Anomalous dimensions via on-shell methods: Operator mixing and leading mass effects

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We elaborate on the application of on shell and unitarity-based methods for evaluating renormalization group coefficients, and generalize this framework to account for the mixing of operators with different dimensions and leading mass effects. We derive a master formula for anomalous dimensions stemming from the general structure of operator mixings, up to two-loop order, and show how the Higgs low-energy theorem can be exploited to include leading mass effects. A few applications on the renormalization properties of popular effective field theories showcase the strength of the proposed approach, which drastically reduces the complexity of standard loop calculations. Our results provide a powerful tool to interpret experimental measurements of low-energy observables, such as flavor violating processes or electric and magnetic dipole moments, as induced by new physics emerging above the electroweak scale.

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I. INTRODUCTION

The Standard Model (SM) of particle physics has passed unclashed in several experimental tests in all its sectors. The lack of heavy new physics (NP) at the LHC has firmly established the SM as a very successful theory describing the fundamental interactions of nature up to the TeV scale. However, it is a common belief that the SM has to be regarded as the low-energy description of a more fundamental theory emerging at a large, yet unknown, energy scale Λ. New interactions can be then described by an effective field theory (EFT) containing nonrenormalizable operators that are invariant under the SM gauge group. EFTs provide a very powerful and model-independent approach to NP which does not rely on the details of the underlying (unknown) high-energy theory but just on its symmetries.

Predictions for physical processes are obtained by evaluating matrix elements of the EFT Lagrangian at energy scales accessible by collider experiments. Therefore, the high-scale Lagrangian needs to be evolved from the scale Λ down to the experimental scale $E \ll \Lambda$. Such a program can be carried out by computing the anomalous dimension matrix of the higher-dimension operators which control both the multiplicative renormalization of operators as well as their mixing effects. In particular, the latter provide important information on how experimental bounds from one operator impact the Wilson coefficients of other operators. This makes the evaluation of EFT anomalous dimension matrices a crucial ingredient for interpreting experimental results. A systematic and comprehensive computation of the one-loop anomalous dimension matrix has been carried out for a number of relevant EFTs, such as the Standard Model EFT [1–4] or the axionlike particle EFT [5,6], exploiting diagrammatic and functional methods.

Recently, the calculation of anomalous dimensions has been addressed also employing on shell and unitarity-based techniques for scattering amplitudes [7–16]. One of the most intriguing features of this approach is to make manifest hidden structures with the appearance of nontrivial zeros in the anomalous dimension matrix. The origin of these vanishing elements has been traced back to selection rules [17], helicity [18], operator lengths [19], and angular momentum conservation [20].

Anomalous dimensions can be extracted from the ultraviolet divergent part of amplitudes exploiting the generalized unitarity method [21–26] for assembling scattering amplitudes from their unitarity cuts (see also [27], for review). Therefore, unitarity cuts give a direct access to the renormalization-scale dependence. In Ref. [7], it was remarkably observed that anomalous dimensions can be directly related to unitarity cuts. In particular, the discontinuities of form factors of EFT operators can be calculated via phase-space integrals and are related to the corresponding anomalous dimensions. This method has been shown to be particularly effective for computing anomalous dimensions at two-loop order [11].

So far, the method proposed in Ref. [7], has been applied to derive the anomalous dimensions of nonrenormalizable massless theories including only mixing effects among operators of the same dimension. However, the nontrivial inclusion of mixings among operators with different dimensions and leading mass effects, which are of paramount importance in many popular EFT extensions of the SM, has not been discussed so far in this framework. The main motivation of this Letter is to fill this gap. In particular, we generalize the method of Ref. [7] providing a master formula which includes the most general operator mixing contributions up to two-loop order. Moreover, we show how to include leading mass effects, still working in the massless limit, by exploiting the Higgs low-energy theorem [28,29]. Few applications of our methods are illustrated by means of the renormalization of the axionlike particle EFT [30–32] and the low-energy EFT of the SM below the electroweak scale (LEFT) [4].

