Optimal impulse control problems with time delays: an illustrative example *

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Abstract

For impulse control systems described by a measure driven differential equation, it is customary to interpret the state trajectory corresponding to an impulse control, specified by a measure, as the limit of state trajectories associated with some sequence of conventional controls approximating the measure. It is known that, when the measure is vector valued, it is possible that different choices of approximating sequences for the measure give rise to different limiting state trajectories. If the measure is scalar valued, however, there is a unique limiting trajectory. Now consider impulse control systems, in which the right side of the measure driven differential equation depends on both the current and delayed states. In recent work by the authors it has been shown that, for such impulse control systems with time delay, the state trajectory corresponding to a given measure may be non-unique, even when the measure is scalar valued. It was also shown that each limiting state trajectory can be identified with the unique state trajectory associated with some measure together with a family of 'attached controls'. (The attached controls capture the nature of the measure approximation.) The authors also derived a maximum principle governing minimizers for a general class of impulse optimal control problems with time delay, in which the domain of the optimization problem comprises measures coupled with a family of 'attached controls'. The purpose of this paper is both to illustrate, by means of an example, this newly discovered non-uniqueness phenomenon and to provide the first application of the new maximum principle, to investigate minimizers for scalar input impulse optimal control problems with time delay, in circumstances when limiting state trajectories associated with a given measure control are not unique. The example is an optimal control problem, for which the underlying control system is a forced harmonic oscillator, with scalar

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impulse control, in which the control gain is a nonlinear function of the current and delayed states.

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1. Introduction

In a recent paper the authors provided necessary conditions of optimality for optimal impulse control problems, in which the underlying control system takes the form

$$(\text{DIS}) \begin{cases} dx(t) = f(x(t), x(t-h))dt + g(x(t), x(t-h))d\mu(t),\ t \in [0,T], \\ \text{where } \mu \text{ is a non-negative Borel measure on } [0,T], \text{ and} \\ x \text{ satisfies the boundary condition } x(0) = x_0 \text{ and } x(t) = \xi_0 \text{ for } t < 0\,. \end{cases}$$

Here, the data comprise positive constants T and h, functions $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, vectors $x_0 \in \mathbb{R}^n$ and $\xi_0 \in \mathbb{R}^n$.

In this formulation of impulse control systems, the state x may be discontinuous at times that are atoms of the 'impulse control' μ. When μ is absolutely continuous w.r.t. Lebesgue measure, in which case it can be written $d\mu(t) = u(t)dt$ for some conventional, non-negative valued control $u \in L^1$, the state variable x is governed by the delay differential equation

$$\begin{cases} \dot{x}(t) = f(x(t), x(t-h)) + g(x(t), x(t-h))u(t),\ t \in [0,T], \\ x(0) = x_0 \text{ and } x(t) = \xi_0 \text{ for } t < 0\,. \end{cases} \tag{1.1}$$

To set the historical context, consider first the delay free case, in which the impulse control system takes the form:

$$(\text{DF}) \begin{cases} dx(t) = f(x(t))dt + g(x(t))d\mu(t),\ t \in [0,T], \\ \text{where } \mu \text{ is a non-negative Borel measure on } [0,T], \text{ and} \\ x \text{ satisfies } x(0) = x_0. \end{cases}$$

(We have used the same symbols for f and g as before, when these functions do not depend on the delayed argument.) In the case when μ is absolutely continuous, (i.e. $d\mu(t) = u(t)dt$, for some integrable function u), the state x is governed by the differential equation

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t),\ t \in [0,T]. \tag{1.2}$$

The framework (DF) has been widely used in the study of control systems for which the dynamic constraint is affine w.r.t. the control variable [3, 2, 5, 8, 9, 13, 14, 15, 16, 17, 18, 19, 20]. The definition of state trajectory, consistent with aerospace applications (see e.g. [1], [12], [13]) in which the measure is an idealization of high intensity control pulses of short duration, is that it should be the limit of some sequence of state trajectories associated with a sequence $\{u_i\}$ of absolutely continuous controls such that $u_i(t)dt \to d\mu(t)$ in a weak* sense. We find that, for an arbitrary impulse control μ, all limits of state trajectories corresponding to approximating absolutely continuous controls $u_i(t)dt$, in this sense, are the same. Consistent with this approach, we define a state trajectory $x : [0,T] \to \mathbb{R}^n$ for (DF), corresponding to a measure control μ, to be a $\mathbb{R}^n$ valued function of bounded variation such that $x(0) = x_0$ and

$$x(t) = x_0 + \int_0^t f(x(t'))dt' + \int_{[0,t]} g(x(t'))d\mu^c(t') \\ + \sum_{r\in[0,t]} (\zeta^r(1) - x^-(r)) \quad \text{for } t \in (0,T].$$

Here, μ^c is the continuous component of μ, $x^-(r)$ is the left limit of x at r and, for each $r \in [0,T]$, $\zeta^r : [0,1] \to \mathbb{R}^n$ is the solution to the differential equation

$$\begin{cases} \dot{\zeta}^r(s) = \mu(\{r\})g(\zeta^r(s)), \text{ a.e. } s \in [0,1] \\ \zeta^r(0) = x^-(r)\,. \end{cases}$$

(We define $x^-(r) := x_0$, if $r = 0$.) Notice that the sum in the above relation is, in effect, countable since the term $\zeta^r(1) - x^-(r) = 0$ (and therefore does not contribute to the sum) when r is not an atom of μ, and since μ has at most a countable number of atoms. Discontinuities in the state equation occur only at times that are atoms of the control measure and the discontinuity in the state at a particular atom is governed by a differential equation that models the 'instantaneous' evolution of the state at the jump time.

As is well-documented (see e.g. [5], [13], [18]) the property 'all sequences of absolutely controls approximating the reference impulse control μ are associated with sequences of state trajectories converging to the same state in the limit' depends critically on the assumption that the measure μ is scalar valued. If the delay-free model is extended to include a k-vector valued impulse control $\mu = (\mu_1, \ldots, \mu_k)$, thus

$$\begin{cases} dx(t) = \tilde{f}(x(t))dt + \displaystyle\sum_{i=1}^{k} \tilde{g}_i(x(t))d\mu_i(t), \quad t \in [0,T] \\ \text{in which } \mu_i,\ i = 1,\ldots,k, \text{ are non-negative Borel measures on } [0,T], \text{ and} \\ x(0) = x_0\,, \end{cases}$$

the uniqueness property no longer holds: different approximations of the vector impulse controls by absolutely continuous controls can give rise to different state trajectories in the limit, unless we impose rather stringent 'commutativity' hypotheses on the $\tilde{g}_i$'s [5]. In this situation, the family of state trajectories associated with a given vector valued impulse control μ can be parameterized by collections of control functions attached to each point of discontinuity in the distribution of μ, which determine the evolution of the state during the jumps.

Now consider, once again, time-delayed impulse control systems, for scalar valued impulse controls. The properties of delay-free impulse control systems would suggest that, here also, the limits of all sequences of approximating absolutely continuous state trajectories for a nominal impulse control would be the same, and would provide a suitable interpretation of 'state trajectory' for this impulse control. But, on the contrary, we instead encounter novel behavior: the limits are no longer unique and, to represent all possible limiting state trajectories, we must introduce collections of attached controls that govern the instantaneous evolution of the state over each of the jumps.

