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A conjecture related to the nilpotency of groups with isomorphic non-commuting graphs

Valentina Grazian [a], Carmine Monetta [b,*]

[a] *Department of Mathematics and Applications, University of Milano – Bicocca, Via Roberto Cozzi 55, 20125 Milano, Italy*
[b] *Department of Mathematics, University of Salerno, via Giovanni Paolo II 132, 84084 Fisciano (SA), Italy*

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ABSTRACT

In this work we discuss whether the non-commuting graph of a finite group can determine its nilpotency. More precisely, Abdollahi, Akbari and Maimani conjectured that if G and H are finite groups with isomorphic non-commuting graphs and G is nilpotent, then H must be nilpotent as well (Conjecture 2). We characterize the structure of such an H when G is a finite AC-group, that is, a finite group in which all centralizers of non-central elements are abelian. As an application, we prove Conjecture 2 for finite AC-groups whenever $|Z(G)| \geq |Z(H)|$.

1. Introduction

Given a finite group G, one can consider the graph whose vertices are (some) elements of G and whose edges reflect a certain structure property of G: we refer to [4] for a survey on the topic. Such a technique has proven to be a valuable tool to study various aspects

* Corresponding author.
E-mail addresses: valentina.grazian@unimib.it (V. Grazian), cmonetta@unisa.it (C. Monetta).

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of finite groups. In this work we will focus our attention on the non-commuting graph of the non-abelian finite group G, that is, the graph Γ_G whose vertices are the non-central elements of G and in which two vertices are joined by an edge if they do not commute. Notice that if G and H are finite groups with isomorphic non-commuting graphs $\Gamma_G \cong \Gamma_H$, then $|G| - |Z(G)| = |H| - |Z(H)|$. In [2] the authors studied what structural properties can be deduced from the assumption that two finite groups have isomorphic non-commuting graphs, posing (among others) the following two conjectures:

Conjecture 1. *[2, Conjecture 1.1] Let G and H be two non-abelian finite groups such that $\Gamma_G \cong \Gamma_H$. Then $|G| = |H|$.*

Conjecture 2. *[2, Conjecture 3.25] Let G be a finite non-abelian nilpotent group and H be a group such that $\Gamma_G \cong \Gamma_H$. Then H is nilpotent.*

Conjecture 2 appears also in the Kourovka Notebook [7], stated as Problem 16.1. In [2, Theorem 3.24] the authors proved that Conjecture 1 implies Conjecture 2:

Theorem 1.1. *Let G be a finite non-abelian nilpotent group and H be a group such that $\Gamma_G \cong \Gamma_H$. If $|G| = |H|$ then H is nilpotent.*

Conjecture 1 has been proven to be true when G is a p-group ([1, Theorem 1.2]), a dihedral group ([2, Proposition 3.19]) or a simple group ([5, Theorem A]), where in the latter case we can even conclude that $G \cong H$ ([12, Theorem 1.1]). Moreover, in [2, Proposition 3.3] the authors showed that it is sufficient to prove Conjecture 1 for a specific class of finite groups: the class of finite groups in which all centralizers of non-central elements are abelian, called AC-groups. However, in [8], the author constructed two AC-groups $G = P \times A$ and $H = Q \times B$ with isomorphic non-commuting graphs, where P is a non-abelian 2-group of order 2^{10}, Q is a non-abelian 5-group of order 5^6, A and B are abelian groups satisfying $2^5 \cdot |A| = 2^2 \cdot 5^3 \cdot |B|$, and in general, $|G| = 2^{10}|A| \neq 5^6|B| = |H|$. Nevertheless, Conjecture 2 is not affected by this example and is in fact still open. One of the major progresses made toward its proof is the following result:

Theorem 1.2. *[3, Theorem 2.4] Let G be a finite non-abelian nilpotent group and suppose $\Gamma_G \cong \Gamma_H$ for a group H. If G has at least two distinct non-abelian Sylow subgroups and $|Z(G)| \geq |Z(H)|$ then $|G| = |H|$, (and so H is nilpotent by Theorem 1.1).*

Theorem 1.2 suggests that, in order to prove Conjecture 2, it is necessary to study the class of finite non-abelian nilpotent groups having a unique non-abelian Sylow subgroup, that is, finite groups of the form $G = P \times A$, where p is a prime, $P \in \mathrm{Syl}_p(G)$ is non-abelian and A is an abelian p'-group. Taking inspiration from these considerations, we pose the following conjecture:

Conjecture 3. *Let p be a prime and suppose $G = P \times A$ is a finite group where $P \in \mathrm{Syl}_p(G)$ is non-abelian and A is an abelian p'-group. If $\Gamma_G \cong \Gamma_H$ for a finite group H and $|Z(G)| \geq |Z(H)|$ then $H = Q \times B$, where q is a prime, $Q \in \mathrm{Syl}_q(H)$ is non-abelian and B is an abelian q'-group. In particular, H is nilpotent.*

If Conjecture 3 is true, then, combining it with Theorem 1.2, we can immediately deduce Conjecture 2 as long as $|Z(G)| \geq |Z(H)|$:

Theorem 1.3. *Let G be a finite non-abelian nilpotent group and suppose $\Gamma_G \cong \Gamma_H$ for a group H. If Conjecture 3 holds and $|Z(G)| \geq |Z(H)|$ then H is nilpotent.*

As for Conjecture 3, it is known to be true if G is a p-group (by [1, Theorem 1.2]) and if the graph Γ_G is regular (as this means that the conjugacy classes of non-central elements of G have only one size and we conclude by [6, Theorem 1]).

In this work, we prove Conjecture 3 when G is an AC-group. More precisely, we obtain the following stronger result:

Theorem A. *Let p be a prime and let $G = P \times A$ be a finite non-abelian nilpotent AC-group, where $P \in \mathrm{Syl}_p(G)$ is non-abelian and A is an abelian p'-group. Let H be a group such that $\Gamma_G \cong \Gamma_H$. Then H is a finite AC-group and either*

(1) $H = Q \times B$, where q is a prime, $Q \in \mathrm{Syl}_q(H)$ is non-abelian and B is an abelian q'-group; or
(2) $|Z(H)| > |Z(G)|$, $[P\colon Z(P)] > p^4$, none of the maximal subgroups of G is abelian (so P does not have maximal nilpotency class) and $H/Z(H)$ is a Frobenius group with Frobenius kernel and complement $F/Z(H)$ and $K/Z(H)$, respectively, where K is an abelian subgroup of H, $Z(F) = Z(H)$, and $[F\colon Z(H)] = q^t$ for some prime $q \neq p$.

Given that all finite nilpotent AC-groups are of the form described in Conjecture 3 (see Theorem 2.7), we can now state our contribution to Conjecture 2 as a direct consequence of Theorem A:

Corollary 1.4. *Let G be a finite non-abelian nilpotent AC-group and H be a group such that $\Gamma_G \cong \Gamma_H$. If $|Z(G)| \geq |Z(H)|$ then H is nilpotent.*

2. Finite AC-groups

In this section we describe some properties of AC-groups, starting by recalling their definition:

Definition 2.1. A finite group G is said to be an AC-group if all centralizers in G of non-central elements of G are abelian.

The structure of non-abelian AC-groups was deeply investigated by Schmidt ([11]) and Rocke ([9] and [10]). We report some useful properties of such groups.

First of all note that the centralizers of non-central elements of a finite AC-group G are exactly the maximal abelian subgroups of G.

Lemma 2.2. *[10, Lemma 3.2] The following are equivalent for a finite non-abelian group G.*

(1) G is an AC-group.
(2) If $[x, y] = 1$, then $C_G(x) = C_G(y)$, whenever $x, y \in G\backslash Z(G)$.
(3) If $[x, y] = [x, z] = 1$, then $[y, z] = 1$, whenever $x \in G\backslash Z(G)$.
(4) If A and B are subgroups of G and $Z(G) < C_G(A) \leq C_G(B) < G$, then $C_G(A) = C_G(B)$.

Corollary 2.3. *Let G be a finite non-abelian AC-group and let $x, y \in G\backslash Z(G)$. Then the following are equivalent*

(1) $[x, y] \neq 1$;
(2) $C_G(x) \cap C_G(y) = Z(G)$;
(3) $C_G(x) \neq C_G(y)$.

