



Semi-linear parabolic equations on homogenous Lie groups arising from mean field games

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Abstract

The existence and the uniqueness of solutions to some semilinear parabolic equations on homogeneous Lie groups, namely, the Fokker–Planck equation and the Hamilton–Jacobi equation, are addressed. The anisotropic geometry of the state space plays a crucial role in our analysis and creates several issues that need to be overcome. Indeed, the ellipticity directions span, at any point, subspaces of dimension strictly less than the dimension of the state space. Finally, the above results are used to obtain the short-time existence of classical solutions to the mean field games system defined on an homogeneous Lie group.

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1 Introduction

In the last decades, the existence and the regularity of solutions to linear, semi-linear and fully nonlinear equations defined on sub-Riemannian structures have attracted a lot of attention. Besides their own interests, these equations arise from several models such as diffusion process, control theory and human vision (we refer, for instance, to [5, 19]). In all these settings such equations have in common the fact that their ellipticity directions span, at any point, subspaces of dimension strictly less than the dimension of the state space and all the remaining directions are recovered from commutators. It follows that the underlying geometric structure of the state space is of anisotropic type and this plays a crucial role in the analysis of solutions to elliptic and parabolic equations.

We recall that the classical results on the subelliptic Laplacian operator and the relative results on the fundamental solution of the subelliptic heat equation can be found in [29, 30, 43].

In the elliptic case, several results are present in literature on divergence form operators both for linear and semi-linear equations: Harnack’s inequality, regularity results, existence and size estimates of the Green’s function can be found, for instance, in [25, 35]. To the authors’ knowledge, only few results are available for non-divergence form operators, for which we refer to [9, 11, 42]. The parabolic counterpart, despite its great relevance from the applicative viewpoint, has been less investigated, and we mention [3] and references therein. A general and self-contained introduction to the subject can be found in [3, 8, 12].

In this work we focus our attention on two semi-linear parabolic equations posed on an homogenous Lie group: Hamilton–Jacobi equation and Fokker–Planck equation, both arising from mean field games (MFG) theory. In particular, we are interested in showing the existence of classical solutions to such equations in order to obtain, as a consequence, the existence of classical solutions to the MFG system. In the framework of evolutive noncoercive MFG, we quote here the results in [1, 2, 14, 36, 37, 39] but in these cases a key assumption is that the coefficients in the dynamics grow at most linearly. Up to the authors’ knowledge, no results are available for parabolic MFG problem in unbounded domain for general Lie groups.

Next, we give an overview of the main results proved in this paper and the strategy of proofs.

1.1 Main results and strategy of proof

1.1.1 Fokker–Planck equation

The first problem we address is the existence of classical solutions to the Fokker–Planck equation

$$\begin{cases} \partial_t \rho - \sigma \Delta_G \rho - \operatorname{div}_G(b(t, x)\rho) = 0, & (t, x) \in (0, T) \times \mathbb{R}^d \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}^d \end{cases} \quad (1.1)$$

where $T > 0$, $\sigma > 0$, Δ_G and div_G are respectively the horizontal Laplacian and the horizontal divergence with respect of a family of left invariant smooth vector fields $\{X_i\}_{i=1,\dots,m}$ associated to a homogeneous Lie group (see Sect. 2.1) and the drift b is regular and bounded w.r.t. these vector fields. Note that, with such regularity assumption on b , Eq. (1.1) can be written as

$$\partial_t \rho - \sigma \Delta_G \rho - \operatorname{div}_G(b(t, x)\rho) - b(t, x)\nabla_G \rho = 0 \quad (t, x) \in (0, T) \times \mathbb{R}^d.$$

We show the existence of a global classical solution to (1.1) which, besides its own interests, it will be a fundamental tool to study the Hamilton–Jacobi equation looking at (1.1) as the dual of such equation for a suitable choice of the drift b . It will also play a crucial role in the study of MFG. In this case the model we have in mind is obtained taking

$$b(t, x) = \gamma |\nabla_G u|^{\gamma-2} \nabla_G u$$

where $\gamma \geq 2$, u is the solution to a Hamilton–Jacobi equation and ∇_G is the horizontal gradient.

We also find uniqueness of the solution of the Fokker–Planck equation by showing uniqueness of the solution to a general linear equation with bounded coefficients. In our opinion this result has its own interest. Let us remark that in [16] a similar result is obtained with a different approach under stronger assumptions on the coefficients and for a different linear subelliptic equation.

Moreover we also find an Hölder regularity result w.r.t. the flat Wasserstein metric d_0 which will be defined in Sect. 3.2.1.

Throughout this paper the space $C_G^{1,\nu}(\mathbb{R}^d)$, defined in Sect. 2.1, is the Hölder space w.r.t. the horizontal derivatives and the metric associated to the group.

Theorem 1.1 *Consider a nonnegative continuous function $\rho_0 \in L^\infty(\mathbb{R}^d)$, with $\int_{\mathbb{R}^d} \rho_0(x) dx = 1$ and for some $\delta \in (0, 1]$ a drift function*

$$b \in C([0, T]; (C_G^{1,\delta}(\mathbb{R}^d))^m) \text{ with } \sup_{t \in [0, T]} \|b(t, \cdot)\|_{(C_G^{1,\delta}(\mathbb{R}^d))^m} < +\infty.$$

Then, Eq. (1.1) has a unique classical bounded solution $\rho \in C((0, T); C_{G,loc}^{2,\delta}(\mathbb{R}^d))$. Moreover, the following hold.

- (i) $0 \leq \rho \leq \|\rho_0\|_{L^\infty(\mathbb{R}^d)}$ and $\int_{\mathbb{R}^d} \rho(t, x) dx = 1$ for every $t \in [0, T]$.
(ii) For every $\varphi \in C([0, T]; W_G^{1,2}(\mathbb{R}^d))$ with $\partial_t \varphi \in L^2(0, T; (W_G^{1,2}(\mathbb{R}^d))')$ we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \rho(x, t) \varphi(x, t) dx - \int_{\mathbb{R}^d} \rho_0(x) \varphi(x, 0) dx - \iint_{(0,t) \times \mathbb{R}^d} \partial_t \varphi(t, x) \rho(t, x) dx dt \\ & + \iint_{(0,t) \times \mathbb{R}^d} \nabla_G \varphi(t, x) (\sigma \nabla_G \rho(t, x) + b(t, x) \rho) dx dt = 0. \end{aligned}$$

- (iii) There exists a constant $C_\rho \geq 0$ such that

$$d_0(\rho_t, \rho_s) \leq C_\rho |t - s|^{\frac{1}{2}} \quad \forall t, s \in [0, T]$$

where ρ_t is the measure associated to the density $\rho(t, \cdot)$.

The existence part of the statement is obtained approximating problem (1.1) by truncation and exploiting some regularity results which strongly rely on the Hörmander condition. The uniqueness part of the statement is obtained finding a suitable subsolution used as a test function related to the fundamental solution of the horizontal heat equation.

Note that, as proved in [6, 41], the measure $\{\rho_t\}_{t \in [0, T]}$ represents the law of the stochastic process

$$d\xi_t = \sum_{j=1}^m X_j(\xi_t) \circ dB_t^j + X_0(\xi_t) dt$$

where $\circ$ denotes the Stratonovich stochastic integral, $\{B^j\}_{j=1,\dots,m}$ are d dimensional independent Brownian motions and X_0 is the drift term explicitly provided in [41]. However, since the data can grow more than linearly, we are not able to provide a bound on the moments of such a process which is a key result to get the solution in the non degenerate case.

1.1.2 Hamilton–Jacobi equation

We are interested in studying the well-posedness of the following Hamilton–Jacobi equation

$$\begin{cases} \partial_t u(t, x) - \sigma \Delta_G u(t, x) + |\nabla_G u(t, x)|^\gamma = F(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.2)$$

with $\gamma \geq 2$ under the assumptions

$$F \in C^0([0, T]; C_G^1(\mathbb{R}^d)) \cap L^2((0, T) \times \mathbb{R}^d) \text{ with } \sup_{t \in [0, T]} \|F(t, \cdot)\|_{C_G^1(\mathbb{R}^d)} < +\infty \quad (1.3)$$

$$u_0 \in W_G^{2, \infty}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d), \quad u_0 \geq 0. \quad (1.4)$$

To do this we will study a general semi-linear parabolic equation of the following type

$$\begin{cases} \partial_t u(t, x) - \sigma \Delta_G u(t, x) = f(t, x, \nabla_G u(t, x)), & (t, x) \in (0, T) \times \mathbb{R}^d \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (1.5)$$

From the structure of (1.2) we assume the following on $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}$.

(HP) $f(\cdot, x, p) \in C([0, T])$, $f(t, \cdot, p) \in C^1(\mathbb{R}^d)$ with $\sup_{t \in [0, T]} \|f(t, \cdot, p)\|_{C_G^1(\mathbb{R}^d)} < +\infty$ and there exist $C_f \geq 0$, $\gamma \geq 2$ such that

$$\begin{aligned} f(t, x, p) &\leq C_f(1 + |p|^\gamma), \quad (t, x, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^m, \\ |f(t, x, p) - f(t, x, q)| &\leq C_f(|p|^{\gamma-1} + |q|^{\gamma-1})|p - q|, \\ t &\in [0, T], \quad x \in \mathbb{R}^d, \quad p, q \in \mathbb{R}^m. \end{aligned}$$

(HP') there exists $C_f \geq 0$ such that, for every $i = 1, \dots, m$, $t \in [0, T]$, $x \in \mathbb{R}^d$ and $p, q \in \mathbb{R}^m$, there hold

$$\begin{aligned} |X_i f(t, x, p)| &\leq C_f(1 + |p|^\gamma), \\ |\partial_{p_i} f(t, x, p)| &\leq C_f(1 + |p|^{\gamma-1}), \\ |X_i (f(t, x, p) - f(t, x, q))| &\leq C_f(|p|^{\gamma-1} + |q|^{\gamma-1})|p - q|, \\ |\partial_{p_i} (f(t, x, p) - f(t, x, q))| &\leq C_f(|p|^{\gamma-2} + |q|^{\gamma-2})|p - q|. \end{aligned}$$

The first result is the existence and uniqueness of a classical solution of (1.5) for small times.

Theorem 1.2 *Let f satisfy **(HP)** and **(HP')** and let $u_0 \in W_G^{2, \infty}(\mathbb{R}^d)$. Then, there exists $T_0 > 0$ such that for any $T \leq T_0$ there exists a unique classical solution $u \in C([0, T], W_G^{2, \infty}(\mathbb{R}^d)) \cap C_{G, loc}^{1+\frac{\alpha}{2}, 2+\alpha}((0, T] \times \mathbb{R}^d)$ to (1.5) such that*

$$\sup_{t \in [0, T]} \|u(t)\|_{W_G^{2, \infty}(\mathbb{R}^d)} \leq \kappa(T_0)$$

for some $\kappa(T_0) > 0$.

In order to prove the above small-time existence of classical solutions to the semi-linear Hamilton–Jacobi equation (1.5) we use the Duhamel formula and the decay estimate of the subelliptic heat semigroup (Lemma 4.1).

Next, we state the second main result on the global in time existence of classical solutions to the Hamilton–Jacobi equation (1.2). To this end, we establish an estimate of the horizontal gradient of u using the Bernstein method suitably adapted to sub-Riemannian framework. In order to apply this method, we need an extra technical assumption for a set of smooth right-invariant vector fields $\{Y_1, \dots, Y_m\}$ (see Sect. 2) and the property $[X_i, Y_j] = 0$ (see [38, Lemma 2.1]).

