

ON THE SHEAFYNESS PROPERTY OF SPECTRA OF BANACH RINGS

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ABSTRACT. Let R be a non-Archimedean Banach ring, satisfying some mild technical hypothesis that we will specify later on. We prove that it is possible to associate to R a homotopical Huber spectrum $\mathrm{Spa}^h(R)$ via the introduction of the notion of derived rational localization. The spectrum so obtained is endowed with a derived structural sheaf $\mathcal{O}_{\mathrm{Spa}^h(R)}$ of simplicial Banach algebras for which the derived Čech-Tate complex is strictly exact. Under some hypothesis, we can prove that there is a canonical morphism of underlying topological spaces $|\mathrm{Spa}(R)| \rightarrow |\mathrm{Spa}^h(R)|$ that is a homeomorphism in some well-known examples of non-sheafy Banach rings, where $\mathrm{Spa}(R)$ is the usual Huber spectrum of R . This permits the use of the tools from derived geometry to understand the geometry of $\mathrm{Spa}(R)$ in cases when the classical structure sheaf $H^0(\mathcal{O}_{\mathrm{Spa}(R)})$ is not a sheaf.

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1. INTRODUCTION

Geometry of abstract commutative Banach rings. A well-known limitation of analytic geometry with respect to algebraic geometry is that it lacks an abstract approach. By this

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we mean that given a general (commutative) Banach ring¹ R it is not clear, not even in the case when R is a Banach k -algebra over a complete valued field, how to interpret R as a space of functions on some meaningful geometrical object. More precisely, it is not clear what the analytic version of Grothendieck's definition of the spectrum of a commutative ring is. Several notions of spectra for Banach rings have been proposed, like the Berkovich spectrum of R , here denoted by $\mathcal{M}(R)$. But besides the extreme success of this notion in the classical setting of non-Archimedean geometry, Mihara showed in [25] that there exist Banach k -algebras for which the (natural) definition of the structural sheaf on $\mathcal{M}(R)$ does not even give a well-defined pre-sheaf. Another possibility is to associate to R an affinoid pair (R, R°) in the sense of Huber and to consider the adic spectrum $\mathrm{Spa}(R, R^\circ) = \mathrm{Spa}(R)$. In this case, it is possible to show that the structural pre-sheaf is always well defined but it lacks the sheaf property in general (*cf.* [25] and [11] for counter-examples, or check Section 5 where an example is worked out in detail).

In [25] and [11] some conditions on R that ensure the sheafyness of the structural pre-sheaf on $\mathrm{Spa}(R)$ are given. One drawback of these conditions is that they are difficult to check in practice, and another one is that there is an increasing need for more general results for some applications of the theory (*cf.* [16] for example). In this work, we propose a solution to this problem based on the theory of quasi-abelian categories and on the methods developed in [4]. More precisely, we propose to use the methods of derived geometry in order to obtain a derived structural pre-sheaf that satisfies descent in the derived setting (the descent condition is the derived version of the sheafyness condition). This is done compatibly with the classical theory of Huber spaces and therefore provides an extension of it to a broader class of Banach rings that do not seem treatable with the classical methods.

Our main results. We now summarize the main ideas of this paper. Suppose that R is a Banach A -algebra, where A is a non-Archimedean strongly Noetherian Tate ring (for simplicity the reader can consider the case when A is a non-Archimedean complete valued field, that is already very important for applications). If R is an affinoid A -algebra (*cf.* Definition 4.1), the classical theory of Huber spaces gives a structure sheaf of Banach algebras on $\mathrm{Spa}(R)$. The topological space $\mathrm{Spa}(R)$ is defined as a suitable subset of the set of Krull valuations on R and its topology is defined by considering an analytic version of the principal Zariski localizations of algebraic geometry. These analytic localizations are called *rational localizations* and, roughly speaking, consists of subsets determined by inequalities of the form

$$U_{f_0, f_1, \dots, f_n} = U\left(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}\right) = \{x \in \mathrm{Spa}(R) \mid |f_1(x)| \leq |f_0(x)|, \dots, |f_n(x)| \leq |f_0(x)|\}$$

by elements $f_0, \dots, f_n \in R$ that generate the unit ideal in R . The structure pre-sheaf is defined to take values

$$\mathcal{O}_{\mathrm{Sp}(R)}(U_{f_0, f_1, \dots, f_n}) = \frac{R\langle X_1, \dots, X_n \rangle}{(f_0 X_1 - f_1, \dots, f_0 X_n - f_n)},$$

where the quotient is computed in the category of Banach rings and therefore if the ideal is not closed, the quotient must be computed by taking the closure of the ideal.

One of the main ideas of this paper is to replace the notion of rational localization with its derived analog, which we call *derived rational localization*. Very basically, this means to replace the quotients described so far in the definition of rational localization by the *homotopical*

¹We will limit our discussion to the case when R is non-Archimedean for simplicity, but similar problems are present also in the Archimedean theory. Actually, in the Archimedean setting, there are more complications that are subjects of further investigations. We refer to [6] for a detailed analysis of the problems appearing when working with Archimedean Banach rings.

quotients. Concretely, this means that in both situations we are dealing with a morphism of Banach R -modules

$$R\langle X_1, \dots, X_n \rangle^n \rightarrow R\langle X_1, \dots, X_n \rangle$$

whose image is the ideal $(f_0X_1 - f_1, \dots, f_0X_n - f_n)$ and instead of computing the cokernel of this map as done classically, we compute its cone in the derived category of Banach R -modules. To achieve this a mixture of functional analysis and homological algebra must be used because the Banach algebra structure is the relevant structure and the closedness properties of the image of morphisms and ideals play an important role. It is not possible to work out a properly working theory considering only the underlying algebraic categories. Also, the use of the completed tensor product is needed and hence the use of categories of Banach modules and similar functional analytic structures equipped with an algebraic structure compatible with an analytic structure is required. In our work, we found it convenient to use the class of bornological modules over R . This is a natural and simple extension, in our opinion, of the notion of Banach module over R . We will devote Section 3 to recall the main features of this theory. We also underline that this class does not form an abelian category but they form what is called a *quasi-abelian category*. Luckily, the theory of quasi-abelian categories has been well-established in the last 20 years or so, mainly through the work of Schneiders [27]. We will recall the basic features of the theory in Section 2 of the paper and explain how we used it for doing derived geometry.

In [8] it has been proved in the specific case when A is a non-Archimedean valued field, and R an affinoid algebra over A that affinoid localizations² are precisely characterized by a homological property that is called *homotopical epimorphism*. In particular, this implies that, under such hypotheses, the notions of derived rational localization and rational localization agree. Thus, our first main task in this work is to generalize these results to the case when R is an affinoid algebra over a strongly Noetherian Tate ring. In order to prove this result we need to add the hypothesis that A has a topologically nilpotent uniform unit, *i.e.* a unit $u \in A^\times$ such that $|u^n| = |u|^n$ for all $n \in \mathbb{Z}$. We conjecture that both the hypothesis that A is a Tate ring and that it has a uniform unit can be removed with better strategies of proof. Under such hypotheses, we prove that all derived rational localizations of A -affinoid algebras are non-derived at the beginning of Section 4. This ensures the compatibility of the geometrical notions from the theory of Huber spaces and the notions from the derived geometry of bornological algebras we developed in our previous works (see [4]). So, we introduce the derived version of the notion of classical rational localization and we check that in the specific case of affinoid A -algebras the derived and classical notions coincide (see Proposition 4.15). Then, to analyze the general case we write

$$R \cong \varinjlim_{i \in I}^{\leq 1} R_i$$

where R_i are affinoid algebras over A and $\varinjlim_{i \in I}^{\leq 1}$ means the colimit in the contracting category of Banach algebras (see Section 3 for the definition of the contracting category and the computation of limits and colimits in this category). We underline that, contrary to the algebraic situation, the functor $\varinjlim_{i \in I}^{\leq 1}$ is not exact, therefore it should be expected that computations that for R_i give a result that is concentrated in degree 0 may have higher cohomology when the functor $\varinjlim_{i \in I}^{\leq 1}$ is applied. This is a possible interpretation of the sheafyness issue that appears in the passage from (analytically) finite dimensional algebras over A to infinite dimensional ones. So, for any Banach ring both the notion of rational localization and derived rational localization

²These are just finite unions of rational localizations.

make sense and under the mentioned hypothesis we are able to check that derived rational covers always give a cover in the sense of the machinery of derived geometry introduced in Section 2 (see Theorem 4.31). This permits us to associate to R a homotopical version of the Huber spectrum that we denote $\mathrm{Spa}^h(R)$ and endow it with the structure of derived scheme relative to derived geometry over bornological modules, we would call these objects derived analytic spaces (see Theorem 4.35). Of course, if R is affinoid we have a canonical homeomorphism of underlying topological spaces $|\mathrm{Spa}(R)| \cong |\mathrm{Spa}^h(R)|$, but we do not know how far this comparison of spectra extends as we do not know any counter-example.

After proving this main abstract result we dedicate a section of the paper to working out in full detail a classical example of a non-sheafy Banach ring. So, this is a relatively simple example of Banach algebra over a valued field k for which it is known that Huber's theory does not work and the classical definition of structure pre-sheaf is not a sheaf. We show that in this particular case, there are rational localizations that are not derived localizations, that their pathological behaviour is precisely the source of non-sheafyness and that their derived counterparts fix this issue. So, we recall the computations showing that the Čech-Tate complex of this cover is not exact. After that, we compute the derived version of the same constructions and we check explicitly that the derived Čech-Tate complex of the same cover is strictly exact, as implied by the abstract theory developed before. We also see how the appearance of higher cohomology groups on open subsets of the structural sheaf is precisely what is needed to compensate for the non-injectivity of the restriction map on the H^0 -part.

Structure of the paper. The paper is structured as follows. In Section 2 we recall some basic language of the theory of quasi-abelian categories and some fundamental results from [4] that are used throughout the whole paper. We also recall how the homotopy Zariski topology is defined and how it can be used to define derived analytic spaces over any given Banach ring R . In Section 3 we recall the definitions of the quasi-abelian categories that are used in this paper, the categories of Banach modules, the contracting categories of Banach modules, and the categories of bornological modules over a fixed Banach (or bornological) ring. In Section 4 we prove our main results. We first prove that rational localizations of affinoid A -algebras, with A any strongly Noetherian Tate ring, are open localizations for the homotopy Zariski topology giving a homological characterization of the Huber spectrum, generalizing the results of [8]. Then, we show how to associate to any Banach A -algebra R (satisfying some mild hypothesis) a ∞ -site $\mathrm{Spa}^h(R)$ but differently from the case when R is an affinoid A -algebra, the space $\mathrm{Spa}^h(R)$ is equipped with a structure sheaf that makes it into a derived analytic space. We also describe how one can find a continuous map $\mathrm{Spa}(R) \rightarrow \mathrm{Spa}^h(R)$ when R is defined over a valued field. We conclude Section 4 by showing how these results can be extended to more general bornological rings.

Finally, in Section 5 we perform some explicit computations of the derived structural sheaf for spectra for which the standard definitions do not give a well-defined structural sheaf. We compute in detail the structural sheaf on a Laurent cover of a well-known non-sheafy Banach ring discussed by Buzzard-Verberkmoes ([11]) and Huber ([13]). We show that in this case $\mathrm{Spa}^h(R) \cong \mathrm{Spa}(R)$ so that our main results define a derived structural sheaf directly on $\mathrm{Spa}(R)$ and we compare the derived structure with the non-derived one.

We conclude the paper discussing some possible and conjectural generalizations of the results of this paper, mainly via the use of the theory of reified spaces introduced by Kedlaya in [18].

Notation and conventions. In this work we use standard conventions and terminology in addition to the following

- the term ring always means commutative ring with unit, if not otherwise stated;
- ring homomorphisms are always supposed to preserve identities;
- if \mathbf{C} is a category, we adopt the common abuse of notation of identifying \mathbf{C} with its class of objects; therefore, $X \in \mathbf{C}$ means that X is an object of \mathbf{C} ;
- we suppose the existence of an uncountable strongly inaccessible cardinal and we fix one which will bound the class of morphisms of all our basic categories; this allows us to always consider categories of functors avoiding set-theoretic issues;
- we will use cohomological indexing for chain complexes, *i.e.* the differentials increase the degree.

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2. RELATIVE ALGEBRAIC GEOMETRY OVER QUASI-ABELIAN CATEGORIES

In this section we recall some basic notions from the theory of quasi-abelian categories (*cf.* [27]) and we recall how they have been used in [4] (and also in [8], [3], [2], [7], [9]) for defining derived geometry over the category of (simplicial) commutative algebras over a symmetric monoidal quasi-abelian category. Currently, the most general version of this kind of theory goes beyond the theory of quasi-abelian categories and is developed in [20]. In this work we will never leave the setting of the theory of quasi-abelian categories.

2.1. Quasi-abelian categories. Recall that a *pre-abelian category* is an additive category for which all morphisms have a kernel and a cokernel. Any morphism in such a category fits in the following canonical diagram

$$(2.0.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(f) & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & \text{Coker}(f) & \longrightarrow & 0 \\ & & & & \downarrow & & \uparrow & & & & \\ & & & & \text{Coim}(f) & \longrightarrow & \text{Im}(f) & & & & \end{array}$$

where $\text{Coim}(f) = \text{Coker}(\text{Ker}(f) \rightarrow A)$ and $\text{Im}(f) = \text{Ker}(B \rightarrow \text{Coker}(f))$. The morphism $f : A \rightarrow B$ is called *strict* if the canonical morphism $\text{Coim}(f) \rightarrow \text{Im}(f)$ is an isomorphism. We recall that a morphism is strict if and only if it can be written as a composition of a strict epimorphism followed by a strict monomorphism (*cf.* [27, Remark 1.1.2 (c)]). We say that a short exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

is *strictly exact* or that it is a *strict short exact sequence* if both f and g are strict morphisms. Such short exact sequences are also called *kernel-cokernel sequences* in literature.

A *quasi-abelian category* is a pre-abelian category such that the family of strict short exact sequences forms a Quillen exact structure. The definition of quasi-abelian category can be restated by saying that it is a pre-abelian category such that strict monomorphisms are stable by pushouts and strict epimorphisms are stable by pullbacks. Hence, the property of being quasi-abelian is an intrinsic property of a category. We remark that for all quasi-abelian categories the canonical morphism $\text{Coim}(f) \rightarrow \text{Im}(f)$ is always a bimorphism, *i.e.* it is both an epimorphism

and a monomorphism (cf. [27, Corollary 1.1.5]). But this latter property does not characterize quasi-abelian categories.

In this section \mathbf{C} denotes a quasi-abelian category. Later on we will also suppose that \mathbf{C} has a closed symmetric monoidal structure that we denote by $(-)\otimes(-) : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ and $\underline{\text{Hom}}(-, -) : \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{C}$. We give some key examples of quasi-abelian categories focusing on those relevant to this paper.

- Example 2.1.** (1) The category of abelian groups is clearly quasi-abelian (as it is abelian), has a standard symmetric tensor product, and an internal hom. In the same fashion, the category of modules over a commutative ring is (quasi-)abelian with a symmetric tensor product and an internal hom.
- (2) Let R be a non-Archimedean Banach ring. We denote by $\mathbf{Ban}_R^{\text{na}}$ the category of non-Archimedean Banach R -modules with bounded morphisms, *i.e.* homomorphism $\phi : X \rightarrow Y$ such that $|\phi(x)| \leq C|x|$ for a fixed $C > 0$. This category is quasi-abelian, closed symmetric monoidal (a structure that we describe in more detail later on), but it is not complete nor cocomplete (cf. Section 3.1 of [2]).
- (3) Let R be any (commutative) Banach ring. The category \mathbf{Ban}_R of Banach R -modules with bounded morphisms is quasi-abelian, closed symmetric monoidal, but not complete nor cocomplete. Notice that if R is non-Archimedean the category \mathbf{Ban}_R contains more objects than the category $\mathbf{Ban}_R^{\text{na}}$ and their monoidal structures do not agree on $\mathbf{Ban}_R^{\text{na}}$.
- (4) Let R be a Banach ring. If R is non-Archimedean, then the category $\mathbf{Ban}_R^{\leq 1, \text{na}}$ of ultrametric Banach R -modules with contracting homomorphisms, *i.e.* bounded homomorphism $\phi : X \rightarrow Y$ such that $|\phi(x)| \leq |x|$, is quasi-abelian. The category $\mathbf{Ban}_R^{\leq 1, \text{na}}$ has a closed symmetric monoidal structure and is complete and cocomplete. Notice that it is possible to consider also the category $\mathbf{Ban}_R^{\leq 1}$, *i.e.* the category of all Banach R -modules with contracting morphisms, but this category is not additive.
- (5) To remedy the fact that arbitrary limits and colimits do not exist in \mathbf{Ban}_R we will consider the category $\mathbf{Ind}(\mathbf{Ban}_R)$ and its subcategory \mathbf{Born}_R of (complete) bornological modules. These are complete and cocomplete closed symmetric monoidal quasi-abelian categories, whose definition will be recalled in detail in Section 3.3.

We now recall some notions and results about exactness of additive functors between quasi-abelian categories.

Definition 2.2. *An additive functor $F : \mathbf{C} \rightarrow \mathbf{D}$ between two quasi-abelian categories is called left exact if for any exact sequence*

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

where f and g are strict morphisms, the sequence

$$0 \rightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact with $F(f)$ strict. The functor F is called strictly left exact if for any exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

where f is a strict morphism, the sequence

$$0 \rightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact with $F(f)$ strict. Dually one defines the notions of right exactness and strictly right exactness. We say that F is exact (resp. strictly exact) if it is both left and right exact (resp. both strictly left and strictly right exact).

Definition 2.2 can be restated by saying that a functor is left exact if and only if it preserves kernels of strict morphisms and it is strictly left exact if and only if it preserves kernels of all morphisms³. The dual statement holds for right exact and strictly right exact functors.

The next proposition should be well-known to experts but we cannot find a reference in literature where it is stated in this form.

Proposition 2.3. *Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be a functor between quasi-abelian categories.*

- (1) *If F is right exact, then it is exact if and only if it preserves strict monomorphisms.*
- (2) *If F is strictly right exact, then it is strictly exact if and only if it maps strict monomorphisms to strict monomorphisms and monomorphisms to monomorphisms.*

Dually for left exact and strictly left exact functors.

Proof. (1) Left exact functors clearly preserve strict monomorphisms, so we only need to prove the converse. We first notice that F maps strict morphisms to strict morphism because the class of strict morphism is precisely the class of morphism that can be written as a composition of a strict epimorphism followed by a strict monomorphism, and F preserves both classes of morphisms. So, let us consider a strict morphism f and write it as $f = m \circ e$ with e a strict epimorphism and m a strict monomorphism. Then, we have the exact sequence of strict morphisms

$$0 \rightarrow \text{Ker}(f) \rightarrow B \xrightarrow{f} C$$

that is mapped to the sequence

$$0 \rightarrow F(\text{Ker}(f)) \rightarrow F(B) \xrightarrow{F(f)} F(C).$$

It is easy to check that $\text{Ker}(f) \cong \text{Ker}(e)$ and since $F(m)$ is a strict monomorphism then we also have that $\text{Ker}(F(f)) \cong \text{Ker}(F(e))$. Thus, it is enough to consider the sequence

$$0 \rightarrow F(\text{Ker}(e)) \rightarrow F(B) \xrightarrow{F(e)} F(\text{Coim}(f)) \rightarrow 0$$

in which $F(e)$ is a strict epimorphism because F is right exact. Moreover, the sequence is exact in the middle because F is right exact and the fact that the morphism $F(\text{Ker}(e)) \rightarrow F(B)$ is a strict monomorphism implies that the sequence is actually a strict short exact sequence. All together, this shows that $F(\text{Ker}(f)) \cong \text{Ker}(F(f))$ as claimed.

- (2) Strictly left exact functors clearly preserve strict monomorphisms and monomorphisms, so we only need to prove the converse. Conversely, consider now an exact sequence

$$0 \rightarrow A \rightarrow B \xrightarrow{f} C$$

where $A \cong \text{Ker}(f)$ but f is not necessarily a strict morphism. Since F preserves strict monomorphisms and is right exact, the sequence

$$0 \rightarrow F(\text{Im}(f)) \rightarrow F(C) \rightarrow F(\text{Coker}(f)) \rightarrow 0$$

is strictly exact and thus the isomorphism $F(\text{Coker}(f)) \cong \text{Coker}(F(f))$ implies $\text{Im}(F(f)) \cong F(\text{Im}(f))$. Since F preserves monomorphisms we can deduce that the induced morphism

³Notice that we slightly changed the terminology of [27] where what we called strictly exact functor is called strongly exact functor. This comes with no harm.

$F(\text{Coim}(f)) \rightarrow \text{Coim}(F(f))$ is a monomorphism. Indeed, in the canonical commutative diagram of canonical maps

$$(2.3.1) \quad \begin{array}{ccc} \text{Coim}(F(f)) & \longrightarrow & \text{Im}(F(f)) \\ \uparrow & & \downarrow \\ F(\text{Coim}(f)) & \longrightarrow & F(\text{Im}(f)) \end{array}$$

the horizontal maps are bismorphisms and the right vertical map is an isomorphism, implying that the left vertical map is a monomorphism. But since both $\text{Coim}(F(f))$ and $F(\text{Coim}(f))$ are quotients of $F(B)$, this implies that they are actually isomorphic, and, as before, this implies that $F(A) \cong \text{Ker}(F(f))$ because we have proven that the sequence

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow \text{Coim}(F(f)) \rightarrow 0$$

is strictly exact. □

It is not enough for a strictly right exact functor to maps strict monomorphisms to strict monomorphisms to deduce strict left exactness. Our main example of a functor that is strictly right exact and left exact without being strictly left exact is the completion functor $\widehat{(-)} : \mathbf{Nr}_R \rightarrow \mathbf{Ban}_R$ from the category of normed modules over a Banach ring R to the category of Banach R -modules. This functor is strictly right exact because it is left adjoint to the embedding $\mathbf{Ban}_R \rightarrow \mathbf{Nr}_R$ and it can be checked that it preserves strict monomorphisms. But it is also easy to find examples of (non-strict) monomorphism that are not preserved by $\widehat{(-)}$.

It is possible to associate to \mathbf{C} its derived category $D(\mathbf{C})$ and we now recall its definition as well as the definitions of the categories $D^-(\mathbf{C})$, $D^+(\mathbf{C})$ and $D^{\leq 0}(\mathbf{C})$. Let $K(\mathbf{C})$ be the homotopy category of \mathbf{C} , *i.e.* the localization of the category of unbounded complexes of objects of \mathbf{C} by homotopy equivalences. This is a triangulated category. Let us consider the full subcategory of $N \subset K(\mathbf{C})$ of objects that have a representative that is a strictly exact complex. Recall, that a complex $\cdots \xrightarrow{d_{n-1}} C_{n-1} \xrightarrow{d_n} C_n \xrightarrow{d_{n+1}} C_{n+1} \rightarrow \cdots$ is said to be *strictly exact* if $\text{Ker}(d_n) \cong \text{Im}(d_{n-1})$ for all n and all d_n 's are strict morphisms. It is possible to prove that N is a thick triangulated subcategory of $K(\mathbf{C})$, see [27, Corollary 1.2.15]. We thus define

$$D(\mathbf{C}) = \frac{K(\mathbf{C})}{N}$$

where the quotient denotes the Verdier quotient of $K(\mathbf{C})$ by N . This Verdier quotient is equivalent to inverting the morphisms in $K(\mathbf{C})$ whose cone is a strictly exact complex. We remark that in the case when \mathbf{C} is abelian then $D(\mathbf{C})$ is the derived category of \mathbf{C} in the usual sense, *i.e.* the definition we gave of the derived category is compatible with the standard definition in the case when \mathbf{C} is abelian. The category $D^-(\mathbf{C})$ (resp. $D^+(\mathbf{C})$, resp. $D^{\leq 0}(\mathbf{C})$) can be defined as the full subcategory of objects of $D(\mathbf{C})$ whose objects have a representative that is eventually zero on the right (resp. on the left, resp. in degree ≤ 0) or by a suitable modification the definition of $D(\mathbf{C})$.

By [19, Corollary 5.18] the Dold-Kan correspondence⁴ holds for all additive categories with kernels, in particular, it holds for quasi-abelian categories. This implies that we can describe the category $D^{\leq 0}(\mathbf{C})$ as a localization of the category $\mathbf{Simp}(\mathbf{C})$ of simplicial objects of \mathbf{C} endowed with a suitable model structure. We will say more about this model structure later on. We will

⁴The Dold-Kan correspondence for \mathbf{C} is an explicit adjoint pair equivalence $\mathbf{Simp}(\mathbf{C}) \rightleftarrows \mathbf{Ch}^{\leq 0}(\mathbf{C})$.

often represent objects of $D^{\leq 0}(\mathbf{C})$ by simplicial objects when studying derived geometry relative to \mathbf{C} .

