

Critical Mass Phenomena and Blow-up Behaviors of Ground States in Stationary Second Order Mean-field Games Systems with Decreasing Cost

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Abstract

This paper is devoted to the study of Mean-field Games (MFG) systems in the mass critical exponent case. We firstly establish the optimal Gagliardo-Nirenberg type inequality associated with the potential-free MFG system. Then, under some mild assumptions on the potential function, we show that there exists a critical mass M^* such that the MFG system admits a least energy solution if and only if the total mass of population density M satisfies $M < M^*$. Moreover, the blow-up behavior of energy minimizers are captured as $M \nearrow M^*$. In particular, given the precise asymptotic expansions of the potential, we establish the refined blow-up behavior of ground states as $M \nearrow M^*$. While studying the existence of least energy solutions, we establish new local $W^{2,p}$ estimates of solutions to Hamilton-Jacobi equations with superlinear gradient terms.

KEYWORDS: Mean-field Games, Maximal Regularities, Variational Approach, Ground States, Constrained Minimization Problem, Blow-up Profiles

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1 Introduction

A very important topic in the fields of economics, finance and management is how participants in the market maximize their utilities and minimize costs. Whereas, in various economical and financial problems, researchers have to consider a huge number of players, which brings many challenges to numerical and theoretical studies since they are forced to tackle numerous coupled equations from the analytic perspective. More specifically, it is very difficult to determine the value function of each player one by one. In this situation, it is necessary to propose some simplified model that can be investigated qualitatively by using mathematical tools. Motivated by this, scholars borrowed ideas from statistical physics and developed the theory of Mean-field games (MFG) [28, 29, 34, 35].

1.1 Mean-field Games Theory and Systems

As mentioned above, MFG models proposed by Lasry et al. [35] and Huang et al. [32] independently in 2007, are well-used to describe complex decision processes involving a huge number of homogeneous agents, which are a class of backward-forward parabolic equations consisting of Hamilton-Jacobi equations and Fokker-Planck equations. In the setting described below, their mathematical form reads as

$$\begin{cases} u_t = -\Delta u + H(\nabla u) - V[x] - f(m), & x \in \mathbb{R}^n, t > 0, \\ m_t = \Delta m + \nabla \cdot (\nabla H(\nabla u)m), & x \in \mathbb{R}^n, t > 0, \\ u|_{t=T} = u_T, m|_{t=0} = m_0, & x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where m represents the population density and u denotes the value function of a typical player. Here m_0 is the initial data of density and u_T is the terminal data of value function u . In particular, $H : \mathbb{R}^n \rightarrow \mathbb{R}$ is the so-called Hamiltonian and $V(x)$ denotes a potential function.

To illustrate the MFG theory, we assume that for $i = 1, \dots, N$, the dynamics of the i -th participant is governed by the following controlled stochastic differential equation (SDE):

$$dX_t^i = -\gamma_t^i dt + \sqrt{2}dB_t^i, \quad X_0^i = x^i \in \mathbb{R}^n, \quad (1.2)$$

where x^i are the initial states, γ_t^i represent the control terms and B_t^i are independent Brownian motions. For simplification, we assume that all agents are indistinguishable and follow the same game process, then drop “ i ” in (1.2). The goal of each player is to minimize the following average cost :

$$J(\gamma_t) := \mathbb{E} \int_0^T [L(\gamma_t) + V(X_t) + f(m(X_t))]dt + u_T(X_T), \quad (1.3)$$

where L is the so-called Lagrangian, which is the Legendre transform of H satisfying $H(p) = \sup_{\gamma \in \mathbb{R}^n} (p\gamma - L(\gamma))$. Lasry et al. applied the dynamic programming principle to study (1.3) and formulated the time-dependent system (1.1).

One popular research topic in the study of (1.1) is the global well-posedness and long-time dynamics of (1.1), see [7, 9, 10, 15, 23–25]. Numerical techniques including the finite difference method are also useful on the analysis of solutions to the backward-forward system (1.1) and we refer the readers to [1, 2, 8, 11].

We next consider the stationary problem of (1.1), which serves as a paradigm to describe the distribution of Nash equilibria of infinite-horizon differential games among numerous players. The mathematical form of the stationary problem of (1.1) is

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f(m) + V(x), & x \in \mathbb{R}^n, \\ \Delta m + \nabla \cdot (m \nabla H(\nabla u)) = 0, & x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} m dx = M > 0, \end{cases} \quad (1.4)$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier. Here a triple (m, u, λ) is defined as the solution to (1.4) and constant $M > 0$ is the total mass of population. System (1.4) models in the market, the long time collective behavior of a huge number of homogeneous agents whose goals are to minimize a cost that depends on the population distribution m . One can understand the solution to (1.4) as follows: firstly, the optimal control of a typical agent is inputted into the Fokker-Planck equation in (1.4), yielding m . In the long-time regime, this has to coincide with the population density. Then, the optimal control is outputted via the HJB equation. In particular, the Nash equilibrium is attained when the input coincides with the output.

We would like to point out that Hamiltonian H is convex, and we assume that it has the following typical form:

$$H := C_H |p|^{r'}, \quad \exists r' > 1, C_H > 0. \quad (1.5)$$

By using the definition, the corresponding Lagrangian is given by

$$L = C_L |\gamma|^{r'}, \quad r = \frac{r'}{r' - 1} > 1, C_L = \frac{1}{r} (r' C_H)^{\frac{1}{1-r'}} > 0, \quad (1.6)$$

where r is the conjugate number of r' .

Focusing on the stationary problem (1.4), some works appeared in the last few years [12, 14, 22, 26, 39]. It is worthy mentioning that if the interaction between agents is assumed to be of congestion type, i.e. the cost is monotone increasing, then the uniqueness of the stationary solution can be proved [22, 26, 35]. However, if the cost is monotone decreasing, (1.4) may admit many solutions, and the analysis of existence becomes more challenging if the cost is also unbounded. Motivated by this, Cesaroni and the first author [12] employed the variational structure possessed by (1.4) and showed the existence and concentration behaviors of ground states via the direct method under some assumptions on f , V and H .

Before stating the technical conditions on the functions f , V and H , we discuss the strong relationship between (1.4) and nonlinear Schrödinger equations. In fact, whenever it is possible to trivialize the Fokker-Planck equation in (1.4) as

$$\nabla m + m C_H |\nabla u|^{r'-2} \nabla u = 0 \quad \text{a.e., } x \in \mathbb{R}^n, \quad (1.7)$$

then setting $v := m^{\frac{1}{r}}$, one finds from the u -equation in (1.4) that

$$\begin{cases} -\mu \Delta_r v + [f(v^r) + V(x) - \lambda] v^{r-1} = 0, & x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} v^r dx = M, \quad v > 0, \quad \mu = \left(\frac{r}{C_H}\right)^{r-1}, \end{cases} \quad (1.8)$$

where Δ_r is the r -Laplacian and given by $\Delta_r v = \nabla \cdot (|\nabla v|^{r-2} \nabla v)$. The v -equation is the famous Schrödinger equation with the mass constraint and there are rich literatures [30, 31] on the study of stationary problem (1.8) via the following useful variational structures:

$$\mathcal{F}(v) := \int_{\mathbb{R}^n} \left[\frac{\mu}{r} |\nabla v|^r + F(v) + \frac{1}{r} V(x) v^r \right] dx, \quad (1.9)$$

where $F(v)$ denotes the anti-derivative of $f(v^r) v^{r-1}$. We would like to mention that when $r = 2$ and $f(v^r) = -v^{r\alpha}$ in (1.8), the properties of the ground states to (1.8) associated with (1.9) are well-understood. As

shown in [33], it is well-known that under the subcritical Sobolev exponent case, the solution to (1.8) is radially symmetric, unique up to scaling and translation, has the exponential decay property if $\lambda < 0$, etc. It is necessary to point out that (1.7) is not expected to hold in general [13], but it is true for instance in the quadratic case $r' = 2$ or if one considers radial solutions. This strong connection when $r' = 2$ is seen also at the variational level: the ground states to (1.4) and (1.8) are the same under some general assumptions on $f(m)$ and V given in (1.4), see the detailed discussion in Appendix A.

Inspired by the discussion above, Cirant et al. [12] proved the existence of least energy solutions to (1.4) via the variational method with some technical assumptions imposed on H , V and f . In detail, Hamiltonian H is assumed to satisfy $H \in C^2(\mathbb{R}^n \setminus \{0\})$; moreover, there exist $C_H > 0$, $K > 0$ and $r' > 1$ such that

$$C_H|p|^{r'} - K \leq H \leq C_H|p|^{r'}, \quad (1.10)$$

and

$$\nabla H(p) \cdot p - H(p) \geq K^{-1}|p|^\gamma - K \text{ and } |\nabla H(p)| \leq K|p|^{\gamma-1}. \quad (1.11)$$

In addition, they suppose that the potential V is locally Hölder continuous and satisfy

$$C_V^{-1}(\max\{|x| - C_V, 0\})^b \leq V(x) \leq C_V(1 + |x|)^b, \quad \exists b, C_V > 0. \quad (1.12)$$

On the other hand, the local coupling $f : [0, +\infty) \rightarrow \mathbb{R}$ is imposed to be locally Lipschitz continuous, which satisfies

$$-C_f m^\alpha - K \leq f(m) \leq -C_f m^\alpha + K, \quad \exists C_f, K, \alpha > 0. \quad (1.13)$$

In particular, the author were concerned with the mass subcritical exponent case, which is

$$0 < \alpha < \frac{r}{n}. \quad (1.14)$$

To illustrate (1.14), we firstly state the following energy functional associated with (1.4):

$$\mathcal{E}(m, w) := \int_{\mathbb{R}^n} \left[mL\left(-\frac{w}{m}\right) + V(x)m + F(m) \right] dx, \quad (1.15)$$

where $F(m) := \int_0^m f(s)ds$ for $m \geq 0$ and $F(m) = 0$ for $m \leq 0$. As shown in [12], Lagrangian L is defined by

$$L\left(-\frac{w}{m}\right) := \begin{cases} \sup_{p \in \mathbb{R}^n} \left(-\frac{pw}{m} - H(p)\right), & m > 0, \\ 0, & (m, w) = (0, 0), \\ +\infty, & \text{otherwise,} \end{cases} \quad (1.16)$$

and $mL\left(-\frac{w}{m}\right)$ is the Legendre transform of $mH(p)$. In addition, the admissible set \mathcal{K}_M is defined as

$$\begin{aligned} \mathcal{K}_M := & \left\{ (m, w) \in (L^1(\mathbb{R}^n) \cap W^{1, \hat{q}}(\mathbb{R}^n)) \times L^1(\mathbb{R}^n) \right. \\ & \text{s. t. } \int_{\mathbb{R}^n} \nabla m \cdot \nabla \varphi dx = \int_{\mathbb{R}^n} w \cdot \nabla \varphi dx, \forall \varphi \in C_c^\infty(\mathbb{R}^n), \\ & \left. \int_{\mathbb{R}^n} V(x)m dx < +\infty, \int_{\mathbb{R}^n} m dx = M > 0, m \geq 0 \text{ a.e.} \right\}, \end{aligned} \quad (1.17)$$

where \hat{q} is given by

$$\hat{q} := \begin{cases} \frac{n}{n-r+1} & \text{if } r < n, \\ \in \left(\frac{2n}{n+2}, n\right) & \text{if } r = n, \\ r & \text{if } r > n. \end{cases} \quad (1.18)$$

With these preliminaries, to find the ground states of (1.4) associated with energy (1.15), we consider the following problem:

$$e_{\alpha, M} := \inf_{(m, w) \in \mathcal{K}_M} \mathcal{E}(m, w). \quad (1.19)$$

We would like to remark that $e_{\alpha, M} < +\infty$. To show this, one sets $(m_s, w_s) = \left(ce^{-|x|}, -\frac{xe^{-|x|}}{|x|} \right)$ with c determined by $\int_{\mathbb{R}^n} m dx = M$ and $w_s = \nabla m_s$, then can straightforward check $(m_s, w_s) \in \mathcal{K}_{\alpha, M}$ and $\mathcal{E}(m_s, w_s) < +\infty$. With the upper bound of $e_{\alpha, M}$, we claim that (1.14) is the necessary condition to guarantee that $\mathcal{E}(m, w)$ is bounded from below for all $M > 0$. In this case, $e_{\alpha, M}$ given in (1.19) is well-defined. Indeed, if $\alpha > \frac{r}{n}$, for any $(\bar{m}, \bar{w}) \in \mathcal{K}_{\alpha, M}$,

$$\mathcal{E}(\bar{m}_\delta, \bar{w}_\delta) \rightarrow -\infty,$$

as $\delta \rightarrow 0$, in which $(\bar{m}_\delta, \bar{w}_\delta) := (\delta^{-n} \bar{m}(\delta^{-1}x), \delta^{-(n+1)} \bar{w}(\delta^{-1}x)) \in \mathcal{K}_{\alpha, M}$ since $\delta^{-n} \int_{\mathbb{R}^n} \bar{m}(\delta^{-1}x) dx \equiv M$. It follows that $e_{\alpha, M}$ is not well-defined when $\alpha > \frac{r}{n}$. As a consequence, Cirant et al. [12] discussed the ground states to (1.4) associated with energy (1.15) under condition (1.14). In particular, the authors [12] assumed H, V and f satisfy (1.10), (1.11), (1.12) and (1.13), respectively. Moreover, they showed the ground states to (1.4) are in fact classical solutions by applying the regularization techniques and standard elliptic estimates. The concentration behaviors of the ground states were also established in the vanishing viscosity limit sense under the mass subcritical exponent case (1.14) in [12].

In this paper, we shall investigate the qualitative behaviors of ground states to (1.4) with $n \geq 2$ and focus on the *mass critical exponent case* $\alpha = \alpha^* := \frac{r}{n}$. More specifically, we aim to discuss the existence and asymptotic profiles of ground states to (1.15). This critical case is much more delicate than the subcritical one, and many arguments used in [12] break down. It is worthy mentioning that there exists the other critical exponent $\alpha = \frac{r}{n-r}$ arising from Sobolev embedding on analyzing the regularity of m -component to (1.4). Recently, some results in the mass supercritical case up to the Sobolev critical exponent were completed by the first author and his collaborators [38], in which the domain is assumed to be bounded and Neumann boundary conditions are imposed on (m, u) . More specifically, it is shown that energy (1.15) admits local minimizers with the mass supercritical exponent.

Now, we state our main results in the following subsection.

1.2 Main results

This paper extends the results in [12] to the mass critical case, and allows for much more general potentials V . In detail, we consider $\alpha = \alpha^* := \frac{r}{n}$, H is given by (1.5), $f(m) := -m^\alpha$ and locally Hölder continuous potential V satisfies some of the following conditions

$$(V1). \quad \inf_{x \in \mathbb{R}^n} V(x) = 0 \leq V(x) \in L_{\text{loc}}^\infty(\mathbb{R}^n).$$

$$(V2). \quad \text{there exist constants } C_1, C_2, K > 0 \text{ and } b, \delta > 0 \text{ such that}$$

$$C_1(1 + |x|^b) \leq V(x) \leq C_2 e^{\delta|x|}, \quad |x| \geq K; \quad (1.20a)$$

$$0 < C_1 \leq \frac{V(x+y)}{V(x)} \leq C_2, \quad \text{for all } |x| \geq K \text{ with } |y| < 2; \quad (1.20b)$$

$$\sup_{v \in [0,1]} V(vx) \leq C_2 V(x) \text{ for } |x| \geq K. \quad (1.20c)$$

$$(V3). \quad |\mathcal{Z}| = 0 \text{ with } \mathcal{Z} := \{x \in \mathbb{R}^n \mid V(x) = 0\}.$$

Then, we are able to classify the ground states to (1.4) by using the variational method and introducing some critical mass threshold M^* . A vital step here is to study the regularity of the solution after the minimizer is obtained. In fact, the regularity of solutions to the Fokker-Planck equation is well understood [6] and some

useful results are stated in Section 3. Whereas, focusing on the Hamilton-Jacobi equation, the quasi-linear structure may cause difficulties. There are some references on the regularity of the Hamilton-Jacobi equation that deserve discussion. Consider the following equation:

$$-\Delta u + |\nabla u|^{r'} = f, \quad (1.21)$$

where $r' > 1$. A natural question is the $W^{2,p}$ regularity of the solution to (1.21) provided that $f \in L^p$ for some $p > 1$. P.-L. Lions conjectured that for all $p > \frac{n}{r}$, the L^p estimate should hold for the D^2u of (1.21) when f is assumed to satisfy $f \in L^p$. This conjecture is confirmed when the periodic boundary condition is imposed on equation (1.21) [16]. For the case of $1 < r' \leq 2$, the global estimates were discussed in [5, 20, 21, 27] with Dirichlet or Neumann boundary conditions. There are a few results on the local L^p estimates for D^2u ; these have been obtained by the first author [17] and Verzini, in the superquadratic case $r' > 2$. However, the local L^p regularity in the case of $r' \leq 2$ is open to our knowledge. It turns out that in our analysis of the MFG system in the mass critical case, the local maximal regularity estimate plays a crucial role. Motivated by this, we obtain the following result:

Theorem 1.1. *Let $p > \frac{n}{r}$, $r' \geq \frac{n}{n-1}$, $f \in L^p(\Omega)$ and assume $u \in W^{2,p}(\Omega)$ solves*

$$-\Delta u + C_H |\nabla u|^{r'} = f \text{ in } \Omega,$$

in the strong sense. Then for each $M > 0$ and $\Omega' \subset\subset \Omega$, we have

$$\|\nabla u\|_{L^p(\Omega')} + \|D^2u\|_{L^p(\Omega')} \leq C,$$

where $\|f\|_{L^p(\Omega)} \leq M$ and the constant $C = C(M, \text{dist}(\Omega', \partial\Omega), n, p, C_H, r) > 0$.

Theorem 1.1 is an analogue of Calderón-Zygmund regularity for the linear second order elliptic equation. We would like to remark that the Lipschitz gradient estimate of the solution can be obtained [37] when $p > n$, see also [] for previous results. For the proof of Theorem 1.1, we shall perform a blow-up analysis which is inspired by the strategy shown in [17], but tailored to Morrey spaces rather than Hölder spaces.

Next, we come back to the MFG system, and focus on the existence of ground states under the critical mass exponent case. To begin with, we consider the following constrained minimization problem:

$$e_{\alpha^*, M} = \inf_{(m,w) \in \mathcal{K}_M} \mathcal{E}(m, w), \quad \alpha^* := \frac{r}{n}, \quad (1.22)$$

where the energy $\mathcal{E}(m, w)$ is given by

$$\mathcal{E}(m, w) = C_L \int_{\mathbb{R}^n} \left| \frac{w}{m} \right|^r m \, dx + \int_{\mathbb{R}^n} V(x) m \, dx - \frac{n}{n+r} \int_{\mathbb{R}^n} m^{1+\frac{r}{n}} \, dx, \quad (1.23)$$

and \mathcal{K}_M is given in (1.17). Before studying the existence of minimizers to (1.22), one must establish the optimal Gagliardo-Nirenberg type inequality subject to the mass critical exponent involving the Lagrangian term in energy (1.15). This is in fact crucial to understand whether the infimum in (1.22) is finite or not. More precisely, we focus on the following minimization problem:

$$\Gamma_\alpha := \inf_{(m,w) \in \mathcal{A}} \frac{\left(C_L \int_{\mathbb{R}^n} m \left| \frac{w}{m} \right|^r \, dx \right)^{\frac{n\alpha}{r}} \left(\int_{\mathbb{R}^n} m \, dx \right)^{\frac{(\alpha+1)r-n\alpha}{r}}}{\int_{\mathbb{R}^n} m^{\alpha+1} \, dx}, \quad \alpha \in \left(0, \frac{r}{n} \right], \quad (1.24)$$

where

$$\begin{aligned} \mathcal{A} := & \left\{ (m, w) \in (L^1(\mathbb{R}^n) \cap W^{1,\hat{q}}(\mathbb{R}^n)) \times L^1(\mathbb{R}^n) \right. \\ & \left. \text{s. t. } \int_{\mathbb{R}^n} \nabla m \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^n} w \cdot \nabla \varphi \, dx, \forall \varphi \in C_c^\infty(\mathbb{R}^n), \int_{\mathbb{R}^n} |x|^b m \, dx < +\infty, m \geq, \neq 0 \text{ a.e.} \right\}, \end{aligned} \quad (1.25)$$

with \hat{q} defined as (1.18) and $b > 0$. We would like to remark that problem (1.24) is scaling invariant under the scaling $(t^\beta m(tx), t^{\beta+1} w(tx))$ for any $t > 0$ and $\beta > 0$. As a consequence, it is easy to see that (1.24) is equivalent to the following minimization problem

$$\Gamma_\alpha = \inf_{(m,w) \in \mathcal{A}_M} \frac{\left(C_L \int_{\mathbb{R}^n} m \left| \frac{w}{m} \right|^r dx \right)^{\frac{n\alpha}{r}} \left(\int_{\mathbb{R}^n} m dx \right)^{\frac{(\alpha+1)r-n\alpha}{r}}}{\int_{\mathbb{R}^n} m^{\alpha+1} dx}, \quad \alpha \in \left(0, \frac{r}{n} \right], \quad (1.26)$$

where

$$\mathcal{A}_M := \left\{ (m, w) \in \mathcal{A}, \int_{\mathbb{R}^n} m dx = M > 0 \right\}. \quad (1.27)$$

We first prove there exist minimizers to (1.26) for any $\alpha \in (0, \frac{r}{n})$ thanks to Theorem 1.3 in [12]. Then, using an approximation argument, we show that Γ_α can be also attained when $\alpha = \frac{r}{n}$, provided that $M = M^*$ is suitably chosen. The result is summarized as

Theorem 1.2. *For any $r > 1$, assume that $\alpha = \alpha^* := \frac{r}{n}$ in (1.24), then we have Γ_{α^*} is finite and attained by some minimizer $(m_{\alpha^*}, w_{\alpha^*}) \in \mathcal{A}$. Correspondingly, there exists a solution $(m_{\alpha^*}, u_{\alpha^*}) \in W^{1,p}(\mathbb{R}^n) \times C^2(\mathbb{R}^n)$, $\forall p > 1$, to the following potential-free MFG systems:*

$$\begin{cases} -\Delta u + C_H |\nabla u|^{r'} - \frac{r}{nM^*} = -m^{\alpha^*}, & x \in \mathbb{R}^n, \\ \Delta m + C_{Hr'} \nabla \cdot (m |\nabla u|^{r'-2} \nabla u) = 0, & x \in \mathbb{R}^n, \\ w = -C_{Hr'} m |\nabla u|^{r'-2} \nabla u, \int_{\mathbb{R}^n} m dx = M^*, \end{cases} \quad (1.28)$$

where

$$M^* := [\Gamma_{\alpha^*}(\alpha^* + 1)]^{\frac{1}{\alpha^*}}. \quad (1.29)$$

In particular, there exists some positive constants c_1 and c_2 such that $0 < m_{\alpha^*}(x) \leq c_1 e^{-c_2|x|}$.

Note that working on the whole Euclidean space here involves non trivial compactness issues. These were solved in [12] by a concentration-compactness argument; here we follow a different strategy based on Pohozaev type identities. With the aid of Theorem 1.2, we analyze the boundedness of $e_{\alpha^*, M}$ from below and obtain the sharp existence of minimizers for (1.22), which is shown in the following theorem:

Theorem 1.3. *Assume that V satisfies (V1)-(V2) and M^* is defined by (1.29), then the following alternatives hold:*

- (i). *If $0 < M < M^*$, problem (1.22) admits at least one minimizer $(m_M, w_M) \in W^{1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \forall p > 1$, which satisfies for some $\lambda_M \in \mathbb{R}$,*

$$\begin{cases} -\Delta u_M + C_H |\nabla u_M|^{r'} + \lambda_M = V(x) - m_M^{\frac{r}{n}}, \\ \Delta m_M + C_{Hr'} \nabla \cdot (m_M |\nabla u_M|^{r'-2} \nabla u_M) = 0, \\ w_M = -C_{Hr'} m_M |\nabla u_M|^{r'-2} \nabla u_M, \int_{\mathbb{R}^n} m_M dx = M < M^*. \end{cases} \quad (1.30)$$

- (ii). *If $M > M^*$, there is no minimizer of problem (1.22).*

- (iii). *If $M = M^*$, when $1 < r \leq n$ and potential V satisfies also (V3), or when $r > n$, then problem (1.22) has no minimizer.*

It is worthy mentioning that while discussing the case of $M < M^*$ in Theorem 1.3, we need to apply the maximal regularity shown in Theorem 1.1 to obtain the uniformly boundedness of m -component in L^∞ and the detailed discussion is shown in Section 5. Theorem 1.3 implies that in the mass critical exponent case, the existence of minimizers to (1.22) depends on the total mass of population density m and there exists a mass threshold M^* explicitly given by (1.29). In particular, when $M = M^*$, the constrained problem (1.22) is not attained.

To further understand the delicate case $M = M^*$, we explore the blow-up behaviors of minimizers as $M \nearrow M^*$ and obtain that

Theorem 1.4. Assume that $V(x)$ satisfies either (V1) – (V3) for $1 < r \leq n$, or (V1) – (V2) for $r \geq n$. Let (m_M, w_M) be the minimizer of $e_{\alpha^*, M}$ obtained in Theorem 1.3 with $0 < M < M^*$ and $\mathcal{Z} := \{x \in \mathbb{R}^n \mid V(x) = 0\}$ with $|\mathcal{Z}| = 0$. Then, we have

(i).

$$\varepsilon_M = \varepsilon := \left(C_L \int_{\mathbb{R}^n} \left| \frac{w_M}{m_M} \right|^r m_M dx \right)^{-\frac{1}{r}} \rightarrow 0 \text{ as } M \nearrow M^*. \quad (1.31)$$

(ii). Let $\{x_\varepsilon\}$ be one of the global minimum points of u_M , then $\text{dist}(x_\varepsilon, \mathcal{Z}) \rightarrow 0$ as $M \nearrow M^*$, where $\mathcal{Z} = \{x \in \mathbb{R}^n \mid V(x) = 0\}$. Moreover,

$$u_\varepsilon := \varepsilon^{\frac{2-r'}{r-1}} u_M(\varepsilon x + x_\varepsilon), \quad m_\varepsilon := \varepsilon^n m_M(\varepsilon x + x_\varepsilon), \quad w_\varepsilon := \varepsilon^{n+1} w_M(\varepsilon x + x_\varepsilon), \quad (1.32)$$

satisfies up to a subsequence,

$$u_\varepsilon \rightarrow u_0 \text{ in } C_{\text{loc}}^2(\mathbb{R}^n), \quad m_\varepsilon \rightarrow m_0 \text{ in } L^p(\mathbb{R}^n) \forall p \in [1, \hat{q}^*), \quad \text{and } w_\varepsilon \rightarrow w_0 \text{ in } L^{\hat{q}}(\mathbb{R}^n), \quad (1.33)$$

where (m_0, w_0) is a minimizer of (1.24), and (u_0, m_0, w_0) satisfies (1.28). In particular, when V satisfies (1.12), let \bar{x}_ε be any one of global maximum points of m_M , then

$$\limsup_{\varepsilon \rightarrow 0} \frac{|\bar{x}_\varepsilon - x_\varepsilon|}{\varepsilon} < +\infty. \quad (1.34)$$

With mild assumptions imposed on potential V , Theorem 1.4 exhibits the basic blow-up behaviors of ground states as $M \nearrow M^*$. The reader may observe that when $r \leq n$ we need stronger information on V . Moreover, by imposing some more specific local asymptotics on the potential $V(x)$, we can get the refined blow-up behaviors of ground states, which is

Theorem 1.5. Suppose all conditions shown in Theorem 1.4 hold. Assume that V has $l \in \mathbb{N}_+$ distinct zeros given by $\{P_1, \dots, P_l\}$ and $\exists a_i > 0, q_i > 0$ and $d > 0$ such that

$$V(x) = a_i |x - P_i|^{q_i} + O(|x - P_i|^{q_i+1}), \quad 0 < |x - P_i| \leq d, \quad i = 1, \dots, l.$$

Denote

$$Z := \{P_i \mid q_i = q, i = 1, \dots, l\} \text{ and } Z_0 := \{P_i \mid q_i \in Z \text{ and } \mu_i = \mu, i = 1, \dots, l\},$$

where $q := \max\{q_1, \dots, q_l\}$ and $\mu := \min\{\mu_i \mid P_i \in Z, i = 1, \dots, l\}$ with

$$\mu_i := \min_{y \in \mathbb{R}^n} H_i(y), \quad H_i(y) := \int_{\mathbb{R}^n} a_i |x + y|^{q_i} m_0(x) dx \quad i = 1, \dots, l.$$

Let $(m_\varepsilon, w_\varepsilon, u_\varepsilon)$ be the convergent subsequence, (m_0, w_0, u_0) be the corresponding limit and x_ε be the global minimum point of each u_M chosen in Theorem 1.4. Then $x_\varepsilon \rightarrow P_i \in Z_0$. Moreover, as $M \nearrow M^*$,

$$\frac{e_{\alpha^*, M}}{\frac{q+r}{q} \left(\frac{q\mu}{r} \right)^{\frac{r}{r+1}} \left[1 - \left(\frac{M}{M^*} \right)^{\frac{r}{n}} \right]^{\frac{q}{r+q}}} \rightarrow 1,$$

and

$$\frac{\varepsilon}{\left[\frac{r}{q\mu} \left[1 - \left(\frac{M}{M^*} \right)^{\frac{r}{n}} \right] \right]^{\frac{1}{r+q}}} \rightarrow 1, \text{ as } M \nearrow M^*, \quad (1.35)$$

where $e_{\alpha^*, M}$ and $\varepsilon = \varepsilon_M$ are defined in (1.22) and (1.31), respectively. In addition, up to a subsequence,

$$\frac{x_\varepsilon - P_i}{\varepsilon_M} \rightarrow y_0 \text{ with } P_i \in Z_0 \text{ and } H_i(y_0) = \inf_{y \in \mathbb{R}^n} H_i(y) = \mu. \quad (1.36)$$

Theorem 1.5 captures the precise blow-up behaviors of ground states arising from problem (1.22), in which the flattest minima of V are selected.

The rest of this article is organized as follows: In Section 2, we establish maximal regularities of Hamilton-Jacobi equations with subquadratic gradient terms. Section 3 is devoted to some preliminaries for the proof of the existence of ground states. In Section 4, we formulate the optimal Gagliardo-Nirenberg type inequality, which is Theorem 1.2. Then in Section 5, we applied this essential inequality to derive the critical mass phenomenon shown in Theorem 1.3 under the mass critical exponent case. Finally, in Section 6, we investigate the blow-up behaviors of ground states obtained in Theorem 1.3 and finish the proof of Theorem 1.4 and 1.5. Without of confusion, we define constant $C > 0$ is generic, which may change line to line.

2 Local Maximal Regularity of Hamilton-Jacobi Equations

In this section, we focus on the following elliptic PDEs involving gradient terms:

$$-\Delta u + C_H |\nabla u|^{r'} = f \text{ in } \Omega, \quad (2.1)$$

where $r' > 1$, $C_H > 0$ are constants and $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Our goal is to study the $W^{2,p}$ regularities of solutions to (2.1) if we assume that $f \in L^p(\Omega)$ with $p > \frac{n}{r'}$. To begin with, we give some preliminary notations and results.

2.1 Preliminaries

The Morrey space $L^{r',s}(\mathbb{R}^n)$ with $r' \in [1, \infty)$ and $s \in [0, n]$ is defined as a functional space consisting of all measurable functions $u : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying

$$\|u\|_{L^{r',s}(\mathbb{R}^n)}^{r'} := \sup_{R>0; x \in \mathbb{R}^n} R^s \int_{B_R(x)} |u|^{r'} dy < +\infty, \quad (2.2)$$

where

$$\int_{B_R(x)} |u|^{r'} dy := \frac{1}{|B_R(x)|} \int_{B_R(x)} |u|^{r'} dy.$$

We have facts that when $s = n$, the Morrey space $L^{r',n}(\mathbb{R}^n)$ coincides with the Lebesgue space $L^{r'}(\mathbb{R}^n)$ and the Morrey space $L^{r',0}(\mathbb{R}^n)$ coincides with $L^\infty(\mathbb{R}^n)$. Correspondingly, if Ω is a bounded domain, we define $u \in L^{r',s}(\Omega)$ as the space of $u : \Omega \rightarrow \mathbb{R}$ satisfying

$$\|u\|_{L^{r',s}(\Omega)}^{r'} := \sup_{0 < R < \text{diam} \Omega; x \in \Omega} R^s \int_{B_R(x) \cap \Omega} |u|^{r'} dy < +\infty.$$

With the definition of Morrey spaces, we recall the following improved Gagliardo-Nirenberg inequality involving Morrey's norm in the whole space \mathbb{R}^n :

Lemma 2.1. *Let $1 \leq p < n$, $1 \leq r' < p^* := \frac{np}{n-p}$, and $q = r'(\frac{n}{p} - 1)$. Then there exists a constant $C > 0$ depending on n and p such that for any $\frac{p}{p^*} \leq \theta < 1$,*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}^\theta \|u\|_{L^{r',q}(\mathbb{R}^n)}^{1-\theta} \text{ for all } u \in W^{1,p}(\mathbb{R}^n).$$

Proof. See the proof of Theorem 2 in [41]. □

With the aid of Lemma 2.1, we next formulate the Gagliardo-Nirenberg's inequality involving Morrey norms in the ball B_R . To this end, we recall the following extension Theorem and interpolation inequalities.