Proof. Suppose $[x, y] \neq 1$. Note that $C_G(x) \cap C_G(y) = C_G(\langle x, y\rangle)$. By Lemma 2.2(4) if $Z(G) < C_G(\langle x, y\rangle)$ then $C_G(\langle x\rangle) = C_G(\langle x, y\rangle) = C_G(\langle y\rangle)$. Thus $y \in C_G(x)$, a contradiction. Hence we must have $Z(G) = C_G(\langle x, y\rangle) = C_G(x) \cap C_G(y)$, proving that (1) implies (2). Clearly (2) implies (3), since $Z(G) < Z(G)\langle x\rangle \leq C_G(x)$. Finally, (3) implies (1) by Lemma 2.2(2). □

It is clear that the property of being an AC-group is preserved by subgroups:

Lemma 2.4. *[10, Remark 3.3(c)] If G is a finite AC-group and $H \leq G$ then H is a finite AC-group.*

Notation 2.5. If G is a finite non-abelian group, set

$$\mathcal{C}(G) = \{C_G(x) \mid x \in G\backslash Z(G)\}.$$

For a finite non-abelian AC-group G, we can compute the order of $\mathcal{C}(G)$:

Lemma 2.6. *Suppose that G is a finite non-abelian AC-group Then*

$$|\mathcal{C}(G)| = -[G\colon Z(G)] + 1 + \sum_{C \in \mathcal{C}(G)} [C\colon Z(G)].$$

In particular, if $G/Z(G)$ is a p-group then $|\mathcal{C}(G)| \equiv 1 \mod p$.

Proof. By Corollary 2.3 we have $C_G(x) \cap C_G(y) = Z(G)$ whenever $C_G(x) \neq C_G(y)$. Therefore

$$|G| = \sum_{C \in \mathcal{C}(G)} |C| - |\mathcal{C}(G)||Z(G)| + |Z(G)|$$

that, dividing all by $|Z(G)|$, gives the result. □

In [11, Satz 5.12] Schmidt classified finite non-abelian solvable AC-groups, also indicating the order of the set $\mathcal{C}(G)$:

Theorem 2.7. *Let G be a finite non-abelian solvable AC-group. Then G satisfies one of the following properties:*

(1) G is non-nilpotent and it has an abelian normal subgroup N of prime index; moreover $|\mathcal{C}(G)| = [N\colon Z(G)] + 1$;
(2) $G/Z(G)$ is a Frobenius group with Frobenius kernel and complement $F/Z(G)$ and $K/Z(G)$, respectively, and F and G are abelian subgroups of G; moreover $|\mathcal{C}(G)| = [F\colon Z(G)] + 1$;
(3) $G/Z(G)$ is a Frobenius group with Frobenius kernel and complement $F/Z(G)$ and $K/Z(G)$, respectively, K is an abelian subgroup of G, $Z(F) = Z(G)$, and $F/Z(F)$ is of prime power order; moreover $|\mathcal{C}(G)| = [F\colon Z(G)] + |\mathcal{C}(F)|$;
(4) $G/Z(G) \cong \mathrm{Sym}(4)$ and V is a non-abelian subgroup of G such that $V/Z(G)$ is the Klein 4-group of $G/Z(G)$; moreover $|\mathcal{C}(G)| = 13$;
(5) $G = P \times A$, where P is an AC-subgroup of prime power order and A is an abelian group.

Remark 2.8. Note that if G is a finite nilpotent AC-group, then G is of type (5) of Theorem 2.7, that is, $G = P \times A$, where $P \in \mathrm{Syl}_p(G)$ for some prime p and A is an abelian subgroup of G, that we can assume to be of p'-order. In particular, $G/Z(G) \cong P/Z(P)$ is a p-group, so from Lemma 2.6 we deduce that $|\mathcal{C}(G)| \equiv 1 \mod p$, and for all $x \in G \backslash Z(G)$ we have $C_G(x) = C_P(x) \times A$.

We conclude this section with some properties of AC-groups of prime power order.

Lemma 2.9. *Let P be a finite non-abelian AC-group and suppose that P is a p-group for some prime p. Let $x \in Z_2(P) \backslash Z(P)$. Set $[P\colon Z(P)] = p^n$ and $[C_P(x)\colon Z(P)] = p^s$. If c denotes the nilpotency class of P, then*

(1) $P' \leq C_P(x) \trianglelefteq P$ (in particular P' is abelian);
(2) if $c > 2$ then $C_P(x)$ is the unique normal centralizer of P and its order is maximum among the non-central element-centralizers of G (in particular $Z_2(P) \leq C_P(x) = C_P(Z_2)$);