Theorem 1.3 *Let $F : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfy (1.3) and let u_0 satisfy (1.4). Let $\{Y_1, \dots, Y_m\}$ be a set of right-invariant smooth vector fields on $\mathbb{R}^d$ and assume that*

$$\|Y_j F\|_{L^\infty([0, T] \times \mathbb{R}^d)} + \|Y_j u_0\|_{L^\infty(\mathbb{R}^d)} < \infty. \quad (1.6)$$

Let $T > 0$ be arbitrary. Then, there exists a classical solution $u \in C_{G, loc}^{1+\frac{\alpha}{2}, 2+\alpha}((0, T] \times \mathbb{R}^d)$ to (1.2).

To obtain the global in time existence, we first show that the solution is bounded in space and time—uniformly w.r.t. the second order coefficient σ —(Proposition 4.4) and that also its horizontal gradient is bounded globally in time and locally in space (Proposition 4.6)—uniformly w.r.t. σ . The common main technical point of the above results is the use of the duality between Hamilton–Jacobi equation and Fokker–Planck equation: given a suitable choice of the drift function and the corresponding solution to the Fokker–Planck equation we deduce properties for the solution u . This duality property have been investigated in several other settings, see for instance [17, 23, 28]. Moreover, in order to get the estimate on the horizontal gradient we use the Bernstein method adapted to the sub-Riemannian framework. More precisely, one cannot apply a vector field generating the Lie group to the equation because, otherwise, one will get some extra terms involving commutators which are difficult to deal with. In order to overcome this issue, we first differentiate the equation by the family of smooth right-invariant vector fields introduced in the statement recalling that given a left-invariant and a right-invariant vector fields X and Y , respectively, we have $[X, Y] = 0$ (see [38, Lemma 2.1]). Hence, we obtain a local bound for the gradient of the solution on the distribution generated by such right-invariant vector fields. Finally, since the right-invariant distance is locally Lipschitz equivalent to the left-invariant one, we get the desired estimate. We observe that a novel and non-standard regularization techniques have been used in [18] in order to study the regularity of viscosity solutions to some nonlinear PDEs, which is however different from the method we propose here that is strictly related to the nature of the Hamilton–Jacobi equation we are interested in.

We conclude this part, observing that the assumption $\gamma \geq 2$ is needed in order to gain regularity on the solution to the Hamilton–Jacobi equation when the initial data is regular enough. In a forthcoming paper, we will address a similar problem investigating the sub-quadratic case with merely local integrability of the solution. In this case, the integrability assumptions of the drift of the transport equation and the duality approach still allow us to obtain classical solutions to the Fokker–Planck equation, to the Hamilton–Jacobi equation and, consequently, to the MFG system.

1.1.3 Mean field games

We conclude this work investigating the existence for small times of solutions to the MFG system

$$\begin{cases} -\partial_t u(t, x) - \sigma \Delta_G u(t, x) + |\nabla_G u(t, x)|^\gamma = F_{\text{MFG}}[\rho_t](x), & (t, x) \in (0, T) \times \mathbb{R}^d \\ \partial_t \rho - \sigma \Delta_G \rho - \operatorname{div}_G(\gamma |\nabla_G u(t, x)|^{\gamma-2} \nabla_G u(t, x) \rho) = 0, & (t, x) \in (0, T) \times \mathbb{R}^d \\ u(T, x) = u_T(x), \quad \rho(0, x) = \rho_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (1.7)$$

Such a system couples the equations we have previously studied, i.e., the Hamilton–Jacobi equation in (1.2) in which the potential F_{MFG} is a strongly regularizing nonlocal term which depends on the distribution ρ and the Fokker–Planck equation in (1.7) whose drift is defined by the optimal feedback associated with (1.2). In the following, we consider a coupling function $F_{\text{MFG}} : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow \mathbb{R}$ and we assume the following:

(MFG) the map $\rho \mapsto F_{\text{MFG}}[\rho](x)$ is Lipschitz continuous w.r.t. d_0 from $\mathcal{P}(\mathbb{R}^d)$ to $C_G^1(\mathbb{R}^d)$ and the function $F_{\text{MFG}}[\rho](\cdot)$ is bounded in $C_G^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ uniformly w.r.t. ρ

where d_0 is the Fortet–Mourier distance introduced in (3.15).

MFG theory, introduced in [32–34], is devoted to the study of differential games with a very large number of interacting agents. A typical model is described by a system of PDEs: a backward-in-time Hamilton–Jacobi equation whose solution is the value function for the generic player (and also provides their optimal choices) and a forward-in-time Fokker–Planck equation which describes how the distribution of individuals changes. The two equations are coupled in a way that takes into account both the state of a single agent and how he/she is influenced by the others. The system of PDEs describes a model with a continuum of players which is, clearly, not realistic. However, the solution of such a system is expected to capture the behavior of Nash equilibria for differential N-players game as the number of agents goes to infinity. For an extensive and detailed introduction to the subject we refer to [7, 15, 27] and references therein.

The motivation for studying our model is the following: heuristically, given $X_1, \dots, X_m$ smooth vector fields on $\mathbb{R}^d$, each single player can move only along the directions generated by the set of vector fields. In [21, 24] stationary subelliptic MFG systems have been studied in the torus. We also cite the papers [2, 14, 36, 37, 39], where, in the whole space, the generic player has some forbidden directions because it follows either a dynamic generated by the vector fields on Heisenberg group, or a Grushin dynamic or it controls its acceleration. In particular, the results in [36, 37] are obtained under the key assumption that the coefficients of the vector fields X_i grow at most linearly. Up to the authors’ knowledge, no results are available for parabolic MFG problem in unbounded Lie groups. Clearly, the unboundedness of the state space, eventually with unbounded vector fields, gives rise to several difficulties to overcome.

From the optimization point of view, a generic player wants to minimize the cost

$$\mathbb{E} \left[\int_t^T (C|\alpha(s)|^{\gamma^*} + F_{\text{MFG}}[\rho_s](\xi(s)) ds + u_T(\xi(T)) \right]$$

where γ^* is the conjugate index of γ and $\xi(\cdot)$ denotes the stochastic process that solves

$$d\xi(s) = \alpha(s)ds + \sum_{j=1}^m X_j(\xi(s)) \circ dB_j$$

where α is the control chosen by the player while $\circ$ and B_j are as before.

We prove a small time existence result of a classical solution. Note that we cannot prove the existence for any time T because of the lack of compactness in the results of Theorem 1.3.

Theorem 1.4 *Assume $\gamma \geq 2$. Let $u_T \in W^{2,\infty}(\mathbb{R}^d)$, $\rho_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ such that $\rho_0 \geq 0$ and $\int_{\mathbb{R}^d} \rho_0(x) dx = 1$ and let F_{MFG} satisfies **(MFG)**. Then, there exists $T_0 > 0$ such that for any $T \leq T_0$ there exists a classical solution $(u, \rho) \in C((0, T); C_{\text{G,loc}}^{2+\alpha, 1+\frac{\alpha}{2}}(\mathbb{R}^d)) \times C((0, T); C_{\text{G,loc}}^{2,v}(\mathbb{R}^d))$ to (1.7).*

2 Sub-Riemannian setting

Let $(G, *)$ be a homogeneous Lie group and let $\{\delta_\lambda\}_{\lambda>0}$ be a family of dilations which are automorphisms of the group, i.e.,

$$\delta_\lambda(x * y) = \delta_\lambda(x) * \delta_\lambda(y)$$

for all $x, y \in G$ and $\lambda > 0$. Let $\mathfrak{g}$ be the Lie algebra associated with the group G . The dilations of the group induce a direct sum decomposition on $\mathfrak{g}$, i.e.,

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k.$$

In particular, V_1 is called the horizontal layer and its elements are left-invariant vector fields. We can identify $(G, *)$ with $\mathbb{R}^d$ via the so-called exponential map

$$\exp : \mathfrak{g} \rightarrow G$$

which turns out to be a diffeomorphism. Given a basis $X_1, \dots, X_d$ adapted to the stratification, any $x \in G$ can be written in a unique way as

$$x = \exp(x_1 X_1 + \cdots + x_d X_d)$$

and one can identify x with $(x_1, \dots, x_d)$ and G with $(\mathbb{R}^d, \circ)$ where the group law is given by the Baker–Campbell–Hausdorff formula.

Hence, hereafter we work on $\mathbb{R}^d$ and we consider an orthonormal basis $X_1, \dots, X_m$ of the horizontal layer V_1 where the vector fields $X_1, \dots, X_m$ are assumed to be smooth. We assume that such a family of vector fields satisfies the Hörmander condition, i.e.,

$$\text{Lie}(X_1, \dots, X_m)(x) = \mathbb{R}^d \quad \text{for all } x \in \mathbb{R}^d.$$

and to be homogeneous of degree one w.r.t. the family of dilations.

We recall that a vector field X is left-invariant when, for any f sufficiently regular and for any $\eta \in \mathbb{R}^d$ fixed, there holds

$$X(f(\eta \circ \xi)) = (Xf)(\eta \circ \xi) \quad \forall \xi \in \mathbb{R}^d \quad (2.1)$$

and similarly for right-invariant vector fields.

Moreover, given $a, b \in \mathbb{R}$ we say that an absolutely continuous curve $\gamma : [a, b] \rightarrow \mathbb{R}^d$ is horizontal if there exist $\alpha_1, \dots, \alpha_m \in L^1(a, b)$ such that

$$\dot{\gamma}(t) = \sum_{j=1}^m \alpha_j(t) X_j(\gamma(t)), \quad \text{a.e. } t \in [a, b]$$

and the length of γ is defined as

$$\ell(\gamma) = \int_a^b |\alpha(t)| \, dt.$$

Under the Hörmander condition, a well-known result by Chow states that any two points on $\mathbb{R}^d$ can be connected by an horizontal curve. Hence, the definition of Carnot–Carathéodory distance, denoted by d_{CC} , is well-posed, i.e.,

$$d_{CC}(x, y) = \inf\{\ell(\gamma) : \gamma \text{ is an horizontal curve joining } x \text{ to } y\}.$$

One can prove a variational interpretation of the above distance as

$$\inf\{T > 0 : \exists \gamma : [0, T] \rightarrow \mathbb{R}^d, \text{ horizontal and joining } x \text{ to } y \text{ with } |\dot{\gamma}(t)| \leq 1\}.$$

Note that, the Carnot–Carathéodory distance is not equivalent to the Euclidean one. Indeed, it is well-known that, for any K compact set, there exists a constant $C > 0$ such that, for any y and x in K , we have

$$\frac{1}{C} |x - y| \leq d_{CC}(x, y) \leq C |x - y|^{\sigma(x)}$$

where $\sigma(x) \in \mathbb{N}$ is the nonholonomic degree at $x \in \mathbb{R}^d$, that is, the maximum of the degrees of the iterated brackets occurring to fulfill the Hörmander condition. Using

the family of dilations and the sub-Riemannian distance one can define a norm on $\mathbb{R}^d$ tailored from the Lie group,

$$\|x\|_{\text{CC}} = d_{\text{CC}}(0, x).$$

However, from the homogeneity of the vector fields $X_1, \dots, X_m$ and the stratification of $\mathbb{R}^d$, one can define an homogeneous norm $\|\cdot\|_{\text{G}}$ and the homogeneous dimension Q of the group as

$$\|x\|_{\text{G}} = \left(\sum_{j=1}^k |y_j|^{\frac{2k!}{j}} \right)^{\frac{1}{2k!}} \quad \text{and} \quad Q = \sum_{j=1}^k j \dim V_j$$

where $y_j = \pi_j(x)$ and $\pi_j : \mathbb{R}^d \rightarrow \mathbb{R}^{\dim V_j}$ denotes the projection onto the j -layer V_j .

