Università degli Studi di Padova

Dipartimento di Matematica "Tullio Levi-Civita"

Corso di Dottorato di Ricerca in Scienze Matematiche
Curricolo Matematica
Ciclo XXXVII

ON P-ADIC UNIFORMIZATION OF ABELIAN VARIETIES

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Anno accademico 2024

Luận án này dành cho 50 năm của Mẹ
và 60 năm của Ba.
Pentru a 70-a aniversare a profesorului meu.

Chờ mưa tạnh ta trải trăng làm chiếu
Nghìn năm sau hoa trắng trổ trên đồi.
-Tuệ Sỹ

Abstract

This thesis focuses on exploring a new p-adic uniformization of abelian varieties with semistable reduction, called conjugate uniformization. While the case of good reduction has been previously addressed by Iovita–Morrow–Zaharescu through Fontaine integration, we extend their work by introducing a logarithmic version of Fontaine integration, building upon their approach. Under a mild condition, we generalize their main results to abelian varieties with semistable reduction, adapting the techniques employed by Iovita and colleagues. Furthermore, we investigate Raynaud uniformization, identifying a class of abelian varieties that meet the necessary conditions, leading to the uniformization established by Iovita–Morrow–Zaharescu.

Acknowledgements

This thesis is deeply indebted to the mathematical expertise and guidance of my advisor, Adrian Iovita. His generosity with his time and ideas, unwavering patience with my work, and articulate explanations of intricate mathematical concepts have been instrumental in shaping my understanding and progress.

I am also profoundly grateful to my co-advisor, Nicola Mazzari, for his invaluable guidance, constant support, and encouragement during my time in Padova. He introduced me to the fascinating realm of p-adic Hodge theory and has been a steadfast guide in its exploration. The foundational concept of logarithmic Fontaine integration stems from his insights.

I am profoundly grateful to the PhD program within the Mathematics Department at the esteemed University of Padova for graciously providing the funding that made my doctoral studies possible. Over the course of the past three years, this academic odyssey has been a source of immense enrichment and personal growth, an experience I hold in the highest regard. I extend my deepest appreciation to Prof. Giovanni Colombo and Loretta Dalla Costa for their unwavering guidance and steadfast support throughout every twist and turn of my journey. Furthermore, I wish to express my gratitude to the dedicated professors and staff at the University of Padova, particularly those within the number theory group, whose firm assistance has been pivotal in navigating the challenges inherent to a PhD program. Their collective expertise and encouragement have been instrumental in my academic development and the successful completion of my doctoral studies.

I would like to thank Professor Heer Zhao and Professor Veronika Ertl-Bleimhofer for serving as my referees. Their careful reading and insightful feedback have significantly improved my thesis.

Special mention is due to Professor Chikara Nakayama and Professor Heer Zhao for their enlightening discussions and insightful guidance on logarithmic Abelian varieties, which is the main ingredient in my thesis. To all those who have played a role, whether large or small, in this transformative chapter of my academic journey, I extend my sincerest thanks. Your contributions have left an indelible mark on my scholarly pursuits and have helped shape the researcher I am today.

I would also like to extend my heartfelt appreciation to my extensive network of friends and collaborators, among whom Vu Anh Nguyen, Ngoc Bach Tran, Bao Tran, Dat Pham, Manh-Hung Le, Sang Hong Van Nguyen, Minh-Tri Le, Thanh Vuong Dang, Huy Dang, Quang Khai Nguyen, Khanh Hung Nguyen, Suraj Dash, Arshay Sheth, Luca Mastella, Marco Baracchini, Pietro Vanni, Runlei Xiao, and numerous others stand out prominently. Their constant encouragement, genuine kindness, and invaluable assistance have been instrumental throughout my academic journey. Each interaction has enriched my experience, leaving an indelible mark on my growth and development. To all those who have stood by me, offering solidarity and camaraderie, I am deeply grateful for the privilege of learning and evolving alongside you.

Finally, I am deeply indebted to my beloved family, whose unwavering love and unyielding support have been the bedrock upon which I have built my journey of achievement. Their persistent belief in my potential has infused me with the courage to pursue my dreams relentlessly. Through every triumph and trial, their presence has been my guiding light, illuminating the path forward with love, encouragement, and faith. To my parents, I offer my sincerest gratitude for being the driving force behind my accomplishments and for shaping me into the person I am today. To my beloved Chuchi, your presence and solidarity mean the world to me.

Contents

Introduction

0.1 Uniformization of abelian varieties over complex numbers

Consider an abelian variety A over $\mathbb{C}$. The group $A(\mathbb{C})$ is a compact complex Lie group. A celebrated theorem of Riemann asserts that $A(\mathbb{C})$ has the form of a torus (cf. [1]).

Theorem 0.1.1 (Complex uniformization of abelian varieties)**.** *Let A be a g-dimensional abelian variety over $\mathbb{C}$. Then there exists a full-rank lattice Λ inside $\mathbb{C}^g$ such that $\mathrm{A}(\mathbb{C}) \cong \mathbb{C}^g/\Lambda$ as Lie groups.*

Riemann's theorem is grounded in a theory of integration over complex varieties, originating from Abel's work in the 19th century. Specifically, let $\mathrm{H}_1(\mathrm{A}(\mathbb{C}), \mathbb{Z})$ denote the first homology group of the complex variety $A(\mathbb{C})$, and $\mathrm{H}^0(\mathrm{A}, \Omega^1_{\mathrm{A}})$ represent the space of Kähler differential forms on A. Integrating a form $\omega \in \mathrm{H}^0(\mathrm{A}, \Omega^1_{\mathrm{A}})$ along a class $\gamma \in \mathrm{H}_1(\mathrm{A}(\mathbb{C}), \mathbb{Z})$ establishes a perfect pairing, as detailed in [1].

$$
\begin{aligned}
\int : \mathrm{H}_1(\mathrm{A}(\mathbb{C}), \mathbb{Z}) \times \mathrm{H}^0(\mathrm{A}, \Omega^1_{\mathrm{A}}) &\to \mathbb{C} \\
(\gamma, \omega) &\mapsto \int_\gamma \omega
\end{aligned}
$$

Note that the dual of $\mathrm{H}^0(\mathrm{A}, \Omega^1_{\mathrm{A}})$ is the Lie algebra $\mathrm{Lie}(\mathrm{A}(\mathbb{C}))$ of $\mathrm{A}(\mathbb{C})$. The pairing gives rise to a morphism

$$
\begin{aligned}
\phi : \mathrm{H}_1(\mathrm{A}(\mathbb{C}), \mathbb{Z}) &\to \mathrm{Lie}\,\mathrm{A}(\mathbb{C}) \\
\gamma &\mapsto \left(\omega \mapsto \int_\gamma \omega\right)
\end{aligned}
$$

It turns out that the map ϕ is injective, with its cokernel precisely being $\mathrm{A}(\mathbb{C})$.

Consequently, we achieve the complex uniformization of the abelian variety A.

$$\mathrm{A}(\mathbb{C}) \cong \frac{\mathrm{Lie}\,\mathrm{A}(\mathbb{C})}{\phi(\mathrm{H}_1(\mathrm{A}(\mathbb{C}), \mathbb{Z}))} \cong \mathbb{C}^g/\Lambda \tag{1}$$

The converse of the Riemann theorem is true for 1-dimensional case; that is, every $\mathbb{C}/\Lambda$ corresponds to some elliptic curve. This insight has been instrumental in constructing the classification space, or moduli space, of elliptic curves, which plays a pivotal role in various fields, including the theory of modular forms.

Utilizing Riemann's theorem, we can investigate the rational points of a complex abelian variety. Through the isomorphisms $\mathbb{C}^g \cong \mathbb{R}^{2g}$ and $\Lambda \cong \mathbb{Z}^{2g}$, we can regard A as a real torus $(\mathbb{R}/\mathbb{Z})^{2g}$. The set of torsion points of A corresponds to $(\mathbb{Q}/\mathbb{Z})^{2g}$. To elaborate, suppose $\lambda_1, \ldots, \lambda_{2g}$ form a $\mathbb{Z}$-basis for Λ. A torsion point $P \in A$ is then associated with a class modulo Λ of a vector

$$r_1\lambda_1 + \cdots + r_{2g}\lambda_{2g} \in \mathbb{R}^{2g}$$

Here, r_i are rational numbers. Assume we fix the expression, if P has an exact order N, then each r_i has the common denominator N. This observation serves as a foundation in various developments concerning arithmetic questions about abelian varieties, as seen in [2]. Another noteworthy application of the uniformization of complex elliptic curves is the construction of Heegner points, as discussed in [3]. These points played a pivotal role in proving the Birch and Swinnerton-Dyer conjecture for rank 1, as demonstrated in [4].

0.2 Uniformization of abelian varieties over p-adic numbers

Originally introduced by the German school in algebraic number theory to explore the approximation of integers modulo a specific prime, p-adic numbers have since found applications across various mathematical disciplines. To gain insights about a scheme over $\mathbb{Z}$, it is often more approachable to complete it at a prime number p. The underlying philosophy is that by understanding enough information for a range of primes p, we can grasp the global structure of the object in question. This perspective prompts us to consider the uniformization of abelian varieties over a p-adic field.

In this section, consider A as an abelian variety over a finite extension K of

$\mathbb{Q}_p$, and let $V_p(A)$ denote the rational Tate module associated with A.

0.2.1 The Raynaud uniformization

The initial solution to the uniformization question was provided by Tate in his seminal work [5]. He introduced a theory over p-adic fields akin to complex analytic spaces, termed rigid analytic spaces. In this groundbreaking research, Tate demonstrated that an elliptic curve with multiplicative reduction can be uniformized as the rigid analytic space $\mathbb{C}_p^*/q^{\mathbb{Z}}$, where $\mathbb{C}_p$ serves as the p-adic counterpart of $\mathbb{C}$ and $q \in \mathbb{C}_p$ with a valuation $v_p(q) > 1$. It resembles the classical case because we can view the torus $\mathbb{C}^2/\mathbb{Z}^2$ as $\mathbb{C}^*/q^{\mathbb{Z}}$. Tate developed a theory of analytic spaces over fields equipped with a complete non-Archimedean valuation, drawing inspiration from Grothendieck's ideas. In this framework, the quotient $\mathbb{C}_p^*/q^{\mathbb{Z}}$ is well-defined. The research field was later expanded and refined by contributions from mathematicians like Grauert, Remmert, Kiehl, Raynaud, and others (see [6]).

In 1970, Raynaud proposed a program addressing the uniformization of abelian varieties, as documented in [7]. Instead of the uniformizing space $\mathbb{C}^g$, Raynaud introduced an extension G of an abelian variety B (with good reduction) by a split affine torus T. The abelian variety A is then represented as a rigid geometric quotient G/Λ, where Λ is a lattice in G of rank equal to $\dim T$, determined by the semi-abelian reduction of A. A key element in Raynaud's proof is Grothendieck's semi-stable reduction theorem, as outlined in [8]. This work spurred the development of an extensive theory of analytic spaces over p-adic fields, providing a natural interpretation for the quotient G/Λ.

An important application of the Raynaud uniformization is the Coleman–Iovita monodromy theorem [9]. More precisely, the Raynaud uniformization gives rise to the following two exact sequences of Tate modules of T, G, B, A

$$0 \to V_p(T) \to V_p(G) \to V_p(B) \to 0 \tag{2}$$

and

$$0 \to V_p(G) \to V_p(A) \to \Lambda \otimes \mathbb{Q}_p \to 0 \tag{3}$$

Denote by $N : \mathbf{D}_{\mathrm{st}}(V_p(A)) \to \mathbf{D}_{\mathrm{st}}(V_p(A))$ the monodromy operator of $\mathbf{D}_{\mathrm{st}}V_p(A)$.

Then the Coleman–Iovita monodromy theorem states that

$$\ker N = \mathbf{D}_{\mathrm{cris}}(V_p(G)) = \mathbf{D}_{\mathrm{cris}}(V_p(A))$$

Consequently, A has good reduction if and only if $\ker N = 0$ if and only if $V_p(A)$ is crystalline.

0.2.2 The Iovita–Morrow–Zaharescu uniformization

Fontaine introduced a strategy to reconstruct the $\mathbb{C}_p$ points of A using only the torsion points of $A(\overline{K})$, as detailed in [10]. Specifically, let $\mathrm{Lie}(A)$ denote the Lie algebra of A. The logarithm yields the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A_{\mathrm{tor}}(\overline{K}) \arrow[r] \arrow[d] & A(\overline{K}) \arrow[r, "\log_A"] \arrow[d, hook] & \mathrm{Lie}(A)(\overline{K}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{\mathrm{tor}}(\mathbb{C}_p) \arrow[r] & A(\mathbb{C}_p) \arrow[r, "\log_A"] & \mathrm{Lie}(A)(\mathbb{C}_p) \arrow[r] & 0
\end{tikzcd}$$

here $A_{\mathrm{tor}}(\overline{K})$ and $A_{\mathrm{tor}}(\mathbb{C}_p)$ represent the torsion subgroups of $A(\overline{K})$ and $A(\mathbb{C}_p)$, respectively. We can further decompose the torsion subgroup into p-power torsion, denoted as $A_{p\text{-tors}}(\overline{K})$, and the prime-to-$p$ torsion, denoted as $A_{p'\text{-tors}}(\overline{K})$.

$$A_{\mathrm{tor}}(\overline{K}) = A_{p\text{-tors}}(\overline{K}) \oplus A_{p'\text{-tors}}(\overline{K})$$

Fontaine constructed a G_K-equivariant section

$$s : A(\mathbb{C}_p) \to A_{p'\text{-tors}}(\overline{K})$$

of the natural inclusion $A_{p'\text{-tors}}(\overline{K}) \subset A(\mathbb{C}_p)$ defined as follows. There exists an $\mathcal{O}_K$-lattice Λ in $\mathrm{Lie}(A)(K)$ and a map

$$\exp_\Lambda : \mathcal{O}_{\mathbb{C}_p} \otimes_{\mathcal{O}_K} \Lambda \to A(\mathbb{C}_p)$$

such that $\log_A(\exp_\Lambda(x)) = x$ for $x \in \mathcal{O}_{\mathbb{C}_p} \otimes_{\mathcal{O}_K} \Lambda$. For $x \in A(\mathbb{C}_p)$, for sufficiently large n we have $p^n \log_A(x) \in \mathcal{O}_{\mathbb{C}_p} \otimes_{\mathcal{O}_K} \Lambda$ and we define

$$s(x) := p^{-n}\pi\left(\frac{x^{p^n}}{\exp_\Lambda(p^n \log_A(x))}\right),$$

where $\pi : A_{\mathrm{tor}}(\overline{K}) \to A_{p'\text{-tors}}(\overline{K})$ is the canonical projection. We denote the kernel of s by $A^{(p)}(\mathbb{C}_p)$. We then have the decomposition

$$A(\mathbb{C}_p) = A^{(p)}(\mathbb{C}_p) \oplus A_{p'\text{-tors}}(\overline{K})$$

Let $A^{(p)}(\overline{K}) := A^{(p)}(\mathbb{C}_p) \cap A(\overline{K})$, we also get

$$A(\overline{K}) = A^{(p)}(\overline{K}) \oplus A_{p'\text{-tors}}(\overline{K})$$

Fontaine claims that we can recover $A^{(p)}(\overline{K})$, and so also $A^{(p)}(\mathbb{C}_p)$ from the knowledge of $A_{p\text{-tors}}(\overline{K})$ (see [10, Proposition 8.6]).

Recently, Iovita, Morrow, and Zaharescu [11] investigated the Fontaine integration [12] to obtain a new form of p-adic uniformization for abelian varieties with good reduction, resembling the classical complex uniformization. More precisely, let $T_p(A)$ denote the Tate module associated to A, Fontaine defined a G_K-equivariant map called Fontaine integration

$$\varphi_A : T_p(A) \to \mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1)$$

Assuming that the abelian variety possesses good reduction and $T_p(A)^{I_K} = 0$, where I_K is the inertial subgroup of G_K, Iovita et al. demonstrated the injectivity of the Fontaine integration.

Moreover, the authors extended Fontaine integration to a broader space known as the p-adic universal cover of A, denoted by $\mathcal{B}_A$. We have $T_p(A) \subset \mathcal{B}_A$, and the cokernel is precisely $A(\overline{K})$. This leads to the fundamental exact sequence of G_K-modules:

$$0 \to T_p(A) \to \mathcal{B}_A \to A(\overline{K}) \to 0 \tag{4}$$

Under the same aforementioned assumptions, they were able to identify the kernel of the extended Fontaine integration, which manifests as the prime-to-p torsion points of $A(\overline{K})$. We encapsulate the aforementioned discussion through the following commutative diagram

$$\begin{tikzcd}
 & & \ker\varphi_A \arrow[r, equal] \arrow[d] & A_{p'-tor}(\overline{K}) \arrow[d] & \\
0 \arrow[r] & T_p(A) \arrow[r] \arrow[d, equal] & \mathcal{B}_A \arrow[r] \arrow[d, "\varphi_A"] & A(\overline{K}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_p(A) \arrow[r, "\varphi_A"] & \mathrm{Lie}(A)(K)\otimes_K \mathbb{C}_p(1) \arrow[r] & \mathrm{Lie}(A)(K)\otimes_K \mathbb{C}_p(1)/\varphi_A(T_p(A)) \arrow[r] & 0
\end{tikzcd}$$

It follows that we obtain an inclusion called Iovita–Morrow–Zaharescu's uniformization map (see [11, Theorem A])

$$\iota_A : \mathrm{A}^{(p)}(\overline{K}) := \mathrm{A}(\overline{K}) / \,\mathrm{A}_{\text{p'-tors}}(\overline{K}) \hookrightarrow \frac{\mathrm{Lie}(A)\otimes_K \mathbb{C}_p(1)}{\varphi_A(T_p\,\mathrm{A})}$$

Moreover, one can recover $\mathrm{A}^{(p)}(\overline{K})$ from the triple (see [11, Theorem B])

$$(T_p(A),\ \mathrm{Lie}(A)(K)\otimes_K \mathbb{C}_p(1),\ \varphi_A : T_p(A) \to \mathrm{Lie}(A)(K)\otimes_K \mathbb{C}_p(1))$$

Through the uniformization map, we can effectively correlate the p-divisible points on both ends, represented by

$$\iota_A(A(\overline{K})[p^\infty]) = \frac{\mathrm{Lie}(A)(K)\otimes_K \mathbb{C}_p(1)}{\varphi_A(T_p(A))}[p^\infty]$$

This perspective unveils a new insight into the exploration of non-torsion points within $A(\overline{K})$, which are projected onto a specific category of good elements within $\frac{\mathrm{Lie}(A)(K)\otimes_K \mathbb{C}_p(1)}{\varphi_A(T_p(A))}$.

0.2.3 Main results

This thesis studies the p-adic uniformization of abelian varieties akin to the approach of Iovita–Morrow–Zaharescu but with a focus on the semi-stable reduction scenario. The primary objective of our research is to establish a framework that extends the method employed in cases of good reduction. We introduce the concept of logarithmic Fontaine integration, which can be viewed as an extension of Fontaine integration tailored for semi-stable reduction cases, following the approach of Iovita et al.

Theorem A (cf. 5.1.16)**.** *Let A be a semi-stable abelian variety over K. There* *exists a G_K-equivariant morphism called logarithmic Fontaine integration*

$$\varphi_A : \mathcal{B}_A \to \mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1)$$

Under the hypothesis $T_p(A)^{I_K} = 0$, the kernel of the logarithmic Fontaine integration is the prime-to-p torsion points of $\mathrm{A}(\overline{K})$, denoted by $\mathrm{A}_{p'\text{-}tors}(\overline{K})$.

Central to our approach is the theory of logarithmic abelian varieties, as developed by Kato et al. ([13], [14], [15], [16]). Our proof of the aforementioned theorem follows the methodology outlined in [11].

The primary application of our thesis lies in achieving p-adic uniformization for abelian varieties with semi-stable reduction. It is the generalization of theorem B in [11].

Theorem B (cf. 6.3.6)**.** *Let K be a finite extension of $\mathbb{Q}_p$. Let A be an abelian variety over K with semi-stable reduction. Assume that $T_p(A)^{I_K} = 0$, we obtain an inclusion called p-adic uniformization map*

$$\iota_A : \mathrm{A}^{(p)}(\overline{K}) := \mathrm{A}(\overline{K}) / \mathrm{A}_{p'\text{-}tors}(\overline{K}) \hookrightarrow \frac{\mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1)}{\varphi_A(T_p\,\mathrm{A})}$$

The image of ι_A corresponds to semi-stable elements in $\frac{\mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1)}{\varphi_A(T_p\,\mathrm{A})}$.

Moreover, one can recover $\mathrm{A}^{(p)}(\overline{K})$ from the triple

$$(T_p(A),\ \mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1),\ \varphi_A : T_p(A) \to \mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1))$$

In addition, we derive a result concerning the hypothesis $T_p(A)^{I_K} = 0$. For elliptic curves with good reduction, this condition is equivalent to the absence of complex multiplication for E. In this thesis, we employ p-adic Hodge theory to identify conditions under which this assumption holds. We introduce the following definition.

Definition 0.2.1. *Given a non-split exact sequence of I_K-modules*

$$0 \to M \to N \to P \to 0$$

We say the sequence is I_K strongly non-split if for any I_K-submodule P' of P, the pull-back sequence

$$0 \to M \to N' \to P' \to 0$$

is also non-split, where $N' := N \times_P P'$.

We obtain the following result.

Theorem C (cf. 6.2.11)**.** *If* $T \neq 0$ *and assuming further that* $0 \to T_p(T) \to T_p(G) \to T_p(B) \to 0$ *is* I_K *strongly non-split, then* $T_p(A)^{I_K} = 0$.

Our argument relies on the monodromy theorem à la Coleman–Iovita.

0.3 Outline of the construction

The first main ingredient of our process is the logarithmic Fontaine differential forms developed by Mazzari in [17]. More precisely, they are logarithmic differential forms $\Omega^{\log} := \Omega_{(\mathcal{O}_{\overline{K}},M)/(\mathcal{O}_K,N)}$ where M and N are certain logarithmic structures considered in section 3.1. A notable property of these logarithmic differential forms is that their Tate modules naturally contain the log derivatives of the generators of the Tate module of certain Tate curves (see Proposition 3.2.2).

Proposition 0.3.1. *Let denote* $\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ *by* Ω*. We have a natural inclusion* $V_p(\Omega) \hookrightarrow V_p(\Omega^{\log})$ *which uniquely splits as a homomorphism of* $\mathbb{C}_p$*-representations of* G_K.

As a motivating example, we provide the p-adic uniformization of multiplicative groups and Tate curves in a very explicit way.

The second crucial input of our construction is the notion of logarithmic abelian variety introduced and developed by Kato et al (see [13], [14], [15], [16], [18]). They are sheaves of abelian groups on (fs/S), the category of fine and saturated logarithmic schemes over the standard logarithmic trait $S := (\operatorname{Spec} \mathcal{O}_K, (1 \mapsto \pi)^a)$ with étale topology (see section 4.3 for more details). In the context of semi-stable reduction, they serve as substitutes for Néron models. More precisely, there is an equivalence between the category of log abelian varieties over $\mathcal{O}_K$ with the standard log structure to the category of semi-stable abelian varieties over K.

From now on, for any abelian variety A over K, we denote its corresponding logarithmic abelian variety over $\mathcal{O}_K$ as $\mathcal{A}$. It follows that

$$A(\overline{K}) = \varinjlim_{L/K\text{finite}} A(L) = \varinjlim_{L/K\text{finite}} \mathcal{A}(\mathcal{O}_L, (1 \mapsto \pi_L)^a)$$

Hence, for any $x \in A(\overline{K})$, there exists a finite field extension L of K such that $x \in \mathcal{A}(\mathcal{O}_L, (1 \mapsto \pi_L)^a)$. We then have the following commutative diagram of morphisms of sheaves on (fs/S) (see 5.1)

$$\begin{tikzcd} x : (\operatorname{Spec} \mathcal{O}_L, (1 \mapsto \pi_L)^a) \arrow[r] \arrow[dr] & \mathcal{A} \arrow[d] \\ & S. \end{tikzcd}$$

We can define the pull-back of x locally on a certain subcategory of (fs/S) (see 5.1.4)

$$x^* : \omega_{\mathcal{A}/S} \to x_* \omega_{D_L/S}$$

where $\omega_{\mathcal{A}/S}$ is the sheaf of differential forms of $\mathcal{A}$ (see 4.6.8 for the definition). Then for any $x \in A(\overline{K})$, we have a perfect pairing

$$\begin{aligned} A(\overline{K}) \times H^0(\mathcal{A}, \omega_{\mathcal{A}/S}) &\to \Omega^{\log} \\ (x, \omega) &\mapsto x^* \omega \end{aligned}$$

where $H^0(\mathcal{A}, \omega_{\mathcal{A}/S})$ is the global sections of $\omega_{\mathcal{A}/S}$. The pairing is compatible with the multiplication by p, we hence obtain the logarithmic version of the Fontaine integration (see 5.1.8)

$$\begin{aligned} (,)_A^{\log} : \mathcal{B}_A \times H^0(\mathcal{A}, \omega_{\mathcal{A}/S}) &\to V_p(\Omega^{\log}) \\ ((x_n)_n, \omega) &\mapsto (x_n^* \omega)_n \end{aligned}$$

We can prove that $H^0(\mathcal{A}, \omega_{\mathcal{A}/S}) = H^0(A, \omega_{A/K})$ and the dual of $H^0(A, \omega_{A/K})$ is $\mathrm{Lie}(A)(K)$. Thus we get the Fontaine integration in the following form (see 5.1.16).

Proposition 0.3.2. *Let A be an abelian variety over K. We have a G_K-equivariant map*

$$\varphi_A^{\log} : \mathcal{B}_A \to \mathrm{Lie}(A)(K) \otimes_K V_p(\Omega^{\log})$$

The splitting $V_p(\Omega^{\log}) \to V_p(\Omega) \cong \mathbb{C}_p(1)$ yields $\mathcal{B}_A \to \mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1)$.

$$\begin{tikzcd} \mathcal{B}_A \arrow[r] \arrow[dr, dotted] & \mathrm{Lie}(A)(K) \otimes_K V_p(\Omega^{\log}) \arrow[d, dotted, bend left] \\ & \mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1) \arrow[u] \end{tikzcd}$$

0.4 Other constructions

In [19], Howe, Morrow and Wear employ the p-divisible group associated to a 1-motive to define the conjugate p-adic uniformization to p-adic formal semi-abelian schemes or p-divisible groups over the ring of integers of a p-adic field. For semi-stable curves, Katz and Litt [20] develop the theory of p-adic iterated integrals by using the unipotent log rigid fundamental group. This approach unifies several earlier constructions by Coleman, Berkovich, and Vologodsky.

0.5 What lies ahead?

While the realm of p-adic uniformization has been extensively investigated in specific areas, there are still many avenues that remain largely unexplored. As a result, numerous intriguing questions have surfaced that await answers.

The primary question at hand is whether we can remove the condition $T_p(A)^{I_K} = 0$ or demonstrate the injectivity of the Fontaine integration without depending on this condition. For example, in the context of Tate elliptic curves, we can directly establish this result.

Secondly, can we broaden the scope of the current construction to encompass semi-abelian varieties beyond formal semi-abelian varieties? Investigating p-adic uniformization of Shimura varieties could offer valuable insights. Such uniformization could have promising applications in arithmetic.

Lastly, the uniformization map may induce a functor from a particular subcategory of Galois modules to a category characterized by triples

$$(T_p(A),\ \mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1),\ \varphi_A : T_p(A) \to \mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1))$$

A pivotal question here is whether this functor is fully faithful. This aligns with the significant results established in [21, Theorem B] and [22, Theorem 0.1].

Structure of the thesis

In Chapter 1, we revisit the classical construction of Fontaine integration and its extended version as developed by Iovita–Morrow–Zaharescu. Additionally, we explore an alternative construction using 1-motives introduced by Howe–Morrow–Wear.

In Chapter 2, we present the fundamental concepts of logarithmic geometry. Specifically, we provide explicit computations of certain logarithmic structures that will be instrumental in subsequent chapters.

In Chapter 3, we introduce the module of logarithmic Fontaine differential forms $\Omega_{(\mathcal{O}_{\overline{K}},M)/(\mathcal{O}_K,N)}$, where M and N are specific logarithmic structures introduced in Chapter 2. As an illustrative application, we offer the p-adic uniformization of multiplicative groups and Tate curves, following the approach of Iovita–Morrow–Zaharescu.

In Chapter 4, we recall the theory of logarithmic abelian varieties, as introduced by Kajiwara–Kato–Nakayama. This theory plays a pivotal role in the subsequent construction of logarithmic Fontaine integration in Chapter 5.

In Chapter 5, we construct the logarithmic version of Fontaine integration, extending the foundational work of Iovita et al.

In Chapter 6, we identify a class of abelian varieties that meet the criterion $T_p(A)^{I_K} = 0$. Then we employ the methodology outlined by Iovita–Morrow–Zaharescu to establish the p-adic uniformization of abelian varieties with semistable reduction, subject to a mild condition. All results of this chapter are original.

Convention

Throughout this thesis, we will adhere to the following notations, unless stated otherwise.

1. For simplicity, we fix a rational prime $p > 2$ and a finite extension K of $\mathbb{Q}_p$. Denote by $\mathcal{O}_K$ the ring of integers of K, by π a uniformizer of $\mathcal{O}_K$ and by $k = \mathcal{O}_K/\pi$ its residue field. Let $\overline{K}$ be a fixed algebraic closure of K and $\mathcal{O}_{\overline{K}}$ be the ring of integers of $\overline{K}$. Let $\mathbb{C}_p = \widehat{\overline{K}}$ be the completion of $\overline{K}$ with respect to the unique extension of the normalized p-adic valuation on $\mathbb{Q}_p$.