The recent paper [11] provides a framework for studying the multiplicity of state trajectories associated with a given impulse control, in the presence of time delays. A definition of state trajectory is given, consistent with the limits of state trajectories associated with conventional controls. [11] also provides necessary conditions of optimality for optimal control problems associated with the impulse control system (DIS).

The purpose of this paper is two-fold. First, we provide an example of an optimal impulse control problem with time delays, exhibiting the non-uniqueness phenomena described above. Second, we give the first application of the maximum principle for such problems [11], when we must take account of the non-uniqueness of state trajectories associated with the minimizing impulse control. The underlying control system in the example is a forced harmonic oscillator with impulse input, for which the gain depends on both the current and delayed state.

The paper is structured as follows. In Section 2 we introduce an optimal impulse control problem, involving a forced harmonic oscillator with state and delayed state-dependent gain, and identify a candidate optimal control. Section 3 provides a precise definition of impulse processes for (DIS). In Section 4 we formulate a general optimal impulse control problem and state necessary conditions of optimality in the form of a maximum principle, derived in [11]. Finally, in Section 5 we show that the candidate optimal process for forced harmonic oscillator problem of Section 2 actually satisfies these optimality conditions.

1.1. Notation

Given a real number $T > 0$, we write $W^{1,1}([0,T];\mathbb{R}^k)$, $L^1([0,T];\mathbb{R}^k)$, $L^\infty([0,T];\mathbb{R}^k)$, $BV([0,T];\mathbb{R}^k)$, for the set of absolutely continuous, Lebesgue integrable, essentially bounded, and bounded variation $\mathbb{R}^k$-valued functions on $[0,T]$, respectively. We write $\|\cdot\|_\infty$ to denote the ess-sup norm in $[0,T]$. The left and right limits of a function $\sigma : [S,T] \to \mathbb{R}^k$ at a point $t \in [S,T]$ are written $\sigma^-(t)$ and $\sigma^+(t)$, respectively. We interpret $\sigma^-(t) = \sigma(S)$ and $\sigma^+(t) = \sigma(T)$ if $t = S$ or $t = T$, respectively.

We denote by $C^\oplus(0,T)$ the set of Borel non-negative scalar valued measures on $[0,T]$ (from now on we will refer to such μ simply as measures). For any $\mu \in C^\oplus(0,T)$, we use both the notations $\|\mu\|_{TV}$ and $\int_{[0,T]} d\mu(t)$ for the total variation

of μ. In the following, μ-*a.e.* means "almost everywhere w.r.t. μ", and when we do not specify μ we implicitly refer to Lebesgue measure. We denote the Dirac measure concentrated at $t \in [0,T]$ by $\delta_{\{t\}}$. Given a sequence $\{\mu_i\} \subset C^{\oplus}(0,T)$ and $\mu \in C^{\oplus}(0,T)$, as customary we write $\mu_i \rightharpoonup^* \mu$ if $\lim_i \int_{[0,T]} \Phi(t) d\mu_i(t) = \int_{[0,T]} \Phi(t) d\mu(t)$ for all continuous functions $\Phi : [0,T] \to \mathbb{R}$.
The *limiting normal cone* $N_C(\bar{x})$ to a closed set $C \subseteq \mathbb{R}^k$ at $\bar{x} \in \mathbb{R}^k$ (see [7]), is

$$N_C(\bar{x}) := \left\{ \eta \in \mathbb{R}^k : \exists x_i \xrightarrow{C} \bar{x},\, \eta_i \to \eta \;\text{ s.t. }\; \limsup_{x \to x_i} \frac{\eta_i \cdot (x - x_i)}{|x - x_i|} \le 0 \;\; \forall i \right\},$$

in which the notation $x_i \xrightarrow{C} \bar{x}$ means that $(x_i)_i \subset C$.

2. A Forced Harmonic Oscillator with Delayed State-Dependent Gain

In this section we introduce, with preliminary comments, an optimal control problem that we regard as an example of optimal impulse control problems with time delay, exhibiting the behavior discussed in Section 1. The problem is that of finding an impulse control which maximizes the terminal velocity of a forced harmonic oscillator for which the gain is a function both of the state and the delayed state.

2.1. The optimal control problem

$$(\mathrm{E}) \begin{cases} \text{Minimize } J(x) := -x_2(2\pi) \\ \text{over } x \in BV([-\pi, 2\pi]; \mathbb{R}^2) \text{ and } \mu \in C^{\oplus}(0, 2\pi) \text{ satisfying} \\ \begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dt + \begin{bmatrix} x_1(t) x_1(t-\pi) \\ 0 \end{bmatrix} d\mu(t) \\ ||\mu||_{TV} \le 1, \\ (x_1(0), x_2(0)) = (0,1), \text{ and } (x_1(t), x_2(t)) = (1,0) \text{ for } t \in [-\pi, 0)\,. \end{cases}$$

As earlier remarked, the underlying control system in this problem is a forced harmonic oscillator, in which the control gain is a nonlinear function of the current and delayed states. When μ is absolutely continuous w.r.t. Lebesgue measure, i.e. $d\mu(t) = u(t)dt$ for some conventional, non-negative valued control $u \in L^1$ the state variable x is governed by the delay differential equation

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} x_1(t) x_1(t-\pi) \\ 0 \end{bmatrix} u(t) \\ (x_1(0), x_2(0)) = (0,1), \text{ and } (x_1(t), x_2(t)) = (1,0) \text{ for } t \in [-\pi, 0)\,. \end{cases} \tag{2.1}$$

2.2. A class of candidate minimizing impulse controls

In this subsection we identify sequences of classical controls, converging in the weak* topology to measures that are candidate optimal measure controls for problem (E). We also illustrate that, for a fixed measure control, different approximating sequences of classical controls can give rise to distinct limiting state trajectories.

The free response of the harmonic oscillator x^{free}, obtained by setting $\mu = 0$, is given by

$$x^{\text{free}}(t) = (\sin(t), \cos(t)) \quad \text{for } t \in [0, 2\pi] \,.$$

Because the nonlinear gain $g(x_1(t), x_1(t-\pi)) = x_1(t)x_1(t-\pi)$ is monotone in each variable, we can expect the impulse control that maximizes the terminal velocity $x_2(2\pi)$ to be discrete and to be concentrated at the times when, for the free response, the absolute value of $x_1^{\text{free}}(t)$ is maximized, namely at times $t = \pi/2$ and $3\pi/2$. Given the total variation constraint, this implies the optimal impulse control $\bar{\mu}$ is of the form

$$\mu^\gamma = \gamma\, \delta_{\{\pi/2\}} + (1-\gamma)\delta_{\{3\pi/2\}}, \tag{2.2}$$

for some value of the parameter $\gamma \in [0, 1]$.