(3) if $[P\colon C_P(x)] \geq p^2$ *then* $P/Z(P)$ *has exponent* p*;*
(4) if $[P\colon C_P(x)] \geq p^2$ *then* $c \leq p$ *and* $Z_i(P) \leq C_P(x)$ *for every* $1 \leq i \leq c-1$*;*
(5) if $y \in P \backslash C_P(x)$ *and* $[C_P(y)\colon Z(P)] = p^t$ *then* $n \geq s+t$ *(in particular if* $c > 2$ *then* $n \geq 2t$*).*

Proof. Let $x \in Z_2(P) \backslash Z(P)$. Then x commutes with $P' = [P, P]$ and so $P' \leq C_P(x)$, implying that $C_P(x) \trianglelefteq P$ and proving (1). By [10, Lemma 3.8] if $c > 2$ then $C_P(x)$ is the unique normal centralizer of P and by [1, Proposition 2.7] the order of $C_P(x)$ is maximal. To complete the proof of part (2), note that this, combined with part (1), implies $C_P(x) = C_P(y)$ for every $x, y \in Z_2(P) \backslash Z(P)$. Part (3) is [10, Theorem 3.13(b)] and (4) is a combination of [10, Lemma 3.14(c) and Lemma 3.15]. Finally, since $C_P(x) \cap C_P(y) = Z(G)$ by Corollary 2.3 and $C_P(x) \trianglelefteq P$ by part (1), we deduce that $C_P(y)/Z(P) \cong C_P(x)C_P(y)/C_P(x) \leq P/C_P(x)$. Hence $p^t \leq p^{n-s}$, giving $n \geq s+t$. In particular, if $c > 2$ then $s \geq t$ by part (2) and so $n \geq 2t$. □

3. Non-commuting graphs of finite AC-groups

In this section we analyse the properties of finite AC-groups having isomorphic non-commuting graphs. We begin recalling the definition of non-commuting graph and some considerations on groups having isomorphic non-commuting graphs.

Definition 3.1. If G is a finite group, then the non-commutative graph of G, denoted by Γ_G, is the graph whose vertices are the elements of $G \backslash Z(G)$ and in which two vertices x and y are adjacent if and only if $xy \neq yx$.

Remark 3.2. Suppose G and H are finite groups with a graph-isomorphism $\phi\colon \Gamma_G \to \Gamma_H$. First of all, the graphs Γ_G and Γ_H have the same number of vertices, so

$$|G| - |Z(G)| = |H| - |Z(H)|. \tag{1}$$

For $x \in G \backslash Z(G)$, set $\Gamma_G(x) = \{y \in G \backslash Z(G) \mid y \text{ is adjacent to } x\} = G \backslash C_G(x)$. Then $|\Gamma_G(x)| = |\Gamma_H(\phi(x))|$, implying

$$|G| - |C_G(x)| = |H| - |C_H(\phi(x))|. \tag{2}$$

Combining Equations (1) and (2) we also obtain

$$|C_G(x)| - |Z(G)| = |C_H(\phi(x))| - |Z(H)|. \tag{3}$$

Equations (1), (2) and (3) will be used in the rest of this work without further reference.

We now focus on the non-commuting graph of finite AC-groups. Note that the property of having abelian centralizers can be easily read from the non-commuting graph of a finite

group G. Indeed, G is an AC-group if and only if Γ_G is a complete multipartite graph (see Lemma 2.2). In particular, we immediately get the following (with the help of [2, Lemma 3.1] to prove that H is finite):

Lemma 3.3. *Let G be a finite non-abelian AC-group and H be a group such that $\Gamma_G \cong \Gamma_H$. Then H is a finite AC-group.*

Recall that the clique number $w(\Gamma)$ of a graph Γ is the largest possible number of vertices of a complete subgraph of Γ. The fact that distinct elements x and y of $G\backslash Z(G)$ are adjacent in Γ_G if and only if $C_G(x) \neq C_G(y)$, proves the next result:

Lemma 3.4. *Let G be a finite AC-group and H be a group such that $\Gamma_G \cong \Gamma_H$. Then*

$$|\mathcal{C}(G)| = w(\Gamma_G) = w(\Gamma_H) = |\mathcal{C}(H)|.$$

From now on we focus on our main goal, that is understanding the structure of H when G is a nilpotent AC-group.