2. For a tower of extensions $K \subset L \subset \overline{K}$, we denote by G_K and G_L the absolute Galois groups of K and L respectively.

3. For any $\mathbb{Z}_p$-module V with G_K-action, we set $V(n) = V \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes n}$, the n-th Tate twist of V.

4. For any abelian group M, we define

$$T_pM = \varprojlim_{x \mapsto px} M[p^n] = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p]/\mathbb{Z}, M)$$

   here the inverse limit is over $\mathbb{Z}$-modules each kill by some power of p. We also set

$$V_p(M) = \varprojlim_{x \mapsto px} M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p], M)$$

5. Let A be an abelian variety over K and $T_p(A) = T_p(A(\overline{K}))$ (resp. $V_p(A) = V_p(A(\overline{K}))$) be the Tate module (resp. rational Tate module) associated to A.

Chapter 1

Fontaine integration and p-adic uniformization of abelian varieties

In his seminal paper [12], Jean-Marc Fontaine introduced a morphism, now known as Fontaine integration, to establish the Hodge-Tate decomposition theorem for abelian varieties. Building on this, Iovita, Morrow, and Zaharescu explored the integration in [11] and discovered a new form of p-adic uniformization for abelian varieties over p-adic fields, resembling classical complex uniformization called conjugate uniformization. In this section, we revisit the construction of Fontaine integration and study the contributions of Iovita, Morrow, and Zaharescu. Additionally, we discuss an alternative approach presented by Howe, Morrow, and Wear in [19].

1.1 Fontaine integration

Consider an abelian variety A over a p-adic field K, with $\operatorname{Lie} A$ representing the Lie algebra of A. The p-adic Tate module associated with A is denoted by $T_p(A)$. Fontaine introduced a G_K-equivariant map

$$\varphi : T_p(A) \to \operatorname{Lie} A(K) \otimes_K \mathbb{C}_p(1)$$

The construction is as follows. Let $\mathcal{A}$ be a proper model of A, i.e. a proper scheme $\mathcal{A} \to \operatorname{Spec}(\mathcal{O}_K)$ such that $\mathcal{A} \times_{\operatorname{Spec}(\mathcal{O}_K)} \operatorname{Spec}(K)$ is isomorphic to A.

$$\begin{array}{ccc} A & \xrightarrow{\quad g \quad} & \mathcal{A} \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} \mathcal{O}_K \end{array}$$

Let $u \in A(\overline{K}) = \mathcal{A}(\mathcal{O}_{\overline{K}})$, where the equality stems from the properness of $\mathcal{A}$. Then, u corresponds to a morphism of schemes over $\mathcal{O}_K$.

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{O}_{\overline{K}}) \arrow[r, "u"] \arrow[dr] & \mathcal{A} \arrow[d] \\
& \mathrm{Spec}(\mathcal{O}_K)
\end{tikzcd}$$

For any $\omega \in H^0(\mathcal{A}, \Omega_{\mathcal{A}})$ where $\Omega_{\mathcal{A}} := \Omega_{\mathcal{A}/\mathcal{O}_K}$, the pull-back $u^*(\omega) \in \Omega := \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$. Consequently, this defines a pairing known as the Fontaine pairing.

$$\begin{aligned}
\langle , \rangle : H^0(\mathcal{A}, \Omega_{\mathcal{A}}) \times \mathcal{A}(\mathcal{O}_{\overline{K}}) &\to \Omega \\
(\omega, u) &\mapsto u^*\omega
\end{aligned}$$

The Fontaine pairing $\langle , \rangle$ is $\mathcal{O}_K$-linear with respect to ω and G_K-equivariant with respect to u.

Lemma 1.1.1. *We have*

$$H^0(A, \Omega_A) \cong K \otimes_{\mathcal{O}_K} H^0(\mathcal{A}, \Omega_{\mathcal{A}})$$

Proof. The lemma would follow simply from [23, Chap. III, Prop.9.3]. We can give a slightly different proof as follows. Denote by $g : A \to \mathcal{A}$ is the morphism given by the base extension, we have that $H^0(A, \Omega_A) \cong g^* H^0(\mathcal{A}, \Omega_{\mathcal{A}})$ [23, Proposition 8.1]. For every affine open $U = \mathrm{Spec}\, R$ of $\mathcal{A}$, we have $V = U \times_{\mathrm{Spec}\, \mathcal{O}_K} \mathrm{Spec}\, K = \mathrm{Spec}(R \otimes_{\mathcal{O}_K} K)$ is an affine open of A. It follows that

$$H^0(V, \Omega_A) = H^0(V, \Omega_V) = \Omega_{R \otimes K/K} = H^0(U, \Omega_{\mathcal{A}}) \otimes_{\mathcal{O}_K} K$$

here we used the compatibility of tensor product and differentials. Since it holds for every affine open and the glueing is unique, we have $H^0(A, \Omega_A) \cong K \otimes_{\mathcal{O}_K} H^0(\mathcal{A}, \Omega_{\mathcal{A}})$ □

The pairing is $\mathbb{Z}$-linear with respect to $u \in \mathcal{A}(\mathcal{O}_{\overline{K}})$, up to a p-power, as follows.

Proposition 1.1.2 ([12], Proposition 3)**.** *1. We can identify $H^0(A, \Omega_A)$ with $K \otimes_{\mathcal{O}_K} H^0(\mathcal{A}, \Omega_{\mathcal{A}})$ via the map $\omega \mapsto 1 \otimes \omega$. Furthermore, there exists an integer $r \geq 0$ such that the restriction of the map to $p^r H^0(\mathcal{A}, \Omega_{\mathcal{A}})$ is injective.*

2. *Given the integer r in (1), the Fontaine pairing is $\mathbb{Z}$-linear in u, i.e., for $\omega \in p^r H^0(\mathcal{A}, \Omega_{\mathcal{A}})$ and $u_1, u_2 \in \mathcal{A}(\mathcal{O}_{\overline{K}})$ we have*

$$\langle \omega, u_1 + u_2 \rangle = \langle \omega, u_1 \rangle + \langle \omega, u_2 \rangle$$

Therefore, we can define a pairing, also denoted by $\langle , \rangle$

$$\begin{aligned} \langle , \rangle : p^r \cdot H^0(\mathcal{A}, \Omega_{\mathcal{A}}) \times \mathcal{A}(\mathcal{O}_{\overline{K}}) \to \Omega \\ (\omega, u) \mapsto u^*(\omega) \end{aligned}$$

The pairing is $\mathcal{O}_K$-linear with respect to ω and $\mathbb{Z}[G_K]$-linear with respect to u. Now, let $u = (u_n)_{n \in \mathbb{N}} \in T_p(A)$ and $\omega \in p^r \cdot H^0(\mathcal{A}, \Omega_{\mathcal{A}})$, then the sequence $(u_n^*(\omega))_{n \geq 0}$ is a sequence of differentials in Ω satisfying $pu_{n+1}^*(\omega) = u_n^*(\omega)$, that is

$$(u_n^*(\omega))_{n \geq 0} \in \varprojlim \left(\Omega \xleftarrow{[p]} \Omega \xleftarrow{[p]} \cdots \Omega \xleftarrow{[p]} \cdots \right) \cong V_p(\Omega) \cong \mathbb{C}_p(1)$$

The latter isomorphism is obtained by the following theorem

Proposition 1.1.3 ([12], Corollaire 1)**.** *The $\mathcal{O}_{\overline{K}}$-module Ω is torsion and p-divisible, with semi-linear action of G_K. The canonical derivation $d : \mathcal{O}_{\overline{K}} \to \Omega$ is surjective. Furthermore, we have*

$$V_p(\Omega) \cong \mathbb{C}_p(1)$$

Hence we get the Fontaine integration

$$\begin{aligned} \varphi : T_p(A) \to \mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1) \\ u = (u_n)_{n \geq 0} \mapsto (\omega \mapsto (u_n^*(\omega))_{n \geq 0}) \end{aligned}$$

explicitly given in coordinates by

$$\varphi(u) = \sum \omega_i^\vee (u_n^*(\omega_i))$$

where $(\omega_i)_{i \in I}$ is an $\mathcal{O}_K$-basis of $p^r H^0(\mathcal{A}, \Omega_{\mathcal{A}})$ and $(\omega_i)_{i \in I}^\vee$ the associated dual basis of $\mathrm{Lie}(A)(K)$.

1.2 p-adic uniformization of abelian varieties with good reduction

Assuming that A has good reduction, we can choose $\mathcal{A}$ to be the Néron model of A, which is an abelian scheme over $\mathcal{O}_K$. Note that the Néron model $\mathcal{A}$ exists for any abelian variety but it is proper if and only if A has good reduction.

We can extend the definition of the Fontaine integration to the universal covering space of $A(\overline{K})$.

Definition 1.2.1. *Let A be an abelian variety with good reduction and let $\mathcal{A}$ be its Néron model. We define the universal covering space of $A(\overline{K})$ to be*

$$\begin{aligned}\mathcal{B}_A &:= \varprojlim \left(A(\overline{K}) \overset{[p]}{\leftarrow} A(\overline{K}) \overset{[p]}{\leftarrow} \cdots A(\overline{K}) \overset{[p]}{\leftarrow} \cdots \right) \\ &= \varprojlim \left(\mathcal{A}(\mathcal{O}) \overset{[p]}{\leftarrow} \mathcal{A}(\mathcal{O}) \overset{[p]}{\leftarrow} \cdots \mathcal{A}(\mathcal{O}) \overset{[p]}{\leftarrow} \cdots \right)\end{aligned}$$

here we write $\mathcal{O}$ for $\mathcal{O}_{\overline{K}}$.

The morphism $\alpha : \mathcal{B}_A \to A(\overline{K})$ sending $(u_n)_{n\in\mathbb{Z}_{\geq 0}}$ to u_0 gives the following fundamental exact sequence of G_K-modules

$$0 \to T_p(A) \to \mathcal{B}_A \overset{\alpha}{\to} A(\overline{K}) \to 0$$

Remark 1.2.2. *The construction of Fontaine integration cannot be straightforwardly extended to $\mathcal{B}_A$ using any arbitrary proper model. This limitation arises because the universal covering space $\mathcal{B}_A$ is not torsion-free, unlike $T_p(A)$. Specifically, $\mathcal{B}_A$ contains p-power torsion points. For instance, if $x \neq 0$ is a p-torsion point, the Fontaine pairing results in $\langle \omega, x\rangle = 0$ for any $\omega \in p^r \cdot H^0(\mathcal{A}, \Omega_{\mathcal{A}})$. In this case, the Néron model, which naturally possesses a group structure, can be employed to define the generalized Fontaine integration.*

Recall from the previous section that we have the Fontaine pairing with the Néron model $\mathcal{A}$

$$\begin{aligned}\langle , \rangle : H^0(\mathcal{A}, \Omega_{\mathcal{A}}) \times \mathcal{A}(\mathcal{O}_{\overline{K}}) &\to \Omega \\ (\omega, u) &\mapsto u^*\omega\end{aligned}$$

Now it is $\mathcal{O}_K$-linear with respect to ω and $\mathbb{Z}[G_K]$-equivariant with respect to u. It follows that, for any $u = (u_n)_{n\in\mathbb{N}} \in \mathcal{B}_A$ and $\omega \in H^0(\mathcal{A}, \Omega_{\mathcal{A}})$, then the sequence

$(u_n^*(\omega))_{n\geq 0}$ is a sequence of differentials in Ω satisfying $pu_{n+1}^*(\omega) = u_n^*(\omega)$, that is

$$(u_n^*(\omega))_{n\geq 0} \in \varprojlim \left(\Omega \xleftarrow{[p]} \Omega \xleftarrow{[p]} \cdots \Omega \xleftarrow{[p]} \cdots \right) = V_p(\Omega) \cong \mathbb{C}_p(1)$$

Taking duality of $H^0(\mathcal{A}, \Omega_{\mathcal{A}})$ yields the Fontaine integration

$$\begin{aligned} \varphi : \mathcal{B}_A &\to \mathrm{Lie}(A)(K) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(1) \\ (u_n)_{n\geq 0} &\mapsto (\omega \mapsto (u_n^*(\omega))_{n\geq 0}) \end{aligned}$$

The first crucial result in [11] is about the kernel of Fontaine integration, which equals to the periodic subgroup of $\mathcal{B}_A$ defined as follows.

Definition 1.2.3. *The periodic subgroup of $\mathcal{B}_A$ is*

$$\begin{aligned} \mathrm{Per}(\mathcal{B}_A) &= \{(u_n)_{n\in\mathbb{N}} \in \mathcal{B}_A : \exists k \in \mathbb{Z}_{>0}, u_n = u_{n+k}\} \\ &= \{(u_0, [p^{k-1}]u_0, ..., [p]u_0, u_0, ...) : u_0 \in A(\overline{K})\} \end{aligned}$$

Theorem 1.2.4 ([11], Theorem A)**.** *Let A be an abelian variety over K with good reduction. Assume further that $T_p(A)^{I_K} = 0$ then we have*

$$\ker \varphi = \mathrm{Per}(\mathcal{B}_A) = A_{p'-tor}(\overline{K})$$

where $A_{p'-tor}(\overline{K})$ is the prime-to-p torsion points of $A(\overline{K})$.

Thus we obtain the following commutative diagram of G_K-modules

$$\begin{tikzcd} & & \mathrm{Per}(\mathcal{B}_A) \arrow[r, equal] \arrow[d] & A_{p'-tor}(\overline{K}) \arrow[d] & \\ 0 \arrow[r] & T_p(A) \arrow[r] \arrow[d, equal] & \mathcal{B}_A \arrow[r] \arrow[d] & A(\overline{K}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & T_p(A) \arrow[r] & \mathrm{Lie}(A)(K) \otimes \mathbb{C}_p(1) \arrow[r] & \left(\mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1)\right)/\varphi(T_p(A)) \arrow[r] & 0 \end{tikzcd}$$

This leads to an injective morphism, known as the p-adic uniformization map of A:

$$\iota_A : A(\overline{K})/A_{p'-tor}(\overline{K}) \hookrightarrow \left(\mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1)\right)/\varphi(T_p(A))$$

We will describe the image of ι_A, which can be viewed as the p-adic uniformization of A. To precisely state the result, we introduce some definitions related to the Galois representations associated with A.

Consider T as a free $\mathbb{Z}_p$-module of rank $2g$, equipped with a continuous G_K-action, and V as a free $\mathcal{O}_K$-module of rank g, where G_K acts trivially on V. Assume there exists an injective $\mathbb{Z}_p$-linear G_K-equivariant map

$$\varphi : T \hookrightarrow V \otimes_{\mathcal{O}_K} \mathbb{C}_p(n)$$

with $n \in \mathbb{Z}$, $n \neq 0$.

Definition 1.2.5. *Let $x \in (V \otimes_{\mathcal{O}_K} \mathbb{C}_p(n))/\varphi(T)$. We say x is algebraic if the orbit $G_K \cdot x \subset (V \otimes_{\mathcal{O}_K} \mathbb{C}_p(n))/\varphi(T)$ is finite. If such an x is algebraic, then there is $L \subset \overline{K}$, $[L : K]$ finite such that $x \in (V \otimes_{\mathcal{O}_K} \mathbb{C}_p(n))/\varphi(T))^{G_L}$*

We consider the good class of algebraic elements

Definition 1.2.6. *Let $x \in (V \otimes_{\mathcal{O}_K} \mathbb{C}_p(n))/\varphi(T)$ be an algebraic element. We say that x is crystalline if one of the following happens*

1. *x is a torsion element, or*
2. *x is non-torsion, and for every L with $[L : K]$ finite such that $x \in (V \otimes_{\mathcal{O}_K} \mathbb{C}_p(n))/\varphi(T))^{G_L}$ we have $\alpha^{-1}(x\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a crystalline G_L-representation, where $\alpha : \mathcal{B}_A \to A(\overline{K})$ maps $(u_n)_{n\geq 0}$ to u_0.*

The main result in [11] is the following.

Theorem 1.2.7 ([11], Theorem B)**.** *We have that an element $x \in (\mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1))\,/\varphi(T_p(A))$ belongs to the image of ι_A if and only if x is crystalline. In particular, one can recover $A^{(p)}(\overline{K}) := A(\overline{K})/A_{p'-tor}(\overline{K})$ from the triple*

$$(T_p(A), \mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1), \varphi : T_p(A) \hookrightarrow \mathrm{Lie}(A)(K) \otimes_K \mathbb{C}_p(1))$$

1.3 The conjugate uniformization

In this section, we provide a brief overview of a recent work by Howe, Morrow, and Wear, which offers an alternative construction of Fontaine integration for p-divisible groups (or p-adic formal semi-abelian schemes). Let G be a p-divisible group (or p-adic formal semi-abelian scheme) over $\mathcal{O}_K$. We recall the functorial Hodge-Tate exact sequence, as introduced in [24].

$$0 \to \mathrm{Lie}\, G \otimes_{\mathcal{O}_K} \mathbb{C}_p(1) \to T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \to \omega_{G[p^\infty]^\wedge} \otimes_{\mathcal{O}_K} \mathbb{C}_p \to 0$$

Remark 1.3.1. *While it is not necessary for k to be perfect to establish the Hodge-Tate filtration (as shown in [25]), we require perfectness to ensure the exact sequence splits, as illustrated by a counter-example in [26].*

Recall that a 1-motive over $\mathcal{O}_K$ is a map

$$\varphi : M \to G$$

where G is a semi-abelian scheme over R and M is a locally constant sheaf which is étale locally a finite rank free abelian group. We attach to $x \in G(\mathcal{O}_K)$ a 1-motive $[\varphi : \mathbb{Z} \to G, 1 \mapsto x]$, this gives rise to a push-out G_x in which we adjoint p-power roots of x

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{-\varphi} & G \\ \downarrow & & \downarrow \\ \mathbb{Z}[1/p] & \longrightarrow & G_x. \end{array}$$

The universal cover of G is the functor on p-adically complete R-algebras given by

$$S \mapsto \hom_{\mathbb{Z}}(\mathbb{Z}[1/p], G(S))$$

We denote it by $\mathcal{B}_G$, then it fits into the following fundamental exact sequence

$$0 \to T_p(G) \to \mathcal{B}_G \to G(\mathcal{O}_{\overline{K}}) \to 0$$

Then we consider the fiber product of the following diagram

$$\begin{array}{ccc} & & \mathcal{B}_G \\ & & \downarrow \\ G(\mathcal{O}_K) & \hookrightarrow & G(\mathcal{O}_{\mathbb{C}_p}) \end{array}$$

We construct an integration map à la Howe–Morrow–Wear (cf. [19, subsection 6.2])

$$I_G : \mathcal{B}_G \times_{G(\mathcal{O}_{\mathbb{C}_p})} G(\mathcal{O}_K) \to T_p(G_x) \to \operatorname{Lie} G \otimes_{\mathcal{O}_K} \mathbb{C}_p(1)$$

as follows. Let $x \in G(\mathcal{O}_K)$ we have a canonical compatible system of p-power roots of $-x$ denoted by $-\tilde{x_{can}} \in \mathcal{B}_{G_x}(\mathcal{O}_K)$ as follows. The map $\mathbb{Z}[1/p] \to G_x$ provides us with a canonical system of p-power roots of $-x$ in $G_x(\mathcal{O}_K)$ via the images of $1/p^n$. Let $\tilde{x} \in \mathcal{B}_G$ be a compatible system of p-power roots of x, the element $\tilde{x} + -\tilde{x_{can}}$ belongs to $T_p(G_x) = T_p(G_x[p^\infty])$, and under the canonical splitting of the Hodge-Tate filtration for $T_p(G_x[p^\infty])$ that is

$$T_p(G_x[p^\infty]) \to \mathrm{Lie}(G_x) \otimes_{\mathcal{O}_K} \mathbb{C}_p(1) = \mathrm{Lie}\, G \otimes_{\mathcal{O}_K} \mathbb{C}_p(1)$$

Thus, we obtain the desired integration map

$$I_G : \mathcal{B}_G \times_{G(\mathcal{O}_{\mathbb{C}_p})} G(\mathcal{O}_K) \to T_p(G_x) \to \mathrm{Lie}\, G \otimes_{\mathcal{O}_K} \mathbb{C}_p(1)$$

Remark 1.3.2. *The integration map above is a lifting of the following map (cf. [19, p. 5.1])*

$$\tilde{I} : G(\mathcal{O}_K) \to (\mathrm{Lie}\, G \otimes_{\mathcal{O}_K} \mathbb{C}_p(1))\, / T_p(G)$$

constructed as the composition of the following two maps.

1. *The homomorphism*

$$G(\mathcal{O}_K) \to \mathrm{Ext}(\mathbb{Q}_p/\mathbb{Z}_p, G[p^\infty]), x \mapsto \mathcal{E}_x[p^\infty]$$

where $\mathcal{E}_x : 0 \to G \to G_x \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$ is the extension attached to

$$[\mathbb{Z} \to G : 1 \mapsto x]$$

2. *The homomorphism*

$$\mathrm{Ext}(\mathbb{Q}_p/\mathbb{Z}_p, G[p^\infty]) \to (\mathrm{Lie}\, G \otimes_{\mathcal{O}_K} \mathbb{C}_p(1))/T_p(G)$$

sending $0 \to G[p^\infty] \to H \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$ to the image of $1 \in \mathbb{Z}_p$ under the following diagram

$$\begin{tikzcd}
0 \arrow[r] & T_p(G) \arrow[r] \arrow[d] & T_p(H) \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[r] \arrow[d] & 0 \\
& \mathrm{Lie}\, G \otimes_{\mathcal{O}_K} \mathbb{C}_p(1) \arrow[r, equal] & \mathrm{Lie}\, G \otimes_{\mathcal{O}_K} \mathbb{C}_p(1) \arrow[r] & (\mathrm{Lie}\, G \otimes_{\mathcal{O}_K} \mathbb{C}_p(1))\, / T_p(G) &
\end{tikzcd}$$

The integration I described above aligns with the Fontaine integration à la Iovita–Morrow–Zaharescu, as stated in the following theorem.

Theorem 1.3.3 ([19], Theorem 6.1)**.** *Let $\mathcal{C}$ be the full subcategory of group-valued functors on $Nilp_{\mathcal{O}_K}$ consisting of functors represented by p-divisible groups over*

$\mathcal{O}_K$ *or by p-adic formal abelian schemes over* $\mathcal{O}_K$. *The natural transformation of functors from* $\mathcal{C}$ *to abelian groups*

$$G \mapsto I_G : \mathcal{B}_G \times_{G(\mathcal{O}_C)} G(\mathcal{O}_K) \to \operatorname{Lie} G \otimes_{\mathcal{O}_K} \mathbb{C}_p(1)$$

is the unique natural transformation that is Galois equivariant and agrees with the canonical splitting of the Hodge-Tate filtration on

$$T_p(G) \hookrightarrow \mathcal{B}_G \times_{G(\mathcal{O}_p} G(\mathcal{O}_K)$$

Chapter 2

Logarithmic geometry

Logarithmic (log for brevity) geometry was pioneered in the 1980s by Fontaine and Illusie. It was further developed by Kato and others with the aim of examining p-adic varieties exhibiting bad reduction. Broadly speaking, by introducing an appropriate log structure, a variety with bad reduction can be transformed into one with good properties. In this chapter, we revisit the fundamental concepts of logarithmic geometry that will be employed later on. The primary references for this chapter are [27] and [28].

2.1 Logarithmic rings

In this section, we will present the foundational theory of logarithmic rings, which play the role of "affine log schemes" in log geometry. We will explicitly showcase several essential log structures.

Definition 2.1.1. *A monoid is a set M equipped with an associative, commutative binary operation (denoted by $+$), and a distinguished element $0 \in M$ which is an identity element, i.e. $m + 0 = m$ for all $m \in M$.*

A morphism of monoids is a map $f : M \to N$ between two monoids such that $f(m_1) + f(m_2) = f(m_1 + m_2)$ for all $m_1, m_2 \in M$ and $f(0) = 0$.

Example 2.1.2. *In this section, we mainly consider the following monoids*

1. *Trivial monoid:* $\{0\}$.
2. *Natural numbers:* $(\mathbb{N}, +)$.
3. *Non-negative rational numbers with p-power denominator:* $(\mathbb{N}[1/p], +)$.

Remark 2.1.3. *For any monoid M, denote by M^{gp} the associated group given by*

$$M^{gp} := M \oplus M/\sim$$

where $(a,b) \sim (c,d)$ if there exists an $m \in M$ such that $a+d+m = b+c+m$.

Definition 2.1.4. *A pre-logarithmic ring (R, M, α) consists of a ring R, a monoid M and a morphism $\alpha : M \to R$ of monoids. We call (R, M, α) a logarithmic ring if α induces an isomorphism $\alpha^{-1}(R^*) \cong R^*$. A morphism of (pre)-log rings $(R, M, \alpha) \to (R', N, \beta)$ satisfies the following commutative diagram*

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "h"] & R \arrow[d, "f"] \\
N \arrow[r, "\beta"] & R'
\end{tikzcd}$$

where h is a morphism of monoids and f is a ring homomorphism.

Remark 2.1.5. *For simplicity, we will refer to the prelog structure $\alpha : M \to R$ as M, leaving the map α and the ring R implicit. A (pre)log ring is a ring with a (pre)log structure.*

Remark 2.1.6. *Given a pre-log structure $\alpha : M \to R$, we can form an associated log structure $M^a \to R$ as follows. Take $M^a := R^* \oplus_{\alpha^{-1}R} M$ to be the pushout of the following diagram*

$$\begin{tikzcd}
\alpha^{-1}R^* \arrow[r] \arrow[d, "\alpha"] & M \arrow[d] \\
R^* \arrow[r] & M^a
\end{tikzcd}$$

in the category of monoids. More precisely, $M^a = \dfrac{R^* \oplus M}{\sim}$ where $(x,y) \sim (x',y')$ iff there exists $z, z' \in \alpha^{-1}(R^*)$ such that $x\alpha(z) = x'\alpha(z')$ and $y\alpha(z) = y'\alpha(z)$.

Example 2.1.7. *For any ring, the trivial monoid $M = 1$ and the prelog structure $\{1\} \hookrightarrow R$ has the associated log structure $M^a = (M^a)^{gp} = R^*$.*

Example 2.1.8 (Canonical log structure)**.** *Let L be a finite extension of K and $\mathcal{O}_L$ be its ring of integers. Consider the canonical log structure associated to the following pre-log structure*

$$\begin{aligned} \mathbb{N} &\to \mathcal{O}_L \\ 1 &\mapsto \pi_L \end{aligned}$$

where π_L is a uniformizer of $\mathcal{O}_L$. We denote by $(1 \mapsto \pi_L)$ this pre-log structure.

We can compute the canonical log structure as follows.

Lemma 2.1.9. *Let L be an extension of K (possibly infinite). The canonical log structure associated to the monoid*

$$\alpha : \mathbb{N} \to \mathcal{O}_L, 1 \mapsto \pi_L$$

where π_L is an element in the maximal ideal of $\mathcal{O}_L$, is given by

$$(1 \mapsto \pi_L)^a = \{u \cdot \pi_L^n : u \in \mathcal{O}_L^*,\ n \in \mathbb{N}\}$$

Proof. By definition, we have

$$(1 \mapsto \pi_L)^a = \frac{\mathcal{O}_L^* \oplus \pi_L^{\mathbb{N}}}{\sim}$$

where $(x, y) \sim (x', y')$ if and only if there exist $z, z' \in \alpha^{-1}(\mathcal{O}_L^*)$ such that $x\alpha(z) = x'\alpha(z')$ and $y\alpha(z) = y'\alpha(z')$. Note that $\alpha^{-1}(\mathcal{O}_L^*) = 1$ because $\pi_L \notin \mathcal{O}_L^*$. Hence, we get

$$(1 \mapsto \pi_L)^a \cong \{u \cdot \pi_L^n : u \in \mathcal{O}_L^*,\ n \in \mathbb{N}\}$$

□

Remark 2.1.10. *If $\mathcal{O}_L$ is a discrete valuation ring (DVR for short), and π_L is a uniformizer of $\mathcal{O}_L$ then every element in $\mathcal{O}_L \setminus \{0\}$ can be written as the form $x = u \cdot \pi_L^n$ for some $n \in \mathbb{N}$, where $u \in \mathcal{O}_L^*$. It follows that*

$$(1 \mapsto \pi_L)^a = \mathcal{O}_L \setminus \{0\} \text{ and } ((1 \mapsto \pi_L)^a)^{gp} = L^*$$

In the case $\mathcal{O}_L$ is not DVR (e.g $L = \overline{K}$) then $(1 \mapsto \pi_L)^a$is not $\mathcal{O}_L$.

We define a refinement of the canonical log structure as follows.

Example 2.1.11. *Let L be an extension of K. Let π_L be a non-zero element in the maximal ideal of $\mathcal{O}_L$. For every fixed $n \in \mathbb{N}$, assume there exists an nth-root of π_L, denoted by $\pi_L^{1/n}$. For every $n \in \mathbb{N}$, we fix a compatible sequence of such $\pi_L^{1/n}$. Consider the following pre-log structure*

$$\begin{aligned} \mathbb{N} &\to \mathcal{O}_L \\ 1 &\mapsto \pi_L^{1/n} \end{aligned}$$

Let denote by $(1 \mapsto \pi_L^{1/n})$ *this pre-log structure*

For every $n \in \mathbb{N}$ and $a \in \mathbb{N}$ we denote by $\pi_L^{a/n} := (\pi_L^{1/n})^a$. We can compute its associated log structure as follows.