Example 2.1 Examples of homogeneous Lie groups are Heisenberg-type groups, Engel group and Martinet group (see, for instance, [40]).

For completeness, we write the following result on homogenous functions of Lie groups which will be useful later.

Remark 2.2 Let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ be a homogeneous function of degree 1 and assume $|g(x)| \leq c$ for any $\|x\|_{\text{G}} \leq 1$. Then $|g(x)| \leq c\|x\|_{\text{G}}$, for all $x \in \mathbb{R}^d$. Indeed, taking $y \in \mathbb{R}^d$ we have

$$|g(y)| = \left| g \left(\delta_{\|y\|_{\text{G}}} \delta_{\frac{1}{\|y\|_{\text{G}}}}(y) \right) \right| = \|y\|_{\text{G}} \left| g \left(\delta_{\frac{1}{\|y\|_{\text{G}}}}(y) \right) \right|.$$

So, we conclude, observing that

$$\left\| \delta_{\frac{1}{\|y\|_{\text{G}}}}(y) \right\|_{\text{G}} \leq \frac{1}{\|y\|_{\text{G}}} \|y\|_{\text{G}} \leq 1.$$

2.1 Subelliptic equations

Let us consider a family of smooth left-invariant vector fields $X_1, \dots, X_m$ as in the beginning of this section, satisfying the Hörmander condition and a function $u : \mathbb{R}^d \rightarrow \mathbb{R}$. We define the horizontal gradient and the horizontal Laplacian of u as

$$\nabla_{\text{G}} u = (X_1 u, \dots, X_m u)^T \in \mathbb{R}^m$$

and respectively

$$\Delta_{\text{G}} u = \sum_{i=1}^m X_i^2 u \in \mathbb{R}.$$

For any vector-valued function $u : \mathbb{R}^d \rightarrow \mathbb{R}^m$, we will consider the divergence operator induced by the vector fields, that is,

$$\operatorname{div}_G u = X_1 u_1 + \cdots + X_m u_m$$

where u_i denotes the i th component of u for $i = 1, \dots, m$. Next, we recall the definition of Hölder space associated with the family of vector fields. For every multi-index $J = (j_1, \dots, j_m) \in \mathbb{N}^m$ let $X^J = X_1^{j_1} \cdots X_m^{j_m}$ and let $|J| = j_1 + \cdots + j_m$ be the length of multi-index.

We introduce $C^0(\mathbb{R}^d)$ as the set of continuous (possibly unbounded) functions on $\mathbb{R}^d$ and we associate the norm $\|u\|_{C^0(\mathbb{R}^d)} = \|u\|_{L^\infty(\mathbb{R}^d)}$.

Moreover for $\alpha \in (0, 1]$, given $u \in C^0(\mathbb{R}^d)$ and $U \subset \mathbb{R}^d$ we define the seminorm

$$[u]_{C_G^{0,\alpha}(U)} = \sup_{\substack{x, y \in U \\ x \neq y}} \frac{|u(x) - u(y)|}{d_{CC}(x, y)^\alpha},$$

where d_{CC} is the Carnot–Carathéodory distance defined before. Note that, for $\alpha = 1$ the above seminorm coincide with the Lipschitz seminorm tailored to the group structure. We introduce

$$C_G^{0,\alpha}(\mathbb{R}^d) = \left\{ u \in C^0(\mathbb{R}^d) : [u]_{C_G^{0,\alpha}(U)} < \infty, \text{ for every compact } U \subset \mathbb{R}^d \right\}$$

and the corresponding norm

$$\|u\|_{C_G^{0,\alpha}(\mathbb{R}^d)} := \|u\|_{C^0(\mathbb{R}^d)} + [u]_{C_G^{0,\alpha}(\mathbb{R}^d)}.$$

Similarly, for $r \in \mathbb{N}$ and $\alpha \in (0, 1]$ we define

$$C_G^{r,\alpha}(\mathbb{R}^d) = \left\{ u \in C^0(\mathbb{R}^d) : X^J u \in C_G^{0,\alpha}(\mathbb{R}^d), \forall |J| \leq r \right\}$$

and the corresponding norm

$$\|u\|_{C_G^{r,\alpha}(\mathbb{R}^d)} := \sum_{0 \leq |J| \leq r} \left(\|X^J u\|_{C^0(\mathbb{R}^d)} + [X^J u]_{C_G^{0,\alpha}(\mathbb{R}^d)} \right).$$

The spaces $C_G^{r,\alpha}(\mathbb{R}^d)$ with the norm $\|\cdot\|_{C_G^{r,\alpha}(\mathbb{R}^d)}$ are Banach spaces for any $r \in \mathbb{N}$ and any $\alpha \in (0, 1]$.

We conclude this preliminary section recalling, also, the definition of horizontal Sobolev spaces. Let $r \in \mathbb{N}$ and $1 \leq p \leq \infty$. We define the space

$$W_G^{r,p}(\mathbb{R}^d) = \left\{ u \in L^p(\mathbb{R}^d) : X^J u \in L^p(\mathbb{R}^d), \forall |J| \leq r \right\}$$

endowed with the norm

$$\|u\|_{W_G^{r,p}(\mathbb{R}^d)} = \left(\sum_{|J| \leq r} \int_{\mathbb{R}^d} |X^J u(x)|^p dx \right)^{\frac{1}{p}},$$

for $r \in [1, \infty)$ and similarly for $p = \infty$.

We denote by $C_{G,\text{loc}}^{r,\alpha}(\mathbb{R}^d)$ and $W_{G,\text{loc}}^{r,p}(\mathbb{R}^d)$ the horizontal Hölder space and horizontal Sobolev space, respectively, as in the above manner with $\mathbb{R}^d$ replaced by any compact subset Ω .

In order to have a self-contained presentation of the work, next we recall the main results used below on linear parabolic subelliptic equations and a Sobolev embedding result.

Theorem 2.3 [31, Theorem 1] *Let $p \in (1, \infty)$ and Ω an open subset of $\mathbb{R}^d$. If $f \in L^p([0, T] \times \Omega)$, then the problem*

$$\begin{cases} \partial_t u - \Delta_G u = f \\ u(0, x) = 0 \end{cases}$$

admits a solution $u \in C([0, T]; L^p(\Omega))$ such that $\partial_t u, \Delta_G u \in L^p([0, T] \times \Omega)$ and

$$\|\partial_t u\|_{L^p([0,T] \times \Omega)} + \|\Delta_G u\|_{L^p([0,T] \times \Omega)} \leq C_p \|f\|_{L^p([0,T] \times \Omega)}$$

where C_p depends only on p and on the holomorphic constant of the semi-group $e^{t\Delta_G}$.

Theorem 2.4 [26, Theorem 1.4] *Let $\Omega \subset \mathbb{R}^d$ and let U be a compact subset of $(0, T] \times \Omega$. Let $Q + 2 < p < 2(Q + 2)$. Then, there exists a positive constant C , depending only on U , Ω , T and p such that for $\alpha = \frac{1}{p}(p - (Q + 2))$ and for every $u \in W^{1,p}([0, T]; W_G^{2,p}(\Omega))$ we have*

$$\|u\|_{C_G^{1,\alpha}(U)} \leq C \left(\|\partial_t u\|_{L^p((0,T] \times \Omega)} + \|u\|_{W_G^{2,p}((0,T] \times \Omega)} \right).$$

(Recall that Q is the homogeneous dimension of the group defined before).

3 Fokker–Planck equation

3.1 Existence of solutions

We consider the Fokker–Planck equation

$$\begin{cases} \partial_t \rho - \sigma \Delta_G \rho - \text{div}_G(b(t, x)\rho) = 0, & (t, x) \in (0, T) \times \mathbb{R}^d \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}^d \end{cases} \quad (3.1)$$

where $b \in C([0, T]; (C_G^{1,\delta}(\mathbb{R}^d))^m)$, for some $\delta \in (0, 1)$.

Proposition 3.1 *Under the assumptions of Theorem 1.1 Eq. (3.1) has a classical bounded solution $\rho \in C((0, T); C_{G,loc}^{2,\delta}(\mathbb{R}^d))$. Moreover,*

$$0 \leq \rho \leq \|\rho_0\|_{L^\infty(\mathbb{R}^d)}, \quad \text{and} \quad \int_{\mathbb{R}^d} \rho(t, x) dx = 1, \quad \forall t \in [0, T]. \quad (3.2)$$

Proof Consider the initial-boundary value problem

$$\begin{cases} \partial_t \rho_R - \sigma \Delta_G \rho_R - \operatorname{div}_G(b(t, x) \rho_R) = 0, & (t, x) \in (0, T) \times B_R \\ \rho_R(0, x) = \rho_0(x), & x \in B_R \\ \rho_R(t, x) = 0, & (t, x) \in (0, T) \times \partial B_R \end{cases} \quad (3.3)$$

where B_R is the ball of radius R w.r.t. the Carnot–Carathéodory distance. We shall first solve problem (3.3) establishing several properties of ρ_R and after, letting $R \rightarrow \infty$, we obtain a solution to problem (3.1) with the desired properties.

Invoking Lions’ Theorem (see [13, theorem X.9]), we infer that there is a unique function $\rho_R \in L^2(0, T; H_{0,G}^1(B_R)) \cap C([0, T]; L^2(B_R))$ with $\partial_t \rho_R \in L^2(0, T; H_G^{-1}(B_R))$ such that

$$\begin{cases} \int_{B_R} \partial_t \rho_R(t) v dx + \\ \int_{B_R} (\sigma \nabla_G \rho_R(t) + b(t) \rho_R(t)) \nabla_G v = 0, & v \in H_{0,G}^1(B_R), \text{ a.e. } t \in [0, T] \\ \rho_R(0, x) = \rho_0(x), & x \in B_R \end{cases} \quad (3.4)$$

where $H_{0,G}^1(B_R)$ is the closure of $C_{G,0}^1(B_R)$ in $W^{1,2}(B_R)$ and $H_G^{-1}(B_R)$ is its dual. Observe that ρ_R is a distributional solution to

$$\partial_t \rho_R - \sigma \Delta_G \rho_R = (\operatorname{div}_G b) \rho_R + b \nabla_G \rho_R$$

where the right hand side belongs to $L^2((0, T) \times B_R)$. The results in [43, Theorem 18] ensures that: $\partial_t \rho_R$, $X_i \rho_R$ and $X_i X_j \rho_R$ belong to $L^2((0, T) \times B_R)$. In particular, we deduce that the differential equation in (3.3) is satisfied for a.e. $(t, x) \in (0, T) \times B_R$.

We claim that $\rho_R \in C([0, T] \times B_R)$ and that, for every domain $\Omega \subset (0, T) \times B_R$, there exist a constant $K(\Omega, R)$ (depending on the assumptions, on Ω and on R) and a constant $K'(\Omega)$ (depending on the assumptions and on Ω), such that

$$\|\rho_R\|_{C_G^{2,\delta}(\Omega)} \leq K(\Omega, R) \quad \text{and} \quad \|\rho_R\|_{C_G^{2,\delta}(\Omega)} \leq K'(\Omega) \quad \text{for } R \text{ sufficiently large} \quad (3.5)$$

(recall that δ is the Hölder exponent of $\nabla_G b$). Indeed, consider a sequence $\{b_n\}_n$ of drifts such that $b_n \in C^\infty$ and b_n uniformly converges to b in $[0, T] \times B_R$ as $n \rightarrow \infty$. Therefore, by the same arguments as before, problem (3.3) with b replaced by b_n has a solution $\rho_{R,n}$. Applying iteratively [43, Theorem 18], we infer that $\rho_{R,n} \in C^\infty$ and, by standard comparison principle, we get $\|\rho_{R,n}\|_{L^\infty([0,T] \times \mathbb{R}^d)} \leq \|\rho_0\|_{L^\infty(\mathbb{R}^d)}$.

