lesssim R^{\frac{1}{2}} \|g_k(\rho)\|_{L_{\rho d\rho}^2(I)}, \quad (33)$$

then the result will follow by interpolation. To estimate (32) we need to split the cases $p \in [2, 4)$ and $p \in [4, +\infty]$.

If $p < 4$ we observe that, from Minkowski and Hausdorff-Young inequalities and Lemma 4, we have⁷

⁶ Notice that since $\rho \in [\frac{1}{2}, 1]$ then $\rho R \leq 1$ and $\rho^\gamma \leq 1$.

⁷ We observe that since $\rho \in [\frac{1}{2}, 1]$ then $R\rho \geq 1$.

$$\begin{aligned}
& \left\| \int_0^{+\infty} e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{dr}^p([R, 2R])} = \left\| \mathcal{F}_{t \mapsto \rho}(\psi_k(r\rho) g_k(\rho) \rho \chi_{\mathbb{R}^+}(\rho)) \right\|_{L_{dr}^p([R, 2R]) L_t^p} \\
& \lesssim \left\| \left\| \psi_k(r\rho) g_k(\rho) \right\|_{L_{d\rho}^{p'}(I)} \right\|_{L_{dr}^p([R, 2R])} \\
& \lesssim \left(\int_I |g_k(\rho)|^{p'} \left(\int_R^{2R} |\psi_k(r\rho)|^p dr \right)^{\frac{1}{p}} d\rho \right)^{\frac{1}{p'}} \lesssim R^{\frac{1}{p}-\frac{1}{2}} \|g_k(\rho)\|_{L_{\rho d\rho}^{p'}(I)}.
\end{aligned}$$

The same holds also for $\psi'_k(r\rho)$. Then, from the embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, we get

$$\begin{aligned}
& \left\| \int_0^\infty e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{dr}^\infty([R, 2R])} = \left\| \int_0^\infty e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{dr}^\infty([R, 2R])} \\
& \lesssim \left(\left\| \int_0^\infty e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{dr}^p([R, 2R])}^p + \left\| \int_0^\infty e^{it\rho} \psi'_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{dr}^p([R, 2R])}^p \right)^{\frac{1}{p}} \\
& \lesssim R^{\frac{1}{p}-\frac{1}{2}} \|g_k(\rho)\|_{L_{\rho d\rho}^2(I)}.
\end{aligned}$$

If $p \geq 4$, we use the embedding $W^{1,\tilde{q}}(\Omega) \hookrightarrow L^\infty(\Omega)$, $\tilde{q} = 4\varepsilon$ with $\varepsilon \sim 1$ ⁸ and proceeding as before we get

$$\left\| \int_0^\infty e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{dr}^\infty([R, 2R])} \lesssim R^{\frac{1}{4\varepsilon}-\frac{1}{2}} \|g_k(\rho)\|_{L_{\rho d\rho}^2(I)}.$$

The estimate (33) comes easily observing that $R\rho \geq 1$ then from Lemma 4 we have

$$\left(\int_R^{2R} |\psi_k(r\rho)|^2 r dr \right)^{\frac{1}{2}} \leq CR^{\frac{1}{2}}.$$

Lastly, we can estimate the $L_{dr}^\infty([R, 2R])$ norm as

$$\begin{aligned}
& \left\| \int_0^\infty e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^\infty L_{dr}^\infty([R, 2R])} \lesssim R^{\gamma-1} \|g_k(\rho)\|_{L_{\rho d\rho}^2(I)}, \quad \text{for } R \leq 1, \\
& \left\| \int_0^\infty e^{it\rho} \psi_k(r\rho) g_k(\rho) \rho d\rho \right\|_{L_t^p L_{dr}^\infty([R, 2R])} \lesssim R^{\delta(p)} \|g_k(\rho)\|_{L_{\rho d\rho}^2(I)}, \quad \text{for } R \geq 2,
\end{aligned}$$

⁸ This means that the next estimates hold for any $\varepsilon \in (\frac{1}{2}, 1)$ but we want ε near 1.

where for the former estimate we use the embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$. $\square$

Step 3: We remove assumptions on the localizations on the solution in order to get the generalized Strichartz estimates with the indexes p, q in the range of admissibility given by the following

Lemma 5. *Let*

$$p \in (2, +\infty], \quad 6 < q < \frac{2}{\frac{1}{2} - \gamma_{\frac{1}{2}}}, \quad \frac{1}{q} + \left(\frac{1}{p} - \frac{1}{2}\right) \left(1 - \frac{2}{q}\right) < 0.$$

Then

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L_t^p L_{rdr}^q L_\theta^2} \leq C \|u_0\|_{\dot{H}_{\mathcal{D}_\nu}^s}$$

where $s = 1 - \frac{1}{p} - \frac{2}{q}$ and $\gamma_{\frac{1}{2}} = \sqrt{\frac{1}{4} - \nu^2}$. Moreover, if u_0 is Dirac-non radial we can take $q \in (6, +\infty]$.