II. THE METHOD OF FORM FACTORS

In this section, we review the method of Ref. [7]. The fundamental objects we deal with are the form factors of local gauge-invariant operators $\mathcal{O}_i$ of the Lagrangian $\mathcal{L}_{\mathrm{EFT}} = \sum_i c_i \mathcal{O}_i / \Lambda^{[\mathcal{O}_i]-4}$. These form factors are defined as

$$F_i(\mathbf{n}; q) = \frac{1}{\Lambda^{[\mathcal{O}_i]-4}} \langle \mathbf{n} | \mathcal{O}_i(q) | 0 \rangle, \tag{1}$$

which is a matrix element between an outgoing on shell state $\langle \mathbf{n}| = \langle 1^{h_1}, \ldots, n^{h_n}|$ and an operator $\mathcal{O}_i$ that injects an additional off shell momentum q. In dimensional regularization, form factors depend on the renormalization scale μ and satisfy the Callan-Symanzik equation

$$\left(\delta_{ij} \mu \frac{\partial}{\partial \mu} + \frac{\partial \beta_i}{\partial c_j} - \delta_{ij} \gamma_{i,\mathrm{IR}} + \delta_{ij} \beta_g \frac{\partial}{\partial g} \right) F_i = 0, \tag{2}$$

where g collectively denotes the couplings related to the renormalizable operators of our Lagrangian, while $\gamma_{i,\mathrm{IR}}$ is the infrared anomalous dimension. The renormalization of the operator $\mathcal{O}_i$ induced by $\mathcal{O}_j$ is described by

$$\beta_i(\{c_k\}) \equiv \mu \frac{\mathrm{d} c_i}{\mathrm{d}\mu} = \gamma_{i \leftarrow j} c_j, \tag{3}$$

where c_i are the Wilson coefficients of the effective Lagrangian $\mathcal{L}_{\mathrm{EFT}}$.

Exploiting the analyticity of form factors, unitarity, and the CPT theorem, it can be shown that an elegant relation exists linking the action of the dilatation operator (D) to the action of the S matrix (S) on form factors [7]:

$$e^{-i\pi D} F_i^* = S F_i^*, \tag{4}$$

where $S = \mathbf{1} + i\mathcal{M}$ while $D = \sum_i p_i \cdot \partial/\partial p_i$ (the sum is over all particles i). For a massless theory, in dimensional regularization, the latter reduces to $D \simeq -\mu \partial_\mu$.

This allows one to connect Eqs. (2) and (4), thus relating the infrared and ultraviolet anomalous dimensions to the scattering matrix phase when applied to a form factor. In particular, at one-loop order, it has been found that

$$\left(\frac{\partial \beta_i^{(1)}}{\partial c_j} - \delta_{ij} \gamma_{i,\mathrm{IR}}^{(1)} + \delta_{ij} \beta_g^{(1)} \frac{\partial}{\partial g} \right) F_i^{(0)} = -\frac{1}{\pi} (\mathcal{M} F_j)^{(1)}, \tag{5}$$

where the right-hand side of Eq. (5) corresponds to a sum over all one-loop two-particle unitarity cuts

$$\begin{aligned}(\mathcal{M} F_j)^{(1)}(1, \ldots, n) = \sum_{k=2}^{n} \sum_{\{\ell_1, \ell_2\}} \int d\mathrm{LIPS}_2 \\ \times \sum_{h_1, h_2} F_j^{(0)}(\ell_1^{h_1}, \ell_2^{h_2}, k+1, \ldots, n) \\ \times \mathcal{M}^{(0)}(1, \ldots, k; \ell_1^{h_1}, \ell_2^{h_2}),\end{aligned} \tag{6}$$

where $\mathcal{M}(\mathbf{n}; \mathbf{m}) = \langle \mathbf{n} | \mathcal{M} | \mathbf{m} \rangle$ and $d\mathrm{LIPS}_2$ is the (two-particle) Lorentz invariant phase-space measure. The corresponding cut integral can be evaluated employing different parametrizations, by angular integration [7–9,11,12], or via Stokes's theorem [10,26].

III. GENERAL OPERATOR MIXING

The mixing among operators of different dimensions is required in many EFTs in order to capture the leading effects to several observables. This feature can be elegantly included within the method of form factors as we are going to discuss.