As for the case of the derived categories of abelian categories, objects of $D(\mathbf{C})$ are usually studied via projective or injective resolutions. The quasi-abelian version of this process is very similar, compatible with the abelian one, and it is reviewed in what follows.

Definition 2.4. *Let $P \in \mathbf{C}$. We say that P is projective if the functor $\mathrm{Hom}(P, -)$ is exact (in the sense of Definition 2.2).*

More explicitly, using Proposition 2.3, $P \in \mathbf{C}$ is projective if $\mathrm{Hom}(P, -)$ sends strict epimorphisms to surjections. We say that \mathbf{C} has enough projective objects if for any $X \in \mathbf{C}$ there exists a strict epimorphism $P \rightarrow X$ with P projective. If \mathbf{C} has enough projective objects and is complete and cocomplete, then [4, Theorem 3.7] implies that on $\mathbf{Simp}(\mathbf{C})$, the quasi-abelian category of simplicial objects on \mathbf{C} , there is a symmetric monoidal combinatorial model structure whose homotopy category is equivalent to $D^{\leq 0}(\mathbf{C})$, via the quasi-abelian Dold-Kan correspondence.

Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be an additive functor between quasi-abelian categories. We now recall how to define its left and right derived functors. A subcategory $\mathbf{P}_F \subset \mathbf{C}$ is called F -projective if

- (1) for any $X \in \mathbf{C}$ there exists a strict epimorphism $P \rightarrow X$ with $P \in \mathbf{P}_F$;
- (2) for any strictly exact sequence

$$0 \rightarrow P \rightarrow P' \rightarrow P'' \rightarrow 0$$

with $P', P'' \in \mathbf{P}_F$ one has that $P \in \mathbf{P}_F$;

- (3) for any strictly exact sequence

$$0 \rightarrow P \rightarrow P' \rightarrow P'' \rightarrow 0$$

in \mathbf{P}_F , one has that

$$0 \rightarrow F(P) \rightarrow F(P') \rightarrow F(P'') \rightarrow 0$$

is strictly exact in \mathbf{D} .

In a dual fashion, one can define the notion of F -injective category \mathbf{I}_F . In the case when F admits an F -projective category (resp. F -injective category) it has a left derived functor $\mathbb{L}F : D^-(\mathbf{C}) \rightarrow D^-(\mathbf{D})$ (resp. right derived functor $\mathbb{R}F : D^+(\mathbf{C}) \rightarrow D^+(\mathbf{D})$) defined using F -projective resolutions (resp. F -injective resolutions). An object $P \in \mathbf{C}$ is said F -acyclic if

$$\mathbb{L}F(P) \cong F(P) \quad (\text{resp. } \mathbb{R}F(I) \cong F(I))$$

in $D^-(\mathbf{D})$ (resp. $D^+(\mathbf{D})$). In general, the class of F -acyclic objects may not form an F -projective (resp. F -injective) class.

- Example 2.5.**
- (1) The category of abelian groups has enough projective objects. Indeed, every free abelian group is projective and every abelian group is a quotient of a free abelian group.
 - (2) Let R be a Banach ring. The category \mathbf{Ban}_R has enough projective objects. We will give a proof of this fact in Proposition 3.11.
 - (3) Let R be a non-Archimedean Banach ring. The category $\mathbf{Ban}_R^{\leq 1, \text{na}}$ does not have enough projective objects, in contrast with the category $\mathbf{Ban}_R^{\text{na}}$ which, similarly to \mathbf{Ban}_R , has enough projective objects.
 - (4) Both categories \mathbf{Born} and $\mathbf{Ind}(\mathbf{Ban})$ have enough projective objects. Moreover, they are derived equivalent, *i.e.* $D(\mathbf{Ind}(\mathbf{Ban}_R)) \cong D(\mathbf{Born}_R)$, and we will show in Proposition 3.19 that the equivalence preserves the monoidal structure (a fact for which we

cannot find a reference in literature). We will prefer to work with \mathbf{Born}_R but, from the perspective of the derived geometry that will be introduced in the following pages, these two categories are essentially equivalent.

2.2. The left-heart of a quasi-abelian category. The derived category $D(\mathbf{C})$ has a t-structure called the *left t-structure* (of course there exists also a right t-structure, but we only use the left t-structure in this work) whose associated truncation functors are denoted by $\tau_L^{\leq n}$ and $\tau_L^{\geq n}$. The explicit definition of this t-structure is not important for our discussion as we will need only the properties that we discuss in this section⁵. The heart of the left t-structure is denoted by $\mathrm{LH}(\mathbf{C})$ and it is an abelian category. Moreover, one has that $D(\mathbf{C}) \cong D(\mathrm{LH}(\mathbf{C}))$, *i.e.* \mathbf{C} and $\mathrm{LH}(\mathbf{C})$ are derived equivalent. We also notice that the left t-structure gives the correct notion of the cohomology of an object $X \in D(\mathbf{C})$, given by

$$\mathrm{LH}^n(X) = \tau_L^{\leq n}(\tau_L^{\geq n}(X)) \in \mathrm{LH}(\mathbf{C})$$

because $X \cong 0$ in $D(\mathbf{C})$ if and only if $\mathrm{LH}^n(X) \cong 0$ for all n .

Proposition 2.6. *The objects of $\mathrm{LH}(\mathbf{C})$ can be described as complexes of objects of \mathbf{C} of the form $[0 \rightarrow E \rightarrow F \rightarrow 0]$ (with F in degree 0), where $E \rightarrow F$ is a monomorphism. The morphisms of such complexes are commutative squares localized by the multiplicative system generated by the ones that are simultaneously cartesian and cocartesian.*

Proof. This is in [27, Corollary 1.2.20]. Explicitly, given two objects $[0 \rightarrow E_0 \rightarrow F_0 \rightarrow 0]$ and $[0 \rightarrow E_1 \rightarrow F_1 \rightarrow 0]$ and a morphism $f = (f_E, f_F)$ defined by

$$(2.6.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & E_0 & \xrightarrow{d_0} & F_0 & \longrightarrow & 0 \\ & & \downarrow f_E & & \downarrow f_F & & \\ 0 & \longrightarrow & E_1 & \xrightarrow{d_1} & F_1 & \longrightarrow & 0 \end{array},$$

then $\mathrm{cone}(f)$ is the complex

$$0 \rightarrow E_0 \xrightarrow{\begin{pmatrix} -d_0 \\ f_E \end{pmatrix}} F_0 \oplus E_1 \xrightarrow{(f_F, d_1)} F_1 \rightarrow 0$$

from which it follows immediately that $\mathrm{cone}(f)$ is strictly exact if and only if the commutative square of (2.6.1) is both cartesian and cocartesian. \square

One important property of $\mathrm{LH}(\mathbf{C})$, for our scopes, is the following.

Proposition 2.7. *The category \mathbf{C} is a reflective subcategory of $\mathrm{LH}(\mathbf{C})$. The embedding functor $i : \mathbf{C} \rightarrow \mathrm{LH}(\mathbf{C})$ sends an object of \mathbf{C} to a complex concentrated in degree 0, and its adjoint sends an object $[0 \rightarrow E \rightarrow F \rightarrow 0]$ in $\mathrm{LH}(\mathbf{C})$ to the cokernel $\mathrm{Coker}(E \rightarrow F)$ computed in \mathbf{C} (it is easy to check that the cokernel does not depend on the representative using Proposition 2.6, cf. [27, Lemma 1.2.25]).*

Proof. [27, Corollary 1.2.20]. \square

The left adjoint of the embedding functor $i : \mathbf{C} \rightarrow \mathrm{LH}(\mathbf{C})$ is denoted by $c : \mathrm{LH}(\mathbf{C}) \rightarrow \mathbf{C}$ and is called the *classical part functor*.

Corollary 2.8. *The embedding functor $i : \mathbf{C} \rightarrow \mathrm{LH}(\mathbf{C})$ preserves monomorphisms.*

⁵The interested reader can find the details about the t-structures on $D(\mathbf{C})$ in Section 1.2 of [27].

Corollary 2.9. *With the same notation of Proposition 2.6, the morphism f is a monomorphism if and only if the sequence*

$$0 \rightarrow E_0 \xrightarrow{\begin{pmatrix} -d_0 \\ f_E \end{pmatrix}} F_0 \oplus E_1 \xrightarrow{(f_F, d_1)} F_1$$

is exact and $\begin{pmatrix} -d_0 \\ f_E \end{pmatrix}$ is a strict morphism, whereas f is an epimorphism if and only if (f_F, d_1) is a strict epimorphism.

Proof. To simplify notation let us write $\mathcal{E}_0 = [0 \rightarrow E_0 \rightarrow F_0 \rightarrow 0]$ and $\mathcal{E}_1 = [0 \rightarrow E_1 \rightarrow F_1 \rightarrow 0]$. By [27, Proposition 1.2.32] the embedding functor $i : \mathbf{C} \rightarrow \text{LH}(\mathbf{C})$ induces a derived equivalence that identifies the left t-structure on $D(\mathbf{C})$ with the canonical t-structure on $D(\text{LH}(\mathbf{C}))$. This means that f is a monomorphism if and only if $\text{LH}^{-1}(\text{cone}(f)) = 0$ and f is an epimorphism if and only if $\text{LH}^0(\text{cone}(f)) = 0$, as the left heart cohomology coincides with the usual cohomology of complexes of $\text{LH}(\mathbf{C})$ as an abelian category. In the proof of Proposition 2.6 we described $\text{cone}(f)$ as the complex

$$0 \rightarrow E_0 \xrightarrow{\begin{pmatrix} -d_0 \\ f_E \end{pmatrix}} F_0 \oplus E_1 \xrightarrow{(f_F, d_1)} F_1 \rightarrow 0$$

where F_1 is placed in degree 0. By [27, Corollary 1.2.20] the left heart cohomology of $\text{cone}(f)$ at degree n is zero if and only if $\text{cone}(f)$ is strictly exact in degree n , that is precisely the condition in the statement of the corollary. \square

Proposition 2.6 suggests an interpretation of the objects of $\text{LH}(\mathbf{C})$ as quotients of objects of \mathbf{C} as follows. In \mathbf{C} the canonical maps from kernels of morphisms are strict monomorphisms, thus the computation of a cokernel always yields a strict short exact sequence. But if we consider a generic monomorphism $E \rightarrow F$ in \mathbf{C} , then this monomorphism fits into the short exact sequence

$$0 \rightarrow i(E) \rightarrow i(F) \rightarrow [0 \rightarrow E \rightarrow F \rightarrow 0] \rightarrow 0$$

in $\text{LH}(\mathbf{C})$. In this sense, we think of $[0 \rightarrow E \rightarrow F \rightarrow 0]$ as the quotient of F by E , that is in general different from its quotient in \mathbf{C} which is given by its classical part. We further emphasize this point with an example from functional analysis.

Example 2.10. Consider the quasi-abelian category \mathbf{Ban}_R of Banach modules over a Banach ring R . Strict monomorphisms in this case are given by inclusions of closed subspaces and monomorphisms just by injective morphisms. Thus, if $E \rightarrow F$ is a monomorphism, then in \mathbf{Ban}_R

$$\text{Coker}(E \rightarrow F) \cong \frac{F}{\overline{E}}$$

where \overline{E} is the closure of the image of E in F . Whereas the object $[0 \rightarrow E \rightarrow F \rightarrow 0]$ of $\text{LH}(\mathbf{Ban}_R)$ can be thought of as the quotient of F by E , without the closure operation. These quotients will be important later on.

We will use the following property several times.

Lemma 2.11. *Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be a left derivable functor between quasi-abelian categories. Then, $\mathbb{L}F$ is isomorphic to $\mathbb{L}(\text{LH}^0(\mathbb{L}F))$ and the restriction of $\text{LH}^0(\mathbb{L}F) : \text{LH}(\mathbf{C}) \rightarrow \text{LH}(\mathbf{D})$ is isomorphic to F if and only if F is regular⁶ and right exact. The dual statement holds to right derivable functors.*

Proof. This is proved in [27, Proposition 1.3.15]. \square

⁶We follow the terminology of [27, Page 8] where a functor is called regular if it maps strict morphisms to strict morphisms.

It is natural to ask what is the relation between the left t-structure and the notions of exactness introduced so far, *i.e.* to find conditions that permit to check when a functor $F : \mathbf{C} \rightarrow \mathbf{D}$ “derives trivially” *i.e.* it has $\mathrm{LH}^n(D(F)) = 0$ for all $n \neq 0$, where $D(F)$ denote the derived functor of F , in terms of the exactness properties of F .

Proposition 2.12. *Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be a right exact functor between quasi-abelian categories and assume that F is left derivable to a functor $\mathbb{L}F : D^-(\mathbf{C}) \rightarrow D^-(\mathbf{D})$. Then, $\mathrm{LH}^n(\mathbb{L}F) = 0$ for all $n \neq 0$ if and only if F is strictly left exact.*

Proof. Suppose that F is strictly left exact. By [27, Proposition 1.3.15] the functor $\mathrm{LH}^0(\mathbb{L}F)$ is right exact, therefore to check that it is left exact it is enough to check that it sends monomorphisms (of $\mathrm{LH}(\mathbf{C})$) to monomorphisms (of $\mathrm{LH}(\mathbf{D})$). The left and right exactness of F imply that it preserve strict monomorphisms and strict epimorphisms and hence F is a regular functor. This permits to apply Lemma 2.11 to deduce that the restriction of $\mathrm{LH}^0(\mathbb{L}F)$ to \mathbf{C} is isomorphic to F and since F preserves monomorphisms, because it is strictly left exact, then we have that

$$(\mathrm{LH}^0(\mathbb{L}F))([E_0 \rightarrow F_0]) \cong [F(E_0) \rightarrow F(F_0)].$$

By Corollary 2.9 a morphism $f : E = [E_0 \rightarrow F_0] \rightarrow F = [E_1 \rightarrow F_1]$ in $\mathrm{LH}(\mathbf{C})$ is a monomorphism if and only if the associated sequence

$$0 \rightarrow E_0 \xrightarrow{\alpha} F_0 \oplus E_1 \xrightarrow{\beta} F_1$$

is strictly exact at $F_0 \oplus E_1$, *i.e.* the morphism α is the kernel of β where β is a strict morphism (*i.e.* not necessarily strict). Here α and β denotes the morphism described explicitly in Corollary 2.9. We thus have that $(\mathrm{LH}^0(\mathbb{L}F))(f)$ is given, up to canonical isomorphism, by the morphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(E_0) & \xrightarrow{F(d_0)} & F(F_0) & \longrightarrow & 0 \\ & & \downarrow F(f_E) & & \downarrow F(f_F) & & \\ 0 & \longrightarrow & F(E_1) & \xrightarrow{F(d_1)} & F(F_1) & \longrightarrow & 0 \end{array},$$

that is a monomorphism if and only if the sequence

$$0 \rightarrow F(E_0) \xrightarrow{F(\alpha)} F(F_0 \oplus E_1) \xrightarrow{F(\beta)} F(F_1)$$

is strictly exact, again by applying Corollary 2.9. But this sequence is strictly exact because F is strictly left exact, proving that $\mathrm{LH}^0(\mathbb{L}F)$ is an exact functor.

On the other hand, if we suppose that F is not left exact then $\mathrm{LH}^0(\mathbb{L}F)$ is not exact because it does not preserve strict exactness of complexes of $D(\mathbf{C})$, otherwise F would preserve strict exactness too. If we suppose that F is left exact but not strictly left exact then there exists a monomorphism in \mathbf{C} whose kernel is not preserved by F . If $A \rightarrow B$ is such a monomorphism then

$$0 \rightarrow A \rightarrow B \rightarrow [A \rightarrow B] \rightarrow 0$$

is an exact sequence in $\mathrm{LH}(\mathbf{C})$ that is mapped to

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow \mathrm{LH}^0(\mathbb{L}F)([A \rightarrow B]) \rightarrow 0$$

because the restriction of $\mathrm{LH}^0(\mathbb{L}F)$ to \mathbf{C} is isomorphic to F in this case by [27, Proposition 1.3.15]. But by hypothesis $F(A) \rightarrow F(B)$ is not a monomorphism and thus $\mathrm{LH}^0(\mathbb{L}F)$ not exact. \square

Corollary 2.13. *Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be left strictly exact functor and right exact functor. Then, $\mathrm{LH}^n(\mathbb{L}F) = 0$ for all $n \neq 0$ and $\mathrm{LH}^0(F)$ restricts to F on \mathbf{C} .*

Proof. Proposition 2.12 and Lemma 2.11 prove the claim. \square

We stress once more the fact that it is not enough that F is exact for having $\mathrm{LH}^n(\mathbb{L}F) = 0$ for all $n \neq 0$, in contrast with the case of additive functors between abelian categories. So, the terminologies used in the theory of quasi-abelian categories and that of abelian categories are not in perfect agreement. Having vanishing higher order derived functors for the left heart cohomology for a functor between quasi-abelian categories is equivalent to being right exact and strictly left exact and dually having vanishing higher order derived functors for the right heart cohomology is equivalent to being strictly right exact and left exact. The latter claim can be proved but dualizing the argument provided so far for the left heart cohomology. We deduce that for testing the strict exactness of a functor one must compute its higher derived functors both for the left heart and right heart cohomology. The main example that we will meet in applications of an exact functor for which the higher derived functors do not vanish is the completion functor $\widehat{(-)} : \mathbf{Nr}_R \rightarrow \mathbf{Ban}_R$. This functor is strictly right exact and left exact but not strictly left exact. Thus, although $\widehat{(-)}$ is an exact functor for the right heart cohomology it is not for the left heart cohomology and thus it is a non-trivial functor homologically, with non-vanishing higher derived functors for the left heart cohomology. We will also use the following lemma.

Lemma 2.14. *Let \mathbf{C} be a quasi-abelian category with enough projective objects and $F : \mathbf{C} \rightarrow \mathbf{D}$ a right exact functor to another quasi-abelian category. Then, the following diagram of functors is commutative up to natural isomorphism*

$$(2.14.1) \quad \begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ \downarrow i & & \uparrow c \\ \mathrm{LH}^0(\mathbf{C}) & \xrightarrow{\mathrm{LH}^0(F)} & \mathrm{LH}^0(\mathbf{D}) \end{array} .$$

Proof. It is enough to check that the functors F and $c \circ \mathrm{LH}^0(F) \circ i$ have the same values on objects. Let $X \in \mathbf{C}$. We can write

$$X \cong \mathrm{Coker}(P^1 \rightarrow P^0)$$

where the P^i are the first two terms of a projective resolution of X . Notice that the morphism $P^1 \rightarrow P^0$ is strict because it is the composition of the strict epimorphism $P^1 \rightarrow \mathrm{Ker}(P^0 \rightarrow X)$ with the strict monomorphism $\mathrm{Ker}(P^0 \rightarrow X) \rightarrow X$, and the post-composition of a strict epimorphism with a strict monomorphism is a strict morphism. Thus, since F is right exact we have that

$$F(X) \cong \mathrm{Coker}(F(P^1) \rightarrow F(P^0)).$$

By [27, Proposition 1.3.24], the category $\mathrm{LH}^0(\mathbf{C})$ has enough projective objects and the projective objects of $\mathrm{LH}^0(\mathbf{C})$ are isomorphic to objects of the form $i(P)$ for P projective in \mathbf{C} . Therefore, we have that

$$i(X) \cong \mathrm{Coker}(i(P^1) \rightarrow i(P^0))$$

because $i(P^1)$ and $i(P^0)$ are the first two terms of a projective resolution of $i(X)$ (notice that the functor i sends strictly exact sequences to exact sequences, cf. [27, Corollary 1.2.27]). Thus,

$$\mathrm{LH}^0(F)(X) \cong \mathrm{Coker}(\mathrm{LH}^0(F)(i(P^1)) \rightarrow \mathrm{LH}^0(F)(i(P^0)))$$

because $\mathrm{LH}^0(F)$ is right exact and since c is a left adjoint functor we have that

$$\begin{aligned} c(\mathrm{LH}^0(F)(X)) &\cong c(\mathrm{Coker}(\mathrm{LH}^0(F)(i(P^1)) \rightarrow \mathrm{LH}^0(F)(i(P^0)))) \cong \\ &\mathrm{Coker}(c(\mathrm{LH}^0(F)(i(P^1))) \rightarrow c(\mathrm{LH}^0(F)(i(P^0)))) \cong \mathrm{Coker}(F(P^1) \rightarrow F(P^0)) \end{aligned}$$

because

$$\mathrm{LH}^0(F)(i(P)) \cong \mathbb{L}F(P) \cong F(X)$$

for all projective objects of \mathbf{C} . \square

We now consider \mathbf{C} to be closed symmetric monoidal.

Proposition 2.15. *Let $(\mathbf{C}, \overline{\otimes})$ be a closed symmetric monoidal quasi-abelian category with enough projective objects. Suppose also that for all projective objects $P, Q \in \mathbf{C}$ one has that $P \overline{\otimes} Q$ is projective. Then, $\mathrm{LH}(\mathbf{C})$ is a closed symmetric monoidal abelian category equipped with the functors $\mathrm{LH}^0((-) \overline{\otimes}^{\mathbb{L}}(-))$ and $\mathrm{LH}^0(\mathbb{R}\underline{\mathrm{Hom}}(-, -))$.*

Proof. This is Proposition 1.5.3 and Corollary 1.5.4 [27]. \square

Proposition 2.15 has the following important consequence (we refer to the beginning of the next sub-section for the definition of the ∞ -categories associated to \mathbf{C} , $\mathrm{LH}(\mathbf{C})$ and their categories of monoids).

Corollary 2.16. *In the hypothesis of Proposition 2.15 the symmetric monoidal ∞ -categories $\infty\text{-LH}(\mathbf{C})$ and $\infty\text{-}\mathbf{C}$ are monoidally Quillen equivalent. Hence, the ∞ -categories $\infty\text{-}\mathbf{Comm}(\mathrm{LH}(\mathbf{C}))$ and $\infty\text{-}\mathbf{Comm}(\mathbf{C})$ are Quillen equivalent.*

Proof. The fact that the ∞ -categories $\infty\text{-LH}(\mathbf{C})$ and $\infty\text{-}\mathbf{C}$ are Quillen equivalent is just a restatement of the fact that the homotopy categories of both categories are equivalent to $D^{\leq 0}(\mathbf{C})$ and the equivalences are induced by the adjunction $c : \mathrm{LH}(\mathbf{C}) \rightleftarrows \mathbf{C} : i$, discussed in Proposition 2.7. Another way of thinking about this is to notice that the objects of both $D^{\leq 0}(\mathbf{C})$ and $D^{\leq 0}(\mathrm{LH}(\mathbf{C}))$ can be described in terms of the projective objects of \mathbf{C} and $\mathrm{LH}(\mathbf{C})$, respectively. But since \mathbf{C} has enough projective the class of projective objects of \mathbf{C} and $\mathrm{LH}(\mathbf{C})$ agree, cf. [27, Proposition 1.3.24]. It is thus clear that, under the hypothesis of the corollary, this equivalence preserves the tensor products and closed structures. The assertion about the categories of commutative algebras is then a formal consequence of the monoidal Quillen equivalence. \square

In general, the categories $\mathbf{Comm}(\mathbf{C})$ and $\mathbf{Comm}(\mathrm{LH}(\mathbf{C}))$ are not equivalent. Moreover, there seem to be no reason for which in general the adjunction $i : \mathrm{LH}(\mathbf{C}) \rightleftarrows \mathbf{C} : c$ must be a monoidal adjunction without any assumption on \mathbf{C} .

Proposition 2.17. *Suppose that $(\mathbf{C}, \overline{\otimes})$ has enough projective objects, then in the adjunction of Proposition 2.7*

$$(2.17.1) \quad c : \mathrm{LH}(\mathbf{C}) \rightleftarrows \mathbf{C} : i$$

i is a lax monoidal functor and c is a strong monoidal functor.