Lemma 3.5. *Let G be a finite non-abelian nilpotent AC-group and H be a group such that $\Gamma_G \cong \Gamma_H$. Then H is a finite solvable AC-group.*

Proof. The fact that H is a finite AC-group follows from Lemma 3.3. Aiming for a contradiction, suppose H is non-solvable. Then [2, Proposition 3.14] gives that $|G| = |H|$. Thus Theorem 1.1 implies that H is nilpotent, a contradiction. □

Lemma 3.5 tells us that we can suppose that H is one of the group of Theorem 2.7. This will be a crucial ingredient in the proof of Theorem A.

4. Proof of the main theorem

In this section, let G be a finite non-abelian nilpotent AC-group and H be a group such that $\Gamma_G \cong \Gamma_H$.

Notation 4.1. By Theorem 2.7 we can assume $G = P \times A$, where $P \in \mathrm{Syl}_p(G)$ for some prime p and A is an abelian subgroup of G of p'-order. Set $p^n = [G\colon Z(G)] = [P\colon Z(P)]$, $p^r = |Z(P)|$ and $a = |A|$, so $|Z(G)| = |Z(P)||A| = p^r a$ and $|G| = p^{n+r}a$.

By Lemma 3.5, the group H is a finite solvable AC-group, and so it must correspond to one of the groups described in Theorem 2.7. In particular, it is nilpotent if and only if it is of type (5). We will therefore analyze the various possibilities for H as listed in Theorem 2.7. As a first step, we show that if H is not nilpotent then it must be of type (3).

Lemma 4.2. *H is not of type (1).*

Proof. Let q be a prime such that $[H\colon N] = q$. Take $n \in N\backslash Z(H)$ and $h \in H\backslash N$. Since H is a non-nilpotent AC-group, we get $N = C_H(n)$ and $C_H(h) = Z(H)\langle h\rangle$, with $|C_H(h)| = q|Z(H)|$.

Aiming for a contradiction, suppose that there exists an isomorphism of graphs $\Phi : \Gamma_H \to \Gamma_G$. Let $m = \phi(n)$ and $g = \phi(h)$ and set $M = C_G(m)$, $[M\colon Z(G)] = p^t$ and $[C_G(g)\colon Z(G)] = p^u$.

Claim 1: p divides $[N\colon Z(H)]$.

Proof: Note that $G/Z(G) \cong P/Z(P)$ is a p-group. Using Lemma 2.6 we get

$$[N\colon \mathrm{Z}(H)] + 1 = w(\Gamma_H) = w(\Gamma_G) \equiv 1 \mod p$$

Hence $[N\colon \mathrm{Z}(H)] \equiv 0 \mod p$, that is, p divides $[N\colon Z(H)]$.

Claim 2: p^r is the largest power of p dividing $|Z(H)|$.

Proof: Since $\Gamma_H \cong \Gamma_G$ we get

$$|N| - |Z(H)| = |M| - |Z(G)|$$
$$|Z(H)|([N\colon Z(H)] - 1) = p^r a(p^t - 1)$$

Since p divides $[N\colon Z(H)]$ by Claim 1 and $(p, a) = 1$ by assumption, we deduce that p^r divides $|Z(H)|$ and it is the largest power of p dividing it.

Claim 3: $p = q$.

Proof: Let $Q \in \mathrm{Syl}_q(Z(H))$ (possibly $|Q| = 1$). Since $\Gamma_H \cong \Gamma_G$ we get

$$|N| - |C_H(h)| = |M| - |C_G(g)|$$
$$|Q|([N\colon Q] - [Z(H)\colon Q]q) = p^{r+t}a - p^{r+u}a$$

Note that p^{r+1} divides $p^{r+t}a - p^{r+u}a$. If $p \neq q$, then p^{r+1} must divide $([N\colon Q] - [Z(H)\colon Q]q)$. By Claims 1 and 2, we have that p^{r+1} divides $[N\colon Z(H)][Z(H)\colon Q] = [N\colon Q]$. Hence we deduce that p^{r+1} divides $[Z(H)\colon Q]q$ and so it must divide $|Z(H)|$, contradicting Claim 2. Thus we deduce that $p = q$.

Claim 4: p^t divides $[N\colon Z(H)]$.

Proof: Since $\Gamma_H \cong \Gamma_G$ and $p = q$ by Claim 3, we get

$$|H| - |N| = |G| - |M|$$
$$[N\colon Z(H)]|Z(H)|(p-1) = p^r p^t a([G\colon M] - 1)$$

By Claim 2 we deduce that p^t divides $[N\colon Z(H)]$.

