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Ergodic Mean-Field Games with aggregation of Choquard-type

Chiara Bernardini *, Annalisa Cesaroni

Dipartimento di Matematica "Tullio Levi Civita", Università di Padova, Via Trieste 63, 35121 Padova, Italy

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Abstract

We consider second-order ergodic Mean-Field Games systems in the whole space $\mathbb{R}^N$ with coercive potential and aggregating nonlocal coupling, defined in terms of a Riesz interaction kernel. These MFG systems describe Nash equilibria of games with a large population of indistinguishable rational players attracted toward regions where the population is highly distributed. Equilibria solve a system of PDEs where an Hamilton-Jacobi-Bellman equation is combined with a Kolmogorov-Fokker-Planck equation for the mass distribution. Due to the interplay between the strength of the attractive term and the behavior of the diffusive part, we will obtain three different regimes for the existence and non existence of classical solutions to the MFG system. By means of a Pohozaev-type identity, we prove nonexistence of regular solutions to the MFG system without potential in the Hardy-Littlewood-Sobolev-supercritical regime. On the other hand, using a fixed point argument, we show existence of classical solutions in the Hardy-Littlewood-Sobolev-subcritical regime at least for masses smaller than a given threshold value. In the mass-subcritical regime we show that actually this threshold can be taken to be $+\infty$.

MSC: 35J47; 35B33; 49N70; 35Q55

Keywords: Stationary Mean-Field Games; Choquard equation; Riesz potential; Fixed point argument; Pohozaev identity

* Corresponding author.
E-mail addresses: chiara.bernardini@math.unipd.it (C. Bernardini), annalisa.cesaroni@unipd.it (A. Cesaroni).

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1. Introduction

In this paper, we study ergodic Mean-Field Games systems defined in the whole space $\mathbb{R}^N$ with a coercive potential V and attractive nonlocal coupling, defined in terms of a Riesz interaction kernel. More in details, given $M > 0$, we consider elliptic systems of the form

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - \int_{\mathbb{R}^N} \frac{m(y)}{|x-y|^{N-\alpha}} dy \\ -\Delta m - \operatorname{div}(m\nabla u(x)\,|\nabla u(x)|^{\gamma-2}) = 0 \\ \int_{\mathbb{R}^N} m = M, \quad m \geq 0 \end{cases} \quad \text{in } \mathbb{R}^N \tag{1}$$

where $\gamma > 1, \alpha \in (0, N)$ are fixed. Note that the unknowns in the system (1) are the functions u, m and the constant $\lambda \in \mathbb{R}$ which can be interpreted as a Lagrange multiplier, related to the mass constraint $\int_{\mathbb{R}^N} m = M$.

We will assume that the potential V is a locally Hölder continuous coercive function, that is there exist b and C_V positive constants such that

$$C_V^{-1}(\max\{|x| - C_V, 0\})^b \leq V(x) \leq C_V(1 + |x|)^b, \quad \forall x \in \mathbb{R}^N. \tag{2}$$

The assumption of V to be non-negative is not restrictive, we can assume more generally that V is bounded from below and shift appropriately λ.

Finally, the coupling in the system is given through the interaction term $-m * K_\alpha$, where K_α is the Riesz potential of order $\alpha \in (0, N)$ defined as

$$K_\alpha(x) = \frac{1}{|x|^{N-\alpha}}.$$

We assume the Hamiltonian in the system (1) has the form $H(p) = \frac{1}{\gamma}|p|^\gamma$ for sake of simplicity but actually it may be more general, namely we may assume that $H : \mathbb{R}^N \to \mathbb{R}$ is strictly convex, $H \in C^2(\mathbb{R}^N \setminus \{0\})$ and there exist $C_H,\ K > 0$ and $\gamma > 1$ such that $\forall p \in \mathbb{R}^N$, it holds

$$C_H|p|^\gamma - K \leq H(p) \leq C_H|p|^\gamma$$
$$\nabla H(p) \cdot p - H(p) \geq K^{-1}|p|^\gamma - K$$
$$|\nabla H(p)| \leq K|p|^{\gamma-1}.$$

Mean-Field Games have been introduced in the seminal papers of Lasry and Lions [24] and by Huang, Caines and Malhamé [22] in order to describe Nash equilibria of differential games with infinitely many infinitesimal rational players; this led to a broader study, also encouraged by their powerful applications in a wide range of disciplines (equations of this kind arise in Economics, Finance and models of social systems). The key idea underlying the theory comes from Statistical Mechanics and Physics, and consists in a mean-field approach to describe equilibria in a system of many interacting particles. The theory of Mean-Field Games models the behavior of a very large number of rational and indistinguishable players aiming at minimizing a certain cost, by anticipating the distribution of the overall population which result from the actions of all other players. We refer to [19,20] for a general presentation of Mean-Field Games and their applications. In our setting, the dynamics of each player is described by the following controlled stochastic differential equation

$$dX_t = -v_t\, dt + \sqrt{2}\, dB_t$$

where v_t is the controlled velocity and B_t is a standard N dimensional Brownian motion. Each agent chooses v_t in order to minimize the following long time average cost

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \left[\frac{|v_t|^{\gamma'}}{\gamma'} + V(X_t) - K_\alpha * m(X_t)\right] dt$$

where $\gamma' = \frac{\gamma}{\gamma-1}$ is the conjugate exponent of γ and $m(x)$ is the density of population at $x \in \mathbb{R}^N$. In the ergodic setting, the distribution law of each player moving with optimal speed converges as $t \to +\infty$ to an invariant measure μ (independent of the initial position) and μ coincides, in a equilibrium regime for the game, with the density of the population m. From a PDE viewpoint, equilibria of the differential game are encoded by solutions of the system (1), where the Hamilton-Jacobi-Bellman equation takes into account the value of the game λ and the optimal speed $-\nabla u|\nabla u|^{\gamma-2}$ of the optimal control problem of a typical agent, and the Kolmogorov-Fokker-Planck equation gives the density of the overall population m.

In the case when $\gamma = \gamma' = 2$, as pointed out in [24], using the Hopf-Cole transformation $v(x) := e^{-u(x)/2}$ we can reduce the MFG system (1) to a single PDE. In particular we observe that with the previous change of variable, setting $m(x) = v^2(x)$, the MFG system (1) is equivalent to the normalized Choquard equation

$$\begin{cases} -2\Delta v + (V(x) - \lambda)v = (K_\alpha * v^2)v & \\ \int_{\mathbb{R}^N} v^2(x)dx = M, \qquad v > 0 & \end{cases} \qquad \text{in } \mathbb{R}^N, \tag{3}$$

with associated energy

$$\mathcal{E}(v) = \int_{\mathbb{R}^N} 2|\nabla v|^2 + V(x)v^2 dx - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^2(x)\, v^2(y)}{|x-y|^{N-\alpha}} dx\, dy.$$

Choquard-type equations have been intensively studied during the last decades and have appeared in the context of various mean-field type physical models (refer to [26,29,30,33,34] and references therein for a complete overview). Indeed their solutions are steady states of a generalized nonlinear Schrödinger equation, with an attractive interaction potential given in terms of the Riesz interaction kernel, which is therefore weaker and with longer range than the usual power-type potential in nonlinear Schrödinger equation. We recall that the relation between MFG systems and normalized nonlinear elliptic equations has been exploited in the recent work [36] in the case of nonlinear Schrödinger systems.

Going back to our Mean-Field Game system, the two distinctive features of our model are the following: the state space is the whole Euclidean space $\mathbb{R}^N$, and the coupling is aggregative and defined in terms of a Riesz-type interaction kernel. Usually, Mean-Field Game systems are considered in bounded domains, with Neumann or periodic boundary conditions, in order to avoid non-compactness issues. We recall some works in the non compact setting: in particular [3] in the linear-quadratic framework, [37] in the time-dependent case, [18] for regularity results and finally [6], where a system analogous to (1) has been considered, with power-type nonlinearity. In the unbounded setting, the dissipation induced by the Brownian motion has to be compensated by the optimal velocity, which is a priori unknown and depends by the distribution m itself and on the coercive potential V. The coercive potential V describes spatial preferences of agents and hence discourages them to be far away from the origin. Moreover, due to the presence of the Riesz-type interaction potential $-K_\alpha * m$ which represents the coupling between the individual and the overall population, every player of the game is attracted toward regions where the population is highly distributed. Most of the MFG literature focuses on the study of systems with competition, namely when the coupling descourages aggregation: this assumption is essential if one seeks for uniqueness of equilibria, and it is in general crucial in many existence and regularity arguments (see [19]). Focusing MFG systems, namely models with coupling which encourages aggregation, have been studied for instance in [6,7,9,10,17] in the stationary setting.

In this paper we provide existence and nonexistence results of classical solutions solving the MFG system (1), where by *classical solution* we will mean a triple $(u, m, \lambda) \in C^2(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N) \times \mathbb{R}$ for every $p \in (1, +\infty)$. Our focus will be to obtain classical solutions which satisfy some integrability conditions and boundary conditions at ∞ which will be meaningful from the point of view of the game. In particular, we will require some integrability properties of the optimal speed with respect to m, namely

$$m|\nabla u|^\gamma \in L^1(\mathbb{R}^N) \qquad Vm \in L^1(\mathbb{R}^N) \qquad \text{and} \qquad |\nabla m||\nabla u| \in L^1(\mathbb{R}^N). \tag{4}$$

Indeed, if one looks at the Kolmogorov equation, such integrability properties are important to ensure some minimal regularity of m and uniqueness of the invariant distribution itself (see [21,35]). Regularity and boundedness of m is quite crucial in our setting: indeed, due to the aggregating forces, m has an intrinsic tendency to concentrate and hence to develop singularities. Moreover the Lagrange multiplier λ will be uniquely defined as the generalized principal eigenvalue (see for details [4,8,23]): for fixed $m \in L^1(\mathbb{R}^N)$ such that $K_\alpha * m \in C^{0,\theta}(\mathbb{R}^N)$ for some $\theta \in (0, 1)$, we define λ as

$$\lambda := \sup\left\{ c \in \mathbb{R} \,\middle|\, \exists v \in C^2(\mathbb{R}^N) \text{ solving } \Delta v + \frac{1}{\gamma}|\nabla v|^\gamma + c = V(x) - K_\alpha * m \right\}.$$

Once that we know this value exists, it is possible to show that there exists $u \in C^2(\mathbb{R}^N)$ solving the HJB equation with such value λ, and that such solution u is coercive i.e.

$$u(x) \to +\infty \qquad \text{as } |x| \to +\infty \tag{5}$$

and moreover its gradient has polynomial growth (see Section 2 and the references [4,8,23]). Note that (5) is a quite natural "boundary" condition for ergodic HJB equations on the whole space: indeed the optimal speed would give rise to an ergodic process, so in particular, at least heuristically $-\nabla u \cdot x < 0$ for $|x| \to +\infty$, (refer to [21] and references therein, for more information about ergodic problems on the whole space and their characterization in terms of Lyapunov functions). Existence results for such classical solutions will depend on the interplay between the dissipation (i.e. by the diffusive term in the system) and the aggregating forces (described in terms of the Riesz potential K_α and the coercive potential V). So, we get that the MFG system (1) shows three different regimes which correspond to $\alpha \in (0, N - 2\gamma')$, $\alpha \in (N - 2\gamma', N - \gamma')$

and $\alpha \in (N-\gamma', N)$. We will refer to $\alpha = N-2\gamma'$ as the *Hardy-Littlewood-Sobolev-critical exponent* and to $\alpha = N-\gamma'$ as the *mass-critical* (or L^2-critical) *exponent*, in analogy with the regimes appearing in the study of the Choquard equation (3) when $\gamma' = 2$. Obviously if $\gamma' \geq N$, there exists just one regime, which will be the mass-subcritical regime $\alpha \in [0, N)$, whereas if $\frac{N}{2} \leq \gamma' < N$ there will be just 2 regimes.

First of all we observe that for classical solutions to (1) with $V \equiv 0$ and which satisfy (4), a Pohozaev type identity holds (see Proposition 3.2):

$$(2-N)\int_{\mathbb{R}^N} \nabla u \cdot \nabla m \, dx + \left(1-\frac{N}{\gamma}\right)\int_{\mathbb{R}^N} m|\nabla u|^\gamma dx = \lambda N M + \frac{\alpha+N}{2}\int_{\mathbb{R}^{2N}} \frac{m(x)m(y)}{|x-y|^{N-\alpha}} dxdy. \tag{6}$$

Also in presence of the potential a similar identity holds, under the additional integrability condition that $m\nabla V \cdot x \in L^1(\mathbb{R}^N)$. For MFG in the periodic setting with polynomial interaction potential an analogous Pohozaev identity has been proved in [9]. For the case of the Choquard equation we refer to [33] and references therein.

In the *Hardy-Littlewood-Sobolev-supercritical regime* $0 < \alpha < N-2\gamma'$, the Pohozaev identity, together with the fact that $\lambda \leq 0$ (see Lemma 2.11), implies that solutions to the MFG system (1) do not exist. More precisely, we obtain the following nonexistence result.

Theorem 1.1. *Assume that $\alpha \in (0, N-2\gamma')$ and $V \equiv 0$. Then, the MFG system (1) has no classical solutions $(u, m, \lambda) \in C^2(\mathbb{R}^N) \times W^{1,\frac{2N}{N+\alpha}}(\mathbb{R}^N) \times \mathbb{R}$ which satisfy (4) and (5).*

In the case when $N-2\gamma' < \alpha < N$ we obtain existence of classical solutions to the MFG system (1) by means of a Schauder fixed point argument (refer to [2] and see also [9]). More in detail, we consider a regularized version of problem (1), obtained by convolving the Riesz-interaction term with a sequence of standard symmetric mollifiers (see (54) below). Taking advantage of the fixed-point structure associated to the MFG system and exploiting the Schauder Fixed Point Theorem, we show that solutions to the "regularized" version of the MFG system do exist. Then we provide a priori uniform estimates on the solutions to the regularized problem, which allow us to pass to the limit and obtain a classical solution of the MFG system (1).

Theorem 1.2. *Assume that the potential V is locally Hölder continuous and satisfies (2). We have the following results:*

i. if $N-\gamma' < \alpha < N$ then, for every $M > 0$ the MFG system (1) admits a classical solution (u, m, λ);
ii. if $N-2\gamma' < \alpha \leq N-\gamma'$ then, there exists a positive real value $M_0 = M_0(N, \alpha, \gamma, C_V, b)$ such that if $M \in (0, M_0)$ the MFG system (1) admits a classical solution (u, m, λ).