Lemma 2.1.12. *The log structure associated to* $(1 \mapsto \pi_L^{1/n})$ *is*

$$\left(1 \mapsto \pi_L^{1/n}\right)^a = \{u \cdot \pi_L^{a/n} | u \in \mathcal{O}_L^*,\ a \in \mathbb{N}\}$$

Proof. We have $\left(1 \mapsto \pi_L^{1/n}\right)^a = \dfrac{\mathcal{O}_L^* \oplus \pi_L^{\frac{1}{n}\mathbb{N}}}{\sim}$ where $(x, y) \sim (x', y')$ iff there exists $z, z' \in \alpha^{-1}(\mathcal{O}_L^*)$ such that $x\alpha(z) = x'\alpha(z')$ and $y\alpha(z) = y'\alpha(z')$. Again, we see that $\alpha^{-1}(\mathcal{O}_L^*) = 1$ because $\pi_L \notin \mathcal{O}_L^*$. Hence,

$$\left(1 \mapsto \pi_L^{1/n}\right)^a = \{u \cdot \pi_L^{a/n} | u \in \mathcal{O}_L^*,\ a \in \mathbb{N}\}$$

□

We also introduce several examples that will be used later.

Example 2.1.13. *There is a pre-log structure introduced in [17, section 2.1]*

$$Q := \{\pi^{m/p^s}, m \in \mathbb{N}, s \in \mathbb{N}\} = \pi^{\mathbb{N}[1/p]} \hookrightarrow \mathcal{O}_{\overline{K}}$$

Example 2.1.14. *To consider all* n*-th roots of* π*, we consider the monoid* $P = \mathbb{Q}_{>0}$ *and we define the pre-log structure*

$$\begin{aligned} P &\to \mathcal{O}_{\overline{K}} \\ \frac{u}{v} &\mapsto \pi^{u/v} \end{aligned}$$

We note that, the valuation of $\overline{K}$ *is isomorphic to* $\mathbb{Q}$ *and* $\mathbb{Q} \cong (\mathbb{Q}_{>0})^{gp}$*. Then we have*

$$(P^a)^{gp} = \overline{K}^*$$

2.2 Logarithmic schemes

In this section, we outline key definitions of log schemes as presented in [27]. For a more detailed exposition, we refer the reader to [28].

Definition 2.2.1. *A pre-log structure on a scheme X is a sheaf of monoids M on the étale site $X_{\acute{e}t}$ together with a homomorphism of sheaves of monoids*

$$\alpha : M \to \mathcal{O}_X$$

It is further called a log structure if the induced morphism

$$\alpha : \alpha^{-1}\mathcal{O}_X^* \to \mathcal{O}_X^*$$

is an isomorphism.

Remark 2.2.2. *Similar to the log ring situation, for a pre-log structure (M, α) on X, we define its associated log structure M^a to be the push-out of*

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^*) & \longrightarrow & M \\ \downarrow & & \downarrow \\ \mathcal{O}_X^* & \longrightarrow & M^a \end{array}$$

in the category of étale sheaves of monoids on X, endowed with

$$\begin{aligned} M^a &\to \mathcal{O}_X \\ \overline{(a,b)} &\mapsto \alpha(a)b \end{aligned}$$

where $a \in M, b \in \mathcal{O}_X^*$.

The definition of a morphism between log schemes is the following.

Definition 2.2.3. *Let X and Y be schemes endowed with log structures M_X and M_Y respectively. A morphism of log schemes is a morphism $f : X \to Y$ of the underlying schemes together with a homomorphism $f^\dagger : M_Y \to f_*(M_X)$ such that the following diagram*

$$\begin{array}{ccc} M_Y & \longrightarrow & f_*(M_X) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y & \longrightarrow & f_*(\mathcal{O}_X) \end{array}$$

commutes.

We can define the direct image and inverse image of a log structure as follows.

Definition 2.2.4. *Let $f : X \to Y$ be a morphism of schemes. For a log structure M on X, we define the log structure on Y called the direct image of M, to be the fiber product of sheaves*

$$\begin{tikzcd} & f_*(M) \arrow[d] \\ \mathcal{O}_Y \arrow[r] & f_*(\mathcal{O}_X) \end{tikzcd}$$

Definition 2.2.5. *For a log structure M on Y, we define the log structure on X called the inverse image of M and denoted by $f^*(M)$, to be the log structure associated to the pre-log structure $f^{-1}(M)$ endowed with the composite map*

$$f^{-1}(M) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$$

We will consider various kinds of log schemes.

Definition 2.2.6. *A log structure M on a scheme X is called quasi-coherent (resp. coherent) if étale locally on X, there exists a monoid (resp. finitely generated monoid) P and a homomorphism $P_X \to \mathcal{O}_X$ whose associated lo structure is isomorphic to M, where P_X is the constant sheaf of value P.*

Most of the time we will focus on fine and saturated log schemes.

Definition 2.2.7. 1. *A monoid is called integral if the following condition holds*

$$ab = ac \Rightarrow b = c$$

A log structure M on a scheme X is called integral if M is a sheaf of integral monoids.

2. *An integral monoid is saturated if, for any $p \in P^{gp}$ with $np \in P$, we have $p \in P$. A log structure M on a scheme X is called saturated if M is a sheaf of saturated monoids.*

3. *We call a log structure fine if it is coherent and integral.*

Example 2.2.8. *The log ring given by the monoid $\mathbb{N} \to \mathcal{O}_K : 1 \mapsto \pi$ gives rise to a log scheme, denoted by $S := (\operatorname{Spec} \mathcal{O}_K, (1 \mapsto \pi)^a)$. It is fine and saturated Similar for $\left(\operatorname{Spec} \mathcal{O}_{\overline{K}}, (1 \mapsto \pi^{1/n})^a\right)$.*

A fine log scheme can be covered by an "atlas" in the following sense.

Definition 2.2.9. *For a scheme X with a fine log structure M, a chart of M is a homomorphism $P_X \to M$ for a finitely generated integral monoid P which induces $(P_X)^a \cong M$, where P_X is the constant sheaf of value P.*

We will mainly consider strict and étale morphisms defined as follows.

Definition 2.2.10. *Given any morphism between log schemes* $(X, M_X) \to (Y, M_Y)$ *with the morphism of the underlying schemes is* f*. We say the morphism is strict if the induced morphism* $f^*M_Y \to M_X$ *is an isomorphism.*

Definition 2.2.11. *A morphism* $f : (X, M) \to (Y, N)$ *of schemes with fine log structure is called étale if the underlying morphism* $X \to Y$ *is locally of finite presentation and if for any commutative diagram*

$$\begin{array}{ccc} (T', L') & \xrightarrow{s} & (X, M) \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle f} \\ (T, L) & \xrightarrow{t} & (Y, N) \end{array}$$

where i *is a closed immersion, there exists a unique morphism*

$$g : (T, L) \to (X, M)$$

such that $gi = s$ *and* $fg = t$*.*

Chapter 3

Logarithmic differential forms and applications

In this section, we will extend the Fontaine integration [12, section 3] by incorporating logarithmic differential forms, which encompass a richer set of "periods". As illustrative examples, we explicitly demonstrate the p-adic uniformization of multiplicative groups and Tate curves in this section. Our methodology relies on computing the Tate module of log differential forms, a subject previously explored in [17], and the application of Sen-Tate theory.

3.1 Logarithmic Fontaine differential forms

We revisit the definition of logarithmic differential forms as presented in [27].

Definition 3.1.1. *Given a morphism of pre-logarithmic rings* $(A, M, \alpha) \to (B, N, \beta)$ *we define the* B*-module of log differentials* $\Omega_{(B/N)/(A,N)}$ *to be the quotient of the module*

$$\Omega_{B/A} \oplus (B \otimes_{\mathbb{Z}} \operatorname{coker}(M^{gp} \to N^{gp}))$$

by the submodule generated by the elements of the form $(d\beta(n), 0) - (0, \beta(n) \otimes n)$ *for* $n \in N$*. Here* $d : B \to \Omega_{B/A}$ *is the canonical differential map.*

Remark 3.1.2. *There are two natural maps* $d : B \to \Omega_{(B/N)/(A,M)}$ *the usual Kähler differential and* $\operatorname{dlog}_N : N \to \Omega_{(B,N)/(A,M)}$ *the log differential so that we have the relation*

$$\beta(n) \operatorname{dlog}_N(n) = d\beta(n)$$

for all $n \in N$.

Example 3.1.3. *Now take* $A = \mathcal{O}_K$, $B = \mathcal{O}_{\overline{K}}$ *and log structures* M *and* N *to be trivial. The classical result of Fontaine showed that [12, Corollaire 1]*

$$V_p\left(\Omega_{(\mathcal{O}_{\overline{K}},triv)/(\mathcal{O}_K,triv)}\right) \cong \mathbb{C}_p(1).$$

Example 3.1.4. *For an element* q *in the maximal ideal of* $\mathcal{O}_K$, *recall the pre-log structure introduced in [17, Section 2.1]*

$$Q := \{q^{m/p^s}, m \in \mathbb{N}, s \in \mathbb{N}\} = q^{\mathbb{N}[1/p]} \hookrightarrow \mathcal{O}_{\overline{K}}.$$

We get

$$V_p\left(\Omega_{(\mathcal{O}_{\overline{K}},Q)/(\mathcal{O}_K,triv)}\right) \cong \mathbb{C}_p \otimes V_p(E_q)$$

where E_q *is the Tate curve associated to* q.

Recall from Chapter 2 that we consider the monoid P as in 2.1.14 and we define the pre-log structure

$$\begin{aligned} P &\to \mathcal{O}_{\overline{K}} \\ \frac{u}{v} &\mapsto p^{u/v} \end{aligned}$$

We now want to compare the Tate modules of logarithmic differential forms with $A = \mathcal{O}_K$, $B = \mathcal{O}_{\overline{K}}$ and log structures P and Q. Denote by triv the trivial pre-log structure $\{1\} \hookrightarrow \mathcal{O}_K$.

Proposition 3.1.5. *Keep the notation as above. We have the following isomorphism as* G_K*-representations*

$$T_p\left(\Omega_{(\mathcal{O}_{\overline{K}},P)/(\mathcal{O}_K,triv)}\right) \cong T_p\left(\Omega_{(\mathcal{O}_{\overline{K}},Q)/(\mathcal{O}_K,triv)}\right)$$

Proof. By definition we have

$$\Omega_P := \Omega_{(\mathcal{O}_{\overline{K}},P)/(\mathcal{O}_K,\mathrm{triv})} = \frac{\Omega \oplus (\mathcal{O}_{\overline{K}} \otimes_{\mathbb{Z}} \pi^{\mathbb{Q}})}{\sim}$$

and

$$\Omega_Q := \Omega_{(\mathcal{O}_{\overline{K}},Q)/(\mathcal{O}_K,\mathrm{triv})} = \frac{\Omega \oplus (\mathcal{O}_{\overline{K}} \otimes_{\mathbb{Z}} \pi^{\mathbb{Z}[1/p]})}{\sim}$$

where $\mathbb{Z}[1/p] = \{m/p^s : m \in \mathbb{Z}, s \in \mathbb{N}\}$. We can compute the difference between Ω_Q and Ω_P as follows

$$0 \to \Omega_Q \to \Omega_P \to \mathbb{Q}/\mathbb{Z}[1/p] \to 0$$

Consider the multiplication by p^n

$$\begin{tikzcd} 0 \arrow[r] & \Omega_Q \arrow[r] \arrow[d, "\cdot p^n"] & \Omega_P \arrow[r] \arrow[d, "\cdot p^n"] & \mathbb{Q}/\mathbb{Z}[1/p] \arrow[r] \arrow[d, "\cdot p^n"] & 0 \\ 0 \arrow[r] & \Omega_Q \arrow[r] & \Omega_P \arrow[r] & \mathbb{Q}/\mathbb{Z}[1/p] \arrow[r] & 0 \end{tikzcd}$$

Note that $\mathbb{Q}/\mathbb{Z}[1/p]$ is p-torsion-free, then $\ker\left(p^n : \mathbb{Q}/\mathbb{Z}[1/p] \to \mathbb{Q}/\mathbb{Z}[1/p]\right) = 0$, by the Snake lemma we obtain

$$\Omega_Q[p^n] \cong \Omega_P[p^n]$$

Taking the inverse limit, we obtain

$$T_p(\Omega_Q) \cong T_p(\Omega_P)$$

□

Proposition 3.1.6. *Keep the notation as above. We have the following inclusion as G_K-representations*

$$T_p\left(\Omega_{(\mathcal{O}_{\overline{K}},P)/(\mathcal{O}_K,triv)}\right) \subset T_p\left(\Omega_{(\mathcal{O}_{\overline{K}},P)/(\mathcal{O}_K,(1\mapsto\pi))}\right)$$

Proof. By definition we have

$$\Omega_{\mathrm{triv}} := \Omega_{(\mathcal{O}_{\overline{K}},P)/(\mathcal{O}_K,\mathrm{triv})} = \frac{\Omega \oplus (\mathcal{O}_{\overline{K}} \otimes_{\mathbb{Z}} \pi^{\mathbb{Q}})}{\sim}$$

and

$$\begin{aligned} \Omega_{\mathrm{can}} := \Omega_{(\mathcal{O}_{\overline{K}},P)/(\mathcal{O}_K,(1\mapsto\pi))} &= \frac{\Omega \oplus (\mathcal{O}_{\overline{K}} \otimes_{\mathbb{Z}} \mathrm{coker}(\pi^{\mathbb{Z}} \to \pi^{\mathbb{Q}})}{\sim} \\ &= \frac{\Omega \oplus (\mathcal{O}_{\overline{K}} \otimes_{\mathbb{Z}} \pi^{\mathbb{Q}/\mathbb{Z}})}{\sim} \end{aligned}$$

Hence we have

$$0 \to \mathbb{Z} \to \Omega_{\text{triv}} \to \Omega_{\text{can}} \to 0$$

Consider the multiplication by p^n

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\cdot p^n"] & \Omega_{\text{triv}} \arrow[r] \arrow[d, "\cdot p^n"] & \Omega_{\text{can}} \arrow[r] \arrow[d, "\cdot p^n"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \Omega_{\text{triv}} \arrow[r] & \Omega_{\text{can}} \arrow[r] & 0
\end{tikzcd}$$

Note that $\ker(p^n : \mathbb{Z} \to \mathbb{Z}) = 0$, using the Snake lemma and taking the inverse limit, we obtain

$$T_p(\Omega_{\text{triv}}) \subset T_p(\Omega_{\text{can}})$$

□

3.2 The Fontaine integration on multiplicative group

Now, we construct the Fontaine integration for the multiplicative group. Let us consider the following logarithmic differential forms, as discussed in the previous section. Set

$$\Omega^q := \Omega_{(\mathcal{O}_{\overline{K}}, Q)/(\mathcal{O}_K, \text{triv})}$$

and

$$\Omega^{\log} := \Omega_{(\mathcal{O}_{\overline{K}}, P)/(\mathcal{O}_K, (1 \mapsto \pi))}$$

then by previous calculations, we obtain

$$V_p\left(\Omega^q\right) \cong \mathbb{C}_p \otimes V_p(E_q) \subset V_p(\Omega^{\log})$$

Remark 3.2.1. *The logarithmic derivation induces a group G_K-equivariant morphism*

$$\operatorname{dlog} : \overline{K}^* \to \Omega^q$$

defined as follows. If $x \in \mathcal{O}_{\overline{K}}$, then $x = u \cdot p^{a/n}$ with $u \in \mathcal{O}_{\overline{K}}^*$, so that $\operatorname{dlog}(x) =$

$\operatorname{dlog}(u) + a\operatorname{dlog}(p^{1/n})$ *with* $\operatorname{dlog}(u) = \dfrac{du}{u} \in \Omega$ *and* $\operatorname{dlog}(p^{1/n}) \in \Omega^q$. *If* $x \in \overline{K}^* \setminus \mathcal{O}_{\overline{K}}$, *then* $\operatorname{dlog}(x) := -\operatorname{dlog}(x^{-1})$.

For any $x_n \in \mathcal{O}_{\overline{K}}$ *such that* $x = (x_n)^{p^n}$ *then*

$$x^{-1}\left(x \cdot \operatorname{dlog}(x_n)\right) = x^{-1}\left(\frac{x}{p^n}\operatorname{dlog}(x_n^{p^n})\right)$$
$$= x^{-1}\left(\frac{x}{p^n}\operatorname{dlog}(x)\right) = \frac{1}{p^n}\cdot\frac{dx}{x} \in \overline{K}\otimes\Omega$$

This clarifies the observation in [11, section 3]. We will use this observation to explicitly construct the Fontaine integration map for both the multiplicative groups and Tate curves.

There is a relation between these two Tate modules due to the Sen-Tate theory (cf. [24] or [29, Theorem 2.2.7]).

Proposition 3.2.2. *There is a short exact sequence of* $\mathbb{C}_p$*-representations of* G_K

$$0 \to V_p(\Omega) \to V_p(\Omega^q) \to \mathbb{C}_p \to 0$$

that uniquely splits as G_K*-representations. Explicitly, there exists a unique* $\gamma_q \in \mathbb{C}_p \setminus \overline{K}$ *and a restriction* $\rho : V_p(\Omega^q) \to V_p(\Omega)$ *such that* $\rho(\operatorname{dlog}(q)) = \gamma_q \operatorname{dlog}(\epsilon)$. *Moreover, there is also a unique split of the composition of inclusions*

$$V_p(\Omega) \hookrightarrow V_p(\Omega^q) \hookrightarrow V_p(\Omega^{\log}).$$

Proof. Recall that we have the following exact sequence of G_K-modules

$$0 \to \mathbb{Z}_p(1) \to T_p(E_q) \to \mathbb{Z}_p \to 0$$

Tensor with $\mathbb{C}_p$ we obtain the desired exact sequence. The splitting follows from $H^1(G_K, \mathbb{C}_p(1)) = 0$ (cf. [24][Theorem 1]) and the uniqueness follows from $H^0(G_K, \mathbb{C}_p(1)) = \mathbb{C}_p(1)^{G_K} = 0$. The splitting of the inclusion $V_p(\Omega) \hookrightarrow V_p(\Omega^{\log})$ is certainly similar. □

To define the Fontaine integration, we recall the following definition of universal cover in [11, section 2]

Definition 3.2.3. *Let* G *be a group scheme over* K. *The universal covering space* *of $G(\overline{K})$ is defined by*

$$\mathcal{B}_G := \varprojlim \left(G(\overline{K}) \xleftarrow{\cdot p} G(\overline{K}) \xleftarrow{\cdot p} \cdots \right)$$

Remark 3.2.4. *It yields the Fontaine construction in a broader sense induced by* dlog

$$\begin{aligned} \varphi_{\mathbb{G}_m} : \mathcal{B}_{\mathbb{G}_m} &\to V_p(\Omega^q) \xrightarrow{\rho} V_p(\Omega) \cong \mathbb{C}_p(1) \\ (x_n) &\mapsto (\operatorname{dlog}(x_n))_n \mapsto \rho(\operatorname{dlog}(x_n))_n \end{aligned}$$

By the above construction, we then have the uniformization for the multiplicative group as in [11, section 3].

3.3 p-adic uniformization of Tate curve

We can take advantage of the previous parts to construct Fontaine integration for Tate curves explicitly as follows. Firstly, note that two elements $x, y \in \overline{K}^*$ give the same point of $E_q(\overline{K})$ if and only if $x = yq^m$ for some $m \in \mathbb{Z}$. In this case, we have

$$\operatorname{dlog}(x) = \operatorname{dlog}(y) + m \operatorname{dlog}(q)$$

Note that $q \operatorname{dlog}(q) = dq = 0$ for all $q \in \mathcal{O}_K$, hence the differentials $q \operatorname{dlog}(x)$ only depends on the class of x modulo $q^{\mathbb{Z}}$. Define

$$\begin{aligned} \varphi^q : \mathcal{B}_{E_q} &\to V_p(\Omega^q) \\ (x_n)_n &\mapsto \frac{1}{q} \cdot (q \operatorname{dlog}(x_n))_n \end{aligned}$$

We see that φ^q is well-defined. Indeed, let $x = (x_n)_n \in \mathcal{B}_A$ then $x_n / x_{n+1}^p = q^m$ for some $m \in \mathbb{Z}$. It follows that

$$q \operatorname{dlog}(x_n) = q \operatorname{dlog}(x_{n+1}^p q^m) = pq \operatorname{dlog}(x_{n+1})$$

hence $(q \operatorname{dlog}(x_n))_n \in V_p(\Omega^q)$. Then the Fontaine integration is given by the composition

$$\varphi_{E_q} := \rho \circ \varphi^q : \mathcal{B}_{E_q} \to V_p(\Omega^q) \to V_p(\Omega)$$

We have the fundamental exact sequence

$$0 \to T_p(E_q) \to \mathcal{B}_{E_q} \to E_q(\overline{K}) \to 0$$

Lemma 3.3.1. *The Fontaine integration φ_{E_q} restricting on $T_p(E_q)$ is injective.*

Proof. Let $(\epsilon_n)_n$ be the sequence of p-power roots of unity, denote by $\tau = (\mathrm{dlog}(\epsilon_n))_n \in T_p(\Omega)$ and by $\tilde{e}, \tilde{q}$ the generators of $T_p(E_q)$. By Sen-Tate theory, $T_p(E_q)$ splits uniquely after tensor with $\mathbb{C}_p$ but not with $\overline{K}$. Let $\gamma_q \in \mathbb{C}_p \setminus \overline{K}$ be the unique element such that $1 \mapsto \gamma_{\underline{q}}\tilde{e}$ is a G_K-semi-linear section of the projection

$$\begin{aligned} T_p(E_q) \otimes \mathbb{C}_p &\to \mathbb{C}_p \\ a\tilde{\epsilon} + b\tilde{q} &\mapsto b \end{aligned}$$

Then $\varphi(\tilde{\epsilon}) = \tau$ and $\varphi(\tilde{q}) = \gamma\tau$, the injectivity follows. □

Remark 3.3.2. *Above we provided a direct proof of injectivity of φ_{E_q} when we restrict to $T_p(E_q)$, which is different from proofs in [11]. Note that we also have $T_p(E_q)^{I_K} = 0$ but we do not use this fact.*

We claim that periodic paths correspond to prime-to-p torsion points of E_q.

Lemma 3.3.3. *Let $(u_n)_{n\geq 0} \in \mathcal{B}_{E_q}$ be a periodic path in $\mathcal{B}_{E_q}$. Then there is an bijection between $\mathrm{Per}(\mathcal{B}_{E_q})$ and prime-to-p torsion point of $E_q(\overline{K})$*

$$\begin{aligned} \mathrm{Per}(\mathcal{B}_{E_q}) &\to E_q(\overline{K})_{p'-tor} \\ (u_n)_{n\geq 0} &\mapsto u_0 \end{aligned}$$

Proof. **Injectivity**: Let u and u' be periodic paths such that $u_0 = u'_0$. It follows that $u - u' \in T_p(E_q)$. We have $\varphi_{E_q}(u - u') = 0$, hence $u - u' = 0$ by the injectivity of φ_{E_q} on $T_p(E_q)$.

Surjectivity: Now, take $u_0 \in E_q(\overline{K})[m]$ with $(m, p) = 1$. It follows from Euler theorem that $m \mid p^{\psi(m)} - 1$ where ψ is Euler totient function. Therefore we have $[p^{\psi(m)} - 1]u_0 = 0$. Set $u = (u_0, [p^{\psi(m)-1}]u_0, ..., [p]u_0, u_0, ...)$, then $u \in \mathrm{Per}(\mathcal{B}_{E_q})$. □

We need the following technical result.

Lemma 3.3.4. *Let $u \in \mathcal{B}_{E_q}$ such that $u_0 \in E_q(L)$, for L/K a finite extension in $\overline{K}$. If $\varphi_{E_q}(u) = 0$, then $u \in (\mathcal{B}_{E_q})^{G_L}$.*

Proof. Since $x_0 \in E_q(L) = E_q(\overline{K})^{G_L}$, then $g \cdot x - x \in T_p(E_q)$ for all $g \in G_L$. Moreover we have $\varphi_{E_q}(g \cdot x) = g\varphi_{E_q}(x) = 0$, so $\varphi_{E_q}(g \cdot x - x) = 0$. By the injectivity of φ_{E_q} on $T_p(E_q)$ we have $g \cdot x = x$. □

The main theorem of this subsection is as follows. We adapt the method in [11].

Proposition 3.3.5. *The kernel of* $\varphi_{E_q} : \mathcal{B}_{E_q} \to \mathbb{C}_p(1)$ *is* $\mathrm{Per}(\mathcal{B}_{E_q})$.

Proof. The inclusion $\mathrm{Per}(\mathcal{B}_{E_q}) \subseteq \ker(\varphi_{E_q})$ comes from the fact that $V_p(\Omega)$ is torsion-free. Indeed, let $u \in \mathrm{Per}(\mathcal{B}_{E_q}$, then $[p^k]u = u$ for some $k \in \mathbb{N}$. It follows that $p^k\varphi(u) = \varphi(u)$ implies $\varphi(u) = 0$. We claim that $\ker(\varphi_{E_q}) \subseteq \mathrm{Per}(\mathcal{B}_{E_q})$. Now consider three cases

1. Let $u \in \ker(\varphi_{E_q})$ such that $u_0 \in E_q(\overline{K})[n]$ for $n \geq 1$ with $(n, p) = 1$. Then by the lemma 6.3.2 we have $u \in \mathrm{Per}(\mathcal{E}_{\amalg A})$

2. Let $u \in \ker(\varphi_{E_q})$ with $u_0 \in E_q(\overline{K})$ such that u_0 is a torsion point of A of order a power of p, say $u_0 \in E_q(\overline{K})[p^a]$ for some $a \in \mathbb{Z}_{>0}$. Then $p^a u \in T_p(E_q)$ and $\varphi_{E_q}(p^a u) = p^a\varphi_{E_q}(u) = 0$. As $\varphi_{E_q|T_p(E_q)}$ is injecitve, we obtain that $p^a u = 0$ in $\mathcal{B}_{E_q}$ which is p-torsion free, i,e., $\mathcal{B}_{E_q}[p] = 0$, we get $u = 0$.

3. Let $u \in \ker(\varphi_{E_q})$ with $u_0 \in E_q(\overline{K})$ not a torsion point of $E_q(\overline{K})$. Denote $\alpha : \mathcal{B}_{E_q} \twoheadrightarrow E_q(\overline{K})$. Note that $\alpha \otimes 1_{\mathbb{Q}}(u \otimes 1_{\mathbb{Q}_p}) = u_0 \otimes 1 \neq 0$. We have a natural exact sequence of $\mathbb{Q}_p$-vector spaces, with continuous G_K-action

$$0 \to T_p(E_q) \otimes \mathbb{Q}_p := V_p(E_q) \to \mathcal{B}_{E_q} \otimes_{\mathbb{Z}} \mathbb{Q} \to E_q(\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$$

Assume that $u_0 \in E_q(L)$ for some finite extension L of K in $\overline{K}$. Consider the fiber product of G_L-modules

$$\begin{array}{ccc} \mathcal{B}_{E_q,L} & \xrightarrow{\alpha_L} & E_q(L) \\ \downarrow & & \downarrow \\ \mathcal{B}_{E_q} & \xrightarrow{\alpha} & E_q(\overline{K}) \end{array}$$

We get that $\mathcal{B}_{E_q,L} \subset \mathcal{B}_{E_q}$. On the other hand we have $\mathcal{B}_{E_q}^{G_L} \subset \mathcal{B}_{E_q,L}$. Hence $(\mathcal{B}_{E_q,L} \otimes \mathbb{Q})^{G_L} = (\mathcal{B}_{E_q} \otimes_{\mathbb{Z}} \mathbb{Q})^{G_L}$.

Consider the exact sequence of finite $\mathbb{Q}_p$-vector space with continuous G_L-action

$$0 \to V_p(E_q) \to \mathcal{B}_{E_q,L} \otimes \mathbb{Q} \to E_q(L) \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$$

Taking cohomology yields that

$$(\mathcal{B}_{E_q} \otimes \mathbb{Q})^{G_L} \to (E_q(L) \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\partial} H^1(G_L, V_p(E_q))$$

By[30] and [9], we have that ∂ is injective, so $\alpha_L \otimes 1 \left((\mathcal{B}_{E_q} \otimes \mathbb{Q})^{G_L}\right) = 0$. Therefore u is not G_L-invariant. It contradicts with the lemma 6.3.3

□

We have the following commutative diagram

$$\begin{tikzcd}
 & & \mathrm{Per}(\mathcal{B}_{E_q}) \arrow[r] \arrow[d] & E_Q(\overline{K})_{p'-tor} \arrow[d] & \\
0 \arrow[r] & T_p(E_q) \arrow[r] \arrow[d] & \mathcal{B}_{E_q} \arrow[r] \arrow[d] & E_q^w(\overline{K}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_p(E_q) \arrow[r] & \mathbb{C}_p(1) \arrow[r] & \mathbb{C}_p(1)/\varphi(T_p(E_q)) \arrow[r] & 0
\end{tikzcd}$$

Therefore we have injective, continuous and G_K-equivariant map

$$\iota_{E_q} : E_q(\overline{K})^{(p)} := E_q(\overline{K})/E_q(\overline{K})_{p'-tor} \hookrightarrow \mathbb{C}_p(1)/\varphi_{E_q}(T_p(E_q))$$

We are now ready to describe the image of the uniformization map ι which turns out to be the semi-stable class of algebraic elements as follows.

Definition 3.3.6. *Let $x \in (V \otimes_{\mathcal{O}_K} \mathbb{C}_p(n))/\varphi(T)$ be an algebraic element. We say that x is semi-stable if one of the following happens*

1. *x is a torsion element, or*

2. *x is non-torsion, and $\alpha^{-1}(x\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a semi-stable G_L-representation, for every L with $[L : F]$ finite such that $x \in (V \otimes_{\mathcal{O}_K} \mathbb{C}_p(n))/\varphi(T))^{G_L}$*

Theorem 3.3.7. *We have that an element $x \in \mathbb{C}_p(1)/T_p(E_q)$ belongs to the image if ι_{E_q} if and only if x is semi-stable.*

Proof. Note that $\mathbb{C}_p(1)$ is a divisible group and the torsion subgroup of $\mathbb{C}_p(1)/\varphi(T_p(E_q))$ is $\mathbb{Q}_p(1)/\varphi(T_p(A))$. We have the following isomorphisms of $\mathbb{Z}_p$ and G_F-modules

$$\begin{aligned}\mathbb{C}_p(1)/T_p(E_q)[p^n] &\cong \left(\frac{1}{p^n}\varphi(T_p(E_q))/\varphi(T_p(E_q))\right)\\ &\cong \varphi\left((\frac{1}{p^n}T_p(E_q))/T_p(E_q)\right)\\ &\cong \iota(E_q(\overline{K})[p^n])\end{aligned}$$

Therefore ι_{E_q} induces a G_K-isomorphism between the p-power torsion of two modules. It follows that we can deduce to consider the map induced by ι_{E_q} on the quotients of domain and target by the torsion subgroup.