We are now ready to reach a contradiction. Since p^t divides $[N\colon Z(H)]$ and $|C_H(h)| = p|Z(H)|$, we deduce that $|N| \geq |C_H(h)|$. Recall that

$$|N| - |C_H(h)| = |M| - |C_G(g)|. \tag{4}$$

In particular $|M| \geq |C_G(g)|$, so $p^t \geq p^u$ and Claim 4 implies that p^u divides $[N\colon Z(H)]$. Recalling that $p = q$, from Equation (4) we also get

$$|Z(H)|([N\colon Z(H)] - p) = ap^r p^u(p^{t-u} - 1).$$

Since p^r is the largest power of p dividing $|Z(H)|$, we conclude that p^u divides $([N\colon Z(H)] - p)$ and so p^u divides p. Hence $u = 1$, that is, $|C_G(g)| = p|Z(G)|$. Now

$$\begin{aligned} |C_H(h)| - |Z(H)| &= |C_G(g)| - |Z(G)| \\ |Z(H)|(p-1) &= |Z(G)|(p-1) \end{aligned}$$

implying $|Z(H)| = |Z(G)|$ and so $|H| = |G|$. Therefore by Theorem 1.1 the group H is nilpotent, a contradiction. □

Lemma 4.3. *H is not of type (2) or (4).*

Proof. Aiming for a contradiction, suppose H is of type (2) or (4). Then by [2, Lemma 3.11 and 3.12] we have $|G| = |H|$ and so by Theorem 1.1 we deduce that H is nilpotent, a contradiction. □

Note that Lemmas 4.2 and 4.3 imply that either H is nilpotent or it is as described in part (3) of Theorem 2.7. In order to prove Theorem A we must show that in the latter case we should have $|Z(G)| < |Z(H)|$, with some extra properties.

Theorem 4.4. *Suppose H is of type (3) with $[F\colon Z(H)] = q^f$, where $F/Z(H)$ is the kernel of $H/Z(H)$, q is a prime and $f \geq 1$ is an integer. Then*

(1) $p \neq q$, that is, the Sylow q-subgroups of G are abelian (possibly trivial);
(2) $|Z(H)| > |Z(G)|$;
(3) $[P\colon Z(P)] > p^4$;
(4) none of the maximal subgroups of G is abelian (in particular P does not have maximal nilpotency class).

Proof. Set $|Z(H)| = bq^u$ for some integers $b, u \geq 1$ with $(b, q) = 1$, so $|F| = bq^{u+f}$. Let $K/Z(H)$ denote a Frobenius complement of the group $H/Z(H)$. Then $|K| = [K\colon Z(H)]|Z(H)| = cbq^u$, for some integer $c \geq 2$ such that c divides $q^f - 1$ (and so in particular $(c, q) = 1$). Take $k \in K\backslash Z(H)$, so $K = C_H(k)$, and $x_1, \ldots, x_v \in F\backslash Z(H)$, one for each size of H-conjugacy classes, so $C_H(x_i) = C_F(x_i)$. In particular

$$|C_H(k)| = |K| = c|Z(H)| \quad \text{and} \quad |C_H(x_i)| = q^{t_i}|Z(H)|$$

for distinct integers $t_i \geq 1$. Note that from $(c, q) = 1$ we see that $|C_H(k)| \neq |C_H(x_i)|$.

Aiming for a contradiction, suppose there exists an isomorphism of graphs $\phi\colon \Gamma_H \to \Gamma_G$. Let $g = \Phi(k)$, $M = C_G(g)$ and $y_i = \phi(x_i)$ for every $1 \leq i \leq v$. Note that $G/Z(G)$ is a p-group, so we can set $[M\colon Z(G)] = p^m$ for some integer $m \geq 1$ and $[C_G(y_i)\colon Z(G)] = p^{s_i}$ for some distinct integers $s_i \geq 1$. Also, $|K|-|C_H(x_i)| = |M|-|C_G(y_i)|$, so $|M| \neq |C_G(y_i)|$, that is, $m \neq s_i$ for every $i \geq 1$.

Claim 1: $p \neq q$, so part (1) holds.