Fix $\gamma \in [0, 1]$. There are different ways of constructing ordinary approximating controls $\{\alpha^\gamma_\varepsilon\}_\varepsilon$ for μ^γ, in the sense that $\alpha^\gamma_\varepsilon(t)dt \rightharpoonup^* d\mu^\gamma(t)$ as $\varepsilon \downarrow 0$. We focus on two of them: for $\varepsilon > 0$ define

$$(1): \quad u^\gamma_\varepsilon(t) := \varepsilon^{-1}\gamma\, \chi_{[\pi/2-\varepsilon, \pi/2]}(t) + \varepsilon^{-1}(1-\gamma)\, \chi_{[3\pi/2, 3\pi/2+\varepsilon]}(t), \quad t \in [0, 2\pi]$$

and

$$(2): \quad v^\gamma_\varepsilon(t) := \varepsilon^{-1}\gamma\, \chi_{[\pi/2, \pi/2+\varepsilon]}(t) + \varepsilon^{-1}(1-\gamma)\, \chi_{[3\pi/2-\varepsilon, 3\pi/2]}(t), \quad t \in [0, 2\pi].$$

Here, χ_I denotes the indicator function of the interval I.

Notice that the u^γ_ε control is obtained by approximating the impulse at time $\pi/2$ by a pulse (of short duration) occurring *before* this impulse time and approximating the impulse at time $3\pi/2$ by a pulse occurring *after* this impulse time. On the other hand, v^γ_ε control is obtained by approximating the impulse at time $\pi/2$ by a pulse occurring *after* this impulse time and approximating the impulse at time $3\pi/2$ by a pulse occurring *before* this impulse time.

Write x^γ_ε and y^γ_ε for the corresponding state trajectories, obtained by solving the differential equation (2.1), following insertion of the relevant control into the right side.

Proposition 2.1.

$$x^\gamma_\varepsilon(t) \to x^\gamma(t) \text{ and } y^\gamma_\varepsilon(t) \to y^\gamma(t) \text{ as } \varepsilon \downarrow 0, \text{for all } t \in [0, 2\pi]\backslash(\{\pi/2\} \cup \{3\pi/2\}),$$

where

$$x^\gamma(t) := \begin{cases} (\sin(t), \cos(t)) & \textit{for } t \in [0, \pi/2) \\ e^\gamma(\sin(t), \cos(t)) & \textit{for } t \in [\pi/2, 3\pi/2] \\ e^{\gamma+(1-\gamma)e^\gamma}(\sin(t), \cos(t)) & \textit{for } t \in (3\pi/2, 2\pi] \end{cases}$$

and

$$y^\gamma(t) := \begin{cases} (\sin(t), \cos(t)) & \textit{for } t \in [0, \pi/2] \\ e^\gamma(\sin(t), \cos(t)) & \textit{for } t \in (\pi/2, 3\pi/2) \\ e(\sin(t), \cos(t)) & \textit{for } t \in [3\pi/2, 2\pi] \end{cases}.$$

Proof. Consider the controls u_ε^γ and v_ε^γ and their corresponding trajectories x_ε^γ and y_ε^γ, respectively. Notice that when the control $u_\varepsilon^\gamma(t)$ is active at times t near $3\pi/2$, the delayed state component $x_{1\varepsilon}^\gamma(t-\pi)$ has already jumped from 1 to approximately e^γ. On the other hand, when the $v_\varepsilon^\gamma(t)$ control is active at times t near $3\pi/2$, the delayed state component $y_{1\varepsilon}^\gamma(t-\pi)$ remains approximately at its initial value 1. (These statements are approximate, because of the small deviations away from zero, of the slow moving state components $x_{2\varepsilon}^\gamma$, $y_{2\varepsilon}^\gamma$ at the impulse times). Thus, the evolutions of the state components $x_{1\varepsilon}^\gamma$ and $y_{1\varepsilon}^\gamma$ over the time interval near $3\pi/2$, when the corresponding controls are non-zero, are approximately governed by the differential equations

$$\begin{cases} \dot{x}_{1\varepsilon}^\gamma(t) = \varepsilon^{-1}(1-\gamma)e^\gamma x_{1\varepsilon}^\gamma(t), & \text{a.e. } t \in [3\pi/2, 3\pi/2+\varepsilon] \\ x_{1\varepsilon}^\gamma(t)(3\pi/2) = -e^\gamma \end{cases}$$

and

$$\begin{cases} \dot{y}_{1\varepsilon}^\gamma = \varepsilon^{-1}(1-\gamma)y_{1\varepsilon}^\gamma(t), & \text{a.e. } t \in [3\pi/2-\varepsilon, 3\pi/2] \\ y_{1\varepsilon}^\gamma(3\pi/2-\varepsilon) = -e^\gamma. \end{cases}$$

We can thereby show that the values of $x_\varepsilon^\gamma(t)$ and $y_\varepsilon^\gamma(t)$ at times near $3\pi/2$ when the controls cease to be active, are $x_\varepsilon^\gamma(3\pi/2+\varepsilon) = -e^{\gamma+(1-\gamma)e^\gamma}(1,0)+O_1(\varepsilon)$ and $y_\varepsilon^\gamma(3\pi/2) = -e^{\gamma+(1-\gamma)}(1,0)+O_2(\varepsilon)$, for terms $O_1(\varepsilon)$ and $O_2(\varepsilon)$ that vanish as $\varepsilon \downarrow 0$. Since we know $x_\varepsilon^\gamma(0) = y_\varepsilon^\gamma(0) = (0,1)$ we have obtained estimates (which are exact in the limit as $\varepsilon \downarrow 0$) for the values of x_ε^γ and y_ε^γ at the initial times of subintervals between impulses. Taking account of the fact that x_ε^γ and y_ε^γ evolve according to simple harmonic motion between the impulse times, we see that the limiting state trajectories x^γ and y^γ are as in the proposition statement. □

Discussion:

(a): *Non-Uniqueness:* The limiting state trajectories, resulting from approximating the discrete measure μ in two different ways, are distinct, for any $\gamma \in (0,1)$. Thus, the example reveals that different ways of approximating a given scalar valued measure control by classical controls can give rise to different limiting state trajectories, owing to the presence of a time delays in the control system differential equation.

(b): *Optimality:* The example motivates us to pose the optimization problem (E) over elements $(\mu, \{u_i\})$, comprising a measure $\mu \in C^\oplus$ and a sequence $\{u_i\}$ of non-negative, integrable functions u_i such that $u_i(t)dt \rightharpoonup^* d\mu(t)$

and the sequence of corresponding state trajectories $\{x_i = (x_{1i}, x_{2i})\}$ converges to some function (called the limiting state trajectory), almost everywhere and at the right end-point. We evaluate the cost of over a couple $(\mu, \{u_i\})$ as

$J(\{\mu, \{u_i\}) := -\lim_i x_{2i}(2\pi)$.

Now consider the approximating controls $\{u^\gamma_\varepsilon\}_{\varepsilon>0}$ and $\{v^\gamma_\varepsilon\}_{\varepsilon>0}$ of Prop. 2.1. For $i = 1, 2, \dots$, write $u^\gamma_i := u^\gamma_{\varepsilon=i^{-1}}$ and $v^\gamma_i := v^\gamma_{\varepsilon=i^{-1}}$.