Suppose $x \in \mathbb{C}_p(1)/T_p(E_q)$ algebraic. Consider the exact sequence of p-adic G_L-representations

$$\begin{tikzcd} 0 \arrow[r] & V_p(E_q) \arrow[r] \arrow[d] & \mathcal{B}_{E_q} \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r] \arrow[d] & \overline{K}^*/q^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \varphi(V_p(E_q)) \arrow[r] & \mathbb{C}_p(1) \arrow[r] & \mathbb{C}_p(1)/\varphi(V_p(E_q)) \arrow[r] & 0 \end{tikzcd}$$

Taking G_L-continous cohomology yeilds that

$$\begin{tikzcd} 0 \arrow[r] & L^\times/q^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r, "\delta_{E_q}"] \arrow[d, "\iota_{E_q}"] & H^1(G_L, V_p(E_q)) \arrow[d, "="] \\ 0 \arrow[r] & \mathbb{C}_p(1)/\varphi(V_p(E_q)) \arrow[r] & H^1(G_L, V_p(E_q)) \end{tikzcd}$$

By Kummer theory the image of δ_{E_q} is $H^1_f(G_K, V_p(E_q))$ and by [9] it is $H^1_{st}(L, V_p(E_q))$. Therefore an element $x \in \mathbb{C}_p(1)/T_p(E_q)$ belongs to the image of ι_{E_q} if and only if x is semi-stable $\square$

Chapter 4

Logarithmic abelian varieties

In this section, we introduce the theory of log abelian varieties following the approach of Kajiwara-Kato-Nakayama (cf. [13]). These objects represent a specific type of degeneration of abelian varieties. Unlike the usual geometry, where degenerations of abelian varieties exhibit singularities and lack group laws, in the refined context of logarithmic geometry, Kato and his collaborators have successfully constructed a category of proper and smooth group objects known as log abelian varieties.

Convention

We will follow the standard notations in [13]. We denote by $S := (\operatorname{Spec} \mathcal{O}_K, (1 \mapsto \pi)^a)$ the standard log point, where $(1 \mapsto \pi)^a$ is the canonical log structure defined in chapter 2. Let (fs/S) denote the category of fine saturated log schemes over S. We regard (fs/S) as a site endowed with the étale topology, i.e., a covering $(U_i \to U)_i$ of an object $U \in (\mathrm{fs}/S)$ means a covering of the underlying scheme of U by étale schemes U_i over U which are endowed with the inverse images of the log structure of U. We mainly consider the logarithmic schemes coming from finite extension L of K (e.g. $(\operatorname{Spec} L, \mathrm{triv})$ and $(\operatorname{Spec} \mathcal{O}_L, (1 \mapsto \pi_L)^a)$).

4.1 Various logarithmic multiplicative groups according to Kajiwara–Kato–Nakayama

Before introducing the concept of log abelian varieties, let us consider various multiplicative groups from a functorial perspective. Classically, the multiplicative group $\mathbb{G}_m := \operatorname{Spec} K[x,y]/(xy-1)$ represents the following functor

$$\begin{aligned}\mathbb{G}_m : (\text{Scheme}\,/K) &\to (\text{Groups})\\ T &\mapsto \mathcal{O}_T^*(T) = \Gamma(T, \mathcal{O}_T^*)\end{aligned}$$

where (Groups) is the category of groups. For every log scheme T, denote by M_T its log structure, consider a naive log multiplicative group $\mathbb{G}_{m,\log}$

$$\begin{aligned}\mathbb{G}_{m,\log} : (\text{fs}/S) &\to (\text{Groups})\\ T &\mapsto \Gamma(T, M_T^{gp})\end{aligned}$$

Example 4.1.1. *Let $T = S$, we have*

$$\mathbb{G}_{m,\log}(S) = ((1 \mapsto \pi)^a)^{gp} = \{u \cdot \pi^n : u \in \mathcal{O}_K^*,\ n \in \mathbb{Z}\} = K^*$$

We have that $\mathbb{G}_{m,\log}$ is not representable, as noted in [31, Section 2.2.7]. Let q be an element in the maximal ideal of $\mathcal{O}_K$. We will examine the sub-functor $\mathbb{G}_{m,\log}^{(q)}$ of $\mathbb{G}_{m,\log}$.

$$\begin{aligned}\mathbb{G}_{m,\log}^{(q)} : (\text{fs}/S) &\to (\text{Groups})\\ T &\mapsto \{x \in \Gamma(T, M_T^{gp}) | \exists m, n :\ q^m | x | q^n\}\end{aligned}$$

Here $a|b$ means that $a^{-1}b \in M_T$.

For a deeper exploration of these diverse log multiplicative groups, readers are encouraged to consult the insightful survey [32]. Next, we will compute points on these log multiplicative groups.

Proposition 4.1.2. *Let L be a finite extension K with a uniformizer $\pi_L \notin K$. We have*

$$\mathbb{G}_{m,\log}^{(\pi_L)}(\operatorname{Spec}\mathcal{O}_L, (1 \mapsto \pi_L)^a) = ((1 \mapsto \pi_L)^a)^{gp} = \mathbb{G}_{m,\log}(\operatorname{Spec}\mathcal{O}_L, (1 \mapsto \pi_L)^a) = L^*$$

Proof. We have $((1 \mapsto \pi_L)^a)^{gp} = \{u \cdot \pi_L^n | u \in \mathcal{O}_L^*,\ n \in \mathbb{Z}\}$ (see 2.1.9). For any $x = u \cdot \pi_L^n$, choose $m \in \mathbb{Z}$ such that $mm < n$, then $u \cdot \pi_L^n \cdot \pi_L^m \in (1 \mapsto \pi_L)^a$. By definition, we have $\pi_L^m | x$. Thus we obtain

$$\begin{aligned}\mathbb{G}_{m,\log}^{(\pi_L)}(\operatorname{Spec}\mathcal{O}_L, (1 \mapsto \pi_L)^a) &= \{x \in ((1 \mapsto \pi_L)^a)^{gp} | \exists m, n :\ \pi_L^m | x | \pi_L^n\}\\ &= ((1 \mapsto \pi_L)^a)^{gp}\end{aligned}$$

Since $\mathcal{O}_L$ is DVR then $((1 \mapsto \pi_L)^a)^{gp} = L^*$. $\square$

Let us consider the monoid $P = \mathbb{Q}_{>0}$ and its associated log structure as in 2.1.14

$$\begin{aligned} P = \mathbb{Q}_{>0} &\to \mathcal{O}_{\overline{K}} \\ \frac{u}{v} &\mapsto \pi^{u/v} \end{aligned}$$

Proposition 4.1.3. *For any finite extension K of $\mathbb{Q}_p$ with a uniformizor π we have*

$$\mathbb{G}_{m,\log}^{(\pi)}(\operatorname{Spec}\mathcal{O}_{\overline{K}}, P^a) := \varinjlim_n \mathbb{G}_{m,\log}^{(\pi)}(\operatorname{Spec}\mathcal{O}_{\overline{K}}, (1 \mapsto \pi^{1/n})^a) = \overline{K}^*$$

Proof. We have $\left(\left(1 \mapsto \pi^{1/n}\right)^a\right)^{gp} = \{u \cdot \pi^{a/n} \mid u \in \mathcal{O}_{\overline{K}}^*,\ a \in \mathbb{Z}\}$2.1.12. For any $x = u \cdot \pi^{a/n}$, we choose $m \in \mathbb{Z}$ such that $a - mn > 0$, we have $\pi^m | x$. Hence

$$\mathbb{G}_{m,\log}^{(\pi)}(\operatorname{Spec}\mathcal{O}_{\overline{K}}, (1 \mapsto \pi^{1/n})^a) = \left(\left(1 \mapsto \pi^{1/n}\right)^a\right)^{gp}$$

Because the valuation group of $\overline{K}$ is $\mathbb{Q}$, every element of $\overline{K}$ can be written as $u \cdot p^{a/n}$ with $a \in \mathbb{Z}$, $n \in \mathbb{N}$ we obtain

$$\mathbb{G}_{m,\log}^{(\pi)}(\operatorname{Spec}\mathcal{O}_{\overline{K}}, P^a) := \varinjlim_n \mathbb{G}_{m,\log}^{(\pi)}(\operatorname{Spec}\mathcal{O}_{\overline{K}}, (1 \mapsto \pi^{1/n})^a) = \overline{K}^*$$

$\square$

Remark 4.1.4. *Keep the notation as in the above proposition, since* $\mathbb{G}_{m,\log}^{(\pi)}(\operatorname{Spec}\mathcal{O}_{\overline{K}}, (1 \mapsto \pi^{1/n})^a) = \left(\left(1 \mapsto \pi^{1/n}\right)^a\right)^{gp} = \mathbb{G}_{m,\log}(\operatorname{Spec}\mathcal{O}_{\overline{K}}, (1 \mapsto \pi^{1/n})^a)$ *we also have*

$$\mathbb{G}_{m,\log}(\operatorname{Spec}\mathcal{O}_{\overline{K}}, P^a) := \varinjlim_n \mathbb{G}_{m,\log}(\operatorname{Spec}\mathcal{O}_{\overline{K}}, (1 \mapsto \pi^{1/n})^a) = \overline{K}^*$$

4.2 Logarithmic 1-motives

We recall briefly the theory of log 1-motive defined in [13, Section 2] which is crucial for the construction of the log abelian varieties. Let G be a commutative group scheme over the underlying scheme of S which is an extension of an abelian scheme B by a torus T

$$1 \to T \to G \to B \to 1$$

Here we identify G with a sheaf of abelian groups on the site (fs/S): For an fs log scheme U over S, $G(U)$ is defined by forgetting the log structure of U.

Denote the hom sheaf in (fs/S) by $\mathcal{H}\mathrm{om}$, and denote by $X := \mathcal{H}\mathrm{om}(T, \mathbb{G}_m)$ the character group sheaf of T, which is a locally constant sheaf of finitely generated free $\mathbb{Z}$-modules on (fs/S). Let $T_{\log} := \mathcal{H}\mathrm{om}(X, \mathbb{G}_{m,\log})$. Recall that for any fs log scheme U, in particular the log structure of U is integral, over S we have

$$\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U^\times) \subset \mathbb{G}_{m,\log}(U) = \Gamma(U, M_U^{\mathrm{gp}})$$

by the universal property of integral log structure (see [28, p. I.1.3]). Hence we have a canonical embedding $T \hookrightarrow T_{\log}$. We define $G_{\log}$ as the pushout of the following diagram in the category of sheaves of abelian groups on (fs/S)

$$\begin{array}{ccc} T & \longrightarrow & G \\ \downarrow & & \downarrow \\ T_{\log} & \longrightarrow & G_{\log} \end{array}$$

Then we have an exact sequence

$$1 \to T_{\log} \to G_{\log} \to B \to 1$$

Example 4.2.1. *If $G = T = \mathbb{G}_m$ then $X = \mathcal{H}\mathrm{om}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$, it follows that*

$$G_{\log} = T_{\log} = \mathcal{H}\mathrm{om}(\mathcal{H}\mathrm{om}(\mathbb{G}_m, \mathbb{G}_m), \mathbb{G}_{m,\log}) = \mathcal{H}\mathrm{om}(\mathbb{Z}, \mathbb{G}_{m,\log}) \cong \mathbb{G}_{m,\log}$$

Definition 4.2.2. *A log 1-motive M over S consists of the following data*

1. *A locally constant sheaf Y of finitely generated free $\mathbb{Z}$-modules on (fs/S).*
2. *A commutative group scheme G over the underlying scheme of S which is an extension of an abelian scheme B by a torus.*
3. *A homomorphism $Y \to G_{\log}$*

Example 4.2.3. *Let $X = Y = \mathbb{Z}$ and let $q \in \mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$. In [13], the authors define log Tate curves via the following log 1-motive*

$$M = [\mathbb{Z} \to \mathbb{G}_{m,\log} : 1 \mapsto q]$$

We denote M by $[Y \to G_{\log}]$ seen as a complex of sheaves of abelian groups with Y of degree -1 and with $G_{\log}$ of degree 0.

$$\begin{tikzcd} & & Y \arrow[d] & & \\ 1 \arrow[r] & T_{\log} \arrow[r] & G_{\log} \arrow[r] & B \arrow[r] & 1 \end{tikzcd}$$

Definition 4.2.4. *Let $M = [u : Y \to G_{\log}]$ and $M' = [u' : Y' \to G'_{\log}]$ be log 1-motives over S. A morphism $h : M \to M'$ is a homomorphism of complexes*

$$\begin{tikzcd} Y \arrow[r, "u"] \arrow[d, "h_{-1}"] & G_{\log} \arrow[d, "h_0"] \\ Y' \arrow[r, "u'"] & G'_{\log} \end{tikzcd}$$

We also need to define the dual of a log 1-motive $M^* = [X \to G^*_{\log}]$ as follows. Recall that $X := \mathcal{H}om(T, \mathbb{G}_m)$ and let B^* be the dual abelian scheme of B, and $T^* := \mathcal{H}om(Y, \mathbb{G}_m)$. Denote by (F, h) an extension F of B by $\mathbb{G}_m$ over a fs log scheme U and a homomorphism $h : Y \to F$ such that the composition $Y \to F \to B$ coincides with $Y \to G_{\log} \to B$. We define G^* as the sheafification of the presheaf

$$\begin{aligned} (\mathrm{fs}/S) &\to (\mathrm{AbGroups}) \\ U &\mapsto \{\mathrm{pairs}(F, h)\}/\cong \end{aligned}$$

Then G^* is representable by an fs log scheme over S. There is a natural exact sequence of sheaves

$$1 \to T^* \to G^* \to B^* \to 1$$

Similarly, we can have the sheaf of commutative groups $G^*_{\log}$ associated to G^* as follows. Denote by $(F_{\log}, h_{\log})$ an extension F of B by $\mathbb{G}_m$ over an fs log scheme U and a homomorphism $h_{\log} : Y \to F_{\log}$ such that the composition $Y \to F_{\log} \to B_{\log} = B$ coincides with $Y \to G_{\log} \to B_{\log} = B$. Then $G^*_{\log}$ is defined to be the sheafification of the presheaf

$$\begin{aligned} (\mathrm{fs}/S) &\to (\mathrm{AbGroups}) \\ U &\mapsto \{\mathrm{pairs}(F_{\log}, h_{\log})\}/\cong \end{aligned}$$

The homomorphism $X \to G^*_{\log}$ is defined as follows. For a section x of X, let F be the extension of B by $\mathbb{G}_m$ obtained as the pushout of $1 \to T \to G \to B \to 1$

with respect to $x : T \to \mathbb{G}_m$. Let h be the composition

$$Y \to G_{\log} \to F_{\log}$$

Then the desired homomorphism $X \to G^*_{\log}$ is defined by associating to x the class of $(F_{\log}, h_{\log})$. We can now define polarization on a log 1-motive.

Definition 4.2.5. *Let $M = [Y \to G_{\log}]$ be a log 1-motive over S. A polarization on M is a homomorphism*

$$h : M \to M^* = [X \to G^*_{\log}]$$

of log 1-motives satisfying the following conditions

1. *The induced homomorphism $B \to B^*$ is a polarization on B.*
2. *The induced homomorphism $\phi : Y \to X$ is injective and each stalk of its cokernel is finite.*
3. *For any $s \in S$ and any nontrivial $y \in Y_{\overline{s}}$, the element $\langle \phi(y), y \rangle_{\overline{s}}$ of $M^{gp}_{S,\overline{s}}/\mathcal{O}^*_{S,\overline{s}}$ is nontrivial and belongs to $M_{S,\overline{s}}/\mathcal{O}^*_{S,\overline{s}}$. Here*

$$\langle , \rangle : X \times Y \to M^{gp}_S/\mathcal{O}^*_S$$

   is the canonical pairing induced by

$$Y \to G_{\log} \to G_{\log}/G \cong T_{\log}/T \cong \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$$

4. *The homomorphism $T_{\log} \to T^*_{\log}$ induced by $G_{\log} \to G^*_{\log}$ coincides with the one induced by ϕ.*

M is called pointwise polarizable if the pullback of M to $fs/\overline{s}$ admits a polarization for every $s \in S$.

Remark 4.2.6. *The existence of a polarization implies that $Y \to G_{\log}/G$ is injective.*

Example 4.2.7. *Let $X = Y = \mathbb{Z}$ and let q be a locally nipotent element in the maximal ideal of $\mathcal{O}_K$. Then the log 1-motive*

$$M = [Y = \mathbb{Z} \to \mathbb{G}_{m,\log} : 1 \mapsto q]$$

have the dual log 1*-motive given by*

$$M^* = [X = \mathbb{Z} \to \mathbb{G}_{m,\log} : 1 \mapsto q]$$

and the identity map $M \to M^*$ *is a polarization on* M*.*

Definition 4.2.8. *Define a subgroup sheaf* $\mathcal{H}\mathrm{om}(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$ *of* $\mathcal{H}\mathrm{om}(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ *by*

$$\begin{aligned} (fs/S) &\to (AbGroups) \\ U &\mapsto \qquad \{\varphi : \text{for any } \ x \in X, \text{locally } \exists y_1, y_2 \in Y \\ &\qquad\qquad \text{such that } \langle x, y_1 \rangle \, |\varphi(x)| < x.y_2 >\} \end{aligned}$$

Here, for $a, b \in M_U^{gp}/\mathcal{O}_U^*$*, we say that* $a|b$ *if* $a^{-1}b \in M_U/\mathcal{O}_U^*$*.*

Define a subgroup sheaf $G_{\log}^{(Y)}$ of $G_{\log}$ as the inverse image of $\mathcal{H}\mathrm{om}(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$ by the canonical homomorphism

$$G_{\log} \to G_{\log}/G \cong \mathcal{H}\mathrm{om}(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$$

4.3 Logarithmic abellian varieties

We are now set to define the initial type of log abelian varieties, as presented in [13, Section 3].

Definition 4.3.1. *A log abelian variety with constant degeneration over* S *is a sheaf of abelian groups on* (fs/S) *which is ismorphic to the quotient sheaf* $G_{\log}^{(Y)}/Y$ *for a pointwise polarisable log* 1*-motive* $[u : Y \to G_{\log}]$ *over* S*.*

Example 4.3.2. *Consider* $X = Y = \mathbb{Z}$ *and* $q \in \Gamma(S, M_S) \setminus k^\times$*, log* 1*-motives* M *and* M^* *as above. The log Tate curve associated to* q *is given by the quotient*

$$\mathcal{E}_q := \mathcal{H}\mathrm{om}(\mathbb{Z}, \mathbb{G}_{m,\log})^{(\mathbb{Z})}/\mathbb{Z}$$

Here the subsheaf $\mathcal{H}\mathrm{om}(\mathbb{Z}, \mathbb{G}_{m,\log})^{(\mathbb{Z})}$ *of* $\mathcal{H}\mathrm{om}(\mathbb{Z}, \mathbb{G}_{m,\log}) \cong \mathbb{G}_{m,\log}$ *is identified with*

$$\mathbb{G}_{m,\log}^{(q)} := \{x \in \mathbb{G}_{m,\log} | \exists m, n \in \mathbb{Z} : q^m |x| q^n\}$$

Theorem 4.3.3. *[13, Theorem 3.4] The functor*

$$[u : Y \to G_{\log}] \mapsto G_{\log}^{(Y)}/Y$$

induces an equivalence from the category of pointwise polarization log 1*-motive over* S *to that of log abelian varieties with constant degeneration over* S*.*

Remark 4.3.4. *For any* $s \in S$*, let* $\overline{s}$ *be the spectrum of a separable closure of the residue field of* s *endowed with the pullback log structure from* S*. The category* $(fs/\overline{s})$ *is the category of fs log schemes* $V \times_S \overline{s}$ *where* $V \in (fs/S)$*. Hence we can consider the pullback of a sheaf on* (fs/S) *to* $(fs/\overline{s})$*.*

We need a notion of diagonal morphism to be finite for the definition of log abelian varieties.

Definition 4.3.5. *Let* $f : F_1 \to F_2$ *be a morphism of contravariant functors from* (fs/S) *to the category of sets. We say* f *is represented by finite morphisms if for any object* U *of* (fs/S) *and any* $a : U \to F_2$ *(i.e.* $a \in F_2(U)$ *), the fiber product* $U \times_{F_2} F_1$ *in the category of fiber product of presheaves defined by* a *is represented by an fs log scheme over* U *whose underlying scheme is finite over that of* U*.*

We can provide the definition of log abelian varieties as follows (cf. [13, Section 4]).

Definition 4.3.6. *A log abelian variety over an fs log scheme* S *is a sheaf* $\mathcal{A}$ *of abelian groups on* (fs/S) *satisfying the following three conditions*

1. *For any* $s \in S$*, let* $\overline{s}$ *be the spectrum of a separable closure of the residue field of* s *endowed with the pullback log structure from* S*. Then there is a polarizable log* 1*-motive* $[Y \to G_{\log}]$ *over* $\overline{s}$ *such that the pullback of* $\mathcal{A}$ *to* $(fs/\overline{s})$ *is isomorphic to* $G_{\log}^{(Y)}/Y$*.*

2. *Étale locally on* S*, there are a semiabelian group scheme* G *over* S*, finitely generated free* $\mathbb{Z}$*-modules* X *and* Y*, an admissible pairing* $\langle , \rangle : X \times Y \to M_S^{gp}/\mathcal{O}_S^*$*, and an exact sequence*

$$0 \to G \to \mathcal{A} \to \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}/\overline{Y} \to 0$$

   of sheaves of abelian groups, where $\overline{Y}$ *is the image of* Y *in* $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$*.*

3. *The diagonal morphism* $\mathcal{A} \to \mathcal{A} \times_S \mathcal{A}$ *is represented by finite morphisms.*

Remark 4.3.7 ([13], Section 4.2)**.** *Let* V *be an fs log scheme over* S*. Then the diagonal* $\mathrm{Mor}_S(, V) \to \mathrm{Mor}_S(, V) \times \mathrm{Mor}_S(, V)$ *is represented by finite morphisms if and only if the underlying scheme of* V *is separated over that of* S*.*

The semi-abelian scheme G in fact exists globally on S and is uniquely determined by $\mathcal{A}$. We call G the semi-abelian part of $\mathcal{A}$. Denote by $\tilde{\mathcal{A}} := G_{\log}^{(Y)}$, we can summarize our definition in the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \overline{Y} \arrow[r, "="] \arrow[d] & \overline{Y} \arrow[d] & \\
0 \arrow[r] & G \arrow[r] \arrow[d, "="] & \tilde{\mathcal{A}} \arrow[r] \arrow[d] & \mathrm{Hom}(X, (\mathbb{G}_{m,\log}/\mathbb{G}_m))^{(Y)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \arrow[r] & \mathcal{A} \arrow[r] \arrow[d] & \mathrm{Hom}(X, (\mathbb{G}_{m,\log}/\mathbb{G}_m))^{(Y)}/\overline{Y} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Lastly, we briefly mention a particular subsheaf $\mathcal{A}^{(\Sigma)}$ of $\mathcal{A}$, referred to as a model of $\mathcal{A}$ (cf. [14, Section 2]).

Theorem 4.3.8. *[14, Theorem 8.1] If $\mathcal{A}$ is a general log abelian variety over S. Assume that there exist an exact sequence*

$$0 \to G \to A \to \mathcal{H}\mathrm{om}(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}/\overline{Y} \to 0$$

with a homomorphism $S \to M_S/\mathcal{O}_S^*$, and an admissible pairing $X \times Y \to S^{gp}$ as in [14, Section 2.4, 2.5]. Let $C \subset \mathrm{hom}(S, \mathbb{N}) \times \mathrm{hom}(X, \mathbb{Z})$ be as in [14, Section 2.2], and let $\sum$ be a fan in C. If $\sum$ is stable under the action of Y, then there is a model $\mathcal{A}^{(\Sigma)}$ of $\mathcal{A}$ which is an algebraic space over S, i.e an algebraic space over the underlying scheme of S endowed with an fs log structure over S.

Theorem 4.3.9. *[14, Corollary 4.2] Assume the conditions as the above theorem. If $\mathcal{A}$ is a log abelian variety with constant degeneration over S then there is a model $\mathcal{A}^{(\Sigma)}$ of $\mathcal{A}$ which is a proper scheme over S.*

4.4 Weak logarithmic abellian varieties

In this section, we introduce a generalization of logarithmic abellian varieties known as weak logarithmic abellian varieties, as presented in [14, Section 1]. To start, we define a face of a monoid.

Definition 4.4.1. *Let M be a monoid. A face of a monoid is a submonoid F such that whenever $n + m \in F$, we have both n and m are in F.*

Let X and Y be finitely generated free $\mathbb{Z}$-modules, let M be an fs monoid, and let

$$\langle . \rangle : X \times Y \to M^{gp}$$

be a $\mathbb{Z}$-bilinear form. For a face σ of M, define M_σ to be the subgroup of M consisting of all elements x of X such that $\langle x, Y \rangle$ is contained in σ^{gp}.

Definition 4.4.2. *We say that $\langle . \rangle$ is admissible (precisely, M-admissible) if the following condition is satisfied: for any face σ of M and any homomorphism*

$$N : M \to \mathbb{R}_{\geq 0}$$

if we write the face $\sigma \cap N^{-1}(0)$ of M as τ, then the pairing of $\mathbb{R}$-linear spaces induced by N

$$(X_\sigma / X_\tau)_{\mathbb{R}} \times (Y_\sigma / Y_\tau)_{\mathbb{R}} \to \mathbb{R}$$

is nondegenerate.

Remark 4.4.3. *Equivalently, it is admissible if the following condition holds. For any face σ of S and any homomorphism $N : \sigma \to \mathbb{R}_{\geq 0}$, the induced pairing of $\mathbb{R}$-linear spaces by N*

$$(X_\sigma / X_\tau)_{\mathbb{R}} \times (Y_\sigma / Y_\tau)_{\mathbb{R}} \to \mathbb{R}$$

iw nondegenerate, where τ is the face $N^{-1}(0)$ of σ.

Definition 4.4.4. *Let X and Y be finitely generated free $\mathbb{Z}$-modules, and let*

$$\langle , \rangle : X \times Y \to (\mathbb{G}_{m,\log} / \mathbb{G}_m)$$

be a $\mathbb{Z}$-bilinear form. We say that $\langle , \rangle$ is admissible if, for any $s \in S$, the induced pairing

$$X \times Y \to (M_S^{gp} / \mathcal{O}_S^*)_{\overline{s}}$$

is $(M_S / \mathcal{O}_S^*)_{\overline{s}}$-admissible.

Definition 4.4.5. *We say a log 1-motive*

$$[Y \to G_{\log}]$$

over S admissible (resp. nondegenerate) if the canonical pairing

$$X \times Y \to \mathbb{G}_{m,\log}/\mathbb{G}_m$$

is admissible (resp. nondegenerate [13, p. 4.11.2]), where $X = \mathcal{H}\mathrm{om}(T, \mathbb{G}_m)$.

The admissible pairing gives a homomorphism

$$Y \to \mathcal{H}\mathrm{om}(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$$

We denote the subsheaf of groups $\mathcal{H}\mathrm{om}(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$ by $\mathcal{Q}$. Recall that $G_{\log}^{(Y)} \subset G_{\log}$ is the inverse image of $\mathcal{Q}$ under $G_{\log} \to G_{\log}/G = \mathcal{H}\mathrm{om}(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$.

Definition 4.4.6. *A weak log abelian variety over an fs log scheme S is a sheaf of abelian groups $\mathcal{A}$ on (fs/S) satisfying the following three conditions*

1. *For any $s \in S$, the pullback of $\mathcal{A}$ to $(fs/\bar{s})$ is isomorphic to $G_{\log}^{(Y)}/Y$ for an admissible and nodegenerate log 1-motive $[Y \to G_{\log}]$.*

2. *Étale locally on S, there are a semiabelian scheme G over S, finitely generated free $\mathbb{Z}$-modules X and Y, an admissible pairing*

$$\langle , \rangle : X \times Y \to (\mathbb{G}_{m,\log}/\mathbb{G}_m)_S$$

on (fs/S), and an exact sequence

$$0 \to G \to \mathcal{A} \to \mathcal{Q}/\overline{Y} \to 0$$

where $\overline{Y}$ denotes the image of Y in $\mathcal{Q}$.

3. *The diagonal morphism $\mathcal{A} \to \mathcal{A} \times \mathcal{A}$ is represented by finite morphisms.*

Remark 4.4.7. *If we substitute "admissible and nondegenerate" in condition 1 with "polarizable", we arrive at the definition of a log abelian variety. Given that polarizability implies admissibility and nondegeneracy, a log abelian variety is indeed a weak log abelian variety.*

4.5 Weak log abelian varieties over complete discrete valuation rings

We fix an fs log structure N on $\mathrm{Spec}(K)$, and let M be its direct image log structure on $\mathrm{Spec}(\mathcal{O}_K)$.

Remark 4.5.1. *If N is trivial, then M is the standard log structure on $S :=$ $\mathrm{Spec}(\mathcal{O}_K)$. Otherwise M is not an fs log structure, M is the filtered union of fs log structure contained in M whose restriction on S is N.*

We define the following categories following [14, Section 3] and [18, p. C.2] for corrections.