Proof: Since $\Gamma_H \cong \Gamma_G$ we have

$$|H| - |K| = |G| - |M| \Longrightarrow q^u bc(q^f - 1) = p^{m+r}a(p^{n-m} - 1) \tag{5}$$

and

$$|C_H(x_i)| - |Z(H)| = |C_G(y_i)| - |Z(G)| \Longrightarrow q^u b(q^{t_i} - 1) = p^r a(p^{s_i} - 1). \tag{6}$$

Aiming for a contradiction, suppose $p = q$. Then $(p, b) = (p, c) = (p, a) = 1$. From Equation (5) we deduce that $p^u = p^{m+r}$ and from Equation (6) we get $p^u = p^r$, a contradiction. Therefore $p \neq q$.

Claim 2: for every $d \leq r$ we have that p^d divides bc if and only if p^d divides b.

Proof: By assumption p^d divides p^r and by Claim 1 we have $p \neq q$. We conclude by the following equality:

$$|K| - |Z(H)| = |M| - |Z(G)| \Longrightarrow q^u(bc - b) = p^r a(p^m - 1) \tag{7}$$

Claim 3: p^{r+1} does not divide bc and q^{u+1} does not divide a. In particular $(p, c) = 1$ and so $(p, [H\colon Z(H)]) = 1$.

Proof: Recall that $|K| - |C_H(x_i)| \neq 0$ since c does not divide q, and so $|M| \neq |C_g(y)|$. Note that

$$|K| - |C_H(x_i)| = |M| - |C_G(y_i)| \Longrightarrow q^u(bc - bq^{t_i}) = p^r a(p^m - p^{s_i}). \tag{8}$$

From Claim 1 we know $p \neq q$ and by Equation (8) we get that p^{r+1} divides $(bc - bq^{t_i})$. Aiming for a contradiction, suppose p^{r+1} divides bc. Then p^{r+1} must divide b and by Equation (7) we deduce that p^{r+1} divides $p^r a(p^m - 1)$, a contradiction. Therefore p^{r+1} does not divide bc.

Similarly, if q^{u+1} divides a, then Equation (8) implies that q divides bc, reaching again a contradiction.

Finally, if p divides c, then p does not divide $(c-1)$ and by Equation (7) we deduce that p^r divides b. But then p^{r+1} divides bc, a contradiction.

Claim 4: We have

$$p^m(p^{n-m}-1)(c-1) = (p^m-1)c(q^f-1). \tag{9}$$

In particular

(1) $n > 2m$;
(2) $c < p^m < q^f < p^{n-m}$; and
(3) $|Z(H)| > |Z(G)|$, so part (2) holds.

Proof: From Equation (5) we get

$$|Z(H)| = q^u b = \frac{p^{r+m}a(p^{n-m}-1)}{c(q^f-1)}.$$

Substituting this into Equation (7) we obtain

$$p^m(p^{n-m}-1)(c-1) = (p^m-1)c(q^f-1),$$

as wanted.

Recall that c divides q^f-1, so $c \leq q^f-1$. Aiming for a contradiction, suppose $c = q^f-1$. Then Equation (9) becomes

$$p^m(p^{n-m}-1)(c-1) = (p^m-1)c^2,$$

that we can rewrite as

$$(p^n - p^m(1+c)+c)c = p^m(p^{n-m}-1).$$

In particular p^m divides $(p^n - p^m(1+c)+c)c$. However, p^m divides p^n while $(p,c)=1$ by Claim 3, and we reach a contradiction. Thus $c < q^f-1$.

As a consequence, we have $c-1 \leq q^f-1$ and from Equation (9) we deduce that we must have $p^m(p^{n-m}-1) \geq (p^m-1)c$ and so

$$c \leq \frac{p^m(p^{n-m}-1)}{(p^m-1)}.$$

Rewrite Equation (9) as

$$(p^{n-m}-1)(p^m c - p^m) = (p^m c - c)(q^f-1)$$

and notice that

$$c > p^m \Leftrightarrow p^m c - p^m > p^m c - c \Leftrightarrow p^{n-m} - 1 < q^f - 1 \Leftrightarrow p^{n-m} < q^f.$$

Aiming for a contradiction suppose $n \leq 2m$. Then $p^{n-m} \leq p^m$ and by Equation (9) p^m divides $q^f - 1$, so $p^{n-m} \leq p^m \leq q^f - 1$. Hence by what we noted above we obtain $c \geq p^m$. On the other hand, $c \leq \frac{p^m(p^{n-m}-1)}{(p^m-1)} \leq p^m$. Thus the only option is $c = p^m$, contradicting Claim 3. Therefore $n > 2m$.