Moreover, the results in [10, Theorem 1.1] (with $k = 0$) ensure that $\rho_{R,n}$ fulfills (3.5) with constants K and K' , both independent of n . Letting $n \rightarrow \infty$, we accomplish the proof of our claim (3.5).

Since ρ_R is continuous and $\operatorname{div}_G b$ is bounded, standard comparison principle entails

$$0 \leq \rho_R(t, x) \leq \|\rho_0\|_{L^\infty(\mathbb{R}^d)} \quad \forall (t, x) \in [0, T] \times B_R$$

and that, for each (t, x) , the value $\rho_R(t, x)$ is nondecreasing with respect to R , i.e.,

$$\rho_{R_1}(t, x) \geq \rho_{R_2}(t, x) \quad \forall R_1 \geq R_2, (t, x) \in [0, T] \times B_{R_2}.$$

From the properties proved so far we obtain that for each (t, x) there exists the limit $\lim_{R \rightarrow \infty} \rho_R(t, x)$ which we denote by $\rho(t, x)$ and, clearly, $0 \leq \rho \leq \|\rho_0\|_{L^\infty(\mathbb{R}^d)}$. By a standard diagonalization process, using (3.5), $\rho \in C_{G, \text{loc}}^{2, \delta}(\Omega)$ for every domain $\Omega \subset (0, T) \times \mathbb{R}^d$.

We now proceed with the proof of the second part of (3.2). First, we consider again the approximating problem (3.3). We integrate the equation on $[0, T] \times B_R$, we use the divergence theorem and we note that, by the standard Hopf Lemma, $\partial \rho_R / \partial \nu \leq 0$ where ν is the outward pointing normal to ∂B_R . Hence we get that $\rho_R(t) \in L^1(B_R)$ independently of $R \geq 0$ for any $t \in (0, T]$. Moreover, letting $R \uparrow \infty$ we deduce $\rho(t) \in L^1(\mathbb{R}^d)$ for any $t \in (0, T]$. Let us now consider a function $\xi \in C_c^\infty(\mathbb{R}^d)$ such that $\xi(x) = 1$ for any $x \in B_1$ and $\xi(x) = 0$ for any $x \in \mathbb{R}^d \setminus B_2$, and define $\xi_R(x) = \xi\left(\delta_{\frac{1}{R}}(x)\right)$ for each $R > 0$. Hence, multiplying (3.1) by ξ_R and integrating by parts we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \rho(t, x) \xi_R(x) \, dx + \iint_{(0, t) \times \mathbb{R}^d} (-\sigma \Delta_G \xi_R(x) + b \nabla \xi_R(x)) \rho(s, x) \, ds \, dx \\ &= \int_{\mathbb{R}^d} \rho_0(x) \xi_R(x) \, dx. \end{aligned}$$

Observing that

$$\Delta_G \xi_R(x) = \frac{1}{R^2} \Delta_G \xi\left(\delta_{\frac{1}{R}}(x)\right) \quad \text{and} \quad \nabla_G \xi_R(x) = \frac{1}{R} \nabla_G \xi\left(\delta_{\frac{1}{R}}(x)\right),$$

by dominated convergence theorem as $R \uparrow \infty$, we conclude

$$\int_{\mathbb{R}^d} \rho(t, x) \, dx = \int_{\mathbb{R}^d} \rho_0(x) \, dx \quad (0 \leq t \leq T).$$

□

Proposition 3.2 *Under the assumptions of Theorem 1.1, let ρ be the solution of (3.1) constructed in Proposition 3.1. Then:*

(i) there exists a constant K depending on b , σ and $\|\rho_0\|_{L^\infty(\mathbb{R}^d)}$ such that

$$\begin{aligned} \int_{\mathbb{R}^d} |\rho(t, x)|^2 dx &\leq K \|\rho_0\|_{L^2(\mathbb{R}^d)}^2 \quad \forall t \in [0, T], \\ \int_{[0, T] \times \mathbb{R}^d} |\nabla_G \rho(t, x)|^2 dx dt &\leq K \|\rho_0\|_{L^2(\mathbb{R}^d)}^2. \end{aligned} \quad (3.6)$$

(ii) For every $\varphi \in C([0, T]; W_G^{1,2}(\mathbb{R}^d))$ with $\partial_t \varphi \in L^2(0, T; (W_G^{1,2}(\mathbb{R}^d))')$, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} \rho(x, t) \varphi(x, t) dx - \int_{\mathbb{R}^d} \rho_0(x) \varphi(x, t) dx \\ &+ \iint_{(0, t) \times \mathbb{R}^d} \nabla_G \varphi(x, s) (\sigma \nabla_G \rho(x, s) + b \rho(x, s)) dx ds \\ &- \iint_{(0, t) \times \mathbb{R}^d} \partial_t \varphi(x, s) \rho(x, s) dx ds = 0. \end{aligned} \quad (3.7)$$

Proof For $R > 0$, consider the solution ρ_R to (3.3) found before. For simplicity of notation, we shall denote by K a constant which may change from line to line but which always depends only on the assumptions (in particular it is independent of R). Assume for the moment that, for every $t \in [0, T]$ and $R > 0$, there holds

$$\begin{aligned} &\frac{d}{dt} \left(\int_{B_R} |\rho_R(t, x)|^2 dx \right) + \sigma \int_{B_R} |\nabla_G \rho_R(t, x)|^2 dx \\ &\leq \frac{\|b\|_{L^\infty([0, T] \times \mathbb{R}^d)}^2}{\sigma} \int_{B_R} |\rho_R(t, x)|^2 dx. \end{aligned} \quad (3.8)$$

From (3.8), we deduce

$$\frac{d}{dt} \left(\int_{B_R} |\rho_R(t, x)|^2 dx \right) \leq \frac{\|b\|_{L^\infty}^2}{2\sigma} \int_{B_R} |\rho_R(t, x)|^2 dx.$$

Hence, by Gromwall's lemma, we infer

$$\int_{B_R} |\rho_R(t, x)|^2 dx \leq e^{\frac{\|b\|_{L^\infty}^2}{2\sigma} t} \int_{B_R} \rho_0^2(x) dx \leq e^{\frac{\|b\|_{L^\infty}^2}{2\sigma} t} \|\rho_0\|_2^2 \quad \forall t \in [0, T], \quad R > 0. \quad (3.9)$$

So, as $R \uparrow \infty$ we obtain the former estimate in (3.6). On the other hand, integrating (3.8), we have

$$\begin{aligned} &\int_{B_R} |\rho_R(T, x)|^2 dx - \int_{B_R} |\rho_0(x)|^2 dx + \frac{\sigma}{2} \int_{[0, T] \times B_R} |\nabla_G \rho_R(t, x)|^2 dx dt \\ &\leq \frac{\|b\|_{L^\infty}^2}{2\sigma} \int_{[0, T] \times B_R} |\rho_R(t, x)|^2 dx dt \leq e^{\frac{\|b\|_{L^\infty}^2}{2\sigma} T} \|\rho_0\|_{L^2(\mathbb{R}^d)}^2 \end{aligned}$$

where the last inequality is due to relation (3.9). Again by (3.9), we deduce that there exists a constant K (independent of R and ρ_0) such that

$$\int_{[0,T] \times B_R} |\nabla_G \rho_R(t, x)|^2 dx dt \leq K \|\rho_0\|_{L^2(\mathbb{R}^d)}^2.$$

As $R \uparrow \infty$, we obtain the latter estimate in (3.6). It remains to prove estimate (3.8). To this end, using ρ_R as test function for (3.3), we get

$$\begin{aligned} & \frac{d}{dt} \left(\int_{B_R} |\rho_R(t, x)|^2 dx \right) \\ & + 2\sigma \int_{B_R} |\nabla_G \rho_R(x, t)|^2 dx + 2 \int_{B_R} b(t, x) \rho_R(x, t) \nabla_G \rho_R(x, t) dx = 0. \end{aligned}$$

Using Hölder inequality on the last term, we get (3.8). Finally, from the same reasoning we also get (3.7). $\square$

3.2 Uniqueness and regularity

We recover the uniqueness of solutions to (3.1) by showing the uniqueness of the classical solution to the general linear equations with bounded coefficients of the form

$$\begin{cases} \partial_t \rho - \sigma \Delta_G \rho + B \cdot \nabla_G \rho + Q \rho = 0 & (t, x) \in (0, T) \times \mathbb{R}^d \\ \rho(0, x) = \rho_0 & x \in \mathbb{R}^d. \end{cases} \quad (3.10)$$

Let us remark that in [16] a similar result is obtained with a different approach under stronger assumptions on the coefficients and for a different linear subelliptic equation.

Proposition 3.3 *Let $\rho_0 \in L^\infty(\mathbb{R}^d)$. Let B and Q be bounded continuous functions on $[0, T] \times \mathbb{R}^d$ and, moreover, assume that B has a continuous and bounded horizontal gradient. For $j = 1, 2$, let $\rho_j \in C((0, T); C_{G,loc}^{2,v}(\mathbb{R}^d))$ be two classical solutions to (3.10) such that for some positive constant β we have*

$$\int_0^T \int_{\mathbb{R}^d} |\rho_j(t, x)| e^{-\beta(\|x\|_G^2 + 1)} dx dt < \infty.$$

Then, $\rho_1 = \rho_2$.

The proof of the proposition is postponed after the following technical lemma.