Lemma 2.2 (C.f. Lemma 1.5 in [36]). *Let Ω be a Lipschitz bounded domain and assume $f \in W^{1,p}(\Omega)$ with some $1 \leq p \leq +\infty$. Then there exists a bounded linear extension $T : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^n)$ such that*

$$\|Tf\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|f\|_{W^{1,p}(\Omega)}, \quad \forall f \in W^{1,p}(\Omega),$$

where C is a constant depending on n , p and Ω .

Lemma 2.3 (C.f. Lemma 2.1 in [36]). *Let Ω be a bounded Lipschitz domain and $u \in W^{1,s}(\Omega)$, $1 \leq s < +\infty$. Then for any $\epsilon > 0$, $p \geq 1$ with*

$$\frac{n}{p} > \frac{n}{s} - 1, \quad q > 1,$$

there exists a constant C depending on n , q , r , ϵ , Ω such that

$$\|u\|_{L^p(\Omega)} \leq \epsilon\|\nabla u\|_{L^s(\Omega)} + C\|u\|_{L^q(\Omega)}.$$

Now, we are ready to establish the improved Gagliardo-Nirenberg inequality in the bounded domain, which is

Lemma 2.4. *Let p, q, r and θ satisfy the assumptions of Lemma 2.1. Assume $u \in W^{1,p}(B_R) \cap L^{r',q}(B_R)$ with $R > 0$, then there exists a constant $C > 0$ depending on n and p such that,*

$$\|u\|_{L^{p^*}(B_R)} \leq C\|\nabla u\|_{L^p(B_R)}^\theta \|u\|_{L^{r',q}(B_R)}^{1-\theta} + C\|u\|_{L^{r',q}(B_R)}. \quad (2.3)$$

Proof. We consider the case of $R = 1$. By using Lemma 2.2, we extend u to $Tu \in W^{1,p}(\mathbb{R}^n) \cap L^{r',q}(\mathbb{R}^n)$ for $u \in W^{1,p}(B_1)$. Then we invoke Lemma 2.1 to get

$$\begin{aligned} \|u\|_{L^{p^*}(B_1)} &\leq \|Tu\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla Tu\|_{L^p(\mathbb{R}^n)}^{1-\theta} \|Tu\|_{L^{r',q}(\mathbb{R}^n)}^\theta \\ &\leq C[\|u\|_{L^p(B_1)} + \|\nabla u\|_{L^p(B_1)}]^{1-\theta} \|u\|_{L^{r',q}(B_1)}^\theta, \end{aligned} \quad (2.4)$$

where we have used the fact that $\|Tu\|_{L^{r',q}(\mathbb{R}^n)} \leq C\|u\|_{L^{r',q}(B_1)}$. Moreover, from the definition of Morrey space, one can easily deduce that $\|u\|_{L^{r'}(B_1)} \leq C\|u\|_{L^{r',q}(B_1)}$, it then follows from Lemma 2.3 that

$$\|u\|_{L^p(B_1)} \leq \epsilon\|\nabla u\|_{L^p(B_1)} + C\|u\|_{L^{r',q}(B_1)}.$$

This together with (2.4) gives that

$$\begin{aligned} \|u\|_{L^{p^*}(B_1)} &\leq C[\epsilon\|\nabla u\|_{L^p(B_1)} + C\|u\|_{L^{r',q}(B_1)} + \|\nabla u\|_{L^p(B_1)}]^{1-\theta} \|u\|_{L^{r',q}(B_1)}^\theta \\ &\leq C\|\nabla u\|_{L^p(B_1)}^{1-\theta} \|u\|_{L^{r',q}(B_1)}^\theta + C\|u\|_{L^{r',q}(B_1)}, \end{aligned}$$

where $C > 0$ is some constant depending on p . Finally, we perform the scaling argument to obtain the desired estimate (2.3). \square

A vital ingredient in the proof of Theorem 1.1 is the following Harnack type's inequality:

Lemma 2.5. *Let Ω be a bounded Lipschitz domain and $f \in L_{loc}^p(\Omega)$ for some $p \geq 1$. Assume that $u \in W_{loc}^{1,r'}(\Omega)$ is a solution of the following equation in the sense of distributions:*

$$-\Delta u + |\nabla u|^{r'} \leq f, \quad \text{in } \Omega \subset \mathbb{R}^n, \quad (2.5)$$

where $r' > 1$. Then for $B_R \subset \Omega$, we have

$$\int_{B_{R/2}} |\nabla u|^{r'} dx \leq KR^{n-\hat{r}},$$

where $\hat{r} := \max\{\frac{n}{p}, r\}$ and constant K depends on r' , p , n , and $\|f\|_{L^p(B_R)}$.

Proof. We refer the readers to Lemma 2.3 in [19]. □

It is necessary to establish the following improved Poincaré inequality.

Lemma 2.6. *Let $R \geq 1$ and $\gamma > 1$, then for any $v \in W^{1,\gamma}(B_R)$ satisfying*

$$\int_{B_1} v \, dx = 0, \quad (2.6)$$

there exists constant $C = C(R) > 0$ such that,

$$\|v\|_{L^\gamma(B_R)} \leq C \|\nabla v\|_{L^\gamma(B_R)}. \quad (2.7)$$

Proof. First of all, by using the standard Poincaré inequality, one obtains

$$\|v\|_{L^\gamma(B_1)} \leq C \|\nabla v\|_{L^\gamma(B_1)}, \quad (2.8)$$

where C is a constant depending on γ . Then for some $R_1 > 1$ which will be chosen later on, we find there exists some $C > 0$ independent of R_1 such that

$$\begin{aligned} \|v\|_{L^\gamma(B_{R_1})} &\leq \left\| v - \int_{B_{R_1}} v \, dx \right\|_{L^\gamma(B_{R_1})} + \left| \int_{B_{R_1}} v \, dx \right| |B_{R_1}|^{\frac{1}{\gamma}} \\ &\leq CR_1 \|\nabla v\|_{L^\gamma(B_{R_1})} + \left| \int_{B_{R_1}} v \, dx - \int_{B_1} v \, dx \right| |B_{R_1}|^{\frac{1}{\gamma}} + \left| \int_{B_1} v \, dx \right| |B_{R_1}|^{\frac{1}{\gamma}} \\ &= CR_1 \|\nabla v\|_{L^\gamma(B_{R_1})} + \left| \int_{B_{R_1}} v \, dx - \int_{B_1} v \, dx \right| |B_{R_1}|^{\frac{1}{\gamma}}, \end{aligned}$$

where we have used the condition (2.6). On the other hand, one gets

$$\begin{aligned} &|B_{R_1}|^{\frac{1}{\gamma}} \left| \int_{B_{R_1}} v \, dx - \int_{B_1} v \, dx \right| \\ &= |B_{R_1}|^{\frac{1}{\gamma}} \left| \frac{1}{|B_{R_1}|} \int_{B_{R_1}} v \, dx - \frac{1}{|B_{R_1}|} \int_{B_1} v \, dx + \frac{1}{|B_{R_1}|} \int_{B_1} v \, dx - \frac{1}{|B_1|} \int_{B_1} v \, dx \right| \\ &\leq \frac{1}{|B_{R_1}|^{\frac{1}{\gamma}}} \|v\|_{L^\gamma(B_{R_1})} |B_{R_1} \setminus B_1|^{\frac{1}{\gamma}} + \left| \frac{1}{|B_{R_1}|} - \frac{1}{|B_1|} \right| \|v\|_{L^\gamma(B_1)} |B_{R_1}| \\ &\leq \left(\frac{1}{2}\right)^{\frac{1}{\gamma}} \|v\|_{L^\gamma(B_{R_1})} + \frac{2}{|B_1|} |B_{R_1}| \|v\|_{L^\gamma(B_1)}, \end{aligned}$$

where R_1 is chosen as $R_1 = 2^{\frac{1}{\gamma}}$ such that

$$\frac{R_1^n - 1}{R_1^n} \leq \frac{1}{2}.$$

Thus, we further obtain from (2.8) that

$$\begin{aligned} \|v\|_{L^\gamma(B_{R_1})} &\leq CR_1 \|\nabla v\|_{L^\gamma(B_{R_1})} + \left(\frac{1}{2}\right)^{\frac{1}{\gamma}} \|v\|_{L^\gamma(B_{R_1})} + \frac{2}{|B_1|} |B_{R_1}| \|v\|_{L^\gamma(B_1)} \\ &\leq CR_1 \|\nabla v\|_{L^\gamma(B_{R_1})} + \left(\frac{1}{2}\right)^{\frac{1}{\gamma}} \|v\|_{L^\gamma(B_{R_1})} + C |B_{R_1}| \|\nabla v\|_{L^\gamma(B_1)}. \end{aligned}$$

Let $C(R_1) := \frac{C \max\{R_1, |B_{R_1}|\}}{1 - (\frac{1}{2})^{\frac{1}{\gamma}}}$, then it follows that

$$\|v\|_{L^\gamma(B_{R_1})} \leq C(R_1) \|\nabla v\|_{L^\gamma(B_{R_1})}. \quad (2.9)$$

Next, we let $R_2 > R_1$ which will be chosen later on such that

$$\begin{aligned}
\|v\|_{L^\gamma(B_{R_2})} &\leq \left\| v - \int_{B_{R_2}} v dx \right\|_{L^\gamma(B_{R_2})} + \left\| \int_{B_{R_2}} v dx \right\|_{L^\gamma(B_{R_2})} \\
&\leq CR_2 \|\nabla v\|_{L^\gamma(B_{R_2})} + \left| \int_{B_{R_2}} v dx \right| |B_{R_2}|^{\frac{1}{\gamma}} \\
&\leq CR_2 \|\nabla v\|_{L^\gamma(B_{R_2})} + \left| \int_{B_{R_2}} v dx - \int_{B_{R_1}} v dx + \int_{B_{R_1}} v dx \right| |B_{R_2}|^{\frac{1}{\gamma}} \\
&\leq CR_2 \|\nabla v\|_{L^\gamma(B_{R_2})} + \left| \int_{B_{R_2}} v dx - \int_{B_{R_1}} v dx \right| |B_{R_2}|^{\frac{1}{\gamma}} + \left| \int_{B_{R_1}} v dx \right| |B_{R_2}|^{\frac{1}{\gamma}}, \tag{2.10}
\end{aligned}$$

where $C > 0$ is independent of R_1 and R_2 . In light of (2.9) and Hölder's inequality, one finds

$$\left| \int_{B_{R_1}} v dx \right| |B_{R_2}|^{\frac{1}{\gamma}} \leq \frac{|B_{R_2}|^{\frac{1}{\gamma}}}{|B_{R_1}|} |B_{R_1}|^{\frac{1}{\gamma}} \|v\|_{L^\gamma(B_{R_1})} \leq \frac{|B_{R_2}|}{|B_{R_1}|} \|v\|_{L^\gamma(B_{R_1})}. \tag{2.11}$$

Similarly, we have

$$\begin{aligned}
&\left| \int_{B_{R_2}} v dx - \int_{B_{R_1}} v dx \right| |B_{R_2}|^{\frac{1}{\gamma}} \\
&= \left| \frac{1}{|B_{R_2}|} \int_{B_{R_2}} v dx - \frac{1}{|B_{R_2}|} \int_{B_{R_1}} v dx + \frac{1}{|B_{R_2}|} \int_{B_{R_1}} v dx - \frac{1}{|B_{R_1}|} \int_{B_{R_1}} v dx \right| |B_{R_2}|^{\frac{1}{\gamma}} \\
&= \left| \frac{1}{|B_{R_2}|^{\frac{1}{\gamma'}}} \int_{B_{R_2} \setminus B_{R_1}} v dx \right| + \left| \frac{1}{|B_{R_2}|} - \frac{1}{|B_{R_1}|} \right| \left| \int_{B_{R_1}} v dx \right| |B_{R_2}|^{\frac{1}{\gamma}} \\
&\leq \|v\|_{L^\gamma(B_{R_2})} \frac{|B_{R_2} \setminus B_{R_1}|^{\frac{1}{\gamma'}}}{|B_{R_2}|^{\frac{1}{\gamma'}}} + \|v\|_{L^\gamma(B_{R_1})} \frac{2|B_{R_2}|}{|B_{R_1}|}. \tag{2.12}
\end{aligned}$$

Choosing R_2 such that $R_2^n = 2R_1^n$, Then $\frac{R_2^n - R_1^n}{R_2^n} \leq \frac{1}{2}$, and it follows from (2.11) and (2.12) that

$$\left| \int_{B_{R_2}} v dx - \int_{B_{R_1}} v dx \right| |B_{R_2}|^{\frac{1}{\gamma}} + \left| \int_{B_{R_1}} v dx \right| |B_{R_2}|^{\frac{1}{\gamma}} \leq \left(\frac{1}{2}\right)^{\frac{1}{\gamma}} \|v\|_{L^\gamma(B_{R_2})} + 6\|v\|_{L^\gamma(B_{R_1})}.$$

This combines with (2.9) and (2.10) indicate that, there exists $C(R_2) > 0$ such that

$$\|v\|_{L^\gamma(B_{R_2})} \leq C(R_2) \|\nabla v\|_{L^\gamma(B_{R_2})}.$$

By performing the iteration argument, one can obtain for any $R > 0$, the conclusion (2.7) holds. \square

We collect the following Calderón-Zygmund estimates for linear second order elliptic equations:

Lemma 2.7 (C.f. [18]). *Define Ω as a bounded domain. Assume $u \in W_{loc}^{2,p}(\Omega) \cap L^p(\Omega)$ with $1 < p < \infty$ be a strong solution of*

$$-\Delta u = g \text{ in } \Omega.$$

Then we have for each $B_R \subset\subset \Omega$ with $R \leq \delta$ and every $\sigma \in (0, 1)$,

$$\|D^2 u\|_{L^p(B_{\sigma R})} \leq \frac{C}{(1-\sigma)^2 R^2} (R^2 \|g\|_{L^p(B_R)} + \|u\|_{L^p(B_R)}),$$

where constants $\delta > 0$ and $C = C(n, p, \delta)$.

2.2 Weighted Morrey Norm Estimates and $W^{2,p}$ Theories

This subsection is devoted to the local gradient estimates and $W^{2,p}$ regularity of solutions to (2.1). It is worthy mentioning that for the superquadratic case, i.e. $r' > 2$, the local Hölder and maximal regularities were established in [17]. We follow the ideas shown in [17] to study the case of $1 < r' \leq 2$. More precisely, our strategy is to perform the blow-up analysis and apply the Liouville type's results to derive the contradiction, now through the analysis of weighted Morrey estimates. In general, by using Lemma 2.5, we can obtain that the solution u of equation (2.1) satisfies $\nabla u \in L^{r',r}(\Omega)$ provided that $p > \frac{n}{r}$. If $r' > 2$, i.e. $r' > r$, one further has $u \in C_{\text{loc}}^{0,\alpha}(\Omega)$ for $\alpha = 2 - r$ by Morrey's Lemma. However, if $1 < r' \leq 2$, we need to argue in (weighted) Morrey spaces. To this end, we first establish the following key lemma:

Lemma 2.8. *Let $R > 0$, $r' \geq \frac{n}{n-1}$, $g \in L^p(B_{2R})$ with $n > p > \frac{n}{r}$ and $v \in W^{2,p}(B_{2R})$ satisfy*

$$|\Delta v| \leq C_H |\nabla v|^{r'} + |g|, \text{ a.e. in } B_{2R}, \quad (2.13)$$

where $C_H > 0$ is a constant. Then if

$$\|g\|_{L^p(B_{2R})} + \|\nabla v\|_{L^{r',q}(B_{2R})} \leq K,$$

where $q = r'(\frac{n}{p} - 1)$, there exists $C > 0$ depending on K and R such that

$$\|D^2 v\|_{L^p(B_R)} \leq C.$$

Proof. For any $0 < \rho < 2R$, we let $\tilde{v} = v - \int_{B_\rho} v \, dx$. Then we have from (2.13) that

$$|\Delta \tilde{v}| \leq C_H |\nabla \tilde{v}|^{r'} + |g|, \text{ a.e. in } B_{2R}.$$

By employing the Poincaré inequality and Hölder's inequality, we obtain

$$\|\tilde{v}\|_{L^p(B_\rho)} \leq C\rho \|\nabla \tilde{v}\|_{L^p(B_\rho)} \leq C\rho^2 \|\nabla \tilde{v}\|_{L^{p^*}(B_\rho)}, \quad (2.14)$$

where $C > 0$ is independent of $\rho > 0$.

Since $\tilde{v} \in W^{2,p}(B_{2R})$, we apply Gagliardo-Nirenberg inequality (2.3) to get

$$\|\nabla \tilde{v}\|_{L^{p^*}(B_\rho)} \leq C \|D^2 \tilde{v}\|_{L^p(B_\rho)}^\theta \|\nabla \tilde{v}\|_{L^{r',q}(B_\rho)}^{1-\theta} + \|\nabla \tilde{v}\|_{L^{r',q}(B_\rho)} \text{ for any } \frac{p}{p^*} \leq \theta < 1, \quad (2.15)$$

where $q = r'(\frac{n}{p} - 1)$ and $C > 0$ is independent of ρ . From (2.14) and (2.15), we see that there exists $C = C(K) > 0$ independent of ρ such that

$$\begin{aligned} \|\tilde{v}\|_{L^p(B_\rho)} &\leq C\rho^2 \|D^2 \tilde{v}\|_{L^p(B_{2\rho})}^\theta \|\nabla \tilde{v}\|_{L^{r',q}(B_\rho)}^{1-\theta} + \rho^2 \|\nabla \tilde{v}\|_{L^{r',q}(B_\rho)} \\ &\leq C\rho^2 \|D^2 \tilde{v}\|_{L^p(B_\rho)}^{\theta r'} + C\rho^2, \end{aligned}$$

and

$$\begin{aligned} \|\nabla \tilde{v}\|_{L^{p^*}(B_\rho)}^{r'} &\leq C \|D^2 \tilde{v}\|_{L^p(B_\rho)}^{\theta r'} \|\nabla \tilde{v}\|_{L^{r',q}(B_\rho)}^{(1-\theta)r'} + C \|\nabla \tilde{v}\|_{L^{r',q}(B_\rho)}^{r'} \\ &\leq C (\|D^2 \tilde{v}\|_{L^p(B_\rho)}^{\theta r'} + 1), \end{aligned}$$

where we have used $\|\nabla \tilde{v}\|_{L^{r',q}(B_\rho)} \leq K$.

We next fix $R < \rho < 2R$ and apply Lemma 2.7 on equation (2.13) to obtain

$$\begin{aligned} \|D^2 \tilde{v}\|_{L^p(B_{\sigma\rho})} &\leq \frac{C}{(1-\sigma)^2 R^2} \left(R^2 \|\nabla \tilde{v}\|^{r'} + \|g\|_{L^p(B_\rho)} + \|\tilde{v}\|_{L^p(B_\rho)} \right) \\ &\leq \frac{C}{(1-\sigma)^2} \left(R^{r' - \frac{n}{p}(r'-1)} \|\nabla \tilde{v}\|_{L^{p^*}(B_\rho)}^{r'} + \frac{1}{R^2} \|\tilde{v}\|_{L^p(B_\rho)} + 1 \right) \\ &\leq \frac{C}{(1-\sigma)^2} \left(R^{r' - \frac{n}{p}(r'-1)} \|D^2 \tilde{v}\|_{L^p(B_\rho)}^{\theta r'} + R^{r' - \frac{n}{p}(r'-1)} + \|D^2 \tilde{v}\|_{L^p(B_\rho)}^{\theta r'} + 1 \right) \\ &\leq \frac{C}{(1-\sigma)^2} \left(\max\{1, R^{r' - \frac{n}{p}(r'-1)}\} \|D^2 \tilde{v}\|_{L^p(B_\rho)}^{\theta r'} + 1 + R^{r' - \frac{n}{p}(r'-1)} \right) \end{aligned}$$

where $\sigma \in (0, 1)$, and θ is chosen such that $0 < \theta r' < 1$. Noting that $D^2\tilde{v} = D^2v$, we have

$$\|D^2v\|_{L^p(B_{\sigma\rho})} \leq \frac{C}{(1-\sigma)^2} \left(\max\{1, R^{r'-\frac{n}{p}(r'-1)}\} \|D^2v\|_{L^p(B_\rho)}^{\theta r'} + 1 + R^{r'-\frac{n}{p}(r'-1)} \right).$$

Now, we find if R satisfies

$$R^{r'-\frac{n}{p}(r'-1)} \|D^2v\|_{L^p(B_\rho)}^{\theta r'} \leq 1 + R^{r'-\frac{n}{p}(r'-1)},$$

we complete the proof of our desired conclusion. Otherwise, we arrive at

$$\|D^2v\|_{L^p(B_{\sigma\rho})} \leq \frac{E}{(1-\sigma)^2} \|D^2v\|_{L^p(B_\rho)}^{\theta r'},$$

where $0 < \theta r' < 1$ and E is defined as

$$E := C \max\{1, R^{r'-\frac{n}{p}(r'-1)}\}, \quad r' - \frac{n}{p}(r'-1) > 0.$$

Then we follow the argument below (3.6) in Proposition [17] and perform the iteration argument to complete the proof. \square

Remark 2.1. We would like to point out that the condition $r' \geq \frac{n}{n-1}$ in Lemma 2.8 is imposed to guarantee that $q \leq n$, which satisfies the range of s given in the definition of Morrey space (2.2). When $r' < \frac{n}{n-1}$, the Morrey scale is not anymore natural, and we expect ∇u to be controlled in some Hölder space. We are not going to investigate here this kind of analysis.

On the other hand, we require the following Liouville type results:

Lemma 2.9. Assume $w \in W_{loc}^{2,p}(\mathbb{R}^n)$ with $p > \frac{n}{r}$ solves the following equation:

$$-\Delta w + h|\nabla w|^{r'} = 0 \text{ in } \mathbb{R}^n, \quad (2.16)$$

where $h \geq 0$ is a constant. If w satisfies

$$\nabla w \in L^{r',q}(\mathbb{R}^n), \text{ with some } q \in (0, n], \quad (2.17)$$

then we have $w \equiv C$ with $C \in \mathbb{R}$ being some constant.

Proof. If $h > 0$, the conclusion is stated in Lemma 2.5 of [17]. We remark that by the standard bootstrap argument, one has $w \in C^3(\mathbb{R}^n)$ when $h > 0$ since $p > \frac{n}{r}$. Next, we consider $h = 0$ and simplify equation (2.16) as

$$\Delta w = 0 \text{ in } \mathbb{R}^n.$$

Next, we estimate $\nabla w(x_0)$ for arbitrary $x_0 \in \mathbb{R}^n$. To this end, define $\tilde{w}(x) := w(x) - w(x_0)$, then \tilde{w} satisfies

$$\Delta \tilde{w} = 0 \text{ in } \mathbb{R}^n \text{ and } \int_{B_R(x_0)} \tilde{w} dx = \tilde{w}(x_0) = 0 \text{ for any } R > 0.$$

By using the gradient estimates for harmonic functions, we find from Hölder's inequality, Poincaré's inequality and (2.17) that

$$\begin{aligned} |\nabla w(x_0)| &= |\nabla \tilde{w}(x_0)| \leq \frac{C}{R^{n+1}} \|\tilde{w}\|_{L^1(B_R(x_0))} \leq \frac{C}{R^{n+1}} \|\tilde{w}\|_{L^{r'}(B_R(x_0))} R^{\frac{n}{r}} \\ &\leq \frac{C}{R^{n+1}} R \|\nabla \tilde{w}\|_{L^{r'}(B_R(x_0))} R^{\frac{n}{r}} \leq \frac{C}{R^{n-\frac{n}{r}}} \left(\frac{1}{R^{q-n}} \right)^{\frac{1}{r'}} = \frac{C}{R^{\frac{q}{r}}}, \end{aligned}$$

where $C > 0$ is a constant independent of R . In light of the definition of q in (2.17), one obtain $|\nabla w(x_0)| = 0$ by letting $R \rightarrow +\infty$ in the above estimate. Thus, we have $w \equiv C$ for some constant $C \in \mathbb{R}$ since x_0 is arbitrary. \square

We are ready to show the weighted Morrey estimates satisfied by solutions of (2.1), which is

Theorem 2.1. *Let $p > \frac{n}{r}$ and assume $u \in W^{2,p}(\Omega)$ solves*

$$-\Delta u + C_H |\nabla u|^{r'} = f \text{ in } \Omega \subset \mathbb{R}^n,$$

in the strong sense, where $r' \geq \frac{n}{n-1}$ and there exists a constant $M > 0$ such that $\|f\|_{L^p(\Omega)} \leq M$. Then

$$\sup_{B_{2R}(\hat{x}) \subset \Omega} R^q \int_{B_R(\hat{x})} |\nabla u|^{r'} dx (\text{dist}(B_R(\hat{x}), \partial\Omega))^{r-q} \leq C,$$

where $C = C(M, n, p, r', \Omega)$ is a positive constant. In particular, if $p < n$, $q = r'(\frac{n}{p} - 1)$; otherwise if $p \geq n$, q is any number satisfying $q \in (0, r)$.

Proof. To prove this theorem, we argue by contradiction and assume sequences $(f_k)_k \subset L^q(\Omega)$ and $(u_k)_k \subset W^{2,p}(\Omega)$ satisfying for every n ,

$$-\Delta u_k + C_H |\nabla u_k|^{r'} = f_k, \quad x \in \Omega, \quad (2.18)$$

and $\sup_k \|f_k\|_{L^p(\Omega)} \leq M$. Moreover,

$$\sup_{B_{2R}(\hat{x}) \subset \Omega} R^q \int_{B_R(\hat{x})} |\nabla u_k|^{r'} dx (\text{dist}(B_R(\hat{x}), \partial\Omega))^{r-q} := L_k \rightarrow +\infty \text{ as } k \rightarrow +\infty. \quad (2.19)$$

Consequently, there exist x_k and R_k , $k = 1, 2, \dots$, such that

$$\frac{L_k}{2} \leq \text{dist}(B_{R_k}(x_k), \partial\Omega)^{r-q} R_k^q \int_{B_{R_k}(x_k)} |\nabla u_k|^{r'} dx \leq L_k,$$

where $B_{2R_k}(x_k) \subset \Omega$. Define

$$w_k(y) := \frac{R_k^{q/r'}}{R_k M_k^{1/r'}} u_k(x_k + R_k y), \quad (2.20)$$

where

$$y \in \frac{\Omega - x_k}{R_k} := \Omega_k, \quad M_k := R_k^q \int_{B_{R_k}(x_k)} |\nabla u_k|^{r'} dx.$$

In light of $p > \frac{n}{r}$, we obtain from Lemma 2.5 that

$$R_k^r \int_{B_{R_k}(x_k)} |\nabla u_k|^{r'} dx \leq C \quad (2.21)$$

for some constant $C > 0$ independent of k . Moreover, thanks to (2.19), we obtain

$$\left[\frac{\text{dist}(B_{R_k}(x_k), \partial\Omega)}{R_k} \right]^{r-q} \geq \frac{1}{2} \frac{L_k}{R_k^r \int_{B_{R_k}(x_k)} |\nabla u_k|^{r'} dx} \rightarrow +\infty. \quad (2.22)$$

As a consequence, we see that $R_k \rightarrow 0$ and $M_k \rightarrow +\infty$.

We next claim that for any $s > 0$ and $\hat{y} \in \mathbb{R}^n$, the following estimate holds for k large enough:

$$\limsup_k s^q \int_{B_s(\hat{y})} |\nabla w_k(y)|^{r'} dy \leq C < \infty, \quad (2.23)$$

where the constant $C > 0$ independent of k , s and \hat{y} . To begin with, one can see from (2.22) that $\Omega_k \rightarrow \mathbb{R}^n$ as $k \rightarrow +\infty$, hence $\hat{y} \in \Omega_k$ for k large enough. Noting that $B_{R_k}(x_k) \subset \Omega$, we further find from (2.22) that $B_{R_k s}(x_k + R_k \hat{y}) \subset \Omega$, then apply the triangle inequality to get

$$\text{dist}(B_{R_k s}(x_k + R_k \hat{y}), \partial\Omega) \geq \text{dist}(B_{R_k}(x_k), \partial\Omega) - (|\hat{y}| + s - 1)R_k.$$

By using (2.22) again, we deduce that for k large enough,

$$\begin{aligned} s^q \int_{B_s(\hat{y})} |\nabla w_k(y)|^{r'} dy &= \frac{(R_k s)^q}{M_k} \int_{B_{R_k s}(x_k + R_k \hat{y})} |\nabla u_k(x)|^{r'} dx \leq \frac{L_k}{M_k [\text{dist}(B_{R_k s}(x_k + R_k \hat{y}), \partial\Omega)]^{r-q}} \\ &\leq 2 \left(\frac{\text{dist}(B_{R_k}(x_k), \partial\Omega)}{\text{dist}(B_{R_k s}(x_k + R_k \hat{y}), \partial\Omega)} \right)^{r-q} \leq 2 \left(\frac{\text{dist}(B_{R_k}(x_k), \partial\Omega)}{\text{dist}(B_{R_k}(x_k), \partial\Omega) - (|\hat{y}| + s - 1)R_k} \right)^{r-q} \xrightarrow{k} 2, \end{aligned}$$

which finishes the proof of claim (2.23).

On the other hand, invoking (2.20), we have

$$\int_{B_1(0)} |\nabla w_k(y)|^{r'} dy = 1. \quad (2.24)$$

Moreover, one obtains from (2.18) that w_k satisfy

$$-\Delta w_k + \frac{M_k^{1/r}}{R_m^{q/r-1}} |\nabla w_k|^{r'} = \frac{R_k^{q/r'+1}}{M_k^{1/r'}} f_k(x_k + R_k y) := \hat{f}_k, \quad (2.25)$$

where

$$\left(\frac{M_k^{1/r}}{R_k^{q/r-1}} \right)^r = \frac{M_k}{R_k^{q-r}} = R_k^r \int_{B_{R_k}(x_k)} |\nabla u_k|^{r'} dx \leq C < \infty \quad (2.26)$$

by (2.21), and it follows from the boundness of f in $L^p(\Omega)$ and $\frac{q}{r'} - \frac{n}{p} + 1 = 0$ that

$$\begin{aligned} \|\hat{f}_k\|_{L^p(\Omega_k)} &= \left\| \frac{R_k^{q/r'+1}}{M_k^{1/r'}} f_k(x_k + R_k y) \right\|_{L^p(\Omega_k)} \\ &= \frac{R_k^{q/r'+1}}{M_k^{1/r'}} \left(\int |f_k(x_k + R_k y)|^p dy \right)^{\frac{1}{p}} = \frac{R_k^{q/r'+1}}{M_k^{1/r'}} \frac{1}{R_k^{\frac{n}{p}}} \|f_k\|_{L^p(\Omega)} \\ &= \frac{R_k^{q/r' - \frac{n}{p} + 1}}{M_k^{1/r'}} \|f_k\|_{L^p(\Omega)} = \frac{1}{M_k^{1/r'}} \|f_k\|_{L^p(\Omega)} \rightarrow 0 \text{ as } k \rightarrow +\infty. \end{aligned} \quad (2.27)$$

In light of (2.25), we obtain for any $R > 0$,

$$|\Delta w_k| \leq \frac{M_k^{\frac{1}{r}}}{R_k^{\frac{q}{r}-1}} |\nabla w_k|^{r'} + \frac{R_k^{\frac{q}{r}+1}}{M_k^{\frac{1}{r'}}} |f_k|, \text{ in } B_{2R}. \quad (2.28)$$

Using (2.23), (2.26) and (2.27), one can invoke Lemma 2.8 to get

$$\|D^2 w_k\|_{L^p(B_R)} \leq C(R) < \infty$$

for some $C(R) > 0$ independent of k . Moreover, we can deduce from Lemma 2.4 and (2.23) that

$$\|\nabla w_k\|_{L^p(B_R)} \leq C \|\nabla w_k\|_{L^{p^*}(B_R)} \leq C \|D^2 w_k\|_{L^p(B_R)}^{\theta} \|\nabla w_k\|_{L^{r',q}(B_R)}^{1-\theta} + C \|\nabla w_k\|_{L^{r',q}(B_R)} \leq C(R) < \infty. \quad (2.29)$$

Without loss of generality, we may fix $\int_{B_1(0)} w_k dx = 0$, and then obtain from Lemma 2.6 that

$$\|w_k\|_{L^r(B_R)} \leq C \|\nabla w_k\|_{L^r(B_R)} \leq C(R) < \infty. \quad (2.30)$$

Therefore, by using a standard diagonal argument, we can obtain from (2.28), (2.29) and (2.30) that

$$w_k \rightharpoonup w_\infty \text{ in } W_{\text{loc}}^{2,p}(\mathbb{R}^n).$$

Thus $|\nabla w_k| \rightharpoonup |\nabla w_\infty|$ in $W_{\text{loc}}^{1,p}(\mathbb{R}^n) \hookrightarrow L_{\text{loc}}^s(\mathbb{R}^n)$ for all $s \in [p, p^*)$. Noting that $r'p \in (p, p^*)$ by $p > \frac{n}{r}$, we thus deduce that

$$C_H |\nabla w_k|^{r'} \rightharpoonup h_\infty |\nabla w_\infty|^{r'} \text{ in } L_{\text{loc}}^p(\mathbb{R}^n) \text{ for some } h_\infty \geq 0,$$

where (2.26) is used.