Using the formulae of Prop. 2.1, we can evaluate the costs of the elements $(\mu^\gamma, \{u^\gamma_i\})$ and $(\mu^\gamma, \{v^\gamma_i\})$. They are:

$J(\mu^\gamma, \{u^\gamma_i\}) = -x^\gamma_2(2\pi) = -e^{\gamma+(1-\gamma)e^\gamma}$, $\;J(\{\mu^\gamma, \{v^\gamma_i\}) = -y^\gamma_2(2\pi) = -e.$

We see that, for any $\gamma \in (0, 1)$, $(\mu^\gamma, \{u^\gamma_i\})$ has lower cost than $(\mu^\gamma, \{v^\gamma_i\})$. We can therefore discard $(\mu, \{v^\gamma_i\})$ as a candidate minimizer.

Attention then focuses on the element $(\mu^\gamma, \{u^\gamma_i\})$, as a possible minimizer. As we have noted, $(\mu^\gamma, \{u^\gamma_i\})$ has cost $-\exp\{c(\gamma)\}$ where

$$c(\gamma) = \gamma + (1-\gamma)e^\gamma .$$

Since the exponential function is strictly increasing, and $c(\gamma)$ achieves its maximum over $\gamma \in [0, 1]$ at an interior point, the maximum point $\bar\gamma$ satisfies the stationary point condition

$$\frac{d\,c}{d\gamma}(\bar\gamma) = 1 - e^{\bar\gamma} + (1-\bar\gamma)e^{\bar\gamma} = 1 + \bar\gamma e^{\bar\gamma} = 0.$$

We may conclude that $\bar\gamma$ is the unique number in $(0, 1)$ satisfying

$$e^{\bar\gamma} = \frac{1}{\bar\gamma} .$$

We write

$$\bar\mu := \mu^{\bar\gamma} \text{ and } \bar u_i := u^{\bar\gamma}_i, \text{ for } i = 1, 2, \dots$$

Our simple direct analysis has identified an element $(\bar\mu, \{\bar u_i\})$ that minimizes the cost over elements with the structure $\{(\mu^\gamma, \{u^\gamma_i\}) : \gamma \in [0, 1]\}$, as above.

The above comments provoke some fundamental questions, concerning the appropriate representation of couples $(\mu, \{u_i\})$, comprising the measure μ and a sequence of nonnegative and integrable functions $\{u_i\}$ converging to μ in the weak* topology, as 'controls' in an extended optimal control problem, concerning the definition of state trajectories corresponding to such controls and concerning the nature of necessary conditions of optimality that can be derived.

The authors' recent paper [11], which provides a framework for the study of optimal impulsive control problems with time delay and the derivation of necessary conditions of optimality, addresses these questions. We shall show that the forced harmonic oscillator problem introduced in this section conforms to this framework and that the couple $(\bar{\mu}, \{\bar{u}_i\})$ can be interpreted as the an extremal for the extended optimal control problem, in the sense that it satisfies first order necessary conditions of optimality.

The Figure shows the candidate minimizing state trajectory components.

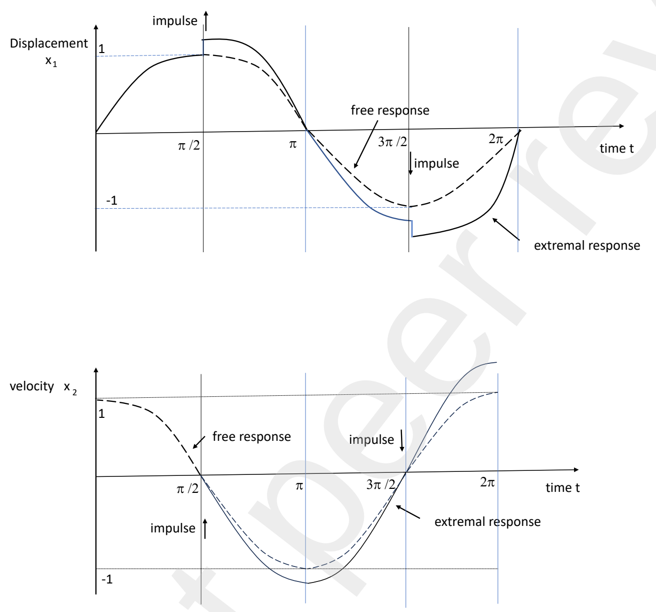

Figure 1: Extremal State Trajectory

3. Impulse Controls and Extended Processes

Return to the impulse control system description of the introduction:

$$
\text{(DIS)} \begin{cases} dx(t) = f(x(t), x(t-h))dt + g(x(t), x(t-h))d\mu(t),\ t \in [0, T], \\ \text{where } \mu \text{ is a non-negative Borel measure on } [0, T], \text{ and} \\ x \text{ satisfies the boundary condition } x(0) = x_0 \text{ and } x(t) = \xi_0 \text{ for } t < 0\,, \end{cases}
$$

in which $f, g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, x_0 and $\xi_0 \in \mathbb{R}^n$, and $h > 0$, $T > 0$. Assume:

(H1): f and g are C^1 functions and, given $K > 0$, there exists $c > 0$ such that, for any nonnegative $u \in L^1$ satisfying $\int_0^T u(t')\, dt' \leq K$, there is a unique solution $x \in W^{1,1}([0, T]; \mathbb{R}^n)$ to the delay differential equation (1.1) satisfying $\|x\|_\infty \leq c$.

(H2): $T = Nh$, for some integer $N > 0$.

3.1. Definitions

Let us recall the precise definitions of impulse control and corresponding extended trajectories introduced in [11].

Definition 3.1. (Impulse and Strict Sense Controls)
An *impulse control* $(\mu, \{w^r\}_{r\in[0,h]})$ consists of a measure $\mu \in C^{\oplus}(0,T)$ and a set of measurable functions $w^r = (w_1^r, \ldots, w_N^r) : [0,1] \to [0, +\infty[^N$, elements in which are parameterized by $r \in [0,h]$, with the properties:

(i): For each $r \in [0,h]$, $\displaystyle\sum_{i=1}^{N} w_i^r(s) = \sum_{i=1}^{N} \int_0^1 w_i^r(s)ds$, a.e. $s \in [0,1]$;

(ii): For each $r \in]0,h[$, $\displaystyle\int_0^1 w_i^r(s)ds = \mu(\{r + (i-1)h\})$, for $i = 1, \ldots, N$.

(iii): $\displaystyle\int_0^1 [w_i^h(s) + w_{i+1}^0(s)]ds = \mu(\{ih\})$, for $i = 0, \ldots, N$. Here, we interpret $w_0^h \equiv 0$ and $w_{N+1}^0 \equiv 0$.

We say '*strict sense control*' to mean an impulse control $(\mu, \{w^r\}_{r\in[0,h]})$, in which $d\mu(t) = u(t)dt$ for some non-negative function $u \in L^1(0,T)$. The name 'strict sense control' is also used for this purpose. The family of 'attached controls' $\{w^r\}_{r\in[0,h]}$ describes, in some sense, the different ways of approximating the measure μ by the limit of some sequence of strict sense controls. Note that an attached control $\{w^r\}$ is non-zero a.e. only if either r, or any of its translations by the delay interval, $r+(i-1)h$ for $i = 2, \ldots, N$, is not an atom of μ. We see that, if u is a strict sense control, i.e. $d\mu(t) = u(t)dt$ for some non-negative $u \in L^1$, then the attached controls are all zero a.e..