1. Let C_0 be the category of weak log abelian varieties over $\mathcal{O}_K$ endowed with the log structure M, which is defined to be the inductive limit of the categories of weak log abelian varieties for fs log structures M_i.

2. Let C_1 be the category of admissible log 1-motives over $\mathcal{O}_K$ with respect to the log structure M, which is defined similarly as above, i.e., the inductive limit of the categories of weak log abelian varieties for fs log structure M_i, where $(M_i)_i$ is fs log structures contained in M whose restriction on S is N.

3. Let (WLAV/K) be the category of weak log abelian varieties over K with respect to the log structure N.

4. Let C_2 be the full subcategory of (WLAV/K) consisting of objects have the following property: if $[Y \to G_{\log}]$ denotes the corresponding log 1-motive over K and $0 \to T \to G \to B \to 0$ denotes the exact sequence with T a torus and B an abelian variety, then Y and T are unramified and B is of semistable reduction.

5. Define $C_1^{[pol]}$ be the following subcategory of C_1. Let $[Y_1 \to G_{\log}]$ be an object of C_1. Let $Y_{(1)}$ be the kernel of the cannonical homomorphism $Y_1 \to Y_2$, where Y_2 is the stalk of $\overline{Y}_1$ at the generic point. Let $T_2 \subset T$ be the torus corresponding to the stalk X_2 of $\overline{X}_1$ at the generic point, where T is the torus part of G. Let $G_{(1)} = G/T_2$. The composition

$$Y_{(1)} \to Y_1 \to G_{\log} \to G_{(1),\log}$$

is an admissible log 1-motive. Then, $[Y_1 \to G_{\log}]$ belongs to $C_1^{[pol]}$ if and only if this log 1-motive $[Y_{(1)} \to G_{(1),\log}]$ is formally polarizable.

6. For $i = 0, 1, 2$, let C_i^{ptpol} be the full subcategory of C_i consisting of formally polarizable objects, and let C_i^{pol} be the full subcategory of $\mathcal{C}_i$ consisting of polarizable objects. In particular, C_0^{ptpol} is the category of log abelian varieties over $\mathcal{O}_K$.

We also define the following functors $\alpha_i : C_0 \to C_i$ for $i = 1, 2$.

1. For any weak log abelian variety over $\mathcal{O}_K$ with respect to M, then for each $n \geq 1$, we have a weak log abelian variety over $\mathcal{O}_K/m_K^n$, and the corresponding admissible log 1-motive over $\mathcal{O}_K/m_K^n$, where m_K is the maximal ideal of $\mathcal{O}_K$. This family of admissible log 1-motives for $n \geq 1$ defines an admissible log 1-motive over $\mathcal{O}_K$, hence a functor $\alpha_1 : C_0 \to C_1$
2. The functor α_2 is the evident functor.

Remark 4.5.2. *For $i = 0, 1, 2$, we have*

$$C_i^{pol} \subset C_i^{ptpol} \subset C_i$$

and, for $i = 1, 2$ the functor α_i sends C_0^{pol} (resp., C_0^{ptpol}) into C_i^{pol} (resp., C_i^{ptpol}).

$$\begin{tikzcd}
C_1 \arrow[r, phantom, "\supset"] & C_1^{[pol]} & \supset & C_1^{pol} \\
& C_0 \arrow[u, "\alpha_1"] \arrow[d, "\alpha_2"'] \arrow[r, phantom, "\supset"] & C_0^{ptpol} \arrow[u] \arrow[d] \arrow[r, phantom, "="] & C_0^{pol} \arrow[u] \arrow[d] \\
& C_2 \arrow[r, phantom, "\supset"] & C_2^{ptpol} \arrow[r, phantom, "="] & C_2^{pol}
\end{tikzcd}$$

and $C_1^{[pol]} \cong C_2$.

Using this diagram the authors can prove the following remarkable result (see [16, Section 3] and [18, p. C.2] for corrections).

Theorem 4.5.3. *We have an equivalence from the category of log abelian varieties over $\mathcal{O}_K$ with the standard log structure to the category of abelian varieties over K with semi-stable reduction.*

Remark 4.5.4. *Given a semi-stable abelian variety A over K, let $\mathcal{A}$ over $(\mathcal{O}_K, (1 \mapsto \pi)^a)$ be its corresponding log abelian variety. For any finite extension L/K, we have $A(L) = \mathcal{A}(\mathcal{O}_L, (1 \mapsto \pi_L)^a)$. Indeed, it is shown that $\mathcal{A}(L, triv) = \mathcal{A}(\mathcal{O}_L, (1 \mapsto \pi_L)^a)$ where the log structure triv is the trivial log structure (cf. [14, Theorem 17.1]). The equivalence yields*

$$A(L) = \mathcal{A}(L, triv) = \mathcal{A}(\mathcal{O}_L, (1 \mapsto \pi_L)^a)$$

.

4.6 Differential forms of log abellian varieties

In this section, we detail the construction of the sheaf of differential forms for a log abelian variety. To begin, we revisit the definition of the sheaf of differential forms for log schemes as outlined in [27].

Definition 4.6.1. *Let $\alpha : M \to \mathcal{O}_X$ and $\beta : N \to \mathcal{O}_Y$ be pre-log structures, and $f : (X, M) \to (Y, N)$ be a morphism. Then, we define the $\mathcal{O}_X$-module $\omega_{(X,M)/(Y,N)}$ to be the quotient of*

$$\omega_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M^{gp})$$

where $\omega_{X/Y}$ is the usual relative differential module, divided by the $\mathcal{O}_X$-submodule generated locally by local sections of the following forms

1. *$(d\alpha(a), 0) - (0, \alpha(a) \otimes a)$ with $a \in M$*
2. *$(0, 1 \otimes a)$ with $a \in \operatorname{im}(f^{-1}(N) \to M)$.*

If the following commutative diagram of log schemes is cartesian

$$\begin{array}{ccc} (X', M') & \xrightarrow{f} & (X, M) \\ \downarrow & & \downarrow \\ (Y', N') & \longrightarrow & (S, N) \end{array}$$

then we have an isomorphism

$$f^* \omega_{(X,M)/(S,N)} \cong \omega_{(X',M')/(Y',N')}$$

Let $\mathcal{A}$ be a log abelian variety over an fs log scheme S. To define the sheaf of differential forms on log abelian varieties we need to introduce several categories. Firstly, we recall the category $(\mathrm{fs}/\mathcal{A})$ from [15, Section 2.1].

Definition 4.6.2. *The étale site $(fs/\mathcal{A})$ consists of pairs (U, α), where U is an fs log scheme over S and α is an element of $\mathcal{A}(U)$. A covering $((U_i, \alpha_i) \to (U, \alpha))_i$ of an object $U \in (fs/\mathcal{A})$ is a covering in (fs/S) such that α_i maps to α.*

Secondly, we introduce the site composed of log étale objects. Let us begin by revisiting various types of morphisms of log schemes.

Definition 4.6.3. *Let S be an fs log scheme. A log algebraic space F over S in the first sense is an algebraic space over the underlying scheme of S endowed with an fs log structure over S [14, Section 10.1]. We say that F is log étale over S if, for any fs log scheme T which is strictly étale over F, the composite $T \to F \to S$ is log étale.*

We take the following definitions from [18, Section 3.9].

Definition 4.6.4. *The small site $(l\acute{e}t/\mathcal{A})$ is the full subcategory of $(fs/\mathcal{A})$ consisting of all log étale objects. An object $(V \to \mathcal{A})$ in $(fs/\mathcal{A})$, (by definition, an fs log scheme over S with a point in $\mathcal{A}(V)$) is called log étale if for any fs log scheme W and any morphism $W \to \mathcal{A} \to S$, the fiber product $W \times_{\mathcal{A}} V$ is a log étale log algebraic space over W.*

Here $W, \mathcal{A}, V$ are sheaves on (fs/S). The fiber product concerned is the one defined in the category of the sheaves on (fs/S).

Remark 4.6.5. *As noted in [18, Section 3.10], a set of topological generators of $(l\acute{e}t/\mathcal{A})$ comprises log étale fs log schemes over models of weak log abelian varieties that are isogenous to $\mathcal{A}$.*

Definition 4.6.6. *The site $(fsl\acute{e}t/\mathcal{A})$ consists of pairs (U, V) where U is an object of (fs/S) and (V) is an object of $(l\acute{e}t/\mathcal{A}_U)$, where $\mathcal{A}_U := \mathcal{A} \times_S U$. Morphisms from (U', V') to (U, V) is a compatible pair of morphisms $(U' \to U)$ and $(V' \to V)$. Coverings are the $((U_i, V_i) \to (U, V))_i$ such that every $U_i \to U$ is strictly étale and that $(V_i \to V)_i$ is a strict étale covering.*

We can summarize the definition in the following diagram, seen as a diagram of sheaves over (fs/S).

$$\begin{tikzcd}
V \arrow[d] \arrow[dr, dashed] & \\
\mathcal{A}_U \arrow[r] \arrow[d] & U \arrow[d] \\
\mathcal{A} \arrow[r] & S
\end{tikzcd}$$

Remark 4.6.7. *As noted in [18, Section 3.10], a set of generators for* $(fslét/\mathcal{A})$ *can be provided, consisting of pairs* (U, V) *where* U *is an fs log scheme over* S*, and* V *is a log étale fs log scheme over a model of a weak log abelian variety that is isogenous to* $\mathcal{A}_U$*. Consequently, we can define* $\omega_{V/U}$ *as the standard sheaf of log differential forms.*

We have the following definition of differential forms.

Definition 4.6.8. *Let* $\mathcal{A}$ *be a log abelian variety over an fs log scheme* S*. Let* $\mathcal{O}_{\mathcal{A}}$ *be the sheaf on* $(fslét/\mathcal{A})$*, defined locally on pairs* (U, V)*, by*

$$\mathcal{O}_{\mathcal{A}}((U, V)) = \Gamma(V, \mathcal{O}_V)$$

(Note that $\mathcal{O}_V$ *is nothing but the usual structure sheaf of a scheme* V*). Define a sheaf of* $\mathcal{O}_{\mathcal{A}}$*-modules* $\omega_{\mathcal{A}/S}$ *by*

$$\omega_{\mathcal{A}/S}((U, V)) = \Gamma(V, \omega_{V/U})$$

where $\omega_{V/U}$ *is the sheaf of relative differential forms of log schemes.*

Definition 4.6.9. *Let* S *be an fs log scheme. For a group sheaf* G *on* (fs/S)*, we define the Lie sheaf* $\mathrm{Lie}(G)$ *on* (fs/S) *by*

$$\mathrm{Lie}(G)(U) = \ker\left(G\left(U[\epsilon]/(\epsilon^2)\right) \to G(U)\right)$$

where $U[\epsilon]/(\epsilon^2)$ *denotes the fs log scheme whose underlying space is that of* U*, whose structure sheaf is* $\mathcal{O}_U[\epsilon]/(\epsilon^2)$ *and whose log structure is the pullback of that of* U*.*

Example 4.6.10 ([18], Section 3.1)**.** *We have*

$$\mathrm{Lie}(\mathbb{G}_{m,\log}) = \mathrm{Lie}(\mathbb{G}_m)$$

Indeed it comes from the definition of a morphism of log schemes. Indeed, let T *be an fs scheme with the associated log structure* $\alpha : M_T \to \mathcal{O}_T$ *and* $f : T[\epsilon]/\epsilon^2 \to T$*. We have the following commutative diagram*

$$\begin{array}{ccc} f^*(M_T) & \longrightarrow & M_T \\ \downarrow & & \downarrow \\ f^*(\mathcal{O}_T) & \longrightarrow & \mathcal{O}_T \end{array}$$

It follows that

$$\begin{aligned}\mathrm{Lie}(\mathbb{G}_{m,\log})(T) &= \ker(f^*(M_T))^{gp}(T) \to (M_T)^{gp}(T)) \\ &= \ker(\mathcal{O}^*_{T[\epsilon]/\epsilon^2}(T) \to \mathcal{O}^*_T(T)) = \mathrm{Lie}(\mathbb{G}_m)\end{aligned}$$

Remark 4.6.11. *We naturally define an action of $\mathcal{O}_S$ on $\mathrm{Lie}(G)$ as follows: for $a \in \mathcal{O}_S(U)$, the action of a on $\mathrm{Lie}(G)(U)$ is determined by the morphism induced by $\epsilon \mapsto a\epsilon$. In general, this action represents the action of $\mathcal{O}_S$ as a multiplicative monoid. However, in many instances—such as when G is represented by an fs log scheme over S or G is a log abelian variety—$\mathrm{Lie}(G)$ becomes an $\mathcal{O}_S$-module through this action.*

Assume that $\mathrm{Lie}(G)$ is an $\mathcal{O}_S$-module, define

$$\mathrm{coLie}(G) := \mathrm{Hom}_{\mathcal{O}_S}(\mathrm{Lie}(G), \mathcal{O}_S)$$

Chapter 5

Logarithmic Fontaine integration for semi-stable abelian varieties

In this chapter, we construct a logarithmic version of the Fontaine integration for abelian varieties with semi-stable reduction. The construction in this part is new and original.

5.1 Construction

Let A be an abelian variety over K, then

$$A(\overline{K}) = \varinjlim_{L/K \text{finite}} A(L)$$

here the colimit runs over finite extensions $L \subset L'$ of K inside $\overline{K}$. Recall that we denote by $(1 \mapsto \pi_L)^a$ the canonical log structure on $\mathcal{O}_L$ for any finite extension L of K

$$\begin{aligned} \mathbb{N} &\to \mathcal{O}_L \\ 1 &\mapsto \pi_L \end{aligned}$$

Lemma 5.1.1. *Given a semi-stable abelian variety A over K, let $\mathcal{A}$ over $(\mathcal{O}_K, (1 \mapsto \pi)^a)$ be its corresponding log abelian variety* (4.5.3)*. Then we have*

$$A(\overline{K}) = \varinjlim_{L/K \mathit{finite}} \mathcal{A}(\mathcal{O}_L, (1 \mapsto \pi_L))$$

where the colimit runs over $(\mathcal{O}_L, (1 \mapsto \pi_L)^a) \to (\mathcal{O}_{L'}, (1 \mapsto \pi_{L'})^a)$ is induced by *the inclusion $\mathcal{O}_L \subset \mathcal{O}_{L'}$ maps $\pi_L \in \mathcal{O}_L$ to itself and induces a morphism of log structures, i.e., we obtain the following commutative diagram*

$$\begin{tikzcd}
(1 \mapsto \pi_L)^a = \mathcal{O}_L \setminus \{0\} \arrow[r, hook] \arrow[d, "\subset"] & \mathcal{O}_L \arrow[d, hook] \\
(1 \mapsto \pi_{L'})^a = \mathcal{O}_{L'} \setminus \{0\} \arrow[r, hook] & \mathcal{O}_{L'}
\end{tikzcd}$$

Proof. By 2.1.10, the log structure $(1 \mapsto \pi_L)^a$ on $\mathcal{O}_L$ can be identified with the inclusion $\mathcal{O}_L \setminus \{0\} \hookrightarrow \mathcal{O}_L$ for any finite extension L/K. It follows that we can consider the colimit running over $(\mathcal{O}_L, (1 \mapsto \pi_L)^a) \to (\mathcal{O}_{L'}, (1 \mapsto \pi_{L'})^a)$ defined as above for $L \subset L'$ being finite extensions of K inside $\overline{K}$. In addition, it was shown in 4.5.4 that

$$A(L) = \mathcal{A}(\mathcal{O}_L, (1 \mapsto \pi_L)^a)$$

for all finite extension L of K inside $\overline{K}$. Hence

$$A(\overline{K}) = \varinjlim_{L/K\text{finite}} A(L) = \varinjlim_{L/K\text{finite}} \mathcal{A}(\mathcal{O}_L, (1 \mapsto \pi_L)^a)$$

□

This implies that for any $x \in A(\overline{K})$, there exists a finite field extension L of K such that $x \in \mathcal{A}(\mathcal{O}_L, (1 \mapsto \pi_L)^a)$.

Next, we construct the pull-back map for the log abelian variety. We recall the sheaf of differential forms for the log abelian variety $\mathcal{A}$ over $S := (\operatorname{Spec} \mathcal{O}_K, (1 \mapsto \pi)^a)$ (see 4.6.8). Locally, on a pair (U, V) from a basis of $(\text{fslét}/\mathcal{A})$, it is given by

$$\omega_{\mathcal{A}/S} : (\text{fslét}/\mathcal{A}) \to (\mathcal{O}_{\mathcal{A}} - \text{modules}) : (U, V) \mapsto \Gamma(V, \omega_{V/U})$$

here we can assume that U and V are log schemes over S, $\omega_{V/U}$ is the usual sheaf of differential form of log schemes.

Remark 5.1.2. *For any finite extension L of K, let $D_L = (\operatorname{Spec} \mathcal{O}_L, (1 \mapsto \pi_L)^a)$. A morphism $x : D_L \to \mathcal{A}$ of sheaves on (fs/S) and the universal property of the fibre product gives us a morphism $D_{L,U} := D_L \times_S U \to \mathcal{A}_U$*

$$\begin{tikzcd}
D_{L,U} \arrow[drr, bend left] \arrow[dr, dotted] \arrow[dd, bend right] & & \\
& \mathcal{A}_U \arrow[r] \arrow[d] & U \arrow[d] \\
D_L \arrow[r, "x"] & \mathcal{A} \arrow[r] & S
\end{tikzcd}$$

It follows that if $(U, V) \in$ (fslét/$\mathcal{A}$), define $V_{D_L} := V \times_U D_{L,U}$, then (U, V_{D_L}) is also in the site (fslét/$\mathcal{A}$)

$$\begin{tikzcd}
V_{D_L} \arrow[d] \arrow[dr, dashed] & \\
\mathcal{A}_U \arrow[r] \arrow[d] & U \arrow[d] \\
\mathcal{A} \arrow[r] & S
\end{tikzcd}$$

Definition 5.1.3. *For any $x \in A(\overline{K})$, let L be a finite extension of K such that $x \in \mathcal{A}(\mathcal{O}_L, (1 \mapsto \pi_L)^a)$ (i.e. a morphism $x : D_L \to \mathcal{A}$). Define a sheaf $x_*\omega_{D_L/S}$*

$$x_*\omega_{D_L/S} : (\text{fslét}/\mathcal{A}) \to (\mathcal{O}_\mathcal{A} - \textit{modules})$$

locally at a pair (U, V) in the basis of (fslét/$\mathcal{A}$) given by

$$x_*\omega_{D_L/S}(U, V) := \omega_{\mathcal{A}/S}((U, V_{D_L})) = \Gamma(V_{D_L}, \omega_{V_{D_L}/U})$$

The morphism $x : D_L \to \mathcal{A}$ fit into the following commutative diagram of sheaves on (fs/S)

$$\begin{tikzcd}
x : (\operatorname{Spec} \mathcal{O}_L, (1 \mapsto \pi_L)^a) \arrow[r] \arrow[dr] & \mathcal{A} \arrow[d] \\
& S = (\operatorname{Spec} \mathcal{O}_K, (1 \mapsto \pi)^a)
\end{tikzcd}$$

Then we have the pull-back morphism as follows.

Lemma 5.1.4. *For any $x \in A(\overline{K})$, let L be a finite extension of K such that $x \in \mathcal{A}(D_L)$. We define the pull-back*

$$x^* : \omega_{\mathcal{A}/S} \to x_*\omega_{D_L/S}$$

locally on basis pair (U, V) in the site (fslét/$\mathcal{A}$) given by the morphism

$$x^*(U, V) : \omega_{\mathcal{A}/S}(U, V) \to x_*\omega_{D_L/S}(U, V_{D_L})$$

coming from the functoriality of differential forms of log schemes.

Proof. Recall $V_{D_L} := V \times_U D_{L,U}$ and (U, V_{D_L}) is also in the site (fslét$/\mathcal{A}$)

$$\begin{tikzcd}
V_{D_L} \arrow[rr] \arrow[d] & & V \arrow[dl] \arrow[d] \arrow[dd, bend left] \\
D_L \arrow[r, "x"] \arrow[dr] & \mathcal{A} \arrow[d] & \mathcal{A}_U \arrow[l] \arrow[d] \\
& S & U \arrow[l]
\end{tikzcd}$$

Thus the morphism $x^*(U, V) : \omega_{\mathcal{A}/S}(U, V) \to x_*\omega_{D_L/S}(U, V_{D_L})$ comes from the usual morphism of log differential forms of log schemes $\omega_{V/U} \to \omega_{V_L/U}$

$$\begin{tikzcd}
V_{D_L} \arrow[r] \arrow[dr] & V \arrow[d] \\
& U
\end{tikzcd}$$

It is compatible with any morphism of pairs $(U, V) \to (U', V')$

$$\begin{tikzcd}
\omega_{\mathcal{A}/S}(U, V) \arrow[r] \arrow[d] & x_*\omega_{D_L/S}(U, V_{D_L}) \arrow[d] \\
\omega_{\mathcal{A}/S}(U', V') \arrow[r] & x_*\omega_{D_L/S}(U', V'_{D_L})
\end{tikzcd}$$

□

We recall the following definition from [16, Sections 3.9 and 3.10].

Definition 5.1.5. *Let $\mathcal{A}$ be a log abelian variety. Define $\mathcal{H}^q(\mathcal{A}, -)$ to be the q-th derived functor of the direct image functor*

$$(\mathit{fslét}/\mathcal{A}) \to (\mathit{fs}/S) : F \mapsto (U \mapsto \varprojlim_{V/\mathcal{A}_U} F((U, V))$$

Hence we can define the global sections of $\mathcal{H}^0(\mathcal{A}, \omega_{\mathcal{A}/S})$ and $\mathcal{H}^0(\mathcal{A}, x_*\omega_{D_L/S})$ for any $x \in A(\overline{K})$ to be $\Gamma(S, \mathcal{H}^0(\mathcal{A}, \omega_{\mathcal{A}/S})) := H^0(\mathcal{A}, \omega_{\mathcal{A}/S})$ and $\Gamma(S, \mathcal{H}^0(\mathcal{A}, x_*\omega_{D_L/S})) := H^0(D_L, x_*\omega_{D_L/S})$, respectively.

Remark 5.1.6. *We owe Prof. Heer Zhao the following observation. The global sections $H^0(\mathcal{A}, \omega_{\mathcal{A}/S})$ (resp. $H^0(D_L, x_*\omega_{D_L/S})$) can be computed as the global section of its restriction to the small Zariski site of S: for any Zariski open U of S, it is $(U \mapsto \varprojlim_{V/\mathcal{A}_U} \Gamma(V, \omega^1_{V/U})$ (resp. $(U \mapsto \varprojlim_{V/\mathcal{A}_U} \Gamma(V_{D_L}, \omega^1_{V_{D_L}/U}))$. Note that, with $U = S$, the model $\mathcal{A}_U$ is a log algebraic space in the first sense.*

We now can define the pull-back map associated to $x \in A(\overline{K})$ as classical construction as follows.

Proposition 5.1.7. *Let A be an abelian variety over K, and let $\mathcal{A}$ be its associated log abelian variety over $\mathcal{O}_K$, Then for any $x \in A(\overline{K})$, we have the pull-back pairing*

$$(,)_A^{\log} : A(\overline{K}) \times H^0(\mathcal{A}, \omega_{\mathcal{A}/S}) \to \Omega_{(\operatorname{Spec}\mathcal{O}_{\overline{K}},P^a)/(\mathcal{O}_K,(1\mapsto\pi)^a)}$$
$$(x,\omega) \mapsto x^*\omega$$

where P is the pre-log structure $P = \mathbb{Q}_{>0} \to \mathcal{O}_{\overline{K}}$ as in 2.1.14.

Proof. Denote by $\Omega^{\log} := \Omega_{(\mathcal{O}_{\overline{K}},P)/(\mathcal{O}_K,(1\mapsto\pi)^a)}$ with P as above. Taking the global section, the above lemma yields

$$\begin{aligned} x^* : H^0(\mathcal{A}, \omega_{\mathcal{A}/S}) \to & H^0(D_L, x_*\omega_{D_L/S}) \\ &= \Omega_{(\operatorname{Spec}\mathcal{O}_L,(1\mapsto\pi_L))/(\operatorname{Spec}\mathcal{O}_K,(1\mapsto\pi))} \\ &\to \Omega_{(\operatorname{Spec}\mathcal{O}_{\overline{K}},(1\mapsto\pi_L))/(\operatorname{Spec}\mathcal{O}_K,(1\mapsto\pi))} \end{aligned}$$

where the last morphism is induced by $\mathcal{O}_L \to \mathcal{O}_{\overline{K}}$. Now we show by definition that there is a morphism

$$\Omega_{(\operatorname{Spec}\mathcal{O}_{\overline{K}},(1\mapsto\pi_L))/(\operatorname{Spec}\mathcal{O}_K,(1\mapsto\pi))} \to \Omega^{\log}$$

Indeed, by 2.1.10 we have

$$\begin{aligned} \Omega_{(\operatorname{Spec}\mathcal{O}_{\overline{K}},(1\mapsto\pi_L))/(\operatorname{Spec}\mathcal{O}_K,(1\mapsto\pi))} &= \frac{\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \oplus (\mathcal{O}_{\overline{K}} \otimes \operatorname{coker}(((1\mapsto\pi)^a)^{gp} \to (1\mapsto\pi_L)^{gp})}{\sim} \\ &= \frac{\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \oplus (\mathcal{O}_{\overline{K}} \otimes L^*/K^*)}{\sim} \end{aligned}$$

On the other hand, we see the group associated to the log structure P is $\overline{K}^*$, so that

$$\begin{aligned} \Omega^{\log} &= \frac{\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \oplus (\mathcal{O}_{\overline{K}} \otimes \operatorname{coker}(((1\mapsto\pi)^a)^{gp} \to P^{gp})}{\sim} \\ &= \frac{\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \oplus (\mathcal{O}_{\overline{K}} \otimes \overline{K}^*/K^*)}{\sim} \end{aligned}$$

Hence the morphism $L^*/K^* \to \overline{K}^*/K$ yields the morphism of differential forms

$$\Omega_{(\operatorname{Spec}\mathcal{O}_{\overline{K}},(1\mapsto\pi_L))/(\operatorname{Spec}\mathcal{O}_K,(1\mapsto\pi))} \to \Omega^{\log}$$

□

Proposition 5.1.8. *Let A be an abelian variety over K, and let $\mathcal{A}$ be its associated log abelian variety over $\mathcal{O}_K$. Denote by $\mathcal{B}_A$ the universal cover of A. We obtain the logarithmic Fontaine pairing on $\mathcal{B}_A$, also denoted by $(,)_A^{\log}$, given by*

$$(,)_A^{\log} : \mathcal{B}_A \times H^0(\mathcal{A}, \omega_{\mathcal{A}/S}) \to V_p(\Omega_{(\operatorname{Spec}\mathcal{O}_{\overline{K}},P)/S})$$
$$(x,\omega) \mapsto (x_n^*\omega)_n$$

Proof. From the pairing

$$A(\overline{K}) \times H^0(\mathcal{A}, \omega_{\mathcal{A}/S}) \to \Omega_{(\operatorname{Spec}\mathcal{O}_{\overline{K}},P)/S}$$
$$(x,\omega) \mapsto x^*\omega$$

we can obtain the logarithmic Fontaine pairing as follows. Let $x = (x_n)_{n\in\mathbb{N}} \in \mathcal{B}_A$. For a fixed $n \in \mathbb{N}$, we can choose a finite extension L/K such that both x_n and x_{n+1} belong to $A(L) = \mathcal{A}(\mathcal{O}_L, (1 \mapsto \pi_L)^a) \subset \mathcal{A}(\mathcal{O}_{\overline{K}}, (1 \mapsto \pi_L)^a)$

$$\begin{tikzcd}
D_L := (\operatorname{Spec}\mathcal{O}_{\overline{K}}, (1 \mapsto \pi_L)^a) \arrow[d, "x_{n+1}"] \arrow[dr, "x_n"] & \\
\mathcal{A} \arrow[r, "p"] & \mathcal{A}
\end{tikzcd}$$

Let $\omega \in H^0(\mathcal{A}, \omega_{\mathcal{A}/S})$, the pull-back $x_n^*(\omega) \in \Omega_{(\operatorname{Spec}\mathcal{O}_{\overline{K}},P)/S}$ for any $n \in \mathbb{N}$. Moreover, by the construction of the pull-back, we have the following commutative diagram (locally at any basis pair U, V of (fslét$/\mathcal{A}$))

$$\begin{tikzcd}
V_{D_L} \arrow[rr] \arrow[d] & & V \arrow[dl] \arrow[d] \arrow[dd, bend left] \\
D_L \arrow[r, "x"] \arrow[dr] & \mathcal{A} \arrow[d] & \mathcal{A}_U \arrow[l] \arrow[d] \\
& S & U \arrow[l]
\end{tikzcd}$$

where $\mathcal{A}_U := \mathcal{A} \times_S U$ and $V_{D_L} := V \times_U D_{L,U}$. Note that the multiplication by p induces the morphism $p \times \mathrm{id} : \mathcal{A}_U \to \mathcal{A}_U$. We see that

$$\begin{tikzcd} & V_{D_L} \arrow[dl, "x_{n+1}"'] \arrow[dr, "x_n"] & & \\ \mathcal{A}_U \arrow[rr, "p\times\mathrm{id}"] & & \mathcal{A}_U \arrow[r] & U \end{tikzcd}$$

By construction of the pull-back, we find that

$$\begin{tikzcd} \omega_{\mathcal{A}/S}(U,V) = \omega_{V/U} \arrow[r, "x_{n+1}^*"] \arrow[dr, "x_n"] & x_*\omega_{D_L/S}(U,V_{D_L}) = \omega_{V_{D_L}/U} \arrow[d, "p"] \\ & x_*\omega_{D_L/S}(U,V_{D_L}) = \omega_{V_{D_L}/U} \end{tikzcd}$$

Hence taking the global section we have $px_{n+1}^*(\omega) = x_n^*(\omega)$ for all $\omega \in H^0(\mathcal{A}, \omega_{\mathcal{A}/S})$. Thus we obtain the logarithmic Fontaine pairing

$$\begin{aligned}(\,,\,)_A^{\log} : \mathcal{B}_A \times H^0(\mathcal{A}, \omega_{\mathcal{A}/S}) &\to V_p(\Omega_{(\operatorname{Spec}\mathcal{O}_{\overline{K}},P)/S}) \\ (x,\omega) &\mapsto (x_n^*\omega)_n\end{aligned}$$

□

Now we are ready to obtain the Fontaine integration. We need the following property of log abelian variety to reduce our base S to a log scheme which is finite type over $\mathbb{Z}$.