Moreover in both cases there exists a constant $C > 0$ such that

$$|\nabla u(x)| \leq C(1+|x|)^{\frac{b}{\gamma}} \qquad u(x) \geq C|x|^{\frac{b}{\gamma}+1} - C^{-1},$$

where $C = C(C_V, b, \gamma, N, \lambda, \alpha)$, $\sqrt{m} \in W^{1,2}(\mathbb{R}^N)$ and it holds

$$m|\nabla u|^\gamma \in L^1(\mathbb{R}^N), \qquad mV \in L^1(\mathbb{R}^N), \qquad |\nabla u||\nabla m| \in L^1(\mathbb{R}^N).$$

Note that in the mass-subcritical case, solutions to the MFG exist for every mass M, whereas in the mass-supercritical case and mass-critical case (namely for $\alpha \in (N-2\gamma', N-\gamma']$) we provide existence just for sufficiently small masses, below some threshold M_0, due to the fact that in this case the interaction attractive potential is stronger than the diffusive part.

The Hardy-Littlewood-Sobolev critical exponent is not covered by our analysis. Indeed it is possible to prove existence of solutions to the regularized problem also in this case, for sufficiently small masses (see Theorem 4.7). Nevertheless in order to pass to the limit in the regularization, we need to obtain a priori L^∞ bounds on solutions m_k to the regularized problem, starting from uniform bounds in $L^{\frac{2N}{N+\alpha}} \cap L^1$. This is not possible at the critical level $\alpha = N-2\gamma'$, due to critical rescaling properties of the Sobolev critical exponent: a priori uniform L^∞ bounds on m_k only hold in the range when we have a uniform bound in L^q, for $q > \frac{N}{\gamma'+\alpha}$ (see Theorem 2.12) and $\frac{2N}{N+\alpha} > \frac{N}{\gamma'+\alpha}$ only in the Hardy-Littlewood-Sobolev subcritical regime. One way to circumvent this difficulty would be to obtain at the critical level $\alpha = N-2\gamma'$, by using regularity estimates on the viscous Hamilton-Jacobi equation and on the Fokker Planck equation and a smallness condition on $\|m\|_{\frac{N}{N-\gamma'}}$, a priori uniform bounds on m in L^q for some $q > \frac{N}{N-\gamma'}$, in order to be able to apply Theorem 2.12. This kind of result has been obtained recently in [11] for MFG in bounded domains with Neumann boundary conditions, and with a nonlinear Schrödinger type potential. This problem is related to the maximal regularity of solutions to viscous Hamilton-Jacobi equation $-\Delta u + |\nabla u|^\gamma = f(x)$ (see [12,13,16]). When $m \in L^{\frac{N}{N-\gamma'}}$, then by Hardy-Littlewood-Sobolev inequality (refer to Theorem 2.6) $K_\alpha * m \in L^{\frac{N}{\gamma'}}$, which is a critical threshold in this setting.

To understand better this difference and also the deep analogy with normalized Choquard-type equations, it will be useful to analyze the problem from a variational point of view. Existence of solutions to the normalized Choquard equations has been first investigated using variational methods by E.H. Lieb [27] and P.-L. Lions [29,31], while more recently Li-Ye [25] studied existence of positive solutions to (3) by using a minimax procedure and the concentration-compactness principle. As Lasry and Lions first pointed out in [28], solutions to (1) correspond to critical points of the following energy

$$\mathcal{E}(m,w) := \begin{cases} \int\limits_{\mathbb{R}^N} \frac{m}{\gamma'} \left|\frac{w}{m}\right|^{\gamma'} + V(x)\, m\, dx - \frac{1}{2} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{m(x)m(y)}{|x-y|^{N-\alpha}} dx\, dy & \text{if } (m,w) \in \mathcal{K}_M, \\ +\infty & \text{otherwise} \end{cases} \tag{7}$$

where $w := -m|\nabla u|^{\gamma-2}\nabla u$ and the constraint set is defined as

$$\begin{aligned} \mathcal{K}_M := \Big\{ (m,w) \in (L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)) \times L^1(\mathbb{R}^N) \quad \text{s.t.} \quad \int\limits_{\mathbb{R}^N} m\, dx = M, \quad m \geq 0 \text{ a.e.} \\ \int\limits_{\mathbb{R}^N} m(-\Delta\varphi)\, dx = \int\limits_{\mathbb{R}^N} w \cdot \nabla\varphi\, dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N) \Big\} \end{aligned} \tag{8}$$

with

$$q := \begin{cases} \frac{N}{N-\gamma'+1} & \text{if } \gamma' < N \\ \gamma' & \text{if } \gamma' \geq N \end{cases}. \tag{9}$$

If $N-\gamma' < \alpha < N$, so in the *mass-subcritical regime*, the energy $\mathcal{E}$ is bounded from below, indeed using elliptic regularity results for the Kolmogorov equation (see Proposition 2.4 below, in this case $1 < \frac{2N}{N+\alpha} < 1 + \frac{\gamma'}{N}$ hence we can use (18)), the Hardy-Littlewood-Sobolev inequality (see (24)) and the fact that $V \geq 0$, we get

$$\mathcal{E}(m,w) \geq C_1 \varepsilon^{\gamma'} \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{2\gamma'}{N-\alpha}} - C_2 \|m\|_{L^\beta(\mathbb{R}^N)}^2$$

where we denoted $\beta = \frac{2N}{N+\alpha}$. Hence, $\inf_{(m,w)\in\mathcal{K}_M} \mathcal{E}(m,w)$ is well defined and by means of classical direct methods and compactness arguments, it is possible to construct global minimizers. Then a linearization argument and a convex duality theorem allow us to show that minimizers (m,w) of $\mathcal{E}$ correspond to solutions to the MFG system (1) (for more details we refer to [6,7]). In the *mass-critical regime*, namely for $\alpha = N-\gamma'$, the energy is bounded from below just for sufficiently small masses M, and we may construct in this range global minimizers. In the *mass-supercritical regime*, namely for $0 < \alpha < N-\gamma'$, the energy is not bounded from below in general so no global minimum can be found. Nonetheless some compactness of sequences with finite energy is still available in the *Hardy-Littlewood-Sobolev-subcritical regime* $N-2\gamma' \leq \alpha < N-\gamma'$. In particular, we may consider a minimization problem adding a smallness constraint on the $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ norm of m and we may show that if the total mass of m is sufficiently small, then the constrained minimizers are actually local free minimizers of the problem. This procedure would provide solutions to the Mean-Field Game which should coincide with the solutions we obtained in Theorem 1.2 for $\alpha \in (N-2\gamma', N-\gamma')$ by using Schauder Fixed Point Theorem and imposing a smallness condition on the $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ norm of m. A similar procedure for constructing local minimizers has recently been developed for MFG in bounded domains with Neumann boundary conditions and local aggregative interaction potential of polynomial type (i.e. with a nonlinear Schrödinger type potential), we refer to [11]. Moreover, since the energy is becoming more and more negative as the $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ norm of m increases (as it can be observed by a simple rescaling argument), then we expect that with a nontrivial adaptation of the mountain-pass theorem, it should be possible to construct in the Hardy-Littlewood-Sobolev-subcritical regime $N-2\gamma' < \alpha < N-\gamma'$ also solutions to the MFG with a min-max procedure (analogously to what is done in the case of normalized Choquard equation, see [25]). We plan to investigate this issue in a forthcoming paper.

Finally we leave open the problem of existence of classical solutions to the MFG system for $\alpha \in [N-2\gamma', N)$ when $V \equiv 0$. Using the variational approach, and an appropriate adaptation of the concentration-compactness Lions theorem, one of the author provided existence of solution in the mass subcritical regime $\alpha \in (N-\gamma', N)$ as global minimizers of the energy (7) with $V \equiv 0$ among competitors with appropriate integrability condition at infinity, see [5]. We expect that in the supercritical mass regime $\alpha \in (N-2\gamma', N-\gamma')$ local minimizers of the free energy are not present, but that it could be possible to construct a critical point of the energy by means of concentration-compactness arguments together with a min-max procedure. We plan to investigate this issue in a forthcoming paper.

The paper is organized as follows. Section 2 contains some preliminary results. In particular we recall regularizing properties of the Riesz interaction kernel, some a priori elliptic estimates for solutions to the Kolmogorov equation, a priori gradient estimates for solutions to the Hamilton Jacobi Bellman equation and finally uniform L^∞ bounds for m, solution to (1). In Section 3 we provide the Pohozaev identity and the proof of the non-existence result Theorem 1.1. Section 3 contains the proof of the existence result Theorem 1.2.

In what follows, $C, C_1, C_2, K_1, \dots$ denote generic positive constants which may change from line to line and also within the same line. Moreover γ' denotes the conjugates exponent of γ, that is $\gamma' = \frac{\gamma}{\gamma-1}$.

2. Preliminaries

2.1. Regularity results for the Kolmogorov equation

Lemma 2.1. *Let $u \in C^{2,\theta}(\mathbb{R}^N)$ and $m \in W^{1,2}(\mathbb{R}^N)$ be a solution (in the distributional sense) to*

$$-\Delta m(x) - \operatorname{div}\left(m(x)\,\nabla u(x)\,|\nabla u|^{\gamma-2}\right) = 0 \quad \text{in } \mathbb{R}^N, \tag{10}$$

where $\gamma > 1$ is fixed. Then, $m \in C^{2,\alpha}(\mathbb{R}^N)$. Moreover, if $m \geq 0$ and $m \not\equiv 0$, then $m(x) > 0$ for any $x \in \mathbb{R}^N$.

Proof. If $\gamma \geq 2$, then m solves

$$-\Delta m - b(x)\cdot\nabla m(x) - m(x)\operatorname{div} b(x) = 0$$

where $b(x) := |\nabla u|^{\gamma-2}\nabla u(x) \in C^{1,\theta}(\mathbb{R}^N)$ and $\operatorname{div} b(x) \in C^{0,\theta}(\mathbb{R}^N)$. By elliptic regularity (see e.g. [15, Theorem 8.24]) we get that $m \in C^{0,\alpha}$. Denoting by $f := m\nabla u|\nabla u|^{\gamma-2}$ we have $-\Delta m = \operatorname{div} f$ where $f \in C^{0,\alpha}$, then by [15, Theorem 4.15] we get that $m \in C^{1,\alpha}$ and hence

$$-\Delta m = \operatorname{div}\left(m\nabla u|\nabla u|^{\gamma-2}\right) \in C^{0,\min\{\alpha,\theta\}}$$

so $m \in C^{2,\min\{\alpha,\theta\}}$, and iterating we finally obtain that $m \in C^{2,\theta}$. If $1 < \gamma < 2$, $b(x)$ is just an Hölder continuous function, hence m is a weak solution of equation (10). In this case, we can replace $b(x)$ with $b_\varepsilon(x) := \nabla u(x)(\varepsilon + |\nabla u|^2)^{\frac{\gamma}{2}-1}$ and m_ε is a-posteriori a classical solution to the approximate equation

$$-\Delta m - \operatorname{div}\left(m(x)\,b_\varepsilon(x)\right) = 0.$$

We can conclude letting $\varepsilon \to 0$. If $m \geq 0$ on $\mathbb{R}^N$, we also have that m satisfy

$$-\Delta m - b(x)\cdot\nabla m(x) - \left(\operatorname{div} b(x)\right)^+ m(x) \leq 0,$$

since $\int_{\mathbb{R}^N} m\,dx = M > 0$, the Strong Minimum Principle (refer e.g. to [15, Theorem 8.19]) implies that $m > 0$ in $\mathbb{R}^N$ (indeed m can not be equal to 0, unless it is constant, which is impossible). □

We will use the following result (proved in [6, Proposition 2.4]) which takes advantage of some classical elliptic regularity results of Agmon [1].

Proposition 2.2. *Let $m \in L^p(\mathbb{R}^N)$ for $p > 1$ and assume that for some $K > 0$*

$$\left| \int_{\mathbb{R}^N} m \Delta\varphi \, dx \right| \leq K \|\nabla\varphi\|_{L^{p'}(\mathbb{R}^N)}, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

Then, $m \in W^{1,p}(\mathbb{R}^N)$ and there exists a constant $C > 0$ depending only on p such that

$$\|\nabla m\|_{L^p(\mathbb{R}^N)} \leq C K.$$

We prove now some a priori estimates for solutions to the Kolmogorov equation. Let us fix $p \in (1, +\infty)$ and $M > 0$.

Proposition 2.3. *Let us consider a couple $(m, w) \in (L^p(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N)) \times L^1(\mathbb{R}^N)$ which solves weakly*

$$-\Delta m + \operatorname{div} w = 0, \quad in \ \mathbb{R}^N.$$

Assume also that $\int_{\mathbb{R}^N} m(x)\, dx = M$, $m \geq 0$ a.e. and

$$E := \int_{\mathbb{R}^N} m \left|\frac{w}{m}\right|^{\gamma'} dx < +\infty. \tag{11}$$

Then, we have that

$$m \in W^{1,r}(\mathbb{R}^N)$$

for r such that $\frac{1}{r} = \left(1 - \frac{1}{\gamma'}\right)\frac{1}{p} + \frac{1}{\gamma'}$ (i.e. $r = \frac{p\gamma'}{\gamma'+p-1}$) and there exists a constant C, depending on r, such that

$$\|m\|_{W^{1,r}(\mathbb{R}^N)} \leq C(E + M)^{\frac{1}{\gamma'}} \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}}. \tag{12}$$

Proof. By definition of weak solution we have

$$-\int_{\mathbb{R}^N} m \, \Delta\varphi \, dx = \int_{\mathbb{R}^N} w \cdot \nabla\varphi \, dx, \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^N),$$

using Hölder inequality (since $\frac{1}{r} = \left(1 - \frac{1}{\gamma'}\right)\frac{1}{p} + \frac{1}{\gamma'}$, it holds $\frac{1}{p\gamma} + \frac{1}{\gamma'} + \frac{1}{r'} = 1$) we obtain

$$\left| \int_{\mathbb{R}^N} m \, \Delta\varphi \, dx \right| \leq \int_{\mathbb{R}^N} |w| \, |\nabla\varphi| dx = \int_{\mathbb{R}^N} \left(\left|\frac{w}{m}\right|^{\gamma'} m\right)^{\frac{1}{\gamma'}} m^{\frac{1}{\gamma}} |\nabla\varphi| \, dx$$

$$\leq \left(\int_{\mathbb{R}^N} \left| \frac{w}{m} \right|^{\gamma'} m \, dx \right)^{\frac{1}{\gamma'}} \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}} \|\nabla \varphi\|_{L^{r'}(\mathbb{R}^N)}$$

and hence

$$\left| \int_{\mathbb{R}^N} m \, \Delta \varphi \, dx \right| \leq E^{\frac{1}{\gamma'}} \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}} \|\nabla \varphi\|_{L^{r'}(\mathbb{R}^N)}.$$