We remove the assumption of the frequency localization of the solution, to get the Strichartz estimates for (p, q) admissible for Lemma 5; Thanks to (27), it suffices to show that

$$\sum_{k \in \mathbb{Z} + \frac{1}{2}} \left\| \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho)](r) \right\|_{L_t^p L_{rdr}^q}^2 \leq C \|u_0\|_{\dot{H}_{\mathcal{D}_\nu}^s}^2.$$

Let N, R be dyadic numbers and $\phi \in C_c^\infty([\frac{1}{2}, 1])$ such that $\sum_{N \in 2^{\mathbb{Z}}} \phi(\rho N^{-1}) = 1$. Then, for $q < +\infty$, from the embedding $l^2 \hookrightarrow l^q$ and scaling we have the following

$$\begin{aligned} & \sum_{k \in \mathbb{Z} + \frac{1}{2}} \left\| \mathcal{P}_k^{-1} [e^{-it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho)](r) \right\|_{L_t^p L_{rdr}^q}^2 \\ &= \sum_{k \in \mathbb{Z} + \frac{1}{2}} \left\| \sum_{N \in 2^{\mathbb{Z}}} \mathcal{P}_k^{-1} [e^{-it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho) \phi(\frac{\rho}{N})](r) \right\|_{L_t^p L_{rdr}^q}^2 \\ &= \sum_{k \in \mathbb{Z} + \frac{1}{2}} \left\| \left(\sum_{R \in 2^{\mathbb{Z}}} \left\| \sum_{N \in 2^{\mathbb{Z}}} \mathcal{P}_k^{-1} [e^{-it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho) \phi(\frac{\rho}{N})](r) \right\|_{L_{rdr}^q([R, 2R])}^q \right)^{\frac{1}{q}} \right\|_{L_t^p}^2 \quad (34) \\ &\lesssim \sum_{k \in \mathbb{Z} + \frac{1}{2}} \sum_{R \in 2^{\mathbb{Z}}} \left\| \sum_{N \in 2^{\mathbb{Z}}} \mathcal{P}_k^{-1} [e^{-it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho) \phi(\frac{\rho}{N})](r) \right\|_{L_t^p L_{rdr}^q([R, 2R])}^2 \\ &\lesssim \sum_{k \in \mathbb{Z} + \frac{1}{2}} \sum_{R \in 2^{\mathbb{Z}}} \left(\sum_{N \in 2^{\mathbb{Z}}} \left\| \mathcal{P}_k^{-1} [e^{-it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho) \phi(\frac{\rho}{N})](r) \right\|_{L_t^p L_{rdr}^q([R, 2R])} \right)^2 \\ &= \sum_{k \in \mathbb{Z} + \frac{1}{2}} \sum_{R \in 2^{\mathbb{Z}}} \left(\sum_{N \in 2^{\mathbb{Z}}} N^{2 - \frac{1}{p} - \frac{2}{q}} \left\| \mathcal{P}_k^{-1} [e^{-it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(N\rho) \phi(\rho)](r) \right\|_{L_t^p L_{rdr}^q([NR, 2NR])} \right)^2. \end{aligned}$$

Moreover, from Lemma 3 we can continue the chain of inequalities and we get

$$\begin{aligned} &\lesssim \sum_{k \in \mathbb{Z} + \frac{1}{2}} \sum_{R \in 2^{\mathbb{Z}}} \left(\sum_{N \in 2^{\mathbb{Z}}} N^{2 - \frac{1}{p} - \frac{2}{q}} Q(NR) \|(\mathcal{P}_k u_{0,k})(N\rho)\phi(\rho)\|_{L^2_{\rho d\rho}} \right)^2 \\ &= \sum_{k \in \mathbb{Z} + \frac{1}{2}} \sum_{R \in 2^{\mathbb{Z}}} \left(\sum_{N \in 2^{\mathbb{Z}}} N^{1 - \frac{1}{p} - \frac{2}{q}} Q(NR) \|(\mathcal{P}_k u_{0,k})(\rho)\phi(\frac{\rho}{N})\|_{L^2_{\rho d\rho}} \right)^2 \end{aligned}$$

where

$$Q(NR) = \begin{cases} (NR)^{\gamma - \frac{1}{2} + \frac{2}{q}} & \text{if } NR \leq 1, \\ (NR)^{\frac{1}{q} + \delta(p)(1 - \frac{2}{q})} & \text{if } NR \geq 2. \end{cases}$$

We note that if we take p, q such that

$$\gamma - \frac{1}{2} + \frac{2}{q} > 0, \quad \frac{1}{q} + \delta(p)(1 - \frac{2}{q}) < 0, \quad (35)$$

then

$$\sup_{R \in 2^{\mathbb{Z}}} \sum_{N \in 2^{\mathbb{Z}}} Q(NR) < +\infty, \quad \sup_{N \in 2^{\mathbb{Z}}} \sum_{R \in 2^{\mathbb{Z}}} Q(NR) < +\infty.$$

Recall that $\gamma = \sqrt{k^2 - \nu^2} \geq \sqrt{2}$ if $|k| > \frac{1}{2}$. Then, the first condition in (35) becomes: $\gamma_{\frac{1}{2}} - \frac{1}{2} + \frac{2}{q} > 0$.

Let $A_{N,k} = N^{1 - \frac{1}{p} - \frac{2}{q}} \|(\mathcal{P}_k u_{0,k})(\rho)\phi(\frac{\rho}{N})\|_{L^2_{\rho d\rho}}$, we use the Schur test Lemma:

$$\left(\sum_{R \in 2^{\mathbb{Z}}} \left(\sum_{N \in 2^{\mathbb{Z}}} Q(NR) A_{N,k} \right)^2 \right)^{\frac{1}{2}} = \sup_{\|B_R\|_{l^2} \leq 1} \sum_{R \in 2^{\mathbb{Z}}} \sum_{N \in 2^{\mathbb{Z}}} Q(NR) A_{N,k} B_R$$