In summary, one arrives at the following limiting problem

$$-\Delta w_\infty + h_\infty |\nabla w_\infty|^{r'} = 0 \text{ in } \mathbb{R}^n,$$

where $w_\infty \in W_{\text{loc}}^{2,p}(\mathbb{R}^n)$ with $p > \frac{n}{r}$, and in particular, $|\nabla w_\infty| \in L^{r',q}(\mathbb{R}^n)$ followed by (2.23). By invoking Lemma 2.9, we find w_∞ is a constant and thus $\nabla w_\infty \equiv 0$, which contradicts (2.24). This completes the proof of our theorem. \square

With the aid of Theorem 2.1, we can prove Theorem 1.1, which is

Proof of Theorem 1.1:

Proof. The argument is similar as the proof of Theorem 1.3 in [17] and we give the sketch of proof here for the completeness.

We first consider the case for $\frac{n}{r} < p < n$. Choosing Ω'' such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, by invoking Theorem 2.1, one has

$$\|\nabla u\|_{L^{r',q}(\Omega'')} \leq C, \quad q = r' \left(\frac{n}{p} - 1 \right).$$

where $C = C(M, \text{dist}(\Omega'', \partial\Omega), n, p, C_H, r) > 0$ and $q < r$. In addition, for $\bar{\Omega}'$, we have there exists a finite cover $\{B_R(x_k)\}_k$ such that $\bar{\Omega}' \subset \cup_k B_R(x_k)$ and $B_{2R}(x_k) \subset \Omega''$ for any k . With the aid of Lemma 2.8, one obtains

$$\|D^2 u\|_{L^p(\Omega')} \leq C,$$

where $C = C(M, \text{dist}(\Omega', \partial\Omega''), n, p, C_H, r) > 0$ is some constant depends on . Moreover, thanks to Lemma 2.4, we can get the gradient estimate of u . By fixing the average of u , one finally gets the desired conclusion.

If $p \geq n$, we pick up some $\frac{n}{r} < q < n$ at first, then follow the discussion shown above to arrive at

$$\|D^2 u\|_{L^q(\Omega')} \leq C,$$

where $C > 0$ is some constant. Next, we perform the bootstrap argument and complete the proof of this theorem since $q > \frac{n}{r}$. \square

Remark 2.2. We remark that our results of L^p estimates shown in Theorem 1.1 also hold for $r' > 2$, which covers the results of Theorem 1.3 in [17]. Indeed, we find when $r' \geq 2$, solution u still satisfies $\nabla u \in L_{\text{loc}}^{r',r}(\Omega)$ provided with $p > \frac{n}{r}$. By establishing the key Lemma 2.5 and performing the blow-up argument, we show that $\nabla u \in L_{\text{loc}}^{r',q}(\Omega)$ with $q < r$ given in Theorem 2.1, which is a higher regularity compared to the conclusion shown in Theorem 1.2 of [17]. In addition, we give a unified argument to prove the L^p maximal regularity of the solution to (2.1) for the case of $\frac{n}{n-1} \leq r' \leq 2$ and $r' > 2$.

Next, we focus on the existence and asymptotic behaviors of least energy solutions to (1.4) under with critical mass exponent.

3 Existence and Regularities: Hamilton-Jacobi and Fokker-Planck equations

In this section, we shall state some key lemmas and important properties satisfied by the solutions to (1.4). First of all, we collect existence and regularity results of Hamilton-Jacobi-Bellman (HJB) equations and Fokker-Planck equations, which are summarized in Subsection 3.1 and 3.2, respectively.

3.1 Hamilton-Jacobi Equations

Consider the following form of Hamilton-Jacobi equations:

$$-\Delta u_k + C_H |\nabla u_k|^{r'} + \lambda_k = V_k(x) + f_k(x), \quad x \in \mathbb{R}^n, \quad (3.1)$$

where $r' > 1$ is fixed, C_H is a given positive constant independent of k and (u_k, λ_k) denote the solutions to (3.1). Focusing on the regularities of u_k , one has

Lemma 3.1. *Suppose that $f_k \in L^\infty(\mathbb{R}^n)$ satisfies $\|f_k\|_{L^\infty} \leq C_f$, $|\lambda_k| \leq \lambda$, and the potential functions $V_k(x) \in C_{\text{loc}}^{0,\theta}(\mathbb{R}^n)$ with $\theta \in (0, 1)$ satisfy $0 \leq V_k(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$, and $\exists R > 0$ sufficiently large such that*

$$0 < C_1 \leq \frac{V_k(x+y)}{V_k(x)} \leq C_2, \quad \text{for all } k \text{ and all } |x| \geq R \text{ with } |y| < 2, \quad (3.2)$$

where the positive constants C_f , λ , R , C_1 and C_2 are independent of k . Let $(u_k, \lambda_k) \in C^2(\mathbb{R}^n) \times \mathbb{R}$ be a sequence of solutions to (3.1). Then, for all k ,

$$|\nabla u_k(x)| \leq C(1 + V_k(x))^{\frac{1}{r'}}, \quad \text{for all } x \in \mathbb{R}^n, \quad (3.3)$$

where constant C depends on C_H , C_1 , C_2 , λ , r , n and C_f .

In particular, if there exist $b \geq 0$ and $C_F > 0$ independent of k , such that following conditions hold on V_k

$$C_F^{-1}(\max\{|x| - C_F, 0\})^b \leq V_k(x) \leq C_F(1 + |x|)^b, \quad \text{for all } k \text{ and } x \in \mathbb{R}^n, \quad (3.4)$$

then we have

$$|\nabla u_k| \leq C(1 + |x|)^{\frac{b}{r'}}, \quad \text{for all } k \text{ and } x \in \mathbb{R}^n, \quad (3.5)$$

where constant C depends on C_H , C_F , b , λ , r , n and C_f .

Proof. The approach what we shall employ is based on Theorem 2.5 in [12]. Indeed, when V_k satisfy (3.4), (3.5) hold and the arguments are stated in [12].

Next, we focus on the proof of (3.3) for the more general V_k satisfying (3.2). It is shown in (2-6) of [12] that if

$$|-\Delta v + |\nabla v|^{r'}| \leq K \text{ in } B_2(0) \text{ with positive constant } K,$$

then

$$\|\nabla v\|_{L^r(B_1(0))} \leq \tilde{C}, \quad \forall r \in [1, \infty], \quad (3.6)$$

where positive constant \tilde{C} depends on K , r , n and C_H . For any fixed $x_0 \in \mathbb{R}^n$, let $\delta = (1 + V_n(x_0))^{-\frac{1}{r}}$ and define

$$v_k(y) = \delta^{\frac{2-r'}{r'-1}} u_k(x_0 + \delta y),$$

then we have v_k solves

$$-\Delta v_k + C_H |\nabla v_k|^{r'} = \delta^r [V_k(x_0 + \delta y) - f_k(x_0 + \delta y) - \lambda_k].$$

Since $\delta > 0$ is sufficiently small and $|x_0| > R$, one finds from (3.2) that

$$\delta^r |V_k(x_0 + \delta y) - f_k(x_0 + \delta y) - \lambda_k| \leq \frac{\bar{C} V_k(x_0 + \delta y) + C_f + \lambda}{1 + V_k(x_0)} \leq \hat{C} \text{ for } |y| \leq 2,$$

where the positive constants \bar{C} and \hat{C} are independent of k . Then it follows from (3.6) that

$$\|\nabla v_k(y)\|_{L^\infty(B_1(0))} \leq \bar{C}$$

for some $\bar{C} > 0$. In particular, choosing $y = 0$, we arrive at

$$|\nabla u_k(x_0)| = \delta^{-\frac{1}{r'-1}} |\nabla v_k(0)| \leq \bar{C} (1 + V_k(x_0))^{\frac{1}{r'}},$$

which gives the desired estimate (3.3). In addition, noting that $u_k \in C^2(\mathbb{R}^n)$, we further obtain the desired conclusion (3.3). \square

Besides the gradient estimates of u_k , we also have the following results for the lower bounds of u_k :

Lemma 3.2. *Suppose all conditions in Lemma 3.1 holds. Let u_k be a family of C^2 solutions and assume that $u_k(x)$ are bounded from below uniformly. Then there exist positive constants C_3 and C_4 independent of k such that*

$$u_k(x) \geq C_3 V_k^{\frac{1}{r'}}(x) - C_4, \quad \forall x \in \mathbb{R}^n, \text{ for all } k. \quad (3.7)$$

In particular, if the following conditions hold on V_k

$$C_F^{-1} (\max\{|x| - C_F, 0\})^b \leq V_k(x) \leq C_F (1 + |x|)^b, \quad \text{for all } k \text{ and } x \in \mathbb{R}^n, \quad (3.8)$$

where constants $b > 0$ and C_F are independent of k , then we have

$$u_k(x) \geq C_3 |x|^{1+\frac{b}{r'}} - C_4, \quad \text{for all } k, x \in \mathbb{R}^n. \quad (3.9)$$

If $b = 0$ in (3.8) and there exist $R > 0$ and $\hat{\delta} > 0$ independent of k such that

$$f_k + V_k - \lambda_k > \hat{\delta} > 0 \text{ for all } |x| > R, \quad (3.10)$$

then (3.9) also holds.

Proof. When V_k is assumed to satisfy (3.8) or (3.10) holds with $b = 0$, the proofs are the same as in [12], Theorem 2.6.

Next, we focus on the case of general potential V_k satisfying (3.2). Since u_k are bounded from below uniformly, we assume that $u_k(x) \geq 0$. Note that we only need to prove the conclusion for $|x|$ large since it can be shown straightforward if there exists $R_0 > 0$ independent of k such that $|x| < R_0$. When $|x|$ is sufficiently large, we argue by contradiction and assume up to a subsequence, there exists $|x_l| \rightarrow +\infty$ such that

$$\lim_{l \rightarrow +\infty} \frac{u_{k_l}(x_l)}{V_{k_l}^{\frac{1}{r'}}(x_l)} = 0. \quad (3.11)$$

Let

$$v_l(x) := \frac{1}{\mu_l} u_{k_l}(x_l + x), \quad \text{where } \mu_l := V_{k_l}^{\frac{1}{r'}}(x_l) \rightarrow +\infty.$$

Then, we have from (3.1) that $v_l(x)$ solves

$$-\mu_l^{1-r'} \Delta v_l(x) + C_H |\nabla v_l(x)|^{r'} = \mu_l^{-r'} [f_{k_l}(x_l + x) + V_{k_l}(x_l + x) - \lambda_{k_l}].$$

Note that

$$\mu_l^{-r'} |f_{k_l}(x_l + x) - \lambda_{k_l} + V_{k_l}(x_l + x)| \geq \frac{V_{k_l}(x_l + x) - C_f - \lambda}{V_{k_l}(x_l)} \geq \delta > 0, \text{ for } |x| < 2,$$

where we have used (3.2) and $\delta > 0$ independent of k . On the other hand, Lemma 3.1 implies

$$|\nabla u_{k_l}(x)| \leq C \left[1 + V_{k_l}^{\frac{1}{r'}}(x) \right].$$

Moreover, v_l satisfies

$$|\nabla v_l(x)| = \frac{1}{V_{k_l}^{\frac{1}{r'}}(x_l)} |\nabla u_{k_l}(x_l + x)| \leq \frac{C[1 + V_{k_l}(x_l + x)]}{V_{k_l}^{\frac{1}{r'}}(x_l)} \leq \delta_2, \text{ for any } |x| < 2, \quad (3.12)$$

where $C > 0$ and $\delta_2 > 0$ are constants independent of k . In particular, (3.11) implies $v_l(0) = \frac{u_{k_l}(x_l)}{\mu_l} \rightarrow 0$, and thus, $v_l(x) \leq 3\delta_2$, $\forall |x| < 2$. Letting $l \rightarrow \infty$ and invoking (3.12) one applies Arzelà-Ascoli theorem to get $v_l \rightarrow v \geq 0$ uniformly in $B_2(0)$. Thus, v is a solution to

$$|\nabla v|^{r'} \geq \delta > 0 \text{ in } B_2(0), \quad v \geq 0,$$

in a viscosity sense. Noting that $h(x) = \delta^{\frac{1}{r'}}(2 - |x|)$ is a viscosity solution to $|\nabla h|^{r'} = \delta$ in $B_2(0)$ with $h(x) = 0$ on $\partial B_2(0)$. By comparison,

$$v(x) \geq h(x), \text{ for any } x \in B_2(0).$$

As a consequence, $v(0) \geq 2\delta^{\frac{1}{r'}} > 0$ which reaches a contradiction to $v_l(0) \rightarrow v(0) = 0$ as $l \rightarrow \infty$. This completes the proof of (3.7). \square

For the existence of the classical solution to (3.1), we have the following results:

Lemma 3.3. *Suppose $V_k + f_k$ are locally Hölder continuous and bounded from below uniformly in k . Define*

$$\bar{\lambda}_k := \sup\{\lambda \in \mathbb{R} \mid (3.1) \text{ has a solution } u_k \in C^2(\mathbb{R}^n)\}. \quad (3.13)$$

Then

- (i). $\bar{\lambda}_k$ are finite for every k and (3.1) admits a solution $(u_k, \lambda_k) \in C^2(\mathbb{R}^n) \times \mathbb{R}$ with $\lambda_k = \bar{\lambda}_k$ and $u_k(x)$ being bounded from below (may not uniform in k). Moreover,

$$\bar{\lambda}_k = \sup\{\lambda \in \mathbb{R} \mid (3.1) \text{ has a subsolution } u_k \in C^2(\mathbb{R}^n)\}.$$

- (ii). If V_k satisfies (3.4) with $b > 0$, then u_k is unique up to constants for fixed k and there exists a positive constant C independent of k such that

$$u_k(x) \geq C|x|^{\frac{b}{r'}+1} - C, \quad \forall x \in \mathbb{R}^n. \quad (3.14)$$

In particular, if $V_k \equiv 0$, $b = 0$ in (1.12) and there exists $\sigma > 0$ independent of k such that

$$f_k - \lambda_k \geq \sigma > 0, \text{ for } |x| > K_2, \quad (3.15)$$

where $K_2 > 0$ is a large constant independent of k , then (3.14) also holds.

- (iii). If V_k satisfies (1.20b) with V replaced by V_k and positive constants C_1, C_2 and δ independent of k , then there exist uniformly bounded from below classical solutions u_k to problem (3.1) satisfying estimate (3.7).

Proof. The proof is the same as Theorem 2.7 shown in [12]. \square

It is worthy mentioning that if locally Hölder continuous potential functions V_k satisfies $C_1 e^{\delta|x|} \leq V_k \leq C_2 e^{\delta|x|}$ for some $C_1, C_2 > 0$ independent of k , (1.20) with V replaced by V_k also holds for V_k . With a-priori estimates and existence results of solutions to HJ equations given by (3.1), we next discuss the regularities of solutions to Fokker-Planck equations, which is exhibited in Subsection 3.2.

3.2 Fokker-Planck Equations

Before stating the gradient estimates satisfied by solutions to Fokker-Planck equations, we recall the following key lemma for any function $m \in L^p(\mathbb{R}^n)$:

Lemma 3.4. *Suppose $p > 1$ and $m \in L^p(\mathbb{R}^n)$ such that*

$$\left| \int_{\mathbb{R}^n} m \Delta \varphi \, dx \right| \leq N \|\nabla \varphi\|_{L^{p'}(\mathbb{R}^n)} \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n),$$

where $N > 0$ is a positive constant. Then we have $m \in W^{1,p}(\mathbb{R}^n)$ and

$$\|\nabla m\|_{L^p(\mathbb{R}^n)} \leq C_p N,$$

where C_p is a positive constant depending only on p .

Proof. See Proposition 2.4 in [12]. □

Now, we are concerned with the following Fokker-Planck equations:

$$-\Delta m + \nabla \cdot w = 0, \quad x \in \mathbb{R}^n, \quad (3.16)$$

where w is given and m denotes the solution. By invoking Lemma 3.4, we can obtain the crucial a-priori estimates satisfied by m . To begin with, we recall that \hat{q} is defined as (1.18), and set $\hat{q}^* = \frac{n\hat{q}}{n-\hat{q}}$ if $\hat{q} < n$, and $\hat{q}^* = +\infty$ if $\hat{q} \geq n$. Choose $\beta \in [\hat{q}, \hat{q}^*]$ such that

$$\frac{1}{\hat{q}} = \frac{1}{r} + \frac{1}{r'\beta}. \quad (3.17)$$

Then one can deduce from (1.18) that

$$\beta = \begin{cases} \hat{q}^*, & \text{if } r < n, \\ \in (\hat{q}, \hat{q}^*) & \text{if } r = n, \\ \infty, & \text{if } r > n. \end{cases} \quad (3.18)$$

Set

$$0 < \mathcal{S}_{\hat{q},r}^{-1} := \inf_{m \in W^{1,\hat{q}}(\mathbb{R}^n)} \frac{\|\nabla m\|_{L^{\hat{q}}(\mathbb{R}^n)}^\theta \|\nabla m\|_{L^{\hat{q}}(\mathbb{R}^n)}^{1-\theta}}{\|m\|_{L^\beta(\mathbb{R}^n)}} < \infty, \quad \text{where } \theta \in [0, 1] \text{ satisfying } \frac{1}{\beta} = \theta \left(\frac{1}{\hat{q}} - \frac{1}{n} \right) + 1 - \theta. \quad (3.19)$$

Then we have the following lemma which addresses the regularity of solutions for equation (3.16).

Lemma 3.5. *Assume $(m, w) \in (L^1(\mathbb{R}^n) \cap W^{1,\hat{q}}(\mathbb{R}^n)) \times L^1(\mathbb{R}^n)$ is a solution to (3.16) and*

$$\Lambda_r := \int_{\mathbb{R}^n} |m| \left| \frac{w}{m} \right|^r \, dx < \infty.$$

Then, we have $w \in L^1(\mathbb{R}^n) \cap L^{\hat{q}}(\mathbb{R}^n)$ and there exists constant $C = C(\Lambda_r, \|m\|_{L^1(\mathbb{R}^n)}) > 0$ such that

$$\|m\|_{W^{1,\hat{q}}(\mathbb{R}^n)}, \|w\|_{L^1(\mathbb{R}^n)}, \|w\|_{L^{\hat{q}}(\mathbb{R}^n)} \leq C.$$

More precisely, we have

$$\|\nabla m\|_{L^{\hat{q}}(\mathbb{R}^n)} \leq \mathcal{S}_{\hat{q},r}^{\frac{1}{r'-\theta}} \left(C_{\hat{q}} \Lambda_r^{\frac{1}{r}} \right)^{\frac{r'}{r'-\theta}} \|m\|_{L^1(\mathbb{R}^n)}^{\frac{1-\theta}{r'-\theta}}, \quad \|m\|_{L^{\hat{q}}(\mathbb{R}^n)} \leq \mathcal{S}_{\hat{q},r}^{\frac{1}{r'-\theta}} \left(C_{\hat{q}} \Lambda_r^{\frac{1}{r}} \right)^{\frac{\theta}{r'-\theta}} \|m\|_{L^1(\mathbb{R}^n)}^{\frac{1-\theta}{r'-\theta} + \frac{1}{r}}. \quad (3.20)$$

and

$$\|w\|_{L^1(\mathbb{R}^n)} \leq \Lambda_r^{\frac{1}{r}} \|m\|_{L^1(\mathbb{R}^n)}^{\frac{r-1}{r}}, \quad \|w\|_{L^{\hat{q}}(\mathbb{R}^n)} \leq \Lambda_r^{\frac{1}{r}} (\mathcal{S}_{\hat{q},r})^{\frac{1}{r'-\theta}} \left(C_{\hat{q}} \Lambda_r^{\frac{1}{r}} \right)^{\frac{\theta}{r'-\theta}} \|m\|_{L^1(\mathbb{R}^n)}^{\frac{1-\theta}{r'-\theta}}, \quad (3.21)$$

where

$$\theta = \frac{nr(\hat{q}-1)}{(r-1)(nq-n+q)} = \begin{cases} 1, & \text{if } r < n, \\ \frac{n^2(\hat{q}-1)}{(n-1)(n\hat{q}-n+\hat{q})} & \text{if } r = n, \\ \frac{nr}{nr-n+r}, & \text{if } r > n. \end{cases} \quad (3.22)$$

Proof. We refer the readers to Lemma 2.8 in [12] and Proposition 2.5 in [38]. For the sake of completeness, we give the proof of this lemma as follows:

Let $\beta \in [\hat{q}, \hat{q}^*]$ satisfy (3.17) and (3.18). Then we deduce from (3.16) that

$$\left| \int_{\mathbb{R}^n} \nabla m \cdot \nabla \varphi dx \right| = \left| \int_{\mathbb{R}^n} w \cdot \nabla \varphi dx \right| \leq \Lambda_r^{\frac{1}{r}} \|m\|_{L^\beta(\mathbb{R}^n)}^{\frac{1}{r}} \|\nabla \varphi\|_{L^{\hat{q}'}(\mathbb{R}^n)} \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^n). \quad (3.23)$$

In view of Lemma 3.4, one has there exists $C_{\hat{q}} > 0$ such that

$$\|\nabla m\|_{L^{\hat{q}}(\mathbb{R}^n)} \leq C_{\hat{q}} \Lambda_r^{\frac{1}{r}} \|m\|_{L^\beta(\mathbb{R}^n)}^{\frac{1}{r}}. \quad (3.24)$$

Moreover, we apply (3.19) to get

$$\|m\|_{L^\beta(\mathbb{R}^n)} \leq \mathcal{S}_{\hat{q},r} \|\nabla m\|_{L^{\hat{q}}(\mathbb{R}^n)}^\theta \|m\|_{L^1(\mathbb{R}^n)}^{1-\theta} \text{ where } \frac{1}{\beta} = \theta \left(\frac{1}{\hat{q}} - \frac{1}{n} \right) + 1 - \theta. \quad (3.25)$$

In light of (3.17), one obtains

$$\theta = \frac{nr(\hat{q}-1)}{(r-1)(nq-n+q)} \quad (3.26)$$

Invoking (3.24) and (3.25), we find the following inequalities hold:

$$\|m\|_{L^\beta(\mathbb{R}^n)} \leq (\mathcal{S}_{\hat{q},r})^{\frac{r'}{r'-\theta}} \left(\mathcal{S}_{\hat{q},r} C_{\hat{q}} \Lambda_r^{\frac{1}{r}} \right)^{\frac{r'\theta}{r'-\theta}} \|m\|_{L^1(\mathbb{R}^n)}^{\frac{(1-\theta)r'}{r'-\theta}} \quad (3.27)$$

and

$$\|\nabla m\|_{L^{\hat{q}}(\mathbb{R}^n)} \leq \mathcal{S}_{\hat{q},r}^{\frac{1}{r'-\theta}} \left(C_{\hat{q}} \Lambda_r^{\frac{1}{r}} \right)^{\frac{r'}{r'-\theta}} \|m\|_{L^1(\mathbb{R}^n)}^{\frac{1-\theta}{r'-\theta}}. \quad (3.28)$$

Then letting $\tau := \frac{1-\frac{1}{\hat{q}}}{1-\frac{1}{\beta}} = \frac{1}{r'}$ since (3.17) holds, we apply Hölder's inequality to obtain

$$\|m\|_{L^{\hat{q}}(\mathbb{R}^n)} \leq \|m\|_{L^\beta(\mathbb{R}^n)}^\tau \|m\|_{L^1(\mathbb{R}^n)}^{1-\tau} \leq \mathcal{S}_{\hat{q},r}^{\frac{1}{r'-\theta}} \left(C_{\hat{q}} \Lambda_r^{\frac{1}{r}} \right)^{\frac{\theta}{r'-\theta}} \|m\|_{L^1(\mathbb{R}^n)}^{\frac{1-\theta}{r'-\theta} + \frac{1}{r'}}. \quad (3.29)$$

Moreover, using (1.18) and (3.26) again, we obtain (3.22)

From (3.28) and (3.29) we obtain (3.20).

Now, we focus on the estimates of w . Noting that for any $\nu \in [1, \hat{q}]$, we have $\frac{r}{\nu} > \frac{r}{\hat{q}} > 1$. Then, by Hölder's inequality,

$$\int_{\mathbb{R}^n} |w|^\nu dx = \int_{\mathbb{R}^n} |w|^\nu |m|^{-\frac{(r-1)\nu}{r}} |m|^{\frac{(r-1)\nu}{r}} dx \leq \left(\int_{\mathbb{R}^n} |m| \left| \frac{w}{m} \right|^r dx \right)^{\frac{\nu}{r}} \left(\int_{\mathbb{R}^n} |m|^{\frac{r-1}{r-\nu}} dx \right)^{\frac{r-\nu}{r}},$$

which implies

$$\|w\|_{L^\nu(\mathbb{R}^n)} \leq \Lambda_r^{\frac{1}{r}} \|m\|_{L^{\frac{r-1}{r-\nu}}(\mathbb{R}^n)}^{\frac{r-1}{r}} \text{ for all } \nu \in [1, \hat{q}]. \quad (3.30)$$

Choosing $\nu = \hat{q}$, it follows from (3.17) and (3.27) that

$$\|w\|_{L^{\hat{q}}(\mathbb{R}^n)} \leq \Lambda_r^{\frac{1}{r}} \|m\|_{L^\beta(\mathbb{R}^n)}^{\frac{r-1}{r}} \leq \Lambda_r^{\frac{1}{r}} (\mathcal{S}_{\hat{q},r})^{\frac{1}{r'-\theta}} \left(C_{\hat{q}} \Lambda_r^{\frac{1}{r}} \right)^{\frac{\theta}{r'-\theta}} \|m\|_{L^1(\mathbb{R}^n)}^{\frac{1-\theta}{r'-\theta}}. \quad (3.31)$$

Taking $\nu = 1$ in (3.30), together with (3.31), we obtain (3.21). We complete the proof of this lemma. \square

By the same arguments of (3.17), (3.23) and (3.24), we have the following corollary:

Corollary 1. Assume that $(m, w) \in (L^1(\mathbb{R}^n) \cap L^{1+\alpha}(\mathbb{R}^n) \cap W^{1,q}(\mathbb{R}^n)) \times L^1(\mathbb{R}^n)$ be the solution to (3.16) with

$$\frac{1}{q} = \frac{1}{r} + \frac{1}{r'(1+\alpha)}.$$

Then for $\alpha \in (0, \frac{r}{n}]$, there exists a positive constant C depending only on n and α such that

$$\|\nabla m\|_{L^q(\mathbb{R}^n)} \leq C \left(m \left| \frac{w}{m} \right|^r dx \right)^{\frac{1}{r}} \|m\|_{L^{1+\alpha}}^{\frac{1}{r}}. \quad (3.32)$$

Moreover, there exists a positive constant C only depending on r, n and α such that

$$\|m\|_{L^{1+\alpha}(\mathbb{R}^n)}^{1+\alpha} \leq C \left(\int_{\mathbb{R}^n} m dx \right)^{\frac{(\alpha+1)r-n\alpha}{r}} \left(\int_{\mathbb{R}^n} m \left| \frac{w}{m} \right|^r dx \right)^{\frac{n\alpha}{r}}. \quad (3.33)$$

Next, we turn our attention to system (1.4), a coupled system consisting of a HJ equation and a Fokker-Planck Equation. Indeed, with some assumptions imposed on population density m and Lagrange multiplier λ , we have the following lemma for the decay property of m :

Lemma 3.6 (C.f. Proposition 5.3 in [12]). Assume that $(u, \lambda, m) \in C^2(\mathbb{R}^n) \times \mathbb{R} \times (W^{1,p}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))$ with u bounded from below, $p > n$ and $\lambda < 0$ is the solution of the following potential-free problem

$$\begin{cases} -\Delta u + C_H |\nabla u|^{r'} + \lambda = -m^\alpha, & x \in \mathbb{R}^n, \\ \Delta m + C_H r' \nabla \cdot (m |\nabla u|^{r'-2} \nabla u) = 0, & x \in \mathbb{R}^n. \end{cases} \quad (3.34)$$

Then, there exist $\kappa_1, \kappa_2 > 0$ such that

$$m(x) \leq \kappa_1 e^{-\kappa_2 |x|} \text{ for all } x \in \mathbb{R}^n. \quad (3.35)$$

Proof. Since $p > n$, we see that $m \in W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\theta}(\mathbb{R}^n)$ for some $\theta \in (0, 1)$, which indicates $m \rightarrow 0$ as $|x| \rightarrow +\infty$. In addition, noting $-\lambda > 0$, we obtain

$$\liminf_{|x| \rightarrow \infty} (-m^\alpha - \lambda) \geq -\frac{\lambda}{2} > 0, \quad (3.36)$$

which satisfies (3.15). Now, we fix $u(0) = 0 \leq u(x)$ for $x \in \mathbb{R}^n$ and deduce from (3.5) with $b = 0$ that

$$|\nabla u(x)| \leq C_1, \quad x \in \mathbb{R}^n \text{ for some } C_1 > 0. \quad (3.37)$$

To show (3.35), we consider the Lyapunov function $\Phi(x) = e^{\kappa u}$ with $0 < \kappa < -\frac{\lambda}{4}$. By using the u -equation in (3.34) we obtain from (3.36) and (3.5) that, $\exists R > 0$ large enough such that for $|x| > R$,

$$\begin{aligned} -\Delta \Phi + C_H r' |\nabla u|^{r'-2} \nabla u \cdot \nabla \Phi &= \kappa (C_H (r' - 1) |\nabla u|^{r'} - \lambda - \kappa |\nabla u|^2 - m^\alpha) \Phi \\ &\geq \kappa (-\lambda - \kappa |\nabla u|^2 - m^\alpha) \Phi \geq -\frac{\kappa \lambda}{4} \Phi. \end{aligned}$$

Then by using (3.37), as shown in [12], we finish the proof of (3.35). \square

We next collect the Pohozaev identities satisfied by the solution to (3.34) in the following lemma:

Lemma 3.7 (C.f. Proposition 3.1 in [14]). Let (u, λ, m) satisfy the assumptions of Lemma 3.6 and denote $w = -C_H r' m |\nabla u|^{r'-2} \nabla u$. Then the following identities hold:

$$\begin{cases} \lambda \int_{\mathbb{R}^n} m dx = -\frac{(\alpha+1)r-n\alpha}{(\alpha+1)r} \int_{\mathbb{R}^n} m^{\alpha+1} dx, \\ C_L \int_{\mathbb{R}^n} m \left| \frac{w}{m} \right|^r dx = \frac{n\alpha}{(\alpha+1)r} \int_{\mathbb{R}^n} m^{\alpha+1} dx = (r' - 1) C_H \int_{\mathbb{R}^n} m |\nabla u|^{r'} dx. \end{cases} \quad (3.38)$$

Proof. From Lemma 3.6 we see that $m \leq \kappa_1 e^{-\kappa_2|x|}$ is exponential decay. In addition, one can obtain from (3.37) that there exists $R > 0$ such that $|u| \leq C|x|$ for $|x| > R$. It is necessary to mention that if $1 < r' < 2$, the Fokker-Planck equation holds in the weak sense. In this case, we take the approximation argument and let $H_\epsilon(p) := C_H(\epsilon + |p|^2)^{\frac{r'}{2}}$ to approximate H given by (1.5). After performing the computations on m_ϵ , we take the limit $\epsilon \rightarrow 0$ to obtain our desired conclusion.