The crucial observation is that, in the neighborhood of the Gaussian fixed point ($*$), where $c_i = 0 \ \forall \ i$, the renormalization group equations for the Wilson coefficients c_i can be Taylor expanded as

$$\begin{aligned}\mu \frac{\mathrm{d} c_i}{\mathrm{d}\mu} &= \sum_{n>0} \frac{1}{n!} \gamma_{i \leftarrow j_1, \ldots, j_n} c_{j_1} \ldots c_{j_n} \\ &= \gamma_{i \leftarrow j} c_j + \frac{1}{2} \gamma_{i \leftarrow j,k} c_j c_k + \ldots,\end{aligned} \tag{7}$$

where

$$\gamma_{i \leftarrow j_1, \ldots, j_n} = \left. \frac{\partial^n \beta_i}{\partial c_{j_1} \ldots \partial c_{j_n}} \right|_* \tag{8}$$

has a perturbative expansion in the couplings of the leading order Lagrangian: $\gamma_{i \leftarrow j_1, \ldots, j_n} = \sum_{\ell > 0} \gamma_{i \leftarrow j_1, \ldots, j_n}^{(\ell)}$.

Focusing on the most relevant case of a double operator insertion, the key object we have to evaluate is $\gamma_{i \leftarrow j,k}$. A particularly convenient way to write it is the following:

$$\gamma_{i\leftarrow j,k} = \frac{\partial^2 \beta_i}{\partial c_j \partial c_k}\bigg|_* = \frac{\partial}{\partial c_k}\bigg|_{c_k=0} \frac{\partial \beta_i}{\partial c_j}\bigg|_{*,c_k\neq 0}. \tag{9}$$

In fact, the last equality of Eq. (9) enables us to generalize the master formula of Eq. (5) by simply differentiating it with respect to a Wilson coefficient and then evaluating the result at the Gaussian fixed point, where $c_k = 0$.

At one-loop order, we obtain the following expression:

$$\left(\gamma^{(1)}_{i\leftarrow j,k} - \delta_{ij}\frac{\partial \gamma^{(1)}_{i,\mathrm{IR}}}{\partial c_k}\bigg|_* + \delta_{ij}\frac{\partial \beta^{(1)}_g}{\partial c_k}\bigg|_* \frac{\partial}{\partial g}\right) F_i|^{(0)}_* = -\frac{1}{\pi}\frac{\partial}{\partial c_k}\bigg|_{c_k=0} (\mathcal{M}F_j)|^{(1)}_{*,c_k\neq 0}, \tag{10}$$

which represents an important result of this Letter. Since we are interested in mixing of operators with different dimensions, hereafter, we focus on the case where $j, k \neq i$:

$$\gamma^{(1)}_{i\leftarrow j,k} F_i|^{(0)}_* = -\frac{1}{\pi}\frac{\partial}{\partial c_k}\bigg|_{c_k=0} (\mathcal{M}F_j)|^{(1)}_{*,c_k\neq 0}, \tag{11}$$

which can be generalized at two-loop order by properly expanding Eq. (4) at the desired order. We find

$$\gamma^{(2)}_{i\leftarrow j,k} F_i|^{(0)}_* = -\frac{1}{\pi}\frac{\partial}{\partial c_k}\bigg|_{c_k=0} (\mathrm{Re}\mathcal{M}\mathrm{Re}F_j)|^{(2)}_{*,c_k\neq 0} - \gamma^{(1)}_{i\leftarrow j}\frac{\partial}{\partial c_k}\bigg|_{c_k=0} \mathrm{Re}F_i|^{(1)}_{*,c_k\neq 0} - \gamma^{(1)}_{i\leftarrow j,k}\mathrm{Re}F_i|^{(1)}_*. \tag{12}$$

The extension of Eqs. (10), (11), and (12) with multiple operator insertions $\gamma_{i\leftarrow j_1,\ldots,j_n}$ is straightforward.

IV. LEADING MASS EFFECTS

As a natural consequence of operator mixing, chirality-violating and preserving operators do generally mix under the renormalization flow. In the case of EFTs defined below the electroweak scale, the required chirality flip proceeds through a fermion mass insertion. However, such a mass dependence cannot be directly implemented in the massless method of Ref. [7].