Proof. Since c is left adjoint it is enough to show that c is strongly monoidal as then i is automatically lax monoidal. Since \mathbf{C} has enough projective objects, the class of projective objects is $\overline{\otimes}$ -acyclic (cf. [27, Remark 1.3.21]) and the class of projective objects of \mathbf{C} and $\mathrm{LH}(\mathbf{C})$ agree (cf. [27, Proposition 1.3.24]), then every object of $\mathrm{LH}(\mathbf{C})$ can be written as a complex of the form

$$[E \rightarrow P] \cong \mathrm{Coker}(i(E) \rightarrow i(P))$$

where $E, P \in \mathbf{C}$ and P is a projective object. We denote by $\overline{\otimes}^{\mathbb{L}}$ and $\overline{\otimes}^{\mathrm{LH}}$ the monoidal structures of $D(\mathbf{C})$ and of $\mathrm{LH}(\mathbf{C})$ respectively. Consider two such objects $X = [E \rightarrow P]$ and $X' = [E' \rightarrow P']$ then

$$X \overline{\otimes}^{\mathrm{LH}} X' \cong \mathrm{LH}^0(X \overline{\otimes}^{\mathbb{L}} X').$$

We thus have that

$$X \overline{\otimes}^{\text{LH}} X' \cong \text{Coker}(\text{Coker}(i(E) \overline{\otimes}^{\text{LH}} i(E') \rightarrow i(E) \overline{\otimes}^{\text{LH}} i(P')) \rightarrow \text{Coker}(i(P) \overline{\otimes}^{\text{LH}} i(E') \rightarrow i(P) \overline{\otimes}^{\text{LH}} i(P')))$$

where we have just commuted the cokernels with the functor $\overline{\otimes}^{\text{LH}}$. In order to prove the claim we have to show that

$$c(X \overline{\otimes}^{\text{LH}} X') \cong c(X) \overline{\otimes} c(X').$$

Since c is a left adjoint functor it commutes with cokernels, so

$$c(X \overline{\otimes}^{\text{LH}} X') \cong \text{Coker}(\text{Coker}(c(i(E) \overline{\otimes}^{\text{LH}} i(E')) \rightarrow c(i(E) \overline{\otimes}^{\text{LH}} i(P'))) \rightarrow \text{Coker}(c(i(P) \overline{\otimes}^{\text{LH}} i(E')) \rightarrow c(i(P) \overline{\otimes}^{\text{LH}} i(P')))).$$

We can now repeatedly apply Lemma 2.14 to deduce the isomorphisms

$$c(i(E) \overline{\otimes}^{\text{LH}} i(E')) \cong E \overline{\otimes} E', \quad c(i(E) \overline{\otimes}^{\text{LH}} i(P')) \cong E \overline{\otimes} P',$$

$$c(i(P) \overline{\otimes}^{\text{LH}} i(E')) \cong P \overline{\otimes} E', \quad c(i(P) \overline{\otimes}^{\text{LH}} i(P')) \cong P \overline{\otimes} P'.$$

We thus get that

$$c(X \overline{\otimes}^{\text{LH}} X') \cong \text{Coker}(E \rightarrow P) \overline{\otimes} \text{Coker}(E' \rightarrow P') \cong c(X) \overline{\otimes} c(X')$$

because the functor $\overline{\otimes}$ commutes with cokernels. \square

We notice that it is easy to prove that the functor $i : \mathbf{C} \rightarrow \text{LH}(\mathbf{C})$ is strongly monoidal in the case when $\overline{\otimes}$ is a regular functor (*i.e.* preserves the strictness of morphisms). Indeed, in this case, we can directly apply [27, Proposition 1.3.15] to deduce that $i(X) \overline{\otimes}^{\text{LH}} i(X') \cong i(X \overline{\otimes} X')$ for all $X, X' \in \mathbf{C}$. Unfortunately, in cases of interest, the monoidal structure is not regular. We show this fact in the case of Banach spaces over \mathbb{C} , for the sake of simplicity. Consider the Banach space $C([0, 1])$ of continuous functions on the real interval $[0, 1]$. The Banach-Mazur Theorem gives a strict monomorphism $\ell^2 \rightarrow C([0, 1])$, where ℓ^2 is the space of square summable sequences. Notice that ℓ^2 is reflexive and it is not a direct factor of $C([0, 1])$. Now let us consider the morphism

$$\phi : \ell^2 \widehat{\otimes} \ell^2 \rightarrow C([0, 1]) \widehat{\otimes} \ell^2.$$

Since Hilbert spaces have the approximation property, the morphism ϕ is injective. Let us show that it is not strict. Suppose that it is strict. This implies that it is a strict monomorphism and hence

$$\phi^* : \text{Hom}(C([0, 1]) \widehat{\otimes} \ell^2, \mathbb{C}) \rightarrow \text{Hom}(\ell^2 \widehat{\otimes} \ell^2, \mathbb{C})$$

is surjective. By the hom-tensor adjunction, we have the isomorphisms

$$\text{Hom}(C([0, 1]) \widehat{\otimes} \ell^2, \mathbb{C}) \cong \text{Hom}(C([0, 1]), \ell^2), \quad \text{Hom}(\ell^2 \widehat{\otimes} \ell^2, \mathbb{C}) \cong \text{Hom}(\ell^2, \ell^2)$$

where we used the fact that ℓ^2 is isomorphic to its dual. This implies that the strict monomorphism $\ell^2 \rightarrow C([0, 1])$ has a left inverse $C([0, 1]) \rightarrow \ell^2$, but this is impossible because ℓ^2 is not a direct factor of $C([0, 1])$. Nevertheless, it is easy to show that the monoidal structure on $\mathbf{Ban}_k^{\text{na}}$, for k a non-Archimedean field, is regular because it is a strictly exact functor. We conjecture that the monoidal structure on $\mathbf{Ban}_R^{\text{na}}$ is a regular functor for all non-Archimedean Banach rings.

2.3. The homotopy Zariski topology. Henceforth we will use the language of ∞ -categories. The theory of ∞ -categories can be embodied by several different concrete models (for example using weak Kan simplicial complexes) and the resulting theory is independent of the particular chosen model. In this work, we adopt a model-free approach, and thus we just speak of ∞ -categories and we use the formal properties of the resulting theory. One of the main use of the concept of ∞ -category is that it permits better handling of the operation of localization of a category and gives tools for computing objects, and morphism in the localized category. Moreover, the theory permits the characterization of objects by universal properties that are not accessible by looking at the localized category as a classical category. The main example the reader should keep in mind is the localization of the category of chain complexes of an abelian category that gives rise to the derived category. The resulting category is triangulated and all monomorphisms are split, a fact that implies that the cone construction is a limit or a colimit in the classical sense. When the category of chain complexes of an abelian category is enriched with a suitable structure of ∞ -category, then the cone construction becomes a cokernel in the ∞ -categorical sense.

One way of constructing an ∞ -category is via a *model category*. A model category \mathbf{C} is a category equipped with an extra-structure that we do not specify in detail here, see [22, Definition 1.1.3] for a precise definition of model category. Amongst the data needed to specify a model structure, there is a class of morphisms W whose elements are called *weak equivalences*. The formal inversion of the morphisms in W gives the homotopy category $\mathbf{Ho}(\mathbf{C}) = \mathbf{C}[W^{-1}]$. For example, in the case of chain complexes discussed so far, the class of weak equivalences is the class of quasi-isomorphisms of chain complexes. In general, there is a precise construction, called *simplicial localization*, that associates to any model category an ∞ -category and this construction is functorial in an appropriate sense. Although not all ∞ -categories can be obtained via this construction, in this paper only ∞ -categories coming from model categories will be considered and for all the purposes of this paper, the terms model category and ∞ -category can be considered synonymous. It is important to keep in mind the difference between the use of model categories and ∞ -categories. Model categories usually permit to do explicit computations that can be hardly accessible via the abstract machinery of ∞ -categories. Whereas, ∞ -categories give the correct meaning and theoretical interpretation to the computations to these computations that the theory of model categories cannot provide. Indeed, in a precise sense, the theory of ∞ -categories is a faithful extension of the theory of 1-categories whereas the theory of model categories is not. It is therefore fruitful to use both theories to exploit their respective advantages.

We now describe the model categories that appear in this work. Once we fix a quasi-abelian category \mathbf{C} that has all limits and colimits, with enough projective objects we can associate to it the model category obtained by equipping the category of simplicial objects on \mathbf{C} , denoted $\mathbf{Simp}(\mathbf{C})$, with the projective model structure (*cf.* [20, Theorem 4.6] for an explicit description of the projective model structure in the more general context of exact categories). The ∞ -category obtained as the simplicial localization of this model category is denoted $\infty\text{-}\mathbf{C}$. Via the quasi-abelian version of the Dold-Kan correspondence, it is possible to identify simplicial objects of \mathbf{C} with chain complexes concentrated in negative degrees and the weak equivalences of the projective model structure on simplicial objects with the strict quasi-isomorphisms of chain complexes introduced so far. We will therefore switch between the simplicial objects and chain complexes as needed. We find it more convenient to do computations with chain complexes but, when \mathbf{C} is equipped with a symmetric monoidal structure the Dold-Kan correspondence is not symmetric monoidal. In this situation, the correct model category for $\infty\text{-}\mathbf{C}$ is the category

of simplicial objects when considering algebras over $\infty\text{-}\mathbf{C}$. Therefore, it turns out that the homotopy category of $\infty\text{-}\mathbf{C}$ is equivalent to a full subcategory of $D(\mathbf{C})$ that we denote $D^{\leq 0}(\mathbf{C})$.

We now suppose that \mathbf{C} is equipped with a symmetric monoidal structure $\overline{\otimes} : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$. We suppose that $\overline{\otimes}$ that is a left Quillen bifunctor for the projective model structure introduced so far on \mathbf{C} , *i.e.* the pair $(\mathbf{C}, \overline{\otimes})$ is a symmetric monoidal model category. This implies that $\infty\text{-}\mathbf{C}$ becomes a closed symmetric monoidal ∞ -category and the symmetric monoidal structure on $\infty\text{-}\mathbf{C}$ induces a closed monoidal structure on $D^{\leq 0}(\mathbf{C})$. Imposing some further conditions on \mathbf{C} we obtain what is called a *HAG context* (*i.e.* a Homotopical Algebraic Geometry Context). We refer to [29] for the precise list of technical assumptions needed on \mathbf{C} , all of which are satisfied by the category of bornological modules we will use later on, we only recall one of the most important: we suppose that the monoidal structure $\overline{\otimes}$ satisfies the *monoid and the symmetric monoid axioms* when extended to simplicial objects. This condition implies that the ∞ -category $\mathbf{Comm}(\infty\text{-}\mathbf{C})$ of commutative algebras can be described as the ∞ -category associated with the model category of commutative simplicial monoids over $(\mathbf{C}, \overline{\otimes})$ where the weak equivalences are the same as the ones of the projective model structure, *i.e.* a morphism of commutative simplicial monoids is a weak equivalence if and only if it is a weak equivalence of the underlying simplicial objects of $\mathbf{Simp}(\mathbf{C})$. More precisely, the model structure on $\mathbf{Simp}(\mathbf{C})$ is transferred to $\mathbf{Comm}(\mathbf{Simp}(\mathbf{C}))$ via the adjunction $\mathbf{Simp}(\mathbf{C}) \rightleftarrows \mathbf{Comm}(\mathbf{Simp}(\mathbf{C}))$.

In the theory of derived geometry with respect to \mathbf{C} , the category $\mathbf{Comm}(\infty\text{-}\mathbf{C})$ has the same role as the category of commutative rings in algebraic geometry or the category of commutative simplicial rings in derived algebraic geometry. It is therefore natural to give the following definition for the category of affine derived schemes over \mathbf{C} .

Definition 2.18. *The ∞ -category $\mathbf{Comm}(\infty\text{-}\mathbf{C})^{\text{op}}$ is called the category of affine ∞ -schemes over \mathbf{C} and denoted $\mathbf{Aff}(\infty\text{-}\mathbf{C})$. Its homotopy category is called category of affine derived schemes and denoted $\mathbf{dAff}(\mathbf{C})$.*

Having defined the category $\mathbf{Aff}(\infty\text{-}\mathbf{C})$ we can proceed to define schemes and stacks over \mathbf{C} using the functor of points point of view. These will be a specific kind of ∞ -functors $\mathbf{Aff}(\infty\text{-}\mathbf{C})^{\text{op}} \rightarrow \infty\text{-}\mathbf{Sets}$, where $\infty\text{-}\mathbf{Sets}$ is the ∞ -category of spaces, *i.e.* the ∞ -category obtained from the category of simplicial sets equipped with the standard model structure. To do this we need to equip $\mathbf{Aff}(\infty\text{-}\mathbf{C})$ with a Grothendieck topology. There are many possible choices of Grothendieck topologies, some of which are direct generalizations of the most common topologies used in algebraic geometry, like the Zariski topology and the étale topology. We limit the discussion to the homotopical analog of the Zariski topology because it is the only one relevant to this work.

By definition, a Grothendieck topology on $\mathbf{Aff}(\infty\text{-}\mathbf{C})$ is the datum of a Grothendieck topology on the homotopy category $\mathbf{dAff}(\mathbf{C})$. Thus, we have just to deal with the common concept of Grothendieck topology. For doing that, by the assumptions that \mathbf{C} is a HAG context, we notice that for any object $A \in \mathbf{Comm}(\infty\text{-}\mathbf{C})$ it makes sense to consider the symmetric monoidal ∞ -category of A -modules, denoted by $(\infty\text{-}\mathbf{Mod}_A, \overline{\otimes}_A)$ whose homotopy category is denoted by $(D^{\leq 0}(\mathbf{Mod}_A), \overline{\otimes}_A^{\perp})$. If $A \in \mathbf{Comm}(\infty\text{-}\mathbf{C})$ we denote the corresponding object of $\mathbf{Aff}(\infty\text{-}\mathbf{C})$ by $\text{Spec}(A)$. The same notation will be used for derived affine schemes, *i.e.* for objects of the homotopy category of $\mathbf{Aff}(\infty\text{-}\mathbf{C})$ because the two categories have the same class of objects. We are ready to define the homotopy Zariski topology.

Definition 2.19. A morphism $\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(B)$ in $\mathbf{Aff}(\infty\text{-}\mathbf{C})$ is called (formal) homotopy Zariski open immersion⁷ if the induced morphism

$$(2.19.1) \quad A \overline{\otimes}_B A \rightarrow A$$

is an equivalence in $\infty\text{-}\mathbf{Mod}_A$ (i.e. an isomorphism in the homotopy category). A family $\{\mathrm{Spec}(A_i) \rightarrow \mathrm{Spec}(B)\}_{i \in I}$ of homotopy Zariski open immersions is a cover if there exists a finite subfamily $J \subset I$ such that the pullback functors

$$\{(-) \overline{\otimes}_B A_i : \infty\text{-}\mathbf{Mod}_B \rightarrow \infty\text{-}\mathbf{Mod}_{A_i}\}_{i \in J}$$

form a conservative family of functors.

A morphism $B \rightarrow A$ whose dual $\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(B)$ is a homotopy Zariski open immersion is called a *homotopy Zariski open localization*.

Proposition 2.20. The homotopy Zariski open immersions and covers define a Grothendieck topology on $\mathbf{Aff}(\infty\text{-}\mathbf{C})$ that we call the homotopy Zariski topology.

Proof. By definition, a topology on $\mathbf{Aff}(\infty\text{-}\mathbf{C})$ is a usual Grothendieck topology on its homotopy category. The axioms of Grothendieck topology are straightforward (we skip the details) to check because equivalences are clearly homotopy Zariski open immersion, homotopy pullbacks of affine schemes (corresponding to tensor products of algebras) clearly preserve homotopy Zariski open immersion (see Proposition 2.27 below) and covers and finally, it is also easy to check that the composition of two homotopy Zariski open immersions is a homotopy Zariski open immersion and the composition of covers give a cover. \square

The notion of (formal) homotopy Zariski open immersion of Definition 2.19 is an analog of the notion of formal Zariski open immersion of algebraic geometry. So, it does not require any finite presentation for the morphisms. This creates the issue that the family of homotopy Zariski open immersions is huge and difficult to describe. The focus of this work will be to find for any $X \in \mathbf{Comm}(\infty\text{-}\mathbf{C})$, in the case when \mathbf{C} is the quasi-abelian category of bornological modules, a subfamily of homotopy Zariski open immersion that will provide the association to X of a site of reasonable size, similar to the definition of the small Zariski or small étale site of scheme theory.

Given a morphism $f : B \rightarrow A$ in $\mathbf{Comm}(\infty\text{-}\mathbf{C})$, the condition of equation (2.19.1) can usually be checked effectively through explicit computations. We will perform these computations in the specific context of analytic geometry in later sections. Instead, the cover condition of Definition 2.19 is not stated in directly computable terms. So, we would like to find a formulation of the conservativity condition that is more amenable to computations. For doing so, we need to recall the following constructions and lemmas. Consider a linear diagram of objects

$$\cdots \rightarrow M_n \rightarrow M_{n+1} \rightarrow \cdots$$

in $D(\mathbf{C})$ (or more generally $D(\mathbf{Mod}_A)$ for $A \in \mathbf{Comm}(\infty\text{-}\mathbf{C})$). The colimit of this diagram does not exist in $D(\mathbf{C})$, in general, and is substituted by the homotopy-colimit. This latter object can be described at the level of ∞ -categories, but we limit our discussion to $D(\mathbf{C})$. The abstract definition of the homotopy-colimit is given as the cofiber (i.e. the cone) of the canonical morphism

$$\bigoplus_{n \in \mathbb{N}} M_n \rightarrow \bigoplus_{n \in \mathbb{N}} M_n$$

⁷We usually omit the word “formal” as in this work only formal homotopy Zariski open immersions are considered.

induced by the inductive system. But in our specific situation, the objects M_n can be represented by complexes of objects of \mathbf{C} , and if all the maps of the system can be represented by morphisms of the complexes M_n we obtain a double complex whose total complex computes the homotopy-colimit. In formulas we have

$$\operatorname{hocolim}_{n \in \mathbb{N}} M_n \cong \operatorname{Tot} (\cdots \rightarrow M_n \rightarrow M_{n+1} \rightarrow \cdots)$$

in $D(\mathbf{C})$. We should mention that the double complex construction can be given in two different fashions, one using direct sums and another using direct products. In this case, the one with the direct sums must be taken, although in this work (almost) all the computations will be done with finite complexes and the systems will always be eventually zero. Thus only finitely many direct sums will be computed. We need the following lemma.

Lemma 2.21. *Let $\{\operatorname{Spec}(A_i) \rightarrow \operatorname{Spec}(B)\}_{i \in I}$ be a cover for the homotopy Zariski topology, then for any $M \in \infty\text{-Mod}_B$ there is a quasi-isomorphism*

$$\begin{aligned} \operatorname{Tot} (0 \rightarrow \prod_{i \in I} A_i \overline{\otimes}_B^{\mathbb{L}} M \rightarrow \prod_{(i,j) \in I^2} A_i \overline{\otimes}_B^{\mathbb{L}} A_j \overline{\otimes}_B^{\mathbb{L}} M \rightarrow \cdots) &\cong \\ \operatorname{Tot} (0 \rightarrow \prod_{i \in I} A_i \overline{\otimes}_B^{\mathbb{L}} M \rightarrow \prod_{i < j \in I} A_i \overline{\otimes}_B^{\mathbb{L}} A_j \overline{\otimes}_B^{\mathbb{L}} M \rightarrow \cdots) & \end{aligned}$$

between the derived Čech-Tate complex of the cover and the alternating derived Čech-Tate complex.

Proof. An easy way of proving this lemma is by using Mitchell's Embedding Theorem. We notice that once we fix a cofibrant replacement of M and all A_j the $\operatorname{Tot}(-)$ functor is applied to an explicit bicomplex. This bicomplex can be seen as an object in the category of complexes over $D^-(\mathbf{C})$, we denote it $\mathbf{Ch}^+(D^-(\mathbf{C}))$. The category $D^-(\mathbf{C})$ is additive and idempotent complete, so by Quillen's embedding Theorem it can be embedded into an abelian category once $D^-(\mathbf{C})$ is equipped with the split exact structure. Then, Mitchell's Theorem implies that we can embed the resulting abelian category into a category of modules over a ring. We notice that since our set I is finite and the embedding functors are additive the embeddings preserve finite products. Therefore, the condition that $B \rightarrow A_i$ is a homotopy Zariski open embedding (and hence $A_i \widehat{\otimes}_B^{\mathbb{L}} A_i \rightarrow A_i$ is an isomorphism in $D^-(\mathbf{C})$) implies that the resulting complexes are isomorphic, respectively, to the Čech complex and the alternating Čech complex of a sheaf valued in the category of modules over a ring, and hence they are homotopic by classical computations. Since the $\operatorname{Tot}(-)$ of homotopic complexes are quasi-isomorphic complexes of $D^-(\mathbf{C})$, we obtain the claimed quasi-isomorphism. \square

Remark 2.22. The use of Quillen's Embedding Theorem and Mitchell's Embedding Theorem in Lemma 2.21 is a bit disappointing. It is possible to write a more explicit proof of the lemma but it requires entering some detailed computations that do not fit well into this paper. We plan to write them down in future work.

We can finally reformulate the condition of being a cover for the homotopy Zariski topology as follows.

Theorem 2.23. *[Derived Tate's acyclicity] Let $\{\operatorname{Spec}(A_i) \rightarrow \operatorname{Spec}(B)\}_{i \in I}$ be a finite family of homotopy Zariski open embeddings. Then, the family is a cover if and only if the associated augmented derived Čech-Tate complex*

$$(2.23.1) \quad \operatorname{Tot} (0 \rightarrow B \rightarrow \prod_{i \in I} A_i \rightarrow \prod_{i,j \in I} A_i \overline{\otimes}_B^{\mathbb{L}} A_j \rightarrow \cdots).$$

is strictly acyclic.

Proof. If the complex of (2.23.1) is strictly exact and $M \rightarrow M'$ is a morphism in $\infty\text{-Mod}_B$ such that $M \otimes_B^{\mathbb{L}} A_i \rightarrow M' \otimes_B^{\mathbb{L}} A_i$ is an equivalence for all i , then

$$\begin{aligned} M &\cong M \otimes_B^{\mathbb{L}} \text{Tot} \left(\prod_{i \in I} A_i \rightarrow \prod_{i, j \in I} A_i \otimes_B^{\mathbb{L}} A_j \rightarrow \cdots \right) \cong \text{Tot} \left(\prod_{i \in I} M \otimes_B^{\mathbb{L}} A_i \rightarrow \prod_{i, j \in I} M \otimes_B^{\mathbb{L}} A_i \otimes_B^{\mathbb{L}} A_j \rightarrow \cdots \right) \cong \\ &\cong \text{Tot} \left(\prod_{i \in I} M' \otimes_B^{\mathbb{L}} A_i \rightarrow \prod_{i, j \in I} M' \otimes_B^{\mathbb{L}} A_i \otimes_B^{\mathbb{L}} A_j \rightarrow \cdots \right) \cong M'. \end{aligned}$$

The converse implication is an application of the Barr-Beck Theorem. An easy way to see this is to notice that the condition of strict exactness of the augmented derived Čech-Tate complex⁸ is equivalent to the condition of $\prod_{i \in I} A_i$ admitting descent of [24, Proposition 3.20]. \square

Theorem 2.23 gives a computational way of checking the cover condition of Definition 2.19 but still the complex of equation (2.23.1) is not easy to compute in practice. Much easier is to use the alternating version of the complex.

Corollary 2.24. *With the same notation and hypothesis of Theorem 2.23, the augmented alternating derived Čech-Tate complex*

$$(2.24.1) \quad \text{Tot} \left(0 \rightarrow B \rightarrow \prod_{i \in I} A_i \rightarrow \prod_{i < j \in I} A_i \otimes_B^{\mathbb{L}} A_j \rightarrow \cdots \right).$$

is strictly acyclic if and only if the family of morphisms is a cover.

Proof. The corollary is just a combination of Theorem 2.23 and Lemma 2.21. \square

2.4. The homotopy Zariski site. In the previous section, we have proved that the notions of homotopy Zariski open immersion and homotopy Zariski cover define a Grothendieck topology on $\infty\text{-Aff}(\mathbf{C})$. This means that by fixing any object $X \in \infty\text{-Aff}(\mathbf{C})$, we can restrict this topology to the slice category of objects over X . Using the duality $\infty\text{-Aff}(\mathbf{C}) = \infty\text{-Comm}(\mathbf{C})^{\text{op}}$ this is the same as contravariantly associating an ∞ -site to any object of $\infty\text{-Comm}(\mathbf{C})$. We give a name to this site.

Definition 2.25. *Let $A \in \infty\text{-Comm}(\mathbf{C})$ we define the homotopy Zariski ∞ -site associated to A as the ∞ -site obtained by restricting the homotopy Zariski topology of $\infty\text{-Aff}(\mathbf{C})$ to the slice category over $\text{Spec}(A)$. We denote it by Zar_A .*

In our setting the category \mathbf{C} is small with respect to a bigger strongly inaccessible cardinal than the one we fixed from the beginning. Therefore, the next theorem holds true.