Since $\|m\|_{L^1(\mathbb{R}^N)} = M$ and $m \in L^p(\mathbb{R}^N)$, by interpolation we get

$$\|m\|_{L^r(\mathbb{R}^N)} \leq \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}} M^{\frac{1}{\gamma'}} \tag{13}$$

therefore $m \in L^r(\mathbb{R}^N)$. From Proposition 2.2 with $K = E^{\frac{1}{\gamma'}} \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}}$, we obtain that $m \in W^{1,r}(\mathbb{R}^N)$ and there exists a constant $C > 0$, depending on r, such that

$$\|\nabla m\|_{L^r(\mathbb{R}^N)} \leq C \, E^{\frac{1}{\gamma'}} \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}}. \tag{14}$$

By (13) and (14), we can conclude that

$$\|m\|_{W^{1,r}(\mathbb{R}^N)} \leq \left(M^{\frac{1}{\gamma'}} + C E^{\frac{1}{\gamma'}} \right) \|m\|_{L^p}^{\frac{1}{\gamma}} \leq C(E+M)^{\frac{1}{\gamma'}} \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}}. \quad \square$$

Proposition 2.4. *Under the assumption of Proposition 2.3, we have the following results:*

i) if $1 < p < 1 + \frac{\gamma'}{N}$ then, there exists $\delta_1 = \frac{1}{p-1} \left(\frac{\gamma'}{N} + 1 - p \right)$ such that

$$\|m\|_{L^p(\mathbb{R}^N)}^{(1+\delta_1)p} \leq C \, M^{(1+\delta_1)p-1} \, E \tag{15}$$

where C is a constant depending on N, γ and p;

ii) if $\gamma' < N$ and $1 < p \leq \frac{N}{N-\gamma'}$ then, there exists $\delta_2 = \frac{1}{p-1} \frac{\gamma'}{N}$ and a constant C depending on N, γ and p such that

$$\|m\|_{L^p(\mathbb{R}^N)}^{p\delta_2} \leq C(E+M) M^{p\delta_2 - 1}. \tag{16}$$

Proof. i) The proof of (15) follows from [6, Lemma 2.8]. ii) As before let $\frac{1}{r} = \frac{1}{p}\left(1 - \frac{1}{\gamma'}\right) + \frac{1}{\gamma'}$, if $\gamma' < N$ then $r < \gamma' < N$, so by Gagliardo-Niremberg inequality and (12) we get

$$\|m\|_{L^{r^*}(\mathbb{R}^N)} \leq C \|m\|_{L^p(\mathbb{R}^N)}^{\frac{1}{\gamma}} (E+M)^{\frac{1}{\gamma'}} \tag{17}$$

where $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{N}$ and C is a constant depending on N, p and γ'. One can observe that $\frac{1}{r^*} - \frac{1}{p} = \frac{pN-N-p\gamma'}{p\gamma' N} \leq 0$, that is $r^* \geq p$, by interpolation there exists $\theta \in (0, 1]$ such that

$$\|m\|^{\frac{1}{\theta}}_{L^p(\mathbb{R}^N)} \leq M^{\frac{1-\theta}{\theta}} \|m\|_{L^{r^*}(\mathbb{R}^N)}$$

and from (17) we get that

$$\|m\|^{\left(\frac{1}{\theta}-\frac{1}{\gamma}\right)\gamma'}_{L^p(\mathbb{R}^N)} \leq C(E+M)M^{\frac{1-\theta}{\theta}\gamma'}.$$

By simple computations we have that

$$\left(\frac{1}{\theta} - \frac{1}{\gamma}\right)\gamma' = \frac{\gamma'}{N}\frac{p}{p-1}$$

and

$$\left(\frac{1}{\theta} - 1\right)\gamma' = \frac{\gamma'}{N}\frac{p}{p-1} - 1$$

denoting by δ_2 the quantity $\frac{1}{p-1}\frac{\gamma'}{N}$, we finally obtain (16). □

Remark 1. In the following we will use (15) and (16) in the case when $p = \frac{2N}{N+\alpha}$. It will be useful to observe that if $\gamma' \geq N$ then $1 < \frac{2N}{N+\alpha} < 2 \leq 1 + \frac{\gamma'}{N}$, hence estimate (15) holds. In the case when $\gamma' < N$, if $N - \gamma' \leq \alpha < N$ then, $1 < \frac{2N}{N+\alpha} < 1 + \frac{\gamma'}{N}$ and hence from (15) we get that

$$\|m\|^{\frac{2\gamma'}{N-\alpha}}_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq CM^{\frac{2\gamma'}{N-\alpha}-1}E; \tag{18}$$

whereas if $N - 2\gamma' \leq \alpha < N - \gamma'$, we may use estimate (16), which gives us

$$\|m\|^{\frac{2\gamma'}{N-\alpha}}_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq C(E+M)M^{\frac{2\gamma'}{N-\alpha}-1}. \tag{19}$$

Finally, we recall the following a priori elliptic regularity result (see [6, Proposition 2.8, Corollary 2.9]).

Proposition 2.5. *Let*

$$q := \begin{cases} \frac{N}{N-\gamma'+1} & \text{if } \gamma' < N \\ \gamma' & \text{if } \gamma' \geq N \end{cases}$$

and let $(m, w) \in (L^q(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \times L^1(\mathbb{R}^N)$ *be a weak solution to*

$$-\Delta m + \operatorname{div} w = 0, \quad in\ \mathbb{R}^N$$

with $\int_{\mathbb{R}^N} m(x)\,dx = M$, $m \geq 0$ *a.e. and*

$$E := \int_{\mathbb{R}^N} m \left|\frac{w}{m}\right|^{\gamma'} dx < +\infty.$$

Then the following hold:

i)

$$m \in L^\beta(\mathbb{R}^N), \quad \forall \beta \in \left[1, \frac{N}{N-\gamma'}\right) \qquad (\forall \beta \in [1, +\infty),\ if\ \gamma' \geq N)$$

and there exists a constant C depending on N, β and γ′ such that

$$\|m\|_{L^\beta(\mathbb{R}^N)} \leq C(E + M);$$

ii)

$$m \in W^{1,\ell}(\mathbb{R}^N), \quad \forall \ell < q$$

and there exists a constant C depending on N, ℓ and γ′ such that

$$\|m\|_{W^{1,\ell}(\mathbb{R}^N)} \leq C(E + M).$$

Proof. From Proposition 2.3 we have

$$m \in W^{1,r_0}(\mathbb{R}^N) \quad \text{for} \quad \frac{1}{r_0} = \left(1 - \frac{1}{\gamma'}\right)\frac{1}{q} + \frac{1}{\gamma'}.$$

Case $\gamma' < N$. Since $1 < r_0 < \gamma' < N$, by Sobolev embedding theorem and interpolation, we get that

$$m \in L^\beta(\mathbb{R}^N) \quad \forall \beta \leq q_1 \tag{20}$$

where q_1 is the Sobolev critical exponent, i.e.

$$q_1 := \frac{N r_0}{N - r_0} = \frac{qN\gamma'}{N\gamma' - N + q(N - \gamma')},$$

(notice that $q_1 > q$ since $q < \frac{N}{N-\gamma'}$). From (20), using Proposition 2.3 again, we have

$$m \in W^{1,\ell}(\mathbb{R}^N) \qquad \forall \ell \leq r_1 = \frac{q_1\gamma'}{\gamma' - 1 + q_1}.$$

As before, by Sobolev embedding theorem and interpolation, we have that

$$m \in L^{\beta}(\mathbb{R}^N) \qquad \forall \beta \leq q_2 = \frac{q_1 N\gamma'}{N\gamma' - N + q_1(N-\gamma')}.$$

Iterating the previous argument, we observe that $q_{j+1} = f(q_j)$ where $f(s) := \frac{sN\gamma'}{N\gamma'-N+s(N-\gamma')}$. Since f is an increasing function if $s < \frac{N}{N-\gamma'}$ and it has a fixed point for $\bar{s} = \frac{N}{N-\gamma'}$, we obtain that

$$m \in L^{\beta}(\mathbb{R}^N), \quad \forall \beta < \frac{N}{N-\gamma'}$$

and

$$m \in W^{1,\ell}(\mathbb{R}^N), \quad \forall \ell < \frac{N}{N-\gamma'+1}.$$

Moreover, for any fixed $\beta < \frac{N}{N-\gamma'}$, taking $r = r(\beta)$ such that $\frac{1}{r} = \left(1 - \frac{1}{\gamma'}\right)\frac{1}{\beta} + \frac{1}{\gamma'}$, from estimate (12) and the Sobolev embedding theorem (notice that $r^* > \beta$) we get that there exists a constant C depending on N and r such that

$$\|m\|_{L^{\beta}(\mathbb{R}^N)} \leq C(E+M)^{\frac{1}{\gamma'}} \|m\|^{\frac{1}{\gamma}}_{L^{\beta}(\mathbb{R}^N)}. \tag{21}$$

and hence

$$\|m\|_{L^{\beta}(\mathbb{R}^N)} \leq C_1(E+M). \tag{22}$$

Putting (22) in (12) we obtain

$$\|m\|_{W^{1,\ell}(\mathbb{R}^N)} \leq C_2(E+M).$$

Case $\gamma' = N$. Since $r_0 < \gamma' = N$, we can apply the Sobolev embedding theorem and with the same argument as before we obtain

$$q_{j+1} = \frac{N}{N-1} q_j.$$

Obviously $q_{j+1} > q_j$, by iteration we get that

$$m \in L^{\beta}(\mathbb{R}^N), \quad \forall \beta < +\infty$$

and

$$m \in W^{1,\ell}(\mathbb{R}^N), \quad \forall \ell < \gamma'.$$

The estimates on the norms follow in the same way as the previous case.

Case $\gamma' > N$. Since $m \in L^{\gamma'}(\mathbb{R}^N)$, by interpolation $m \in L^N(\mathbb{R}^N)$ and we can go back to the previous case. □

2.2. Some properties of the Riesz potential

We recall here some properties of the Riesz potential, which will be useful in the following in order to deal with the Riesz-type interaction term.

Definition 2.1. *Given $\alpha \in (0, N)$ and a function $f \in L^1_{loc}(\mathbb{R}^N)$, we define the Riesz potential of order α of f as*

$$K_\alpha * f(x) := \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}} dy, \quad x \in \mathbb{R}^N.$$

The Riesz potential K_α is well-defined as an operator on the whole space $L^r(\mathbb{R}^N)$ if and only if $r \in \left[1, \frac{N}{\alpha}\right)$. We state now the following well-known theorems (for which refer e.g. to [38, Theorem 14.37] and [28, Theorem 4.3]).

Theorem 2.6 *(Hardy-Littlewood-Sobolev inequality). Let $0 < \alpha < N$ and $1 < r < \frac{N}{\alpha}$. Then for any $f \in L^r(\mathbb{R}^N)$*

$$\|K_\alpha * f\|_{L^{\frac{Nr}{N-\alpha r}}(\mathbb{R}^N)} \le C\|f\|_{L^r(\mathbb{R}^N)}$$

where C is a constant depending only on N, α and r.

Theorem 2.7. *Let $0 < \lambda < N$ and $p, r > 1$ with $\frac{1}{p} + \frac{\lambda}{N} + \frac{1}{r} = 2$. Let $f \in L^p(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$. Then, there exists a sharp constant $C(N, \lambda, p)$ (independent of f and g) such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)\, g(y)}{|x-y|^\lambda} dx\, dy \right| \le C\|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^r(\mathbb{R}^N)}. \tag{23}$$

Remark 2. It follows immediately that if $0 < \alpha < N$ and $f \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, then there exists a sharp constant C, depending only on N and α, such that

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)\, f(y)}{|x-y|^{N-\alpha}} dx\, dy \right| \le C\|f\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}. \tag{24}$$

As shown in [27], in this case the constant C can be computed explicitly and there exist explicit optimizers for (24) (while neither the constant nor the optimizers are known for $p \neq r$, although do exist).

Regarding the L^∞-norm and the Hölder continuity of the Riesz potential, we recall here the following results.

Theorem 2.8. *Let $0 < \alpha < N$, $1 < r \leq +\infty$ be such that $r > \frac{N}{\alpha}$ and $s \in \left[1, \frac{N}{\alpha}\right)$. Then, for every $f \in L^s(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ we have that*

$$\|K_\alpha * f\|_{L^\infty(\mathbb{R}^N)} \leq C_1 \|f\|_{L^r(\mathbb{R}^N)} + C_2 \|f\|_{L^s(\mathbb{R}^N)} \tag{25}$$

where $C_1 = C_1(N, \alpha, r)$ and $C_2 = C_2(N, \alpha, s)$. Moreover, if $0 < \alpha - \frac{N}{r} < 1$ then,

$$K_\alpha * f \in C^{0,\alpha - \frac{N}{r}}(\mathbb{R}^N)$$

and there exists a constant C, depending on r, α and N, such that

$$\frac{\left|K_\alpha * f(x) - K_\alpha * f(y)\right|}{|x-y|^{\alpha - \frac{N}{r}}} \leq C\|f\|_{L^r(\mathbb{R}^N)} \qquad \text{for } x \neq y.$$

Proof. Concerning Hölder regularity results for the Riesz potential, one may refer to [32, Theorem 2.2, p.155] and [14, Theorem 2].

As for (25), notice that

$$\frac{1}{|x|^{N-\alpha}} \in L^p(B_1), \quad \forall p \in \left[1, \frac{N}{N-\alpha}\right)$$

where B_1 is the ball of radius 1 centered at 0 and it is well-known that $\int_{B_1(0)} \frac{1}{|x|^{(N-\alpha)p}} dx = \frac{\omega_N}{N-(N-\alpha)p}$. By Holder inequality we get

$$\begin{aligned}\int\limits_{B_1} \frac{|f(x-y)|}{|y|^{N-\alpha}} dy &\leq \Big(\int\limits_{B_1} |f(x-y)|^r\, dy\Big)^{\frac{1}{r}} \Big(\int\limits_{B_1} \frac{1}{|y|^{(N-\alpha)r'}}\, dy\Big)^{\frac{1}{r'}} \\ &\leq \|f\|_{L^r(\mathbb{R}^N)} \left(\frac{\omega_N}{N-(N-\alpha)r'}\right)^{\frac{1}{r'}} \leq C_1 \|f\|_{L^r(\mathbb{R}^N)}\end{aligned}$$

using the fact that $r' < \frac{N}{N-\alpha}$, since by assumption $r > \frac{N}{\alpha}$. On the other hand

$$\frac{1}{|x|^{N-\alpha}} \in L^p(B_1^c), \quad \forall p \in \left(\frac{N}{N-\alpha}, +\infty\right]$$

hence

$$\int\limits_{\mathbb{R}^N \setminus B_1} \frac{|f(x-y)|}{|y|^{N-\alpha}} dy \leq \left(\int\limits_{\mathbb{R}^N \setminus B_1} |f(x-y)|^s dy\right)^{\frac{1}{s}} \left\| \frac{1}{|y|^{N-\alpha}} dy \right\|_{L^{s'}(B_1^c)} \leq C\|f\|_{L^s(\mathbb{R}^N)},$$

since $(N-\alpha)s' > N$. □

2.3. Some results on the Hamilton-Jacobi-Bellman equation

By a straightforward adaptation of [6, Theorem 2.5 and Theorem 2.6], we obtain some a priori regularity estimates for solutions to some Hamilton-Jacobi-Bellman equations defined on the whole euclidean space $\mathbb{R}^N$. The following propositions are stated under slightly more general assumptions than ones of our problem.