Now, if $c > p^m$ then $q^f > p^{n-m} > p^m$ contradicting the fact that by Equation (9) p^m divides $q^f - 1$. Therefore $c < p^m < q^f < p^{n-m}$. Finally, from Equation (7) we conclude that $|Z(H)| > |Z(G)|$.

Claim 5: for every $i \geq 1$ we have $n > 2s_i$.

Proof: Note that, using once again the fact that $\Gamma_H \cong \Gamma_G$, we have

$$|H| - |C_H(x_i)| = |G| - |C_G(y_i)| \Longrightarrow q^{u+t_i} b(q^{f-t_i} c - 1) = p^{r+s_i} a(p^{n-s_i} - 1). \quad (10)$$

From Equation (10) we get

$$|Z(H)| = q^u b = \frac{p^{r+s_i} a(p^{n-s_i} - 1)}{q^{t_i}(q^{f-t_i} c - 1)}.$$

Substituting this into Equation (12) we obtain

$$\frac{p^{r+s_i} a(p^{n-s_i} - 1)}{q^{t_i}(q^{f-t_i} c - 1)} \cdot (q^f c - 1) = p^r a(p^n - 1)$$

and so

$$p^{s_i}(p^{n-s_i} - 1)(q^f c - 1) = (p^n - 1) q^{t_i}(q^{f-t_i} c - 1).$$

By Claim 1, $p \neq q$. So we deduce that p^{s_i} divides $q^{f-t_i} c - 1$. In particular,

$$p^{s_i} \leq q^{f-t_i} c - 1 \leq q^{f-t_i} c. \quad (11)$$

Since $\Gamma_H \cong \Gamma_G$, we also obtain

$$|H| - |Z(H)| = |G| - |Z(G)| \Longrightarrow q^u b(q^f c - 1) = p^r a(p^n - 1). \quad (12)$$

From Equation (6) we get

$$|Z(H)| = q^u b = \frac{p^r a(p^{s_i} - 1)}{q^{t_i} - 1}.$$

Substituting this into Equation (12) we obtain

$$q^f c = \frac{(p^n - 1)(q^{t_i} - 1)}{p^{s_i} - 1} + 1.$$

Now, $p^n - 1 = (p_i^s - 1)p^{n-s_i} + (p^{n-s_i} - 1)$ and so we deduce

$$\begin{aligned} q^f c &= \frac{((p_i^s - 1)p^{n-s_i} + (p^{n-s_i} - 1))(q^{t_i} - 1)}{p^{s_i} - 1} + 1 \\ &= p^{n-s_i} q^{t_i} + \frac{(p^{n-s_i} - 1)(q^{t_i} - 1)}{p^{s_i} - 1} + 1 - p^{n-s_i} \\ &= p^{n-s_i} q^{t_i} - \frac{(p^{s_i} - q^{t_i})(p^{n-s_i} - 1)}{p^{s_i} - 1}. \end{aligned}$$

By Claim 4 we have $|Z(H)| > |Z(G)|$, so by Equation (6) we conclude that $p^{s_i} > q^{t_i}$. Thus $\frac{(p^{s_i} - q^{t_i})(p^{n-s_i}-1)}{p^{s_i}-1} > 0$ and $q^f c < p^{n-s_i} q^{t_i}$. Combining this with (11) we get

$$p^{s_i} q^{t_i} \leq q^f c < p^{n-s_i} q^{t_i}.$$

Therefore $p^{s_i} < p^{n-s_i}$, implying $n > 2s_i$.

Now, to prove parts (3) and (4) of the statement, recall that for every $i \geq 1$ we have $m \neq s_i$ and Claims 4 and 5 give $n > 2m$ and $n > 2s_i$. Thus we conclude that $n > 4$ and $s_i \neq n - 1 \neq m$, implying that the maximal subgroups of G are not abelian. In particular, by Lemma 2.9(2) we conclude that if P has nilpotency class greater than 2 then $C_P(Z_2(P))$ is not a maximal subgroup of P. Hence P does not have maximal nilpotency class. □

Proof of Theorem A. By Lemma 3.5 the group H is a finite solvable AC-group and so it is of one of the types described in Theorem 2.7. Theorem A is now a direct consequence of Lemmas 4.2 and 4.3 and of Theorem 4.4. □

Data availability

No data was used for the research described in the article.

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