Definition 3.2. (Extended and Strict Sense Processes)
An element $(x, \mu, \{w^r\}_{r\in[0,h]})$ is said to be an *extended process* (with *extended state trajectory* x) if $x \in BV([-h,T];\mathbb{R}^n)$ and $(\mu, \{w^r\}_{r\in[0,h]})$ is an impulse control such that

(i): $x(t) = \xi_0$, for $t < 0$;

(ii): for each $i = 1, \ldots, N$,

$$x(t) = \begin{cases} x_0 & \text{if } i = 1 \\ \zeta_i^0(1) & \text{if } i > 1 \end{cases} \quad \text{for } t = (i-1)h,$$

$$\begin{aligned} x(t) = \zeta_i^0(1) &+ \int_{(i-1)h}^{t} f(x(t'), x(t'-h))dt' + \int_{[(i-1)h,t]} g(x(t'), x(t'-h))d\mu^c(t') \\ &+ \sum_{r\in]0,\, t-(i-1)h]} (\zeta_i^r(1) - x^-((r + (i-1)h)) \quad \text{for } t \in](i-1)h, ih[\end{aligned}$$

and $x(T) = \zeta_N^h(1)$.

Here, for $r \in [0,h]$, the functions $\zeta_1^r, \ldots, \zeta_N^r : [0,1] \to \mathbb{R}^n$ satisfy the system of differential equations

$$\frac{d}{ds}\zeta_i^r(s) \;=\; w_i^r(s)\, g(\zeta_i^r(s), \zeta_{i-1}^r(s)), \text{ a.e. } s \in [0,1], \qquad \text{for } i = 1, \ldots, N\,, \quad (3.1)$$

for which the boundary conditions are

$$\zeta_i^r(0) = \begin{cases} x^-(r + (i-1)h) & \text{if } r \in]0,h] \\ \zeta_{i-1}^h(1) & \text{if } r = 0\,. \end{cases} \quad (3.2)$$

In the above, $\zeta_0^r(s) := \begin{cases} \xi_0 & \text{if } s \in [0,1[\text{ or } r \in [0,h[\\ x_0 & \text{if } s = 1 \text{ and } r = h \end{cases}$ and μ^c is the continuous component of μ. We see that extended state trajectories have bounded variation and are right continuous on $]0,T[$. Observe that jumps in the state trajectory x are determined by the attached controls $\{w^r\}_{r\in[0,h]}$ through the 'instantaneous dynamics', that is, through the differential equations (3.1 together with the boundary conditions (3.2).

When an extended state process $(x, u\,dt, \{w^r \equiv 0\}_{r\in[0,h]})$ corresponds to a strict sense control u, it is written simply as (x,u) and is called a *strict sense process*. The absolutely continuous function x, referred to as a strict sense state trajectory, satisfies the delayed differential equation

$$\dot{x}(t) = f(x(t), x(t-h)) + g(x(t), x(t-h))u(t), \text{ a.e. } t \in [0,T]\,.$$

Remark. In connection with the forced harmonic oscillator example of Section 2.2, for a fixed value of the parameter $\gamma \in (0,1)$, we examined two different ways of approximating the measure $\mu = \gamma\delta_{\pi/2} + (1-\gamma)\delta_{3\pi/2}$ by sequences of absolutely continuous controls, namely $\{u_\varepsilon^\gamma\}_{\varepsilon>0}$ and $\{v_\varepsilon^\gamma\}_{\varepsilon>0}$. Prop. 2.1 provides formulae for the corresponding state trajectories, x^γ and y^γ respectively, in the limit as $\varepsilon \downarrow 0$.

Now consider the impulse controls $(\mu, \{w^r\}_{r\in[0,h]})$ and $(\mu, \{\tilde{w}^r\}_{r\in[0,h]})$, in the sense of Def. 3.1. Here, μ is as in (2.2), $w^r \equiv \tilde{w}^r \equiv 0$ if $r \neq \{\pi/2\}$,

$$w^{r=\frac{\pi}{2}}(s) = \begin{cases} (1,0) & \text{if } 0 \le s \le \gamma \\ (0,1) & \text{if } \gamma < s \le 1 \end{cases} \quad \text{and} \quad \tilde{w}^{r=\frac{\pi}{2}}(t) = \begin{cases} (0,1) & \text{if } 0 \le s \le 1-\gamma \\ (1,0) & \text{if } 1-\gamma < s \le 1. \end{cases}$$

A straightforward calculation leads to the following formulae for the corresponding extended state trajectories x and $\tilde{x}$, in the sense of Def. 3.2, respectively:

$$x(t) := \begin{cases} (\sin(t), \cos(t)) & \text{for } t \in [0, \pi/2) \\ e^\gamma(\sin(t), \cos(t)) & \text{for } t \in [\pi/2, 3\pi/2) \\ e^{\gamma + (1-\gamma)e^\gamma}(\sin(t), \cos(t)) & \text{for } t \in [3\pi/2, 2\pi] \end{cases}$$

and

$$\tilde{x}(t) := \begin{cases} (\sin(t), \cos(t)) & \text{for } t \in [0, \pi/2) \\ e^\gamma(\sin(t), \cos(t)) & \text{for } t \in [\pi/2, 3\pi/2) \\ e(\sin(t), \cos(t)) & \text{for } t \in [3\pi/2, 2\pi] \end{cases}$$

Re-examining the formulae in Prop. 2.1, we see that

$$x(t) = x^\gamma(t) \text{ and } \tilde{x}(t) = y^\gamma(t) \text{ for all } t \in [0, T]\backslash \mathcal{S},$$

in which $\mathcal{S} := \{\pi/2\} \cup \{3\pi/2\}$. We have shown that we can find extended state trajectories (corresponding to the measure μ and suitable attached controls) which are equivalent to the limiting state trajectories obtained from two different sequences of strict sense controls, converging to μ. Here 'equivalence' means 'modulo functions that coincide on the complement of a countable set not including $\{0\} \cup \{T\}$'; in this example the set is $\{\pi/2\} \cup \{3\pi/2\}$.

3.2. Properties of extended trajectories

This subsection brings together important properties of extended trajectories, all of which are proved in [11]. The first concerns existence and uniqueness matters:

Proposition 3.3. *For each impulse control* $(\mu, \{w^r\}_{r\in[0,h]})$*, there exists one and only one extended trajectory* $x \in BV([-h, T]; \mathbb{R}^n)$ *to* (DIS).

