Unisolvence of random Kansa collocation by Thin-Plate Splines for the Poisson equation

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Abstract

Existence of sufficient conditions for unisolvence of Kansa unsymmetric collocation for PDEs is still an open problem. In this paper we make a first step in this direction, proving that unsymmetric collocation matrices with Thin-Plate Splines for the 2D Poisson equation are almost surely nonsingular, when the discretization points are chosen randomly on domains whose boundary has an analytic parametrization.

1 Introduction

Kansa unsymmetric collocation, originally proposed in the mid '80s [17], has become over the years a popular meshless method for the discretization of boundary value problems for PDEs. Despite its wide and successful adoption for the numerical solution of a variety of physical and engineering problems (cf. e.g. [8] with the references therein), a sound theoretical foundation concerning unisolvence of the corresponding linear systems is still missing. Indeed, it was initially thought that known unisolvence results for interpolation, e.g. the basic theorem by Micchelli for Conditionally Positive Definite RBF of order 1 like e.g. Multi-Quadrics [21], would also be valid for Kansa collocation. But later it was shown by Hon and Schaback [16] that there exist point configurations leading to singularity of the collocation matrices, though these are very special and "rare"cases. For this reason greedy and other approaches have been developed to overcome the theoretical problem and ensure invertibility, cf. e.g. [20, 26]. More recently, some meaningful advances have been made concerning overtesting collocation techniques, that are implemented by least-ℓ^∞ or least squares methods; cf., e.g., [6, 7, 9, 25] with the references therein.

On the other hand, in the textbook [14] one can read : "*Since the numerical experiments by Hon and Schaback show that Kansa's method cannot be well-posed for arbitrary center locations, it is now an open question to find sufficient conditions on the center locations that guarantee invertibility of the Kansa matrix*", and the situation does not seem to have changed so far.

In this paper we make a first step in this direction, proving that unsymmetric collocation matrices with Thin-Plate Splines (without polynomial addition) for the 2D Poisson equation are almost surely nonsingular, when the discretization points are chosen randomly on domains with analytic boundary. Though TPS are not the most adopted option for Kansa collocation, they have been often used in the meshless literature, cf. e.g. [8, 10, 29] with the references therein. One of their most relevant features is that they are scale invariant, thus avoiding the delicate matter of the scaling choice with scale dependent RBF, which is still an active research topic, cf. e.g. [3, 19]. On the other hand, the fact that TPS without polynomial addition can guarantee unisolvence in the interpolation framework has been recently recognized experimentally in [24] and theoretically in [2, 12] by random sampling. Still in the framework of almost sure unisolvence, it is also worth quoting a recent result concerning interpolation by Gaussian RBF [11], where randomness is however relevant to the choice of shape parameters.

As we shall see, one of the key aspects is that Thin-Plate Splines $\phi(\|P-A\|_2)$, which correspond to the radial functions

$$\phi(r) = r^{2\nu} \log(r) \ , \ \ \nu \in \mathbb{N} \ , \tag{1}$$

are *real analytic functions* off their center A, due to analyticity of the univariate functions $\log(\cdot)$ and $\sqrt{\cdot}$ in $\mathbb{R}^+$. Analiticity together with the presence of a singularity at the center will be the key ingredients of our unisolvence result by random collocation.

2 Unisolvence of random Kansa collocation

Consider the Poisson equation with Dirichlet boundary conditions (cf. e.g. [13])

$$\begin{cases} \Delta u(P) = f(P) \ , \ P \in \Omega \\ u(P) = g(P) \ , \ P \in \partial\Omega = \gamma([a,b]) \ , \end{cases} \tag{2}$$

where we assume that $\Omega \subset \mathbb{R}^2$ is a domain (an open connected set) whose boundary curve has an analytic parametrization, namely a curve $\gamma : [a,b] \to \mathbb{R}^2$, $\gamma(a) = \gamma(b)$, that is *analytic* and *regular*, i.e. $\gamma'(t) \neq (0,0)$ for every $t \in (a,b)$.