Proposition 2.9. *Assume that $K_\alpha * m \in L^\infty(\mathbb{R}^N)$ and that V satisfies (2), with $b \geq 0$. Let $(u, c) \in C^2(\mathbb{R}^N) \times \mathbb{R}$ be a classical solution to the HJB equation*

$$-\Delta u + \frac{1}{\gamma}|\nabla u(x)|^\gamma + c = V(x) - K_\alpha * m(x) \quad in\ \mathbb{R}^N, \tag{26}$$

for $\gamma > 1$ fixed. Then

- *i. there exists a constant $C_1 > 0$, depending on $C_V, b, \gamma, N, c, \|K_\alpha * m\|_\infty$, such that*

$$|\nabla u(x)| \leq C_1(1 + |x|)^{\frac{b}{\gamma}};$$

- *ii. if u is bounded from below and $b \neq 0$ in (2), then there exists a constant $C_2 > 0$ such that*

$$u(x) \geq C_2|x|^{\frac{b}{\gamma}+1} - C_2^{-1}, \quad \forall x \in \mathbb{R}^N.$$

  The same result holds also in the case when $b = 0$, but we have to require in addition that there exists $\delta > 0$ such that $V(x) - K_\alpha * m(x) - c > \delta > 0$ for $|x|$ sufficiently large.

Proof. The thesis follows applying [6, Theorem 2.5 and Theorem 2.6]. □

Let us define

$$\lambda := \sup\{c \in \mathbb{R} \mid (26) \text{ has a solution } u \in C^2(\mathbb{R}^N)\} \tag{27}$$

Proposition 2.10. *Besides the hypothesis of Proposition 2.9, let us assume also that $V - K_\alpha * m$ is locally Hölder continuous. Then*

- *i) $\lambda < +\infty$ and there exists $u \in C^2(\mathbb{R}^N)$ such that the pair (u, λ) solves (26).*
- *ii) if $b \neq 0$ in (2), u is unique up to additive constants (namely if $(v, \lambda) \in C^2(\mathbb{R}^N) \times \mathbb{R}$ solves (26) then there exists $k \in \mathbb{R}$ such that $u = v + k$) and there exists a constant $K > 0$ such that*

$$u(x) \geq K|x|^{\frac{b}{\gamma}+1} - K^{-1}, \quad \forall x \in \mathbb{R}^N.$$

Proof. It follows by [6, Theorem 2.7]. We may observe also that

$$\lambda = \sup\{c \in \mathbb{R} \mid (26) \text{ has a subsolution } u \in C^2(\mathbb{R}^N)\}. \quad \square$$

Finally, we conclude with an estimate on the Lagrange multiplier λ defined in (27).

Lemma 2.11. *Let $(u,\lambda)\in C^2(\mathbb{R}^N)\times\mathbb{R}$ be a solution to the HJB equation (26). Then*

i if $V\equiv 0$, then $\lambda\le 0$;
ii if V satisfies (2) then $\lambda\le C$ for some constant depending on b, C_V, γ, N.

Proof. The proof is based on the same argument of [9, Lemma 3.3]. Let us consider the function $\mu_\delta(x):=\left(\frac{\delta}{2\pi}\right)^{N/2}e^{\frac{-\delta|x|^2}{2}}$ for $x\in\mathbb{R}^N$ and $\delta>0$. Obviously $\int_{\mathbb{R}^N}\mu_\delta(x)\,dx=1$. From the definition of Legendre transform we get that

$$\frac{1}{\gamma}|\nabla u|^\gamma=\sup_{\alpha\in\mathbb{R}^N}\left(\nabla u\cdot\alpha-\frac{|\alpha|^{\gamma'}}{\gamma'}\right)\ge\nabla u\cdot(\delta x)-\frac{|\delta x|^{\gamma'}}{\gamma'}$$

hence

$$-\Delta u(x)+\nabla u\cdot(\delta x)-\frac{1}{\gamma'}|\delta x|^{\gamma'}+\lambda\le V(x)-m*K_\alpha(x). \tag{28}$$

Multiplying (28) by μ_δ and integrating over B_R we obtain

$$-\int_{B_R}\Delta u(x)\mu_\delta+\int_{B_R}\nabla u\cdot(\delta x)\mu_\delta-\int_{B_R}\frac{1}{\gamma'}|\delta x|^{\gamma'}\mu_\delta+\lambda\int_{B_R}\mu_\delta\le\int_{B_R}(V(x)-m*K_\alpha)\mu_\delta.$$

Integrating by parts (notice that $\int_{B_R}\nabla u\cdot\nabla\mu_\delta=-\int_{B_R}\nabla u\cdot(\delta x)\mu_\delta$) we get

$$-\int_{\partial B_R}\mu_\delta\nabla u\cdot\nu\,d\sigma-\frac{1}{\gamma'}\int_{B_R}|\delta x|^{\gamma'}\mu_\delta\,dx+\lambda\int_{B_R}\mu_\delta\,dx\le\int_{B_R}(V(x)-m*K_\alpha)\mu_\delta\,dx$$

and since $\int_{B_R}m*K_\alpha(x)\mu_\delta(x)\,dx\ge 0$, we have

$$\lambda\int_{B_R}\mu_\delta\,dx\le\int_{\partial B_R}\mu_\delta\nabla u\cdot\nu\,d\sigma+\frac{1}{\gamma'}\int_{B_R}|\delta x|^{\gamma'}\mu_\delta\,dx+\int_{B_R}V(x)\mu_\delta\,dx. \tag{29}$$

For $\delta>0$ fixed, the first integral in the RHS of (29) can be estimated as follows

$$\left|\int_{\partial B_R}\mu_\delta\nabla u\cdot\nu\,d\sigma\right|\le C\delta^{\frac{N}{2}}e^{-\frac{\delta R^2}{2}}\|\nabla u\|_{L^\infty(\partial B_R)}|\partial B_R|\to 0,\qquad\text{as}\quad R\to+\infty$$

using the gradient estimates on ∇u proved in Proposition 2.9. So, sending $R\to+\infty$ in (29) and using (2) we get

$$\lambda\le\frac{1}{(2\pi)^{\frac{N}{2}}}\frac{\delta^{\frac{\gamma'}{2}}}{\gamma'}\int_{\mathbb{R}^N}|y|^{\gamma'}e^{-\frac{|y|^2}{2}}\,dy+\frac{1}{(2\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N}V\left(\frac{y}{\sqrt{\delta}}\right)e^{-\frac{|y|^2}{2}}\,dy.$$

If $V \equiv 0$, then sending $\delta \to 0$ in the previous inequality, we conclude immediately $\lambda \leq 0$. If $V \not\equiv 0$, we may choose $\delta = 1$ in the previous inequality and conclude recalling (2). □

2.4. Uniform a priori L^∞-bounds on m

We state now the following result, which provides uniform a priori L^∞ bounds on m.

Theorem 2.12. *We consider a sequence of classical solutions (u_k, m_k, λ_k) to the following MFG system*

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = W_k(x) - G_{k,\alpha}[m](x) \\ -\Delta m - \operatorname{div}\left(m\nabla u|\nabla u|^{\gamma-2}\right) = 0 & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} m = M, \quad m \geq 0 \end{cases} \tag{30}$$

where $W_k : \mathbb{R}^N \to \mathbb{R}$ satisfies assumption (2) with constant C_V, b independent of k. Let $G_{k,\alpha} : L^1(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$ such that $G_{k,\alpha}[m] \geq 0$ for all $m \in L^1$, with $m \geq 0$, and moreover we assume that there exists $\alpha \in (0, N)$ such that for all $s \in \left[1, \frac{N}{\alpha}\right)$ and $r \in \left(\frac{N}{\alpha}, +\infty\right]$ there holds for $m \in L^s(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$

$$\|G_{k,\alpha}[m]\|_{L^\infty(\mathbb{R}^N)} \leq C_1\|m\|_{L^r(\mathbb{R}^N)} + C_2\|m\|_{L^s(\mathbb{R}^N)} \tag{31}$$

where $C_1 = C_1(N, \alpha, r)$ and $C_2 = C_2(N, \alpha, s)$.

If u_k are bounded from below and satisfy (5), and $m_k \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, with $\|m_k\|_{L^q} \leq C_q$ for some $q > \frac{N}{\alpha+\gamma'}$ then, there exists a positive constant C not depending on k such that

$$\|m_k\|_{L^\infty(\mathbb{R}^N)} \leq C, \quad \forall k \in \mathbb{N}.$$

Proof. We follow the argument of the proof of [6, Theorem 4.1] (refer also to [9] for the analogous result on $\mathbb{T}^N$), but we have to define a different rescaling in this case.

Up to addition of constants we may assume $\inf u_k(x) = 0$.

We assume by contradiction that

$$\sup_{\mathbb{R}^N} m_k = L_k \to +\infty$$

and we define

$$\delta_k := \begin{cases} L_k^{-\beta}, & \text{if } \gamma' \leq N \quad \text{and } q \leq \frac{N}{\gamma'} \\ L_k^{-\frac{1}{\gamma'}}, & \text{if either } \gamma' > N \quad \text{ or } \gamma' \leq N, q > \frac{N}{\gamma'} \end{cases}$$

where $\beta > 0$ (so $\delta_k \to 0$) has to be chosen in the following way. We fix

$$r \in \left(\frac{N}{\alpha}, \frac{Nq}{N - q\gamma'}\right). \tag{32}$$

Note that since $q > \frac{N}{\gamma'+\alpha}$ the interval is not empty. If $q = \frac{N}{\gamma'}$ it is sufficient to choose $\frac{N}{\alpha} < r < +\infty$. Then we choose β such that

$$\frac{1}{\gamma'}\left(1 - \frac{q}{r}\right) \leq \beta < \frac{q}{N}.$$

We rescale (u_k, m_k, λ_k) as follows:

$$v_k(x) := \delta_k^{\frac{2-\gamma}{\gamma-1}} u_k(\delta_k x) + 1, \qquad n_k(x) := L_k^{-1} m_k(\delta_k x), \qquad \tilde{\lambda}_k := \delta_k^{\gamma'} \lambda_k.$$

Observe that $0 \leq n_k(x) \leq 1$ and $\sup n_k = 1$ and moreover that $v_k(x) \geq 1$ for all x. So we obtain that $(v_k, n_k, \tilde{\lambda}_k)$ is a solution to

$$\begin{cases} -\Delta v_k + \frac{1}{\gamma}|\nabla v_k|^\gamma + \tilde{\lambda}_k = V_k(x) - \tilde{g}_k(x) \\ -\Delta n_k - \operatorname{div}(n_k \nabla v_k |\nabla v_k|^{\gamma-2}) = 0 \end{cases}$$

where

$$V_k(x) := \delta_k^{\gamma'} W_k(\delta_k x) \qquad \text{and} \qquad \tilde{g}_k(x) := \delta_k^{\gamma'} G_{k,\alpha}[m_k](\delta_k x).$$

Observe that by assumption (2) there holds

$$C_V^{-1} \delta_k^{\gamma'} (\max\{|\delta_k x| - C_V, 0\})^b \leq V_k(x) \leq C_V (1 + \delta_k^{\gamma'+b} |x|)^b, \quad \forall x \in \mathbb{R}^N.$$

Computing the equation in a minimum point of u_k we obtain $\lambda_k \geq -\|G_{k,\alpha}[m_k]\|_\infty$ and reasoning as in Lemma 2.11, we get that $\lambda_k \leq C$, for some C just depending on γ, C_V, b, so we get

$$-\|\tilde{g}_k\|_\infty = -\delta_k^{\gamma'} \|G_{k,\alpha}[m_k]\|_\infty \leq \tilde{\lambda}_k \leq \delta_k^{\gamma'} C.$$

If $\gamma' > N$ **or** $\gamma' \leq N$ **and** $q > \frac{N}{\gamma'}$ we apply (31) with $r = +\infty$ and $s = 1$ and we get

$$\|\tilde{g}_k\|_\infty \leq \delta_k^{\gamma'} (C_1 L_k + C_2 M) = L_k^{-1} (C_1 L_k + C_2 M) \leq C$$

which in turns gives also that $|\tilde{\lambda}_k| \leq C$. Moreover if $\gamma' > N$ there holds

$$\|n_k\|_{L^1} = \int_{\mathbb{R}^N} n_k(x) dx = \delta_k^{\gamma'-N} \|m_k\|_{L^1} = \delta_k^{\gamma'-N} M \to 0 \qquad \text{and} \quad 0 \leq n_k \leq 1 = \sup n_k$$

if on the other side $\gamma' \leq N$ and $q > \frac{N}{\gamma'}$ we have that

$$\|n_k\|_{L^q} = L_k^{-1} \delta_k^{-\frac{N}{q}} \|m_k\|_{L^q} \leq L_k^{\frac{N}{q\gamma'}-1} C_q \to 0 \qquad \text{and} \quad 0 \leq n_k \leq 1 = \sup n_k.$$

If $\gamma' \leq N$ **and** $q \leq \frac{N}{\gamma'}$ first of all we observe that, since $\beta < \frac{q}{N}$,

$$\|n_k\|_{L^q} = L_k^{-1}\delta_k^{-\frac{N}{q}}\|m_k\|_{L^q} \le L_k^{\beta\frac{N}{q}-1}C_q \to 0 \qquad \text{and} \quad 0 \le n_k \le 1 = \sup n_k.$$

We apply (31) with r as in (32) and $s = 1$ and we get, using interpolation between L^q and L^∞ to estimate the norm $\|m_k\|_{L^r}$, that there holds

$$\|\tilde{g}_k\|_\infty \le \delta_k^{\gamma'}(C_1\|m_k\|_{L^{\frac{N}{N-2\gamma'}}} + C_2C_q) \le L_k^{-\beta\gamma'}(CL_k^{1-\frac{q}{r}} + C_2C_q) \le CL_k^{1-\beta\gamma'-\frac{q}{r}} \le C$$

since $\beta\gamma' > 1 - \frac{q}{r}$. This in turns implies that $|\tilde{\lambda}_k| \le C$.