Lemma 3.4 *Let ρ_j , for $j = 1, 2$, be two solutions to (3.10) such that for some $\beta > 0$ we have*

$$\int_0^T \int_{\mathbb{R}^d} |\rho_j(t, x)| e^{-\beta(\|x\|_G^2 + 1)} dx dt < \infty \quad (3.11)$$

and let

$$\tau_0 = \inf\{t \in [0, T] : \rho_1(\cdot, t) \neq \rho_2(\cdot, t)\}.$$

For $\beta_1 > \beta$ and $\bar{\beta} > 0$, define the function

$$\Phi(t, x) = e^{-(\beta_1 + \bar{\beta}(t - \tau_0))(\|x\|_G^2 + 1)}.$$

Then,

$$\partial_t \Phi + \sigma \Delta_G \Phi + B \cdot \nabla_G \Phi + (\operatorname{div}_G B) \Phi \leq 0, \quad (t, x) \in (\tau_0, \tau) \times \mathbb{R}^d \quad (3.12)$$

for $\tau \in (\tau_0, T]$, with $\tau - \tau_0$ sufficiently small and $\bar{\beta}$ sufficiently large,

$$\begin{aligned} \int_{\tau_0}^{\tau} \int_{\mathbb{R}^d} |\rho_j(t, x)| \Phi(t, x) \, dt dx &< \infty \\ \text{and } \int_{\tau_0}^{\tau} \int_{\mathbb{R}^d} |\rho_j(t, x)| \nabla_G \Phi(t, x) \, dt dx &< \infty. \end{aligned} \quad (3.13)$$

Proof From the homogeneity of the norm $\|\cdot\|_G$ and using Remark 2.2 we have that

$$|X_j(\|x\|_G^2)|^2 \leq C \|x\|_G^2, \quad |\nabla_G(\|x\|_G^2)|^2 \leq C \|x\|_G^2 \quad \text{and} \quad |\Delta_G(\|x\|_G^2)|^2 \leq C$$

for a suitable constant $C \geq 0$. Hence, setting for simplicity $\bar{\beta}_1 = \beta_1 + \bar{\beta}(t - \tau_0)$ we have

$$\begin{aligned} \partial_t \Phi + \sigma \Delta_G \Phi + B \cdot \nabla_G \Phi + (\operatorname{div}_G B) \Phi \\ = (-\bar{\beta}(\|x\|_G^2 + 1) + \sigma \bar{\beta}_1^2 |\nabla_G(\|x\|_G^2)|^2 - \sigma \bar{\beta}_1 \Delta_G(\|x\|_G^2) - \bar{\beta}_1 B \cdot \nabla_G(\|x\|_G^2) + \operatorname{div}_G B) \Phi \\ \leq \left(-\bar{\beta}(\|x\|_G^2 + 1) + \sigma \bar{\beta}_1^2 C \|x\|_G^2 + \sigma \bar{\beta}_1 C + \bar{\beta}_1 \|B\|_{L^\infty} C \|x\|_G + \|\operatorname{div}_G B\|_{L^\infty} \right) \Phi. \end{aligned}$$

The proof of (3.12) is thus complete by choosing $\tau - \tau_0$ sufficiently small and $\bar{\beta}$ sufficiently large. Bounds (3.13) are easy consequences of the choice of β_1 and assumption (3.11). $\square$

Proof of Proposition 3.3. Without any loss of generality, we assume $Q \geq 0$ and, we proceed by contradiction assuming that $\rho_1 \neq \rho_2$. Set

$$\tau_0 = \inf\{t \in [0, T] : \rho_1(\cdot, t) \neq \rho_2(\cdot, t)\}.$$

The continuity of ρ_j , for $j = 1, 2$, ensure that the function $\rho = \rho_1 - \rho_2$ satisfies

$$\begin{cases} \partial_t \rho - \sigma \Delta_G \rho + B \cdot \nabla_G \rho + Q \rho = 0, & (t, x) \in (\tau_0, T) \times \mathbb{R}^d \\ \rho(\tau_0, x) = 0, & x \in \mathbb{R}^d. \end{cases} \quad (3.14)$$

For any $\varepsilon > 0$ define the function

$$w(t, x) = \sqrt{\rho(t, x)^2 + \varepsilon}$$

and observe that the following equalities hold

$$\begin{aligned} \partial_t w &= \frac{1}{w} \rho \partial_t \rho & X_j w &= \frac{1}{w} \rho X_j \rho \\ X_j^2 w &= \frac{\varepsilon}{w^3} (X_j \rho)^2 + \frac{\rho}{w} X_j^2 \rho & \Delta_G w &= \frac{\varepsilon}{w^3} |\nabla_G \rho|^2 + \frac{\rho}{w} \Delta_G \rho. \end{aligned}$$

Therefore, multiplying (3.14) by $\frac{\rho}{w}$ we get

$$\begin{aligned} \partial_t w &= \sigma \Delta_G w - \sigma \frac{\varepsilon}{w^3} |\nabla_G \rho|^2 - B \cdot \nabla_G w - Q \frac{\rho^2}{w} \leq \sigma \Delta_G w \\ &\quad - B \cdot \nabla_G w, \quad (t, x) \in (\tau_0, T] \times \mathbb{R}^d. \end{aligned}$$

So, for any nonnegative test function $v \in C^\infty([\tau_0, T] \times \mathbb{R}^d)$ with bounded support in space and for any $t \in [\tau_0, T]$ there holds

$$\begin{aligned} &\int_{\mathbb{R}^d} w(t, x) v(t, x) dx - \int_{\mathbb{R}^d} w(\tau_0, x) v(\tau_0, x) dx \\ &\leq \int_{\tau_0}^t \int_{\mathbb{R}^d} w(s, x) (\partial_t v(s, x) + \sigma \Delta_G v(s, x) + \operatorname{div}_G(v(s, x) B)) dx ds. \end{aligned}$$

Since $w(\tau_0, \cdot) = \varepsilon$, letting $\varepsilon \downarrow 0$ we deduce

$$\int_{\mathbb{R}^d} |\rho(t, x)| v(t, x) dx \leq \int_{\tau_0}^t \int_{\mathbb{R}^d} |\rho(t, x)| |\partial_t v + \sigma \Delta_G v + \operatorname{div}_G(v(t, x) B)| ds dx.$$

Choose $t \in [\tau_0, \tau]$ and $v = \xi_R \Phi$ where τ and Φ are respectively the constant and the function introduced in Lemma 3.4 and $\xi_R \in C^\infty(\mathbb{R}^d)$ is a cut-off function such that

$$\begin{aligned} \xi_R(x) &= 1, \quad \text{if } |x| \leq R, \quad \xi_R(x) = 0, \quad \text{if } |x| \geq R + 1 \\ &\text{and } \|D\xi_R\|_{L^\infty} + \|D^2\xi_R\|_{L^\infty} \leq 2. \end{aligned}$$

Hence, we get

$$\begin{aligned} &\int_{\mathbb{R}^d} |\rho(t, x)| \xi_R(x) \Phi(t, x) dx \\ &\leq \int_{\tau_0}^t \int_{B_{R+1} \setminus B_R} |\rho(s, x)| \left| (\sigma \Delta_G \xi_R(x) + B \cdot \nabla_G \xi_R(x)) \Phi(s, x) \right. \\ &\quad \left. + 2\sigma \nabla_G \xi_R(x) \cdot \nabla_G \Phi(s, x) \right| ds dx. \end{aligned}$$

Letting $R \uparrow \infty$, by dominated convergence theorem and Lemma 3.4 we have that the right hand side converges to zero and we obtain

$$\int_{\mathbb{R}^d} |\rho(t, x)| \Phi(t, x) dx \leq 0, \quad \forall t \in [\tau_0, \tau]$$

which entails $\rho(t, x) = 0$ in $(\tau_0, \tau) \times \mathbb{R}^d$ contradicting the definition of τ_0 . $\square$

By Proposition 3.3, the classical solution constructed in Proposition 3.1 is unique, hence we proved the following corollary.

Corollary 3.5 *Under the assumptions of Theorem 1.1, there exists a unique bounded classical solution $\rho \in C((0, T); C_{G,loc}^{2,v}(\mathbb{R}^d))$ of (3.1).*

3.2.1 Hölder regularity and flat metric

Next, we prove Hölder regularity of the solution ρ w.r.t. the so-called flat metric d_0 distance here defined. There are many ways to metrize weak convergence of measures and the one we use here is the following (see, for instance, [44]): the bounded Lipschitz distance, also called Fortet-Mourier distance of flat Wasserstein metric tailored to the group structure

$$\begin{aligned} d_0(m, m') \\ = \sup \left\{ \int_{\mathbb{R}^d} f(x) dm(x) - \int_{\mathbb{R}^d} f(x) dm'(x) : f : \mathbb{R}^d \rightarrow \mathbb{R} \text{ s.t. } \|f\|_{C_G^{0,1}(\mathbb{R}^d)} \leq 1 \right\}. \end{aligned} \quad (3.15)$$

Proposition 3.6 *Let ρ be the unique solution to (3.1) constructed in Proposition 3.1. Then, there exists $C_\rho \geq 0$ such that*

$$d_0(\rho_t, \rho_s) \leq C_\rho |t - s|^{\frac{1}{2}} \quad \forall t, s \in [0, T].$$

Proof We argue adapting some ideas of [22, Proposition 6.6]. We consider the smooth function

$$\xi(x) = \begin{cases} \exp \left\{ \frac{1}{\|x\|_G^{2kl} - 1} \right\} & \text{if } \|x\|_G \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $\varepsilon > 0$, let

$$\xi^\varepsilon(x) = \frac{C}{\varepsilon^Q} \xi(\delta_{\varepsilon^{-1}} x) \quad (x \in \mathbb{R}^d)$$

be a smooth mollifier with support in $B(0, \varepsilon)$ and where the constant C is independent of ε and such that $\int \xi^\varepsilon dx = 1$ (we recall that $\delta_{\varepsilon^{-1}}(x)$ denotes the dilation of radius ε^{-1}). Note that, by homogeneity of the norm $\|\cdot\|_G$ we have

$$X_j \xi^\varepsilon(x) = \frac{1}{\varepsilon} X_j \xi(x). \quad (3.16)$$

Let φ be a real valued function with $\|\varphi\|_{C_G^{0,1}(\mathbb{R}^d)} \leq 1$ and let

$$\varphi_\varepsilon(x) = \xi^\varepsilon \star \varphi(x) := \int_{\mathbb{R}^d} \xi^\varepsilon(x \circ y^{-1}) \varphi(y) dy$$

be the convolution based on the operation on the group. Note that, by standard calculus and Lagrange theorem (see [8, Theorem 20.3.1], there holds:

$$\|\varphi - \varphi_\varepsilon\|_{L^\infty} \leq \varepsilon.$$

Then,

$$\int_{\mathbb{R}^d} \varphi_\varepsilon(x) (\rho(t, x) - \rho(s, x)) dx = \int_s^t \int_{\mathbb{R}^d} (\Delta_G \varphi_\varepsilon(x) - b \cdot \nabla_G \varphi_\varepsilon(x)) \rho(z, x) dz dx.$$

First, from (3.16), the left invariance of the vector fields and standard calculus, we obtain

$$\|\Delta_G \varphi_\varepsilon\|_{C^0(\mathbb{R}^d)} \leq \frac{1}{\varepsilon} \|\varphi\|_{C_G^{0,1}(\mathbb{R}^d)}.$$

Hence,

$$\int_{\mathbb{R}^d} \varphi_\varepsilon(x) (\rho(t, x) - \rho(s, x)) dx \leq 2C \frac{1}{\varepsilon} (1 + \|b\|_{L^\infty}) \|\varphi\|_{C_G^{0,1}(\mathbb{R}^d)} |t - s|$$

which yields to

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) (\rho(t, x) - \rho(s, x)) dx &\leq \int_{\mathbb{R}^d} \varphi_\varepsilon(x) (\rho(t, x) - \rho(s, x)) dx + 2\|\varphi - \varphi_\varepsilon\|_{L^\infty} \\ &\leq C \left(\frac{1}{\varepsilon} |t - s| + \varepsilon \right). \end{aligned}$$

In conclusion, minimizing over $\varepsilon > 0$, we get

$$d_0(\rho_t, \rho_s) \leq C_\rho |t - s|^{\frac{1}{2}} \quad 0 \leq s \leq t \leq T.$$

□

Proof of Theorem 1.1. From the above analysis we have that existence, uniqueness, (i) and (ii) follow from Propositions 3.1, 3.2 and 3.3. Finally, (iii) is proved in Proposition 3.6. $\square$

4 Hamilton–Jacobi equation

4.1 Small-time existence of solutions

Throughout this section we assume that assumptions **(HP)** and **(HP’)** are in force and we study Eq. (1.5).

We introduce, for simplicity of notation, the space

$$\mathcal{X}(T) = C([0, T]; W_G^{2,\infty}(\mathbb{R}^d)) \quad (4.1)$$

equipped with the norm

$$\|\varphi\|_{\mathcal{X}(T)} = \sup_{t \in [0, T]} \|\varphi(t)\|_{W_G^{2,\infty}(\mathbb{R}^d)}.$$

Before, for proving the existence of a small-time solution to (1.5) we need the following decay estimate for the heat semi-group $e^{t\Delta_G}$ generated by the horizontal Laplacian.