We test the u -equation in (3.34) against m and integrate it by parts to obtain

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla m \, dx + C_H \int_{\mathbb{R}^n} |\nabla u|^{r'} m \, dx + \lambda \int_{\mathbb{R}^n} m \, dx = - \int_{\mathbb{R}^n} m^{\alpha+1} \, dx. \quad (3.39)$$

Similarly, multiplying the m -equation in (4.4) by u , we integrate it to find

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla m \, dx = -C_H r' \int_{\mathbb{R}^n} m |\nabla u|^{r'} \, dx. \quad (3.40)$$

Subtracting (3.39) from (3.40), we arrive at

$$(1 - r') C_H \int_{\mathbb{R}^n} m |\nabla u|^{r'} \, dx + \lambda \int_{\mathbb{R}^n} m \, dx = - \int_{\mathbb{R}^n} m^{\alpha+1} \, dx. \quad (3.41)$$

We next prove that

$$-n\lambda \int_{\mathbb{R}^n} m \, dx - \frac{n}{\alpha+1} \int_{\mathbb{R}^n} m^{\alpha+1} \, dx + C_H \frac{n-r}{r-1} \int_{\mathbb{R}^n} m |\nabla u|^{r'} \, dx = 0, \quad (3.42)$$

Multiply the u -equation in (3.34) by $\nabla m \cdot x$ and integrate it by parts to get

$$\begin{aligned} \int_{\mathbb{R}^n} (-m^\alpha - \lambda) \nabla m \cdot x \, dx &= - \int_{\mathbb{R}^n} \Delta u (\nabla m \cdot x) \, dx + C_H \int_{\mathbb{R}^n} |\nabla u|^{r'} (\nabla m \cdot x) \, dx \\ &= \underbrace{\int_{\mathbb{R}^n} \nabla u \cdot \nabla (\nabla m \cdot x) \, dx}_{I_1} - C_H \int_{\mathbb{R}^n} \nabla \cdot (|\nabla u|^{r'} x) m \, dx. \end{aligned} \quad (3.43)$$

Test the m -equation in (3.34) against $\nabla u \cdot x$, then we use the integration by parts to obtain

$$-C_H \int_{\mathbb{R}^n} \nabla (|\nabla u|^{r'}) \cdot x m \, dx = \int_{\mathbb{R}^n} \nabla m \cdot \nabla (\nabla u \cdot x) \, dx + C_H r' \int_{\mathbb{R}^n} |\nabla u|^{r'} m \, dx, \quad (3.44)$$

where we have used

$$C_H \nabla (|\nabla u|^{r'}) \cdot x = C_H r' |\nabla u|^{r'-2} u_{x_i} u_{x_i x_j} x_j = C_H r' |\nabla u|^{r'-2} \nabla u \cdot \nabla (\nabla u \cdot x) - r' C_H |\nabla u|^{r'}.$$

For I_1 , we have from the integration by parts that

$$\begin{aligned} \int_{\mathbb{R}^n} \nabla u \cdot \nabla (\nabla m \cdot x) \, dx &= \int_{\mathbb{R}^n} u_{x_i} m_{x_i x_j} x_j \, dx + \int_{\mathbb{R}^n} \nabla u \cdot \nabla m \, dx \\ &= - \int_{\mathbb{R}^n} m_{x_i} u_{x_i x_j} x_j \, dx + (1-n) \int_{\mathbb{R}^n} \nabla u \cdot \nabla m \, dx \\ &= - \int_{\mathbb{R}^n} \nabla m \cdot \nabla (\nabla u \cdot x) \, dx + (2-n) \int_{\mathbb{R}^n} \nabla u \cdot \nabla m \, dx, \end{aligned} \quad (3.45)$$

Combining (3.43), (3.44) and (3.45), one finds

$$\underbrace{\int_{\mathbb{R}^n} (-m^\alpha - \lambda) \nabla m \cdot x \, dx}_{I_2} = C_H (r' - n) \int_{\mathbb{R}^n} |\nabla u|^{r'} m \, dx + (2-n) \int_{\mathbb{R}^n} \nabla u \cdot \nabla m \, dx. \quad (3.46)$$

For I_2 , we integrate by parts again to find

$$I_2 = n \int_{\mathbb{R}^n} \left(\frac{1}{\alpha + 1} m^{\alpha+1} + \lambda m \right) dx. \quad (3.47)$$

By using (3.46) and (3.47), we have shown

$$-n\lambda \int_{\mathbb{R}^n} m dx - \frac{n}{\alpha + 1} \int_{\mathbb{R}^n} m^{\alpha+1} dx + C_H(r' - n) \int_{\mathbb{R}^n} |\nabla u|^{r'} m dx + (2 - n) \int_{\mathbb{R}^n} \nabla u \cdot \nabla m dx = 0,$$

This together with (3.40) indicates (3.42). Since $w = -C_H r' m |\nabla u|^{r'-2} \nabla u$ and $C_L = \frac{1}{r'} (r' C_H)^{\frac{1}{1-r'}}$, one gets

$$C_L \int_{\mathbb{R}^n} m \left| \frac{w}{m} \right|^{r'} dx = C_L (C_H r')^r \int_{\mathbb{R}^n} m |\nabla u|^{r'} dx = (r' - 1) C_H \int_{\mathbb{R}^n} m |\nabla u|^{r'} dx. \quad (3.48)$$

Finally, (3.38) follows directly from (3.41), (3.42) and (3.48). \square

Now, we are ready to show Theorem 1.2, the attainability of problem (1.24). We would like to recall that minimization problem (1.24) is equivalent to (1.26).

4 Gagliardo-Nirenberg Type Inequality: Potential-free MFGs

This section is devoted to the existence of minimizers to problem (1.26). Before studying the mass critical case, we consider the case of $\alpha \in (0, \frac{r}{n})$ and recall that with this condition, Cirant et al. [12] in Theorem 1.3 showed for any $M > 0$, the following minimization problem is attained by pair $(\bar{m}_{\alpha, M}, \bar{w}_{\alpha, M})$,

$$e_{0, \alpha, M} := \inf_{(m, w) \in \mathcal{A}_M} \mathcal{E}_0(m, w) \quad \text{where} \quad \mathcal{E}_0(m, w) = \int_{\mathbb{R}^n} \left(C_L m \left| \frac{w}{m} \right|^{r'} - \frac{1}{\alpha + 1} m^{\alpha+1} \right) dx. \quad (4.1)$$

Moreover, $\bar{m}_{\alpha, M} \in W^{1, p}(\mathbb{R}^n) \forall p \geq 1$ satisfies

$$0 < \bar{m}_{\alpha, M} < c_{1, M} e^{-c_{2, M} |x|} \quad \text{for some } c_{1, M}, c_{2, M} > 0. \quad (4.2)$$

and

$$\exists \bar{u}_{\alpha, M} \in C^2(\mathbb{R}^n) \text{ bounded from below s. t. } \bar{w}_{\alpha, M} = -C_H r' \bar{m}_{\alpha, M} |\nabla \bar{u}_{\alpha, M}|^{r'-2} \nabla \bar{u}_{\alpha, M}. \quad (4.3)$$

In addition, $(\bar{m}_{\alpha, M}, \bar{u}_{\alpha, M})$ solves the following equation with some $\lambda_{\alpha, M} < 0$

$$\begin{cases} -\Delta u + C_H |\nabla u|^{r'} + \lambda = -m^\alpha, & x \in \mathbb{R}^n, \\ \Delta m + C_H r' \nabla \cdot (m |\nabla u|^{r'-2} \nabla u) = 0, & x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} m dx = M, \end{cases} \quad (4.4)$$

and from Lemma 3.7 we have the following Pohozaev identities

$$\begin{cases} \lambda \int_{\mathbb{R}^n} \bar{m}_{\alpha, M} dx = -\frac{(\alpha+1)r-n\alpha}{(\alpha+1)r} \int_{\mathbb{R}^n} \bar{m}_{\alpha, M}^{\alpha+1} dx, \\ C_L \int_{\mathbb{R}^n} \bar{m}_{\alpha, M} \left| \frac{\bar{w}_{\alpha, M}}{\bar{m}_{\alpha, M}} \right|^{r'} dx = \frac{n\alpha}{(\alpha+1)r} \int_{\mathbb{R}^n} \bar{m}_{\alpha, M}^{\alpha+1} dx = (r' - 1) C_H \int_{\mathbb{R}^n} \bar{m}_{\alpha, M} |\nabla \bar{u}_{\alpha, M}|^{r'} dx. \end{cases} \quad (4.5)$$

We next show that $(\bar{m}_{\alpha, M}, \bar{w}_{\alpha, M})$, the minimizer of (4.1), is also a minimizer of problem (1.26).

Lemma 4.1. *For any fixed $\alpha \in (0, \frac{r}{n})$ and $M > 0$, problem (1.24) is attained by $(\bar{m}_{\alpha, M}, \bar{w}_{\alpha, M})$ with $e_{0, \alpha, M} = \mathcal{E}_0(\bar{m}_{\alpha, M}, \bar{w}_{\alpha, M})$. Moreover, we have*

$$\Gamma_\alpha = \frac{n\alpha(-e_{0, \alpha, M})^{\frac{n\alpha-r}{r}} M^{\frac{(\alpha+1)r-n\alpha}{r}} \left(\frac{r-n\alpha}{n\alpha} \right)^{\frac{r-n\alpha}{r}}}{r(1+\alpha)}. \quad (4.6)$$

Proof. For simplicity, we denote

$$G_\alpha(m, w) := \frac{\left(C_L \int_{\mathbb{R}^n} m \left|\frac{w}{m}\right|^r dx\right)^{\frac{n\alpha}{r}} \left(\int_{\mathbb{R}^n} m dx\right)^{\frac{(\alpha+1)r-n\alpha}{r}}}{\int_{\mathbb{R}^n} m^{\alpha+1} dx}, \quad (4.7)$$

and because of (1.26), we rewrite minimization problem (4.1) as

$$\Gamma_\alpha = \inf_{(m,w) \in \mathcal{A}_M} G_\alpha(m, w). \quad (4.8)$$

To show (4.8) is attained by $(\bar{m}_{\alpha,M}, \bar{w}_{\alpha,M})$, we first analyze the lower bound of \mathcal{E}_0 defined by (4.1). For this purpose, we first note that $\mathcal{E}_0(m, w) = G_\alpha(m, w) = \infty$ provided that $\int_{\mathbb{R}^n} m \left|\frac{w}{m}\right|^r dx = +\infty$. Therefore, we just need to consider the case that $(m, w) \in \mathcal{A}_M$ satisfying $\int_{\mathbb{R}^n} m \left|\frac{w}{m}\right|^r dx < \infty$. We define $(m_\mu(x), w_\mu(x)) = (\mu^n m(\mu x), \mu^{n+1} w(\mu x))$ for $\mu \in \mathbb{R}^+ \setminus \{0\}$, then substitute the pair into \mathcal{E}_0 to obtain

$$\begin{aligned} \mathcal{E}_0(m_\mu, w_\mu) &= \mu^r \int_{\mathbb{R}^n} C_L m \left|\frac{w}{m}\right|^r dx - \frac{\mu^{n\alpha}}{\alpha+1} \int_{\mathbb{R}^n} m^{1+\alpha} dx \\ &\geq -\left(\frac{n\alpha}{r}\right)^{\frac{r}{r-n\alpha}} \left[\frac{r-n\alpha}{n\alpha}\right] \left(\frac{1}{\alpha+1} \int_{\mathbb{R}^n} m^{\alpha+1} dx\right)^{\frac{r}{r-n\alpha}} \left(C_L \int_{\mathbb{R}^n} m \left|\frac{w}{m}\right|^r dx\right)^{-\frac{n\alpha}{r-n\alpha}}, \end{aligned} \quad (4.9)$$

where the equality holds if and only if

$$\mu = \left[\frac{n\alpha \int_{\mathbb{R}^n} m^{\alpha+1} dx}{(\alpha+1) C_L \int_{\mathbb{R}^n} m \left|\frac{w}{m}\right|^r dx} \right]^{\frac{1}{r-n\alpha}}.$$

Recall the definition of $e_{0,\alpha,M} := \inf_{(m,w) \in \mathcal{A}_M} \mathcal{E}_0(m, w)$, then we find from (4.9) that

$$-\left(\frac{n\alpha}{r}\right)^{\frac{r}{r-n\alpha}} \left(\frac{r-n\alpha}{n\alpha}\right) \left(\frac{1}{\alpha+1} \int_{\mathbb{R}^n} m^{\alpha+1} dx\right)^{\frac{r}{r-n\alpha}} \left(C_L \int_{\mathbb{R}^n} m \left|\frac{w}{m}\right|^r dx\right)^{-\frac{n\alpha}{r-n\alpha}} \geq e_{0,\alpha,M},$$

which implies

$$\frac{\left(C_L \int_{\mathbb{R}^n} m \left|\frac{w}{m}\right|^r dx\right)^{\frac{n\alpha}{r-n\alpha}}}{\left(\frac{1}{\alpha+1} \int_{\mathbb{R}^n} m^{\alpha+1} dx\right)^{\frac{r}{r-n\alpha}}} \geq (-e_{0,\alpha,M})^{-1} \left(\frac{n\alpha}{r}\right)^{\frac{r}{r-n\alpha}} \left(\frac{r-n\alpha}{n\alpha}\right). \quad (4.10)$$

By using the definition G_α given in (4.7), one obtains from (4.10) that

$$G_\alpha(m, w) = \frac{\left(C_L \int_{\mathbb{R}^n} m \left|\frac{w}{m}\right|^r dx\right)^{\frac{n\alpha}{r}} \left(\int_{\mathbb{R}^n} m dx\right)^{\frac{(\alpha+1)r-n\alpha}{r}}}{\int_{\mathbb{R}^n} m^{\alpha+1} dx} \geq \frac{H_{\alpha,M}}{\alpha+1} M^{\frac{(\alpha+1)r-n\alpha}{r}}, \quad (4.11)$$

where

$$\int_{\mathbb{R}^n} m dx = M, \quad H_{\alpha,M} := \frac{n\alpha}{r} (-e_{0,\alpha,M})^{\frac{n\alpha-r}{r}} \left(\frac{r-n\alpha}{n\alpha}\right)^{\frac{r-n\alpha}{r}}. \quad (4.12)$$

Recall that $(\bar{m}_{\alpha,M}, \bar{w}_{\alpha,M})$ is a minimizer of problem (4.1), then we apply (4.5) to get

$$G_\alpha(\bar{m}_{\alpha,M}, \bar{w}_{\alpha,M}) = \frac{H_{\alpha,M}}{\alpha+1} M^{\frac{(\alpha+1)r-n\alpha}{r}}. \quad (4.13)$$

Combining (4.11) with (4.13), one can see that (4.8) is attained by $(\bar{m}_{\alpha,M}, \bar{w}_{\alpha,M})$. Moreover, noting that $H_{\alpha,M}$ is defined by (4.12), we have (4.6) holds. \square

With the aid of Lemma 4.1, one can use (4.5) to establish the relationship between Lagrange multiplier λ and Γ_α defined by (4.1). Indeed, invoking (4.5) and (4.6), we get that

$$e_{0,\alpha,M} = \frac{n\alpha - r}{(\alpha + 1)r - n\alpha} \lambda M, \quad (4.14)$$

and

$$\lambda M = -S_{\alpha,M} \frac{(\alpha + 1)r - n\alpha}{n\alpha} \left(\frac{n\alpha}{r} \right)^{\frac{r}{r-n\alpha}}, \quad S_{\alpha,M} := \left[\frac{M^{\frac{(\alpha+1)r-n\alpha}{r}}}{(1+\alpha)\Gamma_\alpha} \right]^{\frac{r}{r-n\alpha}}. \quad (4.15)$$

Lemma 4.1 demonstrates that for all $M > 0$, Gagliardo-Nirenberg type inequalities given by (1.26) can be attained under the mass subcritical exponent case $\alpha \in (0, \frac{r}{n})$. In addition, λ , Γ_α and M satisfy (4.15). Next, we shall investigate the mass critical exponent case and prove Theorem 1.2. To begin with, we show that Γ_α defined in (1.26) is uniformly bounded as $\alpha \nearrow \frac{r}{n}$.

Lemma 4.2. *There exist positive constants C_1 and C_2 independent of α such that, for all $\alpha \in (\frac{r}{n} - \epsilon, \frac{r}{n}]$ with $\epsilon > 0$ small,*

$$0 < C_1 \leq \Gamma_\alpha \leq C_2. \quad (4.16)$$

Proof. To establish the upper bound uniformly in α , we set $\tilde{m} = e^{-|x|}$ with $\tilde{w} = \nabla \tilde{m}$. Noting that $(\tilde{m}, \tilde{w}) \in \mathcal{A}$ for any $\alpha \in (0, \frac{r}{n})$, one has

$$\Gamma_\alpha \leq G_\alpha(\tilde{m}, \tilde{w}) = \frac{\left(C_L \int_{\mathbb{R}^n} \tilde{m} \left| \frac{\tilde{w}}{\tilde{m}} \right|^r dx \right)^{\frac{n\alpha}{r}} \left(\int_{\mathbb{R}^n} \tilde{m} dx \right)^{\frac{(\alpha+1)r-n\alpha}{r}}}{\int_{\mathbb{R}^n} \tilde{m}^{\alpha+1} dx} \leq C_2(C_L, r) < +\infty. \quad (4.17)$$

It is left to establish the lower bound satisfied by Γ_α uniformly in α . To this end, we argue by contradiction and assume

$$\liminf_{\alpha \nearrow \frac{r}{n}} \Gamma_\alpha = 0. \quad (4.18)$$

Because of Lemma 4.1, we denote $(m_\alpha, w_\alpha) \in \mathcal{A}$ as a minimizer of problem (1.24). Since (1.24) is invariant under the scaling $s(t^n m(tx), t^{n+1} w(tx))$ for any $s > 0$ and $t > 0$, we normalize m_α to get

$$\int_{\mathbb{R}^n} m_\alpha dx = \int_{\mathbb{R}^n} m_\alpha^{\alpha+1} dx \equiv 1. \quad (4.19)$$

Then it follows from (4.18) and (1.24) that as $\alpha \nearrow \frac{r}{n}$,

$$\int_{\mathbb{R}^n} m_\alpha \left| \frac{w_\alpha}{m_\alpha} \right|^r dx \rightarrow 0. \quad (4.20)$$

We claim that there exists $\alpha_0 \in (0, \frac{r}{n})$ such that

$$q_{\alpha_0} < 1 + \frac{r}{n} \quad \text{and} \quad q_{\alpha_0}^* > 1 + \frac{r}{n}. \quad (4.21)$$

where q_α and q_α^* with $\alpha > 0$ are defined by

$$\frac{1}{q_\alpha} := \frac{1}{r} + \frac{1}{(1+\alpha)r'} \quad \text{and} \quad q_\alpha^* := \begin{cases} \frac{nq_\alpha}{n-q_\alpha}, & q_\alpha < n, \\ +\infty, & q_\alpha \geq n. \end{cases} \quad (4.22)$$

To prove our claim, we choose $\alpha = \alpha^* := \frac{r}{n}$ and compute $q_{\alpha^*} = \frac{n+r}{n+1} < 1 + \frac{r}{n}$. Moreover, one finds if $q_{\alpha^*} < n$, then $q_{\alpha^*}^* = \frac{n(n+r)}{n^2-r} > 1 + \frac{r}{n}$; otherwise if $q_{\alpha^*} \geq n$, then $q_{\alpha^*}^* = +\infty > 1 + \frac{r}{n}$. Hence, by the continuity of q_α and q_α^* with respect to α , we finish the proof of claim (4.21).

With this claim, we invoke Corollary 1 and choose $\alpha = \alpha_0$ in (3.32) to get

$$\|\nabla m_\alpha\|_{L^{q\alpha_0}(\mathbb{R}^n)} \leq \bar{C}_{\alpha_0} \left(\int_{\mathbb{R}^n} m_\alpha \left| \frac{w_\alpha}{m_\alpha} \right|^r dx \right)^{\frac{1}{r}} \|m_\alpha\|_{L^{1+\alpha_0}(\mathbb{R}^n)}^{\frac{1}{r}}. \quad (4.23)$$

Noting that as $\alpha \nearrow \frac{r}{n}$, $1 < 1 + \alpha_0 < 1 + \alpha$. By Hölder's inequality and (4.19), we have

$$\limsup_{\alpha \nearrow \frac{r}{n}} \|m_\alpha\|_{L^{1+\alpha_0}(\mathbb{R}^n)} \leq C, \text{ where } C > 0 \text{ does not depend on } \alpha. \quad (4.24)$$

Invoking Gagliardo-Nirenberg's inequality, we obtain from (4.20), (4.23) and (4.24) that

$$\|m_\alpha\|_{L^{1+\frac{r}{n}}(\mathbb{R}^n)} \leq C_{\alpha_0} \|\nabla m_\alpha\|_{L^{q\alpha_0}(\mathbb{R}^n)}^\theta \|m_\alpha\|_{L^1(\mathbb{R}^n)}^{1-\theta} \leq \tilde{C}_{\alpha_0} \left(\int_{\mathbb{R}^n} m_\alpha \left| \frac{w_\alpha}{m_\alpha} \right|^r dx \right)^{\frac{\theta}{r}} \rightarrow 0, \text{ as } \alpha \nearrow \frac{r}{n},$$

where $\theta \in (0, 1)$ and $\tilde{C}_{\alpha_0} > 0$ are independent of α . Recall (4.19) and thanks to Hölder's inequality, one has

$$\|m_\alpha\|_{L^{\alpha+1}(\mathbb{R}^n)} \leq \|m_\alpha\|_{L^1(\mathbb{R}^n)}^{1-\theta_\alpha} \|m_\alpha\|_{L^{1+\frac{r}{n}}(\mathbb{R}^n)}^{\theta_\alpha} = \|m_\alpha\|_{L^{1+\frac{r}{n}}(\mathbb{R}^n)}^{\theta_\alpha} \rightarrow 0, \text{ as } \alpha \nearrow \frac{r}{n},$$

where constants $\theta_\alpha \rightarrow 1$, which reaches a contradiction to (4.19). Thus, there exists $C_1 > 0$ independent of α such that

$$0 < C_1 \leq \Gamma_\alpha. \quad (4.25)$$

Combining (4.25) with (4.17), we complete the proof of (4.16). \square

With the uniform boundedness of Γ_α , we next establish the uniform L^∞ bound of m_α as $\alpha \nearrow \frac{r}{n}$, which is

Lemma 4.3. *Let $(u_\alpha, \lambda_\alpha, m_\alpha) \in C^2(\mathbb{R}^n) \times \mathbb{R} \times W^{1,p}(\mathbb{R}^n)$, $\forall p > 1$ be the solution of*

$$\begin{cases} -\Delta u + C_H |\nabla u|^{r'} + \lambda = -m^\alpha, & x \in \mathbb{R}^n, \\ -\Delta m - C_{Hr'} \nabla \cdot (m |\nabla u|^{r'-2} \nabla u) = 0, & x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} m dx = M_\alpha. \end{cases} \quad (4.26)$$

Define $w_\alpha = -C_{Hr'} m_\alpha |\nabla u_\alpha|^{r'-2} \nabla u_\alpha$. Assume that each u_α is bounded from below and there exists a constant $C > 0$ independent of α , such that

$$\limsup_{\alpha \nearrow \frac{r}{n}} \int_{\mathbb{R}^n} m_\alpha |\nabla u_\alpha|^{r'} dx \leq C, \quad \lim_{\alpha \nearrow \frac{r}{n}} \int_{\mathbb{R}^n} m_\alpha dx = \lim_{\alpha \nearrow \frac{r}{n}} M_\alpha \leq C, \quad \limsup_{\alpha \nearrow \frac{r}{n}} |\lambda_\alpha| \leq C, \quad (4.27)$$

then there exists $C_1 > 0$ independent of α such that

$$\limsup_{\alpha \nearrow \frac{r}{n}} \|m_\alpha\|_{L^\infty(\mathbb{R}^n)} \leq C_1. \quad (4.28)$$

Proof. Motivated by the argument of [12, Theorem 4.1], to prove (4.28), we argue by contradiction and suppose that up to a subsequence,

$$\mu_\alpha := \|m_\alpha\|_{L^\infty(\mathbb{R}^n)}^{-\frac{1}{n}} \rightarrow 0 \text{ as } \alpha \nearrow \frac{r}{n}. \quad (4.29)$$

Since u_α is bounded from below, we fix $0 = u_\alpha(0) = \inf_{x \in \mathbb{R}^n} u_\alpha(x)$ without loss of generality. Define

$$\bar{u}_\alpha := \mu_\alpha^{\frac{2-r'}{r'-1}} u_\alpha(\mu_\alpha x) + 1, \quad \bar{m}_\alpha := \mu_\alpha^n m_\alpha(\mu_\alpha x) \text{ and } \bar{w}_\alpha := \mu_\alpha^{n+1} w_\alpha(\mu_\alpha x), \quad (4.30)$$

then we have from (4.27) and (4.29) that up to a subsequence,

$$\int_{\mathbb{R}^n} \bar{m}_\alpha dx = \int_{\mathbb{R}^n} m_\alpha dx = M_\alpha, \quad \int_{\mathbb{R}^n} \bar{m}_\alpha^{\alpha+1} dx = \mu_\alpha^{\alpha n} \int_{\mathbb{R}^n} m_\alpha^{\alpha+1} dx \rightarrow 0 \quad \text{as } \alpha \nearrow \frac{r}{n}, \quad (4.31)$$

and

$$\int_{\mathbb{R}^n} \bar{m}_\alpha |\nabla \bar{u}_\alpha|^{r'} dx = \mu_\alpha^r \int_{\mathbb{R}^n} m_\alpha |\nabla u_\alpha|^{r'} dx \rightarrow 0 \quad \text{as } \alpha \nearrow \frac{r}{n}. \quad (4.32)$$

Noting the definition of w_α , (1.6) and (4.30), we have

$$C_L \int_{\mathbb{R}^n} \left| \frac{\bar{w}_\alpha}{\bar{m}_\alpha} \right|^r \bar{m}_\alpha dx = (r' - 1) C_H \int_{\mathbb{R}^n} \bar{m}_\alpha |\nabla \bar{u}_\alpha|^{r'} dx \rightarrow 0, \quad \text{as } \alpha \nearrow \frac{r}{n}. \quad (4.33)$$

In light of (4.29), we obtain from (4.30) that

$$\|\bar{m}_\alpha\|_{L^\infty} \equiv 1. \quad (4.34)$$

This together with (4.31) indicates that for any $q > 1 + \alpha$,

$$\int_{\mathbb{R}^n} \bar{m}_\alpha^q dx \leq \left(\int_{\mathbb{R}^n} \bar{m}_\alpha^{1+\alpha} dx \right) \|\bar{m}_\alpha\|_{L^\infty(\mathbb{R}^n)}^{q-\alpha-1} \rightarrow 0 \quad \text{as } \alpha \nearrow \frac{r}{n}. \quad (4.35)$$

On the other hand, by using (4.30), we have from (4.26) that

$$\begin{cases} -\Delta_x \bar{u}_\alpha + C_H |\nabla_x \bar{u}_\alpha|^{r'} + \lambda_\alpha \mu_\alpha^r = -\mu_\alpha^{r-n\alpha} \bar{m}_\alpha^{\alpha+1}, & x \in \mathbb{R}^n, \\ -\Delta_x \bar{m}_\alpha - C_H r' \nabla_x \cdot (\bar{m}_\alpha |\nabla_x \bar{u}_\alpha|^{r'-2} \nabla_x \bar{u}_\alpha) = 0, & x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} \bar{m}_\alpha dx = M_\alpha. \end{cases} \quad (4.36)$$

In light of (4.27) and (4.29), one finds $\lambda_\alpha \mu_\alpha^r \rightarrow 0$ as $\alpha \nearrow \frac{r}{n}$. In addition, thanks to the fact $r \geq n\alpha$, we obtain

$$0 \leq \mu_\alpha^{r-n\alpha} \|\bar{m}_\alpha^{1+\alpha}\|_{L^\infty(\mathbb{R}^n)} \leq \|\bar{m}_\alpha\|_{L^\infty(\mathbb{R}^n)}^{\alpha+1} \leq 1.$$

Thus, one applies Lemma 3.1 on the \bar{u}_α -equation in (4.36) to arrive at

$$\limsup_{\alpha \nearrow \frac{r}{n}} \|\nabla \bar{u}_\alpha\|_{L^\infty(\mathbb{R}^n)} \leq C < \infty. \quad (4.37)$$

Since $\bar{w}_\alpha = -C_H r' \bar{m}_\alpha |\nabla \bar{u}_\alpha|^{r'-2} \nabla \bar{u}_\alpha$, we deduce from (4.37) that

$$\limsup_{\alpha \nearrow \frac{r}{n}} \|\bar{w}_\alpha\|_{L^\infty(\mathbb{R}^n)} \leq C < \infty. \quad (4.38)$$

Now, we focus on Hölder estimates of \bar{m}_α . By using (4.33) and Hölder's inequality, we get as $\alpha \nearrow \frac{r}{n}$,

$$\int_{\mathbb{R}^n} |\bar{w}_\alpha| dx = \int_{\mathbb{R}^n} |\bar{w}_\alpha| \bar{m}_\alpha^{\frac{1-r}{r}} \bar{m}_\alpha^{\frac{r-1}{r}} dx \leq \left(\int_{\mathbb{R}^n} |\bar{w}_\alpha|^r |\bar{m}_\alpha|^{1-r} dx \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^n} \bar{m}_\alpha dx \right)^{\frac{1}{r}} \rightarrow 0. \quad (4.39)$$

Combining (4.38) with (4.39), one has

$$\int_{\mathbb{R}^n} |\bar{w}_\alpha|^p dx \rightarrow 0 \quad \text{as } \alpha \nearrow \frac{r}{n}, \quad \forall 1 < p < +\infty. \quad (4.40)$$

For any $q > n$, by using the \bar{m}_α -equation in (4.36), we obtain as $\alpha \nearrow \frac{r}{n}$,

$$\left| - \int_{\mathbb{R}^n} \bar{m}_\alpha \Delta \varphi dx \right| = \left| \int_{\mathbb{R}^n} \bar{w}_\alpha \cdot \nabla \varphi dx \right| \leq \left(\int_{\mathbb{R}^n} |\bar{w}_\alpha|^q dx \right)^{\frac{1}{q}} \|\nabla \varphi\|_{L^{q'}} \rightarrow 0, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n). \quad (4.41)$$

With (4.40) and (4.41), one applies Lemma 3.5 to get $\|\nabla \bar{m}_\alpha\|_{L^q} \rightarrow 0$ as $\alpha \nearrow \frac{r}{n}$. Combine this with (4.35) we obtain that, $\|\bar{m}_\alpha\|_{W^{1,q}(\mathbb{R}^n)} \rightarrow 0$ as $\alpha \nearrow \frac{r}{n}$. Thanks to Sobolev embedding theorem, one further has for some $\theta' \in (0, 1)$,

$$\|\bar{m}_\alpha\|_{C^{0,\theta'}(\mathbb{R}^n)} \rightarrow 0 \text{ as } \alpha \nearrow \frac{r}{n}. \quad (4.42)$$

Let x_α be a maximum point of \bar{m}_α , i.e., $\bar{m}_\alpha(x_\alpha) = \|\bar{m}_\alpha\|_{L^\infty(\mathbb{R}^n)} = 1$. Then we obtain from (4.42) that there exists $R_1 > 0$ independent of α such that $|\bar{m}_\alpha(x)| \geq \frac{1}{2}, \forall x \in B_{R_1}(x_\alpha)$. It follows that

$$\left(\frac{1}{2}\right)^{1+\frac{r}{n}} |B_{R_1}| \leq \int_{B_{R_1}(x_\alpha)} \bar{m}_\alpha^{1+\frac{r}{n}} dx \leq \int_{\mathbb{R}^n} \bar{m}_\alpha^{1+\frac{r}{n}} dx,$$

which is contradicted to (4.35). This completes the proof of (4.28). \square

Now, we are ready to show Theorem 1.2 by using the approximation argument, which is stated as follows:

Proof of Theorem 1.2:

Proof. Recall that $(\bar{m}_{\alpha,M}, \bar{w}_{\alpha,M}, \lambda_{\alpha,M})$ denotes the minimizer of problem (4.8) for any $M > 0$, which satisfies system (4.4) and estimates (4.2) and (4.3).