which is bounded by

$$\begin{aligned} &\leq C \left(\sum_{R \in 2^{\mathbb{Z}}} \sum_{N \in 2^{\mathbb{Z}}} Q(NR) |A_{N,k}|^2 \right)^{\frac{1}{2}} \left(\sum_{R \in 2^{\mathbb{Z}}} \sum_{N \in 2^{\mathbb{Z}}} Q(NR) |B_R|^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\sup_{R \in 2^{\mathbb{Z}}} \sum_{N \in 2^{\mathbb{Z}}} Q(NR) \sup_{N \in 2^{\mathbb{Z}}} \sum_{R \in 2^{\mathbb{Z}}} Q(NR) \right)^{\frac{1}{2}} \left(\sum_{N \in 2^{\mathbb{Z}}} |A_{N,k}|^2 \right)^{\frac{1}{2}} \left(\sum_{R \in 2^{\mathbb{Z}}} |B_R|^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{N \in 2^{\mathbb{Z}}} |A_{N,k}|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Putting the estimates together, we have obtained

$$\begin{aligned}
\|e^{itD_\nu} u_0\|_{L_t^p L_{rdr}^q L_\theta^2}^2 &\leq C \sum_{k \in \mathbb{Z} + \frac{1}{2}} \sum_{R \in 2^{\mathbb{Z}}} \left(\sum_{N \in 2^{\mathbb{Z}}} Q(NR) A_{N,k} \right)^2 \\
&\leq C \sum_{k \in \mathbb{Z} + \frac{1}{2}} \sum_{N \in 2^{\mathbb{Z}}} N^{2(1 - \frac{1}{p} - \frac{2}{q})} \|(\mathcal{P}_k u_{0,k})(\rho) \phi(\frac{\rho}{N})\|_{L_{\rho d\rho}^2}^2 \\
&\leq C \|u_0\|_{\dot{H}_{D_\nu}^{1 - \frac{1}{p} - \frac{2}{q}}}^2.
\end{aligned}$$

Let us now suppose that u_0 is Dirac-non radial, that is $\mathcal{P}_k u_{0,k} = 0$ for $|k| = \frac{1}{2}$. Then the first condition in (35) is satisfied for all $q \leq +\infty$. Then, the previous estimates prove the claim for any $q \in (6, +\infty)$. If $q = +\infty$, we argue as before; we take N, R dyadic numbers and $\phi \in C_c^\infty([\frac{1}{2}, 1])$ such that $\sum_{N \in 2^{\mathbb{Z}}} \phi(\rho N^{-1}) = 1$. By the immersion $l^2 \hookrightarrow l^\infty$, the Minkowski's inequality and scaling we have

$$\begin{aligned}
&\sum_{k \in \mathbb{Z} + \frac{1}{2}} \left\| \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho)](r) \right\|_{L_t^p L_{dr}^\infty}^2 \\
&\lesssim \sum_{k \in \mathbb{Z} + \frac{1}{2}} \left\| \sum_{N \in 2^{\mathbb{Z}}} \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho) \phi(\frac{\rho}{N})](r) \right\|_{L_t^p L_{dr}^\infty}^2 \\
&\lesssim \sum_{k \in \mathbb{Z} + \frac{1}{2}} \left\| \sup_{R \in 2^{\mathbb{Z}}} \left\| \sum_{N \in 2^{\mathbb{Z}}} \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho) \phi(\frac{\rho}{N})](r) \right\|_{L_{dr}^\infty([R, 2R])} \right\|_{L_t^p}^2 \\
&\lesssim \sum_{k \in \mathbb{Z} + \frac{1}{2}} \left\| \left(\sum_{R \in 2^{\mathbb{Z}}} \left\| \sum_{N \in 2^{\mathbb{Z}}} \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho) \phi(\frac{\rho}{N})](r) \right\|_{L_{dr}^\infty([R, 2R])}^2 \right)^{\frac{1}{2}} \right\|_{L_t^p}^2 \tag{36} \\
&\lesssim \sum_{k \in \mathbb{Z} + \frac{1}{2}} \sum_{R \in 2^{\mathbb{Z}}} \left(\sum_{N \in 2^{\mathbb{Z}}} \left\| \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho) \phi(\frac{\rho}{N})](r) \right\|_{L_t^p L_{dr}^\infty([R, 2R])} \right)^2 \\
&= \sum_{k \in \mathbb{Z} + \frac{1}{2}} \sum_{R \in 2^{\mathbb{Z}}} \left(\sum_{N \in 2^{\mathbb{Z}}} N^{2 - \frac{1}{p}} \left\| \mathcal{P}_k^{-1} [e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(N\rho) \phi(\rho)](r) \right\|_{L_t^p L_{dr}^\infty([NR, 2NR])} \right)^2,
\end{aligned}$$

for every $p > 2$. Then, we conclude using the Schur test Lemma as before.

Step 4: We observe that the set of admissible exponents is not empty if and only if

$$6 < \frac{2}{\frac{1}{2} - \gamma_{\frac{1}{2}}} \iff |\nu| < \frac{\sqrt{2}}{3}.$$

In this case, we have the validity of the Strichartz estimates in the set with blue boundary described in Fig. 2, where we define $q_c := \frac{2}{\frac{1}{2} - \gamma_{\frac{1}{2}}}$ and the corresponding endpoint index

satisfying (35) as p_c , that is $p_c := \frac{\gamma_{\frac{1}{2}} + \frac{1}{2}}{\gamma_{\frac{1}{2}}}$. Then, we can interpolate between the estimate in Lemma 5 with (p, q_c) , $p \in (p_c, +\infty]$ and the standard estimate

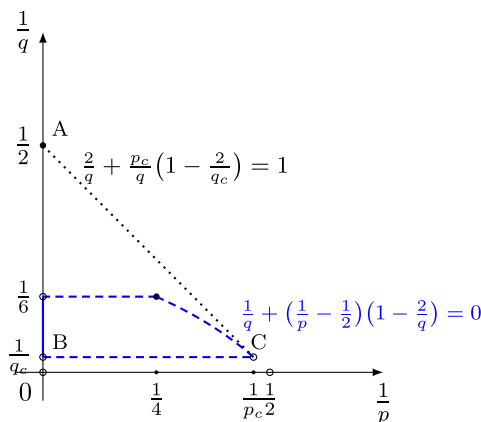


Fig. 2. Case $n = 2$, $\nu = \frac{1}{4}$.