Given $K > 0$, the second concerns compactness properties of the following *reachable set*

$$\mathcal{R}^{\mathrm{e}}_K(T) := \{x(T) : (x, \mu, \{w^r\}_{r\in[0,h]}) \text{ is an extended process for (DIS) s.t. } \|\mu\|_{TV} \le K\},$$

and density properties of the *strict sense reachable set*

$$\mathcal{R}^{\mathrm{s}}_K(T) := \{x(T) : (x, u) \text{ is a strict sense process for (DIS) s.t. } \textstyle\int_0^T u(t)dt \le K\}.$$

We have

Proposition 3.4. *For each* $K > 0$, $\mathcal{R}^{\mathrm{e}}_K(T)$ *is compact and* $\mathcal{R}^{\mathrm{e}}_K(T) = \overline{\mathcal{R}^{\mathrm{s}}_K(T)}$.

Given a family of state trajectories for a control system, an extension of the family is said to be a *relaxation* if the reachable set for the extended family is compact and the reachable set of the original family is a dense subset. In perhaps the best known example, the extension providing the relaxation is obtained by replacing the original velocity set by its convex hull. The preceding proposition justifies interpreting the class of extended state trajectories as a relaxation of the class of strict sense trajectories.

Remark. The proofs in [9] invoke the hypothesis that 'the functions f and g in (DIS) are continuously differentiable and bounded', but they are valid, with minor changes, under the weaker hypothesis (H1). A simple analysis of control system (2.1), in which we establish (with the help of Gronwall's lemma) a bound on state trajectories first over $[0, \pi]$ and then over $[\pi, 2\pi]$, confirms that the forced harmonic oscillator of Section 2 satisfies assumption (H1).

4. Optimal Extended Processes

We next consider a general optimal control problem for which the dynamic constraint is the impulse control system (DIS). The problem has terminal cost and constraints both on the end-points of extended state trajectories and on the total variation of the measure control.

$$
\text{(P)} \begin{cases} \text{Minimize } \Lambda(x(T)) \\ \text{over extended processes } (x,\mu,\{w^r\}_{r\in[0,h]}) \text{ for (DIS), satisfying} \\ \|\mu\|_{TV} \le K \quad \text{and} \quad x(T)\in\mathcal{C}\,. \end{cases}
$$

Here, $K > 0$ is a given real number, $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is a given terminal cost function and $\mathcal{C} \subset \mathbb{R}^n$ is a given subset.

An extended process $(x, \mu, \{w^r\}_{r\in[0,h]})$ for (DIS) is *feasible* if $\|\mu\|_{TV} \le K$ and $x(T) \in \mathcal{C}$. A feasible extended process $(\bar{x}, \bar{\mu}, \{\bar{w}^r\}_{r\in[0,h]})$ for (DIS) that minimizes $\Lambda(x(T))$ over all feasible extended processes $(x, \mu, \{w^r\}_{r\in[0,h]})$ for (DIS) is said to be an *optimal extended process*.

Noting that a feasible extended process $(x, \mu, \{w^r\}_{r\in[0,h]})$ is optimal if and only if x is a feasible state trajectory such that $x(T)$ minimizes Λ over the reachable set $\mathcal{R}^e_K(T) \cap \mathcal{C}$, we deduce from Prop. 3.4:

Theorem 4.1 (Existence of optimal controls)**.** *Let $\mathcal{C} \subset \mathbb{R}^n$ be a closed, nonempty set and let $\Lambda : \mathbb{R}^n \to \mathbb{R}$ be a lower semicontinuous function. Assume that the set of feasible extended process is non-empty. Then, there exists an optimal extended process for problem* (P)*.*

The necessary conditions for $(\bar{x}, \bar{\mu}, \{\bar{w}^r\}_{r\in[0,h]})$ to be an optimal extended process, in the form of a maximum principle below, are proved in [11]. Given a function $\varphi = \varphi(x_1, x_2)$, $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$, we set $\bar{\varphi}(t) := \varphi(\bar{x}(t), \bar{x}(t-h))$ and write $\bar{\varphi}_{x_1}(t)$, $\bar{\varphi}_{x_2}(t)$ for the Jacobian of φ at $(\bar{x}(t), \bar{x}(t-h))$ w.r.t. x_1 and x_2, respectively. For instance, we have $\bar{g}(t) = g(\bar{x}(t), \bar{x}(t-h))$ and $\bar{g}_{x_1}(t) = \frac{\partial g}{\partial x_1}(\bar{x}(t), \bar{x}(t-h))$.

Theorem 4.2 (Maximum Principle)**.** *Assume that $(\bar{x}, \bar{\mu}, \{\bar{w}^r\}_{r\in[0,h]})$ is an optimal extended process for* (P)*. For $i = 1, \dots, N$ and $r \in [0, h]$, let $\zeta_i^r : [0,1] \to \mathbb{R}^n$ be the corresponding functions describing the instantaneous evolution of the state of this process, defined by* (3.1) *and* (3.2)*. Assume that $\mathcal{C} \subseteq \mathbb{R}^n$ is closed and nonempty and $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function. Then, there exist $\lambda \ge 0$, $c \in \mathbb{R}$, $d \ge 0$ and a function $p \in BV([0,T];\mathbb{R}^n)$, right continuous on $]0,T[$, with the following properties: $d = 0$ if $\|\bar{\mu}\|_{TV} < K$ and*

(A): $\lambda + \|p\|_{L^\infty} \neq 0$,

(B): *for each* $i = 1, \dots, N$ *and* $t \in [(i-1)h, ih]$,

$$p(t) = \eta_i^0(1), \text{ when } i = 2, \dots, N \text{ and } t = (i-1)h,$$
$$p(t) = \eta_i^0(1) - \int_{(i-1)h}^t p(t') \cdot \bar{f}_{x_1}(t')dt' - \int_{[(i-1)h,t]} p(t') \cdot \bar{g}_{x_1}(t')d\bar{\mu}^c(t')$$
$$- \int_{(i-1)h}^t p(t'+h) \cdot \bar{f}_{x_2}(t'+h)dt' - \int_{[(i-1)h,t]} p(t'+h) \cdot \bar{g}_{x_2}(t'+h))d\bar{\mu}^c(t'+h)$$
$$- \sum_{r \in]0, t-(i-1)h]} (\eta_i^r(0) - \eta_i^r(1)), \quad \text{for } t \in](i-1)h, ih[,$$

Also,

$$p(T) = \eta_N^h(1) \text{ and } p(t) = 0 \text{ for } t > T.$$

For every $r \in [0, h]$ *and* $i \in \{1, \dots, N\}$, $\eta_i^r : [0,1] \to \mathbb{R}^n$ *is the solution to the differential equation*

$$\begin{aligned} \frac{d\eta_i^r}{ds}(s) = &-\eta_i^r(s) \cdot g_{x_1}(\bar{\zeta}_i^r(s), \bar{\zeta}_{i-1}^r(s))\bar{w}_i^r(s) \\ &-\eta_{i+1}^r(s) \cdot g_{x_2}(\bar{\zeta}_{i+1}^r(s), \bar{\zeta}_i^r(s))\bar{w}_{i+1}^r(s) \quad \text{a.e. } s \in [0,1], \end{aligned} \tag{4.1}$$

for boundary conditions

$$\eta_i^r(0) = \begin{cases} p^-(r + (i-1)h) & \text{if } r \in]0, h] \\ \eta_{i-1}^h(1) & \text{if } r = 0\,. \end{cases} \tag{4.2}$$

In the above relations, we set $\eta_0^h(1) := p(0)$ *and* $\eta_{N+1}^r(s) := 0$ *for all* $r \in [0, h]$ *and* $s \in [0, 1]$.