The rest of the proof follows exactly the same lines of the proof of [6, Theorem 4.1], since we have uniform bounds on $\tilde{\lambda}_k$ and on $\|\tilde{g}_k\|_\infty$, either the L^1 or the L^q norm of n_k vanishing as $k \to +\infty$, whereas $\|n_k\|_\infty = 1$. In particular one shows that if x_k is an approximated maximum point of n_k (that is $n_k(x_k) = 1 - \delta$), then necessarily $\delta_k^{\gamma'+b}|x_k|^b \to +\infty$. If it is not the case, using a priori gradient estimates on v_k as in Proposition 2.9, we get that n_k is uniformly (in k) Holder continuous in the ball $B_1(x_k)$, contradicting the fact that $n_k \ge 0$ and either $\|n_k\|_{L^q} \to 0$ or $\|n_k\|_{L^1} \to 0$. On the other hand, if $\delta_k^{\gamma'+b}|x_k|^b \to +\infty$, we may construct a Lyapunov function for the system, which allows for some integral estimates on n_k showing again a uniform (in k) Holder bound for n_k in $B_{1/2}(x_k)$ and again getting a contradiction. Therefore one concludes that $L_k \to +\infty$ is not possible. □

3. Pohozaev identity and nonexistence of solutions

In this section, we study the MFG system (1) in the case $V \equiv 0$, i.e.

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u(x)|^\gamma + \lambda = -K_\alpha * m(x) \\ -\Delta m - \operatorname{div}\left(m(x)\nabla u(x)\,|\nabla u(x)|^{\gamma-2}\right) = 0 \quad \text{in } \mathbb{R}^N. \\ \int_{\mathbb{R}^N} m = M, \quad m \ge 0 \end{cases} \tag{33}$$

The following Lemma (see Lemma 3.2 in [9]) will be useful in order to control the behavior of m, ∇u, ∇m at infinity.

Lemma 3.1. *Let $h \in L^1(\mathbb{R}^N)$. Then, there exists a sequence $R_n \to \infty$ such that*

$$R_n \int_{\partial B_{R_n}} |h(x)|dx \to 0, \quad \text{as } n \to \infty.$$

In order to prove nonexistence of solutions to the MFG system (33) in the *supercritical regime* $0 < \alpha < N - 2\gamma'$, we need a Pohozaev-type identity.

Proposition 3.2 ***(Pohozaev identity)***. *Let $(u, m, \lambda) \in C^2(\mathbb{R}^N) \times W^{1,\frac{2N}{N+\alpha}}(\mathbb{R}^N) \times \mathbb{R}$ be a solution to (33) such that*

$$m|\nabla u|^\gamma \in L^1(\mathbb{R}^N) \qquad \text{and} \qquad |\nabla m||\nabla u| \in L^1(\mathbb{R}^N).$$

Then, the following equality holds

$$(2-N)\int\limits_{\mathbb{R}^N}\nabla u\cdot\nabla m\,dx+\left(1-\frac{N}{\gamma}\right)\int\limits_{\mathbb{R}^N}m|\nabla u|^\gamma\,dx=\lambda NM+\frac{\alpha+N}{2}\int\limits_{\mathbb{R}^N}m(x)(K_\alpha*m)(x)\,dx. \tag{34}$$

Proof. From Lemma 2.1, we get that m is twice differentiable, so the following computations are justified. Consider the first equation in (33), multiplying each term by $\nabla m\cdot x$ and integrating over $B_R(0)$ for $R>0$, we get

$$-\int\limits_{B_R}\Delta u\,\nabla m\cdot x\,dx+\frac{1}{\gamma}\int\limits_{B_R}|\nabla u(x)|^\gamma\nabla m\cdot x\,dx+\lambda\int\limits_{B_R}\nabla m\cdot x\,dx=-\int\limits_{B_R}K_\alpha*m(x)\nabla m\cdot x\,dx. \tag{35}$$

We take into account each term of (35) separately. Integrating by parts the first term, we have

$$-\int\limits_{B_R}\Delta u\,\nabla m\cdot x\,dx=\int\limits_{B_R}\nabla u\cdot\nabla(\nabla m\cdot x)dx-\int\limits_{\partial B_R}(\nabla u\cdot\nu)(\nabla m\cdot x)\,d\sigma, \tag{36}$$

we observe that

$$\int\limits_{B_R}\nabla u\cdot\nabla(\nabla m\cdot x)dx=\int\limits_{B_R}\sum_{i=1}^N u_{x_i}(\nabla m\cdot x)_{x_i}=\int\limits_{B_R}\nabla u\cdot\nabla m+\int\limits_{B_R}\sum_{i,j}u_{x_i}m_{x_i x_j}x_j$$

and integrating by parts the last term of the previous one, we get

$$\begin{aligned}\int\limits_{B_R}\sum_{i,j}(u_{x_i}x_j)m_{x_i x_j}&=\int\limits_{\partial B_R}\sum_{i,j}u_{x_i}m_{x_i}x_j\cdot\frac{x_j}{R}-\int\limits_{B_R}\sum_{i,j}m_{x_i}u_{x_i x_j}x_j-N\int\limits_{B_R}\sum_i u_{x_i}m_{x_i}\\&=\int\limits_{\partial B_R}(\nabla u\cdot\nabla m)x\cdot\nu d\sigma-\int\limits_{B_R}\nabla m\cdot\nabla(\nabla u\cdot x)+(1-N)\int\limits_{B_R}\nabla u\cdot\nabla m.\end{aligned}$$

Note that $x\cdot\nu=R$ on ∂B_R. Coming back to (36) we obtain

$$\begin{aligned}-\int\limits_{B_R}\Delta u\,\nabla m\cdot x\,dx=&-\int\limits_{B_R}\nabla m\cdot\nabla(\nabla u\cdot x)dx+(2-N)\int\limits_{B_R}\nabla u\cdot\nabla m\,dx\\&+\int\limits_{\partial B_R}(\nabla u\cdot\nabla m)(x\cdot\nu)d\sigma-\int\limits_{\partial B_R}(\nabla u\cdot\nu)(\nabla m\cdot x)\,d\sigma.\end{aligned} \tag{37}$$

Concerning the second and the third term in (35), we get that

$$\frac{1}{\gamma}\int\limits_{B_R}|\nabla u(x)|^\gamma\nabla m\cdot x\,dx=\frac{1}{\gamma}\int\limits_{\partial B_R}|\nabla u|^\gamma m\,x\cdot\nu\,d\sigma-\frac{1}{\gamma}\int\limits_{B_R}m\,\mathrm{div}(|\nabla u|^\gamma x)dx=$$

$$= \frac{1}{\gamma} \int_{\partial B_R} |\nabla u|^\gamma m\, x \cdot \nu \, d\sigma - \frac{1}{\gamma} \int_{B_R} m\, \nabla(|\nabla u|^\gamma) \cdot x \, dx - \frac{N}{\gamma} \int_{B_R} m |\nabla u(x)|^\gamma \, dx \tag{38}$$

and

$$\lambda \int_{B_R} \nabla m \cdot x \, dx = \lambda \int_{\partial B_R} m\, x \cdot \nu d\sigma - \lambda N \int_{B_R} m \, dx. \tag{39}$$

Similarly, multiplying the second equation in (33) by $\nabla u \cdot x$ and integrating over $B_R(0)$ we get

$$\begin{aligned} \int_{B_R} \Delta m \nabla u \cdot x \, dx &= - \int_{B_R} \operatorname{div}(m |\nabla u|^{\gamma-2} \nabla u) \nabla u \cdot x \, dx = \\ &= \int_{B_R} \nabla(\nabla u \cdot x) \cdot (m\, |\nabla u|^{\gamma-2} \nabla u) \, dx - \int_{\partial B_R} (\nabla u \cdot x) m |\nabla u|^{\gamma-2} \nabla u \cdot \nu \, d\sigma = \\ &= \int_{B_R} \frac{1}{\gamma} m \nabla(|\nabla u|^\gamma) \cdot x \, dx + \int_{B_R} m |\nabla u|^\gamma dx - \int_{\partial B_R} (\nabla u \cdot x) m |\nabla u|^{\gamma-2} \nabla u \cdot \nu \, d\sigma \end{aligned} \tag{40}$$

where we have integrated by parts and then used the following identity

$$\frac{1}{\gamma} \nabla(|\nabla u|^\gamma) \cdot x = |\nabla u|^{\gamma-2} \nabla u \cdot \nabla(\nabla u \cdot x) - |\nabla u|^\gamma.$$

Integrating by parts the LHS of (40) we get

$$\begin{aligned} \int_{\partial B_R} (\nabla m \cdot \nu)(\nabla u \cdot x) d\sigma &- \int_{B_R} \nabla m \cdot \nabla(\nabla u \cdot x) dx = \\ &= \int_{B_R} \frac{m}{\gamma} \nabla(|\nabla u|^\gamma) \cdot x \, dx + \int_{B_R} m |\nabla u|^\gamma dx - \int_{\partial B_R} (\nabla u \cdot x) m |\nabla u|^{\gamma-2} \nabla u \cdot \nu \, d\sigma \end{aligned}$$

and then isolating the first term in the second line

$$\begin{aligned} -\frac{1}{\gamma} \int_{B_R} m \nabla(|\nabla u|^\gamma) \cdot x \, dx = \int_{B_R} \nabla m \cdot \nabla(\nabla u \cdot x) dx - \int_{\partial B_R} (\nabla m \cdot \nu)(\nabla u \cdot x) \, d\sigma \\ + \int_{B_R} m |\nabla u|^\gamma dx - \int_{\partial B_R} (\nabla u \cdot x) m |\nabla u|^{\gamma-2} \nabla u \cdot \nu \, d\sigma \end{aligned} \tag{41}$$

plugging (41) in (38) we obtain

$$\frac{1}{\gamma}\int_{B_R}|\nabla u(x)|^\gamma \nabla m\cdot x\,dx=\frac{1}{\gamma}\int_{\partial B_R} m|\nabla u|^\gamma x\cdot\nu\,d\sigma+\int_{B_R}\nabla m\cdot\nabla(\nabla u\cdot x)\,dx+$$
$$-\int_{\partial B_R}(\nabla m\cdot\nu)(\nabla u\cdot x)\,d\sigma+\left(1-\frac{N}{\gamma}\right)\int_{B_R} m|\nabla u|^\gamma\,dx-\int_{\partial B_R}(\nabla u\cdot x)m|\nabla u|^{\gamma-2}\nabla u\cdot\nu\,d\sigma. \tag{42}$$

For what concern the Riesz's potential term, since $m\in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ from Theorem 2.6 it follows that $K_\alpha * m\in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, hence by Hölder inequality

$$\left|\int_{B_R(0)} K_\alpha * m(x)\nabla m\cdot x\,dx\right|\le R\int_{B_R(0)}|K_\alpha * m|\,|\nabla m|\,dx$$
$$\le R\,\|K_\alpha * m\|_{L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)}\,\|\nabla m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)},$$

this proves that the term $-\int_{B_R} K_\alpha * m(x)\nabla m\cdot x\,dx$ is finite. We get

$$-\int_{B_R}K_\alpha * m(x)\nabla m\cdot x\,dx=-\int_{B_R}\int_{\mathbb{R}^N}\frac{m(y)\nabla m(x)\cdot x}{|x-y|^{N-\alpha}}dy\,dx=-\int_{\mathbb{R}^N}\int_{B_R}\frac{m(y)\nabla m(x)\cdot x}{|x-y|^{N-\alpha}}dx\,dy=$$
$$=-\int_{\mathbb{R}^N}\int_{\partial B_R}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}(x\cdot\nu)\,d\sigma(x)\,dy+\int_{\mathbb{R}^N}\int_{B_R}m(x)\mathrm{div}_x\left(\frac{m(y)}{|x-y|^{N-\alpha}}x\right)dx\,dy \tag{43}$$

and furthermore,

$$\int_{\mathbb{R}^N}\int_{B_R}m(x)\,\mathrm{div}_x\left(\frac{m(y)}{|x-y|^{N-\alpha}}x\right)dx\,dy=$$
$$=(\alpha-N)\int_{\mathbb{R}^N}\int_{B_R}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}\frac{(x-y)\cdot x}{|x-y|^2}dx\,dy+N\int_{\mathbb{R}^N}\int_{B_R}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}dx\,dy=$$
$$=\frac{\alpha+N}{2}\int_{\mathbb{R}^N}\int_{B_R}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}dx\,dy+\frac{\alpha-N}{2}\int_{\mathbb{R}^N}\int_{B_R}\frac{m(x)\,m(y)}{|x-y|^{N-\alpha}}\frac{(x+y)\cdot(x-y)}{|x-y|^2}dx\,dy \tag{44}$$

where we used that $\frac{x\cdot(x-y)}{|x-y|^2}=\frac{1}{2}+\frac{(x+y)\cdot(x-y)}{2|x-y|^2}$. Summing up (37), (39), (42), (43) and (44) we get the following identity

$$(2-N)\int_{B_R}\nabla u\cdot\nabla m\,dx+\left(1-\frac{N}{\gamma}\right)\int_{B_R}m|\nabla u|^\gamma\,dx-\lambda N\int_{B_R}m(x)\,dx$$

$$-\frac{\alpha+N}{2}\int\limits_{\mathbb{R}^N}\int\limits_{B_R}\frac{m(x)m(y)}{|x-y|^{N-\alpha}}dx\,dy-\frac{\alpha-N}{2}\int\limits_{\mathbb{R}^N}\int\limits_{B_R}\frac{m(x)m(y)}{|x-y|^{N-\alpha}}\frac{(x+y)\cdot(x-y)}{|x-y|^2}dx\,dy$$
$$=I_{\partial B_R} \tag{45}$$

where

$$I_{\partial B_R}=\int\limits_{\partial B_R}\left[-(\nabla u\cdot\nabla m)-\frac{1}{\gamma}m|\nabla u|^\gamma-\lambda m\right](x\cdot\nu)d\sigma-\int\limits_{\mathbb{R}^N}\int\limits_{\partial B_R}\frac{m(x)m(y)}{|x-y|^{N-\alpha}}(x\cdot\nu)\,d\sigma(x)\,dy$$
$$+\int\limits_{\partial B_R}(\nabla u\cdot\nu)(\nabla m\cdot x)+(\nabla m\cdot\nu)(\nabla u\cdot x)+m|\nabla u|^{\gamma-2}(\nabla u\cdot x)(\nabla u\cdot\nu)\,d\sigma.$$

Now, we let R go to infinity in (45). We observe that (changing variables x and y)

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{m(x)m(y)}{|x-y|^{N-\alpha}}\frac{(x+y)\cdot(x-y)}{|x-y|^2}dx\,dy=0.$$

Moreover

$$|I_{\partial B_R}|\le R\int\limits_{\partial B_R}\big(3|\nabla u|\,|\nabla m|+2m\,|\nabla u|^\gamma+|\lambda|m\big)\,d\sigma+\int\limits_{\mathbb{R}^N}R\int\limits_{\partial B_R}\frac{m(x)m(y)}{|x-y|^{N-\alpha}}d\sigma(x)\,dy$$

since by assumption $|\nabla u|\,|\nabla m|$, $m|\nabla u|^\gamma$ and $m\in L^1(\mathbb{R}^N)$, by Lemma 3.1, we get that for some sequence $R_n\to+\infty$

$$R_n\int\limits_{\partial B_{R_n}}\Big(3|\nabla u|\,|\nabla m|+2m\,|\nabla u|^\gamma+|\lambda|m\Big)d\sigma\to 0,\quad \text{as } n\to+\infty.$$

By means of the same argument, since $m\in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ implies that $G(x):=\int_{\mathbb{R}^N}\frac{m(x)m(y)}{|x-y|^{N-\alpha}}dy\in L^1(\mathbb{R}^N)$ (by Theorem 2.7), we get that there exists a sequence $R_n\to+\infty$ such that

$$R_n\int\limits_{\partial B_{R_n}}G(x)dx\to 0,\quad \text{as } n\to+\infty,$$

which conclude the proof of the Pohozaev-type equality (34). □

We are now in position to prove nonexistence of classical solutions with prescribed integrability and boundary conditions at ∞.