Proposition 5.1.9. *[18, Proposition 2.6] Let $(S_\lambda)_\lambda$ be a filtered projective system of quasicompact and quasiseparated fs log schemes whose transition morphisms are affine and strict. Let $S := \varprojlim S_\lambda$. For a principally polarized log abelian variety A over S, there are an index λ and a principally polarized log abelian variety A_λ over S_λ whose pullback to S is isomorphic to A.*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_\lambda \arrow[d] \\ S \arrow[r] & S_\lambda \end{tikzcd}$$

In particular, we have

$$H^0(\mathcal{A}, \omega_{\mathcal{A}/S}) = H^0(\mathcal{A}_\lambda, \omega_{\mathcal{A}_\lambda/S_\lambda}) \otimes \mathcal{O}_K \text{ and } \mathrm{coLie}(\mathcal{A})(S) = \mathrm{coLie}(\mathcal{A}_\lambda)(S_\lambda) \otimes \mathcal{O}_K$$

The following property plays an important role in the theory of Lie algebra of log abelian varieties.

Proposition 5.1.10. *[18, Proposition 3.12] Let S_λ be an fs log scheme which is of finite type over $\mathbb{Z}$. Let $\mathcal{A}_\lambda$ be principally log abelian variety over S_λ. Then we have*

$$H^0(\mathcal{A}_\lambda, \omega_{\mathcal{A}_\lambda/S_\lambda}) \cong \mathrm{coLie}(\mathcal{A}_\lambda)(S_\lambda)$$

To reach our purpose, we need the following property called absolute Noetherian approximation.

Proposition 5.1.11. *[33, 01Z1] Let S be a quasi-compact and quasi-separated scheme. There exists a directed set I and an inverse system of schemes $S_i, f_{ii'}$ over I such that*

1. *the transition morphisms $f_{ii'}$ are affine*
2. *each S_i is of finite type over $\mathbb{Z}$*
3. *$S = \varprojlim_i S_i$*

Example 5.1.12. *Let R be any commutative ring, we denote by I the set of finite subsets of R ordered by inclusion. Let J be the set of $\mathbb{Z}$-algebras $\mathbb{Z}[S]$ generated by $S \in I$, ordered by inclusion; they form a directed system over I. Taking the colimit in J we obtain*

$$R \cong \varinjlim_{S\in I}(\mathbb{Z}[S])$$

It follows that $\operatorname{Spec} R = \varprojlim_{S\in I} \operatorname{Spec}(\mathbb{Z}[S])$*.*

For inverse limit in the category of log schemes, we take the following definition from [27, Definition 1.6]

Definition 5.1.13. *Let (X_λ, M_λ) be an inverse system, the inverse limit is (X, M) where X is the inverse limit of the system of schemes X_λ, and M is given as follows. Let $p_\lambda : X \to X_\lambda$ be the projection, take the inductive limit M' of the inductive system of sheaves of monoids $p_\lambda^{-1}(M_\lambda)$, and then let M be the log structure associated to the pre-log structure $M' \to \mathcal{O}_X$.*

Remark 5.1.14. *Consider the log scheme $S := (\operatorname{Spec} \mathcal{O}_K, (1 \mapsto \pi)^a)$. Then, as a scheme, S we can be written as $\varprojlim_{\lambda\in\mathbb{N}} S_\lambda$ with $S_\lambda = \operatorname{Spec} R_\lambda$ satisfying the properties of Noetherian approximation lemma. Regarding log structure, consider $p_\lambda : S \to S_\lambda$ be the projection, it corresponds to a morphism of rings $q_\lambda : R_\lambda \to \mathcal{O}_K$, we endow each R_λ with the log structure given by the inverse image of the canonical log structure on $\mathcal{O}_K$, denoted by $q_\lambda^*(((1 \mapsto \pi)^a)$*

$$\begin{tikzcd}
q_\lambda^*(((1 \mapsto \pi)^a) \arrow[r] \arrow[d, "q_\lambda"] & R_\lambda \arrow[d, "q_\lambda"] \\
(1 \mapsto \pi)^a \arrow[r, hook] & \mathcal{O}_K
\end{tikzcd}$$

Proposition 5.1.15. *Let A be an abelian variety over K, and let $\mathcal{A}$ be its associated log abelian variety over $\mathcal{O}_K$. We have*

$$H^0(A, \omega_{A/K}) = K \otimes H^0(\mathcal{A}, \omega_{\mathcal{A}/S})$$

Proof. Since every abelian variety is isogenous to a principally polarized abelian variety [1, Corollary 1 on page 234] and isogeny preserves Lie algebras, we assume that A is a principally polarized abelian variety, and so is $\mathcal{A}$. By the proposition 5.1.9, there exist λ and a principally polarized log abelian variety A_λ over S_λ such that

$$H^0(\mathcal{A}, \omega_{\mathcal{A}/S}) = H^0(\mathcal{A}_\lambda, \omega_{\mathcal{A}_\lambda/S_\lambda}) \otimes \mathcal{O}_K \text{ and } \operatorname{coLie}(\mathcal{A})(S) = \operatorname{coLie}(\mathcal{A}_\lambda)(S_\lambda) \otimes \mathcal{O}_K$$

Note that S_λ is finite type over $\mathbb{Z}$, by 5.1.10 we have

$$H^0(\mathcal{A}_\lambda, \omega_{\mathcal{A}_\lambda/S_\lambda}) = \operatorname{coLie}(\mathcal{A}_\lambda)(S_\lambda)$$

It implies $H^0(\mathcal{A}, \omega_{\mathcal{A}/S}) = \operatorname{coLie}(\mathcal{A})(S)$. The fibre product functor $\mathcal{A} \mapsto A$ yields

$$H^0(A, \omega_{A/K}) \cong \operatorname{coLie}(A)(K) \cong \operatorname{coLie}(\mathcal{A})(S) \otimes K \cong H^0(\mathcal{A}, \omega_{\mathcal{A}/S}) \otimes K$$

□

Proposition 5.1.16. *Let A be an abelian variety over K, and let $\mathcal{A}$ be its associated log abelian variety over $\mathcal{O}_K$. Denote by $\mathcal{B}_A$ the universal cover of A. we obtain the following Fontaine integration induced by $(,)_A^{\log}$*

$$\varphi_A^{\log} : \mathcal{B}_A \to \operatorname{Lie}(A)(K) \otimes V_p(\Omega_{(\operatorname{Spec}\mathcal{O}_{\overline{K}}, P)/S})$$

Proof. By the proposition, we can take the dual of $H^0(A, \omega_{A/K})$ which is $\operatorname{Lie}(A)(K)$. Thus we have the Fontaine integration in the following form

$$\varphi_A^{\log} : \mathcal{B}_A \to \operatorname{Lie}(A)(K) \otimes V_p(\Omega_{(\operatorname{Spec}\mathcal{O}_{\overline{K}}, P)/S})$$

□

5.2 Example: Fontaine integration for Tate curves

We can construct Fontaine integration for Tate curve E_q from the functorial point of view. Now consider the logarithmic Tate curve $\mathcal{E}_q$ as in [13]. Note that the log Tate curve is a constant degeneration log abelian variety. Let $(\mathrm{fs}/S)'$ be the full subcategory of (fs/S) consisting of objects on which m_K is locally nilpotent. The logarithmic Tate curve on $(\mathrm{fs}/S)'$ is $\mathbb{G}_{m,\log}^{(q)}/q^{\mathbb{Z}}$. It is shown that for any finite extension L of K we get

$$E_q(L) = \mathcal{E}_q(\mathcal{O}_L)$$

With the log structure we defined above we also have (with $q = \pi$)

$$\varinjlim_{n} \mathbb{G}_{m,\log}^{(q)}(\mathcal{O}_{\overline{K}}, (1 \mapsto \pi^{1/n})^a/q^{\mathbb{Z}} = \frac{\overline{K}^*}{q^{\mathbb{Z}}} = E_q(\overline{K})$$

There is a canonical way to construct $\mathcal{E}_q$ as follows. Let $X = Y = \mathbb{Z}$, then $T = G = \mathbb{G}_m$, $T_{\log} = G_{\log} = \mathcal{H}\mathrm{om}(\mathbb{Z}, \mathbb{G}_{m,\log}) = \mathbb{G}_{m,\log}$. Let $q \in \Gamma(s, M_s) \setminus K^*$, consider the log 1-motif

$$M = [\mathbb{Z} \to \mathbb{G}_{m,\log} : 1 \mapsto q]$$

The dual log 1-motif of M is itself $M^* = [\mathbb{Z} \to \mathbb{G}_{m,\log} : 1 \mapsto q]$ with the identity map $M \to M^*$ is a polarization on M. The subsheaf $\mathbb{G}_{m,\log}^{(q)}$ of $\mathbb{G}_{m,\log}$ is the inverse image of $\mathbb{G}_{m,\log}^{(\mathbb{Z})} \subset \mathbb{G}_{m,\log}/\mathbb{G}_m$ under the canonical morphism $\mathbb{G}_{m,\log} \to \mathbb{G}_{m,\log}/\mathbb{G}_m$. Note that we have

$$\mathrm{Lie}(\mathbb{G}_{m,\log}) = \mathrm{Lie}(\mathbb{G}_m) = \mathbb{G}_a$$

Where $\mathbb{G}_a$ is the additive group scheme. It follows that

$$\mathrm{Lie}(\mathbb{G}_{m,\log}) = \mathrm{Lie}(\mathbb{G}_{m,\log}^{(\pi)}) = \mathrm{Lie}(\mathbb{G}_m)$$

Hence

$$H^0(E_q, \Omega_{E_q}) = H^0(\mathbb{G}_m, \Omega_{\mathbb{G}_m}) = H^0(\mathbb{G}_{m,\log}^{(q)}, \Omega_{\mathbb{G}_{m,\log}^{(q)}}) = H^0(\mathcal{E}_q, \Omega_{\mathcal{E}_q})$$

For any $x_n \in \overline{K}^*/q^{\mathbb{Z}} = E_q(\overline{K})$ and $\omega \in H^0(\mathcal{E}_q, \Omega_{\mathcal{E}_q})$. The pullback gives us the Fontaine integration as the composition

$$\varphi_{E_q}^{\log} : \mathcal{B}_{E_q} \to \mathrm{Lie}(E_q) \otimes V_p(\Omega^{\log}) \to \mathrm{Lie}(E_q) \otimes V_p(\Omega)$$

Chapter 6

P-adic conjugate uniformization of abelian variety with semi-stable reduction

In this section, we provide a condition so that the Galois invariant $T_p(A)^{I_K}$ is trivial. It follows that we can obtain Iovita-Morrow-Zaharesu's p-adic uniformization of a certain class of abelian varieties. Our approach is based on Raynaud's p-adic uniformization theory [6]. The main ingredient of our argument is Coleman-Iovita's monodromy theorem.

6.1 Some preliminaries

6.1.1 p-adic Hodge theory

In [12], Fontaine constructed a ring $\mathbf{B}_{\mathrm{dR}}^{+}$, the de Rham period ring, which contains all periods of p-adic abelian varieties. It is a complete discrete valuation ring with residue field $\mathbb{C}_p$. There is a uniformizer of $\mathbf{B}_{\mathrm{dR}}^{+}$, denoted by t, that is a p-adic analog of $2\pi i$. The Galois group G_K acts on $\mathbf{B}_{\mathrm{dR}}^{+}$ via the cyclotomic character: $\sigma(t) = \chi(\sigma)t$ for $\sigma \in G_K$. Furthermore, $\mathbf{B}_{\mathrm{dR}}^{+}$ is equipped with a descending filtration by the powers of t

$$\mathbf{B}_{\mathrm{dR}}^{+} \supset F^n\mathbf{B}_{\mathrm{dR}}^{+} := (t^n)$$

Let $\mathbf{B}_{\mathrm{dR}} := \mathbf{B}_{\mathrm{dR}}^{+}[1/t]$. Fontaine also defined other refined period rings (see [34]):

$$\mathbf{B}_{\mathrm{cris}} \subset \mathbf{B}_{\mathrm{st}} \subset \mathbf{B}_{\mathrm{dR}}$$

where $\mathbf{B}_{\mathrm{cris}}$ is the crystalline period ring and $\mathbf{B}_{\mathrm{st}}$ is the semi-stable period ring. They have the following properties:

1. The ring $\mathbf{B}_{\mathrm{cris}}$ is equipped with G_K-action and Frobenius operator φ.
2. The ring $\mathbf{B}_{\mathrm{st}}$ moreover has an action of a $\mathbf{B}_{\mathrm{cris}}$-linear monodromy operator N that commutes with the action of G_K and satisfies $N\varphi = p\varphi N$.

Definition 6.1.1. *For a p-adic representation V, let $\mathbf{D}_{dR}(V) := \left(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}$, $\mathbf{D}_{\mathrm{cris}}(V) := \left(\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}$ and $\mathbf{D}_{\mathrm{st}}(V) := \left(\mathbf{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}$.*

A p-adic representation V is said to be de Rham if $\dim_K \mathbf{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$. We have that $\mathbf{D}_{\mathrm{dR}}(V)$ is a K-vector space with a decreasing filtration

$$\mathbf{D}_{\mathrm{dR}}(V)^i = \left(F^i\mathbf{B}_{\mathrm{dR}}^{+} \otimes V\right)^{G_K} \quad (i \in \mathbb{Z})$$

A p-adic representation V is semi-stable (resp. crystalline) if $\dim_{K_0} \mathbf{D}_{\mathrm{st}}(V) = \dim_{\mathbb{Q}_p} V$ (resp. $\dim_{K_0} \mathbf{D}_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$) where K_0 is the maximal unramified extension of $\mathbb{Q}_p$ contained in K. On $\mathbf{D}_{\mathrm{cris}}(V)$ (resp. $\mathbf{D}_{\mathrm{st}}(V)$) we have the corresponding Frobenius operator φ (resp. Fobenius operator φ and monodromy operator N) induced from $\mathbf{B}_{\mathrm{cris}}$ (resp. $\mathbf{B}_{\mathrm{st}}$). These maps commute with the action of G_K and satisfy $N\varphi = p\varphi N$.

Example 6.1.2 (Fontaine, Coleman-Iovita [9], Breuil)**.** *Let A be an abelian variety over K. Then $V_p(A)$ is de Rham. Moreover, A has good reduction (resp. semi-stable reduction) if and only if $V_p(A)$ is crystalline (resp. semi-stable).*

We have that semi-stable implies crystalline implies de Rham.

Proposition 6.1.3. *A representation V is crystalline if and only if V is semi-stable and $N = 0$ on $\mathbf{D}_{\mathrm{st}}(V)$.*

We recall some definitions and properties from [30].

Definition 6.1.4. *Define*

$$\begin{cases} H_e^1(K,V) = \operatorname{Ker}\left(H^1(K,V) \to H^1\left(K, \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \otimes V\right)\right) \\ H_f^1(K,V) = \operatorname{Ker}\left(H^1(K,V) \to H^1\left(K, \mathbf{B}_{\mathrm{cris}} \otimes V\right)\right) \\ H_g^1(K,V) = \operatorname{Ker}\left(H^1(K,V) \to H^1\left(K, \mathbf{B}_{\mathrm{dR}} \otimes V\right)\right). \end{cases}$$

Proposition 6.1.5 ([30],Corollary 3.8.4)**.** *We have*

$$\dim_{\mathbf{Q}_p} H^1_f(K,V) = \dim_{\mathbf{Q}_p}\left(\mathbf{D}_{dR}(V)/\mathbf{D}_{dR}(V)^0\right) + \dim_{\mathbf{Q}_p} H^0(K,V)$$

and

$$H^1_f(K,V)/H^1_e(K,V) \cong \mathbf{D}_{\mathrm{cris}}(V)/(1-\varphi)\mathbf{D}_{\mathrm{cris}}(V).$$

There is a fundamental exact sequence

$$0 \to \mathbf{Q}_p \to \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^{+} \to 0$$

Tensoring with V and taking the invariants under the action of G_K we get the Bloch-Kato exponential map to be the connecting homomorphism

$$\exp_V : \mathbf{D}_{\mathrm{dR}}(V)/\mathbf{D}_{\mathrm{dR}}(V)^0 \to H^1_e(K,V)$$

It is surjective with kernel $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}/H^0(K,V)$.

Proposition 6.1.6. *Let A be an abelian variety over K. We have*

$$V_p(A)^{G_K} = 0$$

Proof. By the crystalline Weil conjecture [35], 1 is not the eigenvalue of the Frobenius on $\mathbf{D}_{\mathrm{cris}}(V_p(A))$. Thus, $\exp_V$ is an isomorphism and

$$\mathbf{D}_{\mathrm{dR}}(V_p(A))/\mathbf{D}_{\mathrm{dR}}(V_p(A))^0 \cong H^1_e(K,V_p(A)) \cong H^1_f(K,V_p(A))$$

From this, we obtain $\dim_{\mathbf{Q}_p} H^0(K,V_p(A)) = 0$. □

6.1.2 Coleman-Iovita monodromy theorem

To start, we revisit the Raynaud uniformization of semi-stable abelian varieties, with a more in-depth discussion provided in Appendix B. For any semi-stable abelian variety A over K, its Néron model $\mathcal{N}$ is a smooth group scheme over $\mathcal{O}_K$. We denote by $\mathcal{N}_k$ its special fiber. We consider the identity component $\mathcal{N}^\circ$ of $\mathcal{N}$, the identity component $\mathcal{N}_k^\circ$ fits into a canonical short exact sequence of algebraic groups over k

$$0 \to T_k \to \mathcal{N}_k^\circ \to B_k \to 0 \tag{6.1}$$

where T_k is a torus and B_k is an abelian variety. We take $\tilde{A}$ to be the formal completion of $\mathcal{N}^0$ along its special fibre $\mathcal{N}^0_k$. Then the tori T_k lifts uniquely to a formal torus $\tilde{T} \subset \tilde{A}$. We set B to be the quotient of $\tilde{A}$ by $\tilde{T}$.

Assume further that $\tilde{T} = (\tilde{\mathbb{G}}_m)^d$ is split, we enlarged the formal Raynaud extension to an extension of B by the full affine torus $T = (\mathbb{G}_m)^d$ associated to $\tilde{T}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{T} & \longrightarrow & \tilde{A} & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The formal abelian scheme B is in fact algebraizable, so is the second row. We refer to both rows of the diagram as the Raynaud extension associated to the abelian variety A, the rigid analytic group G serves as a uniformizing space for A. The closed immersion $\tilde{T} \to \tilde{A}$ extends uniquely to a rigid analytic group morphism $p_T : T \to A$ and, thus, the open immersion $\tilde{A} \to A$ extends uniquely to a rigid analytic group morphism $p : G \to A$. The kernel of p is a lattice in G whose rank equals the dimension of the torus part T of G. Denote by Γ this lattice, then the analytic group morphism $G/\Gamma \to A$ induced by p is an isomorphism.

Proposition 6.1.7. *We have*

$$V_p(G)^{G_K} = 0$$

Proof. It is known that $V_p(T)$, $V_p(B)$, $V_p(G)$ are crystalline G_K-representations. Hence, we obtain the following exact sequence:

$$0 \to \mathbf{D}_{\text{cris}}(V_p(T)) \to \mathbf{D}_{\text{cris}}(V_p(G)) \to \mathbf{D}_{\text{cris}}(V_p(B)) \to 0$$

Denote by $\varphi_T, \varphi_G, \varphi_B$ the linearized Frobenius on $\mathbf{D}_{\text{cris}}(V_p(T), \mathbf{D}_{\text{cris}}(V_p(G), \mathbf{D}_{\text{cris}}(V_p(B)$, respectively. We prove $V_p(G)^{G_K} = 0$ by using Frobenius eigenvalues. If $q = |k| = p^f$ the cardinality of the residue field, then we have $\varphi_T(x) = \frac{1}{q}x \quad \forall x \in \mathbf{D}_{\text{cris}}(V_p(T))$. On $\mathbf{D}_{\text{cris}}(V_p(B)$, by crystalline Weil conjecture [35] the eigenvalues of φ_B have complex absolute value $q^{-\frac{1}{2}}$. It follows that the eigenvalues of φ_G have complex absolute values $\frac{1}{q}$ or $\frac{1}{\sqrt{q}}$. Now let $0 \neq \alpha \in V_p(G)^{G_K}$ so that $\alpha \otimes 1 \in \mathbf{D}_{\text{cris}}(V_p(G))$, then $\varphi_G(\alpha \otimes 1) = \alpha \otimes 1$, i.e 1 is an eigenvalue of φ_G, a contradiction. $\square$

Let $V_p(T)$, $V_p(G)$, $V_p(B)$, $V_p(A)$ be the corresponding Tate modules of T, G, B, A, respectively. We have the following exact sequence

$$0 \to \mathbf{D}_{\mathrm{st}}(V_p(G)) \to \mathbf{D}_{\mathrm{st}}(V_p(A)) \to \Gamma \otimes K_0 \to 0$$

Additionally, there exists an inclusion map

$$\mathbf{D}_{\mathrm{st}}(V_p(T)) \to \mathbf{D}_{\mathrm{st}}(V_p(G))$$

The monodromy operator $N : \mathbf{D}_{\mathrm{st}}(V_p(A)) \to \mathbf{D}_{\mathrm{st}}(V_p(A))$ can be factored through the following map

$$N' : \Gamma \otimes K_0 \to \mathbf{D}_{\mathrm{st}}(V_p(T)) = \mathbf{D}_{\mathrm{cris}}(V_p(T))$$

Let $a \in \mathbf{D}_{\mathrm{st}}(V_p(A))$, Coleman and Iovita provide an explicit definition for $N(a) \in \mathbf{D}_{\mathrm{st}}(V_p(T))$ [9, Part II, Section4]. Hence, they define the Fontaine monodromy operator

$$N : \mathbf{D}_{\mathrm{st}}(V_p(A)) \to \mathbf{D}_{\mathrm{st}}(V_p(A))$$

The following is Coleman-Iovita's monodromy theorem (see [9, Corollary 4.6]).

Theorem 6.1.8. *The rank the monodromy operator N is* $\mathrm{rank}_{K_0} N = \dim T = \mathrm{rank}_{\mathbb{Z}} \Gamma$

In this paper, we will use the following corollary.

Corollary 6.1.9. *Let N be the monodromy operator as above. Then we have*

$$\ker N = \mathbf{D}_{\mathrm{cris}}(V_p(A)) = \mathbf{D}_{\mathrm{cris}}(V_p(G))$$

6.1.3 Basics on Galois cohomology

Let us recall some basic facts on Galois cohomology.

Lemma 6.1.10. *Let F be a field with discrete topology, and G be a group with finite dimensional F-representation M. Then $H^1(G, M)$ classifies G-extensions of F by M. More precisely, denote by $\rho : G \to \mathrm{GL}_F(M)$ the representation. Given a cohomology class $[c] \in H^1(G, M)$ represented by cocycle $g \mapsto c_g \in M$, then it* *defines a representation of G on $E_c := M \oplus F$*

$$g \mapsto \begin{pmatrix} \rho(g) & c_g \\ 0 & 1 \end{pmatrix}$$

If $(c_g)_{g\in G}$ and $(c''_g)_{g\in G}$ define the same cohomology class, the representations E_c and $E_{c''}$ are isomorphic.

Furthermore, by definition there exists an exact sequence

$$0 \to M \to E_c \to F \to 0$$

then $\delta(1) = [c]$ where δ is the connecting homomorphism

$$\delta : F^G = F \to H^1(G, M)$$

Proof. The cocycle condition gives us $c_g + \rho(g)c_h = c_{gh}$, hence

$$\begin{pmatrix} \rho(g) & c_g \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \rho(g) & c_h \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \rho(gh) & c_{gh} \\ 0 & 1 \end{pmatrix}$$

It follows that the cohomology class $[c]$ defines a representation E_c of G.

Assume that $(c_g)_{g\in G}$ and $(c'_g)_{g\in G}$ define the same cohomology class, then there exists $m \in M$ such that $c_g = c'_g + gm - m$ for all $g \in G$. It follows that

$$\begin{pmatrix} \rho(g) & c_g \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \rho(g) & c'_g \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

Now let $1 \in F$, we lift it to column vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in E_c$, the image $\delta(1)$ is defined by

$$g\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_g \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_g \in M$$

□

Lemma 6.1.11. *Given a non-split exact sequence of finite dimension $\mathbb{Q}_p$-representations of G_K*

$$0 \to M \to N \to \mathbb{Q}_p \to 0 \tag{6.2}$$

Assume further that $M^{I_K} = 0$, then it is non-split as I_K-modules.

Proof. We have the inflation-restriction sequence [36]

$$0 \to H^1(G_K/I_K, M^{I_K}) \to H^1(G_K, M) \to H^1(I_K, M)^{G_K/I_K} \to \\ \to H^2(G_K/H_K, M^{I_K}) \to H^2(G_K, M)$$

It follows that $H^1(G_K, M) \cong H^1(I_K, M)^{G_K/I_K} \hookrightarrow H^1(I_K, M)$, every non-split exact sequence of G_K-modules gives rise to a non-split one as I_K-modules. $\square$

The following lemma is well-known.

Lemma 6.1.12. *Denote the maximal unramified extension of K by L, we have*

$$H^1(I_K, \mathbb{Q}_p(1)) = \varprojlim_n \left(L^\times / L^{\times^{p^n}}\right) \otimes \mathbb{Q}_p$$

In particular, if T is a split torus of dimension r we obtain

$$H^1(I_K, V_p(T) = H^1(I_K, \mathbb{Q}_p(1)^r) = \varprojlim_n \left(L^\times / L^{\times^{p^n}}\right)^r \otimes \mathbb{Q}_p$$

Proof. Consider the following exact sequence

$$1 \to \mu_{p^n} \to \overline{K}^\times \xrightarrow{p^n} \overline{K}^\times \to 1$$

Note that $\mu_{p^n}^{I_K} = 1$ because the K^{ur} contains only order the prime-to-p roots of unity. By Hilber 90 theorem we have $H^1(I_K, \overline{K}^\times) = 1$. Taking I_K galois cohomology we obtain

$$1 \to (K^{ur})^\times \xrightarrow{p^n} (K^{ur})^\times \to H^1(I_K, \mu_{p^n}) \to 1$$

We obtain the usual Kummer isomorphism $L^\times / L^{\times^{p^n}}$. Taking the projective limit gives us the desired result. $\square$

6.2 Galois invariants of Tate modules

6.2.1 Examples

Example 6.2.1. *Let A be a Tate curve, then $G = T$ in the Raynaud uniformization. Its Tate module $V_p(A)$ is the extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ as I_K-modules, i.e.,* *there is an exact sequence of I_K-modules*

$$0 \to \mathbb{Q}_p(1) \to V_p(A) \to \mathbb{Q}_p \to 0 \tag{6.3}$$

It is known that (6.3) *is I_K-non split. Since $\mathbb{Q}_p$ is* 1*-dimensional representation of I_K, it is easy to check that* (6.3) *is also strongly non-split. Note that $\mathbb{Q}_p(1)^{I_K} = 0$ as the cyclotomic action is ramified, we thus have*

$$0 \to V_p(A)^{I_K} \to \mathbb{Q}_p^{I_K} = \mathbb{Q}_p \xrightarrow{\delta} H^1(I_K, \mathbb{Q}_p(1)) \to \cdots$$

Using Kummer theory we see that $H^1(I_K, \mathbb{Q}_p(1)) = \mathbb{Q}_p^2$ and the connecting map δ is injective. Thus $V_p(A)^{I_K} = 0$.

Remark 6.2.2. *More generally, the same method applies to abelian varieties with totally generated reduction (see [37]).*

In the case of an elliptic curve with good reduction, $V_p(A)^{I_K} \neq 0$ is equivalent to the fact that A has no complex multiplication (see [38, A.2.4] or [39]). Thanks to this, we can also provide a higher-dimensional example as follows.

Example 6.2.3. *Let X be a Tate curve over K and let E be an elliptic curve with good reduction having complex multiplication over K. Then $A := X \times E$ is a semi-stable abelian variety with Tate module $V_p(A) = V_p(X) \times V_p(E)$. It follows that $V_p(A)^{I_K} = V_p(E)^{I_K} \neq 0$.*

6.2.2 Proof of the main theorem

We can reduce the triviality of $V_p(A)^{I_K}$ to $V_p(G)^{I_K}$ as follows.