Lemma 4.1 *For any $t \in [0, T]$ and any $\varphi \in L^\infty(\mathbb{R}^d)$ we have that*

$$\|e^{t\Delta_G}\varphi\|_{L^\infty(\mathbb{R}^d)} \leq \|\varphi\|_{L^\infty(\mathbb{R}^d)}, \quad \|X_i e^{t\Delta_G}\varphi\|_{L^\infty(\mathbb{R}^d)} \leq c(T)t^{-\frac{1}{2}}\|\varphi\|_{L^\infty(\mathbb{R}^d)}, \quad (4.2)$$

where $c(T)$ is a constant depending only on T .

Proof Let Γ be the fundamental solution to the heat operator $(\partial_t - \Delta_G)$, found in [9]. Following [9, Theorem 1.2] and by construction of the heat semigroup we get

$$\|e^{t\Delta_G}\varphi\|_{L^\infty(\mathbb{R}^d)} = \|\Gamma(t) \star \varphi\|_{L^\infty(\mathbb{R}^d)} \leq \|\varphi\|_{L^\infty(\mathbb{R}^d)}.$$

Similarly, still from [9, Theorem 1.2] we deduce the second estimate in (4.2). $\square$

Proof of Theorem 1.2. Consider $T > 0$, which will be chosen later on. For any $k > 0$ we denote by $\mathcal{X}_k(T)$ the closed ball of radius k in $\mathcal{X}(T)$, defined in (4.1). Given $k > 0$, we consider the map

$$\Phi : \mathcal{X}_k(T) \rightarrow \mathcal{X}_k(T)$$

defined by

$$\Phi u(t) = e^{t\Delta_G}u_0 + \int_0^t e^{(t-s)\Delta_G} f(s, x, \nabla_G u(s)) \, ds, \quad \forall t \geq 0. \quad (4.3)$$

Next, we show that there exist $T_0 > 0$ and $k > 0$ such that the map Φ is well defined, i.e., $\Phi u \in \mathcal{X}_k(T)$ for $u \in \mathcal{X}_k(T)$ and Φ is a contraction for any $T \leq T_0$.

To do so, let us fix $k > 0$ and $u \in \mathcal{X}_k(T)$ for some $T > 0$. Then, from (4.2) and assumptions **(HP)**, we have that

$$\begin{aligned} \left\| \int_0^t e^{(t-s)\Delta_G} f(s, x, \nabla_G u(s)) \, ds \right\|_{L^\infty} &\leq \int_0^t \|f(s, x, \nabla_G u(s))\|_{L^\infty} \, ds \\ &\leq \int_0^t C_f (1 + \|\nabla_G u(s)\|_{L^\infty}^\gamma) \, ds \leq C_f T (1 + k^\gamma). \end{aligned} \quad (4.4)$$

Moreover, still from (4.2) and assumptions **(HP)** and **(HP')**, we have that

$$\begin{aligned} \left\| \nabla_G \int_0^t e^{(t-s)\Delta_G} f(s, x, \nabla_G u(s)) \, ds \right\|_{L^\infty} &\leq \int_0^t \|\nabla_G e^{(t-s)\Delta_G} f(s, x, \nabla_G u(s))\|_{L^\infty} \, ds \\ &\leq c(T) \int_0^t C_f (t-s)^{-\frac{1}{2}} (1 + \|\nabla_G u(s)\|_{L^\infty}^\gamma) \, ds \leq C_f 2c(T) T^{\frac{1}{2}} (1 + k^\gamma) \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} &\left\| X_i X_j \int_0^t e^{(t-s)\Delta_G} f(s, x, \nabla_G u(s)) \, ds \right\|_{L^\infty} \\ &\leq \int_0^t \left\| X_i e^{(t-s)\Delta_G} X_j f(s, \cdot, \nabla_G u(\cdot, s)) \right\|_{L^\infty} \, ds \\ &\leq c(T) \int_0^t (t-s)^{-\frac{1}{2}} \|X_j f(s, \cdot, \nabla_G u(\cdot, s))\|_{L^\infty} \, ds \\ &\leq c(T) c T^{\frac{1}{2}} (1 + k^\gamma) \end{aligned} \quad (4.6)$$

where $c(T)$ is the constant introduced in (4.2) while c is a constant, that depends on C_f , d and m (in particular, is independent of T and k) and may change from line to line. Moreover, by (4.2), standard arguments entail

$$\begin{aligned} \|e^{t\Delta_G} u_0\|_{L^\infty(\mathbb{R}^d)} &\leq \|u_0\|_{L^\infty(\mathbb{R}^d)}, \\ \|\nabla_G e^{t\Delta_G} u_0\|_{L^\infty(\mathbb{R}^d)} &= \|e^{t\Delta_G} \nabla_G u_0\|_{L^\infty(\mathbb{R}^d)} \leq \|\nabla_G u_0\|_{L^\infty(\mathbb{R}^d)}, \\ \|X_i X_j e^{t\Delta_G} u_0\|_{L^\infty(\mathbb{R}^d)} &\leq \|e^{t\Delta_G} X_i X_j u_0\|_{L^\infty(\mathbb{R}^d)} \leq \|X_i X_j u_0\|_{L^\infty(\mathbb{R}^d)}. \end{aligned} \quad (4.7)$$

By relations (4.4), (4.5), (4.6) and (4.7), for $k > 0$ sufficiently large and T sufficiently small, there holds

$$k > C_f \|u_0\|_{W^{2,\infty}_G(\mathbb{R}^d)} + C_f c (1 + k + k^\gamma) T^{1/2} (T^{1/2} + c(T))$$

then, $\Phi u \in \mathcal{X}_k(T)$ for any $u \in \mathcal{X}_k(T)$.

Next, we proceed to show that Φ is a contraction. Let $u, v \in \mathcal{X}_k(T)$. Then, from (4.2) we have that

$$\begin{aligned} & \left\| \int_0^t e^{(t-s)\Delta_G} f(s, x, \nabla_G u(s)) ds - \int_0^t e^{(t-s)\Delta_G} f(s, x, \nabla_G v(s)) ds \right\|_{L^\infty} \\ & \leq \int_0^t \|f(s, x, \nabla_G u(s)) - f(s, x, \nabla_G v(s))\|_{L^\infty} ds. \end{aligned}$$

So, by **(HP)** we obtain

$$\begin{aligned} & \left\| \int_0^t e^{(t-s)\Delta_G} f(s, x, \nabla_G u(s)) ds - \int_0^t e^{(t-s)\Delta_G} f(s, x, \nabla_G v(s)) ds \right\|_{L^\infty} \\ & \leq \int_0^t C_f (\|\nabla_G u(s)\|_{L^\infty}^{\gamma-1} + \|\nabla_G v(s)\|_{L^\infty}^{\gamma-1}) \|\nabla_G u(s) - \nabla_G v(s)\|_{L^\infty} ds \\ & \leq 2C_f T k^{\gamma-1} \|u - v\|_{\mathcal{X}(T)}. \end{aligned} \quad (4.8)$$

By using similar arguments, one gets

$$\begin{aligned} & \left\| \nabla_G \int_0^t e^{(t-s)\Delta_G} f(s, x, \nabla_G u(s)) ds - \nabla_G \int_0^t e^{(t-s)\Delta_G} f(s, x, \nabla_G v(s)) ds \right\|_{L^\infty} \\ & \leq 4c(T) C_f T^{\frac{1}{2}} k^{\gamma-1} \|u - v\|_{\mathcal{X}(T)}. \end{aligned} \quad (4.9)$$

Furthermore, for any $i, j \in \{1, \dots, m\}$, we have

$$\begin{aligned} A_{ij} &:= \left\| X_i X_j \int_0^t e^{(t-s)\Delta_G} [f(s, x, \nabla_G u(s)) - f(s, x, \nabla_G v(s))] ds \right\|_{L^\infty([0, T] \times \mathbb{R}^d)} \\ &\leq \int_0^t \left\| X_i e^{(t-s)\Delta_G} X_j [f(s, \cdot, \nabla_G u(\cdot, s)) - f(s, \cdot, \nabla_G v(\cdot, s))] \right\|_{L^\infty([0, T] \times \mathbb{R}^d)} ds \\ &\leq 2T^{1/2} c(T) \|X_j [f(s, \cdot, \nabla_G u(\cdot, s)) - f(s, \cdot, \nabla_G v(\cdot, s))]\|_{L^\infty([0, T] \times \mathbb{R}^d)}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} & \|X_j [f(s, \cdot, \nabla_G u(\cdot, s)) - f(s, \cdot, \nabla_G v(\cdot, s))]\|_{L^\infty([0, T] \times \mathbb{R}^d)} \\ & \leq \|X_j [f(s, \cdot, \nabla_G u) - f(s, \cdot, \nabla_G v)]\|_{L^\infty([0, T] \times \mathbb{R}^d)} \\ & \quad + \sum_{i=1}^m \|\partial_{p_i} f(s, \cdot, \nabla_G u(\cdot, s)) - \partial_{p_i} f(s, \cdot, \nabla_G v(\cdot, s))\|_{L^\infty([0, T] \times \mathbb{R}^d)} \\ & \|X_i X_j u\|_{L^\infty([0, T] \times \mathbb{R}^d)} \\ & \quad + \sum_{i=1}^m \|\partial_{p_i} f(s, \cdot, \nabla_G v(\cdot, s))\|_{L^\infty([0, T] \times \mathbb{R}^d)} \|X_i X_j (u - v)\|_{L^\infty([0, T] \times \mathbb{R}^d)}. \end{aligned}$$

Replacing the last inequality in the previous one, by assumption **(HP')**, we get

$$A_{ij} \leq T^{1/2} c(T) c(1 + k^{\gamma-1}) \|u - v\|_{\mathcal{X}(T)} \quad (4.10)$$

where c is a constant, that depends on C_f , d and m (in particular, is independent of T and k) and may change from line to line. Hence, using (4.8), (4.9) and (4.10), we get

$$\|\Phi u - \Phi v\|_{\mathcal{X}(T)} \leq cC_f T^{\frac{1}{2}}(T^{1/2} + 2c(T))(1 + k^{\gamma-1})\|u - v\|_{\mathcal{X}(T)}, \quad \forall u, v \in \mathcal{X}_k(T)$$

and we conclude choosing k such that

$$cC_f T^{\frac{1}{2}}(T^{1/2} + 2c(T))(1 + k^{\gamma-1}) < 1.$$

Thus, from the fixed point theorem we obtain the existence of a unique solution in $\mathcal{X}_k(T)$. Moreover, from the representation formula provided by the contraction argument, i.e.,

$$u(t) = e^{t\Delta_G} u_0 + \int_0^t e^{(t-s)\Delta_G} f(s, x, \nabla_G u(t, x)) ds$$

and the regularity of the fundamental solution (see [9, Theorem 1.2]) we deduce that $u \in C_G^{1+\frac{\alpha}{2}, 2+\alpha}((0, T] \times \mathbb{R}^d)$. $\square$

4.2 Global existence of solutions

In Theorem 1.2 we showed that for a sufficiently small time horizon T there exists a solution to the general semilinear parabolic Eq. (1.5) in $\mathcal{X}(T)$. In this section, we go back considering the Hamilton–Jacobi

$$\begin{cases} \partial_t u(t, x) - \sigma \Delta_G u(t, x) + |\nabla_G u(t, x)|^\gamma = F(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d \end{cases} \quad (4.11)$$

and we prove that there exists a solution for any $T > 0$.