In Kansa collocation (see e.g. [14, 16, 17, 20, 26, 28]) one seeks a function

$$u_N(P) = \sum_{j=1}^{N_I} c_j \, \phi_j(P) + \sum_{k=1}^{N_B} d_k \, \psi_k(P) \ , \ \ N = N_I + N_B \ , \tag{3}$$

where N_I denotes the number of internal collocation points and N_B the number of boundary collocation points, namely

$$\phi_j(P) = \phi(\|P - P_j\|_2) \ , \ \ \{P_1, \dots, P_{N_I}\} \subset \Omega \ , \tag{4}$$

$$\psi_k(P) = \phi(\|P - Q_k\|_2) \ , \ \ \{Q_1, \dots, Q_{N_B}\} \subset \partial\Omega \ , \tag{5}$$

by solving the discrete problem

$$\begin{cases} \Delta u_N(P_i) = f(P_i) \ , \ i = 1, \dots, N_I \\ u_N(Q_h) = g(Q_h) \ , \ h = 1, \dots, N_B \ . \end{cases} \tag{6}$$

The following facts will be used below. Defining $\phi_A(P) = \phi(\|P - A\|)$, we have $\phi_A(B) = \phi_B(A)$ and $\Delta\phi_A(B) = \Delta\phi_B(A)$. In fact, the Laplacian in polar coordinates centered at A (cf. e.g. [13, Ch.2]) is the radial function

$$\Delta\phi_A = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} = 4\nu r^{2(\nu-1)}(\nu \log(r) + 1) \ . \tag{7}$$

Moreover, $\phi_A(A) = 0$ and $\Delta\phi_A(A) = 0$ for $\nu \geq 2$, since $\Delta\phi \to 0$ as $r \to 0$.

Kansa collocation can be rewritten in matrix form as

$$\begin{pmatrix} \Delta\Phi & \Delta\Psi \\ \\ \Phi & \Psi \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \\ \mathbf{g} \end{pmatrix} \tag{8}$$

where the $N \times N$ block matrix is

$$K_N = K_N(\{P_i\}, \{Q_h\}) = \begin{pmatrix} \Delta\Phi & \Delta\Psi \\ \\ \Phi & \Psi \end{pmatrix} = \begin{pmatrix} (\Delta\phi_j(P_i)) & (\Delta\psi_k(P_i)) \\ \\ (\phi_j(Q_h)) & (\psi_k(Q_h)) \end{pmatrix}$$

and $\mathbf{f} = \{f(P_i)\}$, $\mathbf{g} = \{g(Q_h)\}$, $1 \leq i, j \leq N_I$, $1 \leq h, k \leq N_B$. Observe that the square matrices $\Delta\Phi$ and Ψ both have null main diagonal.

We can now state and prove our main result.

Theorem 1 *Let K_N be the TPS Kansa collocation matrix defined above, with $N = N_I + N_B \geq 2$ with $N_I \geq 1$ and $N_B \geq 1$, where $\{P_j\}$ is a sequence of independent uniformly distributed random points in Ω, and $\{Q_h\}$ a sequence of independent uniformly distributed points on $\partial\Omega$. Namely, $\{Q_h\} = \{\gamma(t_h)\}$ with $\{t_h\}$ sequence of independent identically distributed random abscissas in (a, b) with respect to the arclength density $\|\gamma'(t)\|_2/L$, $L = length(\gamma([a, b]))$. Moreover, let Ω satisfy a weak segment condition, i.e. for every $x \in \partial\Omega$ there exist a segment with an extremum in x completely contained in Ω.*

Then for every $N \geq 3$ the matrix K_N is a.s. (almost surely) nonsingular.

We observe that the weak segment condition is implied by standard conditions for PDE domains, like e.g. the weak cone condition and the segment condition, cf. [1]. Before proving Theorem 1 by induction, it is worth proving a Lemma concerning the induction base.