In order to circumvent this issue, we rely on the low-energy Higgs theorem [28,29], which was originally introduced to estimate the properties of a light Higgs boson in analogy to how soft-pion theorems are used to study low-energy pion interactions [33–36].

The key observation is that the fermionic Higgs interactions in the SM can be written in the following form:

$$\mathcal{L}^{\mathrm{int}}_H = -\left(1 + \frac{h}{v}\right)\sum_f m_f \bar{f} f, \tag{13}$$

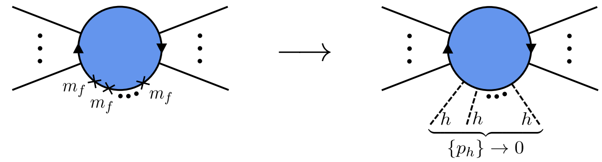

FIG. 1. Diagrammatic representation of our method for massifying amplitudes based on the low-energy Higgs theorem.

where the sum is over all fermions in the theory. In the limit where the Higgs field h has a vanishing four-momentum, $p_h \to 0$, h becomes a constant field, and its effect is equivalent to redefining all mass parameters as $m_f \to m_f(1 + h/v)$. This immediately implies the following low-energy theorem [28,29]:

$$\lim_{\{p_h\}\to 0} \mathcal{M}(A \to B + Nh) = \sum_f \frac{m_f^N}{v^N}\frac{\partial^N}{\partial m_f^N}\mathcal{M}(A \to B), \tag{14}$$

relating the amplitudes of two processes differing by N insertions of zero momentum Higgs bosons. In practice, whenever an amplitude requires N fermion mass insertions not to vanish, we consider an equivalent amplitude entailing N extra massless Higgs fields; see Fig. 1.

The number N can be determined as follows. By dimensional analysis we argue that whenever the anomalous dimension $\gamma_{i\leftarrow j_1,\ldots,j_n}$ vanishes in the limit of massless fermions, possible mass effects must be of order $(m_f/\Lambda)^N$, where

$$N = 4 - [\mathcal{O}_i] + \sum_{k=1}^{n}([\mathcal{O}_{j_k}] - 4) \tag{15}$$

is the superficial degree of divergence associated with the loop diagram under consideration (see the Appendix for a derivation). Notice that the number of needed Higgs insertions coincides with the superficial degree of divergence in Eq. (15) because scaleless integrals vanish in dimensional regularization.

For $N < 0$, $\gamma_{i\leftarrow j_1,\ldots,j_n}$ is trivially zero. For $N \geq 0$, the anomalous dimension is obtained by renormalizing the operator $\mathcal{O}^{Nh}_i = (h/v)^N \mathcal{O}_i/N!$ instead of $\mathcal{O}_i$.

The procedure outlined before is summarized in the algorithm in Fig. 2.

Remarkably, the approximation of setting the Higgs mass to zero is justified in our study since we are interested in the evaluation of anomalous dimensions that are related to the ultraviolet properties of a theory.

V. PHENOMENOLOGICAL APPLICATIONS

In this section, we discuss some applications of our results. First, we illustrate how to separately deal with either the mixing of operators with different dimensions or leading mass effects in axionlike particle EFT [5,6,32].

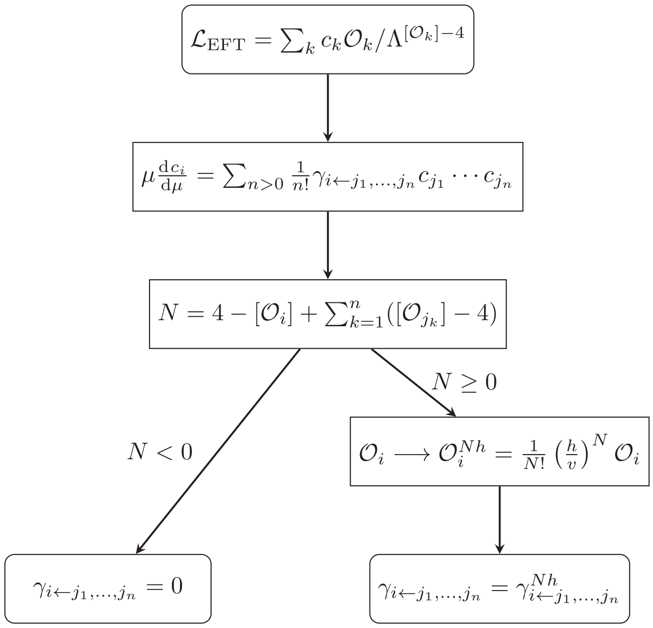

FIG. 2. Algorithm for the computation of anomalous dimensions that require N fermionic mass insertions not to vanish.