In order to prove that such a solution exists for any arbitrary $T > 0$, the key point is the duality feature between the Hamilton–Jacobi equation and the Fokker–Planck equation studied so far. For this reason, we first show that the solution u constructed in Theorem 1.2 taking $f(t, x, p) = F(t, x) - |p|^\gamma$ solves problem (4.11) also in a suitable weak (energy) sense and, then, following a standard procedure (see for instance [17]), we provide the duality relation between the two equations in the sub-Riemannian setting.

Lemma 4.2 *Let F be as in (1.3), u_0 as in (1.4) and let u be the solution of (4.11) found in Theorem 1.2. For any $T \leq T_0$, we have*

- (i) $\nabla_G u \in L^p((0, T) \times \mathbb{R}^d)$ for every $\gamma \leq p < \infty$;
- (ii) $(\partial_t - \sigma \Delta_G) u \in L^2((0, T) \times \mathbb{R}^d)$;

(iii) for any $\varphi \in C([0, T]; L^2(\mathbb{R}^d))$ there holds

$$\begin{aligned} & \int_s^\tau \int_{\mathbb{R}^d} \varphi(t, x) \left(\partial_t u(t, x) - \sigma \Delta_G u(t, x) + |\nabla_G u(t, x)|^\gamma \right) dt dx \\ &= \int_s^\tau \int_{\mathbb{R}^d} \varphi(t, x) F(t, x) dt dx, \quad 0 \leq s \leq \tau \leq T \leq T_0. \end{aligned} \quad (4.12)$$

Proof Recall that the solution u constructed in Theorem 1.2 belongs to $C([0, T], W_G^{2,\infty}(\mathbb{R}^d))$, with $T \leq T_0$ and it is given by

$$u(t) = e^{t\Delta_G} u_0 + \int_0^t e^{(t-s)\Delta_G} (F(s, x) - |\nabla_G u(t, x)|^\gamma) ds \quad \forall t \in [0, T_0]. \quad (4.13)$$

(i). Following the arguments of [4, Theorem B] we get that $u(t) \in L^1(\mathbb{R}^d)$: let $\tilde{u}$ solve $\partial_t \tilde{u} - \sigma \Delta_G \tilde{u} = 0$ with $\tilde{u}(0) = u_0$; we refer to [9, Theorem 1.2] for the representation formula of $\tilde{u}$ and for the regularity of the fundamental solution for the heat equation. By comparison principle, using that $u_0 \geq 0$, we get $0 \leq u \leq \tilde{u}$ and consequently $u(t) \in L^1(\mathbb{R}^d)$. Hence, integrating (4.13) we deduce $\nabla_G u \in L^\gamma((0, T) \times \mathbb{R}^d)$; Since $\nabla_G u \in L^\infty((0, T) \times \mathbb{R}^d)$, by interpolation we conclude (i).

(ii). It is an immediate consequence of point (i) and assumption (1.3).

(iii). Clearly, for all $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ we have

$$\begin{aligned} & \int_s^\tau \int_{\mathbb{R}^d} \varphi(t, x) \left(\partial_t u(t, x) - \sigma \Delta_G u(t, x) + |\nabla_G u(t, x)|^\gamma \right) dt dx \\ &= \int_s^\tau \int_{\mathbb{R}^d} \varphi(t, x) F(t, x) dt dx. \end{aligned}$$

By a standard approximation argument, we infer (4.12). $\square$

In the following lemma we get a useful relation between the solutions u and μ respectively of the Hamilton–Jacobi and Fokker–Planck equations using the duality structure of these equations.

Lemma 4.3 *Let $u \in \mathcal{X}(T_0)$ be a solution to (4.11) as in Theorem 1.2. Let $\tau \in (0, T_0]$. For any $\mu_\tau \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ with $\mu_\tau \geq 0$, let μ be the solution to*

$$\begin{cases} -\partial_t \mu(t, x) - \sigma \Delta_G \mu(t, x) - \operatorname{div}_G(\gamma |\nabla_G u(t, x)|^{\gamma-2} \nabla_G u(t, x) \mu(t, x)) = 0, \\ (t, x) \in (0, \tau) \times \mathbb{R}^d \\ \mu(\tau, x) = \mu_\tau(x), \quad x \in \mathbb{R}^d \end{cases} \quad (4.14)$$

found in Theorem 1.1. Then, for any $s \in (0, \tau)$ we have that

$$\begin{aligned} \int_{\mathbb{R}^d} u(\tau, x) \mu(\tau, x) dx &= \int_{\mathbb{R}^d} u(s, x) \mu(s, x) dx \\ &+ \int_s^\tau \int_{\mathbb{R}^d} (\gamma - 1) |\nabla_G u(t, x)|^\gamma \mu(t, x) dt dx + \int_s^\tau \int_{\mathbb{R}^d} F(t, x) \mu(t, x) dt dx. \end{aligned} \quad (4.15)$$

Proof From the regularity of u and μ , by using u as a test function in (4.14) and, respectively, μ as a test function in (4.12) and taking the sum of the two relations we obtain

$$\begin{aligned} & - \int_s^\tau \int_{\mathbb{R}^d} \partial_t [u(t, x) \mu(t, x)] dx dt \\ & + \int_s^\tau \int_{\mathbb{R}^d} (\gamma |\nabla_G u(t, x)|^\gamma - |\nabla_G u(t, x)|^\gamma) \mu(t, x) dx dt \\ & = - \int_s^\tau \int_{\mathbb{R}^d} F(t, x) \mu(t, x) dx dt. \end{aligned}$$

Hence we get (4.15). $\square$

In the following proposition we prove a key estimate using the duality argument of Lemma 4.3.

Proposition 4.4 *Let $u \in \mathcal{X}(T_0)$ be a solution to (4.11) and let $\tau \in [0, T_0]$. Then, there exists C , depending on T_0 , $\|u_0\|_{L^\infty(\mathbb{R}^d)}$, $\|F\|_{L^\infty([0, T] \times \mathbb{R}^d)}$ (and independent of T_0), such that*

$$\sup_{t \in [0, \tau]} \|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq C. \quad (4.16)$$

Proof First, we prove a bound from above for u . To do so, fix $\tau \in [0, T_0]$ and consider the solution $\mu : [0, \tau] \times \mathbb{R}^d \rightarrow \mathbb{R}$ to the following problem

$$\begin{cases} -\partial_t \mu(t, x) - \sigma \Delta_G \mu(t, x) = 0, & (t, x) \in [0, \tau] \times \mathbb{R}^d \\ \mu(\tau, x) = \mu_\tau(x), & x \in \mathbb{R}^d \end{cases}$$

with $\mu_\tau \in \mathcal{C}_c^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ with $\mu_\tau \geq 0$ and $\|\mu_\tau\|_{1, \mathbb{R}^d} = 1$. By duality arguments, i.e., proceeding as in Lemma 4.3, we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} u(\tau, x) \mu_\tau(x) dx &= \int_{\mathbb{R}^d} u_0(x) \mu(0, x) dx \\ &+ \int_0^\tau \int_{\mathbb{R}^d} F(s, x) \mu(s, x) dx ds - \int_0^\tau \int_{\mathbb{R}^d} |\nabla_G u(s, x)|^\gamma \mu(s, x) dx ds \end{aligned} \quad (4.17)$$

Since, from Theorem 1.1, $\|\mu(t, \cdot)\|_{1, \mathbb{R}^d} = 1$ for any $t \in [0, \tau]$ and $\mu \geq 0$, we get

$$\begin{aligned} & \int_{\mathbb{R}^d} u_0(x) \mu(0, x) dx + \int_0^\tau \int_{\mathbb{R}^d} F(s, x) \mu(s, x) dx ds \\ & - \int_0^\tau \int_{\mathbb{R}^d} |\nabla_G u(s, x)|^\gamma \mu(s, x) dx ds \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} + T \|F\|_{L^\infty([0, \tau] \times \mathbb{R}^d)}. \end{aligned}$$

Hence,

$$\int_{\mathbb{R}^d} u(\tau, x) \mu_\tau(x) dx \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} + T \|F\|_{L^\infty([0, \tau] \times \mathbb{R}^d)}$$

and, thus, by passing to the supremum, over $\mu_\tau \geq 0$ with $\|\mu(t, \cdot)\|_1 = 1$ one deduces

$$u(\tau, x) \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} + T \|F\|_{L^\infty([0, \tau] \times \mathbb{R}^d)}. \quad (4.18)$$

To prove the lower bound for u , we first observe that combining (4.15) and (4.18) we get

$$\int_0^\tau \int_{\mathbb{R}^d} (\gamma - 1) |\nabla_G u(s, x)|^\gamma \mu(s, x) ds dx \leq 2(\|u_0\|_{L^\infty(\mathbb{R}^d)} + T \|F\|_{L^\infty([0, \tau] \times \mathbb{R}^d)}). \quad (4.19)$$

So, again by (4.17) we get

$$\begin{aligned} & \int_{\mathbb{R}^d} u(\tau, x) \mu_\tau(x) dx \geq \int_{\mathbb{R}^d} u_0(x) \mu(0, x) dx \\ & - \frac{2}{\gamma - 1} (\|u_0\|_{L^\infty(\mathbb{R}^d)} + T \|F\|_{L^\infty([0, \tau] \times \mathbb{R}^d)}) + \int_0^\tau \int_{\mathbb{R}^d} F(s, x) \mu(s, x) ds dx \end{aligned}$$

which, by the arbitrariness of μ_τ , yields to

$$u(\tau, x) \geq -\frac{\gamma + 1}{\gamma - 1} (\|u_0\|_{L^\infty(\mathbb{R}^d)} + T \|F\|_{L^\infty([0, \tau] \times \mathbb{R}^d)}). \quad (4.20)$$

Therefore, (4.18) and (4.20) imply (4.16). $\square$

Now we want to prove a local bound on $\nabla_G u$. To do this we need the following remark.

Remark 4.5 Following [38], for any two Riemannian metrics g and $\tilde{g}$, we have that, by the very definition of the gradient, for every smooth function u it holds that $g(D_g u, \cdot) = \tilde{g}(D_{\tilde{g}} u, \cdot)$. So, since any two Riemannian metrics are equivalent, if we denote by $|D_g u|_g^2 = g(D_g u, D_g u)$, and the same for $D_{\tilde{g}} u$, we have that for any compact set $\Omega \subset \mathbb{R}^d$ there exist constants $K_1(\Omega)$, $K_2(\Omega)$ such that

$$K_1(\Omega) |D_g u|_g^2 \leq |D_{\tilde{g}} u|_{\tilde{g}}^2 \leq K_2(\Omega) |D_g u|_g^2. \quad (4.21)$$

In particular, for any two sequences of metrics $\{g_n\}_{n \in \mathbb{N}}, \{\tilde{g}_n\}_{n \in \mathbb{N}}$ such that $g_n \rightarrow g$ and $\tilde{g}_n \rightarrow \tilde{g}$ it can be proved that $K_1(\Omega)$ and $K_2(\Omega)$ in (4.21) can be chosen independently of n , i.e., there exist $K'_1(\Omega), K'_2(\Omega)$ such that

$$K'_1(\Omega) |D_{g_n} u|_{g_n}^2 \leq |D_{\tilde{g}_n} u|_{\tilde{g}_n}^2 \leq K'_2(\Omega) |D_{g_n} u|_{g_n}^2. \quad (4.22)$$

We now recall that, to obtain a bound on $\nabla_G u$, we cannot directly apply the classical Bernstein method to $X_i u$, where u solves the Eq. (4.11), but we have to adapt it because some extra terms, involving commutators, would appear. In order to overcome this issue, we consider the family of right-invariant smooth vector fields $\{Y_1, \dots, Y_m\}$ introduced in (1.6).