Lemma 1 *The assertion of Theorem 1 holds true for $N = 2$ and $N = 3$.*

Proof. For $N = 2$, we have $N_I = 1$ and $N_B = 1$. Assume that Q_1 is chosen on the boundary (randomly or not) and that P_1 is chosen randomly in the interior. Since

$$\det(K_2) = -\phi_1(Q_1)\Delta\psi_1(P_1)$$
$$= -4\nu\|P_1 - Q_1\|_2^{2\nu}\log(\|P_1 - Q_1\|_2)\|P_1 - Q_1\|_2^{2\nu-2}\left(\nu\log(\|P_1 - Q_1\|_2) + 1\right) \ ,$$

being $P_1 \neq Q_1$ the determinant vanishes if and only if P_1 falls on (the intersection with Ω of) the curves $\log(\|P - Q_1\|_2) = 0$ or $\nu\log(\|P - Q_1\|_2) + 1 = 0$, that is on one of the circles

$$\|P - Q_1\|_2^2 = 1 \ \text{ or } \ \|P - Q_1\|_2^2 = \exp(-2/\nu)$$

and this event has null probability, since any algebraic curve is a null set in $\mathbb{R}^2$.

For $N = 3$ we have $N_I = 1$ and $N_B = 2$ or $N_I = 2$ and $N_B = 1$. For $N_I = 1, N_B = 2$, developing the determinant along the first row

$$\det(K_3(P_1, Q_1, Q_2)) = \Delta\psi_1(P_1)\psi_2(P_1)\psi_2(Q_1) + \Delta\psi_2(P_1)\psi_1(P_1)\psi_1(Q_2)$$
$$= \psi_1(Q_2)(\Delta\psi_1(P_1)\psi_2(P_1) + \Delta\psi_2(P_1)\psi_1(P_1)) \ .$$

Now, $\psi_1(Q_2)$ is a.s. nonzero. In fact, given $Q_1 = \gamma(t_1)$, the function $\lambda(t) = \psi_1^2(\gamma(t))$ is analytic in (a, t_1) and in (t_1, b). Then $\psi_1^2(\gamma(t_2))$ is zero iff $t_2 = t_1$ (an event that has null probability), or t_2 falls on the zero set of λ in (a, t_1) or (t_1, b). Again this event has null probability since the zero set of a nonzero univariate analytic function in an open interval is a null set (cf. [18, 22]).

On the other hand, also $\Delta\psi_1(P_1)\psi_2(P_1) + \Delta\psi_2(P_1)\psi_1(P_1)$ is a.s. nonzero. In fact, consider the function $H(P) = \Delta\psi_1(P)\psi_2(P) + \Delta\psi_2(P)\psi_1(P)$ which is analytic in $\mathbb{R}^d \setminus \{Q_1, Q_2\}$. We claim that $H(P)$ is a.s. not identically zero in Ω. Indeed, by the weak segment condition there exists a segment with an extremum in Q_1, say $P(t) = Q_1 + tv$, $t \in (0, \delta)$ and $\|v\|_2 = 1$, completely contained in Ω. Observe that the functions $\psi_2(P(t))$ and $\Delta\psi_2(P(t))$ are a.s. analytic at $t = 0$ and a.s. $\psi_2(P(0)) = \psi_2(Q_1) \neq 0$ and $\Delta\psi_2(P(0)) = \Delta\psi_2(Q_1) \neq 0$. By $H(P) \equiv 0$ we would get

$$u(t) = \Delta\psi_1(P(t))\psi_2(P(t)) = 4\nu t^{2(\nu-1)}(\nu\log(t) + 1)\psi_2(P(t))$$
$$= -\Delta\psi_2(P(t))\psi_1(P(t)) = -t^{2\nu}\log(t)\Delta\psi_2(P(t)) \ , \ \ t \in (0, \delta) \ .$$

But then as $t \to 0^+$ we would get $u(t) \sim ct^{2(\nu-1)}\log(t)$ with $c \neq 0$ and $u(t) \sim dt^{2\nu}\log(t)$ with $d \neq 0$, which is a contradiction since a function cannot have two distinct orders of infinitesimal at the same point. Finally, we get that a.s. $H(P_1) \neq 0$ and thus $\det(K_3(P_1, Q_1, Q_2)) \neq 0$, because the zero set in Ω of the nonzero analytic function $H(P)$ is a null set (cf. [22] for an elementary proof).