Then, we show how to treat these effects simultaneously in the context of the LEFT [4]. The most general effective Lagrangian describing axionlike particle (ALP) interactions with SM fields reads as [30–32]

$$\mathcal{L}_\phi = \frac{\tilde{\mathcal{C}}_\gamma}{\Lambda} \phi F \tilde{F} + \frac{\tilde{\mathcal{C}}_g}{\Lambda} \phi G \tilde{G} + i \mathcal{Y}_P^{ij} \phi \bar{f}_i \gamma_5 f_j + \frac{\mathcal{C}_\gamma}{\Lambda} \phi F F + \frac{\mathcal{C}_g}{\Lambda} \phi G G + \mathcal{Y}_S^{ij} \phi \bar{f}_i f_j, \qquad (16)$$

where ϕ is the ALP field, $\Lambda \gg v \approx 246$ GeV is the EFT cutoff scale, and $f \in \{e, u, d\}$ denotes SM fermions in the mass basis. Moreover, G and F are the QCD and QED field-strength tensors, while $\tilde{G}$ and $\tilde{F}$ are their duals.

The need for an appropriate treatment of leading mass effects is evident in the renormalization of the operator $\mathcal{O}_S = \phi \bar{f} f$ as induced by $\mathcal{O}_\gamma = \phi F F$. Indeed, owing to the chirality mismatch between the two operators, a mass insertion would be necessary to obtain a non-null result. This situation can be handled by renormalizing the operator $\mathcal{O}_S^h = (h/v) \phi \bar{f} f$ instead of $\phi \bar{f} f$,

$$\gamma^{ij(1)}_{S \leftarrow \gamma} F_S^h|_*^{(0)} = -\frac{1}{\pi} (\mathcal{M} F_\gamma)|_*^{(1)}, \qquad (17)$$

which is represented in Fig. 3. Summing up the three contributions of Fig. 3, we obtain[1]

[1]We adopt the spinor-helicity formalism, where the spinor inner product is defined as $\langle ij \rangle = \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta$, where $\epsilon_{\alpha\beta}$ is the $SL(2, \mathbb{C})$ invariant Levi-Civita tensor, $\bar{\sigma}_\mu^{\dot{\alpha}\alpha} p_i^\mu = \tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^\alpha$ and $\bar{\sigma}_\mu^{\dot{\alpha}\alpha} = (\mathbf{1}, \boldsymbol{\sigma})^{\dot{\alpha}\alpha}$.

$$(\mathcal{M} F_\gamma)|_*^{(1)} (1^-_{\bar{f}_i}, 2^-_{\bar{f}_j}, 3_\phi, 4_h) = -\frac{3}{2\pi\Lambda} \frac{m_i}{v} \delta^{ij} e^2 Q_f^2 \langle 12 \rangle. \qquad (18)$$

Using

$$F_S^h|_*^{(0)} (1^-_{\bar{f}_i}, 2^-_{\bar{f}_j}, 3_\phi, 4_h) = \frac{1}{v} \langle 12 \rangle \qquad (19)$$

one can then find the sought-after result,

$$\gamma^{ij(1)}_{S \leftarrow \gamma} = \frac{3}{2\pi^2} \frac{m_i}{\Lambda} \delta^{ij} e^2 Q_f^2, \qquad (20)$$

which agrees with Refs. [5,6,32].