Choosing

$$M = M_\alpha := e^{\frac{r-n\alpha}{(\alpha+1)r-n\alpha}} [(\alpha+1)\Gamma_\alpha]^{\frac{r}{(\alpha+1)r-n\alpha}}$$

in (4.4), one can get from (4.15) that

$$S_{\alpha, M_\alpha} := \left[\frac{M_\alpha^{\frac{(\alpha+1)r-n\alpha}{r}}}{(1+\alpha)\Gamma_\alpha} \right]^{\frac{r}{r-n\alpha}} \equiv e.$$

Then as $\alpha \nearrow \frac{r}{n}$, we find up to a subsequence,

$$\frac{M_\alpha^{\frac{(\alpha+1)r-n\alpha}{r}}}{(\alpha+1)\Gamma_\alpha} \rightarrow 1, \quad \frac{(\alpha+1)r-n\alpha}{r} \rightarrow \frac{r}{n}. \quad (4.43)$$

Without confusing the readers, rewrite $(\bar{m}_{\alpha,M}, \bar{w}_{\alpha,M}, \lambda_{\alpha,M})$ as $(\bar{m}_{\alpha, M_\alpha}, \bar{w}_{\alpha, M_\alpha}, \lambda_{\alpha, M_\alpha})$ since $M = M_\alpha$ depends on α . We also recall from (4.4) that $(\bar{m}_{\alpha, M_\alpha}, \bar{u}_{\alpha, M_\alpha}, \bar{w}_{\alpha, M_\alpha}, \lambda_{\alpha, M_\alpha})$ satisfies

$$\begin{cases} -\Delta u + C_H |\nabla u|^{r'} + \lambda_{\alpha, M_\alpha} = -m^\alpha, & x \in \mathbb{R}^n, \\ \Delta m + C_{Hr'} \nabla \cdot (m |\nabla u|^{r'-2} \nabla u) = 0, & x \in \mathbb{R}^n, \\ w = -C_{Hr'} m |\nabla u|^{r'-2} \nabla u, \int_{\mathbb{R}^n} m dx = M_\alpha. \end{cases} \quad (4.44)$$

By using Lemma 4.2, one can see that as $\alpha \nearrow \frac{r}{n}$, up to a subsequence, $\Gamma_\alpha \rightarrow \bar{\Gamma}_{\alpha^*} := \liminf_{\alpha \nearrow \frac{r}{n}} \Gamma_\alpha > 0$. Moreover,

(4.43) implies $M_\alpha \rightarrow M_{\alpha^*} := M^*$, where

$$M^* = \left[\left(1 + \frac{r}{n}\right) \bar{\Gamma}_{\alpha^*} \right]^{\frac{n}{r}}, \quad \alpha^* = \frac{r}{n}. \quad (4.45)$$

In addition, by using (4.15), we obtain as $\alpha \nearrow \frac{r}{n}$, up to a subsequence,

$$\lambda_{\alpha, M_\alpha} \rightarrow \lambda_{\alpha^*} := -\frac{r}{nM^*}, \quad (4.46)$$

and it follows from (4.5) that

$$\int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} dx = M_\alpha \rightarrow M^* > 0, \quad \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha}^{\alpha+1} dx \rightarrow 1 + \frac{r}{n}, \quad C_L \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} \left| \frac{\bar{w}_{\alpha, M_\alpha}}{\bar{m}_{\alpha, M_\alpha}} \right|^r dx \rightarrow 1. \quad (4.47)$$

According to Lemma 4.3, we obtain from (4.46) and (4.47) that

$$\limsup_{\alpha \nearrow \frac{r}{n}} \|\bar{m}_{\alpha, M_\alpha}\|_{L^\infty(\mathbb{R}^n)} < \infty. \quad (4.48)$$

We then deduce from (3.5) with $b = 0$ that

$$\limsup_{\alpha \nearrow \frac{r}{n}} \|\nabla \bar{u}_{\alpha, M_\alpha}\|_{L^\infty} < \infty. \quad (4.49)$$

Noting from the definition of $\bar{w}_{\alpha, M_\alpha}$, one further has

$$\limsup_{\alpha \nearrow \frac{r}{n}} \|\bar{w}_{\alpha, M_\alpha}\|_{L^\infty} < \infty. \quad (4.50)$$

Similar to the argument of (4.42), we can use (4.47)-(4.50) to get that

$$\limsup_{\alpha \nearrow \frac{r}{n}} \|\bar{m}_{\alpha, M_\alpha}\|_{W^{1,q}(\mathbb{R}^n)} < +\infty \quad \forall q > n, \quad \text{and} \quad \limsup_{\alpha \nearrow \frac{r}{n}} \|\bar{m}_{\alpha, M_\alpha}\|_{C^{0,\tilde{\theta}}(\mathbb{R}^n)} < \infty \quad \text{for some } \tilde{\theta} \in (0, 1). \quad (4.51)$$

since each $\bar{u}_{\alpha, M_\alpha} \in C^2(\mathbb{R}^n)$ is bounded from below, we may assume that $\bar{u}_{\alpha, M_\alpha}(0) = 0 = \inf_{x \in \mathbb{R}^n} \bar{u}_{\alpha, M_\alpha}(x)$. We then deduce from u -equation of (4.44) that $\bar{m}_{\alpha, M_\alpha}^\alpha(0) \geq -\lambda_{\alpha, M_\alpha}$, which together with (4.47) and (4.46) implies that there exist $\delta_1, R > 0$ independent of α such that,

$$\bar{m}_{\alpha, M_\alpha}(x) > \frac{\delta_1}{2} > 0, \quad \text{for } |x| < R. \quad (4.52)$$

We rewrite the u -equation of (4.44) as

$$-\Delta \bar{u}_{\alpha, M_\alpha} = -C_H |\nabla \bar{u}_{\alpha, M_\alpha}|^{r'} + h_\alpha(x) \quad \text{with} \quad h_\alpha(x) := -\lambda_{\alpha, M_\alpha} - \bar{m}_{\alpha, M_\alpha}^\alpha, \quad x \in \mathbb{R}^n \quad (4.53)$$

In light of (4.49), we deduce that $|\bar{u}_{\alpha, M_\alpha}(x)| \leq C(|x| + 1)$ with $C > 0$ independent of α . We then derive from the classical $W^{2,p}$ estimate to derive from (4.48) and (4.49) that, for any $\bar{R} > 0$ and $p > n$,

$$\|\bar{u}_{\alpha, M_\alpha}\|_{W^{2,p}(B_{\bar{R}+1})} \leq C \left(\|\bar{u}_{\alpha, M_\alpha}\|_{L^p(B_{2\bar{R}}(0))} + \|h_\alpha\|_{L^p(B_{2\bar{R}}(0))} + \|\nabla \bar{u}_{\alpha, M_\alpha}\|_{L^p(B_{2\bar{R}}(0))} \right) \leq C_{p, \bar{R}} < \infty,$$

where the constant $C_{p, \bar{R}} > 0$ is independent of α . It then follows from the Sobolev embedding theorem that $\|\bar{u}_{\alpha, M_\alpha}\|_{C^{1,\theta_1}(B_{\bar{R}+1}(0))} \leq C_{\theta_1, \bar{R}} < \infty$ for some $\theta_1 \in (0, 1)$. This combines with (4.51) gives

$$\|\bar{u}_{\alpha, M_\alpha}\|_{C^{0,\theta_2}(B_{\bar{R}+1}(0))} + \|h_\alpha\|_{C^{0,\theta_2}(B_{\bar{R}+1}(0))} \leq C_{\theta_2, \bar{R}} < \infty \quad \text{for some } \theta_2 \in (0, 1)..$$

Then by using Schauder's estimates, we have from (4.53) that

$$\|\bar{u}_{\alpha, M_\alpha}\|_{C^{2,\theta_3}(B_{\bar{R}}(0))} \leq C_{\theta_3, \bar{R}} < \infty, \quad \text{for some } \theta_3 \in (0, 1). \quad (4.54)$$

Now, letting $\bar{R} \rightarrow \infty$ and proceeding the standard diagonalization procedure, we can apply Arzelà-Ascoli theorem to get from (4.51) and (4.54) that there exists $(m_{\alpha^*}, u_{\alpha^*}) \in W^{1,p}(\mathbb{R}^n) \times C^2(\mathbb{R}^n)$ such that

$$\bar{m}_{\alpha, M_\alpha} \rightharpoonup m_{\alpha^*} \quad \text{in } W^{1,p}(\mathbb{R}^n), \quad \text{and} \quad \bar{u}_{\alpha, M_\alpha} \rightarrow u_{\alpha^*} \quad \text{in } C_{\text{loc}}^2(\mathbb{R}^n), \quad \text{as } \alpha \nearrow \frac{r}{n}. \quad (4.55)$$

This together with (4.44) and (4.46) implies that $(m_{\alpha^*}, u_{\alpha^*}) \in W^{1,p}(\mathbb{R}^n) \times C^2(\mathbb{R}^n)$ satisfies

$$\begin{cases} -\Delta u + C_H |\nabla u|^{r'} - \frac{r}{nM^*} = -m_{\alpha^*}^\frac{r}{n}, & x \in \mathbb{R}^n, \\ -\Delta m - r' C_H \nabla \cdot (m |\nabla u|^{r'-2} \nabla u) = 0, & x \in \mathbb{R}^n, \\ w = -C_H r' m |\nabla u|^{r'-2} \nabla u. \end{cases} \quad (4.56)$$

Thanks to (4.52) and Fatou's lemma, one finds that

$$\int_{\mathbb{R}^n} m_{\alpha^*} dx = a \in (0, M^*]. \quad (4.57)$$

Moreover, we deduce from Lemma 3.6 that there exists some $\kappa, C > 0$ such that $m_{\alpha^*}(x) < Ce^{-\kappa|x|}$. In addition, from (4.49) we have $\|\nabla u_{\alpha^*}\|_{L^\infty} < \infty$. Then, by applying Lemma 3.7, Pohozaev identities, one has

$$C_L \int_{\mathbb{R}^n} \left| \frac{w_{\alpha^*}}{m_{\alpha^*}} \right|^r m_{\alpha^*} dx = \frac{n}{n+r} \int_{\mathbb{R}^n} m_{\alpha^*}^{1+\frac{r}{n}} dx. \quad (4.58)$$

Next, we discuss the relationship between $\bar{\Gamma}_{\alpha^*} := \liminf_{\alpha \nearrow \frac{r}{n}} \Gamma_\alpha$ and Γ_{α^*} with $\alpha^* = \frac{r}{n}$. We claim that

$$\bar{\Gamma}_{\alpha^*} = \Gamma_{\frac{r}{n}}. \quad (4.59)$$

To show this, we first note from Lemma 4.1 that

$$\begin{aligned} \Gamma_\alpha &= G_\alpha(\bar{m}_{\alpha, M_\alpha}, \bar{w}_{\alpha, M_\alpha}) = G_{\frac{r}{n}}(\bar{m}_{\alpha, M_\alpha}, \bar{w}_{\alpha, M_\alpha}) \frac{\left(C_L \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} \left| \frac{\bar{w}_{\alpha, M_\alpha}}{\bar{m}_{\alpha, M_\alpha}} \right|^r dx \right)^{\frac{n\alpha}{r}} \left(\int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} dx \right)^{\frac{(\alpha+1)r-n\alpha}{r}} \cdot \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha}^{\frac{r}{n}+1} dx}{\left(C_L \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} \left| \frac{\bar{w}_{\alpha, M_\alpha}}{\bar{m}_{\alpha, M_\alpha}} \right|^r dx \right) \left(\int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} dx \right)^{\frac{r}{n}} \cdot \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha}^{\alpha+1} dx} \\ &\geq \Gamma_{\frac{r}{n}} \frac{\left(C_L \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} \left| \frac{\bar{w}_{\alpha, M_\alpha}}{\bar{m}_{\alpha, M_\alpha}} \right|^r dx \right)^{\frac{n\alpha}{r}} \left(\int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} dx \right)^{\frac{(\alpha+1)r-n\alpha}{r}} \cdot \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha}^{\frac{r}{n}+1} dx}{\left(C_L \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} \left| \frac{\bar{w}_{\alpha, M_\alpha}}{\bar{m}_{\alpha, M_\alpha}} \right|^r dx \right) \left(\int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} dx \right)^{\frac{r}{n}} \cdot \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha}^{\alpha+1} dx}. \end{aligned} \quad (4.60)$$

Invoking (4.47), we deduce that as $\alpha \nearrow \frac{r}{n}$,

$$\frac{\left(C_L \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} \left| \frac{\bar{w}_{\alpha, M_\alpha}}{\bar{m}_{\alpha, M_\alpha}} \right|^r dx \right)^{\frac{n\alpha}{r}} \left(\int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} dx \right)^{\frac{(\alpha+1)r-n\alpha}{r}} \cdot \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha}^{\frac{r}{n}+1} dx}{\left(C_L \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} \left| \frac{\bar{w}_{\alpha, M_\alpha}}{\bar{m}_{\alpha, M_\alpha}} \right|^r dx \right) \left(\int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha} dx \right)^{\frac{r}{n}} \cdot \int_{\mathbb{R}^n} \bar{m}_{\alpha, M_\alpha}^{\alpha+1} dx} \rightarrow 1.$$

Then one takes the limit in (4.60) to get

$$\bar{\Gamma}_{\alpha^*} := \liminf_{\alpha \nearrow \frac{r}{n}} \Gamma_\alpha \geq \Gamma_{\frac{r}{n}}. \quad (4.61)$$

To finish the proof of our claim, it is left to show that the “=” holds in (4.61). On the contrary, if $\Gamma_{\frac{r}{n}} < \bar{\Gamma}_{\alpha^*}$, then by the definition of $\Gamma_{\frac{r}{n}}$, one finds there exists $(\hat{m}, \hat{w}) \in \mathcal{A}$ given in (1.24) such that

$$G_{\frac{r}{n}}(\hat{m}, \hat{w}) \leq \Gamma_{\frac{r}{n}} + \delta < \Gamma_{\frac{r}{n}} + 2\delta < \bar{\Gamma}_{\alpha^*}, \quad (4.62)$$

where $\delta > 0$ is sufficiently small. On the other hand, from the definition of Γ_α , we find

$$\begin{aligned} G_{\frac{r}{n}}(\hat{m}, \hat{w}) &= G_\alpha(\hat{m}, \hat{w}) \frac{\left(C_L \int_{\mathbb{R}^n} \hat{m} \left| \frac{\hat{w}}{\hat{m}} \right|^r dx \right)^{\frac{n\alpha}{r}} \left(\int_{\mathbb{R}^n} \hat{m} dx \right)^{\frac{(\alpha+1)r-n\alpha}{r}} \cdot \int_{\mathbb{R}^n} \hat{m}^{\frac{r}{n}+1} dx}{\left(C_L \int_{\mathbb{R}^n} \hat{m} \left| \frac{\hat{w}}{\hat{m}} \right|^r dx \right) \left(\int_{\mathbb{R}^n} \hat{m} dx \right)^{\frac{r}{n}} \cdot \int_{\mathbb{R}^n} \hat{m}^{\alpha+1} dx} \\ &\geq \Gamma_\alpha \frac{\left(C_L \int_{\mathbb{R}^n} \hat{m} \left| \frac{\hat{w}}{\hat{m}} \right|^r dx \right)^{\frac{n\alpha}{r}} \left(\int_{\mathbb{R}^n} \hat{m} dx \right)^{\frac{(\alpha+1)r-n\alpha}{r}} \cdot \int_{\mathbb{R}^n} \hat{m}^{\frac{r}{n}+1} dx}{\left(C_L \int_{\mathbb{R}^n} \hat{m} \left| \frac{\hat{w}}{\hat{m}} \right|^r dx \right) \left(\int_{\mathbb{R}^n} \hat{m} dx \right)^{\frac{r}{n}} \cdot \int_{\mathbb{R}^n} \hat{m}^{\alpha+1} dx}. \end{aligned} \quad (4.63)$$

Since

$$\frac{\left(C_L \int_{\mathbb{R}^n} \hat{m} \left| \frac{\hat{w}}{\hat{m}} \right|^r dx \right)^{\frac{n\alpha}{r}} \left(\int_{\mathbb{R}^n} \hat{m} dx \right)^{\frac{(\alpha+1)r-n\alpha}{r}} \cdot \int_{\mathbb{R}^n} \hat{m}^{\frac{r}{n}+1} dx}{\left(C_L \int_{\mathbb{R}^n} \hat{m} \left| \frac{\hat{w}}{\hat{m}} \right|^r dx \right) \left(\int_{\mathbb{R}^n} \hat{m} dx \right)^{\frac{r}{n}} \cdot \int_{\mathbb{R}^n} \hat{m}^{\alpha+1} dx} \rightarrow 1 \quad \text{as } \alpha \nearrow \frac{r}{n},$$

we take the limit in (4.62) and (4.63) to obtain

$$\bar{\Gamma}_{\alpha^*} = \liminf_{\alpha \nearrow \frac{r}{n}} \Gamma_{\alpha} \leq \Gamma_{\frac{r}{n}} + \delta < \Gamma_{\frac{r}{n}} + 2\delta \leq \liminf_{\alpha \nearrow \frac{r}{n}} \Gamma_{\alpha},$$

which reaches a contradiction. Now, one has (4.59), i.e., $\bar{\Gamma}_{\alpha^*} = \Gamma_{\frac{r}{n}}$ holds.

Next, we prove $(m_{\alpha^*}, w_{\alpha^*}) \in \mathcal{A}$. Since $(m_{\alpha^*}, w_{\alpha^*})$ solves (4.56) and $m_{\alpha^*} \in C^{0,\theta}(\mathbb{R}^n)$ with $\theta \in (0, 1)$, we conclude from (4.55) and Lemma 3.1 that $u_{\alpha^*} \in C^1(\mathbb{R}^n)$. Then by standard elliptic estimates, the boundedness of $\|\nabla u_{\alpha^*}\|_{L^\infty}$ and the exponentially decaying property of m_{α^*} , one can prove that $(m_{\alpha^*}, w_{\alpha^*}) \in \mathcal{A}$.

On the other hand, we have from (4.47) and (4.61) that

$$\liminf_{\alpha \nearrow \frac{r}{n}} \Gamma_{\alpha} = \frac{n}{n+r} (M^*)^{\frac{r}{n}} = \Gamma_{\frac{r}{n}}, \quad (4.64)$$

where M^* is given in (4.45). Then with $(m_{\alpha^*}, w_{\alpha^*}) \in \mathcal{A}$, it follows from (4.57), (4.58) and (4.64) that

$$\Gamma_{\frac{r}{n}} = \frac{n}{n+r} (M^*)^{\frac{r}{n}} \leq \frac{C_L \int_{\mathbb{R}^n} \left| \frac{w_{\alpha^*}}{m_{\alpha^*}} \right|^r m_{\alpha^*} dx \left(\int_{\mathbb{R}^n} m_{\alpha^*} dx \right)^{\frac{r}{n}}}{\int_{\mathbb{R}^n} m_{\alpha^*}^{1+\frac{r}{n}} dx} = \frac{n}{n+r} a^{\frac{r}{n}} \leq \frac{n}{n+r} (M^*)^{\frac{r}{n}}, \quad (4.65)$$

which indicates $(m_{\alpha^*}, w_{\alpha^*}) \in \mathcal{A}$ is an minimizer of $\Gamma_{\frac{r}{n}}$ as well as

$$\int_{\mathbb{R}^n} m_{\alpha^*} dx = M^* \quad \text{and} \quad \bar{m}_{\alpha, M_{\alpha}} \rightarrow m_{\alpha^*} \quad \text{in } L^1(\mathbb{R}^n) \quad \text{as } \alpha \nearrow \frac{r}{n}.$$

This together with (4.56) indicates (1.28). The proof of Theorem 1.2 is finished. \square

With the proof of Theorem 1.2, we have studied the existence of ground states for potential-free MFG systems and established Gagliardo-Nirenberg type inequality under the mass critical exponent case. In Section 5, 6 and 6.2, we shall apply the inequality to investigate the blow-up behaviors of ground states to problem (1.22) as $M \nearrow M^*$ when $\alpha = \frac{r}{n}$.

5 Existence of Minimizers: Critical Mass Phenomenon

This section is devoted to the proof of Theorem 1.3. More precisely, we intend to prove that the minimization problem (1.22) with energy $\mathcal{E}(m, w)$ being given by (1.23) has a minimizer $(m, w) \in \mathcal{K}_M$ if and only if $M < M^*$, where \mathcal{K}_M is defined by (1.17). In addition, we show that there exists $(u, \lambda) \in C^2(\mathbb{R}^n) \times \mathbb{R}$ such that $(m, u, \lambda) \in W^{1,p}(\mathbb{R}^n) \times C^2(\mathbb{R}^n) \times \mathbb{R}$ is a solution to (1.30) when V is assumed to satisfy (1.20) when $r > 1$. Recall from the definition of Γ_{α^*} given in (1.24) that

$$\int_{\mathbb{R}^n} m^{1+\frac{r}{n}} dx \leq \frac{C_L(1+\frac{r}{n})}{(M^*)^{\frac{r}{n}}} \left(\int_{\mathbb{R}^n} \left| \frac{w}{m} \right|^r m dx \right) \left(\int_{\mathbb{R}^n} m dx \right)^{\frac{r}{n}}, \quad \forall (m, w) \in \mathcal{A}, \quad (5.1)$$

where \mathcal{A} is given by (1.25).

We would like to mention that there exists a threshold of r while proving the existence of minimizers to problem (1.22). Indeed, as shown in Lemma 3.5, if $r > n$, one can show the uniform boundedness of m_k in $L^\infty(\mathbb{R}^n)$ and $C^{0,\theta}(\mathbb{R}^n)$ for some $\theta \in (0, 1)$ if (m_k, w_k) is a minimizing sequence. However, when $r \leq n$, we must follow the procedure shown in [12] to perform the regularization on (1.23) due to the worse regularity of m .

When $r \leq n$, we let $\eta_\epsilon \geq 0$ be the standard mollifier satisfying

$$\int_{\mathbb{R}^n} \eta_\epsilon dx = 1, \quad \text{supp} \eta_\epsilon \in B_\epsilon(0),$$

where $\epsilon > 0$ is sufficiently small, then consider the following auxiliary minimization problem:

$$e_{\epsilon, M} := \inf_{(m, w) \in \mathcal{A}_M} \mathcal{E}_\epsilon(m, w), \quad (5.2)$$

where \mathcal{A}_M is given by (1.27) and

$$\mathcal{E}_\epsilon(m, w) := C_L \int_{\mathbb{R}^n} \left| \frac{w}{m} \right|^r m dx + \int_{\mathbb{R}^n} V(x) m dx - \frac{1}{1 + \frac{r}{n}} \int_{\mathbb{R}^n} (\eta_\epsilon * m)^{1 + \frac{r}{n}} dx. \quad (5.3)$$

With the approximation energy (5.3), we are able to study the existence of minimizers for (1.22) when $r \leq n$ by taking the limit. However, as discussed in [12], it is necessary to study the uniform boundedness of m_ϵ in L^∞ when we assume (m_ϵ, w_ϵ) as a minimizer for (5.3).

Following the strategies shown above, we can prove Conclusion (i) stated in Theorem 1.3 under the case $M < M^*$. We would like to remark that with (1.20b), the assumption (V2) imposed on potential V , the condition $\int_{\mathbb{R}^n} |x|^b m dx < +\infty$ in (1.27) must be satisfied for any minimizer. Next, we state some crucial propositions and lemmas, which will be used in the proof of Theorem 1.3, as follows:

Lemma 5.1. *Let $p^* = \frac{np}{n-p}$ if $1 \leq p < n$ and $p^* = \infty$ if $p \geq n$. Assume that $0 \leq V(x) \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ satisfies $\liminf_{|x| \rightarrow \infty} V(x) = \infty$ and define*

$$\mathcal{W}_{p, V} := \left\{ m \mid m \in W^{1, p}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} V(x) |m| dx < \infty \right\}.$$

Then, the embedding $\mathcal{W}_{p, V} \hookrightarrow L^q(\mathbb{R}^n)$ is compact for any $1 \leq q < p^*$.

Proof. We refer the readers to [42, Theorem XIII.67] or [3, Theorem 2.1] for the detailed discussions. \square

When $r \leq n$, we have the following lemma for the uniform boundedness of $\|m_\epsilon\|_{L^\infty}$:

Lemma 5.2. *Suppose that $V(x)$ is locally Hölder continuous and satisfies (1.20). Let $(u_k, \lambda_k, m_k) \in C^2(\mathbb{R}^n) \times \mathbb{R} \times (L^1(\mathbb{R}^n) \cap L^{1+\alpha^*}(\mathbb{R}^n))$ be solutions to the following systems*

$$\begin{cases} -\Delta u_k + C_H |\nabla u_k|^{r'} + \lambda_k = V - g_k[m_k], & x \in \mathbb{R}^n, \\ \Delta m_k + C_H r' \nabla \cdot (m_k |\nabla u_k|^{r'-2} \nabla u_k) = 0, & x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} m_k dx = M, \end{cases} \quad (5.4)$$

where $\alpha^* = \frac{r}{n}$ with $1 < r \leq n$, $g_k : L^1(\mathbb{R}^n) \mapsto L^1(\mathbb{R}^n)$ with $\theta \in (0, 1)$ satisfies for all $m \in L^p(\mathbb{R}^n)$, $p \in [1, \infty]$ and $k \in \mathbb{N}$,

$$\|g_k[m]\|_{L^p(\mathbb{R}^n)} \leq K \left(\|m^{\alpha^*}\|_{L^p(\mathbb{R}^n)} + 1 \right) \text{ for some } K > 0, \quad (5.5)$$

and

$$\|g_k[m]\|_{L^p(B_R(x_0))} \leq K \left(\|m^{\alpha^*}\|_{L^p(B_{2R}(x_0))} + 1 \right) \text{ for any } R > 0 \text{ and } x_0 \in \mathbb{R}^n. \quad (5.6)$$

Assume that

$$\sup_k \|m_k\|_{L^1(\mathbb{R}^n)} < \infty, \quad \sup_k \|m_k\|_{L^{1+\alpha^*}(\mathbb{R}^n)} < \infty, \quad \sup_k \int_{\mathbb{R}^n} V m_k dx < \infty, \quad \sup_k |\lambda_k| < \infty, \quad (5.7)$$

and for all k , u_k is bounded from below uniformly. Then we have

$$\limsup_{k \rightarrow \infty} \|m_k\|_{L^\infty(\mathbb{R}^n)} < \infty. \quad (5.8)$$

Proof. We first establish the local uniformly estimates of m_k . To begin with, one finds from (5.7) and (5.5) that

$$\|g_k[m]\|_{L^{1+\frac{1}{\alpha^*}}(\mathbb{R}^n)} \leq K\left(\|m^{\alpha^*}\|_{L^{1+\frac{1}{\alpha^*}}(\mathbb{R}^n)} + 1\right) < \infty. \quad (5.9)$$

Then by maximal regularities shown in Theorem 1.1 and the uniformly local Hölder continuity of V , we have

$$\sup_k \|\nabla u_k\|_{L^{1+\frac{1}{\alpha}}_{\text{loc}}(\mathbb{R}^n)} < \infty, \quad (5.10)$$

which implies

$$\sup_k \|\nabla u_k\|_{L^{(1+\frac{1}{\alpha^*})r}_{\text{loc}}(\mathbb{R}^n)}^{r'-1} < \infty \text{ with } \left(1 + \frac{1}{\alpha^*}\right)r > n. \quad (5.11)$$

Focusing on Fokker-Planck equations, one applies Theorem 1.6.5 in [6] to obtain that

$$\sup_k \|m_k\|_{L^\infty_{\text{loc}}(\mathbb{R}^n)} < \infty, \quad (5.12)$$

which is local uniformly estimates satisfied by m_k .

Next, we claim that

$$\lim_{R \rightarrow +\infty} \sup_k \left\| \frac{m_k^{\alpha^*}}{V} \right\|_{L^\infty(\mathbb{R}^n \setminus B_R(0))} = 0. \quad (5.13)$$

To show (5.13), we argue by contradiction and assume there exist $\varepsilon > 0$, $|x_l| \rightarrow +\infty$ and $k_l \rightarrow +\infty$ such that

$$\frac{m_{k_l}^{\alpha^*}}{V}(x_l) \geq \varepsilon. \quad (5.14)$$

Then we define

$$v_l(x) = a_l^{r-2} u_{k_l}(x_l + a_l x), \quad \mu_l(x) = a_l^n m_{k_l}(x_l + a_l x), \quad (5.15)$$

where a_l will be determined later. Upon substituting (5.15) into (5.4), one obtains

$$\begin{cases} -\Delta v_l + C_H |\nabla v_l|^{r'} + a_l^r \lambda_{k_l} = a_l^r V(x_l + a_l x) - a_l^r g_l[a_l^{-r/\alpha^*} \mu_l], & x \in \mathbb{R}^n, \\ \Delta \mu_l + C_H r' \nabla \cdot (|\nabla v_l|^{r'-2} \nabla v_l \mu_l) = 0, & x \in \mathbb{R}^n. \end{cases} \quad (5.16)$$

Choosing $a_l^r = \frac{1}{V(x_l)}$, one finds from (1.20a) that

$$a_l^r = \frac{1}{V(x_l)} \rightarrow 0, \quad (5.17)$$

where $|x_l| \rightarrow \infty$. In light of the assumption (1.20b), we have

$$\|a_l^r V_{k_l}(x_l + a_l x)\|_{L^\infty(B_1(0))} \leq C_2. \quad (5.18)$$

In addition, we combine (5.5) with (5.6) to obtain for l large,

$$\begin{aligned} \|a_l^r g_l[\mu_l a_l^{-r/\alpha^*}]\|_{L^{1+\frac{1}{\alpha^*}}(B_1(0))} &= a_l^r \|g_l[\mu_l a_l^{-r/\alpha^*}]\|_{L^{1+\frac{1}{\alpha^*}}(B_1(0))} \\ &\leq a_l^r K(\|\mu_l^{\alpha^*} a_l^{-r}\|_{L^{1+\frac{1}{\alpha^*}}(B_2(0))} + 1) \leq K\|\mu_l^{\alpha^*}\|_{L^{1+\frac{1}{\alpha^*}}(B_2(0))} + 1. \end{aligned} \quad (5.19)$$

On the other hand, from (5.7) and (5.17), one has the fact that

$$\|\mu_l^{\alpha^*}\|_{L^{1+\frac{1}{\alpha^*}}(B_2(0))}^{1+\frac{1}{\alpha^*}} = a_l^r \|m_{k_l}\|_{L^{1+\alpha^*}(B_{2a_l}(x_l))} \rightarrow 0 \text{ as } l \rightarrow +\infty. \quad (5.20)$$

We combine (5.18), (5.19) with (5.20) and similarly use the maximal regularities shown in [16] to get

$$\|\nabla v_l\|_{L^{1+\frac{1}{\alpha^*}}(B_{1/2}(0))}^{r'} \leq C, \text{ for } l \text{ large,}$$

where constant $C > 0$ independent of l . Then focusing on the second equation of (5.16), we similarly apply the standard elliptic regularity estimates (See Theorem 1.6.5 in [6]) to obtain $\mu_l \in C^{0,\theta}(B_{1/4}(0))$ with $\theta \in (0, 1)$ independent of l . With the local Hölder's continuity of μ_l , we have from (5.14) that

$$\mu_l^{\alpha^*}(0) = m_{k_l}^{\alpha^*}(x_l) a_l^r = \frac{m_{k_l}^{\alpha^*}(x_l)}{V} \geq \varepsilon,$$

which implies there exists $\delta > 0$ such that

$$\mu_l \geq \delta \text{ in } B_R(0),$$

where $R > 0$ is independent of l . Then it follows that if R is small,

$$\int_{\mathbb{R}^n} m_{k_l} V dx \geq \delta a_l^{-\frac{r}{\alpha^*}} \int_{B_{a_l R}(0)} V(x_l + y) dy \geq \frac{\delta}{2} a_l^{-n} V(x_l) |B_{a_l R}(0)| \geq C \delta V(x_l) \rightarrow +\infty,$$

where $C > 0$ is some constant. This is contradicted to (5.7) then finishes the proof of claim (5.13).

With the aid of (5.13), we find there exists constant $C > 0$ such that

$$|V - g_k[m_k]| \leq C(V + 1). \quad (5.21)$$

Proceeding the same argument shown in the proof of Lemma 3.1, one obtains

$$|\nabla u_k| \leq C(1 + V^{\frac{1}{p}}), \quad (5.22)$$

where $C > 0$ is some constant.

Finally, we establish the global uniformly estimate of m in L^∞ , i.e. prove (5.8). To this end, we set $\phi_k = u_k^p$ for $p > 1$ and show ϕ_k are Lyapunov functions. Indeed, since u_k solve HJ equations, we have

$$-\Delta \phi_k + C_H r' |\nabla u_k|^{r'-2} \nabla u_k \cdot \nabla \phi_k = p u_k^{p-1} G_k(x),$$

where $G_k(x)$ are defined by

$$G_k(x) := -(p-1) \frac{|\nabla u_k|^2}{u_k} - C_H |\nabla u_k|^{r'} + C_H r' |\nabla u_k|^{r'} - \lambda_k + V - g_k[m_k]. \quad (5.23)$$

To estimate (5.23), we use (5.21) and (5.13) to find

$$G_k(x) \geq (p-1) |\nabla u_k|^{r'} \left(\frac{C_H(r'-1)}{(p-1)} - \frac{|\nabla u_k|^{2-r'}}{u_k} \right) - \lambda_k + CV \geq 1 \text{ for all } |x| > R,$$

where $C > 0$ and $R > 0$ independent of k . Similarly as results obtained in [40], one gets

$$\sup_k \int_{\mathbb{R}^n} m_k |V|^p dx < +\infty, \text{ and } \sup_k \int_{\mathbb{R}^n} m_k |\nabla u_k|^p dx < +\infty, \quad (5.24)$$

for large $p > 1$. Next, we perform the global estimates of m_k in L^q with any $q > 1$. In fact, we claim that

$$\sup_k \int_{\mathbb{R}^n} m_k^q dx \leq C_q < +\infty, \quad \forall q > 1, \quad (5.25)$$

where $C_q > 0$ is some constant. Thanks to (5.13) and (5.12), one has for any $k, \exists R_0 > 0$ such that

$$m_k(x) \leq C_{R_0} V^{\frac{1}{\alpha^x}}, \quad \forall |x| > R_0, \quad (5.26)$$

where $C_{R_0} > 0$ is some constant. Moreover, (5.26) together with (5.24) gives us

$$\int_{B_{R_0}^c} m_k^q dx = \int_{B_{R_0}^c} m_k m_k^{q-1} dx \leq C_{R_0}^{q-1} \int_{\mathbb{R}^n} m_k V^{\frac{q-1}{\alpha^x}} dx \leq C_{R_0, q}, \quad (5.27)$$

where $C_{R_0, q}$ is some positive constant. Moreover, it follows from (5.12) that

$$\int_{B_{R_0}(0)} m_k^q dx \leq \|m_k\|_{L^\infty(B_{R_0}(0))}^q |B_{R_0}(0)| \leq \tilde{C}_{R_0, q}, \quad (5.28)$$

where $\tilde{C}_{R_0, q}$ is some positive constant. Hence, claim (5.25) holds by invoking (5.27) and (5.28). We next prove for any $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $q > 1$ that

$$\left| \int_{\mathbb{R}^n} m_k |\nabla u_k|^{r'-2} \nabla u_k \nabla \varphi dx \right| \leq C_q \|\nabla \varphi\|_{L^{q'}(\mathbb{R}^n)}, \quad (5.29)$$

where $C_q > 0$ is some constant. Indeed, for any fixed $q > 1$, we use Hölder's inequality to get that

$$\begin{aligned} \int_{\mathbb{R}^n} m_k |\nabla u_k|^{r'-1} |\nabla \varphi| dx &= \int_{\mathbb{R}^n} m_k^{\frac{1}{p'}} m_k^{\frac{1}{p}} |\nabla u_k|^{r'-1} |\nabla \varphi| dx \\ &\leq \left(\int_{\mathbb{R}^n} m_k^{\frac{s}{p'}} dx \right)^{\frac{1}{s}} \cdot \left(\int_{\mathbb{R}^n} m_k |\nabla u_k|^{(r'-1)p} dx \right)^{\frac{1}{p}} \cdot \left(\int_{\mathbb{R}^n} |\nabla \varphi|^{q'} dx \right)^{\frac{1}{q'}}, \end{aligned} \quad (5.30)$$

where p and s are chosen such that $p > \max\{q, \frac{1}{r'-1}\}$ and $s > p'$ with

$$\frac{1}{s} + \frac{1}{p} = \frac{1}{q}, \quad \frac{1}{p'} + \frac{1}{p} = 1. \quad (5.31)$$

We combine (5.24) with (5.25) and obtain from (5.30) that (5.29) holds. On the other hand, we apply the integration by parts on Fokker-Planck equations to get

$$\int_{\mathbb{R}^n} m_k \Delta \varphi dx = C_H r' \int_{\mathbb{R}^n} m_k |\nabla u_k|^{r'-2} \nabla u_k \cdot \nabla \varphi dx. \quad (5.32)$$

Combining (5.32) with (5.29), we use Lemma 3.4 to obtain

$$\sup_k \|\nabla m_k\|_{W^{1, q}(\mathbb{R}^n)} \leq C_q < +\infty, \quad \forall q > 1. \quad (5.33)$$

(5.33) together with (5.25) implies

$$\sup_k \|m_k\|_{W^{1, q}(\mathbb{R}^n)} \leq C_q < +\infty. \quad (5.34)$$

By choosing $q > n$ and using the Sobolev embedding theorem, we find (5.8) holds. \square

We remark that the conclusion stated in Lemma 5.2 also holds if α^* is replaced by α and satisfies $0 < \alpha < \frac{r}{n}$. Furthermore, it is worthy mentioning that our approach employed in the proof of Lemma 5.2 is distinct from the blow-up analysis shown in [12, Theorem 4.1]. With our novel arguments, we are able to relax the polynomial growth assumption on V and analyze the existence and blow-up behaviors of minimizers under a class of V satisfying (1.20) when $r \leq n$.