Proof of Theorem 1.1. We argue by contradiction. Let $(u,m,\lambda)\in C^2(\mathbb{R}^N)\times W^{1,\frac{2N}{N+\alpha}}(\mathbb{R}^N)\times\mathbb{R}$ be a solution to (33) such that $u\to+\infty$ as $|x|\to+\infty$ and it holds

$$m|\nabla u|^\gamma,\quad |\nabla m||\nabla u|\ \in L^1(\mathbb{R}^N).$$

From Proposition 3.2 we have the following Pohozaev-type identity

$$(2-N)\int_{\mathbb{R}^N} \nabla u\cdot\nabla m\,dx+\left(1-\frac{N}{\gamma}\right)\int_{\mathbb{R}^N} m|\nabla u|^\gamma\,dx=\lambda NM+\frac{\alpha+N}{2}\int_{\mathbb{R}^N} m(K_\alpha * m)\,dx. \tag{46}$$

Moreover, we obtain the following identities

$$\int_{\mathbb{R}^N} \nabla u\cdot\nabla m\,dx=-\frac{1}{\gamma}\int_{\mathbb{R}^N} m|\nabla u|^\gamma\,dx-\lambda M-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{m(x)m(y)}{|x-y|^{N-\alpha}}dx\,dy \tag{47}$$

and

$$\int_{\mathbb{R}^N} \nabla u\cdot\nabla m\,dx=-\int_{\mathbb{R}^N} m|\nabla u|^\gamma\,dx. \tag{48}$$

Proof of (47). Multiplying the first equation in (33) by m and integrating over B_R we obtain

$$\int_{B_R} \nabla u\cdot\nabla m\,dx-\int_{\partial B_R} m\,\nabla u\cdot\nu\,d\sigma+\frac{1}{\gamma}\int_{B_R} m|\nabla u|^\gamma\,dx+\lambda\int_{B_R} m\,dx=-\int_{B_R}\int_{\mathbb{R}^N}\frac{m(x)m(y)}{|x-y|^{N-\alpha}}dy\,dx. \tag{49}$$

By Holder's inequality and using the fact that $m|\nabla u|^\gamma\in L^1(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N}|\nabla u|m\,dx\le\left(\int_{\mathbb{R}^N}|\nabla u|^\gamma m\,dx\right)^{\frac{1}{\gamma}}M^{\frac{1}{\gamma'}}<+\infty,$$

hence $|\nabla u|m\in L^1(\mathbb{R}^N)$ and by Lemma 3.1, we get that for some sequence $R_n\to+\infty$

$$\int_{\partial B_{R_n}} m\,\nabla u\cdot\nu\,d\sigma\to 0,\quad \text{as } n\to+\infty.$$

Equality (47) follows letting $R\to\infty$ in (49). *Proof of* (48). For any $s>0$ let us define the set

$$X_s:=\{x\in\mathbb{R}^N\mid u(x)\le s\},$$

and the function

$$v_s(x):=u(x)-s,\quad \forall x\in\mathbb{R}^N.$$

After a translation we may assume $u(0)=0$. In this way, $\cup_{s>0}X_s=\mathbb{R}^N$, every X_s is non-empty and bounded since $u(x)\to+\infty$ as $|x|\to+\infty$. Multiplying the second equation in (33) by v_s and integrating by parts, we get

$$\int_{X_s} \nabla v_s \cdot \nabla m \, dx = -\int_{X_s} m \, |\nabla u|^{\gamma-2} \nabla u \cdot \nabla v_s \, dx,$$

since $\nabla v_s = \nabla u$, we obtain (48) letting $s \to +\infty$.

Plugging (48) in (47) we get

$$\left(1 - \frac{1}{\gamma}\right) \int_{\mathbb{R}^N} m |\nabla u|^\gamma \, dx = \lambda \, M + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{m(x)m(y)}{|x-y|^{N-\alpha}} dx \, dy \tag{50}$$

and hence

$$\int_{\mathbb{R}^N} m |\nabla u|^\gamma \, dx = \lambda \gamma' M + \gamma' \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{m(x)m(y)}{|x-y|^{N-\alpha}} dx \, dy. \tag{51}$$

Using (48) in (46), we have

$$\left(\frac{N}{\gamma'} - 1\right) \int_{\mathbb{R}^N} m |\nabla u|^\gamma \, dx = \lambda N \, M + \frac{\alpha + N}{2} \int_{\mathbb{R}^N} m(K_\alpha * m) dx$$

and finally from (51) we obtain

$$\left(\frac{N - 2\gamma' - \alpha}{2}\right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{m(x)m(y)}{|x-y|^{N-\alpha}} dx \, dy = \gamma' \lambda \, M. \tag{52}$$

Recall that by Lemma 2.11, we have that $\lambda \le 0$ and by assumption $N - 2\gamma' - \alpha > 0$, so we get a contradiction. □

Remark 3. One could observe that the previous proof (with slight changes) holds also in the case when $u \to -\infty$ as $|x| \to +\infty$, hence one may ask why we do not consider this possibility. This is due to the fact that the property of ergodicity for the process is strictly related to the existence of a Lyapunov function (refer to [21]). More in detail, a necessary condition to have an ergodic process is

$$\nabla u \cdot x > 0, \quad \text{for } x \text{ large}$$

(see also [8] and references therein). As a consequence, the case $u \to -\infty$ as $|x| \to +\infty$ is not relevant.

4. Existence of classical solutions to the MFG system

First of all we consider a regularized version of problem (1), namely

$$\begin{cases} -\Delta u + \frac{1}{\gamma} |\nabla u|^\gamma + \lambda = V(x) - K_\alpha * m * \varphi_k(x) \\ -\Delta m - \operatorname{div}\left(m(x) \nabla u(x) \, |\nabla u(x)|^{\gamma-2}\right) = 0 \\ \int_{\mathbb{R}^N} m = M, \quad m \ge 0 \end{cases} \quad \text{in} \quad \mathbb{R}^N \tag{53}$$

where φ_k is a sequence of standard symmetric mollifiers approximating the unit as $k \to +\infty$ (i.e. a sequence of symmetric functions on $\mathbb{R}^N$ such that $\varphi_k \in C_0^\infty(\mathbb{R}^N)$, $\operatorname{supp}\varphi_k \subset \overline{B_{1/k}(0)}$, $\int \varphi_k = 1$ and $\varphi_k \geq 0$). For every k fixed, using Schauder Fixed Point Theorem, we will prove existence of (u_k, m_k, λ_k) solution to (53), and then, exploiting a priori uniform estimates on these solutions, we will show that we may pass to the limit as $k \to +\infty$ and get a solution of the MFG system (1).

4.1. Solution of the regularized problem

We consider (53) with k fixed:

$$\begin{cases} -\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - K_\alpha * m * \varphi(x) \\ -\Delta m - \operatorname{div}\left(m(x)\nabla u(x)\,|\nabla u(x)|^{\gamma-2}\right) = 0 & \text{in} \quad \mathbb{R}^N \\ \int_{\mathbb{R}^N} m = M, \quad m \geq 0 \end{cases} \tag{54}$$

We are going to construct solution to (54) by using the following version of the well-known Schauder Fixed Point Theorem. Construction of solutions to MFG systems by exploiting fixed point arguments is quite classical in the literature, see [2,9,19,24].

Theorem 4.1 *(Corollary 11.2 [15]). Let A be a closed and convex set in a Banach space X and let $\mathcal{F}$ be a continuous map from A into itself such that the image $\mathcal{F}(A)$ is precompact. Then, $\mathcal{F}$ has a fixed point.*

Let $\xi, C > 0$ (which will be chosen later), $M > 0$ and $\bar{p} > \frac{N}{\alpha}$, we define the set

$$A_{\xi,M,C} := \left\{ \mu \in L^{\bar{p}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \;\middle|\; \|\mu\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \xi, \int_{\mathbb{R}^N} \mu\, dx = M, \;\; \mu \geq 0, \int_{\mathbb{R}^N} \mu V(x)\, dx \leq C \right\}. \tag{55}$$

Lemma 4.2. *For any choice of $\xi, M, C > 0$, the set $A_{\xi,M,C} \subset L^{\bar{p}}(\mathbb{R}^N)$ is closed and convex.*

Proof. The set $A_{\xi,M,C}$ is convex since it is intersection of convex sets.

Let now $(\mu_n)_n$ be a sequence in $A_{\xi,M,C}$ which converges in $L^{\bar{p}}$ to $\bar{\mu}$. Obviously $\bar{\mu} \geq 0$ and since $\mu_n \rightharpoonup \bar{\mu}$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ by weak lower semicontinuity of the norm we have that

$$\|\bar{\mu}\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \liminf \|\mu_n\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \xi.$$

From Fatou's Lemma we get also that $\int_{\mathbb{R}^N} \bar{\mu} V(x) \leq \liminf \int_{\mathbb{R}^N} \mu_n V(x) \leq C$. Note that due to the fact that $0 \leq \int_{\mathbb{R}^N} \mu_n V(x) \leq C$, and that V is coercive, see (2), μ_n are uniformly integrable, since for every $R >> 1$, $0 \leq \int_{|x|\geq R} \mu_n dx \leq \frac{C_V}{R^b} \int_{\mathbb{R}^N} V(x)\mu_n dx \leq \frac{CC_V}{R^b}$. Due to the fact that $\mu_n \to \bar{\mu}$ in $L^{\bar{p}}$, we have also that they have uniformly absolutely continuous integrals, so we may apply the Vitali convergence theorem and obtain that $\mu_n \to \bar{\mu}$ in $L^1(\mathbb{R}^N)$ and hence $\int_{\mathbb{R}^N} \bar{\mu}\, dx = M$. This proves that $\bar{\mu} \in A_{\xi,M,C}$, and hence that $A_{\xi,M,C}$ is closed. □

We define the map $F : A_{\xi,M,C} \to C^2(\mathbb{R}^N) \times \mathbb{R}$ which to every element $\mu \in A_{\xi,M,C}$ associates a solution $(u, \bar{\lambda}) \in C^2(\mathbb{R}^N) \times \mathbb{R}$ to the HJB equation

$$-\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - K_\alpha * \mu * \varphi(x), \quad \text{in } \mathbb{R}^N \tag{56}$$

where $\bar{\lambda}$ is defined as in (27) (refer to [4]); and the map G which to the couple $(u, \bar{\lambda})$ associates the function m which solves (weakly)

$$\begin{cases} -\Delta m - \operatorname{div}\left(m(x)\nabla u(x)\,|\nabla u(x)|^{\gamma-2}\right) = 0 \\ \int_{\mathbb{R}^N} m = M, \quad m \geq 0 \end{cases} . \tag{57}$$

We look for a fixed point of the map $\mathcal{F} : \mu \mapsto m$ defined as the composition of F and G, namely $\mathcal{F}(\mu) := G(F(\mu))$.

We are going to show that, once we have fixed M (in an appropriate range), it is possible to choose appropriately ξ and C in such a way that the map $\mathcal{F}$ defined on $A_{\xi,M,C}$ satisfies the assumptions of the Schauder Fixed Point Theorem 4.1. As we will see the regularization with φ in the system (54) is necessary in order to get precompactness of the image of $\mathcal{F}$. We start with some preliminary results.

Proposition 4.3. *Let us consider $\mu \in A_{\xi,M,C}$, $(u, \bar{\lambda}) = F(\mu)$ and $m = G(u, \bar{\lambda}) = \mathcal{F}(\mu)$. Then,*

i) there exists a constant $C > 0$ such that

$$|\nabla u(x)| \leq C(1 + |x|)^{\frac{b}{\gamma}} \tag{58}$$

where C depends on $C_V, b, \gamma, N, \bar{\lambda}, \|K_\alpha * \mu * \varphi\|_\infty$.

ii) the function u is unique up to addition of constants and there exists $C > 0$ such that

$$u(x) \geq C|x|^{\frac{b}{\gamma}+1} - C^{-1}. \tag{59}$$

iii) it holds

$$-K_1 \leq \bar{\lambda} \leq K_2 \tag{60}$$

where K_1 and K_2 are positive constants depending respectively on $\|K_\alpha * \mu * \varphi\|_\infty$ and on C_V, b, γ, N.

iv) *the function m is unique, $m \in (W^{1,1} \cap L^\infty)(\mathbb{R}^N)$, $\sqrt{m} \in W^{1,2}(\mathbb{R}^N)$, $m \in W^{1,p}(\mathbb{R}^N)$ $\forall p > 1$ and it holds*

$$\|\nabla m\|_{L^p(\mathbb{R}^N)} \leq C\|m^{\frac{1}{p}}|\nabla u|^{\gamma-1}\|_{L^p(\mathbb{R}^N)}\|m^{1-\frac{1}{p}}\|_{L^\infty(\mathbb{R}^N)}. \tag{61}$$

Moreover, the following integrability properties are verified

$$m|\nabla u|^\gamma \in L^1(\mathbb{R}^N), \qquad mV \in L^1(\mathbb{R}^N), \qquad |\nabla u|\,|\nabla m| \in L^1(\mathbb{R}^N). \tag{62}$$

Proof. $i)$ Since $\mu * \varphi \in L^1(\mathbb{R}^N) \cap L^{\bar{p}}(\mathbb{R}^N)$ with $\bar{p} > \frac{N}{\alpha}$, by Theorem 2.8 we obtain $K_\alpha * (\mu * \varphi) \in C^{0,\theta}(\mathbb{R}^N)$ for some $\theta \in (0,1)$ and $\|K_\alpha * \mu * \varphi\|_\infty \le C_{N,\alpha,\bar{p}}\|\mu * \varphi\|_{L^{\bar{p}}} + \|\mu * \varphi\|_{L^1} \le C_{N,\alpha,\bar{p}}\|\mu\|_{L^{\bar{p}}} + M$. We can apply therefore Proposition 2.9, which gives us the following estimate

$$|\nabla u(x)| \le C(1+|x|)^{\frac{b}{\gamma}} \tag{63}$$

where C is a constant depending on $C_V, b, \gamma, N, \bar{\lambda}, \|K_\alpha * (\mu * \varphi)\|_\infty$. This proves (58).