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L_t^\infty L_{r_{dr}}^2 L_\theta^2} \leq C \|u_0\|_{L^2}$$

to reach the triangle $\widehat{ABC}$ (Fig. 2). Lastly, the estimates for Dirac-non radial solutions follow from the same argument. $\square$

3.2.1. Strichartz estimates 3D case

We now discuss the 3-dimensional case. We retrace the proof of the 2D case. For the sake of brevity we omit the proofs of the first three steps since the computations are basically the same.

Step 1:

Lemma 6. Let $k \in \mathbb{Z}^*$, $\gamma = \sqrt{k^2 - \nu^2}$, $|\nu| \leq 1$ and $q \in [2, +\infty]$. Then, the following estimates hold

$$\|\psi_k\|_{L^q([R, 2R])} \leq C \times \begin{cases} R^{\gamma + \frac{1}{q} - 1}, & \text{if } R \leq 1, \\ R^{\beta(q)}, & \text{if } R \geq 1, \end{cases}$$

and

$$\|\psi'_k\|_{L^q([R, 2R])} \leq C \times \begin{cases} R^{\gamma + \frac{1}{q} - 2}, & R \leq 1, \\ R^{\beta(q)}, & R \geq 1, \end{cases}$$

where

$$\beta(q) = \begin{cases} \frac{1}{q} - 1 & \text{if } q \in [2, 4), \\ \frac{1}{q} - \frac{5}{6} & \text{if } q \in [4, +\infty] \end{cases}$$

and all the constants are independent of γ, k , but eventually dependent on q .

Step 2:

Lemma 7. Let $(p, q) \in [2, +\infty) \times [2, +\infty)$ and $k \in \mathbb{Z}^*$. Let $I = [\frac{1}{2}, 1]$ and $g_k \in L^2_{\rho^2 d\rho}((0, +\infty))$ such that $\text{supp}(g_k) \subset I$. Then

$$\left\| \mathcal{P}_k^{-1}[e^{it\rho\sigma_3} g_k(\rho)](r) \right\|_{L_t^p L_{r^2 dr}^q([R, 2R])} \leq C \|g_k(\rho)\|_{L^2_{\rho^2 d\rho}(I)} \times \begin{cases} R^{\gamma-1+\frac{3}{q}}, & R \leq 1 \\ R^{\frac{1}{q}+\delta(p)(1-\frac{2}{q})}, & R \geq 2, \end{cases}$$

where

$$\delta(p) = \begin{cases} \frac{1}{p} - 1 & \text{if } p < 4, \\ \frac{1}{4\varepsilon} - 1 & \text{if } p \geq 4 \end{cases}$$

and the constant C is independent of k, ν , but eventually depends on p and q . Moreover, if $q = +\infty$, for all $p \geq 2$ we have

$$\left\| \mathcal{P}_k^{-1}[e^{it\rho\sigma_3} g_k(\rho)](r) \right\|_{L_t^p L_{dr}^\infty([R, 2R])} \leq C \|g_k(\rho)\|_{L^2_{\rho^2 d\rho}(I)} \times \begin{cases} R^{\gamma-\frac{3}{2}}, & R \leq 1 \\ R^{\delta(p)}, & R \geq 2, \end{cases}$$

Notice that in this case we can take $p = 2$ since $\delta(2) < 0$.

Step 3:

Lemma 8. Let

$$p \in [2, +\infty], \quad \frac{10}{3} < q < \frac{3}{1-\gamma_1}, \quad \frac{1}{q} + \left(\frac{1}{p} - 1\right) \left(1 - \frac{2}{q}\right) < 0.$$

Then

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L_t^p L_{r^2 dr}^q L_\theta^2} \leq C \|u_0\|_{\dot{H}_{\mathcal{D}_\nu}^s}$$

where $s = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}$ and $\gamma_1 = \sqrt{1 - \nu^2}$. Moreover, if u_0 is Dirac-non radial we can take $q \in (\frac{10}{3}, +\infty]$.

Proof. Notice that it suffices to show that

$$\sum_{k \in \mathbb{Z}^*} \left\| \mathcal{P}_k^{-1}[e^{it\rho\sigma_3} (\mathcal{P}_k u_{0,k})(\rho)](r) \right\|_{L_t^p L_{r^2 dr}^q}^2 \leq C \|u_0\|_{\dot{H}_{\mathcal{D}_\nu}^s}^2.$$

We proceed as in Lemma 5, with suitable modifications. $\square$

Step 4: In the last step we want to combine the previous estimate with the conservation of mass in order to get the Strichartz estimates in Theorem 2. We notice that $q = 4$ is admissible if and only if