For $N_I = 2, N_B = 1$, we have that by symmetry properties

$$\det(K_3(P_1, P_2, Q_1)) = \Delta\phi_2(P_1)\Delta\psi_1(P_2)\phi_1(Q_1) + \Delta\psi_1(P_1)\Delta\phi_1(P_2)\phi_2(Q_1)$$
$$= \Delta\phi_1(P_2)(\Delta\psi_1(P_2)\phi_1(Q_1) + \Delta\psi_1(P_1))\psi_1(P_2)) \ .$$

Now, by similar considerations to those developed above based on centers and circles, we have that a.s. $\Delta\phi_1(P_2), \phi_1(Q_1), \Delta\psi_1(P_1)) \neq 0$. Then, also the factor $\Delta\psi_1(P_2)\phi_1(Q_1)+\Delta\psi_1(P_1)\psi_1(P_2))$ is a.s. nonzero. Indeed, the analytic function $\Lambda(P) = \phi_1(Q_1)\Delta\psi_1(P) + \Delta\psi_1(P_1)\psi_1(P)$ is not identically zero in Ω, otherwise reasoning as above on a segment entering Ω from Q_1

$$\begin{aligned} v(t) &= \phi_1(Q_1)\Delta\psi_1(P(t)) = 4\nu\phi_1(Q_1)t^{2(\nu-1)}(\nu\log(t)+1) \\ &= -\Delta\psi_1(P_1)\psi_1(P(t)) = -\Delta\psi_1(P_1)t^{2\nu}\log(t) \end{aligned}$$

and $v(t)$ would have two distinct orders of infinitesimal as $t \to 0^+$. Finally, we get that a.s. $\Lambda(P_2) \neq 0$ and thus $\det(K_3(P_1, P_2, Q_1)) \neq 0$, because the zero set in Ω of the nonzero analytic function $\Lambda(P)$ is a null set. □

Proof of Theorem 1. The proof proceeds by complete induction on N. For the induction base, by Lemma 1 we have that $\det(K_N)$ is a.s. nonzero for $N = 2, 3$.

For the inductive step, we consider separately the case where a boundary point is added, for which we define the $(N+1)\times(N+1)$ matrix by adding a new last row and column

$$U(P) = \begin{pmatrix} \Delta\Phi & \Delta\Psi & (\Delta\vec{\phi}(P))^t \\ \Phi & \Psi & (\vec{\psi}(P))^t \\ \vec{\phi}(P) & \vec{\psi}(P) & 0 \end{pmatrix}$$

where $\vec{\phi}(P) = (\phi_1(P), \ldots, \phi_{N_I}(P))$ and $\vec{\psi}(P) = (\psi_1(P), \ldots, \psi_{N_B}(P))$. In this case $K_{N+1} = U(Q_{N_B+1})$, since $\psi_k(Q_h) = \psi_h(Q_k)$ and $\Delta\phi_j(Q_{N_B+1}) = \Delta\psi_{N_B+1}(P_j)$.

Differently, if an interior point is added, we define the $(N+1)\times(N+1)$ matrix by adding an intermediate row and column

$$V(P) = \begin{pmatrix} \Delta\Phi & (\Delta\vec{\phi}(P))^t & \Delta\Psi \\ \Delta\vec{\phi}(P) & 0 & \Delta\vec{\psi}(P) \\ \Phi & (\vec{\psi}(P))^t & \Psi \end{pmatrix}$$

Observe that in this case $K_{N+1} = V(P_{N_I+1})$, since $\psi_k(P_{N_I+1}) = \phi_{N_I+1}(Q_k)$ and $\Delta\phi_j(P_i) = \Delta\phi_i(P_j)$.