$ii)$ Since, by construction, u is a solution to (56) with $\lambda = \bar{\lambda}$ then by Proposition 2.10 $ii)$ it follows uniqueness up to additive constants and (59).

$iii)$ The fact that $\bar{\lambda} \le K_2$ is a direct consequence of Lemma 2.11. Furthermore, if $\bar{x}$ is a minimum point of u, evaluating (56) at $\bar{x}$ we have that

$$\bar{\lambda} \ge V(\bar{x}) - K_\alpha * \mu * \varphi(\bar{x}) \ge -\|K_\alpha * \mu * \varphi\|_\infty \ge -K_1$$

since $V(x) \ge 0$ in $\mathbb{R}^N$.

$iv)$ For $r > 1$, let us consider the function $h(x) := u(x)^r$, one can observe that

$$\begin{aligned} -\Delta h + |\nabla u|^{\gamma-2}\nabla u \cdot \nabla h &= r u^{r-1}\left(-(r-1)\frac{|\nabla u|^2}{u} - \Delta u + |\nabla u|^\gamma\right) \\ &= r u^{r-1}\left(-\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma - (r-1)\frac{|\nabla u|^2}{u} + \frac{1}{\gamma'}|\nabla u|^\gamma\right) \\ &= r u^{r-1}\left(-\bar{\lambda} + V - K_\alpha * \mu * \varphi - (r-1)\frac{|\nabla u|^2}{u} + \frac{1}{\gamma'}|\nabla u|^\gamma\right), \end{aligned}$$

where in the last equality we used the fact that u solves (56). Denoting by

$$H(x) := -\bar{\lambda} + V(x) - K_\alpha * \mu * \varphi(x) - (r-1)\frac{|\nabla u|^2}{u} + \frac{1}{\gamma'}|\nabla u|^\gamma,$$

from (60), (2) and the fact that $K_\alpha * \mu * \varphi \in L^\infty$, we get

$$H(x) \ge (r-1)|\nabla u|^\gamma \left(\frac{1}{\gamma'(r-1)} - \frac{|\nabla u|^{2-\gamma}}{u}\right) + C_V^{-1}|x|^b - C \ge 1, \quad \text{for } |x| > R$$

taking R sufficiently large. Hence, for $|x| > R$

$$-\Delta h + |\nabla u|^{\gamma-2}\nabla u \cdot \nabla h \ge C|x|^{(\frac{b}{\gamma}+1)(r-1)} > 0$$

this means that h is a Lyapunov function for the stochastic process with drift $|\nabla u|^{\gamma-2}\nabla u$. Since m solves (57), it is the density of the invariant measure associated to this process. So, from [35, Proposition 2.3] we get that

$$m|x|^{(\frac{b}{\gamma}+1)(r-1)} \in L^1(\mathbb{R}^N). \tag{64}$$

More in general, since the value of $r > 1$ can be chosen arbitrarily, from (64) we have that for any $q > 0$

$$m|x|^q \in L^1(\mathbb{R}^N), \tag{65}$$

in particular $m|x|^b \in L^1(\mathbb{R}^N)$, so taking into account estimates (63) and (2) we obtain that

$$m|\nabla u|^\gamma \in L^1(\mathbb{R}^N) \qquad \text{and} \qquad mV \in L^1(\mathbb{R}^N).$$

With the same argument (since $|\nabla u|^{p(\gamma-1)}$ has polynomial growth) it follows that

$$m|\nabla u|^{p(\gamma-1)} \in L^p(\mathbb{R}^N), \quad \forall p > 1$$

hence from [35, Corollary 3.2 and Theorem 3.5] we get that

$$m \in W^{1,1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

Moreover, using the fact that m is a weak solution to the Kolmogorov equation and Hölder inequality, we obtain that for any $\phi \in C_0^\infty(\mathbb{R}^N)$ we have

$$\left|\int_{\mathbb{R}^N} m\Delta\phi\, dx\right| \leq \int_{\mathbb{R}^N} m|\nabla u|^{\gamma-1}||\nabla\phi|\, dx \leq \|m^{\frac{1}{p}}|\nabla u|^{\gamma-1}\|_p \|m^{1-\frac{1}{p}}\|_\infty \|\nabla\phi\|_{p'}.$$

Since $m^{\frac{1}{p}}|\nabla u|^{\gamma-1} \in L^p(\mathbb{R}^N)$ and $m^{1-\frac{1}{p}} \in L^\infty(\mathbb{R}^N)$, by Proposition 2.2 we get that

$$m \in W^{1,p}(\mathbb{R}^N), \ \forall p > 1$$

and estimate (61) holds. Finally, from [35, Theorem 3.1] we have that $\sqrt{m} \in W^{1,2}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \frac{|\nabla m|^2}{m} < +\infty$, so using Hölder inequality we obtain

$$\int_{\mathbb{R}^N} |\nabla u||\nabla m| \leq \left\| |\nabla u|\sqrt{m} \right\|_2 \left\| \frac{|\nabla m|}{\sqrt{m}} \right\|_2 < +\infty.$$

Since the function u is unique up to additive constants, ∇u is fixed and hence, by existence of a Lyapunov function, it follows immediately uniqueness of m solution to the Kolmogorov equation. □

We show now that once we fix the mass M (in $(0, +\infty)$ in the mass-subcritical case, or below a certain threshold in the mass-supercritical and mass-critical regime), then we may choose the constant ξ, C in the definition (55) of the set $A_{\xi,M,C}$ such that the map $\mathcal{F}$ maps $A_{\xi,M,C}$ into itself.

Lemma 4.4. *We have the following results:*

i) if $N-\gamma' < \alpha < N$, then for any $M>0$, there exist $\xi, C>0$ such that $\mathcal{F}$ maps $A_{\xi,M,C}$ into itself;
ii) if $N-2\gamma' \le \alpha \le N-\gamma'$ then there exists a positive real value $M_0 = M_0(N,\alpha,\gamma,C_V,b)$ such that if $M \in (0, M_0)$ there exist $\xi, C>0$ such that $\mathcal{F}$ maps the set $A_{\xi,M,C}$ into itself.

Proof. Let $\mu \in A_{\xi,M,C}$, $m = \mathcal{F}(\mu)$ and $(u,\bar\lambda) = F(\mu)$ as above. Since by Proposition 4.3 $iv)$ $m \in L^\infty(\mathbb{R}^N)$, by interpolation it follows that $m \in L^{\bar p}(\mathbb{R}^N)$. Multiplying (56) by m and integrating over B_R, we obtain

$$-\int_{B_R} m\Delta u\,dx + \frac{1}{\gamma}\int_{B_R} m|\nabla u|^\gamma\,dx + \bar\lambda\int_{B_R} m\,dx = \int_{B_R} V(x)m\,dx - \int_{B_R} m(K_\alpha * \mu * \varphi)\,dx$$

and integrating by parts the first term

$$\int_{B_R} \nabla m\cdot\nabla u\,dx - \int_{\partial B_R} m\nabla u\cdot\nu\,d\sigma + \frac{1}{\gamma}\int_{B_R} m|\nabla u|^\gamma\,dx + \bar\lambda\int_{B_R} m\,dx$$
$$= \int_{B_R} V(x)m\,dx - \int_{B_R} m(K_\alpha * \mu * \varphi)\,dx. \quad (66)$$

From the fact that $\int_{\mathbb{R}^N} m = M$ and $m|\nabla u|^\gamma \in L^1(\mathbb{R}^N)$, by Hölder inequality we get that $m|\nabla u| \in L^1(\mathbb{R}^N)$, hence by Lemma 3.1 for some sequence $R_n \to +\infty$ we have that $\int_{\partial B_{R_n}} m\nabla u\cdot\nu\,d\sigma \to 0$. Since $m(K_\alpha * \mu * \varphi) \in L^1(\mathbb{R}^N)$ and (62) holds, letting R go to $+\infty$ in (66) we obtain that

$$\int_{\mathbb{R}^N} \nabla u\cdot\nabla m\,dx = -\frac{1}{\gamma}\int_{\mathbb{R}^N} m|\nabla u|^\gamma\,dx - \bar\lambda M + \int_{\mathbb{R}^N} V(x)m\,dx - \int_{\mathbb{R}^N} m(K_\alpha * \mu * \varphi)\,dx. \quad (67)$$

Moreover, from the fact that m solves (weakly) the Kolmogorov equation in (57), following the proof of identity (48), we have that

$$\int_{\mathbb{R}^N} \nabla u\cdot\nabla m\,dx = -\int_{\mathbb{R}^N} m|\nabla u|^\gamma\,dx. \quad (68)$$

Putting together (67) and (68) we get that

$$\frac{1}{\gamma'}\int_{\mathbb{R}^N} m|\nabla u|^\gamma\,dx + \int_{\mathbb{R}^N} mV\,dx = \bar\lambda M + \int_{\mathbb{R}^N} m(K_\alpha * \mu * \varphi)\,dx. \quad (69)$$

Since $\bar\lambda \le K_2$ (from (60)), using (23) we have

$$\int_{\mathbb{R}^N} m|\nabla u|^\gamma dx \le C_1 M + C_2 \|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \|\mu * \varphi\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}$$
$$\le C_1 M + C_2 \|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \|\mu\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}$$
$$\le C_1 M + C_2 \xi \|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \tag{70}$$

where $C_1 = C_1(\gamma, C_V, b, N)$ and $C_2 = C_2(\alpha, N, \gamma)$.

Choice of ξ. First of all we show that we may choose ξ in such a way that if $\mu \in A_{\xi,M,C}$ then $\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} = \|\mathcal{F}(\mu)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le \xi$.

Let us fix $a := \frac{2\gamma'}{N-\alpha}$. Notice that $a > 2$ if $\alpha > N - \gamma'$, $a = 2$ if $\alpha = N - \gamma'$, $a \in (1,2)$ if $N - 2\gamma' < \alpha < N - \gamma'$ and $a = 1$ when $\alpha = N - 2\gamma'$.

In the case when $N - \gamma' \le \alpha < N$, using estimate (18), we get

$$\|m\|^a_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le C M^{a-1} \int_{\mathbb{R}^N} m|\nabla u|^\gamma dx \tag{71}$$

where C is a constant depending on N, α and γ; whereas if $N - 2\gamma' \le \alpha < N - \gamma'$ using estimate (19), we get

$$\|m\|^a_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le C M^{a-1} \left(\int_{\mathbb{R}^N} m|\nabla u|^\gamma dx + M \right) \tag{72}$$

where C is a constant depending on N, α and γ.

From (70) and either (71) or (72) we obtain that

$$\|m\|^a_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le C_1 M^a + C_2 M^{a-1} \xi \|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}. \tag{73}$$

We define the function

$$f(t) := t^a - C_2 M^{a-1} \xi t - C_1 M^a$$

and observe that (73) is equivalent to $f(\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}) \le 0$. When $a > 1$, $f(\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}) \le 0$ is equivalent to $\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le t_0$, where t_0 is the unique zero of f. So, in order to conclude that $\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \le \xi$ it is sufficient to choose ξ such that $f(\xi) \ge 0$.

Case $N - \gamma' < \alpha < N$. In this case since $a > 2$ and $f(\xi) = \xi^a - C_2 M^{a-1} \xi^2 - C_1 M^a$ then for every fixed $M > 0$, there exists ξ_M such that $f(\xi) \ge 0$ for every $\xi \ge \xi_M$ and we have done.

Case $\alpha = N - \gamma'$. In this case $a = 2$, so arguing as before, and recalling that $f(\xi) = \xi^2 - C_2 M \xi^2 - C_1 M^2$, we get that whenever $M < \frac{1}{C_2} := M_0$ there exists ξ_M such that $f(\xi) \ge 0$ for every $\xi \ge \xi_M$.

Case $N-2\gamma' < \alpha < N-\gamma'$. In this case $a \in (1,2)$. Denote $g(t) := t^a - C_2 M^{a-1} t^2 - C_1 M^a$. We aim to find ξ such that $g(\xi) \geq 0$. This is possible if and only if $g(t_{max}) \geq 0$, where $t_{max} = \left(\frac{a}{2C_2 M^{a-1}}\right)^{\frac{1}{2-a}}$ is the maximum point of g. Evaluating g in this point we get

$$g(t_{max}) = \left(\frac{2-a}{2}\right) C_2^{-\frac{a}{2-a}} \left(\frac{a}{2}\right)^{\frac{a}{2-a}} M^{\frac{a(1-a)}{2-a}} - C_1 M^a.$$

Since $a \in (1,2)$ we have that $\frac{2-a}{2} > 0$ and $\frac{a(1-a)}{2-a} < 0$, hence

$$g(t_{max}) \geq 0$$

provided that M is sufficiently small. We may choose $\xi = t_{max}$ (or more generally ξ in the range of values t such that $g(t) \geq 0$).

Case $\alpha = N-2\gamma'$. Since $a=1$, the function f reads $f(t) = t - C_2\xi t - C_1 M$ and since $f(\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}) \leq 0$ we have, if $\xi < \frac{1}{C_2}$,

$$\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \frac{C_1 M}{1 - C_2\xi}. \tag{74}$$

We look for some condition on M under which we may choose ξ such that $\frac{C_1 M}{1-C_2\xi} \leq \xi$. Observe that this is equivalent to $C_2\xi^2 - \xi + C_1 M \leq 0$. If $M \leq \frac{1}{4C_1C_2}$, then it is sufficient to choose ξ in the range $\left[\frac{1-\sqrt{1-4C_1C_2M}}{2C_2}, \frac{1+\sqrt{1-4C_1C_2M}}{2C_2}\right] \cap \left(0, \frac{1}{C_2}\right)$.

Choice of C. Notice that in each of the previous cases, from (69), (70) and the fact that $\|m\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \leq \xi$, we get

$$\int_{\mathbb{R}^N} mV\,dx \leq C_1 M + C_2\xi^2.$$

So it is sufficient to choose C greater or equal to $C_1 M + C_2\xi^2$.
We can conclude that $\mathcal{F}$ maps the set $A_{\xi,M,C}$ into itself. □

We show now that the image of $\mathcal{F}$ is precompact, that is relatively compact. Here is the main point in which the regularization with the mollifier φ comes into play.