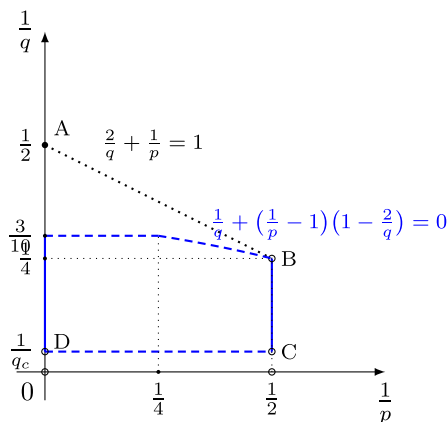


Fig. 3. Case $n = 3$, $\nu = \frac{1}{2}$.

$$4 < \frac{3}{1 - \gamma_1} = q_c \iff |\nu| < \frac{\sqrt{15}}{4}.$$

In this case, we have the validity of the Strichartz estimates in the set with blue boundary described in Fig. 3. Then we interpolate between the estimates in Lemma 8 with exponent $(p, 4)$, $p > 2$ and the $L_t^\infty L_{r^2 dr}^2 L_\theta^2$ -estimate, widening the admissible set up to cover the region $ABCD$ in Fig. 3.

4. Nonlinear application

This Section is dedicated to the proof of the well-posedness results for the system (11) with Dirac-radial initial datum. We start by proving the classical Strichartz estimates for Dirac-radial functions (Proposition 5 below) as well as two Lemmas that we will use throughout the proofs of the nonlinear results.

Proposition 5. Let $|\nu| < \frac{\sqrt{15}}{4}$ and (p, q) such that

$$2 \leq p \leq +\infty, \quad 2 \leq q < q_c, \quad \frac{2}{q} + \frac{1}{p} < 1 \text{ or } (p, q) = (\infty, 2), \quad (37)$$

where $q_c = \frac{3}{1 - \sqrt{1 - \nu^2}}$. Then, there exists a constant $C > 0$ such that for any u_0 Dirac-radial, $u_0 \in \dot{H}^s(\mathbb{R}^3; \mathbb{C}^4)$, the following Strichartz estimates hold

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L_t^p L_x^q} \leq C \|u_0\|_{\dot{H}^s} \quad (38)$$

provided $s = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}$.

Proof. Let u_0 be Dirac-radial. From (24) and observing that the functions Ξ_k are bounded, we get

$$\begin{aligned}
\|u\|_{L_t^p L_x^q} &= \left\| \sum_{|k|=1} \mathcal{P}_k^{-1} [e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)](r) \cdot \Xi_k(\theta) \right\|_{L_t^p L_{r^2 dr}^q L_\theta^q} \\
&\leq \sum_{|k|=1} \|\Xi_k\|_{L_\theta^q} \left\| \mathcal{P}_k^{-1} [e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)](r) \right\|_{L_t^p L_{r^2 dr}^q} \\
&\lesssim \left(\sum_{|k|=1} \left\| \mathcal{P}_k^{-1} [e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)](r) \right\|_{L_t^p L_{r^2 dr}^q}^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{39}$$

Then we can proceed as in the proof of Lemma 8 and we conclude by interpolation with the $L_t^\infty L_x^2$ estimate. $\square$

Remark 8. We observe that Proposition 5 holds the same if the initial datum u_0 is such that, in the decomposition given by (24), $u_{0,k} \neq 0$ only for a finite number of $k \in \mathbb{Z}^*$. In fact, inequality (39) remains true if we sum over a finite number of indexes. Moreover, a similar result can be proved for the 2D case.

Remark 9. We observe that $N(u) = (\omega * \langle \beta u, u \rangle)u$ preserves the Dirac-radiality of u . More precisely, it has been shown in [4] (Lemma 5.5) that if u is a Dirac-radial function, then $\langle \beta u, u \rangle$ is a radial scalar function (in the classical sense). Moreover, we recall that ω is radial, then $\omega * \langle \beta u, u \rangle$ is a radial function which implies that $N(u)$ is still Dirac-radial.

Lemma 9. Let $u \in \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4)$ and $\omega \in L^\gamma(\mathbb{R}^3) \cap L^\alpha(\mathbb{R}^3)$. Then, the following inequality holds

$$\|(\omega * \langle \beta u, u \rangle)u\|_{\dot{H}^s} \lesssim \|\omega\|_{L^\gamma} \|u\|_{\dot{H}^s} \|u\|_{L^{\mu_2}} \|u\|_{L^{p_2}} + \|\omega\|_{L^\alpha} \|u\|_{\dot{H}^s} \|u\|_{L^{2\beta}}^2, \tag{40}$$

where $s \geq 0$, $\alpha, \beta, \gamma \in [1, +\infty]$, $p_2, \mu_2 \in (1, +\infty]$ such that

$$\frac{1}{\gamma} + \frac{1}{p_2} + \frac{1}{\mu_2} = 1, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Proof. Let $D^s := (-\Delta)^{\frac{s}{2}}$. In the following all the estimates have to be thought component by component and then summed back together. From generalized fractional Leibniz rule (see e.g. [16] Thm. 1),

$$\begin{aligned}
\|(\omega * \langle \beta u, u \rangle)u\|_{\dot{H}^s} &= \|D^s[(\omega * \langle \beta u, u \rangle)u]\|_{L^2} \\
&\lesssim \|D^s(\omega * \langle \beta u, u \rangle)\|_{L^{p_1}} \|u\|_{L^{p_2}} + \|\omega * \langle \beta u, u \rangle\|_{L^\infty} \|D^s u\|_{L^2},
\end{aligned} \tag{41}$$

where

$$\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}, \quad p_1 \in (1, +\infty), \quad p_2 \in (1, +\infty]$$

and $p_1 \in [1, +\infty)$ if $s = 0$. We estimate separately:

- Using Young's inequality

$$\|\omega * \langle \beta u, u \rangle\|_{L^\infty} \lesssim \|\omega\|_{L^\alpha} \|\langle \beta u, u \rangle\|_{L^\beta} \lesssim \|\omega\|_{L^\alpha} \|u\|_{L^{2\beta}}^2,$$

where

$$1 = \frac{1}{\alpha} + \frac{1}{\beta}, \quad \alpha, \beta \in [1, \infty];$$

- Using Young's inequality and generalized Leibniz rule

$$\begin{aligned} \|D^s(\omega * \langle \beta u, u \rangle)\|_{L^{p_1}} &= \|\omega * D^s \langle \beta u, u \rangle\|_{L^{p_1}} \lesssim \|\omega\|_{L^\gamma} \|D^s \langle \beta u, u \rangle\|_{L^\mu} \\ &\lesssim \|\omega\|_{L^\gamma} \|D^s u\|_{L^2} \|u\|_{L^{\mu_2}}, \end{aligned}$$

where

$$1 + \frac{1}{p_1} = \frac{1}{\gamma} + \frac{1}{\mu}, \quad \frac{1}{\mu} = \frac{1}{2} + \frac{1}{\mu_2}, \quad \mu, \gamma \in [1, +\infty], \mu_2 \in (1, \infty].$$

Summing up, we obtain the result. $\square$

Lemma 10. Let $u, v \in \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4)$, then

$$\begin{aligned} \|D^s(\langle \beta u, u \rangle - \langle \beta v, v \rangle)\|_{L^\mu} &\lesssim \|D^s(u - v)\|_{L^{\mu_1}} (\|u\|_{L^{\mu_2}} + \|v\|_{L^{\mu_2}}) \\ &\quad + (\|D^s u\|_{L^{\gamma_1}} + \|D^s v\|_{L^{\gamma_1}}) \|u - v\|_{L^{\gamma_2}} \end{aligned}$$

where $s \geq 0$ and

$$\frac{1}{\mu} = \frac{1}{\mu_1} + \frac{1}{\mu_2} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.$$

Proof. We notice that, if u, v are scalar functions, we have

$$D^s(|u|^2 - |v|^2) = D^s(u\bar{u} - v\bar{v} \pm v\bar{u}) = D^s((u - v)\bar{u}) + D^s(v(\bar{u} - \bar{v}))$$

Then we conclude arguing component by component and by the fractional Leibniz rule. $\square$

We can now prove Theorem 4.

Proof. Let $T, M > 0$, we define the space

$$X_{T,M} := \{u \in L_T^\infty H^s : \|u\|_{L_T^\infty \dot{H}^s} \leq M\},$$

that, endowed with the metric

$$d(u, v) = \|u - v\|_{L_T^\infty \dot{H}^s},$$

is a complete metric space, and the map

$$\Phi(u)(t, x) := e^{it\mathcal{D}_\nu} u_0(x) - i \int_0^t e^{i(t-s)\mathcal{D}_\nu} N(u)(s, x) ds. \quad (42)$$

Then, the proof relies on standard contraction argument; we want to find T, M such that $\Phi : X_{T,M} \rightarrow X_{T,M}$ is a contraction map on $(X_{T,M}, d)$.

Step 1: We observe that, since $s < \frac{3}{2}$, $\dot{H}^s \hookrightarrow L^{\frac{6}{3-2s}}$ and that Lemma 9 holds with the choice

$$\begin{cases} \mu_2 = p_2 = 2\beta = \frac{6}{3-2s} = 2p', \\ \alpha = \gamma = \frac{3}{2s} = p. \end{cases}$$

Moreover, since $s \leq 1$ and $|\nu| < \frac{\sqrt{3}}{2}$, $\|u\|_{\dot{H}^s} \simeq \|u\|_{\dot{H}_{\mathcal{D}_\nu}^s}$ (Proposition 2). Then, from Minkowski's inequality and (40), we have

$$\begin{aligned} \|\Phi(u)\|_{L_T^\infty \dot{H}^s} &\lesssim \|u_0\|_{\dot{H}^s} + \int_0^T \|e^{i(t-\tau)\mathcal{D}_\nu} N(u)(\tau, x)\|_{L_T^\infty \dot{H}^s} d\tau \\ &\lesssim \|u_0\|_{\dot{H}^s} + \int_0^T \|N(u)\|_{\dot{H}^s} d\tau \\ &\lesssim \|u_0\|_{\dot{H}^s} + \|\omega\|_{L^p} \int_0^T \|u\|_{\dot{H}^s} \|u\|_{L^{\frac{6}{3-2s}}}^2 d\tau \end{aligned}$$

and, from the Sobolev's embedding, we get

$$\|\Phi(u)\|_{L_T^\infty \dot{H}^s} \lesssim \|u_0\|_{\dot{H}^s} + \|\omega\|_{L^p} T \|u\|_{L_T^\infty \dot{H}^s}^3 \leq C \|u_0\|_{\dot{H}^s} + C \|\omega\|_{L^p} T M^3,$$

for some $C > 0$.