(C): $-p(T) \in \lambda \nabla \Lambda(\bar{x}(T)) + N_{\mathcal{C}}(\bar{x}(T))$,

(D):

(i): $\sum_{j=1}^N p(r + (j-1)h) \cdot \bar{f}(r + (j-1)h) - c = 0$ *for all* $r \in]0, h[$,

and also at $r = 0$ *if* $\sum_{j=1}^N \int_0^1 \bar{w}_j^0(s)\, ds = 0$, *and at* $r = h$ *if* $\sum_{j=1}^N \int_0^1 \bar{w}_j^0(s)\, ds = 0$ *and* $\sum_{j=1}^N \int_0^1 \bar{w}_{j-1}^h(s)\, ds = 0$,

(ii): $p(t) \cdot \bar{g}(t) - d \leq 0$ *for all* $t \in [0, T]$,

(iii): *supp* $\{\bar{\mu}\} \subset \{t \in [0, T] : \quad p(t) \cdot \bar{g}(t) - d = 0\}$,

(iv): *for any* $r \in [0, h]$ *such that* $\sum_{j=1}^N \int_0^1 \bar{w}_j^r(s)\, ds > 0$,

(a): *for* $i = 1, \dots, N$,

$$\eta_i^r(s) \cdot g(\bar{\zeta}_i^r(s), \bar{\zeta}_{i-1}^r(s)) = \max_j \eta_j^r(s) \cdot g(\bar{\zeta}_j^r(s), \bar{\zeta}_{j-1}^r(s))$$
$$\text{for a.e. } s \in [0,1] \text{ such that } \bar{w}_i^r(s) > 0,$$

$$(b):\ \sum_{j=1}^{N}\eta_j^r(s)\cdot f(\bar\zeta_j^r(s),\bar\zeta_{j-1}^r(s)) - c \le \max_j \eta_j^r(s)\cdot g(\bar\zeta_j^r(s),\bar\zeta_{j-1}^r(s)) - d = 0 \quad \textit{for all } s\in[0,1].$$

Comments.

(1): Condition (B) is the integral representation of the costate equation for p. This can also be expressed in infinitesimal form by differentiating the relevant equations at points of differentiability. The resulting generalized 'advance' differential equation (or reverse 'delay' differential equation in time) is consistent with the first order necessary conditions for optimal control problems without impulses,as established, for example, in [4].

(2): Conditions (D), (i) and (D), (iii) reflect the 'constancy of the Hamiltonian' condition appropriate to the impulse control problem. Specifically, the equality in condition (D), (i) is true at $r = 0$ when $t = 0$ is not an atom for $\bar\mu$, and at $r = h$ when all values $t = ih$, $h = 0,\dots,N$, are not atoms for $\bar\mu$.

(3): The requirements of Condition (D), (iv), which is connected to the Weierstrass condition in classical optimal control, complemented by (D), (iii), offer valuable insight into the location of $\bar\mu$'s support. Specifically, (D), (iv) outlines the characteristics of the attached controls that dictate the instantaneous evolution of the state at the atoms of $\bar\mu$.

5. An Extremal for the Forced Harmonic Oscillator Problem with Time Delays

We now return to the optimal control problem (E) of Section 2. First of all we recognize (E) as a special case of (P), in which $n = 2$, $K = 1$, $h = \pi$, $T = 2\pi$, $x_0 = (0,1)$, $\xi_0 = (1,0)$, $\mathcal{C} = \mathbb{R}^n$, $f((x_1,x_2),(z_1,z_2)) = (x_2,-x_1)$ and $g((x_1,x_2),(z_1,z_2)) = (x_1z_1,0)$. In Section 2 (see also the discussion in Section 3), we identified a family of candidate minimizing extended impulse controls $(\mu^\gamma,\{w^{\gamma r}\}_{r\in[0,\pi]})$, parameterized by $\gamma\in(0,1)$, for this problem, namely $\mu^\gamma = \gamma\delta_{\pi/2} + (1-\gamma)\delta_{3\pi/2}$, $w^{\gamma r}\equiv 0$ if $r\neq\pi/2$ and

$$w^{\gamma\frac{\pi}{2}}(s) = \begin{cases}(1,0) & \text{if } 0\le s\le\gamma\\ (0,1) & \text{if } \gamma < s \le 1\,.\end{cases}$$

We also showed that $(\bar\mu,\{\bar w^r\}_{r\in[0,\pi]})$ minimizes the cost within this family, where

$$(\bar\mu,\{\bar w^r\}_{r\in[0,\pi]}) := (\mu^{\bar\gamma},\{w^{\bar\gamma r}\}_{r\in[0,\pi]}). \tag{5.1}$$

Here, $\bar\gamma$ the unique solution on $(0,1)$ to the equation $e^\gamma = \frac{1}{\gamma}$.

In this section we provide more evidence that $(\bar\mu,\{\bar w^r\}_{r\in[0,\pi]})$ is a minimizer over the entire class of feasible extended processes by showing that it is an extremal w.r.t. the necessary conditions of the previous section.

Proposition 5.1. *Let $\bar{\gamma}$, $0 < \bar{\gamma} < 1$, be the unique number such that*

$$e^{\bar{\gamma}} = 1/\bar{\gamma}\,.$$

Then, the extended process $(\bar{x}, \bar{\mu}, \{\bar{w}^r\}_{r\in[0,\pi]})$ as in (5.1) *is an extremal for* (E).

Proof. Take any $\gamma \in (0,1)$. We know from Section 3 that the extended state trajectory x^γ corresponding to $(\mu^\gamma, \{w^{\gamma r}\}_{r\in[0,\pi]})$ is

$$x^\gamma(t) := \begin{cases} (\sin(t), \cos(t)) & \text{for } t \in [0, \pi/2) \\ e^\gamma(\sin(t), \cos(t)) & \text{for } t \in [\pi/2, 3\pi/2) \\ e^{\gamma+(1-\gamma)e^\gamma}(\sin(t), \cos(t)) & \text{for } t \in [3\pi/2, 2\pi] \end{cases}$$

and the arcs describing the instantaneous evolution of the state at times $t = \pi/2$ and $3\pi/2$, are given by

$$\zeta_1^{\gamma\frac{\pi}{2}}(s) = \begin{cases} (e^s, 0) & \text{if } s \in [0, \gamma[\\ (e^\gamma, 0) & \text{if } s \in [\gamma, 1] \end{cases},\ \ \zeta_2^{\gamma\frac{\pi}{2}}(s) = \begin{cases} (-e^\gamma, 0) & \text{if } s \in [0, \gamma[\\ (-e^\gamma e^{(s-\gamma)e^\gamma}, 0) & \text{if } s \in [\gamma, 1] \end{cases}.$$