Concerning the determinants, applying Laplace determinantal rule on the last row of $U(P)$ we see that for every ℓ, $1 \leq \ell \leq N_B$, we get the representation

$$F(P) = \det(U(P)) = \delta_{N-1}\psi_\ell^2(P) + A(P)\psi_\ell(P) + B(P) \tag{9}$$

where

$$|\delta_{N-1}| = |\det(K_{N-1}(\{P_i\}, \{Q_h\}_{h\neq\ell}))|$$

$$A \in \mathrm{span}\{\phi_j, \Delta\phi_j, \psi_k \,;\, 1 \le j \le N_I \,,\, 1 \le k \le N_B \,,\, k \ne \ell\}$$

$$B \in \mathrm{span}\{\phi_i\Delta\phi_j, \psi_k\phi_i, \psi_k\Delta\phi_i, \psi_k\psi_h \,;\, 1 \le i,j \le N_I \,,\, 1 \le k,h \le N_B \,,\, k,h \ne \ell\} \,.$$

Similarly, developing $det(V(P))$ by the (N_I+1)-row we have

$$G(P) = \det(V(P)) = -\det(K_{N-1})(\Delta\phi_{N_I}(P))^2 + C(P)\Delta\phi_{N_I}(P) + D(P) \quad (10)$$

where

$$C \in \mathrm{span}\{\Delta\phi_j, \psi_k, \Delta\psi_k \,;\, 1 \le j \le N_I - 1 \,,\, 1 \le k \le N_B\}$$

$$D \in \mathrm{span}\{\Delta\phi_i\Delta\phi_j, \Delta\phi_i\Delta\psi_h, \psi_k\Delta\phi_i, \psi_k\Delta\psi_h \,;\, 1 \le i,j \le N_I-1 \,,\, 1 \le k,h \le N_B\} \,.$$

First, we prove that G is not identically zero in Ω if $\det(K_{N-1}) \ne 0$ (the latter a.s. holds by inductive hypothesis). Let $P(t) = P_{N_I} + t(1,0)$, $t \in \mathbb{R}$, and $r(t) = \|P(t) - P_{N_I}\|_2 = |t|$. If $G \equiv 0$ then $G(P(t)) \equiv 0$ in neighborhood of $t = 0$. Then, we would locally have

$$u^2(t) = c(t)u(t) + d(t) \,, \quad u(t) = \Delta\phi_{N_I}(P(t)) \,, \quad (11)$$

where $c(t) = C(P(t))/\det(K_{N-1})$ and $d(t) = D(P(t))/\det(K_{N-1})$. Notice that both c and d are analytic in a neighborhood of $t = 0$, since C and D are analytic in a neighborhood of P_{N_I}. By (11) and (7) we get

$$u(t) = 4\nu t^{2(\nu-1)} \left(\nu \log(|t|) + 1\right) \,. \quad (12)$$

Clearly c cannot be identically zero there, otherwise u^2 would be analytic at $t = 0$ and thus would have an algebraic order of infinitesimal as $t \to 0$, whereas by (12) we have $u^2(t) \sim 16\nu^4 t^{4(\nu-1)} \log^2(|t|)$. Hence taking the Maclaurin expansion of c we get $c(t) \sim c_s t^s$ as $t \to 0$ for some $s \ge 0$, the order of the first nonvanishing derivative at $t = 0$. Now, $u^2(t) \sim 16\nu^4 t^{4(\nu-1)} \log^2(|t|)$, whereas by $u^2 \equiv cu + d$ we would have $u^2(t) \sim 4\nu^2 c_s t^{s+2(\nu-1)} \log(|t|) + d_p t^p$, where either $d(0) \ne 0$ and $p = 0$, or $d(0) = 0$ and $p > 0$ (the order of the first nonvanishing derivative at $t = 0$). Then we get a contradiction, since u^2 cannot have two distinct limits or orders of infinitesimal at the same point.

Moreover, G is clearly continuous in Ω and analytic in $\Omega \setminus \{P_1, \dots, P_{N_I}\}$, since all the functions involved in its definition (10) are analytic up to their own center. Consequently, if $\det(K_{N-1}) \ne 0$ by continuity G is not identically zero also in $\Omega \setminus \{P_1, \dots, P_{N_I}\}$.