Proposition 6.2.4. *Let A be an Abelian variety over K with semi-stable reduction and let G be its corresponding semi-Abelian variety in Raynaud's uniformization. We have*

$$V_p(A)^{I_K} = 0 \textit{ if and only if } V_p(G)^{I_K} = 0.$$

Proof. The forward implication is trivial by the short exact sequence (3) from Chapter 0. We claim the converse. Firstly, note that every semi-stable representation as G_K-modules is also semi-stable as I_K-modules. Assume that there is an element $0 \neq x \in (V_p(A))^{I_K}$, set $W = \mathbb{Q}_p x \subseteq V_p(A)$. We notice that W is a crystalline subrepresentation of $V_p(A)$ and $W \cap V_p(G) = 0$. It follows that $W +$

$V_p(G) = W \oplus V_p(G)$ is also a crystalline subrepresentation of $V_p(A)$, (using the short exact sequence (3) to see that $V_p(G) \subset V_p(A)$). Hence $N_{|\mathbf{D}_{\mathrm{cris}}(W \oplus V_p(G))} = 0$ contradicts to the Coleman-Iovita theorem $\ker N = \mathbf{D}_{\mathrm{cris}}(V_p(G))$. $\square$

Corollary 6.2.5. *Let A be an Abelian variety over K with semi-stable reduction and let B be its corresponding Abelian variety with good reduction in Raynaud's uniformization. If $V_p(B)^{I_K} = 0$ then $V_p(A)^{I_K} = 0$. In particular, Abelian varieties with totally degenerate reduction satisfy the hypothesis.*

From now on, we will consider the case

$$V_p(B)^{I_K} \neq 0.$$

We can give a condition so that the Galois invariant $V_p(A)^{I_K}$ is trivial as follows.

Definition 6.2.6. *Let G be a group. Given an exact sequence of G-modules*

$$0 \to M \to N \to P \to 0.$$

We say the sequence is G strongly non-split if for any G-submodule P' of P, the following sequence is also non-split

$$0 \to M \to N' \to P' \to 0$$

where $N' := N \times_P P'$.

Remark 6.2.7. *Let V be a submodule of $V_p(B)$ stable under the action of I_K. Consider the following short exact sequence*

$$0 \to V_p(T) \to E \to V \to 0$$

where $E := V_p(G) \times_{V_p(B)} V$. We do not have this short sequence to be non-split in general.

Remark 6.2.8. *Let G be a group. Given the following short exact sequence of G-modules*

$$0 \to M \to N \to P \to 0.$$

Assume that the sequence is split and denote by $s : P \to N$ the splitting section, then we have $s(P^G) \subset N^G$. It follows that if $P^G \neq 0$ then $N^G \neq 0$.

Lemma 6.2.9. *1. If $V_p(B)^{I_K} \neq 0$ and $V_p(G)^{I_K} = 0$ then the short exact sequence* (2) *from Chapter 0*

$$0 \to V_p(T) \to V_p(G) \to V_p(B) \to 0$$

is non-split as I_K-modules.

2. If $\Gamma \neq 0$ then the short exact sequence of G_K-modules (3) *from Chapter 0*

$$0 \to V_p(G) \to V_p(A) \to \Gamma \otimes \mathbb{Q}_p \to 0.$$

is non-split as G_K-modules.

Proof. 1. It follows by the remark (6.2.8) above.

2. If $\Gamma \neq 0$, we see that $(\Gamma \otimes \mathbb{Q}_p)^{G_K} = \Gamma \otimes \mathbb{Q}_p \neq 0$ but $V_p(A)^{G_K} = 0$, hence the exact sequence is non-split as remark (6.2.8).

□

We will prove that if the exact sequence (2) is I_K-strongly non-slit then $V_p(G)^{I_K} = 0$. Firstly, we claim that in many cases the exact sequence (2) is non-split as I_K-modules.

Proposition 6.2.10. *If the short exact sequence* (3)

$$0 \to V_p(T) \to V_p(G) \to V_p(B) \to 0.$$

is non-split as G_K-modules then it is non-split as I_K-modules.

Proof. We apply the the inflation-restriction sequence [**Was97**] to the exact sequence of G_K-modules $0 \to V_p(T) \to V_p(G) \to V_p(B) \to 0$. We have

$$0 \to H^1(G_K/I_K, V_p(T)^{I_K}) \to H^1(G_K, V_p(T)) \to H^1(I_K, V_p(T))^{G_K/I_K} \to$$
$$\to H^2(G_K/I_K, V_p(T)^{I_K}) \to H^2(G_K, V_p(T)).$$

Note that $V_p(T)^{I_K} = 0$ since the p-adic cyclotomic action is ramified. It follows that $H^1(G_K, V_p(T)) \cong H^1(I_K, V_p(T))^{G_K/I_K}$. Moreover, we see that $H^1(I_K, V_p(T))^{G_K/I_K} \hookrightarrow H^1(I_K, V_p(T))$, i.e., every non-trivial exact sequence of G_K-modules gives rise to a non-split one as I_K-modules. □

We then have an important application.

Corollary 6.2.11. *If $T \neq 0$ and assume further that $0 \to V_p(T) \to V_p(G) \to V_p(B) \to 0$ is I_K-strongly non-split, then $V_p(G)^{I_K} = 0$.*

Proof. Suppose that there exists $0 \neq x \in V_p(B)^{I_K}$ (otherwise $V_p(G)^{I_K} = 0$ directly). Then $\mathbb{Q}_p x \subseteq V_p(B)$ is a trivial representation of I_K. We can choose a basis $\{x_1, \dots, x_i\}$ of $V_p(B)^{I_K}$. Define E_{x_i} to be the pull-back of a basis element x_i under the map $V_p(G) \to V_p(B)$. With this, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_p(T) & \longrightarrow & E_{x_i} & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V_p(T) & \longrightarrow & V_p(G) & \longrightarrow & V_p(B) & \longrightarrow & 0. \end{array}$$

The right vertical arrow takes $1 \to x_i$. The action of I_K on E_{x_i} is given by the cohomology class $[x_i] = \delta(1) \in H^1(I_K, V_p(T))$, where $\delta \ : \mathbb{Q}_p \to H^1(I_K, V_p(T))$ is the connecting homomorphism. Since the top arrow is non-split as I_K-modules, we obtain that $[x_i] \neq 0$. Taking cohomology, we obtain $E_{x_i}^{I_K} = 0$ for all i, and so is $V_p(G)^{I_K}$. □

Remark 6.2.12. *If we tensor the Tate module $V_p(T)$ with $\mathbb{C}_p$ we then $H^1(I_K, V_p(T) \otimes \mathbb{C}_p) = 0$ by Sen–Tate theory (see [24]). It follows that we obtain a unique split exact sequence of I_K-modules.*

$$0 \to V_p(T) \otimes \mathbb{C}_p \to V_p(G) \otimes \mathbb{C}_p \to V_p(B) \otimes \mathbb{C}_p \to 0.$$

The long exact sequence of Galois cohomology gives us $H^1(I_K, V_p(G) \otimes \mathbb{C}_p) = 0$. Hence the following exact sequence is also split.

$$0 \to V_p(G) \otimes \mathbb{C}_p \to V_p(A) \otimes \mathbb{C}_p \to \Gamma \otimes \mathbb{C}_p \to 0.$$

6.3 P-adic conjugate uniformization of abelian varieties with semi-stable reduction

Set $\Omega := \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ and $\Omega^{\log} := \Omega_{(\mathcal{O}_{\overline{K}}, P)/(\mathcal{O}_K, (1 \mapsto \pi)^a)}$ defined in chapter 3. Using Sen-Tate theory (cf. (3.2.2)), the short exact sequence of $\mathbb{C}_p$-representations of G_K

$$0 \to V_p(\Omega) \to V_p(\Omega^{\log}) \to \mathbb{C}_p \to 0$$

uniquely splits as G_K-representations. The construction of log Fontaine integration yields the following G_K-equivariant morphism

$$\varphi_{\log} : \mathcal{B}_A \to \mathrm{Lie}(A) \otimes_K V_p(\Omega^{\log})$$

The splitting $V_p(\Omega^{\log}) \to V_p(\Omega) \cong \mathbb{C}_p(1)$ yields

$$\begin{tikzcd}
\mathcal{B}_A \arrow[r, "\varphi_{\log}"] \arrow[dr, dotted] & \mathrm{Lie}\, A \otimes_K V_p\Omega^{\log} \arrow[d, dotted, bend left] \\
& \mathrm{Lie}\, A \otimes_K \mathbb{C}_p(1) \arrow[u]
\end{tikzcd}$$

We will also denote by $\varphi_{\log}$ the resulting map.

Restricting the map to $T_p(A)$ and tensoring with $\mathbb{C}_p$ we get the G_K-equivariant morphism

$$\varphi_{\log} \otimes_K \mathbb{C}_p : T_p(A) \otimes_K \mathbb{C}_p \to \mathrm{Lie}\, A \otimes_K \mathbb{C}_p(1)$$

By Hodge-Tate decomposition, the map is surjective and $\ker(\varphi_{\log} \otimes_K \mathbb{C}_p) = \mathbb{C}_p^g$ for some g. The first two main results of this section are about the kernel of log Fontaine integration by adapting the method in [11, Appendix].

Theorem 6.3.1. *Let A be an abelian variety with semi-stable reduction. Assume further that $T_p(A)^{I_K} = 0$, then the log Fontaine integration restricted to $T_p(A)$ is injective.*

Proof. Set $S := \ker(\varphi_{\log} \otimes_K \mathbb{Q}_p)$. As $V_p(A)$ is a semi-stable representation of G_K as well as I_K-representation, so is S. Recall that $S \otimes_K \mathbb{C}_p \subset \mathbb{C}_p^g$ for some g, we have the following commutative diagram with the first and last row exact

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] \arrow[d, "\subset"] & V_p(A) \arrow[r, "\varphi_{\log}"] \arrow[d, "\subset"] & \mathrm{Lie}(A) \otimes_K \mathbb{C}_p(1) \arrow[d, "="] & \\
0 \arrow[r] & S \otimes_K \mathbb{C}_p \arrow[r] \arrow[d, "\subset"] & V_p(A) \otimes_K \mathbb{C}_p \arrow[r, "\varphi_{\log}"] \arrow[d, "="] & \mathrm{Lie}(A) \otimes_K \mathbb{C}_p(1) \arrow[d, "="] & \\
0 \arrow[r] & (\mathbb{C}_p)^g \arrow[r] & V_p(A) \otimes_K \mathbb{C}_p \arrow[r, "\varphi_{\log}"] & \mathrm{Lie}(A) \otimes_K \mathbb{C}_p(1) \arrow[r] & 0
\end{tikzcd}$$

Then from Sen-Tate theory, Hodge-Tate weights of S is 0. It follows that the image of the Galois extension $\rho_S : I_K \to \mathrm{End}_{\mathbb{Q}_p}(S)$ is finite. This implies that S is unramified ([40] proposition 9.7). Therefore we obtain $S = S^{I_K} \subset V_p(A)^{I_K} = 0$. $\square$

Lemma 6.3.2. *Let $(u_n) \in \mathcal{B}_A$ be a periodic path in $\mathcal{B}_A$. Then there is a bijection between* $\mathrm{Per}(\mathcal{B}_A)$ *and prime-to-p torsion point of $A(\overline{K})$*

$$\begin{aligned} \mathrm{Per}(\mathcal{B}_A) &\to A(\overline{K}) \\ (u_n)_{n\geq 0} &\mapsto u_0 \end{aligned}$$

Proof. Let u and u' be periodic paths such that $u_0 = u'_0$. It follows that $u - u' \in T_pA$. We have $\varphi_{\log}(u - u') = 0$ by periodity, hence $u - u' = 0$ by the injectivity of $\varphi_{\log}$ on $T_p(A)$.

Now, take $u_0 \in A(\overline{K})[m]$ with $(m,p) = 1$. It follows from the Euler theorem that $m \mid p^{\psi(m)} - 1$ where ψ is the Euler totient function. Therefore we have $[p^{\psi(m)} - 1]u_0 = 0$. Set $u = (u_0, [p^{\psi(m)-1}]u_0, ..., [p]u_0, u_0, ...)$, then $u \in \mathrm{Per}(\mathcal{B}_A)$. ☐

Lemma 6.3.3. *Let $u \in \mathcal{B}_A$ such that $u_0 \in A(L)$, for L/K a finite extension in $\overline{K}$. If $\varphi_{\log}(u) = 0$, then $u \in (\mathcal{B}_A)^{G_L}$.*

Proof. Since $u_0 \in A(L) = A(\overline{K})^{G_L}$, then $g \cdot u - u \in T_p(A)$ for all $g \in G_L$. Moreover we have $\varphi_{\log}(g \cdot u) = g\varphi_{\log}(u) = 0$, so $\varphi_{\log}(g \cdot u - u) = 0$. By the injectivity of $\varphi_{\log}$ on T_pA we have $g \cdot u = u$. ☐

Theorem 6.3.4. *We have*

$$\mathrm{Per}(\mathcal{B}_A) = \ker(\varphi_{\log}) \tag{6.4}$$

Proof. The inclusion $\mathrm{Per}(\mathcal{B}_A) \subseteq \ker(\varphi_{\log})$ since $V_p(\Omega)$ is torsion-free. Indeed, let $u \in \mathrm{Per}(\mathcal{B}_A)$, then $[p^k]u = u$. It follows that $p^k\varphi_{\log}(u) = \varphi_{\log}(u)$ implies $\varphi_{\log}(u) = 0$. We claim that $\ker(\varphi_{\log}) \subseteq \mathrm{Per}(\mathcal{B}_A)$. Now consider three cases:

1. Let $u \in \ker(\varphi_{\log})$ such that $u_0 \in A(\overline{K})[n]$ for $n \geq 1$ with $(n,p) = 1$. Then by the lemma (6.3.2) we have $u \in \mathrm{Per}(\mathcal{B}_A)$.

2. Let $u \in \ker(\varphi_{\log})$ with $u_0 \in A(\overline{K})$ such that u_0 is a torsion point of A of order a power of p, say $u_0 \in A(\overline{K})[p^a]$ for some $a \in \mathbb{Z}_{>0}$. Then $p^a u \in T_p(A)$ and $\varphi_{\log}(p^a u) = p^a\varphi_{\log}(u) = 0$. As $\varphi_{\log\,|T_pA}$ is injecitve, we obtain that $p^a u = 0$ in $\mathcal{B}_A$ which is p-torsion free, i,e., $\mathcal{B}_A[p] = 0$, we get $u = 0$.

3. Let $u \in \ker(\varphi_{\log})$ with $u_0 \in \mathcal{N}(\mathcal{O})$ not a torsion point of $\mathcal{N}(\mathcal{O})$. Denote $\alpha : \mathcal{B}_A \twoheadrightarrow A(\overline{K})$.

Note that $\alpha \otimes_K 1_{\mathbb{Q}}(u \otimes_K 1_{\mathbb{Q}_p}) = u_0 \otimes_K 1 \neq 0$. We have a natural exact sequence of $\mathbb{Q}_p$-vector spaces, with continuous G_K-action

$$0 \to T_p(A) \otimes_K \mathbb{Q}_p := V_p(A) \to \mathcal{B}_A \otimes_{\mathbb{Z}} \mathbb{Q} \to A(\overline{K}) \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$$

Assume that $u_0 \in A(L)$ for some finite extension L of K in $\overline{K}$. Consider the fiber product of G_L-modules

$$\begin{array}{ccc} \mathcal{B}_{\mathcal{N},L} & \xrightarrow{\alpha_L} & A(L) \\ \downarrow & & \downarrow \\ \mathcal{B}_A & \xrightarrow{\alpha} & A(\overline{K}) \end{array}$$

We get that $\mathcal{B}_{A,L} \subset \mathcal{B}_A$. On the other hand we have $\mathcal{B}_A^{G_L} \subset \mathcal{B}_{A,L}$. Hence $(\mathcal{B}_{A,L} \otimes_K \mathbb{Q})^{G_L} = (\mathcal{B}_A \otimes_{\mathbb{Z}} \mathbb{Q})^{G_L}$.

Consider the exact sequence of finite $\mathbb{Q}_p$-vector spaces with continuous G_L-action

$$0 \to V_p(A) \to \mathcal{B}_{A,L} \otimes_{\mathbb{Z}} \mathbb{Q} \to A(L) \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$$

Taking cohomology yields that

$$(\mathcal{B}_A \otimes_{\mathbb{Z}} \mathbb{Q})^{G_L} \to (A(L) \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\partial} H^1(G_L, V_p(A))$$

By Bloch-Kato [30, Example 3.11] we have that this Kummer map ∂ is injective, so $\alpha_L \otimes 1\left((\mathcal{B}_A \otimes_{\mathbb{Z}} \mathbb{Q})^{G_L}\right) = 0$. Therefore u is not G_L-invariant. It contradicts with the lemma (6.3.3).

□

We consider the semi-stable class of algebraic elements [11, Definition 3.6]. For later use, we will consider the following convention: T be a free $\mathbb{Z}_p$-module of rank $m \geq 1$ with a continuous G_F-action, V a free $\mathcal{O}_F$-module of rank $s \geq 1$, where G_F acts trivially on V, and an injective $\mathbb{Z}_p$-linear, G_F-equivariant map $\varphi\colon T \hookrightarrow V \otimes_{\mathcal{O}_F} \mathbb{C}_p(n)$, with $n \in \mathbb{Z}$, $n \neq 0$. We remark that the existence of φ implies $T^{G_K} = 0$.

Definition 6.3.5. *Let $x \in (V \otimes_{\mathcal{O}_K} \mathbb{C}_p(n))/\varphi(T)$ be an algebraic element. Recall that $\alpha : \mathcal{B}_A \to A(\overline{K})$ is the map in the fundamental exact sequence* (4) *in Chapter 0. We say that x is semi-stable if one of the following happens*

1. *x is a torsion element, or*

2. *x is non-torsion, and $\alpha^{-1}(x\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a semi-stable G_L-representation, for every L with $[L:F]$ finite such that $x \in (V \otimes_{\mathcal{O}_K} \mathbb{C}_p(n))/\varphi(T))^{G_L}$.*

We have

$$\begin{tikzcd}
 & & \mathrm{Per}(\mathcal{B}_A) \arrow[r, equal] \arrow[d] & A_{p'-tor}(\overline{K}) \arrow[d] & \\
0 \arrow[r] & T_p(A) \arrow[r] \arrow[d, equal] & \mathcal{B}_A \arrow[r] \arrow[d] & A(\overline{K}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_p(A) \arrow[r] & \mathrm{Lie}(A) \otimes_K \mathbb{C}_p(1) \arrow[r] & \left(\mathrm{Lie}(A) \otimes_K \mathbb{C}_p(1)\right)/\varphi_{\log}(T_p(A)) \arrow[r] & 0
\end{tikzcd}$$

Therefore we have $\iota_A : A(\overline{K})/A_{p'-tor}(\overline{K}) \hookrightarrow \left(\mathrm{Lie}(A) \otimes_K \mathbb{C}_p(1)\right)/\varphi_{\log}(T_p(A))$.

Theorem 6.3.6. *We have that an element $x \in \mathrm{Lie}(A) \otimes_K \mathbb{C}_p(1)/\varphi_{\log}(T_p(A))$ belongs to the image of ι_A if and only if x is semi-stable.*

Proof. Firstly, we have

$$\mathrm{Lie}(A) \otimes_K \mathbb{C}_p(1)/\varphi_{\log}(T_pA)[p^\infty] = \iota_A(A(\overline{K})[p^\infty]$$

Hence every torsion point of $\mathrm{Lie}(A) \otimes_K \mathbb{C}_p(1)/\varphi_{\log}(T_pA)$ is algebraic and thus is semi-stable.

Now we consider non-torsion point x. Set $\mathcal{B}_A^{(p)}(\overline{K}) = \mathcal{B}_A/\mathrm{Per}(\mathcal{B}_A)$. Using the same argument as in [11, Theorem 6.3] we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & V_p(A) \arrow[r] \arrow[d, "="] & \mathcal{B}_A^{(p)}(\overline{K}) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r] \arrow[d] & A^{(p)}(\overline{K}) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[d] & 0 \\
0 \arrow[r] & (\varphi_{\log} \otimes 1)(V_p(A)) \arrow[r] & \mathrm{Lie}(A) \otimes_K \mathbb{C}_p(1) \arrow[r] & \dfrac{\mathrm{Lie}(A) \otimes_K \mathbb{C}_p(1)}{(\varphi_{\log} \otimes 1)(V_p(A))} \arrow[r] & 0 \\
0 \arrow[r] & (\varphi_{\log} \otimes 1)(V_p(A)) \arrow[r] \arrow[u] & \alpha^{-1}(x\mathbb{Q}_p) \arrow[r] \arrow[u] & x\mathbb{Q}_p \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Taking G_K-continuous cohomology yields that

$$\begin{tikzcd}
0 \arrow[r] & A^{(p)}(K) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r, "\delta"] \arrow[d, "\iota_A"] & H^1(K, V_p(A)) \arrow[r] \arrow[d, "="] & \cdots \\
0 \arrow[r] & \left(\dfrac{\mathrm{Lie}(A) \otimes_K \mathbb{C}_p(1)}{V_p(A)}\right)^{G_K} \arrow[r, "="] & H^1(K, V_p(A)) \arrow[r] & 0 \\
0 \arrow[r] & x\mathbb{Q}_p \arrow[r] \arrow[u, hook] & H^1(K, V_p(A)) \arrow[r] \arrow[u, "="'] & \cdots
\end{tikzcd}$$

By Bloch-Kato theory (see [6, Example 3.11 and Proposition 5.4]) the image of δ is $H^1_{st}(L, V_p(A)) := \ker\left(H^1(L, V_p(A)) \to H^1(L, B_{st} \otimes_{\mathbb{Z}} V_p(A))\right)$. Therefore an element $x \in \mathrm{Lie}(A) \otimes_K \mathbb{C}_p(1)/\varphi_{\log}(T_p(A))$ belongs to the image of ι_A if and only if x is semi-stable. $\square$

Appendix A

Abelian varieties

This appendix will provide foundation knowledge of abelian varieties used in the thesis.

A.1 Basic facts

We shall recall the basic definitions of abelian varieties, the main references are [1] and [41].

Definition A.1.1. *A group is a triple* (G, m, i, e) *where* G *is a set together with maps* $m : G \times G \to G$*,* $i : G \to G$*, and a distiguished element* $e \in G$ *sucht that:*

1. $m \circ (m \times \mathrm{Id}_G) = m \circ (\mathrm{Id}_G \times m) : G \times G \times G \to G$
2. $m \circ (e \times \mathrm{Id}_G) = \{e\} \times G \xrightarrow{\sim} G$

   $m \circ (\mathrm{Id}_G \times e) : G \times \{e\} \xrightarrow{\sim} G$
3. *Let* $\pi : G \to \{e\}$ *and* $\Delta_G : G \times G \to G$ *the diagonal map, we obtain* $e \circ \pi = m \circ (\mathrm{Id}_G \times i) \circ \Delta_G = m \circ (i \times \mathrm{Id}_G) \circ \Delta_G$

Let K be a field, we mean a variety over K is a separated K-scheme of finite type which is geometrically integral.

Definition A.1.2. *A group variety over* K *is a* K*-variety* X *together with* K*-morphisms* $m : X \times X \times X \to X$*,* $i : X \to X$*, and a* K*-rational point* $e \in X(K)$ *such that these morphisms satisfy group axioms*

1. $m \circ (m \times \mathrm{Id}) = m \circ (\mathrm{Id} \times m) : X \times X \times X \to X$
2. $m \circ (e \times \mathrm{Id}) = \mathrm{Spec}(K) \times X \xrightarrow{\sim} X$ *and* $m \circ (\mathrm{Id} \times e) : X \times \mathrm{Spec}(K) \xrightarrow{\sim} X$

3. *Let $\pi : X \to \mathrm{Spec}(K)$ the structure morphism and $\Delta_{X/K} : X \times_K X \to X$ the diagonal morphism, we obtain*

$$e \circ \pi = m \circ (\mathrm{Id} \times i) \circ \Delta_{X/K} = m \circ (i \times \mathrm{Id}) \circ \Delta_{X/K}$$

Remark A.1.3. *We can generalize the notion above to group schemes (see [1, Section III]).*

Proposition A.1.4. *([1, Page 95]) Let X be a group variety over a field K. Then X is smooth over K.*

If X is a group variety then the set $X(K)$ of K-rational points naturally inherits the structure of a group.

Remark A.1.5. *Let G be a variety over a field K. Then giving the structure of an K-group variety on G is equivalent to giving a group structure on the sets $G(T) := \mathrm{Hom}_K(T, G)$, functorial for T in the category of K-schemes.*

Definition A.1.6. *An abelian variety is a group variety which, as a variety, is proper.*

The first crucial concept of this section is the notion of Lie algebra associated to an abelian variety which turns out to be (as a set) the tangent space of the abelian variety at the identity.

Definition A.1.7. *Let X be a scheme. For any point $x \in X$, we define the Zariski tangent space T_x to X at x to be the dual of $K(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$.*

Let X be a group variety and Ω_X the sheaf of Kahler differentials on X over K. By a vector field D on X we mean an element in $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$. D is said to be left invariant if the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{D} & \mathcal{O}_X \\ \downarrow{m^*} & & \downarrow{m^*} \\ \mathcal{O}_{X\times X} & \xrightarrow{1\otimes D} & \mathcal{O}_{X\times X} \end{array}$$

where the vertical maps being the pull-back morphisms induced by the multiplication morphism $m : X \times X \to X$. We denote the set of left-invariant vector fields by $\mathrm{Lie}\, X$.

Proposition A.1.8. *([1, Page 94]) Let X be a group variety over a field K and let $T_{X,e}$ be the tangent space at the identity element. The map*

$$\begin{aligned} \operatorname{Lie} X &\to T_{X,e} \\ D &\mapsto D(f) \mod m_e^2 \end{aligned}$$

is an isomorphism.

Given any two vector fields D_1, D_2 on X, considering them as endomorphism of $\mathcal{O}_X$ we see that

$$[D_1, D_2] := D_1 D_2 - D_2 D_1$$

is again a vector field on X. Moreover, if D_1 and D_2 are left-invariant vector fields, then is $[D_1, D_2]$.

Definition A.1.9. *The Lie algebra of a group scheme is the K-vector space of left-invariant vector fields together with the operation $[\cdot, \cdot]$.*

The following proposition called rigidity lemma is important in the theory of abelian variety.

Proposition A.1.10. *Let X, Y and Z be algebraic varieties over a field K. Suppose that X is proper.*

If $f : X \times Y \to Z$ is a morphism with the property that, for some $y \in Y(k)$, the fibre $X \times \{y\}$ is mapped to a point $z \in Z(k)$ then f factors through the projection $\mathrm{pr}_Y : X \times Y \to Y$.

Proof. We may assume that $K = \overline{k}$. Choose a point $x_0 \in X(k)$, and define a morphism

$$\begin{aligned} g : Y &\to Z \\ y &\mapsto f(x_0, y) \end{aligned}$$

Claim: $f = g \circ \mathrm{pr}_Y$. As $X \times Y$ is reduced, it suffices to prove this on K-rational points.

Let $U \subset Z$ be an affine open neighbourhood of z. Since X is complete, the projection $\mathrm{pr}_Y : X \times Y \to Y$ is a closed map (definition). It follows that $V := \mathrm{pr}_Y(f^{-1}(Z - U))$ is closed in Y. By construction, if $P \notin V$ then $f(X \times \{P\}) \subset U$. $\square$

The rigidity lemma shows that up to a translation, every morphism is a homomorphism.

Definition A.1.11. *Let X be a group variety over a field K, and let $x \in X(K)$ be a K-rational point. We define the right translation $t_x : X \to X$*

$$t_x = (X \simeq X \times_K \operatorname{Spec}(K) \overset{\mathrm{Id}_X \times x}{\longrightarrow} X \times_K X \overset{m}{\to} X)$$

the left translation $t_x : X \to X$

$$t_x = (X \simeq \operatorname{Spec}(K) \times_K X \overset{x \times \mathrm{Id}_X}{\longrightarrow} X \times_K X \overset{m}{\to} X$$

Proposition A.1.12. *([1, Page 41]) Let X and Y be abelian varieties and let $f : X \to Y$ be a morphism. Then f us the composition $f = t_{f(e_X)} \circ h$ of a homomorphism $h : X \to Y$ and a translation $t_{f(e_X)}$ over $f(e_X)$ on Y.*

Corollary A.1.13.

1. *If X is a variety over a field K and $e \in X(k)$ then there is at most one structure of an abelian variety on X for which e is the identity element.*

2. *If X is an abelian variety then the group structure on X is commutative.*

A.2 Isogeny

The notion of isogeny which plays an essential role in the theory of abelian variety is given by the following conditions (see [41, Section 5]).

Theorem A.2.1. *Let $f : X \to Y$ be a homomorphism of abelian varieties. Then the following are equivalent*

1. *f is surjective and* $\dim(X) = \dim(Y)$

2. $\ker(f)$ *is a finite group scheme and* $\dim(X) = \dim(Y)$

3. *f is finite, flat and surjective morphism.*

Definition A.2.2. *A homomorphism $f : X \to Y$ of abelian varieties is called an isogeny if f satisfies the three equivalent conditions above.*

Lemma A.2.3. *Let $f : W \to X$ and $h : Y \to Z$ be isogenies of abelian varieties over K. If $g_1, g_2 : X \to Y$ are homomorphisms such that $h \circ g_1 \circ f = h \circ g_2 \circ f$ then $g_1 = g_2$*

We consider several properties of isogeny (see [41, Section 5.1]).

Theorem A.2.4. *Let $f : X \to Y$ be an isogeny. The following are equivalent*

1. *The funtion field $K(X)$ is a separable field extension of $K(Y)$.*
2. *f is an étale morphism.*
3. *$\ker(f)$ is an étale group scheme.*

Definition A.2.5. *An isogeny $f : X \to Y$ is called separable if it satisfies the three equivalent conditions above.*

Theorem A.2.6. *Let $f : X \to Y$ be an isogeny. The following are equivalent*

1. *The funtion field $K(X)$ is a purely inseparable field extension of $K(Y)$.*
2. *f is a purely inseparable morphism.*
3. *$\ker(f)$ is an connected group scheme.*

Definition A.2.7. *An isogeny $f : X \to Y$ is called purely inseparable if it satisfies the three equivalent conditions above.*

Proposition A.2.8. *Every isogeny $f : X \to Y$ can be factorized as $f = h \circ g$, where $g : X \to Z$ is an inseparable isogeny and $h : Z \to Y$ is a separable isogeny.*

This factorization is unique up to isomorphism, in the sense that if $f = h' \circ g' : X \to Z' \to Y$ is a second such factorization then there is an isomorphism $\alpha : Z \xrightarrow{\sim} Z'$ with $g' = \alpha \circ g$ and $h = h' \circ \alpha$

The most important type of isogeny is the following.