As an example of the impact that two insertions of lower-dimensional operators can have on a higher-dimensional one, we will consider here the generation of the Weinberg operator $\mathcal{O}_{\rm W} = f^{abc} G^a_{\mu\rho} G^b{}_\nu{}^\rho \tilde{G}^{c\mu\nu}/3$ as induced by the simultaneous presence of the ALP-gluon couplings $\mathcal{O}_g = \phi G G$ and $\mathcal{O}_{\tilde{g}} = \phi G \tilde{G}$. According to the results of the previous section, the anomalous dimension matrix element for $\mathcal{O}_{\rm W}$ can be extracted by evaluating

$$\gamma^{(1)}_{{\rm W} \leftarrow g, \tilde{g}} F_{\rm W}|_*^{(0)} = -\frac{1}{\pi} \frac{\partial}{\partial \mathcal{C}_g} \bigg|_{\mathcal{C}_g = 0} (\mathcal{M} F_{\tilde{g}})|^{(1)}_{*, \mathcal{C}_g \neq 0}, \qquad (21)$$

where

$$F_{\rm W}|_*^{(0)} (1^-_{g^a}, 2^-_{g^b}, 3^-_{g^c}) = \frac{\sqrt{2}}{\Lambda^2} f^{abc} \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \qquad (22)$$

is the form factor corresponding to the Weinberg operator for three negative-helicity gluons. Remarkably, as shown in Fig. 4, we need to calculate just one contribution, in contrast with the standard diagrammatic method which requires twelve one-loop diagrams. We find

$$(\mathcal{M} F_{\tilde{g}})|^{(1)}_{*, \mathcal{C}_g \neq 0} (1^-_{g^a}, 2^-_{g^b}, 3^-_{g^c}) = \frac{3\sqrt{2}}{\pi \Lambda^2} g_s \mathcal{C}_g f^{abc} \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \qquad (23)$$

yielding in turn

$$\gamma^{(1)}_{{\rm W} \leftarrow g, \tilde{g}} = -\frac{3 g_s}{\pi^2}, \qquad (24)$$

in agreement with the literature [32].

In our last example, we show how to treat simultaneously the methods of the previous sections, by considering the renormalization of the QCD theta term, which is associated with the dimension-four pseudoscalar density $\mathcal{O}_\vartheta = G \tilde{G}$. Focusing on dipole operators, the relevant dimension-five LEFT Lagrangian reads as

$$\mathcal{L}^{(5)} \supset \frac{a_f}{\Lambda} \mathcal{O}_{\rm CM} + \frac{d_f}{\Lambda} \mathcal{O}_{\rm CE}, \qquad (25)$$

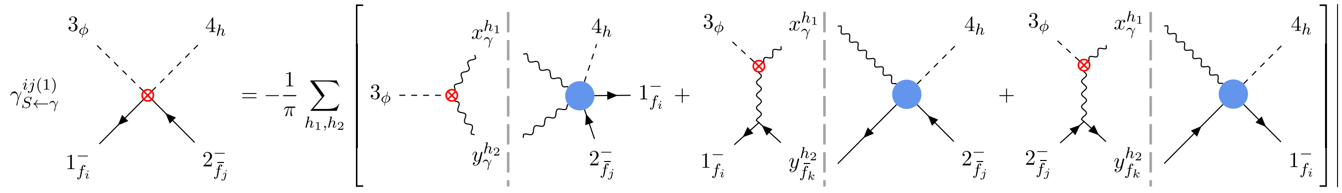

FIG. 3. Diagrams contributing to the renormalization of the $\mathcal{O}_S^h = (h/v)\phi\bar{f}f$ operator as induced by the $\mathcal{O}_\gamma = \phi FF$ one. Here and in the following figures, red and light-blue blobs refer to form factors and amplitudes, respectively.

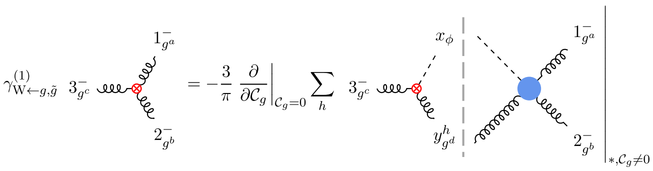

FIG. 4. Diagrams contributing to the renormalization of the Weinberg operator $\mathcal{O}_W = f^{abc}G^a_{\mu\rho}G^{b\rho}_\nu\tilde{G}^{c\mu\nu}/3$ as induced by the operators $\mathcal{O}_g = \phi GG$ and $\mathcal{O}_{\tilde{g}} = \phi G\tilde{G}$. The factor of 3 on the right-hand side stems from the permutation of the gluons.