Proposition 4.6 *Under the assumptions of Theorem 1.3, let $u \in \mathcal{X}(T_0)$ be a solution to (4.11) and let $\tau \in [0, T_0]$. Then, for any compact subset Ω of $\mathbb{R}^d$ there exists $C(\Omega)$, depending on $\|\nabla_G u_0\|_{L^\infty(\mathbb{R}^d)}, \|\nabla_G F\|_{L^\infty([0, T_0] \times \mathbb{R}^d)}$, such that*

$$\sup_{t \in [0, \tau]} \|\nabla_G u(t)\|_{L^\infty(\Omega)} \leq C(\Omega). \quad (4.23)$$

Proof Let $\{X_1, \dots, X_m, X_{m+1}, \dots, X_d\}$ be a smooth completion of $\{X_1, \dots, X_m\}$ built as follows: we re-order $\{X_1, \dots, X_m\}$ so that the associated $d \times m$ matrix has the first m rows given by the $m \times m$ -identity matrix and, for $j = m+1, \dots, d$, we choose $X_j = (0, \dots, 0, 1, 0, \dots, 0)$ where the unique 1 occupies the j -th position. We introduce the Riemann basis $\{X_1^\varepsilon, \dots, X_d^\varepsilon\}$ on $\mathbb{R}^d$ with $X_j^\varepsilon = X_j$ for $j = 1, \dots, m$ and $X_j^\varepsilon = \varepsilon X_j$ for $j = m+1, \dots, d$. We build in a similar manner $\{Y_1^\varepsilon, \dots, Y_d^\varepsilon\}$ as a completion of $\{Y_1, \dots, Y_m\}$. Observe that, for any $i, j = 1, \dots, d$ we have that $[X_i^\varepsilon, Y_j^\varepsilon] = 0$ (see [38, Lemma 2.1]).

We consider the complete Hamilton–Jacobi equation

$$\begin{cases} \partial_t u^\varepsilon(t, x) - \sigma \Delta_\varepsilon u^\varepsilon(t, x) + |\nabla_\varepsilon u^\varepsilon(t, x)|^\gamma = F(t, x), & (t, x) \in (0, T) \times \mathbb{R}^d \\ u^\varepsilon(0, x) = u_0(x), & x \in \mathbb{R}^d \end{cases} \quad (4.24)$$

where Δ_ε and ∇_ε denotes, respectively, the Laplacian and the gradient w.r.t. the vector fields $\{X_1^\varepsilon, \dots, X_d^\varepsilon\}$. Fix $j \in \{1, \dots, d\}$ and set $v_j^\varepsilon = Y_j u^\varepsilon$. By standard regularity theory for elliptic equations, from (4.24) we have that v_j^ε is a classical solution to the equation

$$\begin{aligned} & \partial_t v_j^\varepsilon(t, x) - \sigma \Delta_\varepsilon v_j^\varepsilon(t, x) + \gamma |\nabla_\varepsilon u^\varepsilon(t, x)|^{\gamma-2} \nabla_\varepsilon u^\varepsilon(t, x) \nabla_\varepsilon v_j^\varepsilon \\ & = Y_j F(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d. \end{aligned} \quad (4.25)$$

We proceed with a duality argument as before. To do so, for $\tau > 0$, let μ solve

$$\begin{cases} -\partial_t \mu(t, x) - \sigma \Delta_\varepsilon \mu(t, x) - \operatorname{div}_\varepsilon (\gamma |\nabla_\varepsilon u^\varepsilon(t, x)|^{\gamma-2} \nabla_\varepsilon u^\varepsilon(t, x) \mu(t, x)) = 0, \\ \mu(\tau, x) = \mu_\tau(x) \end{cases} \quad (4.26)$$

where $\operatorname{div}_\varepsilon$ denotes the divergence operator w.r.t. $\{X_1^\varepsilon, \dots, X_m^\varepsilon, X_{m+1}^\varepsilon, \dots, X_d^\varepsilon\}$. Thus, using μ as a test function in the weak formulation of (4.25) and v as a test function in the weak formulation of (4.26) we obtain

$$\sup_{t \in [0, \tau]} \|v_j^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq \|Y_j F\|_{L^\infty([0, \tau] \times \mathbb{R}^d)} + \|Y_j u_0\|_{L^\infty(\mathbb{R}^d)},$$

which is bounded by (1.6).

Applying Remark 4.5 to $\{Y_1^\varepsilon, \dots, Y_m^\varepsilon, Y_{m+1}^\varepsilon, \dots, Y_d^\varepsilon\}$ and $\{X_1^\varepsilon, \dots, X_m^\varepsilon, X_{m+1}^\varepsilon, \dots, X_d^\varepsilon\}$ we can estimate from below the horizontal gradient obtaining

$$\sup_{t \in [0, \tau]} \|X_j^\varepsilon u^\varepsilon\|_{L^\infty(\Omega)} \leq C(\Omega) (\|X_j F\|_{L^\infty([0, \tau] \times \mathbb{R}^d)} + \|X_j u_0\|_{L^\infty(\mathbb{R}^d)})$$

for any compact subset Ω of $\mathbb{R}^d$ and some constant $C(\Omega) \geq 0$. So, as $\varepsilon \downarrow 0$ we have that $X_j^\varepsilon u^\varepsilon \rightarrow X_j u$ if $j \in \{1, \dots, m\}$ and $X_j^\varepsilon u^\varepsilon \rightarrow 0$ if $j \in \{m+1, \dots, d\}$. Hence, this yields to

$$\sup_{t \in [0, \tau]} \|\nabla_G u\|_{L^\infty(\Omega)} \leq C(\Omega) (\|\nabla_G F\|_{L^\infty([0, \tau] \times \mathbb{R}^d)} + \|\nabla_G u_0\|_{L^\infty(\mathbb{R}^d)}).$$

□

Proof of Theorem 1.3. Reasoning as in [20, Sect. 2.1], from Proposition 4.4 and Proposition 4.6 we have that there exists a solution on $[0, T_0 + \varepsilon]$ and thus, a solution $u \in C([0, T]; W_{G, \text{loc}}^{1, \infty}(\mathbb{R}^d))$ for T finite but arbitrary large.

Next, in order to complete the proof of gain regularity of the solution we proceed with a bootstrap argument. For any $R \geq 0$ let u_R be a solution to

$$\begin{cases} \partial_t u_R(t, x) - \sigma \Delta_G u_R(t, x) + |\nabla_G u_R(t, x)|^\gamma = F(t, x), & (t, x) \in (0, T] \times B_R \\ u_R(t, x) = u_0(x), & x \in B_R \\ u_R(t, x) = 0, & x \in \partial B_R. \end{cases}$$

Then, as proved so far we have that $u_R \in C([0, T]; W_G^{1, \infty}(\mathbb{R}^d))$. Then, the same equation can be seen as a subelliptic heat equation with bounded right hand-side. Hence, the right hand-side belongs to $L^p(B_R)$ for any $p \geq 1$ and using Theorem 2.3, we obtain $u_R \in W^{1, p}([0, T]; W_G^{2, p}(B_R))$. Taking $p \geq Q + 2$ applying Theorem 2.4 we gain regularity on the solution, that is, $u_R \in C_G^{1, \alpha}([0, T] \times B_R)$ with $\alpha = \frac{1}{p}(p - (Q + 2))$. So, following again the same reasoning we deduce that for any $R \geq 0$ the

solution u_R belongs to $C_G^{2+\alpha, 1+\frac{\alpha}{2}}([0, T] \times B_R)$. Finally, as $R \uparrow \infty$ by a diagonalization argument the proof is complete. $\square$

5 Application to MFG

The goal of this section is the application of the above results to get the existence for small times of solutions to the following MFG system

$$\begin{cases} -\partial_t u(t, x) - \sigma \Delta_G u(t, x) + |\nabla_G u(t, x)|^\gamma = F_{\text{MFG}}[\rho_t](x), & (t, x) \in (0, T) \times \mathbb{R}^d \\ \partial_t \rho - \sigma \Delta_G \rho - \operatorname{div}_G(\gamma |\nabla_G u(t, x)|^{\gamma-2} \nabla_G u(t, x) \rho) = 0, & (t, x) \in (0, T) \times \mathbb{R}^d \\ u(T, x) = u_T(x), \quad \rho(0, x) = \rho_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (5.1)$$

Note that we cannot prove the existence for any time T because of the lack of compactness in the results of Theorem 1.3.

As it is customary in MFG, the existence result is a consequence of the Schauder's fixed point theorem. However, due to the lack of control of the moments of the measure ρ_t associated to the solution of the Fokker–Planck equation, the strategy is quite different w.r.t. the classical literature.

Proof of Theorem 1.4. We obtain the existence of a classical solution using the Schauder fixed-point theorem. To do so, we endow the space $C([0, T]; W_G^{2,\infty}(\mathbb{R}^d))$ with the topology induced by the uniform convergence and we introduce the map

$$\mathcal{T} : C([0, T]; W_G^{2,\infty}(\mathbb{R}^d)) \rightarrow C([0, T]; W_G^{2,\infty}(\mathbb{R}^d))$$

in the following way: given $u \in \mathcal{C}$, with

$$\mathcal{C} = \left\{ u \in C([0, T]; W_G^{2,\infty}(\mathbb{R}^d)) : \sup_{t \in [0, T]} \|u(t)\|_{W_G^{2,\infty}(\mathbb{R}^d)} \leq \kappa(T_0) \right\}$$

where T_0 and $\kappa(T_0)$ are the constants in Theorem 1.2 and let μ be the unique solution in Theorem 1.1 to

$$\begin{cases} \partial_t \mu - \sigma \Delta_G \mu - \operatorname{div}_G(\gamma |\nabla_G u(t, x)|^{\gamma-2} \nabla_G u(t, x) \mu) = 0, & (t, x) \in (0, T) \times \mathbb{R}^d \\ \mu(0, x) = \rho_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Then, we set $\psi = \mathcal{T}(u)$ as the unique solution constructed in Theorem 1.2 to the Hamilton–Jacobi equation

$$\begin{cases} -\partial_t \psi(t, x) - \sigma \Delta_G \psi(t, x) + |\nabla_G \psi(t, x)|^\gamma = F_{\text{MFG}}[\mu_t](x), & (t, x) \in (0, T) \times \mathbb{R}^d \\ \psi(T, x) = u_T(x), & x \in \mathbb{R}^d. \end{cases}$$

First, we claim that the map $\mathcal{T}$ is well-defined and continuous for any time horizon $T \leq T_0$. Indeed, the vector field $|\nabla_{\mathbf{G}} u|^{\gamma-2} \nabla_{\mathbf{G}} u$ satisfies the assumptions of Theorem 1.1 and the solution to the Fokker–Planck equation is Hölder continuous by Proposition 3.6. Moreover, from Theorem 1.2 we have that $\mathcal{T}(\mathcal{C})$ is a compact subset of $\mathcal{C}$ since the solution to the Hamilton–Jacobi equation is locally bounded in $C_G^{2+\alpha, 1+\frac{\alpha}{2}}([0, T] \times \mathbb{R}^d)$ by [10, Theorem 1.1].

Hence, since all the assumptions of the Schauder fixed point theorem are satisfied, the proof is complete. $\square$

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