We next focus on the case of $r > n$ and give some preliminary results for the proof of Theorem 1.3. First of all, if problem (1.22) is attained by a minimizer (m, w) , then we have

Proposition 5.1. *Assume that V satisfies (V1) and (V2) stated in Subsection 1.2. Let $(m, w) \in \mathcal{K}_M$ be a minimizer to problem (1.22). Then there exists a solution (u, m, λ) to (1.30), which satisfies*

$$|\nabla u(x)| \leq C(1 + V^{\frac{1}{r}}(x)), \quad u(x) \geq CV^{\frac{1}{r}}(x) - C^{-1}, \quad x \in \mathbb{R}^n, \quad (5.35)$$

for some constant $C > 0$.

Proof. The proof is similar as shown in [12, Proposition 3.4]. Define the test function space

$$\mathcal{B} := \left\{ \psi \in C^2(\mathbb{R}^n) \mid \limsup_{|x| \rightarrow \infty} \frac{|\nabla \psi(x)|}{V^{\frac{1}{r}}(x)} < +\infty, \quad \limsup_{|x| \rightarrow \infty} \frac{|\Delta \psi(x)|}{V(x)} < +\infty \right\}. \quad (5.36)$$

By using the condition (1.20c), one can obtain from (5.36) that

$$\limsup_{|x| \rightarrow \infty} \frac{|\psi(x)|}{|x|V^{\frac{1}{r}}(x)} < +\infty. \quad (5.37)$$

We claim that

$$-\int_{\mathbb{R}^n} m \Delta \psi \, dx = \int_{\mathbb{R}^n} w \cdot \nabla \psi \, dx, \quad \text{for all } \psi \in \mathcal{B}. \quad (5.38)$$

Similarly as in [12, Proposition 3.4], we consider a radial smooth cutoff function χ satisfying $\chi(x) = 1$ if $x \in B_1(0)$, and $\chi(x) = 0$ if $x \in B_2^c(0)$. Define $\chi_R(x) := \chi(\frac{x}{R})$, then we have $|\nabla \chi_R(x)| \leq CR^{-1}$ and $|\Delta \chi_R(x)| \leq CR^{-2}$ for some $C > 0$. Noting that $\Delta m = \nabla \cdot w$ in the weak sense, we test the equation against $\psi \chi_R$ with $\psi \in \mathcal{B}$ and integrate it by parts to get

$$-\int_{\mathbb{R}^n} m(\chi_R \Delta \psi + 2\nabla \psi \cdot \nabla \chi_R + \psi \Delta \chi_R) \, dx = \int_{B_{2R}} w \cdot (\chi_R \nabla \psi + \psi \nabla \chi_R) \, dx. \quad (5.39)$$

Since $(m, w) \in \mathcal{K}_M$ is a minimizer of (1.22), we obtain $\int_{\mathbb{R}^n} Vm \, dx < \infty$ and

$$\int_{\mathbb{R}^n} |w|V^{\frac{1}{r}} \, dx = \int_{\mathbb{R}^n} |w|m^{-\frac{1}{r}}m^{\frac{1}{r}}V^{\frac{1}{r}} \, dx \leq \left(\int_{\mathbb{R}^n} \left| \frac{w}{m} \right|^r m \, dx \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^n} mV \, dx \right)^{\frac{r-1}{r}} < \infty.$$

We thus get from (5.36) that

$$\int_{\mathbb{R}^n} |w \nabla \psi| \, dx \leq C \int_{\mathbb{R}^n} |w|V^{\frac{1}{r}} \, dx < \infty, \quad \int_{\mathbb{R}^n} m|\Delta \psi| \, dx \leq C \int_{\mathbb{R}^n} mV \, dx < \infty.$$

Moreover, it follows from (5.36) and (5.37) that

$$\int_{R \leq |x| \leq 2R} m|\psi||\Delta \chi_R| \, dx, \quad \int_{R \leq |x| \leq 2R} m|\nabla \psi||\nabla \chi_R| \, dx \leq \frac{C}{R} \int_{R \leq |x| \leq 2R} mV^{\frac{1}{r}} \, dx \xrightarrow{R \rightarrow \infty} 0,$$

and

$$\int_{\mathbb{R}^n} |w||\psi||\nabla \chi_R| \, dx \leq \frac{C}{R} \int_{R \leq |x| \leq 2R} |w|V^{\frac{1}{r}}|x| \, dx \leq 2C \int_{R \leq |x| \leq 2R} wV^{\frac{1}{r}} \, dx \xrightarrow{R \rightarrow \infty} 0.$$

Upon collecting the above two estimates, we take the limit in (5.39) to prove our claim.

With the claim (5.38), we can follow the subsequent arguments shown in [12, Proposition 3.4] to complete the proof of this proposition and the detailed argument is omitted. \square

With the aid of Proposition 5.1, we study the regularity of m when $r > n$, which is

Proposition 5.2. *Let (m, w, u, λ) be the solution given in Proposition 5.1, then $m \in W^{1,p}(\mathbb{R}^n)$ for all $p > 1$.*

Proof. The proof is based on the argument shown in [12, Proposition 3.6]. In light of (5.35), we may assume that $u(x) \geq 1$ and set $\phi = u^p$ for $p > 1$. Since u solves the HJ equation, we have

$$-\Delta\phi + C_H r' |\nabla u|^{r'-2} \nabla u \cdot \nabla\phi = pu^{p-1}G(x), \quad (5.40)$$

where $G(x)$ is defined by

$$G(x) := -(p-1) \frac{|\nabla u|^2}{u} - C_H |\nabla u|^{r'} + C_H r' |\nabla u|^{r'} - \lambda + V - m^{\frac{r}{n}}. \quad (5.41)$$

Noting that $m \in W^{1,\hat{q}}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ for $r > n$, we find

$$G(x) \geq (p-1) |\nabla u|^{r'} \left(\frac{C_H(r'-1)}{(p-1)} - \frac{|\nabla u|^{2-r'}}{u} \right) - C + V \geq 1 \text{ for all } |x| > R. \quad (5.42)$$

In view of the results obtained in [40], we use (5.35) to get $\int_{\mathbb{R}^n} m|V|^p dx < +\infty$ for all $p > 1$. Then one finds $w = -C_H r' m |\nabla u|^{r'-2} \nabla u \in L^p(\mathbb{R}^n)$ for all $p > 1$ by using (5.35) and the fact $m \in L^\infty(\mathbb{R}^n)$. Finally, it follows from Lemma 3.4 that $m \in W^{1,p}$ for all $p > 1$. \square

With the preliminary results shown above, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3:

Proof. Firstly, we shall show Conclusion (ii) in Theorem 1.3. Recall that $(m_{\alpha^*}, w_{\alpha^*}, u_{\alpha^*})$ given in Theorem 1.2 is a minimizer of problem (1.24) with $\alpha = \alpha^* = \frac{r}{n}$. For the simplicity of notations, we rewrite $(m_{\alpha^*}, w_{\alpha^*}, u_{\alpha^*})$ as (m_*, w_*, u_*) , then define

$$(m_*^t, w_*^t) = \left(\frac{M}{M^*} t^n m_*(t(x-x_0)), \frac{M}{M^*} t^{n+1} w_*(t(x-x_0)) \right) \in \mathcal{K}_M, \quad \forall t > 0, x_0 \in \mathbb{R}^n. \quad (5.43)$$

where the constraint set \mathcal{K}_M and $M^* > 0$ are defined by (1.17) and (1.29), respectively. Recall that $u_* \in C^2(\mathbb{R}^n)$ and m_* decays exponentially as stated in Theorem 1.2, then we employ Lemma 3.7 to get

$$C_L \int_{\mathbb{R}^n} \left| \frac{w_*}{m_*} \right|^r m_* dx = \frac{n}{n+r} \int_{\mathbb{R}^n} m_*^{1+\frac{r}{n}} dx. \quad (5.44)$$

Invoking (5.44), we substitute (5.43) into (1.23) to obtain that if $M > M^*$,

$$\begin{aligned} e_{\alpha^*, M} \leq \mathcal{E}(m_*^t, w_*^t) &= \frac{M}{M^*} \left(C_L t^r \int_{\mathbb{R}^n} \left| \frac{w_*}{m_*} \right|^r m_* dx + \int_{\mathbb{R}^n} V(x) m_* dx \right) - \frac{t^r}{1+\frac{r}{n}} \left(\frac{M}{M^*} \right)^{1+\frac{r}{n}} \int_{\mathbb{R}^n} m_*^{1+\frac{r}{n}} dx \\ &= \frac{M}{M^*} \left[1 - \left(\frac{M}{M^*} \right)^{\frac{r}{n}} \right] \frac{t^r}{1+\frac{r}{n}} \int_{\mathbb{R}^n} m_*^{1+\frac{r}{n}} dx + MV(x_0) + o_t(1) \\ &\rightarrow -\infty \text{ as } t \rightarrow +\infty. \end{aligned} \quad (5.45)$$

It immediately follows that $e_{\alpha^*, M} = -\infty$ for $M > M^*$, there is no minimizer to problem (1.22).

To prove Conclusion (i) in Theorem 1.3, we divide the argument into two cases: $r \leq n$ and $r > n$. For the former case, we first consider the auxiliary problem (5.2), with $\mathcal{E}_\epsilon(m, w)$ being given by (5.3). By using Young's inequality for convolution, we have the fact that

$$\int_{\mathbb{R}^n} m^{1+\frac{r}{n}} dx \geq \int_{\mathbb{R}^n} (m * \eta_\epsilon)^{1+\frac{r}{n}} dx \xrightarrow{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n} m^{1+\frac{r}{n}} dx \text{ for any } m \in L^{1+\frac{r}{n}}(\mathbb{R}^n). \quad (5.46)$$

Then, one finds from (5.1) that

$$\mathcal{E}_\epsilon(m, w) \geq \mathcal{E}(m, w) \geq \left[\left(\frac{M^*}{M} \right)^{\frac{r}{n}} - 1 \right] \frac{n}{n+r} \int_{\mathbb{R}^n} m^{1+\frac{r}{n}} dx + \int_{\mathbb{R}^n} V(x)m dx. \quad (5.47)$$

Next, we show that there exist minimizers to problem (5.2). Let $(m_{\epsilon,k}, w_{\epsilon,k}) \in \mathcal{K}_M$ be a minimizing sequence of (5.2). Noting that if we take

$$(\hat{m}, \hat{w}) := \left(\frac{\|e^{-|\cdot|}\|_{L^1(\mathbb{R}^n)}}{M} e^{-|\cdot|}, \frac{\|e^{-|\cdot|}\|_{L^1(\mathbb{R}^n)}}{M} \frac{x}{|\cdot|} e^{-|\cdot|} \right) \in \mathcal{K}_M,$$

then,

$$e_{\epsilon,M} \leq C_L \int_{\mathbb{R}^n} \left| \frac{\hat{w}}{\hat{m}} \right|^r \hat{m} dx + \int_{\mathbb{R}^n} V(x)\hat{m} dx < +\infty,$$

which implies that there exists $C > 0$ independent of ϵ such that

$$\lim_{k \rightarrow \infty} \mathcal{E}_\epsilon(m_{\epsilon,k}, w_{\epsilon,k}) = e_{\epsilon,M} < C < +\infty. \quad (5.48)$$

Since $M < M^*$, one concludes from (5.1), (5.47) and (5.48) that

$$\sup_{k \in \mathbb{N}^+} \int_{\mathbb{R}^n} m_{\epsilon,k}^{1+\frac{r}{n}} dx \leq C < +\infty, \quad \sup_{k \in \mathbb{N}^+} \int_{\mathbb{R}^n} \left(\left| \frac{w_{\epsilon,k}}{m_{\epsilon,k}} \right|^r m_{\epsilon,k} + V(x)m_{\epsilon,k} \right) dx \leq C < +\infty, \quad (5.49)$$

where $C > 0$ is independent of ϵ . The subsequent argument for proving Conclusion (i) is similar as discussed in [12], for the sake of completeness, we give the proof briefly. Indeed, with the aid of the key Lemma 3.5, we obtain from (5.49) that

$$\sup_{k \in \mathbb{N}^+} \|m_{\epsilon,k}\|_{W^{1,\hat{q}}(\mathbb{R}^n)} \leq C < +\infty \quad \text{and} \quad \sup_{k \in \mathbb{N}^+} \|w_{\epsilon,k}\|_{L^p(\mathbb{R}^n)} \leq C < +\infty, \quad \text{for any } p \in [1, \hat{q}], \quad (5.50)$$

where \hat{q} is given in (1.18) and $C > 0$ is some constant independent of ϵ . As a consequence, there exists $(m_\epsilon, w_\epsilon) \in W^{1,\hat{q}}(\mathbb{R}^n) \times L^{\hat{q}}(\mathbb{R}^n)$ such that

$$(m_{\epsilon,k}, w_{\epsilon,k}) \xrightarrow{k} (m_\epsilon, w_\epsilon) \text{ in } W^{1,\hat{q}}(\mathbb{R}^n) \times L^{\hat{q}}(\mathbb{R}^n). \quad (5.51)$$

In light of the assumption (V1), $\lim_{|x| \rightarrow \infty} V(x) = +\infty$, given in Subsection 1.2, one can deduce from Lemma 5.1 that

$$m_{\epsilon,k} \xrightarrow{k} m_\epsilon \text{ in } L^1(\mathbb{R}^n) \cap L^{1+\frac{r}{n}}(\mathbb{R}^n). \quad (5.52)$$

Therefore, up to a subsequence,

$$\int_{\mathbb{R}^n} (\eta_\epsilon * m_{\epsilon,k})^{1+\frac{r}{n}} dx \xrightarrow{k} \int_{\mathbb{R}^n} (\eta_\epsilon * m_\epsilon)^{1+\frac{r}{n}} dx. \quad (5.53)$$

In addition, thanks to the convexity of $\int_{\mathbb{R}^n} \left| \frac{w}{m} \right|^r m dx$, by letting $k \rightarrow \infty$ in (5.49), we see that there exists $C > 0$ independent of $\epsilon > 0$ such that

$$\int_{\mathbb{R}^n} \left| \frac{w_\epsilon}{m_\epsilon} \right|^r m_\epsilon dx + \int_{\mathbb{R}^n} V(x)m_\epsilon dx \leq \liminf_{k \rightarrow +\infty} \int_{\mathbb{R}^n} \left| \frac{w_{\epsilon,k}}{m_{\epsilon,k}} \right|^r m_{\epsilon,k} dx + \int_{\mathbb{R}^n} V(x)m_{\epsilon,k} dx \leq C < +\infty. \quad (5.54)$$

It then follows that

$$\int_{\mathbb{R}^n} |w_\epsilon| V^{\frac{1}{r'}} dx \leq \left(\int_{\mathbb{R}^n} \left| \frac{w_\epsilon}{m_\epsilon} \right|^r m_\epsilon dx \right)^r \left(\int_{\mathbb{R}^n} V m_\epsilon dx \right)^{r'} \leq C < \infty. \quad (5.55)$$

and

$$\int_{\mathbb{R}^n} |w_\epsilon| dx \leq \left(\int_{\mathbb{R}^n} \left| \frac{w_\epsilon}{m_\epsilon} \right|^r m_\epsilon dx \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^n} m_\epsilon dx \right)^{\frac{r-1}{r}} \leq C < \infty. \quad (5.56)$$

From (5.51), (5.52) and (5.56) we deduce that $(m_\epsilon, w_\epsilon) \in \mathcal{K}_M$. Moreover, it follows from (5.53) and (5.54) that

$$e_{\epsilon, M} = \lim_{k \rightarrow \infty} \mathcal{E}_\epsilon(m_{\epsilon, k}, w_{\epsilon, k}) \geq \mathcal{E}_\epsilon(m_\epsilon, w_\epsilon) \geq e_{\epsilon, M},$$

which indicates $(m_\epsilon, w_\epsilon) \in \mathcal{K}_M$ is a minimizer of problem (5.2). Finally, similarly as the proof of Proposition 3.4 in [12] and the arguments shown in Proposition 5.1 and Proposition 5.2, we apply Lemma 3.3 to obtain that there exists $u_\epsilon \in C^2(\mathbb{R}^n)$ bounded from below (depending on ϵ) and $\lambda_\epsilon \in \mathbb{R}$ such that

$$\begin{cases} -\Delta u_\epsilon + C_H |\nabla u_\epsilon|^{r'} + \lambda_\epsilon = V(x) - (\eta_\epsilon * m_\epsilon)^{\frac{r}{n}} * \eta_\epsilon, \\ \Delta m_\epsilon + C_H r' \nabla \cdot (m_\epsilon |\nabla u_\epsilon|^{r'-2} \nabla u_\epsilon) = 0, \\ w_\epsilon = -C_H r' m_\epsilon |\nabla u_\epsilon|^{r'-2} \nabla u_\epsilon, \int_{\mathbb{R}^n} m_\epsilon dx = M < M^*. \end{cases} \quad (5.57)$$

For each fixed $\epsilon > 0$, we deduce from Lemma 3.1 that, there exists $C_\epsilon > 0$ depends on ϵ such that $|\nabla u_\epsilon(x)| \leq C_\epsilon (1 + V(x))^{\frac{1}{r'}}$. Since $u_\epsilon \in C^2(\mathbb{R}^n)$ satisfies the first equation of (5.57) in classical sense and $(\eta_\epsilon * m_\epsilon)^{\frac{r}{n}} * \eta_\epsilon \in L^\infty(\mathbb{R}^n)$, we deduce that $|\Delta u_\epsilon(x)| \leq C_\epsilon (1 + V(x))$. We claim that

$$|\lambda_\epsilon| \leq C < \infty, \text{ with } C > 0 \text{ independent of } \epsilon > 0. \quad (5.58)$$

By using the m -equation of (5.57), we have from (5.55) that

$$\int_{\mathbb{R}^n} m \Delta u_\epsilon dx = \int_{\mathbb{R}^n} w_\epsilon \cdot \nabla u_\epsilon dx = -C_H r' \int_{\mathbb{R}^n} m_\epsilon |\nabla u_\epsilon|^{r'} \nabla u_\epsilon dx.$$

In addition, we test the u -equation of (5.57) against m_ϵ and integrate to get

$$\begin{aligned} \lambda_\epsilon M &= -(1 - r') C_H \int_{\mathbb{R}^n} m_\epsilon |\nabla u_\epsilon|^{r'} dx + \int_{\mathbb{R}^n} V m_\epsilon dx - \int_{\mathbb{R}^n} (\eta_\epsilon * m_\epsilon)^{1 + \frac{r}{n}} dx \\ &= C_L \int_{\mathbb{R}^n} m_\epsilon \left| \frac{w_\epsilon}{m_\epsilon} \right|^r dx + \int_{\mathbb{R}^n} V m_\epsilon dx - \int_{\mathbb{R}^n} (\eta_\epsilon * m_\epsilon)^{1 + \frac{r}{n}} dx \end{aligned} \quad (5.59)$$

where we have used the fact that $C_L = \frac{1}{r'} (r' C_H)^{\frac{1}{1-r'}}$ in the second equality. Finally, we collect (5.54), (5.55) and (5.59) to find (5.58) holds.

Next, we let $\epsilon \rightarrow 0$ and shall find the minimizer (m_M, w_M) to problem (1.22). Noting $(m_\epsilon, u_\epsilon, \lambda_\epsilon)$ satisfies (5.7) with k replaced by ϵ . We apply Young's inequality to get

$$\sup_k \|(\eta_k * m_k)^{\alpha^*} * \eta_k\|_{L^{1+\frac{1}{\alpha^*}}(\mathbb{R}^n)} \leq \sup_k \|m_k^{\alpha^*}\|_{L^{1+\frac{1}{\alpha^*}}(\mathbb{R}^n)} < \infty,$$

and

$$\sup_k \|(\eta_k * m_k)^{\alpha^*} * \eta_k\|_{L^{1+\frac{1}{\alpha^*}}(B_R(x_0))} \leq \sup_k \|m_k^{\alpha^*}\|_{L^{1+\frac{1}{\alpha^*}}(B_{2R}(x_0))} < \infty, \text{ for any } x_0 \in \mathbb{R}^n \text{ and } R \text{ large.}$$

Then, with the aid of (5.54) and (5.58), we can invoke Lemma 5.2 to derive that

$$\limsup_{\epsilon \rightarrow 0^+} \|m_\epsilon\|_{L^\infty(\mathbb{R}^n)} < \infty. \quad (5.60)$$

Then, by using Lemma 3.1, we find

$$|\nabla u_\epsilon(x)| \leq C(1 + V(x))^{\frac{1}{r'}}, \text{ where } C > 0 \text{ is independent of } \epsilon. \quad (5.61)$$

Since u_ϵ is bounded from below, without loss of generality, we assume that $u_\epsilon(0) = 0$. Thanks to (3.7), one finds that $u_\epsilon(x) \geq C_\epsilon V^{\frac{1}{p}}(x) - C_\epsilon \rightarrow +\infty$ as $|x| \rightarrow +\infty$, which indicates each $u_\epsilon(x) \in C^2(\mathbb{R}^n)$ admits its minimum at some finite point x_ϵ . using (5.58), (5.60) and the coercivity of V , we can deduce from the u -equation of (5.57) that x_ϵ is uniformly bounded with respect to ϵ . It then follows from $u_\epsilon(0) = 0$ and (5.61) that there exists $C > 0$ independent of ϵ such that

$$-C \leq u_\epsilon(x) \leq C|x|(1 + V(x))^{\frac{1}{p}} \text{ for all } x \in \mathbb{R}^n,$$

where we have used (1.20c) in the second inequality. Since u_ϵ are bounded from below uniformly, one can employ Lemma 3.2 to get that $u_\epsilon(x) \geq CV^{\frac{1}{p}}(x) - C$ with $C > 0$ independent of ϵ . In summary, under the assumptions (1.20), we get that,

$$C_1 V^{\frac{1}{p}}(x) - C_1 \leq u_\epsilon \leq C_2 |x|(1 + V(x))^{\frac{1}{p}}, \text{ for all } x \in \mathbb{R}^n. \quad (5.62)$$

where $C_1, C_2 > 0$ are independent of ϵ .

In light of (5.60) and (5.61), we find for any $R > 1$ and $p > 1$,

$$\|w_\epsilon\|_{L^p(B_{2R}(0))} = C_H r' \|m_\epsilon |\nabla u_\epsilon|^{r'-1}\|_{L^p(B_{2R}(0))} \leq C_{p,R} < \infty, \quad (5.63)$$

where the constant $C_{p,R} > 0$ depends only on p, R and is independent of ϵ . By applying Lemma 3.4, we obtain from (5.63) that $\|m_\epsilon\|_{W^{1,p}(B_{2R}(0))} \leq C_{p,R} < \infty$. Taking $p > n$ large enough, we employ Sobolev embedding theorem to get

$$\|m_\epsilon\|_{C^{0,\theta_1}(B_{2R}(0))} \leq C_{\theta_1,R} < \infty \text{ for some } \theta_1 \in (0, 1). \quad (5.64)$$

To estimate u_ϵ , we rewrite the u -equation of (5.57) as

$$-\Delta u_\epsilon = -C_H |\nabla u_\epsilon|^{r'} + f_\epsilon(x) \text{ with } f_\epsilon(x) := -\lambda_\epsilon + V(x) - (\eta_\epsilon * m_\epsilon)^{\frac{r}{n}} * \eta_\epsilon, \quad (5.65)$$

By using (5.60), (5.61) and the fact that V is locally Hölder continuous, we obtain that for any $p > 1$,

$$\|f_\epsilon\|_{L^p(B_{2R}(0))} + \| |\nabla u_\epsilon|^{r'} \|_{L^p(B_{2R}(0))} \leq C_{p,R} < \infty.$$

Then by $W^{2,p}$ estimates, one gets from (5.65) and (5.62) that

$$\|u_\epsilon\|_{W^{2,p}(B_{R+1}(0))} \leq C_{p,R} \left(\|u_\epsilon\|_{L^p(B_{2R}(0))} + \|f_\epsilon\|_{L^p(B_{2R}(0))} + \| |\nabla u_\epsilon|^{r'} \|_{L^p(B_{2R}(0))} \right) \leq \tilde{C}_{p,R} < \infty. \quad (5.66)$$

Taking $p > n$ large enough, we obtain that

$$\|u_\epsilon\|_{C^{1,\theta_2}(B_{R+1}(0))} \leq C_{\theta_2,R} < \infty, \text{ for some } \theta_2 \in (0, 1). \quad (5.67)$$

Combining (5.64) with (5.67), one finds

$$\| |\nabla u_\epsilon|^{r'} \|_{C^{0,\theta_3}(B_{R+1}(0))} + \|f_\epsilon\|_{C^{0,\theta_3}(B_{R+1}(0))} \leq C_{\theta_3,R} < \infty, \text{ for some } \theta_3 \in (0, 1).$$

Then by using Schauder's estimates, we have from (5.65) that

$$\|u_\epsilon\|_{C^{2,\theta_4}(B_R(0))} \leq C_{\theta_4,R} < \infty, \text{ for some } \theta_4 \in (0, 1). \quad (5.68)$$

Now, letting $R \rightarrow \infty$ and proceeding the standard diagonalization procedure, we can apply Arzelà-Ascoli theorem to get that there exists $u_M \in C^2(\mathbb{R}^n)$ such that

$$u_\epsilon \xrightarrow{\epsilon \rightarrow 0^+} u_M \text{ in } C_{\text{loc}}^{2,\theta_5}(\mathbb{R}^n) \text{ for some } \theta_5 \in (0, 1). \quad (5.69)$$

On the other hand, it follows from Lemma 3.5 and (5.54) that, there exists $(m_M, w_M) \in W^{1,\hat{q}}(\mathbb{R}^n) \times (L^1(\mathbb{R}^n) \cap L^{\hat{q}}(\mathbb{R}^n))$ such that

$$m_\epsilon \xrightarrow{\epsilon \rightarrow 0^+} m_M \text{ a.e. in } \mathbb{R}^n, \text{ and } (m_\epsilon, w_\epsilon) \xrightarrow{\epsilon \rightarrow 0^+} (m_M, w_M) \text{ in } W^{1,\hat{q}}(\mathbb{R}^n) \times L^{\hat{q}}(\mathbb{R}^n). \quad (5.70)$$

Moreover, with the help of Lemma 5.1, we have

$$m_\epsilon \xrightarrow{\epsilon \rightarrow 0^+} m_M \text{ in } L^1(\mathbb{R}^n) \cap L^{1+\frac{r}{n}}(\mathbb{R}^n). \quad (5.71)$$

Letting $\epsilon \rightarrow 0^+$ in (5.57), we then conclude from (5.58) and (5.69)-(5.71) that there exists $\lambda_M \in \mathbb{R}$ such that (m_M, u_M, w_M) satisfies (1.30). In particular, we obtain from (5.61) and (5.62) that

$$|\nabla u_M(x)| \leq C(1 + V(x))^{\frac{1}{r}}, C_1|x|^{1+\frac{b}{r}} - C_1 \leq u_M \leq C_2|x|^{1+\frac{b}{r}} + C_2, \text{ for all } x \in \mathbb{R}^n. \quad (5.72)$$

Recall that $m_\epsilon \rightarrow m_M$ a.e. as $\epsilon \rightarrow 0^+$ in \mathbb{R}^n , it follows from (5.60) that $m_M \in L^\infty(\mathbb{R}^n)$. Then, proceeding the same argument as shown in the proof of Proposition 5.2, one can further obtain from (1.30) and (5.72) that

$$w_M = -C_H r' m_M |\nabla u_M|^{r'-2} \nabla u_M \in L^p(\mathbb{R}^n) \text{ and } m_M \in W^{1,p}(\mathbb{R}^n) \text{ for all } p > 1. \quad (5.73)$$

We finally prove that $(m_M, w_M) \in \mathcal{K}_M$ is a minimizer of $e_{\alpha^*,M}$. To this end, we claim that for $M < M^*$, the following conclusion holds:

$$\lim_{\epsilon \rightarrow 0^+} e_{\epsilon,M} = e_{\alpha^*,M}, \quad (5.74)$$

where $e_{\alpha^*,M}$ is given in (1.22). Indeed, in view of (5.46), one can easily find that $\lim_{\epsilon \rightarrow 0^+} e_{\epsilon,M} \geq e_{\alpha^*,M}$. On the other hand, for any $\delta > 0$, we choose $(m, w) \in \mathcal{K}_M$ such that $\mathcal{E}(m, w) \leq e_{\alpha^*,M} + \frac{\delta}{2}$. Then by using (5.46), we deduce that for $\epsilon > 0$ small enough, $\mathcal{E}_\epsilon(m, w) \leq \mathcal{E}(m, w) + \frac{\delta}{2}$. Hence,

$$e_{\epsilon,M} \leq \mathcal{E}_\epsilon(m, w) \leq \mathcal{E}(m, w) + \frac{\delta}{2} \leq e_{\alpha^*,M} + \delta.$$

Letting $\epsilon \rightarrow 0^+$ and then $\delta \rightarrow 0^+$, one has $\lim_{\epsilon \rightarrow 0^+} e_{\epsilon,M} \leq e_{\alpha^*,M}$. Now, we finish the proof of (5.74).

We collect (5.70), (5.71), (5.74) and the convexity of $\int_{\mathbb{R}^n} \left| \frac{w}{m} \right|^r m dx$ to get

$$e_{\alpha^*,M} = \lim_{\epsilon \rightarrow 0^+} e_{\epsilon,M} = \lim_{\epsilon \rightarrow 0^+} \mathcal{E}_\epsilon(m_\epsilon, w_\epsilon) \geq \mathcal{E}(m_M, w_M) \geq e_{\alpha^*,M},$$

which implies $(m_M, w_M) \in \mathcal{K}_M$ is a minimizer of $e_{\alpha^*,M}$. This completes the proof of Conclusion (i) for the case of $r \leq n$.

Now, we consider the case of $r > n$. Compared to the former case of $r \leq n$, since now $W^{1,\hat{q}}(\mathbb{R}^n) \hookrightarrow C^{0,\theta}(\mathbb{R}^n)$ for some $\theta \in (0, 1)$, we can avoid discussing the auxiliary problem (5.2) and consider the minimization problem (1.22) directly. Let $(m_k, w_k) \in \mathcal{K}_M$ be a minimizing sequence of (1.19). Since $M < M^*$, we similarly obtain from (5.47) that

$$\sup_{k \in \mathbb{N}^+} \int_{\mathbb{R}^n} m_k^{1+\frac{r}{n}} dx \leq C < +\infty, \quad \sup_{k \in \mathbb{N}^+} \int_{\mathbb{R}^n} \left(\left| \frac{w_k}{m_k} \right|^r m_k + V(x)m_k \right) dx \leq C < +\infty.$$

Then we use Lemma 3.5 and Lemma 5.1 to conclude that there exists $(m_M, w_M) \in W^{1,\hat{q}}(\mathbb{R}^n) \times (L^1(\mathbb{R}^n) \cap L^{\hat{q}}(\mathbb{R}^n))$ such that

$$m_k \xrightarrow{k} m_M \text{ a.e. in } \mathbb{R}^n, \quad (m_k, w_k) \xrightarrow{k} (m_M, w_M) \text{ in } W^{1,\hat{q}}(\mathbb{R}^n) \times L^{\hat{q}}(\mathbb{R}^n),$$

and

$$m_k \xrightarrow{k} m_M \text{ in } L^1(\mathbb{R}^n) \cap L^{1+\frac{r}{n}}(\mathbb{R}^n).$$

Thus,

$$e_{\alpha^*, M} = \lim_{k \rightarrow \infty} \mathcal{E}(m_k, w_k) \geq \mathcal{E}(m_M, w_M) \geq e_{\alpha^*, M},$$

which indicates that $(m_M, w_M) \in \mathcal{K}_M$ is a minimizer of $e_{\alpha^*, M}$. Moreover, by using Proposition 5.1 and Proposition 5.2, one finds there exists $u_M \in C^2(\mathbb{R}^n)$ such that $(m_M, u_M, \lambda_M) \in W^{1,p}(\mathbb{R}^n) \times C^2(\mathbb{R}^n) \times \mathbb{R}$ is a solution to (1.30).