Step 2: Proceeding as before, we get

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_{L_T^\infty \dot{H}^s} &\lesssim \int_0^T \|N(u) - N(v)\|_{\dot{H}^s} d\tau \\ &\lesssim \int_0^T \left[\|(\omega * (\langle \beta u, u \rangle - \langle \beta v, v \rangle)) u\|_{\dot{H}^s} + \|(\omega * \langle \beta v, v \rangle)(u - v)\|_{\dot{H}^s} \right] d\tau. \end{aligned}$$

$$= \int_0^T (I + II)(\tau) d\tau.$$

We estimate separately I and II ; from Lemma 9 and Young's inequality, we have

$$\begin{aligned} I &\lesssim \|\omega * \langle D \rangle^s (\langle \beta u, u \rangle - \langle \beta v, v \rangle)\|_{L^{\frac{3}{s}}} \|u\|_{L^{\frac{6}{3-2s}}} + \|\omega * (\langle \beta u, u \rangle - \langle \beta v, v \rangle)\|_{L^\infty} \|u\|_{\dot{H}^s} \\ &\lesssim \|\omega\|_{L^p} \|\langle D \rangle^s (\langle \beta u, u \rangle - \langle \beta v, v \rangle)\|_{L^{\frac{3}{3-s}}} \|u\|_{L^{\frac{6}{3-2s}}} \\ &\quad + \|\omega\|_{L^p} \|u - v\|_{L^{\frac{6}{3-2s}}} \| |u| + |v| \|_{L^{\frac{6}{3-2s}}} \|u\|_{\dot{H}^s} \end{aligned}$$

where for the second term we have used that, for two scalars functions f, g , $||f|^2 - |g|^2| \leq |f - g|(|f| + |g|)$.⁹ To estimate the first term we observe that Lemma 10 holds taking

$$\begin{cases} \mu_1 = \gamma_1 = 2, \\ \mu_2 = \gamma_2 = \frac{6}{3-2s}, \end{cases}$$

then by the Sobolev's embedding, we have

$$I \lesssim \|\omega\|_{L^p} \|u - v\|_{\dot{H}^s} \|u\|_{\dot{H}^s} (\|u\|_{\dot{H}^s} + \|v\|_{\dot{H}^s})$$

In addition, we have

$$\begin{aligned} II &\lesssim \|\omega * \langle D \rangle^s \langle \beta v, v \rangle\|_{L^{\frac{3}{s}}} \|u - v\|_{L^{\frac{6}{3-2s}}} + \|\omega * \langle \beta v, v \rangle\|_{L^\infty} \|u - v\|_{\dot{H}^s} \\ &\lesssim \|\omega\|_{L^p} \|v\|_{L^{\frac{6}{3-2s}}} \|v\|_{\dot{H}^s} \|u - v\|_{L^{\frac{6}{3-2s}}} + \|\omega\|_{L^p} \|v\|_{L^{\frac{6}{3-2s}}}^2 \|u - v\|_{\dot{H}^s}. \end{aligned}$$

Putting all the estimates together, we have obtained that there exists $\tilde{C} > 0$ such that

$$\|\Phi(u) - \Phi(v)\|_{L_T^\infty \dot{H}^s} \leq \tilde{C} \|\omega\|_{L^p} \|u - v\|_{L_T^\infty \dot{H}^s} \int_0^T (\|u\|_{\dot{H}^s} + \|v\|_{\dot{H}^s})^2 d\tau.$$

Then, if we choose T, M such that

$$\begin{cases} C \|u_0\|_{\dot{H}^s} \leq \frac{M}{2}, \\ T < \min \left\{ \frac{1}{2C \|\omega\|_{L^p} M^2}, \frac{1}{8\tilde{C} \|\omega\|_{L^p} M^2} \right\}, \end{cases}$$

Φ maps $X_{T,M}$ into itself and it is a contraction on $(X_{T,M}, d)$ and applying the Banach fixed-point theorem we get the claim. $\square$

⁹ Then, for u, v is true that $|\langle \beta u, u \rangle - \langle \beta v, v \rangle| \leq 4|u - v|(|u| + |v|)$.

Now we turn to the proof of Theorem 5; notice that in the proof above we didn't use any kind of Strichartz estimates. We will exploit them in the following.

Proof. The idea is to apply the Banach contraction theorem on a ball in the space $Y_T = L^\infty([0, T]; \dot{H}^s(\mathbb{R}^3)) \cap L^r([0, T]; L^q(\mathbb{R}^3))$, $T > 0$, endowed with the norm

$$\|u\|_{Y_T} := \sup_{t \in [0, T]} \|u\|_{\dot{H}^s(\mathbb{R}^3)} + \|u\|_{L^r([0, T], L^q(\mathbb{R}^3))},$$

for some (r, q) admissible for Proposition 5 so that

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L_t^r L_x^q} \leq C \|u_0\|_{\dot{H}^s},$$

where $s = \frac{3}{2} - \frac{1}{r} - \frac{3}{q}$ is such that $s \leq 1$. We choose $(r, q) = (2p, 2p')$, where p' is the conjugate exponent of p . We claim that this is an admissible couple and that we can find $T, M > 0$ such that the map Φ is as in (42), is a contraction on the complete metric space $(Y_{T, M}, d)$, where

$$Y_{T, M} := \{u \in L_T^\infty \dot{H}^s \cap L_T^{2p} L^{2p'} : \|u\|_{Y_T} \leq M\}, \quad d(u, v) := \|u - v\|_{Y_T}.$$

Let us now prove the claim. We notice that if $p \in (\frac{3}{1+2\sqrt{1-\nu^2}}, +\infty)$ then $(2p, 2p')$ satisfies conditions (37) and $s = \frac{3}{2} - \frac{1}{2p} - \frac{3}{2p'} = \frac{1}{p} \leq 1$. Moreover, estimate (40) holds with the following choice of indexes

$$\begin{cases} \mu_2 = p_2 = 2\beta = 2p', \\ \alpha = \gamma = p. \end{cases}$$