Then, $\det(K_{N+1}) = \det(V(P_{N_I+1})) = G(P_{N_I+1})$ is a.s. nonzero, since the zero set of a not identically zero real analytic function on an open connected set in $\mathbb{R}^d$ is a null set (cf. [22] for an elementary proof). More precisely, denoting by Z_G the zero set of G in Ω, we have that

$$Z_G = (Z_G \cap \{P_1, \dots, P_{N_I}\}) \cup (Z_G \cap (\Omega \setminus \{P_1, \dots, P_{N_I}\})) \,.$$

Hence Z_G is a null set if $G \not\equiv 0$, because the first intersection is a finite set, and the second is the zero set of a not identically zero real analytic function. Considering the probability of the corresponding events and recalling that $\det(K_{N-1}) \ne 0$ (which a.s. holds) implies $G \not\equiv 0$, we can then write

$$\mathrm{prob}\{\det(K_{N+1}) = 0\} = \mathrm{prob}\{G(P_{N_I+1}) = 0\}$$

$$= \text{prob}\{G \equiv 0\} + \text{prob}\{G \not\equiv 0 \ \& \ P_{N_I+1} \in Z_G\} = 0 + 0 = 0 \ ,$$

and this branch of the inductive step is completed.

We turn now to the branch of the inductive step where a boundary point is added. In this case we consider the function F in (9) restricted to the boundary, that is $F(P(t))$ with $P(t) = \gamma(t)$, $t \in (a,b)$, which for every fixed $\ell \in \{1,\ldots,N_B\}$ has the representation

$$F(\gamma(t)) = \det(U(\gamma(t))) = \delta_{N-1}v^2(t) + A(\gamma(t))v(t) + B(\gamma(t))$$

where

$$v(t) = \psi_\ell(\gamma(t)) = r_\ell^{2\nu}(t)\log(r_\ell(t)) \ , \ \ r_\ell(t) = \|\gamma(t) - Q_\ell\|_2 \tag{13}$$

with $Q_\ell = \gamma(t_\ell)$, $t_\ell \in (a,b)$. We claim that if $\delta_{N-1} \neq 0$ (which a.s. holds by inductive hypothesis), $F \circ \gamma$ cannot be identically zero in any of the two connected components of $(a,b) \setminus \{t_1,\ldots,t_{N_B}\}$ (i.e., the subintervals) having t_ℓ as extremum. Otherwise, we would have in a left or right neighborhood of t_ℓ

$$v^2(t) = \alpha(t)v(t) + \beta(t) \ , \tag{14}$$

where $\alpha(t) = A(\gamma(t))/\delta_{N-1}$ and $\beta(t) = B(\gamma(t))/\delta_{N-1}$ are both analytic in a full neighborhood of t_ℓ. Notice that, since $\gamma'(t_\ell) \neq (0,0)$ (the curve is regular), $r_\ell(t) \sim \|\gamma'(t_\ell)\|_2|t-t_\ell|$ which by (13) gives $v(t) \sim \|\gamma'(t_\ell)\|_2^{2\nu}(t-t_\ell)^{2\nu}log(|t-t_\ell|)$ and $v^2(t) \sim \|\gamma'(t_\ell)\|_2^{4\nu}(t-t_\ell)^{4\nu}\log^2(|t-t_\ell|)$ as $t \to t_\ell$. Now α cannot be identically zero in any left or right neighborhood, otherwise $v^2 \equiv \beta$ there and would have an algebraic order of infinitesimal at t_ℓ. Hence taking the Taylor expansion of α we get $\alpha(t) \sim \alpha_s(t-t_\ell)^s$ as $t \to t_\ell$ for some $s \geq 0$, the order of the first nonvanishing derivative at $t = t_\ell$. On the other hand, by $v^2 \equiv \alpha v + \beta$ locally, we would have $v^2(t) \sim \|\gamma'(t_\ell)\|_2^{2\nu}\alpha_s(t-t_\ell)^{s+2\nu}\log(|t-t_\ell|) + \beta_p(t-t_\ell)^p$, where either $\beta(t_\ell) \neq 0$ and $p = 0$, or $\beta(t_\ell) = 0$ and $p > 0$ (the order of the first nonvanishing derivative at $t = t_\ell$). Again we get a contradiction, since v^2 cannot have two distinct limits or orders of infinitesimal at the same point.