Theorem 2.26. *If $A \in \infty\text{-Comm}(\mathbf{C})$, then the ∞ -topos Zar_A defined by the (formal) homotopy Zariski topology has enough points.*

Proof. This is a particular case of Theorem 4.1 of [21] because the (formal) homotopy Zariski topology is finitary (cf. Definition 3.17 of [21]). \square

One issue of Theorem 2.26 is in the word “formal” which we usually ignore in our discussions. The morphisms of $\infty\text{-Aff}_A$ do not have any size restriction, therefore this class of morphisms seems difficult to describe in full generality. One of the main goals of this work is to fix this issue in the case when A is a non-Archimedean Banach ring, or a bornological ring, by finding an explicitly describable canonical sub- ∞ -site of $\infty\text{-Aff}_A$ and relate the ∞ -site we obtain with

⁸Notice that what we called the augmented derived Čech-Tate complex agrees with what in [24] is called *augmented cobar resolution*.

the adic spectrum of A . The next, easy, proposition is a step in proof that the homotopy Zariski topology is well-defined, and thus already used so far. We find it convenient to single this property out for referring to it.

Proposition 2.27. *Let $\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(B)$ in $\infty\text{-}\mathbf{Aff}(\mathbf{C})$ be a homotopy Zariski open immersion, then for any $\mathrm{Spec}(C) \rightarrow \mathrm{Spec}(B)$ the homotopy base change $\mathrm{Spec}(A) \times_{\mathrm{Spec}(B)}^{\mathbb{R}} \mathrm{Spec}(C) \rightarrow \mathrm{Spec}(B)$ is a homotopy Zariski open immersion.*

Proof. By the duality $\infty\text{-}\mathbf{Aff}(\mathbf{C}) \cong \infty\text{-}\mathbf{Comm}(\mathbf{C})^{\mathrm{op}}$, homotopy pullbacks in $\infty\text{-}\mathbf{Aff}(\mathbf{C})$ correspond to homotopy pushouts in $\infty\text{-}\mathbf{Comm}(\mathbf{C})$. The latter are computed via the derived tensor product. Thus, if we have $B \rightarrow A$ is a homotopy Zariski open localization and we are given a $B \rightarrow C$, then

$$(C \overline{\otimes}_B^{\mathbb{L}} A) \overline{\otimes}_C^{\mathbb{L}} (C \overline{\otimes}_B^{\mathbb{L}} A) \cong (A \overline{\otimes}_B^{\mathbb{L}} C) \overline{\otimes}_C^{\mathbb{L}} (C \overline{\otimes}_B^{\mathbb{L}} A) \cong A \overline{\otimes}_B^{\mathbb{L}} C \overline{\otimes}_B^{\mathbb{L}} A \cong C \overline{\otimes}_B^{\mathbb{L}} A \overline{\otimes}_B^{\mathbb{L}} A \cong C \overline{\otimes}_B^{\mathbb{L}} A$$

proving that $C \rightarrow C \overline{\otimes}_B^{\mathbb{L}} A$ is a homotopy Zariski open localization. \square

Since the condition

$$A \overline{\otimes}_B^{\mathbb{L}} A \xrightarrow{\cong} A$$

of Definition 2.19 remind the condition for the map $B \rightarrow A$ being a epimorphism, we will often use the name *homotopy epimorphism* to refer to homotopy Zariski open localizations. We also are interested in understanding homotopy filtered colimits in $\infty\text{-}\mathbf{Comm}(\mathbf{C})$.

Proposition 2.28. *Let $(\mathbf{C}, \overline{\otimes})$ be a closed symmetric monoidal quasi-abelian category as above. Let $\{f_i : A_i \rightarrow B_i\}_{i \in I}$ be a filtered family of homotopy epimorphisms in $\infty\text{-}\mathbf{Comm}(\mathbf{C})$, then*

$$\mathbb{L} \lim_{\rightarrow, i \in I} f_i : \mathbb{L} \lim_{\rightarrow, i \in I} A_i \rightarrow \mathbb{L} \lim_{\rightarrow, i \in I} B_i$$

is a homotopy epimorphism.

Proof. For any i we have that the morphism

$$A_i \overline{\otimes}_{B_i}^{\mathbb{L}} A_i \rightarrow A_i$$

induced by f_i is an isomorphism in the homotopy category. Therefore, we get an equivalence

$$\mathbb{L} \lim_{\rightarrow, i \in I} A_i \overline{\otimes}_{B_i}^{\mathbb{L}} A_i \rightarrow \mathbb{L} \lim_{\rightarrow, i \in I} A_i$$

and since $\overline{\otimes}_{B_i}^{\mathbb{L}}$ commutes with $\mathbb{L} \lim_{\rightarrow}$ and the colimit is filtered we get an isomorphism

$$(\mathbb{L} \lim_{\rightarrow, i \in I} A_i) \overline{\otimes}_{\mathbb{L} \lim_{\rightarrow, i \in I} B_i}^{\mathbb{L}} (\mathbb{L} \lim_{\rightarrow, i \in I} A_i) \rightarrow \mathbb{L} \lim_{\rightarrow, i \in I} A_i$$

because of the functorial isomorphism

$$\mathbb{L} \lim_{\rightarrow, i \in I} (-) \overline{\otimes}_{B_i}^{\mathbb{L}} (-) \cong (-) \overline{\otimes}_{\mathbb{L} \lim_{\rightarrow, i \in I} B_i}^{\mathbb{L}} (-).$$

\square

3. QUASI-ABELIAN CATEGORIES FOR ANALYTIC GEOMETRY

In this section, we consider particular cases of the symmetric monoidal quasi-abelian categories discussed in Section 2 that are relevant in analytic geometry. These categories are the category of Banach modules, the contracting category of Banach modules, and the category of (complete) bornological modules. We now recall their definitions and basic properties.

3.1. The category of Banach modules. Let R be a Banach ring. By this we mean that R is a ring equipped with a Banach norm such that the multiplication and addition maps are bounded morphisms of Banach abelian groups (more precisely the addition is supposed to be a contracting map, *i.e.* the triangle inequality holds). In this work we also suppose R to be non-Archimedean and that Banach modules over R are equipped with a non-Archimedean norm although none of these restrictions are necessary for the theory to work. We will comment more on the differences between the general case and the non-Archimedean case when these occur later on. Therefore, from now on Banach rings or modules are always supposed equipped with a non-Archimedean norm if not stated otherwise. Thus, from here on, to simplify the notation, \mathbf{Ban}_R will denote what in the previous section has been denoted by $\mathbf{Ban}_R^{\text{na}}$.

The category of (non-Archimedean) Banach rings has an initial object given by $\mathbb{Z}_{\text{triv}} = (\mathbb{Z}, |-|_0)$ where $|-|_0$ is the *trivial norm* that assumes the value 1 on all $n \neq 0$ ⁹. The category \mathbf{Ban}_R of *Banach R -modules* is defined as the category of Banach abelian groups¹⁰ equipped with a bounded R -action and bounded morphisms between them. The *completed projective tensor product* of two objects $M, N \in \mathbf{Ban}_R$ is defined as

$$M \widehat{\otimes}_R N = \widehat{(M \otimes_R N, |-|_{M \otimes_R N})}$$

where $\widehat{(-)}$ denotes the separated completion and

$$|x|_{M \otimes_R N} = \inf \left\{ \max |a_i| |b_i| \mid x = \sum a_i \otimes b_i \right\}$$

for any $x \in M \otimes_R N$. We recall the following result.

Proposition 3.1. *The category \mathbf{Ban}_R is quasi-abelian. Moreover, the monoidal structure given by the completed projective tensor product is closed, and its right adjoint is given by the hom-sets equipped with the Banach R -module structure given by the operator norm.*

Proof. cf. [2, Proposition 3.15 and Proposition 3.17]. □

The main drawback of the category \mathbf{Ban}_R is that it does not have any infinite products nor any infinite coproducts. To remedy this issue we will introduce the category of bornological modules. Other choices are possible, but we hope to convince the reader that this is the best choice (known to the authors) for our goals. We now introduce flatness in the context of Banach modules.

Definition 3.2. *We say that a Banach R -module M is flat if the functor $(-)\widehat{\otimes}_R M$ is strictly exact (in the sense of Definition 2.2).*

More explicitly, $M \in \mathbf{Ban}_R$ is flat if the functor $(-)\widehat{\otimes}_R M$ preserves the kernel of any morphism. In the next section, when we will study the contracting category of Banach modules, we will prove that all projective objects of \mathbf{Ban}_R are flat (cf. Proposition 3.11) and in particular $\widehat{\otimes}_R$ -acyclic (as a consequence of Corollary 2.13). Therefore, \mathbf{Ban}_R has enough $\widehat{\otimes}_R$ -acyclic objects and it follows from Proposition 2.17 that the inclusion functor $\mathbf{Ban}_R \rightarrow \text{LH}(\mathbf{Ban}_R)$ is lax monoidal and that its adjoint $\text{LH}(\mathbf{Ban}_R) \rightarrow \mathbf{Ban}_R$ is strongly monoidal.

The next category that we describe is the contracting category of Banach modules.

⁹In the category of all Banach rings the initial object is $\mathbb{Z}_{\text{ar}} = (\mathbb{Z}, |-|_{\infty})$, the ring of integers equipped with the *Euclidean norm*.

¹⁰Recall that we are now restricting ourselves only to non-Archimedean definitions, therefore in this context Banach abelian group means a Banach abelian group equipped with an ultrametric norm.

3.2. The contracting category of Banach modules. Let R be a (non-Archimedean) Banach ring.

Definition 3.3. *The contracting category of Banach R -modules is the subcategory $\mathbf{Ban}_R^{\leq 1} \subset \mathbf{Ban}_R$ where the hom-sets are given by considering only contracting morphisms.*

Notice that the categories $\mathbf{Ban}_R^{\leq 1}$ and \mathbf{Ban}_R have the same class of objects and they only differ for the hom-sets. Moreover, isomorphism classes of objects in $\mathbf{Ban}_R^{\leq 1}$ and \mathbf{Ban}_R differ because in the former category, modules are isomorphic if and only if they are isometrically isomorphic whereas in the latter isomorphic modules are equipped with equivalent norms.

Proposition 3.4. *The category $\mathbf{Ban}_R^{\leq 1}$ is quasi-abelian and is complete and cocomplete. Moreover, the closed monoidal structure of \mathbf{Ban}_R restrict to a well-defined closed monoidal structure on $\mathbf{Ban}_R^{\leq 1}$.*

Proof. This can be checked in the same way one checks that \mathbf{Ban}_R is quasi-abelian noticing that the property that all norms involved must be ultrametric is necessary to ensure that the hom-sets are abelian groups. We omit the details. A proof of the fact that $\mathbf{Ban}_R^{\leq 1}$ has all limits and colimits can be found in [2, Proposition 3.21]. The assertion about the closed monoidal structure immediately follows from the explicit definitions given by the formulas for the norms of both the tensor product and the operator norm. \square

Remark 3.5. The property of $\mathbf{Ban}_R^{\leq 1}$ being quasi-abelian is a very distinctive feature of ultrametric Banach rings. Indeed, if R is not equipped with an ultrametric norm then $\mathbf{Ban}_R^{\leq 1}$ is not an additive category, but besides the lack of additivity, it has all the other properties discussed so far. We do not discuss this version of the theory as the non-additivity of $\mathbf{Ban}_R^{\leq 1}$ would force us to introduce more abstract constructions that will lead us too far astray from the main results of this work.

We give an explicit description of products and coproducts in $\mathbf{Ban}_R^{\leq 1}$.

Proposition 3.6. *Let $\{M_i\}_{i \in I}$ be a small family of objects of $\mathbf{Ban}_R^{\leq 1}$. Then, their coproduct is given by*

$$\coprod_{i \in I}^{\leq 1} M_i = \{(m_i) \in \prod_{i \in I} M_i \mid \lim_{i \in I} |m_i| = 0\},$$

equipped with the sup-norm and the product is given by

$$\prod_{i \in I}^{\leq 1} M_i = \{(m_i) \in \prod_{i \in I} M_i \mid \sup_{i \in I} |m_i| < \infty\},$$

equipped with the sup-norm.

Proof. See [2, Proposition 3.21]. \square

Definition 3.7. *A Banach ring R is said uniform if its norm is equivalent to the spectral semi-norm.¹¹*

For uniform Banach rings we usually suppose that the norm is equal to the spectral norm as this holds up to isomorphism in $\mathbf{Ban}_{\mathbb{Z}_{\text{triv}}}$.

¹¹The definition of uniform ring can be given in several equivalent ways and the one given here is equivalent to any other one the reader may know.

Definition 3.8. We define the (1-dimensional) Tate algebra¹² over \mathbb{Z}_{triv} of radius ρ as

$$T_{\mathbb{Z}_{\text{triv}}}(\rho) = \mathbb{Z}_{\text{triv}}\langle \rho^{-1}T \rangle = \left\{ \sum_{i=0}^{\infty} a_i T^i \in \mathbb{Z}[[T]] \mid \lim_{i \rightarrow \infty} |a_i|_0 \rho^i = 0 \right\}.$$

For any Banach ring R the (1-dimensional) Tate algebra over R of radius ρ is defined as

$$T_{\mathbb{Z}_{\text{triv}}}(\rho) \widehat{\otimes}_{\mathbb{Z}_{\text{triv}}} R = R\langle \rho^{-1}T \rangle = \left\{ \sum_{i=0}^{\infty} a_i T^i \in R[[T]] \mid \lim_{i \rightarrow \infty} |a_i|_0 \rho^i = 0 \right\}.$$

The definition of Tate algebra can be easily generalized to any finite set of variables just by inductively define

$$T_{\mathbb{Z}_{\text{triv}}}(\rho_1, \dots, \rho_n) = \mathbb{Z}_{\text{triv}}\langle \rho_1^{-1}T_1, \dots, \rho_n^{-1}T_n \rangle \doteq \mathbb{Z}_{\text{triv}}\langle \rho_1^{-1}T_1, \dots, \rho_{n-1}^{-1}T_{n-1} \rangle \langle \rho_n^{-1}T_n \rangle.$$

Then, we define

$$T_R(\rho_1, \dots, \rho_n) = T_{\mathbb{Z}_{\text{triv}}}(\rho_1, \dots, \rho_n) \widehat{\otimes}_{\mathbb{Z}_{\text{triv}}} R.$$

We now study flatness properties of projective objects of \mathbf{Ban}_R . We recall from Section 1 of [4] that a normed set is a pointed set (X, \star) equipped with a function $|\cdot|_X : X \rightarrow \mathbb{R}_{\geq 0}$ such that $|x|_X = 0 \iff x = \star$ for $x \in X$. For any normed set $(X, |\cdot|_X)$ we define the *topologically free* Banach R -module

$$(3.8.1) \quad c^0(X) = \prod_{\star \neq x \in X}^{\leq 1} R_{|x|_X}$$

where $R_{|x|_X}$ denotes R considered as a Banach module over itself where the norm has been rescaled by the real number $|x|_X$. For example, if $(X, |\cdot|_X) = (\mathbb{N}, |\cdot|_0)$, where $|n|_0 = 1$ for all $n \in \mathbb{N}$, then

$$c^0(\mathbb{N}) = \left\{ (a_n) \in R^{\mathbb{N}} \mid \lim_{n \rightarrow \infty} |a_n| = 0 \right\}$$

equipped with the max norm.

Proposition 3.9. For any normed set $(X, |\cdot|_X)$ the Banach R -module $c^0(X)$ is projective in \mathbf{Ban}_R .

Proof. It is easy to see that the objects $R_{|x|_X}$ are projective in R and coproducts of projective objects are projective objects. For more details see [2, Lemma 3.26]. \square

Remark 3.10. We warn the reader that the objects $c^0(X)$ are not projective in $\mathbf{Ban}_R^{\leq 1}$. Therefore, \mathbf{Ban}_R and $\mathbf{Ban}_R^{\leq 1}$ are very different from the point of view of homological algebra.

In particular, for any object $M \in \mathbf{Ban}_R$ we can consider

$$c^0(M) = \prod_{0 \neq m \in M}^{\leq 1} R_{|m|}$$

where M is considered as a normed set by forgetting his R -module structure.

Proposition 3.11. The canonical morphism $c^0(M) \rightarrow M$ is a strict epimorphism in $\mathbf{Ban}_R^{\leq 1}$. In particular, \mathbf{Ban}_R has enough projective objects.

Proof. See [2, Lemma 3.27]. \square

¹²Here we discuss only the non-Archimedean version of the theory but with suitable changes one can develop a theory that works uniformly over all Banach rings. See [1], [2], [3] and [4] where all Banach rings are considered and in particular [6] for a more in-depth analysis.

Proposition 3.11 immediately implies the following corollary.

Corollary 3.12. *All projective objects of \mathbf{Ban}_R are direct summands of some $c^0(X)$.*

Proof. Projective objects splits strict epimorphism. Therefore, if P is projective then the strict epimorphism $c^0(P) \rightarrow P$ splits. \square

Now that we know how projective objects of \mathbf{Ban}_R look like we are ready to prove that they are flat.

Proposition 3.13. *In \mathbf{Ban}_R projective objects are flat.*¹³

Proof. By Corollary 3.12 we need to check only that projective objects of the form $c^0(X)$, for some normed set $(X, |\cdot|_X)$, are flat. Since $(-)\widehat{\otimes}_R(-)$ is a left adjoint functor it is a strictly right exact functor. Hence, by Proposition 2.3 to check that $c^0(X)$ is flat we only need to check that the functor $(-)\widehat{\otimes}_R c^0(X)$ preserves monomorphisms and strict monomorphisms.

Let $f : M \rightarrow N$ be a monomorphism in \mathbf{Ban}_R , i.e. this means that f is an injective map. Then, since the functor $\widehat{\otimes}_R$ commutes with contracting colimits (as it is a left adjoint functor)

$$c^0(X)\widehat{\otimes}_R M = \prod_{* \neq x \in X}^{\leq 1} R_{|x|_X} \widehat{\otimes}_R M = \prod_{* \neq x \in X}^{\leq 1} M_{|x|_X}$$

where the notation is self-explanatory enough, and the same for N . Therefore, from this explicit description, it is clear that the induced map

$$\text{Id} \widehat{\otimes} f : c^0(X)\widehat{\otimes}_R M \rightarrow c^0(X)\widehat{\otimes}_R N$$

is injective. Now, let us suppose that f is a strict monomorphism in \mathbf{Ban}_R . We need to show that the induced injective morphism

$$\prod_{* \neq x \in X}^{\leq 1} M_{|x|_X} \rightarrow \prod_{* \neq x \in X}^{\leq 1} N_{|x|_X}$$

has closed image. This follows immediately from the fact that the inclusion $i_x : N_{|x|_X} \rightarrow \prod_{* \neq x \in X}^{\leq 1} N_{|x|_X}$ has a bounded inverse $\pi_x : \prod_{* \neq x \in X}^{\leq 1} N_{|x|_X} \rightarrow N_{|x|_X}$ and therefore every Cauchy sequence of elements of $\prod_{* \neq x \in X}^{\leq 1} M_{|x|_X}$ with respect to the norm on $\prod_{* \neq x \in X}^{\leq 1} N_{|x|_X}$ is sent to a Cauchy sequence in $N_{|x|_X}$ and therefore it converges to an element of $M_{|x|_X}$. \square

Corollary 3.14. *All projective objects of \mathbf{Ban}_R are $\widehat{\otimes}$ -acyclic.*

Proof. Proposition 3.13 implies that for all projective objects $P \in \mathbf{Ban}_R$ the functor $(-)\widehat{\otimes}_R P$ is strictly exact. Therefore, by Corollary 2.13 P is $\widehat{\otimes}$ -acyclic. \square

This corollary immediately implies the following.

Corollary 3.15. *The inclusion functor $\mathbf{Ban}_R \rightarrow \text{LH}(\mathbf{Ban}_R)$ is lax monoidal and its adjoint is strictly monoidal.*

Proof. By Corollary 3.14 \mathbf{Ban}_R has enough $\widehat{\otimes}$ -acyclic objects, then we can apply Proposition 2.17. \square

¹³In our previous related works (like [2], [4] and [9]) flatness of Banach modules has been discussed and the flatness of projective objects has been considered but in those works an object $P \in \mathbf{Ban}_R$ was called flat if the functor $(-)\widehat{\otimes}_R P$ is exact. Therefore, the terminology of this work is not compatible with the one used in those works and the results proved here are strengthenings of the previous ones. From the point of view of this work, the older notion of flatness should be considered as a *weak flatness* and is well-suited for computing the derived functor of $\widehat{\otimes}$, as it was the goal of our previous works. The stronger notion of flatness used in this work is needed for a deeper understanding of the monoidal structures and more refined results like Corollary 3.15.

Proposition 3.13 has the following direct consequence.

Proposition 3.16. *The Banach rings $T_R(\rho_1, \dots, \rho_n)$ are flat over R .*

Proof. It is easy to check that

$$T_R(\rho_1, \dots, \rho_n) \cong c^0((\mathbb{N}^n \cup \{\star\}, | \cdot |_{\rho_1, \dots, \rho_n}))$$

for the pointed normed monoid $(\mathbb{N}^n \cup \{\star\}, | \cdot |_{\rho_1, \dots, \rho_n})$, where $| \cdot |_{\rho_1, \dots, \rho_n}$ is the norm

$$|(m_1, \dots, m_n)|_{\rho_1, \dots, \rho_n} = \rho_1^{m_1} \cdots \rho_n^{m_n}.$$

Therefore $T_R(\rho_1, \dots, \rho_n)$ is a projective Banach R -module, and hence flat by Proposition 3.13. \square

We have already mentioned that the completion functor $\widehat{(-)} : \mathbf{Nr}_R \rightarrow \mathbf{Ban}_R$ is not strictly exact. A consequence of this fact is the following remark.

Remark 3.17. In $\mathbf{Ban}_R^{\leq 1}$ filtered colimits are not strictly exact, in general. Indeed, if $\{M_i\}_{i \in I}$ is a contracting directed system of Banach R -modules, then $\lim_{\substack{\rightarrow \\ i \in I}}^{\leq 1} M_i$ can be computed by applying

the functor $\widehat{(-)}$ to the colimit computed in \mathbf{Nr}_R . Although colimits in \mathbf{Nr}_R are strictly exact, the functor $\widehat{(-)}$ destroys the strict left exactness.

3.3. The category of bornological modules. For a Banach ring¹⁴ R we can consider the category $\mathbf{Ind}(\mathbf{Ban}_R)$, the category of ind-objects of \mathbf{Ban}_R . The category $\mathbf{Ind}(\mathbf{Ban}_R)$ can be defined as the full subcategory of presheaves on \mathbf{Ban}_R whose objects are presheaves that can be written as filtered colimits of representable presheaves. It is easy to see that $\mathbf{Ind}(\mathbf{Ban}_R)$ is quasi-abelian and that it admits all limits and colimits. We call the full subcategory $\mathbf{Born}_R \subset \mathbf{Ind}(\mathbf{Ban}_R)$ consisting of essentially monomorphic objects the category of (*complete*) *bornological R -modules* (we will usually omit the adjective complete as we will only consider complete bornological modules in this work). Recall that an object of $\mathbf{Ind}(\mathbf{Ban}_R)$ is called essentially monomorphic if it can be written as a filtered colimit of representable presheaves whose system morphisms are monomorphisms. Again, it is not hard to see that \mathbf{Born}_R is quasi-abelian and that \mathbf{Born}_R is a reflective subcategory of $\mathbf{Ind}(\mathbf{Ban}_R)$. This implies that also \mathbf{Born}_R has all limits and colimits. Both \mathbf{Born}_R and $\mathbf{Ind}(\mathbf{Ban}_R)$ come naturally equipped with a closed symmetric monoidal structure induced by the one of \mathbf{Ban}_R . In particular, if we write “ $\lim_{\substack{\rightarrow \\ I}} X_i$ ” and “ $\lim_{\substack{\rightarrow \\ J}} Y_j$ ” for objects of

$\mathbf{Ind}(\mathbf{Ban}_R)$ then

$$\lim_{\substack{\rightarrow \\ I}} X_i \widehat{\otimes}_R \lim_{\substack{\rightarrow \\ J}} Y_j = \lim_{\substack{\rightarrow \\ I \times J}} X_i \widehat{\otimes}_R Y_j.$$

We choose to work with \mathbf{Born}_R because it is a more manageable category. Moreover, since \mathbf{Born}_R is a symmetric monoidal category it makes sense to consider the category $\mathbf{Comm}(\mathbf{Born}_R)$ of algebras over \mathbf{Born}_R that we call the category of *bornological algebras over R* . In this way, to any bornological ring it is possible to associate its category of bornological modules that is canonically a closed symmetric monoidal quasi-abelian category. We now give some relevant examples of bornological rings and modules.