Then, proceeding as before, we get

$$\begin{aligned} \|\Phi(u)\|_{\dot{H}^s} &\lesssim \|u_0\|_{\dot{H}^s} + \int_0^T \|e^{i(t-\tau)\mathcal{D}_\nu} N(u)(\tau, x)\|_{L_T^\infty \dot{H}^s} d\tau \\ &\lesssim \|u_0\|_{\dot{H}^s} + \int_0^T \|N(u)\|_{\dot{H}^s} d\tau \\ &\lesssim \|u_0\|_{\dot{H}^s} + \|\omega\|_{L^p} \|u\|_{L_T^\infty \dot{H}^s} \int_0^T \|u(\tau)\|_{L^{2p} L^{2p'}}^2 d\tau \\ &\lesssim \|u_0\|_{\dot{H}^s} + T^{1-\frac{1}{p}} \|\omega\|_{L^p} \|u\|_{L_T^\infty \dot{H}^s} \|u\|_{L_T^{2p} L^{2p'}}^2 \end{aligned}$$

and

$$\|\Phi(u)\|_{L_T^{2p} L^{2p'}} \lesssim \|u_0\|_{\dot{H}^s} + T^{1-\frac{1}{p}} \|\omega\|_{L^p} \|u\|_{L_T^\infty \dot{H}^s} \|u\|_{L_T^{2p} L^{2p'}}^2.$$

Moreover

$$\begin{aligned}
 \|\Phi(u) - \Phi(v)\|_{Y_T} &\lesssim \int_0^T \|N(u) - N(v)\|_{\dot{H}^s} d\tau \\
 &\lesssim \int_0^T \left[\|(\omega * (\langle \beta u, u \rangle - \langle \beta v, v \rangle))u\|_{\dot{H}^s} + \|(\omega * \langle \beta v, v \rangle)(u - v)\|_{\dot{H}^s} \right] d\tau \\
 &= \int_0^T (I + II)(\tau) d\tau.
 \end{aligned}$$

We estimate I, II separately; from Lemma 9 and Young's inequality, we have

$$\begin{aligned}
 I &\lesssim \|\omega * D^s(\langle \beta u, u \rangle - \langle \beta v, v \rangle)\|_{L^{2p}} \|u\|_{L^{2p'}} + \|\omega * (\langle \beta u, u \rangle - \langle \beta v, v \rangle)\|_{L^\infty} \|u\|_{\dot{H}^s} \\
 &\lesssim \|\omega\|_{L^p} \|D^s(\langle \beta u, u \rangle - \langle \beta v, v \rangle)\|_{L^\mu} \|u\|_{L^{2p'}} + \|\omega\|_{L^p} \|\langle \beta u, u \rangle - \langle \beta v, v \rangle\|_{L^{p'}} \|u\|_{\dot{H}^s},
 \end{aligned}$$

where $\mu = (2p)'$. From Lemma 10, we can estimate the L^μ norm as

$$\|D^s(\langle \beta u, u \rangle - \langle \beta v, v \rangle)\|_{L^\mu} \lesssim \|u - v\|_{\dot{H}^s} (\|u\|_{L^{2p'}} + \|v\|_{L^{2p'}}) + \|u - v\|_{L^{2p'}} (\|u\|_{\dot{H}^s} + \|v\|_{\dot{H}^s}).$$

Then, we can continue the chain of inequalities

$$\begin{aligned}
 I &\lesssim \|\omega\|_{L^p} \|u - v\|_{L_T^\infty \dot{H}^s} (\|u\|_{L^{2p'}} + \|v\|_{L^{2p'}}) \|u\|_{L^{2p'}} + \\
 &\quad + \|\omega\|_{L^p} \|u - v\|_{L^{2p'}} [(\|u\|_{L_T^\infty \dot{H}^s} + \|v\|_{L_T^\infty \dot{H}^s}) \|u\|_{L^{2p'}} + \|u\|_{L_T^\infty \dot{H}^s} \|u\| + \|v\|_{L^{2p'}}]
 \end{aligned}$$

We estimate II using again Young's inequality and fractional Leibniz rule, getting

$$II \leq \|\omega\|_{L^p} \|u - v\|_{L_T^\infty \dot{H}^s} \|v\|_{L^{2p'}}^2 + \|\omega\|_{L^p} \|u - v\|_{L^{2p'}} \|u\|_{L_T^\infty \dot{H}^s} \|v\|_{L^{2p'}}.$$

Summing up, there exist two positive constants $C, \tilde{C}$ such that

$$\begin{aligned}
 \|\Phi(u)\|_{Y_T} &\leq C \|u_0\|_{Y_T} + CT^{1-\frac{1}{p}} \|\omega\|_{L^p} \|u\|_{L_T^{2p} L^{2p'}}^3, \\
 \|\Phi(u) - \Phi(v)\|_{Y_T} &\leq \tilde{C} T^{1-\frac{1}{p}} \|\omega\|_{L^p} (\|u\|_{Y_T} + \|v\|_{Y_T})^2 \|u - v\|_{Y_T}.
 \end{aligned}$$

To conclude, if we choose $T, M > 0$ such that

$$\left\{ C \|u_0\|_{\dot{H}^s} \leq \frac{M}{2}, \quad T^{1-\frac{1}{p}} < \min \left\{ \frac{1}{2C \|\omega\|_{L^p} M^2}, \frac{1}{8\tilde{C} \|\omega\|_{L^p} M^2} \right\} \right\},$$

we get the claim. $\square$

Data availability

Data will be made available on request.

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