where $\mathcal{O}_{\rm CM}$ is the chromomagnetic operator while $\mathcal{O}_{\rm CE}$ is the chromoelectric one, defined as

$$\mathcal{O}_{\rm CM} = \bar{f}\sigma^{\mu\nu}T^a f G^a_{\mu\nu}, \qquad \mathcal{O}_{\rm CE} = i\bar{f}\sigma^{\mu\nu}\gamma_5 T^a f G^a_{\mu\nu}. \tag{26}$$

In the limit of massless fermions, the anomalous dimension $\gamma_{\vartheta\leftarrow{\rm CM,CE}}$ vanishes. According to the previous discussion on the low-energy Higgs theorem, possible mass effects must be of order $(m_f/\Lambda)^2$. Therefore, the required double Higgs insertion can be accounted for by introducing the $\mathcal{O}_\vartheta^{2h} = (h^2/2v^2)G\tilde{G}$ operator, as shown in Fig. 5. Then, $\gamma^{(1)}_{\vartheta\leftarrow{\rm CM,CE}}$ can be extracted from

$$\gamma^{(1)}_{\vartheta\leftarrow{\rm CM,CE}} F_\vartheta^{2h}|_*^{(0)} = -\frac{1}{\pi}\frac{\partial}{\partial d_f}\bigg|_{d_f=0} (\mathcal{M}F_{\rm CM})|^{(1)}_{*,d_f\neq 0}, \tag{27}$$

where

$$F_\vartheta^{2h}|_*^{(0)}(1^-_{g^a}, 2^-_{g^b}, 3_h, 4_h) = -\frac{2i}{v^2}\delta^{ab}\langle 12\rangle^2. \tag{28}$$

By taking into account permutations of external particles, namely $(1^-_{g^a} \leftrightarrow 2^-_{g^b})$ and $(3_h \leftrightarrow 4_h)$, we obtain

$$(\mathcal{M}F_{\rm CM})|^{(1)}_{*,d_f\neq 0}(1^-_{g^a}, 2^-_{g^b}, 3_h, 4_h) = \frac{i}{\pi\Lambda^2}\frac{m_f^2}{v^2} d_f \delta^{ab}\langle 12\rangle^2. \tag{29}$$

Inserting the above result in Eq. (27), we find

$$\gamma^{(1)}_{\vartheta\leftarrow{\rm CM,CE}} = \frac{m_f^2}{2\pi^2\Lambda^2}, \tag{30}$$

in agreement with Ref. [4]. More extensive applications are deferred to a companion study [37].

VI. CONCLUSIONS

We have generalized the application of on shell and unitarity-based methods for evaluating renormalization group coefficients, to account for the mixing of operators with different dimensions and leading mass effects, which play a fundamental role in the renormalization program of several effective field theories.

In particular, we have derived a master formula accounting for operator mixings up to two-loop order, and shown how to include leading mass effects, relying on the Higgs low-energy theorem. Our findings have been validated by reproducing well-established results of the literature, relative to popular effective field theories.

Our results can be applied to a number of new physics scenarios, defined above the TeV scale, in order to analyze their impact on low-energy observables (like flavor violating processes, electric and magnetic dipole moments, etc.) occurring at or below the GeV scale. Such a large separation of scales demands the inclusion of running effects at two-loop order to obtain sensible predictions. While this is a very challenging task when approached with standard techniques, on shell and unitarity-based methods offer a simpler, more efficient and elegant way to reach this goal. Our work may constitute an additional milestone for progressing along this direction.