It remains to study the critical case $M = M^*$ and show Conclusion (iii). To this end, we first prove that up to a subsequence,

$$\lim_{M \nearrow M^*} e_{\alpha^*, M} = e_{\alpha^*, M^*} = 0, \quad (5.75)$$

where $\inf_{x \in \mathbb{R}^n} V(x) = 0$ was used, which is stated in (1.20a). By the definition of e_{α^*, M^*} given in (1.22), for any $\delta > 0$, $\exists (m, w) \in \mathcal{A}_{M^*}$ such that

$$e_{\alpha^*, M^*} \leq \mathcal{E}(m, w) \leq e_{\alpha^*, M^*} + \delta. \quad (5.76)$$

Noting that $\frac{M}{M^*}(m, w) \in \mathcal{A}_M$, we have

$$\begin{aligned} e_{\alpha^*, M} &\leq \mathcal{E}\left(\frac{M}{M^*}m, \frac{M}{M^*}w\right) \\ &= \mathcal{E}(m, w) + \left(\frac{M}{M^*} - 1\right) \left[C_L \int_{\mathbb{R}^n} \left| \frac{w}{m} \right|^r m dx + \int_{\mathbb{R}^n} V(x)m dx \right] + \frac{n}{n+r} \left[1 - \left(\frac{M}{M^*}\right)^{1+\frac{r}{n}} \right] \int_{\mathbb{R}^n} m^{1+\frac{r}{n}} dx. \end{aligned} \quad (5.77)$$

By straightforward computation, one has as $M \nearrow M^*$,

$$\left(\frac{M}{M^*} - 1\right) \left[C_L \int_{\mathbb{R}^n} \left| \frac{w}{m} \right|^r m dx + \int_{\mathbb{R}^n} V(x)m dx \right] + \frac{n}{n+r} \left[1 - \left(\frac{M}{M^*}\right)^{1+\frac{r}{n}} \right] \int_{\mathbb{R}^n} m^{1+\frac{r}{n}} dx \rightarrow 0. \quad (5.78)$$

We collect (5.76), (5.77) and (5.78) to find

$$\limsup_{M \nearrow M^*} e_{\alpha^*, M} \leq e_{\alpha^*, M^*} + \delta, \quad \forall \delta > 0. \quad (5.79)$$

Letting $\delta \rightarrow 0$, we obtain from (5.79) that

$$\limsup_{M \nearrow M^*} e_{\alpha^*, M} \leq e_{\alpha^*, M^*}. \quad (5.80)$$

On the other hand, let $(\bar{m}_{\alpha^*, M}, \bar{w}_{\alpha^*, M}) \in \mathcal{A}_M$ be a minimizer of $e_{\alpha^*, M} = \inf_{(m, w) \in \mathcal{A}_M} \mathcal{E}(m, w)$ for any fixed $M \in (0, M^*)$. Then we have $\frac{M^*}{M}(\bar{m}_{\alpha^*, M}, \bar{w}_{\alpha^*, M}) \in \mathcal{A}_{M^*}$ and

$$\begin{aligned} e_{\alpha^*, M^*} &\leq \mathcal{E}\left(\frac{M^*}{M}(\bar{m}_{\alpha^*, M}, \bar{w}_{\alpha^*, M})\right) \\ &= \frac{M^*}{M} \left[C_L \int_{\mathbb{R}^n} \left| \frac{\bar{w}_{\alpha^*, M}}{\bar{m}_{\alpha^*, M}} \right|^r \bar{m}_{\alpha^*, M} dx + \int_{\mathbb{R}^n} V(x)\bar{m}_{\alpha^*, M} dx - \left(\frac{M^*}{M}\right)^{\frac{r}{n}} \left(1 + \frac{r}{n}\right) \int_{\mathbb{R}^n} \bar{m}_{\alpha^*, M}^{1+\frac{r}{n}} dx \right] \\ &\leq \frac{M^*}{M} \mathcal{E}(\bar{m}_{\alpha^*, M}, \bar{w}_{\alpha^*, M}) = \frac{M^*}{M} e_{\alpha^*, M}, \quad \forall M < M^*. \end{aligned}$$

It follows that

$$e_{\alpha^*, M^*} \leq \liminf_{M \nearrow M^*} \frac{M^*}{M} e_{\alpha^*, M} = \lim_{M \nearrow M^*} e_{\alpha^*, M}. \quad (5.81)$$

Combining (5.80) with (5.81), one has

$$\lim_{M \nearrow M^*} e_{\alpha^*, M} = e_{\alpha^*, M^*} \geq 0. \quad (5.82)$$

By using assumptions (V1) and (V2) shown in Subsection 1.2 for potential V , we set $M = M^*$ in (5.45) to get

$$e_{\alpha^*, M^*} \leq \mathcal{E}(m_*^t, w_*^t) = M^* V(x_0) + o_t(1) \rightarrow 0, \text{ if } V(x_0) = 0 \text{ and } t \rightarrow +\infty.$$

Hence $e_{\alpha^*, M^*} \leq 0$, which together with (5.82) implies (5.75).

Now, we argue by contradiction to show Conclusion (iii). Assume that e_{α^*, M^*} has a minimizer $(\hat{m}, \hat{w}) \in \mathcal{A}_{M^*}$, then one applies (5.75) to obtain

$$0 = \mathcal{E}(\hat{m}, \hat{w}) = \int_{\mathbb{R}^n} C_L \left| \frac{\hat{w}}{\hat{m}} \right|^r \hat{m} dx - \frac{1}{1 + \frac{r}{n}} \int_{\mathbb{R}^n} \hat{m}^{1 + \frac{r}{n}} dx + \int_{\mathbb{R}^n} V(x) \hat{m} dx \geq 0,$$

which together with (5.1) implies

$$C_L \int_{\mathbb{R}^n} \left| \frac{\hat{w}}{\hat{m}} \right|^r \hat{m} dx = \frac{1}{1 + \frac{r}{n}} \int_{\mathbb{R}^n} \hat{m}^{1 + \frac{r}{n}} dx \text{ and } \int_{\mathbb{R}^n} V(x) \hat{m} dx = 0. \quad (5.83)$$

As a consequence, we obtain $\text{supp} V(x) \cap \text{supp} \hat{m} = \emptyset$. However, as exhibited in (1.20c), if $r \leq n$, one has $\text{supp} V = \mathbb{R}^n$, which indicates $\hat{m} = 0$ a.e.. Otherwise if $r > n$, we have facts that (\hat{m}, \hat{w}) is a minimizer of e_{α^*, M^*} and $\hat{m} \in W^{1,r}(\mathbb{R}^n) \hookrightarrow C^{0,\theta}(\mathbb{R}^n)$ for some $\theta \in (0, 1)$ by Lemma 3.5. Moreover, in light of Lemma 3.3, one finds there exists a bounded from below solution $\hat{u} \in C^2(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$ such that

$$\begin{cases} -\Delta \hat{u} + C_H |\nabla \hat{u}|^{r'} + \lambda = V(x) - \hat{m}^{\frac{r}{n}}, & x \in \mathbb{R}^n, \\ -\Delta \hat{m} - C_H r' \nabla \cdot (m |\nabla \hat{u}|^{r'-2} \nabla \hat{u}) = 0, & x \in \mathbb{R}^n, \\ \hat{m} \geq 0 \text{ in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} \hat{m} dx = M^* > 0. \end{cases}$$

By standard elliptic estimates, one gets from the Hölder continuity of V that $\hat{u} \in C_{\text{loc}}^{2,\theta_1}(\mathbb{R}^n)$ for some $\theta_1 \in (0, 1)$. Then, we apply the maximum principle as shown in [4] to find $\hat{m} > 0$ in \mathbb{R}^n when $r > n$. This indicates $\int_{\mathbb{R}^n} V(x) \hat{m} dx > 0$, which however contradicts (5.83). This completes the proof of Conclusion (iii). \square

Theorem 1.3 demonstrates that under some technical assumptions on potential V , there exists the critical mass phenomenon to problem (1.22), where $\alpha = \frac{r}{n}$. In detail, there exists $M^* > 0$ such that when $M < M^*$, there exists at least one minimizer; otherwise if $M \geq M^*$, problem (1.22) has no minimizer. A natural question is the asymptotic behaviors of minimizers to (1.22) as $M \nearrow M^*$ and we investigate this in Section 6.

6 Blow-up Behaviors of Ground States as $M \nearrow M^*$

In this section, we discuss the asymptotic profiles of least energy solutions to (1.4) as $M \nearrow M^*$. More precisely, we shall prove Theorem 1.4 and Theorem 1.5. We would like emphasize that the results are parallel to Theorem 1.2 in [12] but they only consider the subcritical mass exponent case.

6.1 Basic Asymptotic Profiles of Ground States

This subsection is devoted to the proof of Theorem 1.4 and the investigation of the rough asymptotics.

Proof of Theorem 1.4:

Proof. To show Conclusion (i), we argue by contradiction and assume that

$$\limsup_{M \nearrow M^*} \int_{\mathbb{R}^n} \left| \frac{w_M}{m_M} \right|^r m_M dx < +\infty.$$

By using Lemma 3.5, one further obtains

$$\limsup_{M \nearrow M^*} \|m_M\|_{W^{1,\hat{q}}(\mathbb{R}^n)}, \limsup_{M \nearrow M^*} \|w_M\|_{L^{\hat{q}}(\mathbb{R}^n)}, \limsup_{M \nearrow M^*} \|w_M\|_{L^1(\mathbb{R}^n)} < +\infty \quad (6.1)$$

It follows that there exists $(m, w) \in W^{1,\hat{q}}(\mathbb{R}^n) \times L^{\hat{q}}(\mathbb{R}^n)$ such that as $M \nearrow M^*$,

$$m_M \rightharpoonup m \text{ in } W^{1,\hat{q}}(\mathbb{R}^n), \text{ and } w_M \rightharpoonup w \text{ in } L^{\hat{q}}(\mathbb{R}^n). \quad (6.2)$$

We next prove that $(m, w) \in \mathcal{K}_{M^*}$ defined by (1.17). Firstly, in light of (6.1), we have the fact that

$$\limsup_{M \nearrow M^*} \int_{\mathbb{R}^n} V(x)m_M dx < +\infty. \quad (6.3)$$

Noting that potential V satisfies (V1), (V2) and (V3) given in Subsection 1.2, we combine (6.2) with (6.3) to obtain from Lemma 5.1 that

$$m_M \rightarrow m \text{ in } L^1(\mathbb{R}^n) \cap L^{1+\frac{r}{n}}(\mathbb{R}^n), \text{ as } M \nearrow M^*, \quad (6.4)$$

which indicates $\int_{\mathbb{R}^n} m dx = M^*$. In addition, we apply (6.2) to get $\Delta m = \nabla \cdot w$ weakly. Then, similarly as in (3.21), we have $w \in L^1(\mathbb{R}^n)$. By summarizing the argument above, we find $(m, w) \in \mathcal{K}_{M^*}$ and $\liminf_{M \nearrow M^*} \mathcal{E}(m_M, w_M) \geq \mathcal{E}(m, w)$ thanks to (6.2) and (6.4). Furthermore, we obtain from (5.75) that

$$e_{\alpha^*, M^*} \geq \mathcal{E}(m, w) \geq e_{\alpha^*, M^*}.$$

Thus, (m, w) is a minimizer of e_{α^*, M^*} , which is a contradiction to Conclusion (iii) in Theorem 1.3. This finishes the proof of Conclusion (i).

(ii). Recall that ε_M , which will be denoted as ε for simplicity, is defined as (1.31). As stated in Conclusion (i) of Theorem 1.3, we have the facts that each $u_M \in C^2(\mathbb{R}^n)$ is bounded from below and satisfies $\lim_{|x| \rightarrow +\infty} u_M(x) = +\infty$. Hence, there exists $x_\varepsilon \in \mathbb{R}^n$ such that $u_M(x_\varepsilon) = \inf_{x \in \mathbb{R}^n} u_M(x)$, which implies $0 = u_\varepsilon(0) = \inf_{x \in \mathbb{R}^n} u_\varepsilon(x)$ thanks to the definition given in (1.32).

From (1.30) and (1.32) we see that $(u_\varepsilon, m_\varepsilon, w_\varepsilon)$ satisfies

$$\begin{cases} -\Delta u_\varepsilon + C_H |\nabla u_\varepsilon|^{r'} + \lambda_M \varepsilon^r = -m_\varepsilon^{\frac{r}{n}} + \varepsilon^r V(\varepsilon x + x_\varepsilon), \\ -\Delta m_\varepsilon - C_H r' \nabla \cdot (m_\varepsilon |\nabla u_\varepsilon|^{r'-2} \nabla u_\varepsilon) = -\Delta m_\varepsilon + \nabla \cdot w_\varepsilon = 0, \\ \int_{\mathbb{R}^n} m_\varepsilon dx = M. \end{cases} \quad (6.5)$$

By (1.31), (5.1) and (5.75), we have

$$C_L \int_{\mathbb{R}^n} \left| \frac{w_\varepsilon}{m_\varepsilon} \right|^r m_\varepsilon dx = \varepsilon^{-r} C_L \int_{\mathbb{R}^n} \left| \frac{w_M}{m_M} \right|^r m_M dx \equiv 1, \quad \int_{\mathbb{R}^n} m_\varepsilon^{1+\frac{r}{n}} dx = \varepsilon^{-r} \int_{\mathbb{R}^n} m_M^{1+\frac{r}{n}} dx \rightarrow \frac{r+n}{n}, \quad (6.6)$$

and

$$\int_{\mathbb{R}^n} V(\varepsilon x + x_\varepsilon) m_\varepsilon dx = \int_{\mathbb{R}^n} V(x) m_M dx \rightarrow 0 \text{ as } M \nearrow M^*. \quad (6.7)$$

Similar to (5.59), we deduce from (6.5) and (6.6) that

$$M \lambda_M = \mathcal{E}(m_M, w_M) - \frac{r}{n+r} \int_{\mathbb{R}^n} m_M^{1+\frac{r}{n}} dx = o(1) - \frac{r}{n+r} \varepsilon^r \int_{\mathbb{R}^n} m_\varepsilon^{1+\frac{r}{n}} dx,$$

which implies

$$\lambda_M \varepsilon^r \rightarrow -\frac{r}{M^* n} \text{ as } M \nearrow M^*. \quad (6.8)$$

In light of the u -equation in (6.5), we apply the maximum principle to obtain

$$\lambda_M \varepsilon^r \geq -m_\varepsilon^{\frac{r}{n}}(0) + \varepsilon^r V(x_\varepsilon) \geq -m_\varepsilon^{\frac{r}{n}}(0), \quad (6.9)$$

which implies

$$m_\varepsilon^{\frac{r}{n}}(0) \geq -\lambda_M \varepsilon^r \geq C > 0, \quad (6.10)$$

where and in the arguments below, $C > 0$ denotes a constant independent of ε , which may change line to line.

We next claim that there exists some constant $C > 0$ such that

$$\varepsilon^r V(x_\varepsilon) \leq C. \quad (6.11)$$

If not, we can find some subsequence $\varepsilon_l \rightarrow 0$ such that $\varepsilon_l^r V(x_{\varepsilon_l}) \rightarrow +\infty$. By using (6.9), one has

$$\frac{m_{\varepsilon_l}^{\alpha^*}(0)}{\varepsilon_l^r V(x_{\varepsilon_l})} \geq C,$$

where $C > 0$ is some constant independent of ε_l . Similarly, let

$$v_l(x) = a_l^{r-2} u_l(a_l x), \quad \mu_l(x) = a_l^n m_l(a_l x), \quad a_l = \frac{1}{\varepsilon_l^r V(x_{\varepsilon_l})},$$

then we have

$$a_l^r = \frac{1}{\varepsilon_l^r V(x_{\varepsilon_l})} \rightarrow 0, \quad a_l^r \varepsilon_l^r V(x_{\varepsilon_l}) = 1.$$

In light of (1.20b), one finds

$$a_l^r \varepsilon_l^r V(a_l \varepsilon_l x + x_{\varepsilon_l}) = \frac{V(a_l \varepsilon_l x + x_{\varepsilon_l})}{V(x_{\varepsilon_l})} \leq C_2,$$

where $C_2 > 0$ is some constant independent of l . Noting that

$$\|\mu_l^{\alpha^*}\|_{L^{1+\frac{1}{\alpha^*}}(B_1(0))}^{1+\frac{1}{\alpha^*}} = a_l^r \|m_{\varepsilon_l}\|_{L^{1+\alpha^*}(B_{a_l}(x_l))} \rightarrow 0 \text{ as } l \rightarrow +\infty,$$

we apply the maximal regularity shown in Theorem 1.1 to obtain

$$\|\nabla v_l\|_{L^{1+\frac{1}{\alpha^*}}(B_{1/2})} \leq C,$$

where $C > 0$ is some constant, which implies $\mu_l \in C^{0,\theta}(B_{1/4}(0))$ with $\theta \in (0, 1)$.

since $\mu_l(0) = a_l^n m_l(0)$, one finds

$$\mu_l^{\alpha^*}(0) = \frac{m_l^{\alpha^*}(0)}{\varepsilon_l^r V(x_{\varepsilon_l})} \geq C > \delta > 0.$$

By using the Hölder's continuity of μ_l , we obtain

$$\mu_l(x) \geq \delta > 0 \text{ in } B_R(0),$$

where $R \in (0, \frac{1}{4})$ is some constant independent of l . We have

$$\begin{aligned} & \int_{\mathbb{R}^n} V(\varepsilon_l x + x_{\varepsilon_l}) m_{\varepsilon_l}(x) dx \\ &= \int_{\mathbb{R}^n} V(\varepsilon_l a_l x + x_{\varepsilon_l}) \mu_l dx \geq \delta \int_{B_R(0)} V(\varepsilon_l a_l x + x_{\varepsilon_l}) dx \geq \frac{\delta}{\varepsilon_l^r} \rightarrow +\infty. \end{aligned}$$

However, as $\varepsilon \rightarrow 0$,

$$\int_{\mathbb{R}^n} V(\varepsilon x + x_\varepsilon) m_\varepsilon(x) dx \rightarrow 0,$$

which is a contradiction. This completes the proof of our claim (6.11).

On the other hand, noting that V satisfies (1.20b), one further finds for $R > 0$ large enough,

$$\varepsilon^r V(\varepsilon x + x_\varepsilon) \leq C_R < +\infty, \text{ for all } |x| \leq 4R, \quad (6.12)$$

where positive constant C_R depends on R but is independent of ε .

Similarly as shown in the proof of Theorem 1.3, we estimate ∇u_ε and rewrite the u -equation of (6.5) as

$$-\Delta u_\varepsilon = -C_H |\nabla u_\varepsilon|^{r'} + g_\varepsilon(x) \text{ with } g_\varepsilon(x) := -\lambda_M \varepsilon^r + \varepsilon^r V(x_\varepsilon + \varepsilon x) - m_\varepsilon^{\frac{r}{n}}. \quad (6.13)$$

Noting that $m^{\alpha^*} \in L^{1+\frac{1}{\alpha^*}}(\mathbb{R}^n)$, we have from the maximal regularity shown in Theorem 1.1 that $|\nabla u|^{r'} \in L_{\text{loc}}^{1+\frac{1}{\alpha^*}}(\mathbb{R}^n)$, i.e. $|\nabla u|^{r'-1} \in L_{\text{loc}}^{(1+\frac{1}{\alpha^*})r'}(\mathbb{R}^n)$, which implies $m \in C_{\text{loc}}^{0,\theta}(\mathbb{R}^n)$ for some $\theta \in (0, 1)$ by using the m -equation in (6.5). Thanks to Lemma 3.1, we further obtain

$$|\nabla u_\varepsilon| \leq C_R < +\infty, \quad \forall |x| < 2R,$$

which implies

$$\| |\nabla u_\varepsilon|^{r'} \|_{L^p(B_{2R}(0))} + \| g_\varepsilon \|_{L^p(B_{2R}(0))} \leq C_{p,R} < +\infty.$$

Then, similarly as the derivation of (5.64) and (5.68), we can obtain that for some $\theta \in (0, 1)$,

$$\| m_\varepsilon \|_{C^{0,\theta}(B_{2R}(0))} \leq C < \infty, \quad (6.14)$$

and

$$\| u_\varepsilon \|_{C^{2,\theta}(B_R(0))} \leq C < \infty. \quad (6.15)$$

In light of (6.10), one has from (6.14) that there exists a constant $R_0 \in (0, 1)$ such that

$$m_\varepsilon(x) \geq C > 0, \quad \forall |x| < R_0. \quad (6.16)$$

Now, we claim that up to a subsequence,

$$\lim_{\varepsilon \rightarrow 0} x_\varepsilon = x_0 \text{ with } V(x_0) = 0. \quad (6.17)$$

If not, one has either $|x_\varepsilon| \rightarrow +\infty$ or $x_\varepsilon \rightarrow x_0$ with $V(x_0) > 0$. In the two cases, we both get $\lim_{x_\varepsilon \rightarrow 0} V(\varepsilon x + x_\varepsilon) \geq A$ a.e. in \mathbb{R}^n for some $A > 0$. Hence, it follows from (6.16) that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} V(\varepsilon x + x_\varepsilon) m_\varepsilon dx \geq \frac{A}{2} \int_{B_{R_0}(0)} m_\varepsilon(x) dx \geq \frac{AC}{2} |B_{R_0}(0)|,$$

which reaches a contradiction to (6.7). Therefore (6.17) is proved.

In light of (6.6), there exists $(m_0, w_0) \in W^{1,\hat{q}}(\mathbb{R}^n) \times (L^1(\mathbb{R}^n) \cap L^{\hat{q}}(\mathbb{R}^n))$ such that

$$(m_\varepsilon, w_\varepsilon) \rightharpoonup (m_0, w_0) \text{ in } W^{1,\hat{q}}(\mathbb{R}^n) \times L^{\hat{q}}(\mathbb{R}^n), \text{ as } \varepsilon \rightarrow 0, \quad (6.18)$$

where $m_0 \neq 0$ by (6.16) and \hat{q} is given by (1.18). Furthermore, thanks to (6.15), we find $u_\varepsilon \rightarrow u_0$ in $C_{\text{loc}}^2(\mathbb{R}^n)$. Moreover, it follows from (6.5) and (6.8) that (m_0, u_0) satisfies

$$\begin{cases} -\Delta u_0 + C_H |\nabla u_0|^{r'} - \frac{r}{M^* n} = -m_0^{\frac{r}{n}}, \\ -\Delta m_0 = -C_H r' \nabla \cdot (m_0 |\nabla u_0|^{r'-2} \nabla u_0) = -\nabla w_0, \\ 0 < \int_{\mathbb{R}^n} m_0 dx \leq M^*, \quad w_0 = -C_H m_0 |\nabla u_0|^{r'-2} \nabla u_0, \end{cases} \quad (6.19)$$

where similarly as in Section 4 (see the proof of (4.65)), we have used Pohozaev identities given in Lemma 3.7 to obtain that (m_0, w_0) is a minimizer of (1.24) and $\int_{\mathbb{R}^n} m_0 dx = M^*$. We thus see that (u_0, m_0, w_0) satisfies (1.28) by recalling (6.19). In addition, we find $m_\varepsilon \rightarrow m_0$ in $L^1(\mathbb{R}^n)$, and then by (6.18), we obtain that

$$m_\varepsilon \rightarrow m_0 \text{ in } L^p(\mathbb{R}^n), \forall p \in [1, \hat{q}^*) \text{ as } \varepsilon \rightarrow 0,$$

which indicates (1.33).

It is left to prove (1.34) when V satisfies (1.12). To this end, we argue by contradiction and assume that, up to a subsequence,

$$\liminf_{\varepsilon \rightarrow 0} \frac{|\bar{x}_\varepsilon - x_\varepsilon|}{\varepsilon} = +\infty. \quad (6.20)$$

Define

$$\begin{cases} \bar{m}_\varepsilon(x) := \varepsilon^n m_M(\varepsilon x + \bar{x}_\varepsilon) = m_\varepsilon\left(x + \frac{\bar{x}_\varepsilon - x_\varepsilon}{\varepsilon}\right), \\ \bar{w}_\varepsilon(x) := \varepsilon^{n+1} w_M(\varepsilon x + \bar{x}_\varepsilon) = w_\varepsilon\left(x + \frac{\bar{x}_\varepsilon - x_\varepsilon}{\varepsilon}\right), \\ \bar{u}_\varepsilon(x) := \varepsilon^{\frac{2-r'}{r'-1}} u_M(\varepsilon x + \bar{x}_\varepsilon) = u_\varepsilon\left(x + \frac{\bar{x}_\varepsilon - x_\varepsilon}{\varepsilon}\right). \end{cases} \quad (6.21)$$

We claim that there exist constants $R_0 > 0$ and $C > 0$ independent of ε such that

$$\bar{m}_\varepsilon(x) \geq C > 0, \quad \forall |x| < R_0. \quad (6.22)$$

In light of (6.21), one finds (6.22) is equivalent to

$$m_\varepsilon(x) \geq C > 0, \quad \forall \left|x - \frac{\bar{x}_\varepsilon - x_\varepsilon}{\varepsilon}\right| < R_0. \quad (6.23)$$

Noting (6.10), we have the fact that

$$\bar{m}_\varepsilon(0) = \|\bar{m}_\varepsilon\|_{L^\infty(\mathbb{R}^n)} = \|m_\varepsilon\|_{L^\infty(\mathbb{R}^n)} > C > 0. \quad (6.24)$$

To show (6.22) or (6.23), we divide our discussions into two cases, $r \leq n$ and $r > n$. Focusing on the case $r \leq n$, we impose (1.12) on the potential V . Noting the first equation in (6.5), one finds \bar{u}_ε satisfies

$$-\Delta \bar{u}_\varepsilon + C_H |\nabla \bar{u}_\varepsilon|^{r'} = \bar{g}_\varepsilon(x) := -\lambda_M \varepsilon^r - \bar{m}_\varepsilon^{\frac{r}{n}} + \varepsilon^r V(\varepsilon x + \bar{x}_\varepsilon). \quad (6.25)$$

Similarly as the argument shown in [12, Theorem 4.1], we consider the following two cases:

Case 1: Assume that there exists some constant $C > 0$ independent of ε such that \bar{x}_ε satisfies

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^r V(\bar{x}_\varepsilon) \leq C < +\infty.$$

Then by using (6.24), one can proceed the same argument shown as (6.13)-(6.14) and (6.16) to obtain the claim (6.22).

Case 2: Assume that \bar{x}_ε satisfies

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^r V(\bar{x}_\varepsilon) = +\infty. \quad (6.26)$$

We define

$$\tilde{m}(x) = \varepsilon^n m_M(\varepsilon x) = m_\varepsilon\left(x - \frac{x_\varepsilon}{\varepsilon}\right), \quad \tilde{u}(x) = \varepsilon^{\frac{2-r'}{r'-1}} u_M(\varepsilon x), \quad \tilde{w}(x) = \varepsilon^{n+1} w_M(\varepsilon x), \quad (6.27)$$

then find from (6.5) that $(\tilde{m}, \tilde{u}, \tilde{w})$ satisfies

$$\begin{cases} -\Delta\tilde{u} + C_H|\nabla\tilde{u}|^{r'} + \lambda_M\varepsilon^r = \varepsilon^r V(\varepsilon x) - \tilde{m}_\varepsilon^{\frac{r}{r-1}}, & x \in \mathbb{R}^n, \\ -\Delta\tilde{m} - C_{Hr'}\nabla \cdot (\tilde{m}|\nabla\tilde{u}|^{r'-2}\nabla\tilde{u}) = 0, & x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} \tilde{m}_\varepsilon dx = M \nearrow M^*, \quad \tilde{w}_\varepsilon = -C_{Hr'}\tilde{m}|\nabla\tilde{u}|^{r'-2}\nabla\tilde{u}. \end{cases} \quad (6.28)$$

Noting that V satisfies (1.12), we employ Lemma 3.1 to get

$$|\nabla\tilde{u}_\varepsilon| \leq C(1 + \sigma_\varepsilon^{\frac{1}{r'}}|x|^{\frac{b}{r'}}), \quad \sigma_\varepsilon := \varepsilon^{r+b}. \quad (6.29)$$

Denote $y_\varepsilon := \frac{x_\varepsilon}{\varepsilon}$ and $\bar{y}_\varepsilon := \frac{\bar{x}_\varepsilon}{\varepsilon}$, which are the minimum and maximum points of $\tilde{u}_\varepsilon(x)$ and $\tilde{m}_\varepsilon(x)$, respectively. Thanks to (6.17), we obtain $|y_\varepsilon| \leq C\varepsilon^{-1}$. Then, it follows from (6.29) that

$$|\tilde{u}_\varepsilon(0)| \leq |\tilde{u}_\varepsilon(y_\varepsilon)| + |y_\varepsilon| \sup_{|y| \leq |y_\varepsilon|} |\nabla\tilde{u}_\varepsilon(y)| \leq 1 + C\varepsilon^{-1} + C\varepsilon^{-1}\sigma_\varepsilon^{\frac{1}{r'}}|y_\varepsilon|^{\frac{b}{r'}} \leq 1 + C\varepsilon^{-1}. \quad (6.30)$$

As a consequence,

$$\tilde{u}_\varepsilon(x) \leq \tilde{u}_\varepsilon(0) + |x| \sup |\nabla\tilde{u}_\varepsilon(x)| \leq 1 + C\varepsilon^{-1} + \sigma_\varepsilon^{\frac{1}{r'}}|x|^{\frac{b}{r'}+1}. \quad (6.31)$$

Invoking (6.26), (6.30) and (6.31), we proceed the same argument shown in [12, Theorem 4.1] to get $\tilde{m}_\varepsilon \in C^{0,\theta}(B_R(\bar{y}_\varepsilon))$ with $\theta \in (0, 1)$ and $R > 0$ independent of ε . Noting that \bar{y}_ε is maximum point of $\tilde{m}_\varepsilon(x)$, we thus obtain from (6.24) and (6.27) that $\tilde{m}_\varepsilon(\bar{y}_\varepsilon) \geq C > 0$. Hence, one finds there exists some $R_0 > 0$ independent of ε such that

$$\tilde{m}_\varepsilon(x) > \frac{C}{2} > 0, \quad \forall |x - \bar{y}_\varepsilon| < R_0.$$

Since $\bar{y}_\varepsilon = \frac{\bar{x}_\varepsilon}{\varepsilon}$, the above estimate together with (6.27) indicates that (6.23) holds.

When $r > n$, with the assumptions (1.20) imposed on potential V , using (6.6) and noting that $W^{1,\hat{q}}(\mathbb{R}^n) \hookrightarrow C^{0,\theta}(\mathbb{R}^n)$ for some $\theta \in (0, 1)$, we deduce from Lemma 3.5 that m_ε is uniformly bounded in $C^{0,\theta}(\mathbb{R}^n)$. Then claim (6.22) follows directly by noting (6.24).

In summary, if V satisfies (1.12) when $r \leq n$ or (1.20) when $r > n$, we obtain (6.22) and (6.23) hold. However, (6.23) together with (6.20) contradicts the fact that m_ε converges strongly to m_0 in $L^1(\mathbb{R}^n)$. Therefore, (1.34) is proved and we complete the proof of Theorem 1.4. \square

Theorem 1.4 indicates that as $M \nearrow M^*$, the ground states (m_M, w_M) to problem (1.22) form some profile with the striking structure, where the leading stone is given by (m_0, w_0) , the minimizer to problem (1.26). In fact, with the precise local expansion of potential V , we are able to capture the asymptotic of scaling factor and the location of concentrated profile, which will be shown in Subsection 6.2.