The result is that $F \circ \gamma$ is a.s. not identically zero in any connected component of $(a,b) \setminus \{t_1,\ldots,t_{N_B}\}$. Then, $\det(K_{N+1}) = \det(U(Q_{N_B+1})) = F(\gamma(t_{N_B+1}))$ is a.s. nonzero. In fact, observe that $F \circ \gamma$ is analytic in $(a,b) \setminus \{t_1,\ldots,t_{N_B}\}$, since F is analytic in $\mathbb{R}^2 \setminus (\{Q_1,\ldots,Q_{N_B}\} \cup \{P_1,\ldots,P_{N_I}\})$. Moreover, denoting by $Z_{F\circ\gamma}$ the zero set of $F \circ \gamma$ in (a,b), we have that

$$Z_{F\circ\gamma} = (Z_{F\circ\gamma} \cap \{t_1,\ldots,t_{N_B}\}) \cup (Z_{F\circ\gamma} \cap ((a,b) \setminus \{t_1,\ldots,t_{N_B}\})) \ .$$

Hence $Z_{F\circ\gamma}$ is a null set if $F \circ \gamma \not\equiv 0$, because the first intersection is a finite set, and the second is the componentwise finite union of the zero sets of a not identically zero real analytic function on each connected component. Considering the probability of the corresponding events and recalling that $\det(K_{N-1}) \neq 0$ (which a.s. holds) implies $F \circ \gamma \not\equiv 0$, we can then write

$$\text{prob}\{\det(K_{N+1}) = 0\} = \text{prob}\{F(Q_{N_B+1}) = 0\}$$

$$= \text{prob}\{F \circ \gamma \equiv 0\} + \text{prob}\{F \circ \gamma \not\equiv 0 \ \& \ t_{N_B+1} \in Z_{F\circ\gamma}\} = 0 + 0 = 0 \ ,$$

and also the boundary branch of the inductive step is completed. $\square$

2.1 Conclusion and remarks

The result of Theorem 1 is a first step towards a theory of Kansa collocation unisolvence, and could be extended in several directions within the random framework. The first extension comes immediately from the fact that a null set has also measure zero for any continuous measure with density (that is, absolutely continuous with respect to the Lebesgue measure). We can state indeed the following

Theorem 2 *The assertion of Theorem 1 holds true if the points $\{P_j\}$ are independent identically distributed with respect any continuous probability measure with density on Ω, say $\sigma \in L^1_+(\Omega)$, and the abscissas $\{t_h\}$ are independent identically distributed with respect any continuous probability measure with density on (a, b), say $w \in L^1_+(a, b)$.*

This extension could be interesting whenever it is known that the solution has steep gradients or other regions where it is useful to increase the discretization density. Concerning the implementation of random sampling with respect to continuous probability densities, we recall the well-known "acceptance-rejection method", cf. e.g. [5, 15, 23] with the references therein.

More difficult but worth of further investigations are:

- extension to $\Omega \subset \mathbb{R}^d$, $d \geq 3$;
- extension to other analitic RBF up to the center, e.g. Radial Powers;
- extension to piecewise analytic or more general boundaries;
- extension to other differential operators and/or boundary conditions.

The latter in particular could be challenging, since the operators involved in the equation and in the boundary conditions may not be radial. Moreover, one might see as final goal a complete classification of RBF that admit an almost surely nonsingular Kansa collocation matrix. Also this task appears challenging. In the present case, we have exploited the peculiar structure of TPS, which are analytic apart from their center, and exactly this real singularity has given a key tool to carry out the proof of Kansa unisolvence by random collocation. A similar approach has been used for the mere interpolation problem by random sampling, where in the inductive step each different RBF has required a special technique to prove that the determinant is a not everywhere null analytic function of the new random point; cf. [2, 27]. For other RBF, in particular those everywhere real analytic, different tools should be adopted, based again on the peculiar structure of the underlying radial functions, for example the presence of complex singularities as with MultiQuadrics (a first attempt in this direction appears in the draft [4]).

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