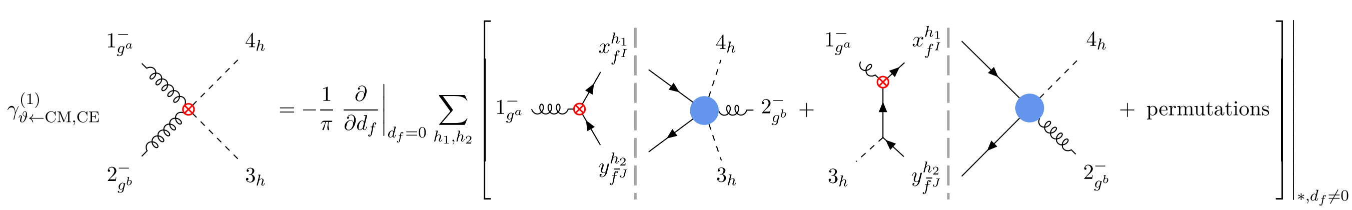

FIG. 5. Diagrams contributing to the renormalization of the operator $\mathcal{O}_\vartheta^{2h} = (h^2/2v^2)G\tilde{G}$ as induced by $\mathcal{O}_{\rm CM}$ and $\mathcal{O}_{\rm CE}$.

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APPENDIX: EFT AND SUPERFICIAL DEGREE OF DIVERGENCE

An n-particle amplitude related to the operator $\mathcal{O}_i$ and entailing m interaction vertices with the operators $\mathcal{O}_{j_1},\ldots,\mathcal{O}_{j_m}$, can be schematically written as

$$\mathcal{M}_n = C_{j_1}\cdots C_{j_m}\int d^4k_1\cdots d^4k_L \frac{N(\{k\},\{p\})}{D(\{k\},\{p\})}\alpha_1\cdots\alpha_n, \tag{A1}$$

where $\{p\}$ are external momenta and:

(i) $C_{j_1},\ldots,C_{j_m}$ are the Wilson coefficients corresponding to the operators $\mathcal{O}_{j_1},\ldots,\mathcal{O}_{j_m}$ such that

$$[C_{j_k}] = 4 - [\mathcal{O}_{j_k}]. \tag{A2}$$

(ii) The L-loop integral has mass dimension

$$\left[\int d^4k_1\cdots d^4k_L \frac{N(\{k\},\{p\})}{D(\{k\},\{p\})}\right] = D + n_\partial \tag{A3}$$

where D is the degree of divergence of $\mathcal{M}_n$ and n_∂ the number of derivatives contained in $\mathcal{O}_i$.

(iii) Depending on the bosonic/fermionic nature of the particle, $\alpha_1,\ldots,\alpha_n\in\{1,\epsilon_\mu,\epsilon_\mu^*,u,v,\bar{u},\bar{v},\ldots\}$. In the fermionic case, writing the Dirac field as

$$\Psi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}(a_k u e^{-ik\cdot x} + b_k^\dagger v e^{ik\cdot x}) \tag{A4}$$

and taking the quantization condition $\{a_k, a_{k'}^\dagger\} = \{b_k, b_{k'}^\dagger\} = (2\pi)^3\delta^{(3)}(k-k')$, one finds that $[a] = [b] = -3/2$ and $[u] = [v] \equiv [\alpha_{\rm fer}] = 1/2$. Similarly, one can also find that $[\alpha_{\rm bos}] = 0$.

By taking into account the above points, we find that

$$[\mathcal{M}_n] = \sum_{k=1}^{m}(4 - [\mathcal{O}_{j_k}]) + n_\partial + D + \frac{1}{2}n_{\rm fer}. \tag{A5}$$

The mass dimension of $\mathcal{M}_n$ can be inferred also from the S-matrix element

$$\langle 0|S|n\rangle = (2\pi)^4\delta^{(4)}(p_1+\cdots+p_n)i\mathcal{M}_n \tag{A6}$$

where $[S] = 0$ and $[|n\rangle] = [(\sqrt{2E}a^\dagger)^n|0\rangle] = -n$ with $n = n_{\rm fer} + n_{\rm bos}$, resulting in

$$[\mathcal{M}_n] = 4 - n. \tag{A7}$$

From Eqs. (A5) and (A7) and taking into account that

$$[\mathcal{O}_i] = n_{\rm bos} + \frac{3}{2}n_{\rm fer} + n_\partial, \tag{A8}$$

it finally follows that

$$D = 4 - [\mathcal{O}_i] + \sum_{k=1}^{m}([\mathcal{O}_{j_k}] - 4). \tag{A9}$$

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