6.2 Refined Asymptotic Profiles of Ground States

In this subsection, we shall discuss the refined blow-up behaviors of the rescaled minimizer $(m_\varepsilon, w_\varepsilon)$. As shown in Section 6, $(m_\varepsilon, w_\varepsilon, u_\varepsilon)$ converges to (m_0, w_0, u_0) with

$$m_\varepsilon \rightarrow m_0 \text{ in } L^p(\mathbb{R}^n) \quad \forall p \in [1, \hat{q}^*), \quad w_\varepsilon \rightarrow w_0 \text{ in } L^{\hat{q}}(\mathbb{R}^n) \text{ and } u_\varepsilon \rightarrow u_0 \text{ in } C_{\text{loc}}^2(\mathbb{R}^n).$$

where (m_0, w_0) is a minimizer of Γ_{α^*} and (u_0, m_0, w_0) satisfies (1.28). In addition, we deduce from Lemmas 3.6 and 3.7 that there exist $\delta_1 > 0$ and $C_{\delta_1} > 0$ such that,

$$m_0(x) \leq C_{\delta_1} C^{-\delta_1|x|}, \quad (6.32)$$

and

$$1 = C_L \int_{\mathbb{R}^n} \left| \frac{w_0}{m_0} \right|^r m_0 dx = \frac{n}{n+r} \int_{\mathbb{R}^n} m_0^{1+\frac{r}{n}} dx. \quad (6.33)$$

Before proving Theorem 1.5, we recall the following assumptions on potential V :

Suppose $V(x)$ has $l \in \mathbb{N}$ distinct zeros defined by $\{P_1, \dots, P_l\} \subset \mathbb{R}^n$; moreover, $\exists a_i > 0, q_i > 0, d > 0$ such that

$$V(x) = a_i |x - P_i|^{q_i} + O(|x - P_i|^{q_i+1}), \quad \text{if } |x - P_i| \leq d. \quad (6.34)$$

Define $q = \max\{q_1, \dots, q_l\}$ and $Z = \{P_i \mid q_i = q, i = 1, \dots, l\}$, then we denote

$$\mu = \min\{\mu_i \mid P_i \in Z, i = 1, \dots, l\} \quad \text{with } \mu_i = \min_{y \in \mathbb{R}^n} H_i(y), \quad H_i(y) = \int_{\mathbb{R}^n} a_i |x + y|^{q_i} m_0(x) dx, \quad (6.35)$$

and define set $Z_0 = \{P_i \mid P_i \in Z \text{ and } \mu_i = \mu, i = 1, \dots, l\}$, which consists of all weighted flattest zeros of $V(x)$. With the notations, we establish the precise upper bound of $e_{\alpha^*, M}$ as $M \nearrow M^*$, which is

Lemma 6.1. *As $M \nearrow M^*$, $e_{\alpha^*, M}$ given in (1.22) satisfies*

$$e_{\alpha^*, M} \leq [1 + o(1)] \frac{q+r}{q} \left(\frac{q\mu}{r} \right)^{\frac{r}{r+q}} \left[1 - \left(\frac{M}{M^*} \right)^{\frac{r}{n}} \right]^{\frac{q}{r+q}}. \quad (6.36)$$

Proof. Choose some $P_i \in Z_0$ and let $y_i \in \mathbb{R}^n$ satisfy

$$\mu = \mu_i = \inf_{y \in \mathbb{R}^n} H_i(y) = H_i(y_i). \quad (6.37)$$

Define

$$m_\tau = \frac{M}{M_*} \tau^n m_0 \left[\tau \left(x - \frac{y_i}{\tau} - P_i \right) \right] \quad \text{and} \quad w_\tau = \frac{M}{M^*} \tau^{n+1} w_0 \left[\tau \left(x - \frac{y_i}{\tau} - P_i \right) \right] \quad \text{for } \tau > 0.$$

Then we find

$$\int_{\mathbb{R}^n} V(x) m_\tau(x) dx = \frac{M}{M^*} \int_{\mathbb{R}^n} V\left(\frac{x+y_i}{\tau} + P_i\right) m_0(x) dx = |\tau|^{q_i} \frac{M}{M^*} \int_{\mathbb{R}^n} \frac{V\left(\frac{x+y_i}{\tau} + P_i\right)}{\left|\frac{x+y_i}{\tau}\right|^{q_i}} |x+y_i|^{q_i} m_0(x) dx. \quad (6.38)$$

Next, we establish the desired estimate for the last equality in (6.38). To begin with, from (1.20a), we see that $\exists \tilde{K} > |P_i|$ such that

$$V(x) < C_2 e^{\delta|x|}, \quad \text{for } |x| > \tilde{K}, \quad (6.39)$$

Define

$$II := \frac{V\left(\frac{x+y_i}{\tau} + P_i\right)}{\left|\frac{x+y_i}{\tau}\right|^{q_i}} |x+y_i|^{q_i} m_0,$$

then we decompose the estimates of II into the following three cases:

Case 1: When $0 \leq \frac{|x+y_i|}{\tau} < d$, we use (6.34) to get

$$|II| = \frac{V\left(\frac{x+y_i}{\tau} + P_i\right)}{\left|\frac{x+y_i}{\tau}\right|^{q_i}} |x+y_i|^{q_i} m_0 \leq 2a_i m_0. \quad (6.40)$$

Case 2: When $d \leq \frac{|x+y_i|}{\tau} \leq 2\tilde{K}$, one has from $\tilde{K} > |P_i|$ that $\left|\frac{x+y_i}{\tau} + P_i\right| \leq 2\tilde{K} + |P_i| \leq 3\tilde{K}$, which implies

$$|II| \leq d^{-q_i} \left[\sup_{|x| < 3\tilde{K}} V(x) \right] m_0(x). \quad (6.41)$$

Case 3: When $\left|\frac{x+y_i}{\tau}\right| \geq 2\tilde{K}$ holds, we find $\left|\frac{x+y_i}{\tau} + P_i\right| \geq 2\tilde{K} - |P_i| \geq \tilde{K}$ and $\left|\frac{x}{\tau}\right| \geq \left|\frac{x+y_i}{\tau}\right| - \left|\frac{y_i}{\tau}\right| \geq \tilde{K}$ provided $\tau > 0$ large enough. As a consequence, we obtain from (6.39) that

$$\frac{V\left(\frac{x+y_i}{\tau} + P_i\right)}{\left|\frac{x+y_i}{\tau}\right|^{q_i}} |x+y_i|^{q_i} = \tau^{q_i} V\left(\frac{x+y_i}{\tau} + P_i\right) \leq C_2 \tau^{q_i} e^{\delta\left|\frac{x+y_i}{\tau} + P_i\right|} \leq C_2 \tau^{q_i} e^{3\delta\left|\frac{x}{\tau}\right|}, \text{ as } \tau \rightarrow +\infty,$$

which together with (6.32) gives that for τ sufficiently large,

$$|II| \leq C_2 C_{\delta_1} \tau^{q_i} e^{3\delta\left|\frac{x}{\tau}\right|} e^{-\delta_1|x|} \leq C_{\delta_1} \tau^{q_i} e^{-\frac{\delta_1}{2}|x|} \leq C_{\delta_1} \tau^{q_i} e^{-\frac{\delta_1}{4}|x|} e^{-\frac{\delta_1}{4}\tau\tilde{K}} \leq C_{\delta_1} e^{-\frac{\delta_1}{4}|x|}. \quad (6.42)$$

Upon collecting (6.40), (6.41) and (6.42), we arrive at

$$|II| \leq \varphi(x) := \begin{cases} 2a_i m_0(x), & \left|\frac{x+y_i}{\tau}\right| < d, \\ d^{-q_i} \left[\sup_{|x| < 3\tilde{K}} V(x) \right] m_0(x), & d < \left|\frac{x+y_i}{\tau}\right| < 2\tilde{K}, \\ C_{\delta_1} e^{-\frac{\delta_1}{4}|x|}, & \left|\frac{x+y_i}{\tau}\right| > 2\tilde{K}. \end{cases}$$

In addition, by using (6.34), one has the fact that $\lim_{\tau \rightarrow +\infty} II = a_i |x+y_i|^{q_i} m_0(x)$ a.e. in \mathbb{R}^n . Hence, we employ the Lebesgue dominated convergence theorem to obtain that

$$\lim_{\tau \rightarrow +\infty} |\tau|^{-q_i} \int_{\mathbb{R}^n} V(x) m_\tau(x) dx = \frac{M}{M^*} \int_{\mathbb{R}^n} a_i |x+y_i|^{q_i} m_0(x) dx = \mu,$$

where we have used the definition of μ given by (6.37). From the definition of (m_τ, w_τ) and (6.33) we have

$$C_L \int_{\mathbb{R}^n} \left| \frac{w_\tau}{m_\tau} \right|^r m_\tau dx - \frac{1}{1 + \frac{r}{n}} \int_{\mathbb{R}^n} m_\tau^{1 + \frac{r}{n}} dx = \frac{M\tau^r}{M^*} \left[1 - \left(\frac{M}{M^*} \right)^{\frac{r}{n}} \right].$$

Let $\tau = \left(\frac{q\mu}{r[1 - (\frac{M}{M^*})^{\frac{r}{n}}]} \right)^{\frac{1}{r+q}} \rightarrow \infty$ as $M \nearrow M^*$, and note that $q_i = q$, we then deduce from the above two estimates that

$$\mathcal{E}(m_\tau, w_\tau) = \frac{M}{M^*} \left[\left(1 - \left(\frac{M}{M^*} \right)^{\frac{r}{n}} \right) \tau^r + \mu \tau^{-q} \right] + o(\tau^{-q}) = \frac{q+r}{q} \left(\frac{q\mu}{r} \right)^{\frac{r}{r+q}} \left[1 - \left(\frac{M}{M^*} \right)^{\frac{r}{n}} \right]^{\frac{q}{r+q}} (1 + o(1)),$$

which completes the proof of the lemma. \square

With the upper bound of $e_{\alpha^*, M}$ given in Lemma 6.1, we shall prove Theorem 1.5, which is

Proof of Theorem 1.5:

Proof. As shown in Theorem 1.4, one has $x_\varepsilon \rightarrow P_i$ for some $1 \leq i \leq l$. Moreover, since (m_M, w_M) denotes the minimizer of (1.22), we obtain

$$\begin{aligned} e_{\alpha^*, M} &= \mathcal{E}(m_M, w_M) = \varepsilon^{-r} C_L \int_{\mathbb{R}^n} \left| \frac{w_\varepsilon}{m_\varepsilon} \right|^r m_\varepsilon dx - \frac{\varepsilon^{-r}}{1 + \frac{r}{n}} \int_{\mathbb{R}^n} m_\varepsilon^{1 + \frac{r}{n}} dx + \int_{\mathbb{R}^n} V(\varepsilon x + x_\varepsilon) m_\varepsilon(x) dx \\ &\geq \varepsilon^{-r} \left[\left(\frac{M^*}{M} \right)^{\frac{r}{n}} - 1 \right] \frac{n}{n+r} \int_{\mathbb{R}^n} m_\varepsilon^{1 + \frac{r}{n}} dx + \int_{\mathbb{R}^n} V(\varepsilon x + x_\varepsilon) m_\varepsilon(x) dx. \end{aligned} \quad (6.43)$$

We have the facts that

$$\int_{\mathbb{R}^n} V(\varepsilon x + x_\varepsilon) m_\varepsilon(x) dx = \varepsilon^{q_i} \int_{\mathbb{R}^n} \frac{V(\varepsilon x + x_\varepsilon)}{|\varepsilon x + x_\varepsilon - P_i|^{q_i}} \left| x + \frac{x_\varepsilon - P_i}{\varepsilon} \right|^{q_i} m_\varepsilon(x) dx, \quad (6.44)$$

and in view of $x_\varepsilon \rightarrow P_i$,

$$\lim_{\varepsilon \rightarrow 0} \frac{V(\varepsilon x + x_\varepsilon)}{|\varepsilon x + x_\varepsilon - P_i|^{q_i}} = \lim_{\varepsilon \rightarrow 0} \frac{a_i |\varepsilon x + x_\varepsilon - P_i|^{q_i} + O(|\varepsilon x + x_\varepsilon - P_i|^{q_i+1})}{|\varepsilon x + x_\varepsilon - P_i|^{q_i}} = a_i, \quad \text{a.e. in } \mathbb{R}^n. \quad (6.45)$$

Now, we claim that

$$q_i = q = \max\{q_1, \dots, q_l\} \text{ and } \limsup_{\varepsilon \rightarrow 0} \left| \frac{x_\varepsilon - P_i}{\varepsilon} \right| \text{ is uniformly bounded.} \quad (6.46)$$

If not, we have either $q_i < q$ or up to a subsequence, $\lim_{\varepsilon \rightarrow 0} \left| \frac{x_\varepsilon - P_i}{\varepsilon} \right| = +\infty$. With the help of (1.33), (6.44) and (6.45), we apply Fatou's lemma to get for any constant $\beta \gg 1$ large enough,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-q} \int_{\mathbb{R}^n} V(\varepsilon x + x_\varepsilon) m_\varepsilon dx = \lim_{\varepsilon \rightarrow 0} \varepsilon^{q_i - q} \int_{\mathbb{R}^n} \frac{V(\varepsilon x + x_\varepsilon)}{|\varepsilon x + x_\varepsilon - P_i|^{q_i}} \left| x + \frac{x_\varepsilon - P_i}{\varepsilon} \right|^{q_i} m_\varepsilon dx \geq \beta \gg 1.$$

As a consequence, we obtain from (6.6) and (6.43) that

$$\begin{aligned} e_{\alpha^*, M} &\geq \varepsilon^{-r} \left[\left(\frac{M^*}{M} \right)^{\frac{r}{n}} - 1 \right] \frac{1}{1 + \frac{r}{n}} \int_{\mathbb{R}^n} m_\varepsilon^{1 + \frac{r}{n}} dx + \beta \varepsilon^q = [1 + o_\varepsilon(1)] \left[\left(\frac{M^*}{M} \right)^{\frac{r}{n}} - 1 \right] \varepsilon^{-r} + \beta \varepsilon^q \\ &\geq (1 + o_\varepsilon(1)) \frac{q+r}{q} \left(\frac{q\beta}{r} \right)^{\frac{r}{r+q}} \left[\left(\frac{M^*}{M} \right)^{\frac{r}{n}} - 1 \right]^{\frac{q}{r+q}}, \quad \text{where } \beta \gg 1. \end{aligned}$$

This contradicts to Lemma 6.1, which finishes the proof of our claim (6.46).

In view of (6.46), we find $\exists y_0 \in \mathbb{R}^n$ such that up to a subsequence,

$$\lim_{\varepsilon \rightarrow 0} \frac{x_\varepsilon - P_i}{\varepsilon} = y_0.$$

We next show y_0 satisfies (1.36). Since $q_i = q$, it follows from (6.34), (6.35), (1.33) and Fatou's lemma that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon^{-q} \int_{\mathbb{R}^n} V(\varepsilon x + x_\varepsilon) m_\varepsilon dx &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} \frac{V\left(\varepsilon \left(x + \frac{x_\varepsilon - P_i}{\varepsilon}\right) + P_i\right)}{|\varepsilon \left(x + \frac{x_\varepsilon - P_i}{\varepsilon}\right)|^q} \left| x + \frac{x_\varepsilon - P_i}{\varepsilon} \right|^q m_\varepsilon dx \\ &\geq \int_{\mathbb{R}^n} a_i |x + y_0|^q m_0 dx \geq \mu, \end{aligned} \quad (6.47)$$

where the last two equalities hold if and only if (1.36) holds. As a consequence, we obtain

$$\begin{aligned} e_{\alpha^*, M} &\geq \varepsilon^{-r} \left[\left(\frac{M^*}{M} \right)^{\frac{r}{n}} - 1 \right] (1 + o(1)) + \varepsilon^q \mu (1 + o(1)) \\ &\geq (1 + o(1)) \frac{q+r}{q} \left(\frac{q\mu}{r} \right)^{\frac{r}{r+q}} \left[\left(\frac{M^*}{M} \right)^{\frac{r}{n}} - 1 \right]^{\frac{q}{r+q}} \\ &= (1 + o(1)) \frac{q+r}{q} \left(\frac{q\mu}{r} \right)^{\frac{r}{r+q}} \left[1 - \left(\frac{M}{M^*} \right)^{\frac{r}{n}} \right]^{\frac{q}{r+q}} \left(\frac{M^*}{M} \right)^{\frac{rq}{n(r+q)}} \\ &= (1 + o(1)) \frac{q+r}{q} \left(\frac{q\mu}{r} \right)^{\frac{r}{r+1}} \left[1 - \left(\frac{M}{M^*} \right)^{\frac{r}{n}} \right]^{\frac{q}{r+q}}, \end{aligned} \quad (6.48)$$

where the “=” holds in the second inequality if and only if

$$\varepsilon = \left[\frac{r}{q\mu} \left[1 - \left(\frac{M}{M^*} \right)^{\frac{r}{n}} \right] \right]^{\frac{1}{r+q}} (1 + o(1)). \quad (6.49)$$

We collect the lower bound (6.48) and the upper bound (6.36) to find all equalities in (6.48) must hold. It follows that all “=” in (6.47) and (6.49) also hold. Therefore, we obtain (1.35) and (1.36), which finishes the proof of Theorem 1.5. \square

Theorem 1.5 demonstrates that when potential V admits typical local expansions e.g. power asymptotic function and the nonlinearity exponent in system (1.4) is critical, there exist ground states to problem (1.22) for $M < M^*$, which concentrate at the location where the potential V is weighted flattest as $M \nearrow M^*$. Moreover, we have captured the asymptotics of scaling factor ε .

7 Discussion

First of all, we have analyzed the $W^{2,p}$ regularity of the solution to the HJ equation (2.1) with subquadratic gradient terms by performing blow-up analysis. With the aid of the regularity results, we have employed the variational method to study the existence of least energy solutions to system (1.4) with the decreasing cost and some technical assumptions imposed on potential V when the nonlinearity exponent in the power local coupling is mass critical. Based on the seminal work of Cirant et al. [12], we have classified the ground states to the associated energy (1.15) in terms of the total mass of population density m when the coupling exponent $\alpha = \frac{r}{n}$. More precisely, we have obtained there exists a critical mass M^* such that if the total mass of density denoted by M satisfying $M < M^*$, problem (1.22) admits minimizers; otherwise, there is no minimizer to (1.22) by using the assumptions (V3) imposed on potential V when the exponent in the local coupling is mass critical. Furthermore, we have showed there exists the concentration phenomenon as $M \nearrow M^*$ and captured the asymptotic behaviors of minimizers. In particular, while proving our main results, Gagliardo-Nirenberg type's inequality corresponding to the ground states for the potential-free MFG system has been established. In our argument, some innovation ideas what we have proposed is the application of Pohozaev identities on proving strongly $L^1(\mathbb{R}^n)$ convergence of population density, which avoids the discussion of concentration compactness as shown in [12]. On the other hand, while studying the existence of least energy solutions, we relax the $|x|^b$ restriction on potential V by applying maximal regularities shown in Theorem 1.1 and [17].

We would like to point out that there are also some interesting questions that deserve investigations in the future. While establishing the optimal inequality in Section 4, we imposed some technical assumption on m , which is the boundedness of $\int_{\mathbb{R}^n} m|x|^b dx$ for sufficiently small $b > 0$. We believe that this condition is unnecessary since our techniques arise from the continuity method but the existence of minimizers to problem (1.24) in the mass subcritical exponent case must be established with this technical assumption on m due to the approaches taken in [12]. To remove this assumption, we must propose some new ideas and it seems significant to understand the properties of principal part in \mathcal{E} given by (1.15) more clearly. Other interesting question is the properties, such as radially symmetry, uniqueness and positivity of ground states, when system (1.4) is potential-free. Since the potential-free system is also coupled, it seems need other ideas rather than moving plane methods to show the symmetry of solutions. One of interesting directions is the extension of the potential functions V . We impose some mild assumptions (1.20) on V , which does not include the super exponential growth case and the logarithmic potential. It seems a challenge to remove the technical assumptions due to the lower bound estimates of the value function u .

Appendix A Ground States of Schrödinger equations and MFGs with $r' = 2$

This Appendix is devoted to the relationship between the ground states to (1.4) and (1.8), which is

Proposition A.1. *Assume $r = r' = 2$, H is given by (1.5), $f(m) = -C_f m^\alpha$ with $\alpha \in (0, \frac{r}{n})$ and V satisfies (1.12) in system (1.4). Let (m_M, w_M) be minimizers to problem (1.19). Correspondingly, $(m_M, u_M, \lambda_M) \in W^{1,p}(\mathbb{R}^n) \times C^2(\mathbb{R}^n) \times \mathbb{R}$ solves (1.4) for any fixed $M > 0$. In addition, let v_M be minimizers to problem*

$$\min_{v \in H^1(\mathbb{R}^n), \int_{\mathbb{R}^n} v^2 dx = M} \mathcal{F}(v) \quad (\text{A.1})$$

with $\mathcal{F}(v)$ defined by (1.9). Then we have that on one hand, $\bar{v}_M := \sqrt{m_M}$ give rise to minimizers of the problem (A.1); on the other hand, $(\bar{m}_M, \bar{w}_M) := (v_M^2, 2v_M \nabla v_M)$ give rise to minimizers of problem (1.19).

Proof. First of all, we show that $\nabla \sqrt{m_M} \in L^2(\mathbb{R}^n)$. To this end, by applying Theorem 3.1 in [40], one obtains

$$\int_{\mathbb{R}^n} \frac{|\nabla m_M|^2}{m_M} dx \leq \hat{C} \int_{\mathbb{R}^n} m_M |\nabla u_M|^2 dx, \quad (\text{A.2})$$

where $\hat{C} > 0$ is some constant. As shown in [12], when potential V satisfies (1.12), u_M at most grows algebraically but m_M has the exponential decay property. It follows from (A.2) that

$$\int_{\mathbb{R}^n} \frac{|\nabla m_M|^2}{m_M} dx \leq \hat{C} \int_{\mathbb{R}^n} m_M |\nabla u_M|^2 dx < +\infty, \quad (\text{A.3})$$

which indicates $\nabla \sqrt{m_M} \in L^2(\mathbb{R}^n)$.

On the other hand, since $(m_M, w_M) \in W^{1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ for any $p > 1$, are minimizers to (1.19) [12], we find for any fixed $M > 0$,

$$II_1 := \int_{\mathbb{R}^n} \frac{|\nabla m_M + g_M|^2}{m_M} dx \quad (\text{A.4})$$

exists with $w_M = \nabla m_M + g_M$ and $\nabla \cdot g_M = 0$ weakly. In addition, invoking (A.3) we have the facts that $\int_{\mathbb{R}^n} \frac{|\nabla m_M|^2}{m_M} dx$ and $\int_{\mathbb{R}^n} \frac{|g_M|^2}{m_M} dx$ are well-defined. Then, it follows that for any $R \rightarrow +\infty$,

$$\lim_{R \rightarrow +\infty} \int_{B_R(0)} \frac{|\nabla m_M + g_M|^2}{m_M} dx = II_1 < +\infty. \quad (\text{A.5})$$

Noting that $m > 0$ for any $x \in \mathbb{R}^n$, by straightforward computation, one has

$$II_1 = \lim_{R \rightarrow +\infty} \left[\int_{B_R(0)} \frac{|\nabla m_M|^2}{m_M} dx + 2 \int_{B_R(0)} \frac{\nabla m_M}{m_M} \cdot g_M dx + \int_{B_R(0)} \frac{|g_M|^2}{m_M} dx \right]. \quad (\text{A.6})$$

where $\frac{\nabla m_M}{m_M} = \nabla \log m_M$ is well-defined. Now, we claim that there exists a sequence $R_k \rightarrow +\infty$ as $k \rightarrow +\infty$ such that

$$\int_{B_{R_k}(0)} (\nabla \log m_M) \cdot g_M dx \rightarrow 0. \quad (\text{A.7})$$

Since $g_M \in L^2(B_R(0))$ and $\nabla \log m_M \in L^2(B_R(0))$ for any large but fixed $R > 0$, we apply the integration by parts to get

$$\int_{B_{R_k}(0)} (\nabla \log m_M) \cdot g_M dx = \int_{\partial B_{R_k}(0)} \log m_M g_M \cdot \nu dS, \quad (\text{A.8})$$

where we have used $\nabla \cdot g_M = 0$ weakly. Next, we shall show that

$$\int_{\partial B_{R_k}(0)} |\log m_M| \cdot |g_M| dS \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (\text{A.9})$$

Indeed, if the following fact holds:

$$\int_{\mathbb{R}^n} |\log m_M| \cdot |g_M| dx < +\infty, \quad (\text{A.10})$$

then we obtain the desired conclusion (A.9) since we have the fact that

$$\int_{\mathbb{R}^n} |\log m_M| \cdot |g_M| dx = \int_0^{+\infty} \left(\int_{\partial B_R(0)} |\log m_M| \cdot |g_M| dS \right) dR.$$

To prove (A.10), we apply Hölder's inequality to get

$$\left(\int_{\mathbb{R}^n} |\log m_M| \cdot |g_M| dx \right)^2 \leq \left(\int_{\mathbb{R}^n} m_M |\log m_M|^2 dx \right) \cdot \left(\int_{\mathbb{R}^n} \frac{|g_M|^2}{m_M} dx \right). \quad (\text{A.11})$$

To estimate the first term in Right Hand Side of (A.11), we use the fact $m_M |\log m_M|^2 \leq C(m_M^2 + \sqrt{m_M})$ with $m_M \geq 0$ and positive constant C to obtain

$$\int_{\mathbb{R}^n} m_M |\log m_M|^2 dx \leq C \left(\int_{\mathbb{R}^n} m_M^2 dx + \int_{\mathbb{R}^n} \sqrt{m_M} dx \right). \quad (\text{A.12})$$

Since $m_M \in L^2(\mathbb{R}^n)$ and $0 < m_M < e^{-\delta|x|}$ for some $\delta > 0$, we have $\sqrt{m_M} \in L^1(\mathbb{R}^n)$, which completes the proof of (A.9). In view of (A.8), one finds as $k \rightarrow +\infty$, the claim (A.7) holds. Focusing on (A.6), we have $g_M \equiv 0$ to guarantee (m_M, w_M) are minimizers to problem (1.19), which implies the energy (1.15) satisfied by (m_M, w_M) is exactly reduced to (1.9) by using the transform $m = v^2$. Now, we have the fact that the infima of problem (1.19) and problem (A.1) are the same.

Conversely, for any minimizer v_M to problem (A.1), it is straightforward to verify that $(\bar{m}_M, \bar{w}_M) := (v_M^2, 2v_M \nabla v_M) \in \mathcal{K}_M$ given by (1.17), which is a minimizer to problem (1.19). It completes the proof of our proposition. \square

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References

- [1] Y. Achdou, F. Camilli, and I. Capuzzo-Dolcetta. Mean field games: numerical methods for the planning problem. *SIAM J. Control Optim.*, 50(1):77–109, 2012.
- [2] Y. Achdou and I. Capuzzo-Dolcetta. Mean field games: numerical methods. *SIAM J. Numer. Anal.*, 48(3):1136–1162, 2010.
- [3] T. Bartsch and Z. Wang. Existence and multiplicity results for some superlinear elliptic problems on \mathbb{R}^N . *Commun. Part. Diff. Eq.*, 20(9-10):1725–1741, 1995.
- [4] C. Bernardini and A. Cesaroni. Ergodic mean-field games with aggregation of choquard-type. *J. Differential Equations*, 364:296–335, 2023.
- [5] M. Betta, R. D. Nardo, A. Mercaldo, and A. Perrotta. Gradient estimates and comparison principle for some nonlinear elliptic equations. *Commun. Pur. Appl. Anal.*, 14(3), 2015.
- [6] V. Bogachev, N. Krylov, M. Röckner, and S. Shaposhnikov. *Fokker–Planck–Kolmogorov Equations*, volume 207. American Mathematical Society, 2022.
- [7] F. Cagnetti, D. Gomes, H. Mitake, and H. Tran. A new method for large time behavior of degenerate viscous Hamilton-Jacobi equations with convex Hamiltonians. *Ann. Inst. H. Poincaré*, 32(1):183–200, 2015.
- [8] F. Camilli and F. Silva. A semi-discrete approximation for a first order mean field game problem. *Netw. Heterog. Media*, 7(2):263–277, 2012.

- [9] P. Cardaliaguet. Long time average of first order mean field games and weak KAM theory. *Dyn. Games Appl.*, 3(4):473–488, 2013.
- [10] P. Cardaliaguet, J. Lasry, P. Lions, and A. Porretta. Long time average of mean field games. *Netw. Heterog. Media*, 7(2):279–301, 2012.
- [11] E. Carlini and F. Silva. A fully discrete semi-Lagrangian scheme for a first order mean field game problem. *SIAM J. Numer. Anal.*, 52(1):45–67, 2014.
- [12] A. Cesaroni and M. Cirant. Concentration of ground states in stationary mean-field games systems. *Anal. PDE*, 12(3):737–787, 2018.
- [13] M. Cirant. A generalization of the Hopf-Cole transformation for stationary mean-field games systems. *C. R. Math. Acad. Sci. Paris*, 353(9):807–811, 2015.
- [14] M. Cirant. Stationary focusing mean-field games. *Commun. Part. Diff. Eq.*, 41(8):1324–1346, 2016.
- [15] M. Cirant and A. Goffi. Maximal L^q -regularity for parabolic Hamilton-Jacobi equations and applications to mean field games. *Ann. PDE*, 7(2):Paper No. 19, 40, 2021.
- [16] M. Cirant and A. Goffi. On the Problem of Maximal L^q -regularity for Viscous Hamilton–Jacobi Equations. *Arch. Ration. Mech. Anal.*, 240:1521–1534, 2021.
- [17] M. Cirant and G. Verzini. Local Hölder and maximal regularity of solutions of elliptic equations with superquadratic gradient terms. *Adv. Math.*, 409:108700, 2022.
- [18] D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order*, volume 224. Springer, 1977.
- [19] A. Goffi. Hölder regularity and liouville properties for nonlinear elliptic inequalities with power-growth gradient terms. *P. Roy. Soc. Edinb. A.*, 153(6):1833–1857, 2023.
- [20] A. Goffi. On the optimal L^q -regularity for viscous hamilton–jacobi equations with subquadratic growth in the gradient. *Communications in Contemporary Mathematics*, page 2350019, 2023.
- [21] A. Goffi and F. Pediconi. Sobolev regularity for nonlinear Poisson equations with Neumann boundary conditions on Riemannian manifolds. *Forum Math.*, 35(2):431–456, 2023.
- [22] D. Gomes and H. Mitake. Existence for stationary mean-field games with congestion and quadratic Hamiltonians. *NoDEA Nonlinear Differential Equations Appl.*, 22(6):1897–1910, 2015.
- [23] D. Gomes, S. Patrizi, and V. Voskanyan. On the existence of classical solutions for stationary extended mean field games. *Nonlinear Anal.*, 99:49–79, 2014.
- [24] D. Gomes, E. Pimentel, and H. Sánchez-Morgado. Time-dependent mean-field games in the superquadratic case. *ESAIM Control Optim. Calc. Var.*, 22(2):562–580, 2016.
- [25] D. Gomes, G. Pires, and H. Sánchez-Morgado. A-priori estimates for stationary mean-field games. *Netw. Heterog. Media*, 7(2):303–314, 2012.
- [26] D. A. Gomes, E. Pimentel, and V. Voskanyan. *Regularity theory for mean-field game systems*. Springer, 2016.
- [27] N. Grenon, F. Murat, and A. Porretta. A priori estimates and existence for elliptic equations with gradient dependent terms. *ANNALI SCUOLA NORMALE SUPERIORE-CLASSE DI SCIENZE*, pages 137–205, 2014.
- [28] O. Guéant. Mean field games and applications to economics. *PhD Thesis (Univ. Paris-Dauphine)*, 2009.
- [29] O. Guéant. A reference case for mean field games models. *J. Math. Pures Appl. (9)*, 92(3):276–294, 2009.
- [30] Y. Guo and R. Seiringer. On the mass concentration for bose–einstein condensates with attractive interactions. *Lett. Math. Phys.*, 104:141–156, 2014.

- [31] Y. Guo, X. Zeng, and H. Zhou. Energy estimates and symmetry breaking in attractive bose–einstein condensates with ring-shaped potentials. *Ann. I. H. Poincaré-an.*, 33(3):809–828, 2016.
- [32] M. Huang, R. Malhamé, and P. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.*, 6(3):221–251, 2006.
- [33] M. Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in \mathbb{R}^n . *Arch. Ration. Mech. Anal.*, 105:243–266, 1989.
- [34] A. Lachapelle, J. Salomon, and G. Turinici. Computation of mean field equilibria in economics. *Math. Models Methods Appl. Sci.*, 20(4):567–588, 2010.
- [35] J. Lasry and P. Lions. Mean field games. *Jpn. J. Math.*, 2(1):229–260, 2007.
- [36] C. Li and K. Zhang. A note on the Gagliardo-Nirenberg inequality in a bounded domain. *Commun. Pure Appl. Anal.*, 21(12):4013–4017, 2022.
- [37] P. Lions. Quelques remarques sur les problèmes elliptiques quasilinéaires du second ordre. *J. Anal. Math.*, 45(1):234–254, 1985.
- [38] M. M. Cirant, A. Cosenza, and G. Verzini. Ergodic mean field games: existence of local minimizers up to the sobolev critical case. *arXiv preprint arXiv:2301.11692*, 2023.
- [39] A. Mészáros and F. Silva. A variational approach to second order mean field games with density constraints: the stationary case. *J. Math. Pures Appl.*, 104(6):1135–1159, 2015.
- [40] G. Metafune, D. Pallara, and A. Rhandi. Global properties of invariant measures. *J. Funct. Anal.*, 223(2):396–424, 2005.
- [41] G. Palatucci and A. Pisante. Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces. *Calc. Var. Partial Differential Equations*, 50(3):799–829, 2014.
- [42] M. Reed. *Methods of modern mathematical physics: Functional analysis*. Elsevier, 2012.