Set $\lambda = 1$. Write p^γ for the costate trajectory and let $\eta_1^{\gamma\frac{\pi}{2}}$, and $\eta_2^{\gamma\frac{\pi}{2}}$ be the arcs describing the instantaneous evolution of the costate at times $t = \pi/2$ and $t = 3\pi/2$. A lengthy but straightforward calculation, in which we solve the costate equation (B) backwards in time (with the right-end boundary condition furnished by the transversality condition (C)) on $[3\pi/2, 2\pi]$, then solve the differential equation (4.1) for $\eta_2^{\gamma\frac{\pi}{2}}$ with right boundary condition (4.2) to obtain the jump and so on, yields the following formulae for these functions:

$$(p_1(t), p_2(t)) = \begin{cases} e^{\gamma+(1-\gamma)e^\gamma}[1 + e^\gamma(1-\gamma)](\sin(t), \cos(t)) & \text{for } t \in [0, \pi/2[, \\ e^{(1-\gamma)e^\gamma}(\sin(t), \cos(t)) & \text{for } t \in [\pi/2, 3\pi/2[, \\ (\sin(t), \cos(t)) & \text{for } t \in [3\pi/2, 2\pi] \end{cases}$$

and

$$\eta_1^{\gamma\frac{\pi}{2}}(s) = \begin{cases} (e^{(1-\gamma)e^\gamma}[1 + e^\gamma(1-\gamma)]e^{\gamma-s}, 0) & \text{if } s \in [0, \gamma[\\ (e^{(1-\gamma)e^\gamma}[1 + e^\gamma(1-\gamma)], 0) & \text{if } s \in [\gamma, 1] \end{cases},$$

$$\eta_2^{\gamma\frac{\pi}{2}}(s) = \begin{cases} (-e^{(1-\gamma)e^\gamma}, 0) & \text{if } s \in [0, \gamma[\\ (-e^{(1-s)e^\gamma}, 0) & \text{if } s \in [\gamma, 1]. \end{cases}$$

With knowledge of these functions, we are now ready to confirm that the extended process $(x^\gamma, \mu^\gamma, \{w^{\gamma r}\}_{r\in[0,\pi]})$ is an extremal, for an appropriate choice of the parameter γ. This involves checking relations (A)-(D) in Prop. 4.2. (A)-(C) are automatically satisfied (independent of our choice of γ), since these relations are used to construct p^γ and $\eta^{\gamma\frac{\pi}{2}} = (\eta_1^{\gamma\frac{\pi}{2}}, \eta_2^{\gamma\frac{\pi}{2}})$. So condition (D) alone requires attention.

Identifying the data (f, g, N, x_0, ξ_0) for (DIS) as that of the forced oscillator problem (E), we can easily show from the above formulae for functions p^γ, $\zeta^{\gamma\frac{\pi}{2}} = (\zeta_1^{\gamma\frac{\pi}{2}}, \zeta_2^{\gamma\frac{\pi}{2}})$ and $\eta^{\gamma\frac{\pi}{2}} = (\eta_1^{\gamma\frac{\pi}{2}}, \eta_2^{\gamma\frac{\pi}{2}})$ that

$$\begin{aligned} &\eta_1^{\gamma\frac{\pi}{2}}(s) \cdot g(\zeta_1^{\gamma\frac{\pi}{2}}(s), \zeta_0^{\gamma\frac{\pi}{2}}(s)) = e^{\gamma+(1-\gamma)e^\gamma}[1 + e^\gamma(1-\gamma)] \quad \text{for all } s \in [0,1] \\ &\eta_2^{\gamma\frac{\pi}{2}}(s) \cdot g(\zeta_2^{\gamma\frac{\pi}{2}}(s), \zeta_1^{\gamma\frac{\pi}{2}}(s)) = e^{(1-\gamma)e^\gamma} e^{2\gamma} \quad \text{for all } s \in [\gamma, 1]\,. \end{aligned}$$

By condition (D)(iv) we deduce that there exists $d \in \mathbb{R}$ such that

$$d = e^{\gamma+(1-\gamma)e^{\gamma}}[1 + e^{\gamma}(1-\gamma)] \text{ for all } s \in [0,\gamma]$$
$$d = e^{(1-\gamma)e^{\gamma}}e^{2\gamma} \quad \text{ for all } s \in [\gamma, 1]$$

This is true if and only if $\gamma = \bar{\gamma}$, where $\bar{\gamma}$ is the unique solution in $(0,1)$ of the equation $[1 + e^{\bar{\gamma}}(1-\bar{\gamma})] = e^{\bar{\gamma}}$, which is equivalent to $e^{\bar{\gamma}} = \bar{\gamma}^{-1}$. The formulae also yield the information that

$$p^{\bar{\gamma}}(t) \cdot \bar{f}(t) = 0, \text{ for all } t \in [0, 2\pi]\,. \tag{5.2}$$

This confirms condition (D)(i), for the choice of parameter $c = 0$. Making use of the relation $[1 + e^{\bar{\gamma}}(1-\bar{\gamma})] = e^{\bar{\gamma}}$, we show also that

$$p^{\bar{\gamma}}(t) \cdot \bar{g}(t) \le e^{\bar{\gamma}+(1-\bar{\gamma})e^{\bar{\gamma}}}[1 + e^{\bar{\gamma}}(1-\bar{\gamma})] \text{ for all } t \in [0, 2\pi]$$

and equality is achieved at the two points $\pi/2$ and $3\pi/2$. We have confirmed conditions (D)(ii) and (iii) when we choose $d = e^{(1-\bar{\gamma})e^{\bar{\gamma}}}e^{2\bar{\gamma}}$.

Finally, we observe that condition (D)(iv)(b) is also satisfied, since

$$\eta_1^{\gamma\frac{\pi}{2}}(s) \cdot f(\zeta_1^{\gamma\frac{\pi}{2}}(s), \zeta_0^{\gamma\frac{\pi}{2}}(s)) \;=\; \eta_2^{\gamma\frac{\pi}{2}}(s) \cdot f(\zeta_2^{\gamma\frac{\pi}{2}}(s), \zeta_1^{\gamma\frac{\pi}{2}}(s)) = 0, \text{ for all } s \in [0,1].$$

$\square$

Remark. The proof of Prop. 5.1 not only establishes that $(\bar{x}, \bar{\mu}, \{\bar{w}^r\}_{r\in[0,\pi]})$ is an extremal for (E). The analysis also identifies, from the maximum principle conditions, the appropriate value of the parameter $\gamma \in [0,1]$, namely $\gamma = \bar{\gamma}$. We have already shown, by means of a simple direct analysis at the end of Section 2, that $(\bar{x}, \bar{\mu}, \{\bar{w}^r\}_{r\in[0,\pi]})$ is a minimizer within the narrow class of extended controls $\{(\mu^{\gamma}, \{w^{\gamma r}\}_{r\in[0,\pi]}) : \gamma \in [0,1]\}$. Prop 5.1 adds extra evidence that $(\bar{x}, \bar{\mu}, \{\bar{w}^r\}_{r\in[0,\pi]})$ is a true minimizer, by showing that it is an extremal. Since, according to Thm. 4.1, there exists a minimizer, this implies that $(\bar{x}, \bar{\mu}, \{\bar{w}^r\}_{r\in[0,\pi]})$ is the minimizer over the class of all extended controls, under the hypothesis that there is a unique extremal.

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