Example 3.18. (1) Banach rings and modules are particular cases of bornological rings and modules.

¹⁴In this subsection there is no reason to restrict the discussion to non-Archimedean Banach rings but we still keep this hypothesis for consistency with the rest of the paper.

- (2) In the case when the base ring is a complete (non-trivially) valued field k our definition of bornological k -modules is equivalent to the classical definition of (complete) bornological spaces of convex type over k (see [26], [1] and [2] for more information about this equivalence of notions).
- (3) Many topological algebras that appear in literature can be seen canonically as bornological algebras as Fréchet algebras, direct limit of Banach algebras, and more. Moreover, for (essentially all of) these algebras the bornological and topological point of view are essentially equivalent.
- (4) In [4] there are several examples of Fréchet-like algebras appearing in arithmetic that are not defined over any base field.

The following property is non-obvious because the inclusion functor $\mathbf{Born}_R \rightarrow \mathbf{Ind}(\mathbf{Ban}_R)$ is not a monoidal functor in general.

Proposition 3.19. *The categories $\mathbf{Ind}(\mathbf{Ban}_R)$ and \mathbf{Born}_R are tensor derived equivalent.*

Proof. The classes of projective objects of \mathbf{Born}_R and $\mathbf{Ind}(\mathbf{Ban}_R)$ agree and, reasoning like Proposition 3.11, one can see that they have enough projective flat objects (*cf.* [2, Lemma 3.29] for a description of the class of projective objects of $\mathbf{Ind}(\mathbf{Ban}_R)$ that, incidentally, are objects of \mathbf{Born}_R). There is a pair of adjoint functors

$$\lim_{\rightarrow} : \mathbf{Ind}(\mathbf{Ban}_R) \rightleftarrows \mathbf{Born}_R : \text{diss}$$

where diss is just the inclusion and \lim_{\rightarrow} computes the direct limit¹⁵ of the ind-objects in \mathbf{Born}_R . Since filtered colimits in \mathbf{Born}_R are strictly exact, the functor \lim_{\rightarrow} canonically induces a derived functor $\mathbb{L}\lim_{\rightarrow}$ that has no higher derived functors for the left t-structure and moreover \lim_{\rightarrow} is a monoidal functor. Therefore, this gives a tensor triangulated equivalence $D^-(\mathbf{Born}_R) \cong D^-(\mathbf{Ind}(\mathbf{Ban}_R))$ because the classes of projective objects of \mathbf{Born}_R and $\mathbf{Ind}(\mathbf{Ban}_R)$ agree and in both cases the existence of enough projective objects imply that the bounded above derived categories are equivalent to the homotopy category of the additive category of projective objects (*cf.* [27, Proposition 1.3.22]). To promote this equivalence to an equivalence between the unbounded derived categories we notice that any object $X \in D(\mathbf{Ind}(\mathbf{Ban}_R))$ can be written as

$$X \cong \lim_{\rightarrow} \tau_L^{\leq n}(X)$$

where $\tau_L^{\leq n}$ denotes the n -th truncation functor for the left t-structure (the same is true for $D(\mathbf{Born}_R)$). Since $\widehat{\otimes}_R^{\mathbb{L}}$ commutes with direct limits and $\mathbb{R}\text{diss}$ is a triangulated functor (actually a triangulated equivalence) we get that for all $X, Y \in D(\mathbf{Ind}(\mathbf{Ban}_R))$

$$\begin{aligned} \lim_{\rightarrow} (X \widehat{\otimes}_R^{\mathbb{L}} Y) &\cong \lim_{\rightarrow} (\lim_{\rightarrow} \tau_L^{\leq n}(X) \widehat{\otimes}_R^{\mathbb{L}} \lim_{\rightarrow} \tau_L^{\leq n}(Y)) \cong \lim_{\rightarrow} \lim_{\rightarrow} (\tau_L^{\leq n}(X) \widehat{\otimes}_R^{\mathbb{L}} \tau_L^{\leq n}(Y)) \cong \\ &\cong \lim_{\rightarrow} \lim_{\rightarrow} (\tau_L^{\leq n}(X)) \widehat{\otimes}_R^{\mathbb{L}} \lim_{\rightarrow} \lim_{\rightarrow} (\tau_L^{\leq n}(Y)) \cong \lim_{\rightarrow} (X) \widehat{\otimes}_R^{\mathbb{L}} \lim_{\rightarrow} (Y). \end{aligned}$$

□

¹⁵It is not difficult to show that the category \mathbf{Born}_R has all limits and colimits. In the case when R is a non-trivially valued field then the result is well-known. For the general case we refer to the appendix of [5] where a general result about the existence of limits and colimits in the category of essentially monomorphic ind-objects is proved.

Proposition 3.19 can be interpreted by saying that the derived geometries relative to $\mathbf{Ind}(\mathbf{Ban}_R)$ and relative to \mathbf{Born}_R (in the sense of Section 2) are equivalent. Proposition 3.16 has the following direct consequences.

Proposition 3.20. *Let $0 \leq \rho \leq \infty$. The bornological algebras*

$$\lim_{\substack{\rightarrow \\ \rho' > \rho}} T_{\mathbb{Z}_{\text{triv}}(\rho') = \mathbb{Z}_{\text{triv}}\langle \rho^{-1}T \rangle^\dagger$$

and

$$\lim_{\substack{\leftarrow \\ \rho' < \rho}} T_{\mathbb{Z}_{\text{triv}}(\rho') = \mathbb{Z}_{\text{triv}}\langle \rho^{-1}T \rangle^\circ$$

are flat over \mathbb{Z}_{triv} .

Proof. Since the functor $\lim_{\substack{\rightarrow \\ \rho' > \rho}}$ is strictly exact, then the first claim follows immediately from Proposition 3.16. The second claim follows from the fact that the projective limit $\lim_{\substack{\leftarrow \\ \rho' < \rho}}$ is nuclear in the sense of Appendix A of [4] and hence it can be written as direct limit of topologically free Banach R -modules and hence it is flat. \square

4. THE HOMOTOPICAL HUBER SPECTRUM OF BANACH AND BORNLOGICAL RINGS

In this section, we show how to enhance the space $\text{Spa}(R)$, for any Banach algebra¹⁶ R over a strongly Noetherian Tate ring¹⁷, to a space that can be equipped with a structural derived sheaf of simplicial Banach algebras. This structural derived sheaf is concentrated in degree 0 and agrees with the usual structural sheaf of $\text{Spa}(R)$ when R lies in specific classes of well-behaved Banach algebras, like the stably uniform studied by Buzzard-Mihara-Verberkmoes. A possible interpretation of the fact that in general one obtains simplicial Banach algebras as sections of the structural sheaf is that it is a consequence of the fact that the functor $\lim_{\substack{\rightarrow \\ \rho' < \rho}}^{\leq 1}$ is not strictly exact.

This section is divided in four parts: in the first part we review (and reinterpret) the classical theory of affinoid algebras over a strongly Noetherian Tate ring A , then we explain how to extend the theory to (almost) any Banach algebra over A , subsequently we show how to construct a homotopical version for the Huber spectrum from this results and finally we show how these results can be further generalized to more general bornological rings.

Before discussing our results, we briefly recall some basic definitions of the theory of the Huber spaces associated to a non-Archimedean Banach ring. Let $(R, |\cdot|)$ be a Banach ring. Let $|\cdot|_{\text{sup}}$ be the spectral norm of R . To the pair $(R, |\cdot|_{\text{sup}})$ one can associate the Huber ring

$$\mathcal{R} = (R, R^\circ)$$

where

$$R^\circ = \{r \in R \mid |r|_{\text{sup}} \leq 1\}.$$

is the set of power-bounded elements of R . This association is functorial and permits to associate to $(R, |\cdot|)$ the affinoid adic space associated to \mathcal{R} that we will denote by $\text{Spa}(\mathcal{R})$ or simply $\text{Spa}(R)$. The points of $\text{Spa}(R)$ are equivalence classes of continuous semi-valuations $v : R \rightarrow \Gamma$

¹⁶We keep considering only non-Archimedean Banach rings.

¹⁷Recall that a Banach ring is called *Tate* if it has a topologically nilpotent unit. We will also refer to such objects as Banach-Tate rings.

to (pointed) ordered abelian groups such that $v(x) \leq 1$ for all $x \in R^\circ$ (where we use the multiplicative notation for Γ). The topology of $\text{Spa}(R)$ is generated by subsets of the form

$$(4.0.1) \quad \{v \in \text{Spa}(R) | v(f_i) \leq v(f_0) \neq 0, f_0, \dots, f_n \in R, (f_0, \dots, f_n) = R\}$$

that are called *rational domains*.

4.1. Localizations of affinoid algebras over strongly Noetherian Tate rings. In this section, we establish some basic facts about affinoid algebras over strongly Noetherian Tate rings. We reinterpret well-known results of Huber in the language of homological algebra over Banach algebras, generalizing our results of [8], [2], proved for affinoid algebras over a valued field. The main result of this section is the interpretation of rational localizations as Koszul commutative dg-algebras with cohomology concentrated in degree 0.

So, in this section, we fix once for all A to be a strongly Noetherian Tate ring. We do not ask A to be defined over a valued field. Recall that a Banach ring is said to be strongly Noetherian if for any $n \in \mathbb{N}$ the Banach algebra $A\langle X_1, \dots, X_n \rangle$ is Noetherian. We are about to give a new perspective to the theory of affinoid rational localizations by presenting them via Koszul resolutions providing simple and explicit flat resolutions of the algebras of analytic functions on rational subdomains of an affinoid adic space. Let us briefly recall what affinoid algebras over A are.

Definition 4.1. *An affinoid algebra over A is a Banach A -algebra R for which there exists an isomorphism of Banach algebras*

$$R \cong \frac{A\langle X_1, \dots, X_n \rangle}{I}$$

where the algebra on the right hand side is equipped with the quotient semi-norm.

The next lemma ensures that Definition 4.1 makes sense.

Lemma 4.2. *Let A be a strongly Noetherian Tate ring, then for all $n \in \mathbb{N}$ the ideals of $A\langle X_1, \dots, X_n \rangle$ are closed.*

Proof. It is well known that the ideals of Noetherian Tate rings are closed, cf. [16, Theorem 2.2.8] and [12]. \square

Since all ideals of $A\langle X_1, \dots, X_n \rangle$ are closed all its quotients have a canonical structure of Banach A -algebras, that are easily to seen to be Tate rings as well. Moreover, notice that the isomorphism of Definition 4.1 is asked to exist in $\mathbf{Comm}(\mathbf{Ban}_A)$, not in $\mathbf{Comm}(\mathbf{Ban}_A^{\leq 1})$, so the algebras considered are isomorphic but not necessarily isometrically isomorphic. We now introduce the notation we will use for the Koszul complexes.

Notation 4.3. Let A be a Banach ring, R a bornological A -algebra and $x \in R$ we denote

$$K_R(x) = [R \xrightarrow{\mu_x} R]$$

the *Koszul complex* of x , where the complex is in degree 0 and -1 and the map μ_x is multiplication by x . For $x_1, \dots, x_n \in R$ we denote

$$K_R(x_1, \dots, x_n) \doteq K_R(x_1) \widehat{\otimes}_A \cdots \widehat{\otimes}_A K_R(x_n).$$

The Koszul complexes $K_R(x_1, \dots, x_n)$ are commutative differential graded algebras on Banach R -modules. One of the goals of this paper is to prove that these complexes can be used to compute the cohomology of the derived structure sheaf on the analytic schemes we have defined applying the abstract theory discussed in Section 2 to the case of the quasi-abelian category of bornological modules over R . But when considering the multiplicative structure on the structure sheaf, some extra care is needed. We explain the problem in the next remark.

Remark 4.4. In Section 2 we used the category of simplicial commutative monoids to define the HAG contexts we considered. The main reason for this choice is that the Dold-Kan correspondence between simplicial objects and chain complexes is not a monoidal functor and it turns out that the tensor product of simplicial objects and that of chain complexes have very different properties. In particular, in the case of bornological modules over R , if the underlying ring of R has characteristic 0, then Dold-Kan is actually symmetric monoidal and both simplicial commutative algebras and commutative differential graded algebras can be used interchangeably. If R does not have characteristic 0, then the category of commutative differential graded algebras is not suitable for homotopy theory. It does not inherit a model structure from the category of chain complexes and therefore it does not even make sense to talk about its homotopy category. More precisely, in this case, the model structure on chain complexes does not satisfy the symmetric monoidal axiom when equipped with the standard tensor product of chain complexes. But the category of simplicial commutative bornological algebras can be equipped with the transferred model structure from the category of simplicial bornological modules and therefore this is the category that we use for defining analytic schemes and stacks. But in the specific case of the objects $K_R(x_1, \dots, x_n)$, these compute the homotopy quotient of the discrete bornological A -algebra $R\langle T_1, \dots, T_n \rangle$ by the ideal (x_1, \dots, x_n) and the homotopy quotient is computed as the homotopy quotient of the underlying modules. Therefore, the complexes $K_R(x_1, \dots, x_n)$ can be used to compute the cohomology (*i.e.* homotopy groups) of the simplicial algebras that represent the homotopy quotient of $R\langle T_1, \dots, T_n \rangle$ by (x_1, \dots, x_n) in all cases, but it does not necessarily have the correct multiplicative structure. For the main results of this paper computing the cohomology of $K_R(x_1, \dots, x_n)$ is enough.

Definition 4.5. We say that the Koszul complex $K_R(x_1, \dots, x_n)$ is regular if $\mathrm{LH}^n(K_R(x_1, \dots, x_n)) = 0$ for all $n \neq 0$. We say that $K_R(x_1, \dots, x_n)$ is strictly regular if it is regular and $\mathrm{LH}^0(K_R(x_1, \dots, x_n)) \in \mathbf{Ban}_R$.

The condition of being Koszul regular means that $K_R(x_1, \dots, x_n)$ is a projective resolution of

$$\mathrm{Coker}(R^n \rightarrow R) \cong \frac{R}{(x_1, \dots, x_n)}$$

where the cokernel is computed in $\mathrm{LH}(\mathbf{Ban}_R)$. Notice that in this case the morphism $R^n \rightarrow R$ may not be strict because it may not have closed image and therefore the quotient is not a Banach ring, but it makes sense as an object of $\mathbf{Comm}(\mathrm{LH}(\mathbf{Ban}_R))$. If $K_R(x_1, \dots, x_n)$ is strictly regular then the morphism $R^n \rightarrow R$ is strict and therefore $K_R(x_1, \dots, x_n)$ is a projective resolution of the Banach ring

$$\frac{R}{(x_1, \dots, x_n)},$$

as in this case the ideal (x_1, \dots, x_n) is closed.

Definition 4.6. Let R be an affinoid algebra over R and $f_0, \dots, f_n \in R$ be such that $(f_0, \dots, f_n) = 1$. We define the associated derived rational localization as the canonical morphism

$$R \rightarrow K_{R\langle X_1, \dots, X_n \rangle}(f_0 X_1 - f_1, \dots, f_0 X_n - f_n) \doteq R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$$

of commutative dg-algebras.

Remark 4.7. We use the notation

$$R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$$

to denote the derived rational localizations in order to distinguish them from the classical (un-derived) rational localizations denoted by

$$R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle = c \left(\text{LH}^0 \left(R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h \right) \right),$$

where $c : \text{LH}(\mathbf{Ban}_R) \rightarrow \mathbf{Ban}_R$ is the classical part functor introduced before Corollary 2.8.

The reader should not be frightened by the appearance of dg-algebras in Definition 4.6. We will soon show that these algebras have cohomology concentrated in degree 0 and agree with the usual definition (up to quasi-isomorphism). But the change of point of view of Definition 4.6 will be important later on when derived rational localizations will not be concentrated in degree 0 for a general Banach ring R .

Inside the class of derived rational localizations the following subclasses are important because of their simplicity that often permits to work out explicit computations.

Definition 4.8. *A derived rational localization of the form*

$$R \rightarrow K_{R\langle X_1, \dots, X_n \rangle} (X_1 - f_1, \dots, X_n - f_n) \doteq R\langle f_1, \dots, f_n \rangle^h$$

with $f_1, \dots, f_n \in R$, is called derived Weierstrass localization. A derived rational localization of the form

$$R \rightarrow K_{R\langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle} (X_1 - f_1, \dots, X_n - f_n, g_1 Y_1 - 1, \dots, g_m Y_m - 1) \doteq R\langle f_1, \dots, f_n, g_1^{-1}, \dots, g_m^{-1} \rangle^h$$

where $f_1, \dots, f_n, g_1, \dots, g_m \in R$, is called derived Laurent localization.

Recall that we keep the hypothesis that A is a strongly Noetherian Banach-Tate ring.

Proposition 4.9. *Derived Weierstrass and derived Laurent localizations of affinoid A -algebras are Koszul strictly regular.*

Proof. Let R be an affinoid A -algebra. As all ideals of $R\langle X_1, \dots, X_n \rangle$ are closed and A is strongly Noetherian, we need only to check that the Koszul complexes that define $R\langle f_1, \dots, f_n \rangle^h$ and $R\langle f_1, \dots, f_n, g_1^{-1}, \dots, g_m^{-1} \rangle^h$ are algebraically exact as morphisms of finite $R\langle X_1, \dots, X_n \rangle$ -modules are automatically strict. By induction we can reduce to the cases $R\langle f^{-1} \rangle^h$ and $R\langle f \rangle^h$. Indeed, suppose that the result is known for $R\langle f_1, \dots, f_n, g_1^{-1}, \dots, g_m^{-1} \rangle^h$. We can then do a double induction on the indexes n and m . So, $R\langle f_1, \dots, f_n, g_1^{-1}, \dots, g_m^{-1} \rangle^h$ is supposed to be quasi-isomorphic to an affinoid A -algebra, let us denote it R' , and hence

$$(4.9.1) \quad R\langle f_1, \dots, f_n, g_1^{-1}, \dots, g_m^{-1} \rangle^h \widehat{\otimes}_R R\langle f \rangle^h \cong [R'\langle X \rangle \xrightarrow{\mu(X-f)} R'\langle X \rangle]$$

in $D^-(\mathbf{Ban}_R)$, and similarly for $R\langle f^{-1} \rangle^h$. But by our inductive hypothesis we know that the complex on the right hand side of (4.9.1) is strictly regular.

Let us consider first $R\langle f^{-1} \rangle^h$. Since the ideal $(fX - 1)$ is closed, we need only to show that the morphism on $R\langle X \rangle$ induced by multiplication by $(fX - 1)$ is injective. This is equivalent to say that the equation

$$(fX - 1)a = 0$$

with $a \in R$ has only $a = 0$ as solution. We can write

$$a = \sum_{n \in \mathbb{N}} a_n X^n$$

with $a_n \in R$ and so the equation

$$(fX - 1) \left(\sum_{n \in \mathbb{N}} a_n X^n \right) = 0$$

gives the set of equations

$$fa_{n-1} - a_n = 0, \quad n \in \mathbb{N}.$$

These equations can be solved recursively starting with

$$a_0 = 0, \quad fa_0 - a_1 = 0 \Rightarrow a_1 = 0, \dots$$

proving the claim. A similar reasoning in the case of the multiplication map induced by $(X - f)$ leads to the system of equations

$$fa_0 = 0, \quad a_{n-1} - fa_n = 0, \quad n \in \mathbb{N}.$$

Therefore, we get that if f is not a zero divisor then $a_0 = 0$ and recursively $a_n = 0$ for all n , whereas if f is a zero divisor then we get that $fa_0 = 0$ has a non-zero solution. Then, solving all the other equations we get the relations

$$a_0 = f^n a_n, \quad \forall n \in \mathbb{N}$$

implying that a_0 must be divisible by all powers of f , i.e. $a_0 \in \bigcap_{n \in \mathbb{N}} (f)^n$. As R is Noetherian we have that $a_n = 0$ for all n . □

We include the next easy lemma for the sake of clarity.

Lemma 4.10. *Let R be an affinoid A algebra. Let $f_1, \dots, f_n \in R$ and $f_0 \in R^\times$, then the derived localization*

$$R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$$

is a derived Weierstrass localization.

Proof. By definition the object

$$R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$$

is the homotopy quotient of the inclusion

$$(f_0 X_1 - f_1, \dots, f_0 X_n - f_n) \hookrightarrow R \langle X_1, \dots, X_n \rangle.$$

But since f_0 is invertible, by multiplying the ideal by f_0^{-1} we get that this equivalent to

$$(X_1 - f_1 f_0^{-1}, \dots, X_n - f_n f_0^{-1}) \hookrightarrow R \langle X_1, \dots, X_n \rangle$$

that is Weierstrass. □

The following lemma is useful for reducing computations about derived rational localizations to computations of derived Laurent localizations.

Lemma 4.11. *Suppose that A has a uniform topologically nilpotent unit, then every derived rational localization of an affinoid A -algebra can be written as a composition of a derived Weierstrass and a derived Laurent localization.*

Proof. We have already proved that derived Weierstrass and derived Laurent localizations are Koszul strictly regular. Therefore, for every $\lambda \in A^\times$ and $f \in A$ we have a quasi-isomorphism

$$R \langle (\lambda f)^{-1} \rangle^h \rightarrow \frac{R \langle X \rangle}{(\lambda f X - 1)},$$

where the object on the right hand side is considered as a complex concentrated in degree 0. Consider a derived rational localization $R \rightarrow R \langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \rangle^h$. Thus, as f_0 is invertible in the

Banach ring $\frac{R\langle X_1, \dots, X_n \rangle}{(f_0 X_1 - f_1, \dots, f_0 X_n - f_n)} = c(\text{LH}^0(R\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \rangle^h))$ (recall that the functor c is strictly monoidal by Corollary 3.15) then

$$\sup_{x \in X} |f_0^{-1}(x)| = M < \infty$$

where we denoted $X = \mathcal{M}(\frac{R\langle X_1, \dots, X_n \rangle}{(f_0 X_1 - f_1, \dots, f_0 X_n - f_n)})$, the Berkovich spectrum. Since A has a uniform topologically nilpotent unit, we can find a $\lambda \in A^\times$ such that

$$\sup_{x \in Y} |\lambda f_0(x)| > M$$

where we denoted $Y = \mathcal{M}(R)$. Therefore,

$$X \subset \mathcal{M}(R\langle (\lambda f_0)^{-1} \rangle^h)$$

and

$$R\langle (\lambda f_0)^{-1} \rangle^h \rightarrow R\langle \frac{\lambda f_1}{\lambda f_0}, \dots, \frac{\lambda f_n}{\lambda f_0} \rangle^h \cong R\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \rangle^h$$

is a derived Weierstrass localization of $R\langle \lambda f_0^{-1} \rangle^h$, because f_0 is invertible in the Banach ring $R\langle \lambda f_0^{-1} \rangle^h$ and hence we can apply Lemma 4.10. In this way we get a morphism of dg-algebras

$$R\langle (\lambda f_0)^{-1} \rangle^h \rightarrow R\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \rangle^h$$

and the composition

$$R \rightarrow R\langle (\lambda f_0)^{-1} \rangle^h \rightarrow R\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \rangle^h$$

shows that every derived rational localization can be written as a composition of a derived Laurent and a derived Weierstrass localization. \square

Remark 4.12. The hypothesis of A having a uniform unit $u \in A^\times$, for which we can suppose $|u^{-1}| > 1$, is necessary to ensure that

$$\lim_{n \rightarrow \infty} |u^{-n} f_0(x)| = \infty$$

that is a necessary step in the strategy of proof of Lemma 4.11. We think that it is possible that better strategies can avoid this assumption.

In particular, Lemma 4.11 shows that all derived rational localizations of affinoid A algebras are strictly Koszul regular. We record this as a corollary.

Corollary 4.13. *In the hypothesis of Lemma 4.11, derived rational localizations of affinoid algebras over A are strictly Koszul regular. Therefore, derived and non-derived localizations agree.*

So, from now on we add the hypothesis that A has a topologically nilpotent uniform unit, in order to be able to apply Corollary 4.13. We are now ready to prove that derived rational localizations of affinoid A -algebras are homotopy Zariski open localizations.