Proposition A.2.9. *([41, Proposition 5.9]) For $n \neq 0$, the morphism $[n]_X$ is an isogeny. If $g = \dim(X)$, we have $\deg([n]_X) = n^{2g}$. If $(\mathrm{char}(K), n) = 1$ then $[n]_X$ is separable. Denote by $X[n] := \ker([n]_X)$ kernel of the multiplication $[n]_X : X \to X$.*

Corollary A.2.10.

1. *If X is an abelian variety over an algebraically closed field K then $X(K)$ is a divisible group. That is, for every $P \in X(K)$ and $n \in \mathbb{Z} \setminus \{0\}$ there exists a point $Q \in X(K)$ such that $n \cdot Q = P$*
2. *If $(\mathrm{char}(K), n) = 1$ then $X[n](K_s) = X[n](\overline{K}) = (\mathbb{Z}/n\mathbb{Z})^{2g}$*

Proposition A.2.11. *If $f : X \to Y$ is an isogeny of degree d then there exists an isogeny $g : Y \to X$ with $g \circ f = [d]_X$ and $f \circ g = [d]_Y$.*

The relation

$$X \sim_K Y := \text{there exists an isogeny } f : X \to Y$$

is an equivalence relation on the set of abelian varieties over K

Definition A.2.12. *Let A be an abelian variety over a field K of characteristic 0, and let p a prime number. Then we define the Tate-p-module, denoted by T_pA, to be the projective limit of the system $\{A[p^n](\overline{K})\}_{n\in\mathbb{Z}_{\geq 0}}$ with respect to the transition maps*

$$p : A[p^{n+1}](\overline{K}) \to A[p^n](\overline{K})$$

A.3 Neron model

Following [42, Section 1.2] we shall introduce the fundamental concept of the Néron model. In this part, we work over a base scheme $S = \operatorname{Spec} R$ where R is a Dedekind domain with the field of fractions K. For example, R can be $\mathbb{Z}$ or $\mathbb{Z}_p$. We say S is of the local case if it is the spectrum of a discrete valuation ring or a field; the general case will be referred to as the global case.

Definition A.3.1. *Let X_K be a K-scheme, we say an S-scheme $\mathcal{X}$ is an S-model of X_K if the generic fiber of $\mathcal{X}$ is isomorphic to X_K*

$$\begin{tikzcd}
X_K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\operatorname{Spec}(K) \arrow[r] & S = \operatorname{Spec}(R)
\end{tikzcd}$$

Definition A.3.2. *Let X_K be a smooth and separated K-scheme of finite type. A Néron model of X_K is an S-model $\mathcal{X}$ which is smooth, separated, and of finite type, and which satisfies the following universal property, called Néron mapping property:*

For each smooth S-scheme $\mathcal{Y}$ and each K-morphism $u_K : \mathcal{Y}_K \to \mathcal{X}_K$ there is a unique S-morphism $u : \mathcal{Y} \to \mathcal{X}$ extending u_K, i.e.,

$$\operatorname{Hom}_K(\mathcal{Y}_K, \mathcal{X}_K) = \operatorname{Hom}_S(\mathcal{Y}, \mathcal{X})$$

Proposition A.3.3. *([42, Proposition 2]) Let $\mathcal{X}$ be a Néron model of its generic fibre X_K. Then*

1. *$\mathcal{X}$ is uniquely determined by X_K, up to canonical isomorphism*

2. *The formation of Néron models commutes with etale base change.*

Theorem A.3.4. *([42, Proposition 8]) Let $\mathcal{A}$ be an abelian scheme over S. Then $\mathcal{A}$ is a Néron model of its generic fibre $\mathcal{A}_K$.*

In the local case, we have the existence of Néron model (see [42, Section 1.3]).

Theorem A.3.5. *Let A_K be an abelian variety over the fraction field K of a discrete valuation R. Then A_K admits a Néron model over R.*

We give a definition of good and bad reduction as follows (see [42, Section 7.4])

Definition A.3.6. *Let A_K be an abelian variety over the fraction field K of a discrete valuation R with residue field k. Let $\mathcal{A}_k := \mathcal{A} \otimes_R k$ the reduction of the Néron model $\mathcal{A}$. Then we say*

1. *A_K has good reduction if the identity component of $\mathcal{A}_k^0$ is an abelian variety; equivalently the Néron model $\mathcal{A}$ is an abelian scheme.*

2. *A_K has semistable reduction if the identity component of $\mathcal{A}_k^0$ is a semiabelian variety.*

The following is the main theorem in [42, Section 7.4].

Theorem A.3.7. *For each abelian variety A_K over the fraction field K of a discrete valuation ring, there exists a finite Galois extension L of K such that $A_L := A \times_K \operatorname{Spec} L$ has semistable reduction.*

Appendix B

Raynaud uniformization

This section will outline the theory of Raynaud uniformization following [6]. For the foundation of rigid geometries and up-to-date treatment of the Raynaud theory, we refer to a book of Lütkebohmert [43].

B.1 Introduction

Let us review the main steps of the proof. Consider an abelian variety A over K as a rigid analytic group which is proper over K because there is a correspondence between schemes of finite type over K and proper rigid analytic spaces over K.

Theorem B.1.1. *[6, Theorem 1.1] There exists a finite separable extension K' of K such that, if we extend the valuation from K to K', replace R by its integral closure in K', and apply the base change $K \to K'$ to A, then there is a unique open subgroup $\tilde{A} \subset A$, which has the structure of a formal group scheme over R and which is an extension of a formal abelian scheme B by a formal torus $\tilde{T}$*

$$0 \to \tilde{T} \to \tilde{A} \to B \to 0$$

We then have the Raynaud uniformization as follows.

Theorem B.1.2. *[6, Theorem 1.2]*

1. *The closed immersion $\tilde{T} \to \tilde{A}$ extends uniquely to a rigid analytic group morphism $p_T : T \to A$ and, thus, the open immersion $\tilde{A} \to A$ extends uniquely to a rigid analytic group morphism $p : E \to A$.*

2. *The kernel of p is a lattice in E whose rank equals the dimension of the torus*

part T of E. Denote by M this lattice, then the analytic group morphism $E/M \to A$ induced by p is an isomorphism.

3. $H^1(E,\mathbb{Z}) = 0$. *Furthermore, any rigid morphism $X \to A$ where $H^1(X,\mathbb{Z}) = 0$ factors through $p : E \to A$.*

B.2 Overview of formal and rigid geometry

Let $K/\mathbb{Q}_p$ be a finite extension, and denote by $\mathcal{O}_K$ its ring of integers. Let $k = \mathcal{O}_K/\mathfrak{m}$ the residue field. We will define categories and functors between them in the following diagram

$$\begin{tikzcd}
\{\text{Schemes over } \mathrm{Spec}(\mathcal{O}_K)\} \arrow[r, "\text{Generic fiber}"] \arrow[d, "\text{Formal completion}"] & \{\text{Schemes over } \mathrm{Spec}(K)\} \arrow[d, "\text{Analytification}"] \\
\{\text{Formal schemes over } \mathrm{Spf}(\mathcal{O}_K)\} \arrow[r, "\text{Rigid generic fiber}"] & \{\text{Rigid spaces over } \mathrm{Spm}(\mathcal{O}_K)\}
\end{tikzcd}$$

Remark B.2.1. *The diagram is not commutative.*

B.2.1 Formal geometry

Definition B.2.2. *Let R be a topological ring. We say R is an adic ring if the topology on R is I-adic for some finitely generated ideal I and R is I-adically complete and separated, that is*

$$R \cong \varprojlim_n R/I^n$$

We call I an ideal of definition.

Given an adic ring R with ideal of definition I, we define the underlying topological space

$$\mathrm{Spf}\, R := \mathrm{Spec}\, R/I$$

The topology of $\mathrm{Spf}\, R$ is generated by

$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}\, R : f \notin \mathfrak{p}, \mathfrak{p}\text{open in } R\}$$

for any $f \in R$. The structure sheaf on that basis is given by

$$\mathcal{O}_{\mathrm{Spf}\, R}(D(f)) := \varprojlim_n (R/I^n)_f = R\left\langle f^{-1}\right\rangle = \varprojlim_n (R/I^n)[T]/(fT-1)$$

Definition B.2.3. *A formal scheme is a locally topologically ring space* $(X, \mathcal{O}_X)$ *such that for any* $x \in X$*, there exists an open neighborhood* U *of* x *satisfying* $(U, \mathcal{O}_{X|U})$ *is isomorphic to* $(\operatorname{Spf} R, \mathcal{O}_{\operatorname{Spf} R})$ *for some adic ring* R*.*

Let X be a locally Noetherian scheme and let $\iota : Z \hookrightarrow X$ be a closed immersion. We associate Z with a sheaf of ideals $\mathcal{I}_Z$ in $\mathcal{O}_X$ as follows. By definition, for any affine open subset $U = \operatorname{Spec} R$ of X, there is an ideal $I \subset R$ such that $\iota^{-1}(U) = \operatorname{Spec} R/I$. Define the presheaf $\mathcal{I}$ by

$$\mathcal{I}(U) = I$$

It turns out that this presheaf is a sheaf. Then, we can define the formal completion $\mathcal{X}$ of X along Z to the formal scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}} = (Z, \varprojlim_n \mathcal{O}_X/\mathcal{I}^n)$.

B.2.2 Rigid analytic geometry

Definition B.2.4. *The* n*-variable Tate algebra over* K *is*

The Tate algebra $K\langle T_1, ..., T_n\rangle$ is equipped with the so-called Gauss norm

$$|| := \max\{|a_i| : i \in \mathbb{Z}_{n\geq 0}\}$$

It turns out the Tate algebra is a Banach K-algebra. Moreover, it is a UFD.

Definition B.2.5. *A topological* K*-algebra* R *is called an affinoid algebra over* K *if there exists* $n \in \mathbb{Z}_{n\geq 0}$ *and a surjective* K*-algebra homomorphism*

$$K\langle T_1, ..., T_n\rangle \to R$$

Given an affinoid algebra R, we define

$$\operatorname{Spm} R := \{\mathfrak{m} \subset R \text{maximal ideals}\}$$

For ant $x \in \operatorname{Spm} R$, we write $\mathfrak{m}_x$ for the corresponding maximal ideal in R. The residue field $k_X := R/\mathfrak{m}_x$ is a finite extension of K, hence we can define

$$|f(x)| := \text{the norm of the image of } f \text{ in } K_x$$

Similar as in the theory of schemes, we would like to define a class of 'distinguished open subsets' that generates the topology on $\operatorname{Spm} R$ as follows. For any $f \in R$

and any $\epsilon \in \mathbb{R}_{>0}$

$$D(f,\epsilon) := \{x \in \mathfrak{m}_x \in \operatorname{Spm} R : |f(x)| \le \epsilon\}$$

Definition B.2.6. *Let R be an affinoid algebra over K*

1. *An open subset $U \subset \operatorname{Spm} R$ is admissible if there exists a topological covering $U = \cup_{i\in I} U_i$ by affinoid subdomains such that for any morphisms of affinoid spaces*

$$f : \operatorname{Spm} S \to \operatorname{Spm} R$$

with image in U, the cover $f^{-1}(U_i)$ of $\operatorname{Spm} S$ admits a finite refinement by affinoid subdomains.

2. *The site of admissible opens $(\operatorname{Spm} R)_{adm}$ consists of admissible open subsets of $\operatorname{Spm} R$ with morphisms given by inclusions. A family of morphisms $\{U_i \hookrightarrow U\}_{i\in I}$ in $(\operatorname{Spm} R)_{adm}$ is a admissible cover if $U = \cup_{i\in I} U_i$ and for any morphism of affinoid spaces*

$$f : \operatorname{Spm} S \to \operatorname{Spm} R$$

with image in U, the cover $f^{-1}(U_i)$ of $\operatorname{Spm} S$ admits a finite refinement by affinoid subdomains.

Definition B.2.7. *A locally ringed G-space $(X, \mathcal{O}_X)$ is a rigid analytic space over K if X admits a covering $\{U_i\}_{i\in I}$ in X_{adm} such that, for each $I \in I$, $(U_i, \mathcal{O}_{X|U_i})$ is isomorphic to an affinoid space over K.*

Definition B.2.8. *Let X be an algebraic variety over K. A rigid analytification of X is a rigid analytic space X^{rig} together with a morphism of locally ringed $G-$spaces*

$$\rho : X^{rig} \to X$$

such that for any rigid analytic space Y over K together with a morphism of locally ringed G-spaces $f : Y \to X$, there exists a unique morphism of rigid analytic spaces $g : Y \to X^{rig}$ such that the diagram

$$
\begin{tikzcd}
X^{rig} \arrow[r] & X \arrow[dl] \\
Y \arrow[u] &
\end{tikzcd}
$$

is commutative.

The process of the rigid analytification is obtained as follows. Let $X = \operatorname{Spec} K[T_1, .., T_n]/\mathfrak{a}$ be an affine variety over K, we get a sequence of morphisms of K-algebras

$$K[T_1, .., T_n]/\mathfrak{a} \to \cdots \to K\left\langle p^m T_1, ..., p^m T_n\right\rangle/\mathfrak{a} \to \\ \to K\left\langle p^{m-1}T_1, ..., p^{m-1}T_n\right\rangle/\mathfrak{a} \to \cdots \to K\left\langle T_1, ..., T_n\right\rangle/\mathfrak{a}$$

which induces a sequence of inclusions on the corresponding sets of maximal ideals

$$\operatorname{Spm} K\left\langle T_1, ..., T_n\right\rangle/\mathfrak{a} \subset \cdots \subset \operatorname{Spm} K\left\langle p^{m-1}T_1, ..., p^{m-1}T_n\right\rangle/\mathfrak{a} \subset \\ \subset \operatorname{Spm} K\left\langle p^m T_1, ..., p^m T_n\right\rangle/\mathfrak{a} \subset \cdots \subset \operatorname{Spec}\operatorname{Max} K[T_1, .., T_n]/\mathfrak{a}$$

As sets, we see that

$$\operatorname{Spec}\operatorname{Max} K[T_1, .., T_n]/\mathfrak{a} = \cup + m \in \mathbb{Z}_{\geq 0} \operatorname{Spm} K\left\langle p^m T_1, ..., p^m T_n\right\rangle/\mathfrak{a}$$

We then define the rigid analytification of affine space $X = \operatorname{Spec} K[T_1, .., T_n]/\mathfrak{a}$ defined as

$$X^{\mathrm{rig}} := \cup_{m\in\mathbb{Z}_{\geq 0}} \operatorname{Spm} K\left\langle p^m T_1, ..., p^m T_n\right\rangle/\mathfrak{a}$$

The general situation can be done by glueing affine pieces.

Theorem B.2.9. *Let X be an algebraic variety over K. Then, the rigid analytification X^{rig} exists uniquely.*

B.2.3 Rigid generic fiber of admissible formal schemes

Definition B.2.10. *Let R be a topological $\mathcal{O}_K$-algebra*

1. *We say R is topologically of finite type if R is isomorphic to $\mathcal{O}_K\left\langle T_1, ..., T_n\right\rangle/\mathfrak{a}$ with respect to the p-adic topology*

2. *We say R is topologically of finite presentation if R is topologically of finite type and the ideal $\mathfrak{a}$ is finitely generated.*

3. *We say R is admissible if R is topologically of finite presentation and is flat over $\mathcal{O}_K$.*

Let $\mathcal{X} = \operatorname{Spf} R$ is an admissible affine formal scheme, then $R = \mathcal{O}_K \langle T_1, .., T_n \rangle / \mathfrak{a}$ for some finitely generated $\mathfrak{a}$ and R is flat over $\mathcal{O}_K$. We define

$$(\operatorname{Spf} R)^{\mathrm{rig}} := \operatorname{Spm} K \langle T_1, .., T_n \rangle / \mathfrak{a}$$

For a general admissible formal scheme $\mathcal{X}$ over $\mathcal{O}_K$ with an affine covering $\mathcal{X} = \cup_{i \in I} \operatorname{Spf} R_i$, we can define the rigid generic fibre of $\mathcal{X}$, denoted by $\mathcal{X}^{\mathrm{rig}}$, as the rigid analytic space over K, obtained by gluing $(\operatorname{Spf} R_i)^{\mathrm{rig}}$'s along $(\operatorname{Spf} R_i \cap \operatorname{Spf} R_j)^{\mathrm{rig}}$'s.

Theorem B.2.11. *[43, Theorem 3.3.3] The functor*

$$rig : \{\textit{Admissible Formal}\mathcal{O}_K - \textit{schemes}\} \to \{\textit{Rigid Analytic K-spaces}\}$$

gives rise to an equivalence of categories between

1. *the category of quasi-compact, quasi-separated admissible formal $\mathcal{O}_K$-schemes, localized by admissible formal blowing-ups, and*

2. *the category of rigid analytic K-spaces which are quasi-compact and quasi-separated.*

B.3 Raynaud uniformization

We now can ready explain the proof of the Raynaud uniformization.

Theorem B.3.1. *[6, Theorem 1.1] There exists a finite separable extension K' of K such that, if we extend the valuation from K to K', replace R by its integral closure in K', and apply the base change $K \to K'$ to A, then there is a unique open subgroup $\tilde{A} \subset A$, which has the structure of a formal group scheme over R and which is an extension of a formal abelian scheme B by a formal torus $\tilde{T}$*

$$0 \to \tilde{T} \to \tilde{A} \to B \to 0$$

Assume that the valuation of K is discrete, we can consider the identity component $\mathcal{A}^0$ of the Neron model of A, and take for $\tilde{A}$ the formal completion of $\mathcal{A}^0$ along its special fibre. The semi-stable reduction theorem of Grothendieck says that, after a finite separable extension of K, we may assume the special fibre of $\mathcal{A}^0$ to be an extension of an abelian variety over k by a torus over k

$$0 \to T_k \to \mathcal{A}_k^0 \to B_k \to 0$$

Then the tori T_k lifts uniquely to a formal torus $\tilde{T} \subset \tilde{A}$. We set B to be the quotient of $\tilde{A}$ by $\tilde{T}$.

Assume further that $\tilde{T} = (\tilde{\mathbb{G}}_m)^d$ is split, we enlarged the formal Raynaud extension to an extension of B by the full affine torus $T = (\mathbb{G}_m)^d$ associated to $\tilde{T}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{T} & \longrightarrow & \tilde{A} & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The formal abelian scheme B is in fact algebraizable, so is the second row. We refer to both rows of the diagram as the Raynaud extension associated to the abelian variety A, the rigid analytic group E serves as a uniformizing space for A.

Remark B.3.2. *The formal Raynaud extension is locally trivial over B if $\tilde{T}$ is split, i.e., there is a formal covering $\{U_i\}$ of B and sections $s_i : U_i \to \tilde{A}$ such that, composed with the group law on $\tilde{A}$, they give rise to open immmersions $U_i \times \tilde{T} \hookrightarrow \tilde{A}$ in the sense of formal rigid spaces.*

Note that the logarithm gives a group homomorphism

$$T(K) \cong (K^\times)^d \to \mathbb{R}^d$$

sending $(x_!, .., x_d)$ to $(-\log|x_1|, ..., -\log|x_d|)$. We can extend this map to $E(K)$ as follows. Locally, it is the map

$$T(K) \times V_i(K) \to T(K) \to \mathbb{R}^d$$

We can glue the local maps together to get a group homomorphism $\sigma : E(K) \to \mathbb{R}^d$ as desired. Note that $\ker \sigma = \tilde{A}$. We call a closed analytic subgroup $M \subset E$ a lattice if it is a constant group space whose K-valued points are mapped bijectively under σ onto a lattice in $\mathbb{R}^d$.

Theorem B.3.3. *[6, Theorem 1.2]*

1. *The closed immersion $\tilde{T} \to \tilde{A}$ extends uniquely to a rigid analytic group morphism $p_T : T \to A$ and, thus, the open immersion $\tilde{A} \to A$ extends uniquely to a rigid analytic group morphism $p : E \to A$.*

2. *The kernel of p is a lattice in E whose rank equals the dimension of the torus part T of E. Denote by M this lattice, then the analytic group morphism $E/M \to A$ induced by p is an isomorphism.*

3. $H^1(E, \mathbb{Z}) = 0$. *Furthermore, any rigid morphism $X \to A$ where $H^1(X, \mathbb{Z}) = 0$ factors through $p : E \to A$.*

Since the formal Raynaud extension is locally trivial, there exists a formal open covering $\{U_i\}$ of B such that we can use $\{U_i \times \tilde{T}\}$ as formal charts for $\tilde{A}$ and $\{U_i \times T\}$ as charts for E. Hence the morphisms $T \to A$ and $\tilde{A} \to A$ give rise to a morphism $p : E \to A$. The uniqueness of p follows from the uniqueness of $T \to A$.

Since $M = \ker p$ is a closed 0-dimensional analytic subset of E, $\sigma(M(K))$ is a lattice in $\mathbb{R}^d$ whose rank is bounded by d. Now consider d points $m_1, .., m_d \in T(K)$ generating a lattice of rank d in T. Their image in A extend to R-valued points of the Neron model $\mathcal{A}$ of A. Since $\mathcal{A}$ is of finite type, its special fibre consists of only finitely many connected components. Hence, we may raise $m_1, .., m_d$ to a suitable power and thereby assume that their images in A extend to $\mathcal{A}^0$ and then $\tilde{A} = \ker \sigma$. Then, we can obtain d elements in $\ker p$ by multiplying m_i with its inverse image in $\tilde{A}$, which generate a lattice of full rank in E. It follows that $p : E \to A$ induces an open immersion $E/\ker p \hookrightarrow A$ between proper rigid analytic spaces, which is an isomorphism since A is connected.

Appendix C

Comparision theorem

Let X be a smooth and projective algebraic variety over the rational numbers $\mathbb{Q}$. For any integer number $n \geq 0$, denote by $H_n\left(X(\mathbb{C}), \mathbb{C}\right)$ the singular homology with complex coefficients of the topological space $X(\mathbb{C})$ and $H^n_{dR}(X_{\mathbb{C}})$ the algebraic de Rham cohomology of $X_{\mathbb{C}}$ (cf. [44]):

$$H^n_{dR}(X_{\mathbb{C}}) := H^n(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}} \to \Omega^1_{X_{\mathbb{C}}} \to \Omega^2_{X_{\mathbb{C}}} \to \cdots)$$

The classical de Rham theorem states that there exists a nondegenerate pairing

$$\begin{aligned} H^n_{dR}(X_{\mathbb{C}}) \times H_n(X(\mathbb{C}), \mathbb{C}) &\to \mathbb{C} \\ (\omega, \gamma) &\mapsto \int_\gamma \omega \end{aligned}$$

The values obtained are often called periods. Equivalently, a period can be seen as an entry of the matrix (in rational bases) of the de Rham isomorphism

$$\mathbb{C} \otimes_{\mathbb{Q}} \mathrm{H}^r_n(X(\mathbb{C}), \mathbb{Q}) \cong \mathbb{C} \otimes_K \mathrm{H}^r_{dR}(X) \tag{C.1}$$

for an algebraic variety X defined over a number field K.

Example C.0.1. *Let γ be the unit circle in the complex plane. We have*

$$\int_\gamma \frac{dz}{z} = 2\pi i$$

To put this integral in our context, take $X = \mathbb{G}_{m,\mathbb{Q}} = \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\}$ the punctured affine space, $\gamma \in H_1(X(\mathbb{C}), \mathbb{C})$, $\omega = \dfrac{dz}{z} \in H^1_{dR}(X_{\mathbb{C}})$.

Now let us pass to the local setting, that is, X will be a variety over a finite extension K of $\mathbb{Q}_p$. The initial motivation of p-adic Hodge theory is the will to design a relevant p-adic analogue of the notion of periods. To this end, our first need is to find a suitable p-adic generalization of the de Rham isomorphism. In the p-adic setting, the singular cohomology is no longer relevant; it has to be replaced by the étale cohomology. Thus, what we need is a ring B allowing for a canonical isomorphism as G_K-modules:

$$B \otimes_{\mathbb{Q}_p} \mathrm{H}^r_{et}(X_{\overline{K}}, \mathbb{Q}_p) \cong B \otimes_K \mathrm{H}^r_{dR}(X) \tag{C.2}$$

when K is now a finite extension of $\mathbb{Q}_p$.

We remark that it was observed by Tate that B cannot be $\mathbb{C}_p$ because the latter does not contain a p-adic analog $(2\pi i)_p$ of $2\pi i$. Indeed, $(2\pi i)_p$ should be a period of $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ and under the Galois action, we must get $\sigma((2\pi i)_p) = \chi(\sigma)(2\pi i)_p$ for all $\sigma \in G_K$ where $\chi : G_K \to \mathbb{Z}_p^*$ is the cyclotomic character. On the other hands, we have

$$\{x \in \mathbb{C}_p : \sigma(x) = \chi(\sigma)x,\ \forall \sigma \in G_K\} = 0$$

which is a special case of a fundamental theorem of Tate (cf. [24]).

Theorem C.0.2. *(de Rham comparison theorem, Faltings, [45]). Let X be a proper and smooth variety over K. There exists a period of isomorphism*

$$\mathbf{B}_{dR} \otimes_{\mathbb{Q}_p} \mathrm{H}^r_{et}(X_{\overline{K}}, \mathbb{Q}_p) \cong \mathbf{B}_{dR} \otimes_K \mathrm{H}^r_{dR}(X)$$

compatible with the Galois action and filtration, where the Hodge filtration on de Rham cohomology is defined by

$$F^i H^n_{dR}(X) := \Im(H^n(X, \Omega^{\geq i}_X) \to H^n_{dR}(X)),\ i \geq 0$$

As corollary, we obtain the Hodge-Tate decomposition.

Theorem C.0.3. *We have a Galois equivariant Hodge-Tate decomposition*

$$H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i \geq 0} H^{n-i}(X, \Omega^i_{X/K}) \otimes_K \mathbb{C}_p(-i)$$

Fontaine (cf. [12, Section 3]) used his integration to prove the Hodge-Tate decomposition for abelian varieties as follows. Recall we have the Fontaine pairing

$$\begin{aligned} \langle , \rangle : p^r \cdot H^0(\mathcal{A}, \Omega_{\mathcal{A}}) \times \mathcal{A}(\mathcal{O}_{\overline{K}}) &\to \Omega \\ (\omega, u) &\mapsto u^*(\omega) \end{aligned}$$

which is O_K-linear with respect to ω and $\mathbb{Z}[G_K]$-linear with respect to u. It yields a O_K-linear homomorphism

$$p^r \cdot H^0(\mathcal{A}, \Omega_{\mathcal{A}}) \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(A(\overline{K}), \Omega)$$

On the other hand, since $A(\overline{K})$ is p-divisible, the functor $V_p(-)$ provides a morphism

$$\mathrm{Hom}_{\mathbb{Z}[G_K]}(A(\overline{K}), \Omega) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}[G_K]}(V_pA, V_p\Omega)$$

It induces a homomorphism of G_K-modules

$$\rho : H^0(A, \Omega_{A/K}) = K \otimes p^r \cdot H^0(\mathcal{A}, \Omega_{\mathcal{A}}) \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(V_pA, V_p\Omega) \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(T_pA, V_p\Omega)$$

where the latter map is given by the inclusion $T_pA \hookrightarrow V_pA$.

Theorem C.0.4. *([12, Theorem 2]) The morphism ρ is an isomorphism. It induces a canonical isomorphism of G_K-modules*

$$\rho' : H^1(A, \mathcal{O}_A) \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(T_pA, \mathbb{C}_p)$$

The comparison theorem yields that there exists a canonical G_K-equivariant isomorphism

$$H^1_{et}(A, \mathbb{Q}_p) \otimes \mathbb{C}_p \cong H^0(A, \Omega^1_{A/K}) \otimes_K \mathbb{C}(-1) \oplus H^1(A, \mathcal{A}) \otimes_K \mathbb{C}_p$$

Thus, we obtain a canonical G_K-equivariant isomorphism

$$T_pA \otimes \mathbb{C}_p \cong (\mathrm{Lie}\, A^\vee)^* \otimes_K \mathbb{C}_p) \oplus (\mathrm{Lie}\, A \otimes_K \mathbb{C}_p(1))$$

Furthermore, Fontaine defined other refined period rings (cf. [34]):

$$\mathbf{B}_{\mathrm{cris}} \subset \mathbf{B}_{\mathrm{st}} \subset \mathbf{B}_{\mathrm{dR}}$$

where $\mathbf{B}_{\mathrm{cris}}$ is the crystalline period ring and $\mathbf{B}_{\mathrm{st}}$ is the semistable period ring. They have the following properties

1. The ring $\mathbf{B}_{\text{cris}}$ is equipped with G_K-action and Frobenius operator φ
2. The ring $\mathbf{B}_{\text{st}}$ moreover has an action of a $\mathbf{B}_{\text{cris}}$-linear monodromy operator N that commutes with the action of G_K and satisfies $N\varphi = p\varphi N$.

The Fontaine rings characterize the reduction of algebraic varieties over p-adic field in the following form.

Theorem C.0.5. *(Semistable comparison theorem, Tsuji, [46]). Let X be a variety over K. There exists a period isomorphism*

$$\alpha_{st} : H^n_{HK}(X_{\overline{K}}) \otimes_{K^{ur}} \mathbf{B}_{st} \cong H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{st},\ n \geq 0$$

where $H^n_{HK}(X_{\overline{K}})$ is the Hyodo-Kato cohomology (cf. [47]). The isomorphism is compatible with the Galois action, Frobenius, monodromy, and with the de Rham period isomorphism, i.e., $\alpha_{st} \otimes \mathbf{B}_{dR} \cong \alpha_{dR}$.

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