Proposition 4.14. *Derived Weierstrass and derived Laurent localizations of affinoid A -algebras are homotopy epimorphisms.*

Proof. Again by induction we only need to discuss the cases $R\langle f \rangle^h$ and $R\langle f^{-1} \rangle^h$. Let R' denote one of these two Koszul complexes, in both cases we need to prove that

$$R' \widehat{\otimes}_R^{\mathbb{L}} R' \cong R'.$$

Let us first show that $R' \widehat{\otimes}_R^{\mathbb{L}} R' \cong R' \widehat{\otimes}_R R'$. By Proposition 4.9 we know that R' is concentrated in degree 0 and by definition the Koszul complex R' is a projective resolution of $c(\mathrm{LH}^0(R')) \cong \mathrm{LH}^0(R')$. Let us fix $R' = R\langle f \rangle^h$, the other case is done similarly. In this case

$$R' = [0 \rightarrow R\langle X \rangle \xrightarrow{\mu(X-f)} R\langle X \rangle \rightarrow 0]$$

therefore

$$R' \widehat{\otimes}_R^{\mathbb{L}} R' \cong [0 \rightarrow \mathrm{LH}^0(R')\langle X \rangle \xrightarrow{\mu(X-f)} \mathrm{LH}^0(R')\langle X \rangle \rightarrow 0]$$

because $R\langle X \rangle$ is flat over R (where we identified f with its image in $\mathrm{LH}^0(R')$). But this is nothing more than the Koszul complex $\mathrm{LH}^0(R')\langle f \rangle^h$ that is again concentrated in degree 0 because $\mathrm{LH}^0(R')$ is an affinoid A -algebra. So, the derived tensor product is concentrated in degree 0 and finally, since $f \in \mathrm{LH}^0(R')^\circ$ we get that $\mathrm{LH}^0(R')\langle f \rangle^h \cong \mathrm{LH}^0(R')$ proving that $R \rightarrow R'$ is a homotopy epimorphism. \square

Proposition 4.15. *Every derived rational localization is a homotopy epimorphism.*

Proof. Since compositions of homotopy epimorphisms are homotopy epimorphisms, the claim follows from Proposition 4.14 and Lemma 4.11. \square

The next proposition shows that the derived intersection of rational subsets of affinoid spaces over A agrees with the underived intersection.

Proposition 4.16. *Let $R \rightarrow R'$ and $R \rightarrow R''$ be two derived rational localization of an affinoid algebra over A , then*

$$R' \widehat{\otimes}_R^{\mathbb{L}} R'' \cong R' \widehat{\otimes}_R R''$$

Proof. Using Lemma 4.11 and induction we can again reduce to the case when both R' and R'' are of the form $R\langle f \rangle^h$ or $R\langle g^{-1} \rangle^h$ for some $f, g \in R$. Explicit computations as in the proof of Proposition 4.14 immediately prove the claim. \square

Proposition 4.16 can be restated geometrically by saying that the derived intersection of two rational open subsets is (quasi-)isomorphic to their underived intersection *i.e.* that the intersection is transversal. This is a property that one expects to hold for open immersions of a topology, at least as far as one is considering topologies that are expected to have some flatness properties. We now check that the topology of $\mathrm{Spa}(R)$ is compatible with the homotopy Zariski topology introduced in Section 2.

Theorem 4.17. *Let R be an affinoid A -algebra. For any (finite) derived rational cover $\{\mathrm{Spa}(R_i)\}_{i \in I}$ of $\mathrm{Spa}(R)$ the derived Čech-Tate complex*

$$(4.17.1) \quad \mathrm{Tot}(0 \rightarrow R \rightarrow \prod_{i \in I} R_i \rightarrow \prod_{i, j \in I} R_i \widehat{\otimes}_R^{\mathbb{L}} R_j \rightarrow \cdots)$$

is strictly exact. In particular, $\{\mathrm{Spa}(R_i)\}_{i \in I}$ being a cover for the homotopy Zariski topology is equivalent to being a cover of $\mathrm{Spa}(R)$.

Proof. Since by Proposition 4.15 derived rational localizations are homotopy Zariski open localizations, then the complex of equation (4.17.1) is the (derived) Čech-Tate complex as considered in Theorem 2.23. Therefore, $\{\mathrm{Spa}(R_i)\}_{i \in I}$ is a cover for the homotopy Zariski topology if and only if (4.17.1) is strictly exact. But since rational localizations of R are strictly Koszul regular

and by Proposition 4.16 we have that $R_i \widehat{\otimes}_R^{\mathbb{I}} R_j \cong R_i \widehat{\otimes}_R R_j$ then the complex (4.17.1) reduces to the usual Čech-Tate complex. Hence, for a family of rational localization it is equivalent being a cover for the homotopy Zariski topology or for the topology of $\mathrm{Spa}(R)$. \square

The results of this section are a generalization of (some of) the main theorems of [8] where only the case when A is a valued field was considered. Our results have the following interpretation. Let R be an affinoid A -algebra, then there is canonical morphism of ∞ -sites

$$\mathrm{Zar}_R \rightarrow \mathrm{Spa}(R)$$

that identifies $\mathrm{Spa}(R)$ as a quotient space of Zar_R via the functor that associates to a rational localization of R with the open Zariski localization it determines.

4.2. Localizations of general Banach rings. As before we fix a base strongly Noetherian Tate ring A . We also suppose A to have a topologically nilpotent uniform unit¹⁸, *i.e.* we suppose that there exists $x \in A^\times$ such that $|x^n| = |x|^n$ for all $n \in \mathbb{Z}$.

We notice that algebras $R \in \mathbf{Comm}(\mathbf{Ban}_A)$ have a multiplication map that is bounded, *i.e.* for which there exists a $C > 0$ such that

$$|xy| \leq C|x||y|$$

for all $x, y \in R$. We recall the following basic lemma that means that in $\mathbf{Comm}(\mathbf{Ban}_A)$ there is no restriction in imposing $C = 1$.

Lemma 4.18. *Let $(R, |-|) \in \mathbf{Comm}(\mathbf{Ban}_A)$, then there exists on R another norm $\|-\|$ that is equivalent to $|-|$ and such that*

$$\|xy\| \leq \|x\|\|y\|$$

for all $x, y \in R$.

Proof. See Proposition 1.2.1/2 of [10]. \square

Let now R be any Banach A -algebra that from now on will be supposed to be equipped with a sub-multiplicative norm.

Proposition 4.19. *For any Banach A -algebra R there exists a topologically free algebra $A\langle\langle r_i \rangle^{-1} X \rangle$, where $X = (X_i)_{i \in I}$ is a vector of variables indexed by a set I , and a strict contracting epimorphism*

$$\varphi : A\langle\langle r_i \rangle^{-1} X_i \rangle \rightarrow R,$$

up to replacing R with a isomorphic Banach A -algebra.

Proof. We recall the universal property of $A\langle\langle r_i \rangle^{-1} X_i \rangle$. First suppose that the morphism $A \rightarrow R$ is a contraction and that the index set I is finite, then

$$\mathrm{Hom}_{\mathbf{Comm}(\mathbf{Ban}_A^{\leq 1})}(A\langle\langle r_i \rangle^{-1} X_i \rangle, R) \cong \prod_{i \in I} R^{\leq r_i}$$

where $R^{\leq r_i}$ is the set of elements of R with norm smaller or equal to r_i . Notice that this universal property differs from the usual universal property of $A\langle\langle r_i \rangle^{-1} X_i \rangle$ considered as an object of $\mathbf{Comm}(\mathbf{Ban}_A)$, in which case X_i can be mapped to any element of R with spectral

¹⁸The hypothesis of A having a topologically nilpotent uniform unit and also the hypothesis of being Tate should not be essential because it should be possible to remove them using the theory of reified spaces introduced by Kedlaya, *cf.* [18]. This would permit us to prove a more general version of Lemma 4.11 that is the only place where we crucially used the hypothesis. For simplicity, we bound our discussion to the case of adic spaces in this work but we will comment more on the use of reified spaces for generalizing the results for this paper in the concluding section.

radius less or equal to r_i . Both universal properties can be proved in a similar way and we give the details for the former one as it is a non-standard result.

We notice that since the structural map $\phi : A \rightarrow R$ is a contraction then any contracting morphism

$$\Phi : A\langle (r_i)^{-1}X_i \rangle \rightarrow R$$

extending Φ is written as

$$\Phi\left(\sum_{n=0}^{\infty} a_{i,n_i} X_i^{n_i}\right) = \sum_{n=0}^{\infty} \phi(a_{i,n_i}) s_i^{n_i}$$

with $s_i \in R^{\leq r_i}$ (otherwise the map is not a contraction). Indeed

$$\begin{aligned} |\Phi\left(\sum_{n=0}^{\infty} a_{i,n_i} X_i^{n_i}\right)| &= \left| \sum_{n=0}^{\infty} \phi(a_{i,n_i}) s_i^{n_i} \right| \leq \max_{n \in \mathbb{N}} \{ |\phi(a_{i,n_i}) s_i^{n_i}| \} \leq \max_{n \in \mathbb{N}} \{ |\phi(a_{i,n_i})| |s_i^{n_i}| \} \leq \\ &\leq \max_{n \in \mathbb{N}} \{ |\phi(a_{i,n_i})| |s_i|^{n_i} \} \leq \max_{n \in \mathbb{N}} \{ |a_{i,n_i}| r_i^{n_i} \} = \left| \sum_{n=0}^{\infty} a_{i,n_i} X_i^{n_i} \right| \end{aligned}$$

because $\phi : A \rightarrow R$ is a contraction and by Lemma 4.18 we suppose (up to isomorphism in $\mathbf{Comm}(\mathbf{Ban}_A)$) that the norm of R is submultiplicative.

As

$$A\langle (r_i)^{-1}X_i \rangle \cong \varinjlim_{F \subset I}^{\leq 1} A\langle (r_f)^{-1}X_f \rangle,$$

where $F \subset I$ runs over all finite subsets, we get that

$$\mathrm{Hom}_{\mathbf{Comm}(\mathbf{Ban}_A)}(A\langle (r_i)^{-1}X_i \rangle, R) \cong \prod_{i \in I} R^{\leq r_i}.$$

Since all elements of R have a finite norm, it is immediate to construct a contracting strict epimorphism $A\langle (|r|^{-1}X_r)_{r \in R} \rangle$ by considering R as the indexing set and sending X_r to $r \in R$.

Finally, suppose that $A \rightarrow R$ is not a contraction. Then, we can find R' such that $R \cong R'$ in $\mathbf{Comm}(\mathbf{Ban}_A)$ and $A \rightarrow R'$ is a contraction and apply the previous reasoning to this morphism. \square

Proposition 4.19 is the analytic analog of the algebraic result that any A -algebra is the quotient of a polynomial algebra (in infinitely many variables).

In order to define the derived structural sheaf of simplicial Banach R -algebras it is convenient to work in the category of bornological rings over A . This is not necessary but the nice properties of the category \mathbf{Born}_A make the construction easy and proofs short and neat. As our definitions will be (a priori) dependent on the choice of a resolution of R by flat A -algebras we prove that such a resolution can be found functorially and later on we show that our definitions will be independent of this choice.

Proposition 4.20. *Let R be a Banach A -algebra. The strict epimorphism of Proposition 4.19 can be chosen to be given by*

$$(4.20.1) \quad A\langle (|r|^{-1}X_r)_{r \in R} \rangle \rightarrow R$$

and can be continued to a simplicial resolution of R by projective Banach A -algebras. Moreover, this resolution is functorial in R .

Proof. It is enough to check that the strict epimorphism (4.20.1) is functorial. We notice that there are adjunctions

$$c^0 : \mathbf{Nr}^{\leq 1} \rightleftarrows \mathbf{Ban}_A^{\leq 1} : U$$

where U is the forgetful functor from the category of Banach A -modules, $\mathbf{Nr}^{\leq 1}$ is the contracting category of normed sets (*cf.* [4, Proposition 2.3]) and c^0 is the topologically-free Banach A -module functor as defined in equation (3.8.1), and

$$S : \mathbf{Ban}_A^{\leq 1} \rightleftarrows \mathbf{Comm}(\mathbf{Ban}_A^{\leq 1}) : V$$

where V is the forgetful functor from the category of Banach A -algebras to the category of A -modules and S is the symmetric algebra functor. By composition we obtain the adjunction $(U \circ V) \dashv (S \circ c^0)$ and

$$S(c^0(U(V(R)))) = A\langle (|r|^{-1}X_r)_{r \in R} \rangle.$$

Then, the morphism (4.20.1) is just obtained as the counit map of the adjunction. \square

If in equation (4.20.1) can be chosen such that $A\langle |r|^{-1}X_r \rangle$ is an A -affinoid algebra for all $r \in R$, then it turns out that we can write

$$(4.20.2) \quad R \cong \underset{i \in I}{\operatorname{colim}^{\leq 1}} R_i$$

for some affinoid A -algebras R_i (notice that since we are using the canonical resolution, the indexing set of this colimit is the underlying set of R itself but we renamed it to I to avoid the confusion of using the same letter to denote the base ring and the indexing set of the colimit).

From now on we will suppose that R satisfies (4.20.2)¹⁹. For such a presentation of R it is clear that for any $f_0, f_1, \dots, f_n \in R$ there exists a cofinal subdiagram of the colimit $J \subset I$ such that $f_0, f_1, \dots, f_n \in R_j$ for all $j \in J$. Therefore, we can consider the idea of defining the rational localization of R by f_0, f_1, \dots, f_n (for a family of elements that generates R as an ideal) as

$$R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle = \underset{j \in J}{\operatorname{colim}^{\leq 1}} R_j \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle.$$

Actually, this definition is equivalent to the usual one used in the theory of Huber spaces but the fact that $\underset{j \in J}{\operatorname{colim}^{\leq 1}}$ is not a strictly exact functor implies that this operation will destroy

the sheafyness properties of the A -affinoid localizations involved. As an intermediate step to remedy to this problem we compute these colimits in \mathbf{Born}_A where colimits are strictly exact. The reason why this is convenient with respect to the straightforward computation of $\underset{j \in J}{\mathbb{L}\operatorname{colim}^{\leq 1}}$

is that in general one does not have that

$$R \cong \mathbb{L} \underset{i \in I}{\operatorname{colim}^{\leq 1}} R_i.$$

Therefore, in general, by computing $\underset{j \in J}{\mathbb{L}\operatorname{colim}^{\leq 1}}$ we would get a derived analytic space X for which

$c(\mathrm{LH}^0(\Gamma(\mathcal{O}_X))) = R$ but $\Gamma(\mathcal{O}_X)$ is not concentrated in degree 0. Instead, to obtain a derived analytic space whose global sections algebra is quasi-isomorphic to R , we take advantage of the

¹⁹This hypothesis is quite weak as it is always satisfied if $A = k$ a valued field (as all affinoid k -algebras, in the sense of Berkovich, are filtered contracting colimits of strictly affinoid algebras). Moreover, we think that this hypothesis can be dropped using the theory of reified spaces.

fact that filtered colimits in \mathbf{Born}_A are strictly exact by replacing the $\text{colim}_{\rightarrow}^{\leq 1}$ with $\text{colim}_{\rightarrow}$ in \mathbf{Born}_A and then we will define the structural sheaf on R via a derived base change.

Definition 4.21. *Let R be a Banach ring over A and let*

$$R \cong \text{colim}_{i \in I}^{\leq 1} R_i$$

as in (4.20.2). Then, we define

$$R_{\text{born}} = \text{colim}_{i \in I} R_i$$

where the colimit is computed in \mathbf{Born}_A .

We now introduce derived rational localizations of R_{born} .

Definition 4.22. *Let R be a Banach ring over A and let $f_0, f_1, \dots, f_n \in R$ be such that $(f_0, \dots, f_n) = (1)$. Then, we define*

$$R_{\text{born}} \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h = K_{R_{\text{born}} \langle X_1, \dots, X_n \rangle} (f_0 X_1 - f_1, \dots, f_0 X_n - f_n)$$

to be the derived rational localization of R_{born} by (f_0, f_1, \dots, f_n) .

Remark 4.23. Since the hypothesis that A has a topologically nilpotent uniform unit implies that also R has a topologically nilpotent uniform unit, it follows from [16, Remark 2.4.7] that the rational subsets of $\text{Spa}(R)$ can always be defined in terms of inequalities of the form

$$\{v \in \text{Spa}(R) \mid v(f_i) \leq v(f_0), (f_0, \dots, f_n) = R\}$$

in place of inequalities of the form of equation (4.0.1).

Similarly to Definition 4.8 one can define Weierstrass and Laurent localizations of R_{born} . We omit the details of these definitions as they are not needed in the following proofs.

Proposition 4.24. *Let $R_{\text{born}} \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$ be a derived rational localization of R_{born} , then*

$$R_{\text{born}} \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h = \lim_{i \in J} \left(R_i \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h \right)$$

where $J \subset I$ is a cofinal subset of I for which $f_0, \dots, f_n \in R_i$ for all $i \in J$.

Proof. As by definition

$$R_{\text{born}} \langle X_1, \dots, X_n \rangle \cong R_{\text{born}} \widehat{\otimes}_A A \langle X_1, \dots, X_n \rangle$$

and $A \langle X_1, \dots, X_n \rangle$ is flat over A (cf. Proposition 3.16) we have that

$$\begin{aligned} \lim_{i \in J} \left(R_i \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h \right) &= \lim_{i \in J} K_{R_i \widehat{\otimes}_A A \langle X_1, \dots, X_n \rangle} (f_0 X_1 - f_1, \dots, f_0 X_n - f_n) \cong \\ &\cong K_{\lim_{i \in J} R_i \widehat{\otimes}_A A \langle X_1, \dots, X_n \rangle} (f_0 X_1 - f_1, \dots, f_0 X_n - f_n) \cong K_{R_{\text{born}} \widehat{\otimes}_A A \langle X_1, \dots, X_n \rangle} (f_0 X_1 - f_1, \dots, f_0 X_n - f_n) \end{aligned}$$

because filtered colimits are strictly exact in \mathbf{Born}_A and commute with tensor products and $A \langle X_1, \dots, X_n \rangle$ is flat. The existence of the subset $J \subset I$ comes from the fact that we identify $I = R$. \square

The fact that filtered colimits in \mathbf{Born}_A are strictly exact permits to deduce easily the following proposition.

Proposition 4.25. *Let*

$$\phi : R_{\text{born}} \rightarrow R_{\text{born}} \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$$

be a derived rational localization, then ϕ is a homotopy epimorphism and $R_{\text{born}} \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$ is strictly Koszul regular.

Proof. Writing

$$R_{\text{born}} \cong \varinjlim_{i \in I} R_i,$$

where R_i are affinoid A -algebra as in (4.20.2), there is a cofinal $J \subset I$ such that $f_0, \dots, f_n \in R_j$ for all $j \in J$. If $(f_0, f_1, \dots, f_n) = 1$ then there exists $g_0, g_1, \dots, g_n \in R$ such that

$$f_0 g_0 + \dots + f_n g_n = 1.$$

Therefore, we there exists a cofinal $J' \subset J$ such that for all $j \in J'$ one has that $f_0, \dots, f_n, g_0, g_1, \dots, g_n \in R_j$ and hence the derived rational localization $R_j \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$ is well defined. Thus by Proposition 4.15 we have that

$$\phi_j : R_j \rightarrow R_j \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$$

is a homotopy epimorphism. Hence by Proposition 2.28 derived rational localizations of R_{born} are homotopy epimorphisms. The fact that $R_{\text{born}} \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$ is strictly Koszul regular follows from Corollary 4.13 and the fact that filtered direct limits are strictly exact in \mathbf{Born}_A . \square

Let $U(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}) \subset \text{Spa}(R)$ be the rational subset determined by the elements $f_0, f_1, \dots, f_n \in R$, then the association $U(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}) \mapsto R_{\text{born}} \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$ does not define a derived pre-sheaf on $\text{Spa}(R)$. Intuitively, this is because $\text{Spa}(R_{\text{born}})$ is the pro-analytic space “ \varprojlim ” $\text{Spa}(R_i)$ whereas $\text{Spa}(R) = \varprojlim \text{Spa}(R_i)$ (cf. [16, Remark 2.6.3]). Nevertheless, R_{born} and its derived rational localizations will be useful for computing the derived rational localizations of R . We will give a more precise description of $\text{Spa}(R_{\text{born}})$ in the next section where we will study adic spectra of multiplicatively convex bornological rings.

Definition 4.26. *Let R be a Banach ring over A and let $f_0, f_1, \dots, f_n \in R$ be such that $(f_0, \dots, f_n) = (1)$. Then, we define*

$$R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h = R_{\text{born}} \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h \widehat{\otimes}_{R_{\text{born}}}^{\mathbb{L}} R$$

to be the derived rational localization of R by (f_0, f_1, \dots, f_n) .

Remark 4.27. Notice that $(-)\widehat{\otimes}_{R_{\text{born}}}^{\mathbb{L}} R$ is not exact and so the derived rational localization may not be concentrated in degree 0.

At this point the derived rational localizations of R are nothing new, if we think of them as Koszul dg-algebras as in the following proposition.

Proposition 4.28. *Let R be a Banach ring over A and let $f_0, f_1, \dots, f_n \in R$ be such that $(f_0, \dots, f_n) = R$. Then,*

$$(4.28.1) \quad R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h \cong K_{R\langle X_1, \dots, X_n \rangle}(f_0 X_1 - f_1, \dots, f_0 X_n - f_n)$$

in $D^{\leq 0}(\mathbf{Ban}_R)$.

Proof. Equation (4.28.1) follows immediately by writing the base change of the complex

$$R_{\text{born}} \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h = K_{R_{\text{born}} \langle X_1, \dots, X_n \rangle} (f_0 X_1 - f_1, \dots, f_0 X_n - f_n).$$

□

Proposition 4.28 tells us that the derived rational localizations we defined in Definition 4.26 are formally the same as the ones defined for affinoid A -algebras so far. The main difference is that for a general Banach ring the Koszul dg-algebra associated with a rational subset is not concentrated in degree 0, in general, and therefore it is not uniquely determined by isomorphism class but it is determined up to quasi-isomorphism. The first step in our study of derived rational localizations of R is to notice that the canonical morphism $R \rightarrow R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$ is an open localization for the homotopy Zariski topology.

Proposition 4.29. *Let R be a Banach ring over A and let $f_0, f_1, \dots, f_n \in R$ be such that $(f_0, \dots, f_n) = R$. Then, the morphism*

$$R \rightarrow R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$$

is a homotopy epimorphism.

Proof. By Proposition 2.27 derived tensor products preserve homotopy epimorphisms. Therefore the claim follows from Proposition 4.25. □

Definition 4.30. *Let R be a Banach ring over A and let $f_0, f_1, \dots, f_n \in R$ be such that $(f_0, \dots, f_n) = (1)$. Then, the standard rational cover of R associated to f_0, \dots, f_n is given by the family of morphisms*

$$\left\{ R \rightarrow R \left\langle \frac{f_0}{f_i}, \dots, \frac{f_n}{f_i} \right\rangle^h \right\}_{0 \leq i \leq n}.$$

Theorem 4.31. *Let R be a Banach ring over A (satisfying the hypothesis introduced so far) and let $f_0, f_1, \dots, f_n \in R$ be such that $(f_0, \dots, f_n) = R$. The standard rational cover of R associated to f_0, \dots, f_n is a cover for the (formal) homotopy Zariski topology.*

Proof. For a cofinal $J \subset I$ the elements f_0, \dots, f_n give a standard rational cover of R_i for all $i \in J$ (this follows by the same reasoning of Proposition 4.25). This implies that the derived Čech-Tate complex is strictly exact as

$$\text{Tot} \left(0 \rightarrow R \rightarrow \prod_j R \left\langle \frac{f_0}{f_j}, \dots, \frac{f_n}{f_j} \right\rangle^h \rightarrow \dots \right)$$

is quasi-isomorphic to

$$\text{Tot} \left(0 \rightarrow \left(\lim_{i \in J} R_i \right) \widehat{\otimes}_{R_{\text{born}}}^{\mathbb{L}} R \rightarrow \prod_j \left(\lim_{i \in J} R_i \left\langle \frac{f_0}{f_j}, \dots, \frac{f_n}{f_j} \right\rangle^h \right) \widehat{\otimes}_{R_{\text{born}}}^{\mathbb{L}} R \rightarrow \dots \right)$$

that is quasi-isomorphic to

$$\text{Tot} \left(\left(\lim_{i \in J} \left(0 \rightarrow R_i \rightarrow \prod_j R_i \left\langle \frac{f_0}{f_j}, \dots, \frac{f_n}{f_j} \right\rangle^h \rightarrow \dots \right) \right) \widehat{\otimes}_{R_{\text{born}}}^{\mathbb{L}} R \right)$$

because J is filtered and colimits commute with tensor products. Therefore, the derived Čech-Tate complex of the standard rational cover induced by f_0, \dots, f_n is quasi-isomorphic to derived tensor product of a direct limit of strictly acyclic complexes, hence it is a strictly acyclic complex. \square

Now, Theorem 4.31 combined with Theorem 2.26 imply that the family of all derived rational localizations of R form a ∞ -site that is a (geometric) quotient of Zar_R . We would like to identify this ∞ -site with the Huber spectrum of R but (probably) in general these two spaces do not agree. We now proceed to the definition of the homotopical Huber spectrum of R and its comparison with the classical Huber spectrum.

4.3. The homotopical Huber spectrum of a Banach ring. We are now ready to define and study the homotopical Huber spectrum. We will define two different flavours of homotopical Huber spectra, strictly related to each other.

Definition 4.32. *We define the homotopical Huber-Zariski spectrum of R , denoted by $\text{Spa}_{\text{Zar}}^h(R)$ as the ∞ -site determined by the family of derived rational localizations (as defined in Definition 4.26) with the same covers of the homotopy Zariski topology (i.e. finite families of conservative derived rational localizations). We define the standard homotopical Huber spectrum of R , denoted by $\text{Spa}_{\text{Rat}}^h(R)$ as the ∞ -site determined by the family of derived rational localizations with covers given by the standard rational covers.*

We mention that the data of an ∞ -Grothendieck topology on an ∞ -category is equivalent to the data of a Grothendieck topology on the homotopy category, therefore, although our statements use the more powerful language of ∞ -categories, in Definition 4.32 and the discussion before we are just considering Grothendieck topologies on the respective homotopy categories (cf. [23, Remark 6.2.2.3] for a discussion of this fact). Therefore, to $\text{Spa}_{\text{Zar}}^h(R)$ and $\text{Spa}_{\text{Rat}}^h(R)$ we can associate classical sites that we denote by $|\text{Spa}_{\text{Zar}}^h(R)|$ and $|\text{Spa}_{\text{Rat}}^h(R)|$.

Theorem 4.31 directly implies that there is a continuous morphism of ∞ -sites

$$\text{Spa}_{\text{Zar}}^h(R) \rightarrow \text{Spa}_{\text{Rat}}^h(R)$$

just given by the identity functor. Although we do not have a counterexample, we do not expect this morphism to be an equivalence of sites in general. But it is in some special cases.

Proposition 4.33. *Let R be an affinoid k -algebra, where k is a non-Archimedean valued field. Then, the canonical map $\text{Spa}_{\text{Zar}}^h(R) \rightarrow \text{Spa}_{\text{Rat}}^h(R)$ is a homeomorphism. Moreover, in this case one has that*

$$\text{Spa}(R) \cong \text{Spa}_{\text{Zar}}^h(R).$$

Proof. This has been proved in [8, Theorem 5.39]. \square

We notice the only step missing in generalizing Proposition 4.33 to the setting of the current paper is a generalization of [8, Lemma 5.32], as all the other main results of *loc.cit.* have been generalized in this work. This amounts to checking that every cover for the homotopy Zariski topology of $\text{Spa}_{\text{Zar}}^h(R)$ is refined by a standard rational cover. We do not know how far such a generalization can hold true.

The next proposition shows that, in general, both $|\text{Spa}_{\text{Zar}}^h(R)|$ and $|\text{Spa}_{\text{Rat}}^h(R)|$ have enough points.

Proposition 4.34. *The topoi associated to $|\text{Spa}_{\text{Zar}}^h(R)|$ and $|\text{Spa}_{\text{Rat}}^h(R)|$ are coherent and hence equivalent to the ones of spectral topological spaces.*

Proof. Since the covers of both sites are determined by finite families of morphisms it is easy to check that they are coherent ∞ -site (cf. Theorem 2.26). It is then easy to check that $|\mathrm{Spa}_{\mathrm{Zar}}^h(R)|$ and $|\mathrm{Spa}_{\mathrm{Rat}}^h(R)|$ are coherent and we can apply the classical Deligne's Theorem to them to deduce that they have enough points. \square

Theorem 4.31 directly implies that the association

$$\mathrm{Spec} \left(R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h \right) \mapsto R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$$

is a derived sheaf of simplicial Banach A -algebras both on $\mathrm{Spa}_{\mathrm{Rat}}^h(R)$ and on $\mathrm{Spa}_{\mathrm{Zar}}^h(R)$. So, combining the previous results we get one main result.

Theorem 4.35. *The ∞ -sites $\mathrm{Spa}_{\mathrm{Rat}}^h(R)$ and $\mathrm{Spa}_{\mathrm{Zar}}^h(R)$ have a canonical ∞ -structural sheaf given by derived rational localizations.*

We will see in Example 5.4 that for some well-known examples of non-sheafy (in the usual sense) Banach rings the spaces $\mathrm{Spa}(R)$, $\mathrm{Spa}_{\mathrm{Rat}}^h(R)$ and $\mathrm{Spa}_{\mathrm{Zar}}^h(R)$ canonically agree. Therefore, in such cases Theorem 4.35 induces a derived structural sheaf of simplicial Banach algebras on $\mathrm{Spa}(R)$.

In [28], Scholze and Clausen have proved results similar to the ones in this section in the context of condensed rings. In particular, [28, Proposition 13.16] proves that any Huber pair has a canonical structure of analytic ring (in the sense of *loc.cit.*) and therefore a structure of analytic space for the topology defined by finite conservative covers of steady localizations (the steady localizations as defined in [28] are the homotopy epimorphisms in the theory of condensed rings and the Grothendieck topology of analytic spaces defined in *loc.cit.* is essentially the homotopy Zariski topology of analytic rings). Therefore, the structure of ∞ -analytic space given to any Huber pair in [28] is the condensed analog of the notion of derived analytic space for the homotopy Zariski topology defined in our previous work [4], that we recalled in Section 2. It is not clear to us how our constructions and the ones of [28] compare but we have been informed via a private communication with Peter Scholze that probably also the derived structural sheaf he has defined can be computed via Koszul complexes.

We do not know how the homotopical Huber spectrum and the usual Huber spectrum of a Banach ring compare in general, but if $A = k$, a non-trivially valued, complete, non-Archimedean field, we can give the following comparison.

Proposition 4.36. *Let R be a k -Banach algebra, then there is a canonical map*

$$\mathrm{Spa}(R) \rightarrow |\mathrm{Spa}_{\mathrm{Zar}}^h(R)|.$$

Proof. Let us write as before $R \cong \varinjlim_{i \in I}^{\leq 1} R_i$ where R_i are k -affinoid algebras. By [16, Remark 2.6.3] we have that

$$\mathrm{Spa}(R) \cong \varprojlim_{i \in I} \mathrm{Spa}(R_i)$$

and by the functoriality of $|\mathrm{Spa}_{\mathrm{Zar}}^h(-)|$ (cf. discussion below) we have that for each i there is a canonical map

$$|\mathrm{Spa}_{\mathrm{Zar}}^h(R)| \rightarrow |\mathrm{Spa}_{\mathrm{Zar}}^h(R_i)|$$

induced by the canonical map $R_i \rightarrow R$. But by Proposition 4.33 we have that

$$|\mathrm{Spa}_{\mathrm{Zar}}^h(R_i)| \cong \mathrm{Spa}(R_i)$$

hence the universal property of the projective limit gives a canonical morphism $\mathrm{Spa}(R) \rightarrow |\mathrm{Spa}_{\mathrm{Zar}}^h(R)|$. \square

One way to obtain a map $\mathrm{Spa}(R) \rightarrow |\mathrm{Spa}_{\mathrm{Zar}}^h(R)|$ in general would be to generalize Proposition 4.33 to affinoid algebras over any strongly Noetherian Banach-Tate ring.

We conclude this section by briefly mentioning that any morphism $\varphi : R \rightarrow S$ of Banach A -algebras induces a map of ringed sites $\varphi^* : |\mathrm{Spa}_{\mathrm{Rat}}^h(S)| \rightarrow |\mathrm{Spa}_{\mathrm{Rat}}^h(R)|$ (where by ringed sites we mean sites with a structural derived sheaf of Banach algebras). Indeed, if $R \rightarrow R \langle \frac{f_0}{f_j}, \dots, \frac{f_n}{f_j} \rangle^h$ is a derived rational localization of R , then $S \rightarrow S \langle \frac{\varphi(f_0)}{\varphi(f_j)}, \dots, \frac{\varphi(f_n)}{\varphi(f_j)} \rangle^h$ is a derived rational localization of S and the square

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R \langle \frac{f_0}{f_j}, \dots, \frac{f_n}{f_j} \rangle^h & \longrightarrow & S \langle \frac{\varphi(f_0)}{\varphi(f_j)}, \dots, \frac{\varphi(f_n)}{\varphi(f_j)} \rangle^h \end{array}$$

is commutative in the category of simplicial algebras because of the explicit definitions of the Koszul complexes. On the other, since the global sections of $|\mathrm{Spa}_{\mathrm{Rat}}^h(R)|$ are precisely R (up to quasi-isomorphism), we have that any morphism of ringed sites $\varphi^* : |\mathrm{Spa}_{\mathrm{Rat}}^h(S)| \rightarrow |\mathrm{Spa}_{\mathrm{Rat}}^h(R)|$ induces a morphism of algebras $R \rightarrow S$. So, if both R and S are concentrated in degree 0 we have that

$$\mathrm{Hom}(R, S) \cong \mathrm{Hom}(|\mathrm{Spa}_{\mathrm{Rat}}^h(S)|, |\mathrm{Spa}_{\mathrm{Rat}}^h(R)|).$$

Similarly one shows that

$$\mathrm{Hom}(R, S) \cong \mathrm{Hom}(|\mathrm{Spa}_{\mathrm{Zar}}^h(S)|, |\mathrm{Spa}_{\mathrm{Zar}}^h(R)|).$$

4.4. Localizations of bornological rings. In this section, we briefly discuss how the results of previous sections can be generalized to some bornological rings that are not necessarily Banach. We keep the notation of the last section where A is a fixed base Tate ring supposed to be strongly Noetherian with a topologically nilpotent uniform unit. All Banach rings (resp. bornological rings) are supposed to be A -Banach algebras (resp. A -bornological algebras). In this section, we keep the hypothesis of Section 4.2 where we assumed that all Banach A -algebras satisfy the hypothesis of having the canonical presentation (4.20.2) essentially equivalent to a filtered colimit of affinoid A -algebras.

Definition 4.37. *Let R be an object of $\mathbf{Comm}(\mathbf{Born}_A)$, then R is called multiplicatively convex if $R \cong \varinjlim_{i \in I} R_i$ for R_i some A -Banach algebras, where the indexing set I is filtered.*

Remark 4.38. We notice that the algebra R_{born} introduced in Definition 4.21 is a special example of a multiplicatively convex bornological A -algebra of a very special kind because

$$R_{\mathrm{born}} = \varinjlim_{i \in I} R_i$$

where R_i are affinoid A -algebras.

The following theorem directly follows from the properties of the functor \varinjlim .

Theorem 4.39. *Let $R \cong \varinjlim_{i \in I} R_i$ be a multiplicatively convex bornological algebra and $f_0, f_1, \dots, f_n \in R$*

be such that $(f_0, \dots, f_n) = R$. Then, the canonical map $R \rightarrow R \langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \rangle^h = K_{R \langle X_1, \dots, X_n \rangle}(f_0 X_1 - f_1, \dots, f_0 X_n - f_n)$ is a homotopy epimorphism and the standard rational covers are covers for the homotopy Zariski topology.

Proof. Suppose that $(f_0, \dots, f_n) = R$, then there exist $g_1, \dots, g_n \in R$ such that

$$f_0 g_0 + \dots + f_n g_n = 1,$$

therefore there exists a $j \in I$ such that $f_i, g_i \in R_j$ for all i . Thus, $(f_0, \dots, f_n) = R_j$ in R_j and hence, thanks to Theorem 4.31, they define the standard rational cover

$$\left\{ R_j \rightarrow R_j \left\langle \frac{f_0}{f_i}, \dots, \frac{f_n}{f_i} \right\rangle^h \right\}_{0 \leq i \leq n}$$

for R_j and for all $j' \in J$ such that $j' \geq j$. Notice that

$$R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h = \varinjlim_{j \in I} R_j \left\langle \frac{f_0}{f_i}, \dots, \frac{f_n}{f_i} \right\rangle^h$$

because \varinjlim commutes with tensor products, where we defined $R\langle X_1, \dots, X_n \rangle = R \widehat{\otimes}_A A\langle X_1, \dots, X_n \rangle$

as the Tate algebras over R . Therefore, $R \rightarrow R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$ is a homotopy epimorphism thanks to Proposition 2.28 and the family

$$\left\{ R \rightarrow R \left\langle \frac{f_0}{f_i}, \dots, \frac{f_n}{f_i} \right\rangle^h \right\}_{0 \leq i \leq n}$$

is a cover for the homotopy Zariski topology because \varinjlim commutes with finite products. \square

Similarly to the case of Theorem 4.31, Theorem 4.39 has the consequence that from the family of derived rational localizations of R we can define two ∞ -site (with enough points) as before: $\mathrm{Spa}_{\mathrm{Zar}}^h(R)$ by considering covers for the homotopy Zariski topology and $\mathrm{Spa}_{\mathrm{Rat}}^h(R)$ by considering standard rational covers.

Another case of interest is when the A -bornological ring R can be written as

$$R \cong \varprojlim_{i \in I} R_i \cong \mathbb{R} \varprojlim_{i \in I} R_i.$$

Theorem 4.40. *Let $R \cong \varprojlim_{i \in I} R_i$, with I directed, and $f_0, f_1, \dots, f_n \in R$ be such that $(f_0, \dots, f_n) = R$. Suppose that for all n one has that*

$$(4.40.1) \quad R\langle X_1, \dots, X_n \rangle \cong \mathbb{R} \varprojlim_{i \in I} (R_i \langle X_1, \dots, X_n \rangle).$$

Then, the canonical map $R \rightarrow R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$ is a homotopy epimorphism and the standard rational covers are covers for the homotopy Zariski topology.

Remark 4.41. We notice that the condition of equation (4.40.1) is often satisfied and easy to check in many situations of interest in applications, *i.e.* for example when R is a multiplicatively convex Fréchet algebra.

Remark 4.42. We also notice that equation (4.40.1) is equivalent to ask that

$$\begin{aligned} \mathbb{R} \varprojlim_{i \in I} (R_i \langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle) &\cong R \langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle \cong \\ &R \langle X_1, \dots, X_n \rangle \widehat{\otimes}_R R \langle Y_1, \dots, Y_m \rangle \cong, \mathbb{R} \varprojlim_{i \in I} (R_i \langle X_1, \dots, X_n \rangle) \widehat{\otimes}_R^{\mathbb{L}} \mathbb{R} \varprojlim_{i \in I} (R_i \langle Y_1, \dots, Y_m \rangle) \end{aligned}$$

so forcing the commutation of $\widehat{\otimes}_R^{\mathbb{L}}$ with $\mathbb{R} \lim$.

Proof. The proof is similar to the proof of Theorem 4.39 once it is noticed that for any $i \in I$ the projection $\pi_i : R \rightarrow R_i$ gives the elements $\pi_i(f_0), \pi_i(f_1), \dots, \pi_i(f_n) \in R_i$ that determine a derived rational localization of R_i . Then, the condition of equation (4.40.1) implies that

$$R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h \cong \mathbb{R} \lim_{\leftarrow, i \in I} \left(R_i \left\langle \frac{\pi_i(f_1)}{\pi_i(f_0)}, \dots, \frac{\pi_i(f_n)}{\pi_i(f_0)} \right\rangle^h \right),$$

because

$$\begin{aligned} R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h &\cong K_R(f_0 X_1 - f_1, \dots, f_0 X_n - f_n) \cong \\ &\mathbb{R} \lim_{\leftarrow, i \in I} K_R(\pi_i(f_0) X_1 - \pi_i(f_1), \dots, \pi_i(f_0) X_n - \pi_i(f_n)) \end{aligned}$$

using equation (4.40.1) and the naturality of the isomorphisms. The only non-trivial thing to check is that $R \rightarrow R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$ is a homotopy epimorphism and this follows from equation (4.40.1) (*cf.* also Remark 4.42) because we can write

$$\begin{aligned} \mathbb{R} \lim_{\leftarrow, i \in I} \left(R_i \left\langle \frac{\pi_i(f_1)}{\pi_i(f_0)}, \dots, \frac{\pi_i(f_n)}{\pi_i(f_0)} \right\rangle^h \right) &\widehat{\otimes}_R \mathbb{R} \lim_{\leftarrow, i \in I} \left(R_i \left\langle \frac{\pi_i(f_1)}{\pi_i(f_0)}, \dots, \frac{\pi_i(f_n)}{\pi_i(f_0)} \right\rangle^h \right) \cong \\ \mathbb{R} \lim_{\leftarrow, i \in I} \left(R_i \left\langle \frac{\pi_i(f_1)}{\pi_i(f_0)}, \dots, \frac{\pi_i(f_n)}{\pi_i(f_0)} \right\rangle^h \widehat{\otimes}_R R_i \left\langle \frac{\pi_i(f_1)}{\pi_i(f_0)}, \dots, \frac{\pi_i(f_n)}{\pi_i(f_0)} \right\rangle^h \right) &\cong \\ \mathbb{R} \lim_{\leftarrow, i \in I} \left(R_i \left\langle \frac{\pi_i(f_1)}{\pi_i(f_0)}, \dots, \frac{\pi_i(f_n)}{\pi_i(f_0)} \right\rangle^h \right). & \end{aligned}$$

□

Again Theorem 4.40 comes with its corollaries about the existence of the ∞ -topoi $\mathrm{Spa}_{\mathrm{Zar}}^h(R)$ and $\mathrm{Spa}_{\mathrm{Rat}}^h(R)$.

5. EXAMPLES AND APPLICATIONS

Before discussing concrete examples we would like to recall the known results about the sheafyness of the structural pre-sheaf of $\mathrm{Spa}(R)$ for a Banach ring R . If R is a uniform (always non-Archimedean) Banach ring one can prove (*cf.* [16] Corollary 2.8.9) that the Čech-Tate complex associated to the Laurent cover $\mathrm{Spa}(R\langle f \rangle) \coprod \mathrm{Spa}(R\langle f^{-1} \rangle) \rightarrow \mathrm{Spa}(R)$

$$(5.0.1) \quad 0 \rightarrow R \rightarrow R\langle f \rangle \times R\langle f^{-1} \rangle \rightarrow R\langle f, f^{-1} \rangle \rightarrow 0$$

is strictly exact for any $f \in R$ by proving that the ideals $(X - f)$ and $(fX - 1)$ are closed in $R\langle X \rangle$. But this is not enough to prove that the Tate complex is strictly exact for any admissible cover, unlike the classical case of rigid geometry, because it is not true that all admissible covers of $\mathrm{Spa}(R)$ can be refined by an intersection of covers of the form $R \rightarrow R\langle f \rangle \times R\langle f^{-1} \rangle$. The best one can achieve is to find a rational cover of $\mathrm{Spa}(R)$ whose elements have the property that every rational cover is refined by a standard Laurent cover. In this way, if we start with a uniform Banach ring R the problem of checking that the Tate complex of an admissible cover is exact is translated into checking that the Tate complex of standard Laurent cover of a rational subdomain of R is exact. The main complication at this point is the fact that rational localizations of a uniform ring may not be uniform and, even more, the Tate complex of a standard Laurent cover of a rational subdomain may not be strictly exact when computed in the category of Banach modules (*cf.* [11], [25]).

A way to avoid the mentioned problem is to suppose that all rational localizations of a given uniform ring are uniform rings, leading to the following definition.

Definition 5.1. *A uniform Banach ring R is called stably uniform if all its rational localizations are uniform Banach rings.*

In this setting one can prove that the usual computations can be carried over, hence obtaining the following result.

Theorem 5.2 (Buzzard-Mihara-Verberkmoes). *Let R be a stably uniform Tate ring. Then, for any rational cover the Čech-Tate complex is strictly acyclic.*

Proof. cf. [11, Theorem 7], [25, Theorem 4.9]. □

Theorem 5.2 has two main drawbacks: one, the condition of being stably uniform, although very general and satisfied by many objects of interest, is not easy to check and the second drawback is that in some applications one may be interested to work with more general Banach rings than the stably uniform ones. We hope that our results will make it possible to overcome these restrictions that the theory of Banach algebras had up to now.

We now show how our results are related to Theorem 5.2 and, in some situations, generalize it.

Theorem 5.3. *Let A be a strongly Noetherian Tate ring with a topologically nilpotent uniform unit and R a Banach A -algebra that can be presented as in equation(4.20.2). If R is stably uniform, then*

$$\mathrm{Spa}(R) \cong |\mathrm{Spa}_{\mathrm{Rat}}^h(R)|,$$

as (Banach) ringed sites.

Proof. Let $f_0, f_1, \dots, f_n \in R$ be a set of elements such that $(f_0, \dots, f_n) = R$ and let us denote by \mathcal{U} the standard derived rational cover associated to them. So, elements $U_i \in \mathcal{U}$ correspond to derived localizations of the form

$$R \rightarrow R \left\langle \frac{f_0}{f_i}, \dots, \frac{f_n}{f_i} \right\rangle^h$$

for $0 \leq i \leq n$. By Theorem 4.31 this form a cover for the homotopy Zariski topology and by Corollary 2.24 this condition is equivalent to the alternating derived Čech-Tate complex being strictly exact. Then, the classical Čech-Tate can be obtained from the bicomplex associated to the derived cover by applying the functor $c \circ \mathrm{LH}^0(-)$, *i.e.* considering the classical part of the 0-th left heart cohomology. By Theorem 5.2 the classical Čech-Tate complex is strictly exact and since the alternating bicomplex is bounded, this can happen if and only if each $R \left\langle \frac{f_0}{f_i}, \dots, \frac{f_n}{f_i} \right\rangle^h$ is concentrated in degree 0. Thus, all derived rational localizations are concentrated in degree 0 and the homotopical Huber spectrum coincides with the the Huber spectrum. □

We will see in the next example that the converse to Theorem 5.3 is not true by showing that there exists a non-sheafy (in the usual sense) Banach algebra for which $\mathrm{Spa}(R) \cong |\mathrm{Spa}_{\mathrm{Rat}}^h(R)|$. In this case, the derived structural sheaf on $|\mathrm{Spa}_{\mathrm{Rat}}^h(R)|$ canonically pulls back to a derived sheaf on R .

Example 5.4. We borrow an example from [11] of a non-sheafy (in the usual sense) Tate ring. This example was actually found by Rost and (probably) presented in a paper for the first time in Huber's paper [13]. In [11] the example is presented in the context of adic rings, by using valuations, and we translate it here in the context of Banach rings, *i.e.* by using norms.

Consider the ring of Laurent polynomials

$$A = \frac{k[T, T^{-1}, Z]}{(Z^2)}$$

over the non-Archimedean non-trivially valued field k and endow it with the norm

$$(5.4.1) \quad \left| \sum_{n \in \mathbb{Z}, m \in \{0,1\}} a_{n,m} T^n Z^m \right| = \max\{\max_{n \in \mathbb{Z}} \{\rho^{-|n|} |a_{n,0}|\}, \max_{n \in \mathbb{Z}} \{\rho^{|n|} |a_{n,1}|\}\}$$

where $0 < \rho < 1$. This function defines a non-Archimedean norm on A that is the norm associated with the valuation discussed in [11]. Let us denote with R the completion of A with respect to the mentioned norm. This is a Banach ring over k and a Tate ring because k is non-trivially valued. The space $\text{Spa}(R)$ is thus well defined and one can consider its standard Laurent cover $U = \text{Spa}(R\langle T \rangle)$ and $V = \text{Spa}(R\langle T^{-1} \rangle)$, where $R\langle T \rangle$ denotes the quotient of $R\langle X \rangle$ by the closure of the ideal $(X - T)$ and similarly $R\langle T^{-1} \rangle$. One can show that

$$R\langle T \rangle \cong \left\{ \sum_{n \in \mathbb{Z}} a_n T^n \in k[[T, T^{-1}]] \mid \lim_{n \rightarrow \infty} |a_n| = 0, \lim_{n \rightarrow -\infty} \rho^n |a_n| = 0 \right\}$$

and

$$R\langle T^{-1} \rangle \cong \left\{ \sum_{n \in \mathbb{Z}} a_n T^n \in k[[T, T^{-1}]] \mid \lim_{n \rightarrow \infty} \rho^{-n} |a_n| = 0, \lim_{n \rightarrow -\infty} |a_n| = 0 \right\}.$$

One way to see this is to notice that the residue norm on

$$A\langle T \rangle = \frac{A\langle X \rangle}{(X - T)}$$

must satisfy for all n the inequality

$$|Z| = |T^n T^{-n} Z| \leq |T^n| |T^{-n} Z| = \rho^n$$

in order to be a well defined ring norm. Therefore (as $\rho < 1$), $|Z| = 0$ and hence Z disappears in the separated completion of $A\langle T \rangle$, that is $R\langle T \rangle$. A similar reasoning proves that Z is in the kernel of the map $R \rightarrow R\langle T^{-1} \rangle$.

Therefore, the Tate complex

$$(5.4.2) \quad 0 \rightarrow R \rightarrow R\langle T \rangle \times R\langle T^{-1} \rangle \rightarrow R\langle T, T^{-1} \rangle \rightarrow 0$$

is not exact because the first map is not injective as Z is mapped to 0. Moreover, it is clear that second map is surjective and its kernel is the algebra

$$\left\{ \sum_{n \in \mathbb{Z}} a_n T^n \in k[[T, T^{-1}]] \mid \lim_{n \rightarrow \infty} \rho^{-n} |a_n| = 0, \lim_{n \rightarrow -\infty} \rho^n |a_n| = 0 \right\}$$

that is a sub-algebra of R . This is one of the main well-known examples where the theory of adic spaces breaks down.

One can get a better insight in the geometry of R , or more precisely $\text{Spa}(R)$, by writing a presentation of R by affinoid k -algebras. It is easy to check that the following presentation holds

$$R \cong \lim_{\substack{\rightarrow \\ n \in \mathbb{N}}}^{\leq 1} \frac{k\langle r^{-1}T_1, r^{-1}T_2, Z, r^{-1}Y_1, r^{-1}Y'_1, r^{-2}Y_2, r^{-2}Y'_2, \dots \rangle}{(T_1 T_2 - 1, Z^2, ZT_1 - Y_1, ZT_2 - Y'_1, ZT_1^2 - Y_2, ZT_2^2 - Y'_2, \dots)} = \lim_{\substack{\rightarrow \\ n \in \mathbb{N}}}^{\leq 1} R_n.$$

This presentation of R as a quotient of a Tate algebra in infinitely many variables has some remarkable properties:

- it is cofinal in the canonical presentation defined in equation (4.20.2), in the sense that it is a subsystem that computes the same colimit;

- all its elements are affinoid k -algebras;
- all the transition maps of the system are isomorphisms in \mathbf{Ban}_k (but not in $\mathbf{Ban}_k^{\leq 1}$ where the limit is computed).

Moreover, it shows that, although R is defined as the completion of a finitely generated algebra over k , $\mathrm{Spa}(R)$ is to be considered as an infinite dimensional analytic space over $\mathrm{Spa}(k)$. Furthermore, the fact that the transition maps of the above contracting inductive system are isomorphisms in \mathbf{Ban}_k implies that

$$\mathrm{Spa}(R_n) \cong \varprojlim_{n \in \mathbb{N}} \mathrm{Spa}(R_n) \cong \mathrm{Spa}(R).$$

Now, by Proposition 4.36, we have canonical morphisms

$$\mathrm{Spa}(R) \rightarrow |\mathrm{Spa}_{\mathrm{Rat}}^h(R)| \rightarrow \mathrm{Spa}(R_n)$$

given by the (derived) base change functors on the site of rational localizations. We need to check that the functor inducing $\mathrm{Spa}(R) \rightarrow |\mathrm{Spa}_{\mathrm{Rat}}^h(R)|$ is an equivalence. It is clearly fully faithful because the composition with the functor inducing $|\mathrm{Spa}_{\mathrm{Rat}}^h(R)| \rightarrow \mathrm{Spa}(R_n)$ is fully faithful. And it is obviously essentially surjective, therefore $\mathrm{Spa}(R) \cong |\mathrm{Spa}_{\mathrm{Rat}}^h(R)|$. Moreover, since rational subsets of $|\mathrm{Spa}_{\mathrm{Rat}}^h(R)|$ are determined by their intersections with $\mathcal{M}(R)$ we can use the same reasoning as in [8, Lemma 5.32] to show that $|\mathrm{Spa}_{\mathrm{Zar}}^h(R)| \cong |\mathrm{Spa}_{\mathrm{Rat}}^h(R)|$.

We put this observation in the form of a proposition.

Proposition 5.5. *The non-sheafy (in the usual sense) Banach k -algebra R satisfies the property*

$$\mathrm{Spa}(R) \cong |\mathrm{Spa}_{\mathrm{Zar}}^h(R)| \cong |\mathrm{Spa}_{\mathrm{Rat}}^h(R)|.$$

In order to better explain what happens in this example we explicitly compute the structural sheaf of $|\mathrm{Spa}_{\mathrm{Rat}}^h(R)|$ on the standard Laurent cover $\{U, V\}$ considered so far. Applying Theorem 4.31 we get that the derived Čech-Tate complex

$$\mathrm{Tot}(0 \rightarrow R \rightarrow R\langle T \rangle^h \times R\langle T^{-1} \rangle^h \rightarrow R\langle T, T^{-1} \rangle^h \rightarrow 0)$$

is strictly exact. We now write down explicitly the complexes appearing in the derived Čech-Tate complex and check that it is strictly exact. This requires some elementary, but subtle, computations involving the Banach ring R .

Proposition 5.6. *The ideal (Z) is not closed in R . Moreover*

$$(Z) = \left\{ \sum_{n \in \mathbb{Z}} a_n Z T^n \mid \lim_{|n| \rightarrow \infty} |a_n| \rho^{-|n|} \rightarrow 0 \right\}.$$

Proof. Since $Z^2 = 0$, the ideal generated by Z is given by all the elements of the form Zf with $f \in R$ a power-series of the form

$$f = \sum_{n \in \mathbb{Z}} a_n T^n$$

with $a_n \in k$. By the definition of the norm of R in equation (5.4.1) such f are precisely given by power-series for which $\lim_{|n| \rightarrow \infty} |a_n| \rho^{-|n|} \rightarrow 0$. If we write

$$R = R_0 \oplus R_1$$

as k -Banach spaces, with R_0 the subspace of power-series without Z terms and R_1 the subspace of power-series with Z , then $(Z) \subset R_1$ properly (because $Z^2 = 0$ and $R_0 \cdot Z \subset R_1$ properly) and it is also dense. Therefore, (Z) is not closed. \square

Proposition 5.6 has the counter-intuitive consequence that in the ring R there are elements that one can write as $Z(\sum a_n T^n)$ but they do not belong to the ideal generated by Z (because the series $\sum a_n T^n$ does not converge if it is not multiplied by Z).

Proposition 5.7. *The ideals $(X - T)$ and $(XT - 1)$ are not closed in $R\langle X \rangle$.*

Proof. Since

$$|T^n Z| = \rho^{|n|}$$

we have that the power-series

$$\sum_{n=0}^{\infty} (-1)^{n+1} Z T^{-n+1} X^n, \quad \sum_{n=0}^{\infty} (-1)^{n+1} Z T^n X^n$$

are elements of $R\langle X \rangle$. So,

$$\begin{aligned} (X - T) \left(\sum_{n=0}^{\infty} (-1)^{n+1} Z T^{-n+1} X^n \right) &= Z(X - T) T^{-1} \left(\sum_{n=0}^{\infty} (-1)^{n+1} T^{-n} X^n \right) = \\ &= Z(X - T) T^{-1} (XT^{-1} - 1)^{-1} = Z(X - T)(X - T)^{-1} = Z \end{aligned}$$

and

$$(TX - 1) \left(\sum_{n=0}^{\infty} (-1)^{n+1} Z T^n X^n \right) = Z(TX - 1)(XT - 1)^{-1} = Z$$

prove that $Z \in (X - T)$ and $Z \in (XT - 1)$. It is then easy to check that

$$(X - T) \cap R = (Z), \quad (TX - 1) \cap R = (Z)$$

proving that the ideals are not closed. \square

Proposition 5.7 has the consequence that the morphisms $\mu_{(X-T)} : R\langle X \rangle \rightarrow R\langle X \rangle$ and $\mu_{(TX-1)} : R\langle X \rangle \rightarrow R\langle X \rangle$ appearing in the Koszul complexes²⁰ $R\langle T \rangle^h$ and $R\langle T^{-1} \rangle^h$ respectively, are not strict. Therefore, we have the following description of the Koszul complexes as objects of LH (**Ban_A**)

$$(5.7.1) \quad R\langle T \rangle^h = [R\langle X \rangle \xrightarrow{\mu_{(X-T)}} R\langle X \rangle]$$

and

$$R\langle T^{-1} \rangle^h = [R\langle X \rangle \xrightarrow{\mu_{(TX-1)}} R\langle X \rangle]$$

as it is easy to check that the morphisms $\mu_{(X-T)}$ and $\mu_{(TX-1)}$ are monomorphisms, *i.e.* injective. We notice that the classical part of $R\langle T \rangle^h$ is just the usual algebra $R\langle T \rangle$ obtained by quotienting $R\langle X \rangle$ by the closure of $(X - T)$ (and similarly for $R\langle T^{-1} \rangle^h$).

By Proposition 4.15 we know that the canonical maps $R \rightarrow R\langle T \rangle^h$ and $R \rightarrow R\langle T^{-1} \rangle^h$ are homotopy epimorphisms. We now check this by explicit computations.

Proposition 5.8. *The canonical maps $R \rightarrow R\langle T \rangle^h$ and $R \rightarrow R\langle T^{-1} \rangle^h$ induce quasi-isomorphisms*

$$R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T \rangle^h \cong R\langle T \rangle^h$$

and

$$R\langle T^{-1} \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T^{-1} \rangle^h \cong R\langle T^{-1} \rangle^h$$

i.e. they are homotopy epimorphisms.

²⁰See Notation 4.3 for our notation about the Koszul complexes.

Proof. By the explicit description given in equation (5.7.1) it is easy to see that

$$R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T \rangle^h \cong [0 \rightarrow R\langle X, Y \rangle \xrightarrow{(-\mu_{(X-T)}, \mu_{(Y-T)})} R\langle X, Y \rangle^2 \xrightarrow{(\mu_{(X-T)}, \mu_{(Y-T)})} R\langle X, Y \rangle \rightarrow 0]$$

by computing the total complex. So, we need to check that the left-heart cohomology of $R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T \rangle^h$ is concentrated in degree 0 and that the LH^0 is isomorphic to $R\langle T \rangle^h$. For sure

$$\mathrm{LH}^n(R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T \rangle^h) \cong 0, \quad n \geq 3$$

trivially, then

$$\mathrm{LH}^2(R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T \rangle^h) \cong 0$$

because $R\langle X, Y \rangle \xrightarrow{(-\mu_{(X-T)}, \mu_{(Y-T)})} R\langle X, Y \rangle^2$ is injective. We need now to prove that

$$\mathrm{Ker} \begin{pmatrix} \mu_{(Y-T)} \\ \mu_{(X-T)} \end{pmatrix} \cong \mathrm{Coim}(\mu_{-(X-T)}, \mu_{(Y-T)}).$$

By the Banach's Open Mapping Theorem for k we just need to prove that the map is bijective. By the usual computations of the algebraic Koszul complexes this amounts to check that

$$\frac{(X-T) \cap (Y-T)}{(X-T)(Y-T)} = 1.$$

The only non trivial thing to check is that $Z \in (X-T)(Y-T)$. But, similarly to Proposition 5.7, one can check that the power-series

$$Z \left(\sum_{n=0}^{\infty} (-1)^{n+1} T^{-n+1} X^n \right) \left(\sum_{n=0}^{\infty} (-1)^{n+1} T^{-n+1} Y^n \right) = Z(X-T)^{-1}(Y-T)^{-1}$$

belongs to $R\langle X, Y \rangle$ proving that $Z = Z(X-T)^{-1}(Y-T)^{-1}(X-T)(Y-T) \in (X-T)(Y-T)$. This proves that

$$\mathrm{LH}^1(R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T \rangle^h) \cong 0$$

Finally, we need to check that

$$\mathrm{LH}^0(R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T \rangle^h) \cong R\langle T \rangle^h.$$

This is equivalent to say that the morphism $R \rightarrow R\langle T \rangle^h$ is an epimorphism in the category of R -algebras in $\mathrm{LH}(\mathbf{Ban}_R)$ and it is easy to check that this is true because $R\langle T \rangle^h$ is a quotient of $R\langle X \rangle$ and the universal property of $R\langle X \rangle$ in $\mathbf{Comm}(\mathrm{LH}(\mathbf{Ban}_R))$ is similar to the universal property in $\mathbf{Comm}(\mathbf{Ban}_R)$. Indeed, as explained in Section 2, a morphism $R\langle X \rangle \rightarrow C = [C_1 \rightarrow C_0]$ is an equivalence class of commutative diagrams

$$\begin{array}{ccc} 0 & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ R\langle X \rangle^h & \longrightarrow & C_0 \end{array}$$

and since $\mathrm{Hom}_{\mathbf{Ban}_R}(R\langle X \rangle, C_0) = (C_0)^\circ$ then $\mathrm{Hom}_{\mathrm{LH}(\mathbf{Ban}_R)}(R\langle X \rangle, C) = (C_0)^\circ / \sim$ where \sim is a suitable equivalence relation (whose explicit description is not important for our purposes).

Therefore, every morphism $\phi : R \rightarrow C$ such that $\phi(T) \in (C_0)^\circ$ extends uniquely to a morphism $R\langle T \rangle^h \rightarrow C$ via the unique morphism that makes the diagram

$$\begin{array}{ccc} R\langle X \rangle & & \\ \downarrow & \searrow^{X \mapsto \phi(T)} & \\ R\langle T \rangle^h & \longrightarrow & C \end{array}$$

commutative, and if $\phi(T) \notin (C_0)^\circ$ then there is no morphism closing the triangle. This shows the claim.

Similar computations can be worked out for the case $R\langle T^{-1} \rangle^h$. \square

Remark 5.9. It is important to remark that the morphism $R \rightarrow R\langle T \rangle = c(R\langle T \rangle^h)$, although it is an epimorphism of Banach algebras, it is not a homotopy epimorphism (similar computations to the ones done above show that $\text{LH}^1(R\langle T \rangle \widehat{\otimes}_R^{\mathbb{L}} R\langle T \rangle) \neq 0$). This is one of the reasons why the usual methods do not work for general Banach rings.

The computations of Proposition 5.8 permit to describe the restriction map

$$R \rightarrow R\langle T \rangle^h \oplus R\langle T^{-1} \rangle^h$$

as the map

$$R \rightarrow [R\langle X \rangle \xrightarrow{\mu(X \rightarrow T)} R\langle X \rangle] \oplus [R\langle X \rangle \xrightarrow{\mu(T \rightarrow X^{-1})} R\langle X \rangle].$$

We notice that, differently to the restriction map of the complex (5.4.2) where

$$\text{Ker}(R \rightarrow R\langle T \rangle \oplus R\langle T^{-1} \rangle) = \overline{(Z)},$$

one has that

$$\text{Ker}(R \rightarrow [R\langle X \rangle \xrightarrow{\mu(X \rightarrow T)} R\langle X \rangle] \oplus [R\langle X \rangle \xrightarrow{\mu(T \rightarrow X^{-1})} R\langle X \rangle]) = (Z),$$

so still there is a kernel when global sections are restricted to the cover U, V , although it is smaller. Then, to understand why the derived Čech-Tate complex is strictly exact we need to compute the derived intersection

$$R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T^{-1} \rangle^h.$$

Proposition 5.10. *The following isomorphisms hold*

$$\begin{aligned} \text{LH}^n(R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T^{-1} \rangle^h) &\cong 0, \quad n \geq 2 \\ \text{LH}^1(R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T^{-1} \rangle^h) &\cong (Z), \\ \text{LH}^0(R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T^{-1} \rangle^h) &\cong \frac{R\langle X, Y \rangle}{(X - T, YT - 1)}. \end{aligned}$$

In particular, $R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T^{-1} \rangle^h$ is not concentrated in degree 0.

Proof. The computation of LH^0 is similar to the computation done in Proposition 5.8. We do not repeat it here. It is more interesting to understand why there is cohomology in degree 1. Again, using the theory of Koszul complexes, we have that

$$\text{LH}^1(R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T^{-1} \rangle^h) \cong \frac{(X - T) \cap (YT - 1)}{(X - T)(YT - 1)}.$$

Now suppose that $Z \in (X - T)(YT - 1)$, this means that the power-series

$$Z(X - T)^{-1}(YT - 1)^{-1}$$

converges in $R\langle X, Y \rangle$ but

$$Z\left(\sum_{n=0}^{\infty} T^{-n+1} X^n\right)\left(\sum_{n=0}^{\infty} T^n Y^n\right) = Z\left(\sum_{n,m=0}^{\infty} T^{-n+1+m} X^n Y^m\right)$$

but for the terms with $n = m + 1$ we get

$$Z\left(\sum_{n=0}^{\infty} X^n Y^{n-1}\right) = \sum_{n=0}^{\infty} Z X^n Y^{n-1}$$

that is not a convergent series, therefore $Z(X - T)^{-1}(YT - 1)^{-1}$ does not belong to $R\langle X, Y \rangle$. Therefore, $Z \notin (X - T)(YT - 1)$. \square

We do not give an elementary proof of the fact that the morphism $R \rightarrow R\langle T \rangle^h \widehat{\otimes}_R^{\mathbb{L}} R\langle T^{-1} \rangle^h$ is a homotopy epimorphism as the computations are quite long and it is not necessary as one of the basic properties of the derived tensor product is the preservation of homotopy epimorphisms (cf. Proposition 2.27).

With this last piece of computation we have an explicit description of the derived Čech-Tate complex of the cover U, V of $\mathrm{Spa}(R)$, that can be written as

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ & & & & & & 0 \\ & & & 0 & \longrightarrow & R\langle X, Y \rangle & \\ & & & \downarrow & & \downarrow & \\ & & & 0 & \longrightarrow & R\langle X \rangle \oplus R\langle Y \rangle & \longrightarrow & R\langle X, Y \rangle^2 \\ & & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & R & \longrightarrow & R\langle X \rangle \oplus R\langle Y \rangle & \longrightarrow & R\langle X, Y \rangle & \longrightarrow & 0 \end{array}$$

then the total complex is given by

$$0 \rightarrow R \oplus R\langle X \rangle \oplus R\langle Y \rangle \oplus R\langle X, Y \rangle \xrightarrow{\alpha} R\langle X \rangle \oplus R\langle Y \rangle \oplus R\langle X, Y \rangle^2 \xrightarrow{\beta} R\langle X, Y \rangle \rightarrow 0.$$

Notice that the morphism β is obviously surjective. The morphism α is given by

$$(r, f, g, h) \mapsto (r + (X - T)f, r + (YT - 1)g, (YT - 1)h - f, -(X - T)h - g)$$

and it is easy to check that it is injective, because

$$r + (X - T)f = 0$$

implies $(X - T)f \in (Z)$ and the same for g . And in such cases there is no h such that

$$(YX - 1)h - f = 0$$

and

$$(X - T)h - g = 0$$

because these equalities imply $h = Z(X - T)^{-1}(YX - 1)^{-1}h'(T)$, with $h'(T) \in R$, but we have seen in the proof of Proposition 5.10 that such an h is not an element of $R\langle X, Y \rangle$. Therefore, using Banach's Open Mapping Theorem for k we just need to check that $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$

set-theoretically to deduce the strict exactness of the sequence. So, suppose that $x \in R\langle X \rangle \oplus R\langle Y \rangle \oplus R\langle X, Y \rangle^2$ is such that

$$\beta(x) = 0.$$

We need to show that $x \in \text{Im}(\alpha)$. If we write $x = (x_1, x_2, x_3, x_4)$ then we have that

$$\beta(x) = x_1 - x_2 + x_3(X - T) + x_4(YT - 1).$$

Comparing term-wise we get that

$$x_3(X - T) + x_4(YT - 1)$$

cannot have any term with $X^n Y^m$ with both $n, m \geq 1$. Therefore,

$$x_3 = (h(YT - 1) + (X - T)^{-1}p + f), \quad x_4 = (-h(X - T) + (YT - 1)^{-1}p + g)$$

with $f \in R\langle X \rangle$ and $g \in R\langle Y \rangle$ and $p \in (Z)$. This implies that $x_1 = (X - T)f$ and $x_2 = g(YT - 1)$ proving that $x \in \text{Im}(\alpha)$, because the map $(X - T)R\langle X \rangle \oplus (YT - 1)R\langle Y \rangle \rightarrow R\langle X, Y \rangle$ has (Z) as kernel.

Roughly speaking, what is happening in this example is that the global sections that restrict to zero in LH^0 are shifted in degree 1 in cohomology, a phenomenon that can be understood only within the framework of derived geometry. If we represent the left-heart cohomology of the double complex that computes the derived Čech-Tate complex we get

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ & & & & & & (Z) \\ & & & & & & \downarrow \\ & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R_0 \oplus (Z) & \longrightarrow & R_T \oplus R_{T^{-1}} & \longrightarrow & R_{T, T^{-1}} \longrightarrow 0 \end{array}$$

where we have written $R = R_0 \oplus (Z)$ (that makes sense as R -modules in $\text{LH}(\mathbf{Ban}_R)$), $R_T = \text{LH}^0(R\langle T \rangle^h)$, $R_{T^{-1}} = \text{LH}^0(R\langle T^{-1} \rangle^h)$ and $R_{T, T^{-1}} = \text{LH}^0(R\langle T, T^{-1} \rangle^h)$. Hence (again roughly speaking), the total complex computes the cohomology of the complex

$$0 \rightarrow R_0 \oplus (Z) \rightarrow R_T \oplus R_{T^{-1}} \oplus (Z) \rightarrow R_{T, T^{-1}} \rightarrow 0$$

that is manifestly exact because (Z) is mapped isomorphically to (Z) , that is a contribution coming from the derived intersection of U and V that is missing in the non-derived Tate-Čech complex.

We conclude Example 5.4 by remarking that from the derived sheaf

$$U \left(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right) \mapsto K_{R\langle X_1, \dots, X_n \rangle}(f_0 X_1 - f_1, \dots, f_0 X_n - f_n)$$

one can recover the structural pre-sheaf defined in Huber's theory by considering the pre-sheaf

$$U \left(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right) \mapsto c(\text{LH}^0(K_{R\langle X_1, \dots, X_n \rangle}(f_0 X_1 - f_1, \dots, f_0 X_n - f_n))).$$

We hope that the computations done in Example 5.4 convinced the reader of the powerfulness of the results proved in Section 4.

6. CONCLUSIONS

We give some comments on the results proved in this paper. The main results we have proved show that for any Banach ring R , or bornological algebra R , satisfying the hypothesis stated in Section 4, defined over a strongly Noetherian Tate ring A there exists a homotopical Huber spectrum $|\mathrm{Spa}_{\mathrm{Rat}}^h(R)|$ canonically associated to R , that refines the usual Huber spectrum $\mathrm{Spa}(R)$ and that is equipped with the structure of a derived analytic space. These results are by no means optimal and we think that future work can lead to generalizations in the following directions.

- It should be possible to remove the hypothesis on A being Tate. We think that the theory of reified spaces introduced by Kedlaya [18] should play an important role in this. As Kedlaya has proved all the basic results of the theory of Huber spaces used in this paper in the context of reified spaces, most of the proofs should translate verbatim for any *really strongly Noetherian* (in the sense of [18, Definition 8.4]) Banach ring A . As $(\mathbb{Z}, |-|_0)$ is really strongly Noetherian and $(\mathbb{Z}, |-|_0)$ is the initial object in the category of ultrametric Banach rings we get that the results of the reified version of the results presented in this paper apply to all non-Archimedean Banach rings, without any restriction. Notice also that in the context of reified spaces, the analog of the hypothesis of equation (4.20.2) is always satisfied and therefore it poses no restriction on R .
- We also think that all Banach rings can be considered, even the non-ultrametric ones. This leads to some complications as the contracting category of Banach modules over a non-ultrametric ring is not additive. Moreover, the correct setting in which the general version of the theory must be written is that of reified space, because one needs to consider polydisks of any real polyradius to obtain the correct constructions. We must underline that the Archimedean situation presents further complications that are not present in the non-Archimedean theory. We refer to the (almost finished at the time of writing) pre-print [6] for an in-depth study of this issue.
- The theory should work also for other kinds of spaces (and maybe sometimes even become easier). For example, Proposition 3.20 tells us that the over-convergent analytic functions on polydisks and the analytic functions on open disks have the same formal properties of the closed disks used in this paper (and in the theory of reified spaces). Therefore, the results of this paper should hold, for example, for (infinite dimensional) dagger analytic spaces with similar proofs.

These directions of inquiry will be investigated in future works.

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