Lemma 4.5. *Let M and ξ, C as given by Lemma 4.4. Then the image $\mathcal{F}(A_{\xi,M,C})$ is precompact.*

Proof. Let us consider a sequence $(m_n)_n \subset \mathcal{F}(A_{\xi,M,C})$, in order to prove that $\mathcal{F}(A_{\xi,M,C})$ is precompact in $A_{\xi,M,C}$, we have to show that $(m_n)_n$ admits a subsequence converging in $L^{\bar{p}}$-norm to a point belonging to $A_{\xi,M,C}$. There exists a sequence $(\mu_n)_n \subset A_{\xi,M,C}$ such that $\mathcal{F}(\mu_n) = m_n$ for every $n \in \mathbb{N}$, considering also $(u_n, \bar{\lambda}_n) = F(\mu_n)$, we have that for every $n \in \mathbb{N}$ the triple $(u_n, m_n, \bar{\lambda}_n)$ is such that

$$\begin{cases} -\Delta u_n + \frac{1}{\gamma}|\nabla u_n|^\gamma + \bar{\lambda}_n = V(x) - K_\alpha * \mu_n * \varphi(x) \\ -\Delta m_n - \operatorname{div}(m_n \nabla u_n |\nabla u_n|^{\gamma-2}) = 0 \\ \int_{\mathbb{R}^N} m_n = M \quad m_n \geq 0 \end{cases}.$$

Note that by Young's convolution inequality $\|\mu_n * \varphi\|_{L^q(\mathbb{R}^N)} \leq \|\mu_n\|_{L^1(\mathbb{R}^N)}\|\varphi\|_{L^q(\mathbb{R}^N)} = M\|\varphi\|_{L^q(\mathbb{R}^N)}$ for every q. Therefore by Proposition 2.8 we get that $K_\alpha * \mu_n * \varphi \in L^\infty \cap C^{0,\theta}$ for some $\theta \in (0,1)$ uniformly in n, that is $\|K_\alpha * \mu_n * \varphi\|_{L^\infty(\mathbb{R}^N)} \leq C$, for some C independent of n. By Proposition 4.3 we have that u_n are bounded from below, that $m_n \in L^\infty$ and that $\bar{\lambda}_n$ are equibounded in n, so we may apply Theorem 2.12 (actually a simpler version, with $W_n(x) = V(x) - K_\alpha * \mu_n * \varphi$ and $G_{k,\alpha} \equiv 0$). So we obtain that there exists a positive constant C not depending on n such that

$$\|m_n\|_{L^\infty(\mathbb{R}^N)} \leq C, \qquad \forall n \in \mathbb{N}. \tag{75}$$

Now we use Proposition 2.5 ii), since $m_n \in L^q(\mathbb{R}^N)$ (where q is defined as in Proposition 2.5) and $E_n \leq C_1 M + C_2\xi^2$ and we get that

$$\|m_n\|_{W^{1,\ell}(\mathbb{R}^N)} \leq C, \quad \forall \ell < q$$

where the constant C does not depend on n. Hence, by Sobolev compact embeddings, $m_n \to \bar{m}$ strongly in $L^s(K)$ for $1 \leq s < q^*$ and for every $K \subset\subset \mathbb{R}^N$. Moreover, using the fact that $\int_{\mathbb{R}^N} m_n V\,dx \leq C$ uniformly in n and (2) we get that for $R > 1$

$$C \geq \int\limits_{\mathbb{R}^N} m_n V\,dx \geq \int\limits_{|x|\geq R} m_n V\,dx \geq C R^b \int\limits_{|x|\geq R} m_n(x)\,dx$$

that is

$$\int\limits_{|x|\geq R} m_n(x)dx \to 0, \text{ as } R \to +\infty.$$

Using also the uniform estimate (75), from the Vitali Convergence Theorem we obtain that up to sub-sequences

$$m_n \to \bar{m} \quad \text{in } L^1(\mathbb{R}^N) \tag{76}$$

and as a consequence $\int_{\mathbb{R}^N} \bar{m}(x)dx = M$. Finally, from (75) and (76), we deduce that $m_n \to \bar{m}$ strongly in $L^{\bar{p}}(\mathbb{R}^N)$. Since $A_{\xi,M,C}$ is closed and by Lemma 4.4 we have that $\mathcal{F}(A_{\xi,M,C}) \subset A_{\xi,M,C}$, we may conclude that $\bar{m} \in A_{\xi,M,C}$. □

Finally we show that $\mathcal{F}$ is continuous.

Lemma 4.6. *Let ξ, M and C as given by Lemma 4.4. Then, the map $\mathcal{F}$ is continuous.*

Proof. Let $(\mu_n)_n$ be a sequence in $A_{\xi,M,C}$ such that $\mu_n \to \tilde{\mu} \in A_{\xi,M,C}$ strongly in $L^{\bar{p}}(\mathbb{R}^N)$. In order to prove that the map $\mathcal{F}$ is continuous, we have to show that $\mathcal{F}(\mu_n) \to \mathcal{F}(\tilde{\mu})$ with respect to the $L^{\bar{p}}$-norm, that is $m_n \to \tilde{m}$ strongly in $L^{\bar{p}}(\mathbb{R}^N)$. We consider the sequence made by the couples $(u_n, \bar{\lambda}_n) \in C^2(\mathbb{R}^N) \times \mathbb{R}$, where $(u_n, \bar{\lambda}_n) = F(\mu_n)$ $\forall n \in \mathbb{N}$, as previously defined. As observed in Lemma 4.5, $K_\alpha * \mu_n * \varphi$ is uniformly bounded in L^∞. So by Proposition 4.3 we have that $\bar{\lambda}_n$ are uniformly bounded, that

$$|\nabla u_n(x)| \leq C(1 + |x|^{\frac{b}{\gamma}}) \qquad \text{uniformly in } n$$

and then consequently

$$|\Delta u_n| \leq C(1 + |x|^b), \qquad \text{uniformly in } n. \tag{77}$$

Up to extracting a subsequence we assume that $\bar{\lambda}_n \to \lambda^{(1)}$. Since u_n is a classical solution to the HJB equation, by classical elliptic regularity estimates applied to $v_n(x) := u_n(x) - u_n(0)$ (refer e.g. to [15, Theorem 8.32]) for any $\theta \in (0, 1]$ and $K \subset\subset \mathbb{R}^N$ we get

$$\|v_n\|_{C^{1,\theta}_{loc}(K)} \leq C \qquad \text{uniformly in } n$$

(notice that the previous estimate holds for $\theta = 1$ thanks to (77)). By Arzelà-Ascoli Theorem, up to extracting a subsequence, we get that

$$v_n \to u^{(1)} \qquad \text{locally uniformly in } C^{1,\theta}$$

and hence

$$\nabla u_n \to \nabla u^{(1)} \qquad \text{locally uniformly in } C^{0,\theta}.$$

Since $\|(\mu_n - \tilde{\mu}) * \varphi\|_{L^{\bar{p}}(\mathbb{R}^N)} \leq 2M\|\varphi\|_{L^{\bar{p}}(\mathbb{R}^N)}$, by Theorem 2.8 we get that

$$\|K_\alpha * \varphi * \mu_n\|_{C^{0,\alpha-N/\bar{p}}} \leq C, \qquad \text{uniformly in } n$$

and

$$\|K_\alpha * \varphi * \mu_n - K_\alpha * \varphi * \tilde{\mu}\|_{L^\infty(\mathbb{R}^N)} \leq C_{N,\alpha,\bar{p}}\|\mu_n - \tilde{\mu}\|_{L^{\bar{p}}(\mathbb{R}^N)} + \|\mu_n - \tilde{\mu}\|_{L^1(\mathbb{R}^N)}.$$

Since $\mu_n \to \tilde{\mu}$ in $L^1(\mathbb{R}^N) \cap L^{\bar{p}}(\mathbb{R}^N)$, then up to subsequences

$$K_\alpha * \varphi * \mu_n \longrightarrow K_\alpha * \varphi * \tilde{\mu} \quad \text{locally uniformly in } \mathbb{R}^N.$$

By stability with respect to locally uniform convergence, we get that $(u^{(1)}, \lambda^{(1)})$ is a solution (in the viscosity sense) to the HJB equation

$$-\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - K_\alpha * \varphi * \tilde{\mu}(x), \quad \text{on } \mathbb{R}^N.$$

Let $(\tilde{u}, \tilde{\lambda}) = F(\tilde{\mu})$, we want to show that $\tilde{\lambda} = \lambda^{(1)}$. Assume by contradiction that $\tilde{\lambda} \neq \lambda^{(1)}$, without loss of generality we can assume that $\lambda^{(1)} < \tilde{\lambda} - 2\varepsilon$ for a certain $\varepsilon > 0$. Then, for n sufficiently large $\bar{\lambda}_n < \tilde{\lambda} - \varepsilon$ and, possibly enlarging n, we have also $\| K_\alpha * \varphi * \mu_n - K_\alpha * \varphi * \tilde{\mu}\|_\infty \leq \varepsilon$. One can observe that

$$-\Delta \tilde{u} + \frac{1}{\gamma}|\nabla \tilde{u}|^\gamma + \tilde{\lambda} - \varepsilon - V(x) + K_\alpha * \varphi * \mu_n(x) \leq 0,$$

i.e. $\tilde{u}$ is a subsolution to the equation

$$-\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \tilde{\lambda} - \varepsilon = V(x) - K_\alpha * \varphi * \mu_n(x). \tag{78}$$

Since by definition (see [6, Theorem 2.7 (i)])

$$\bar{\lambda}_n := \sup\left\{\lambda \in \mathbb{R} \,\middle|\, -\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = V(x) - K_\alpha * \mu_n * \varphi(x) \text{ has a subsolution in } C^2(\mathbb{R}^N)\right\}$$

it must be $\bar{\lambda}_n \geq \tilde{\lambda} - \varepsilon$, which yields a contradiction. Therefore $\tilde{\lambda} = \lambda^{(1)}$. By Proposition 4.3 $ii)$ $\tilde{u}$ is unique up to addition of constants, namely there exists $c \in \mathbb{R}$ such that $\tilde{u} = u^{(1)} + c$, it follows that $\nabla \tilde{u} = \nabla u^{(1)}$. Once we have the sequence of function u_n, we construct the sequence $(m_n)_n \subset \mathcal{F}(A_{\xi,M,C})$ such that for every $n \in \mathbb{N}$ fixed, it holds

$$\begin{cases} -\Delta m_n - \operatorname{div}(m_n \nabla u_n |\nabla u_n|^{\gamma-2}) = 0 \\ \int_{\mathbb{R}^N} m_n = M, \quad m_n \geq 0 \end{cases}.$$

From Lemma 4.5, up to extracting a subsequence

$$m_n \to m^{(1)} \text{ in } L^{\bar{p}}(\mathbb{R}^N)$$

where $m^{(1)} \in A_{\xi,M,C}$. Since $\nabla u_n |\nabla u_n|^{\gamma-2} \to \nabla \tilde{u} |\nabla \tilde{u}|^{\gamma-2}$ locally uniformly in $\mathbb{R}^N$, we get that $m^{(1)}$ is a weak solution to

$$-\Delta m - \operatorname{div}(m \nabla \tilde{u} |\nabla \tilde{u}|^{\gamma-2}) = 0$$

that has $\tilde{m} = \mathcal{F}(\tilde{\mu})$ as unique solution. This proves that $m_n \to \tilde{m}$ in $L^{\bar{p}}(\mathbb{R}^N)$. □

We are ready to prove the following result on existence of solutions to the regularized MFG system (54).

Theorem 4.7. *We get the following results:*

i. *if $N - \gamma' < \alpha < N$ then, for every $M > 0$ the MFG system* (54) *admits a classical solution;*
ii. *if $N - 2\gamma' \leq \alpha \leq N - \gamma'$ then, there exists a positive real value $M_0 = M_0(N, \alpha, \gamma, C_V, b)$ such that if $M \in (0, M_0)$ the MFG system* (54) *admits a classical solution.*

Proof. From Lemma 4.2, Lemma 4.4, Lemma 4.5 and Lemma 4.6 assumptions of Theorem 4.1 are verified, hence the map $\mathcal{F}$ has a fixed point m_φ. The fixed point m_φ together with the couple $(u_\varphi, \bar{\lambda}_\varphi) = F(m_\varphi)$ obtained solving the Hamilton-Jacobi-Bellman equation with Riesz potential term equal to $K_\alpha * m_\varphi * \varphi$, will be a solution to the MFG system (54). □

4.2. Limiting procedure

Let $(\varphi_k)_k$ be a sequence of standard symmetric mollifiers approximating the unit as $k \to +\infty$. For every $k \in \mathbb{N}$ (under the additional assumption that the constraint mass M is sufficiently small in the case when $N - 2\gamma' < \alpha \leq N - \gamma'$) from Theorem 4.7 we can construct a classical solution $(u_k, m_k, \bar{\lambda}_k)$ to the corresponding regularized MFG system (53). Our aim now is passing to the limit as $k \to +\infty$ and prove that $(u_k, m_k, \bar{\lambda}_k)$ converges to a solution of the MFG system (1).

We need some preliminary apriori estimates.

Lemma 4.8. *Let $\alpha \in (N - 2\gamma', N)$ and $(u_k, m_k, \bar{\lambda}_k)$ be a solution to the regularized MFG system (53) as constructed in Theorem 4.7. Then, there exist C_1, C_2, C_3 positive constants independent of k such that*

$$\|m_k\|_{L^\infty(\mathbb{R}^N)} \leq C_1, \quad \forall k \in \mathbb{N}$$
$$|\bar{\lambda}_k| \leq C_2$$

and

$$|\nabla u_k| \leq C_3(1 + |x|^{\frac{b}{\gamma}}) \qquad |\Delta u_k| \leq C_3(1 + |x|^b). \tag{79}$$

Proof. Note that if $m \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ by Theorem 2.8, we have that

$$\begin{aligned}\|K_\alpha * \varphi_k * m\|_{L^\infty(\mathbb{R}^N)} &\leq C_{N,\alpha,r,s}(\|\varphi_k * m\|_{L^r(\mathbb{R}^N)} + \|\varphi_k * m\|_{L^s(\mathbb{R}^N)}) \\ &\leq C_1\|m\|_{L^r(\mathbb{R}^N)} + C_2\|m\|_{L^s(\mathbb{R}^N)}\end{aligned}$$

for every $r \in \left(\frac{N}{\alpha}, +\infty\right]$ and $s \in \left[1, \frac{N}{\alpha}\right)$. So, since $m_k \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\|m_k\|_{\frac{2N}{N+\alpha}} \leq \xi$, and u_k are bounded from below, we may apply Theorem 2.12 with $W_k \equiv V$, $G_{k,\alpha}[m] = K_\alpha * \varphi_k * m$ and $q = \frac{2N}{N+\alpha} > \frac{N}{\alpha+\gamma'}$ and conclude the uniform L^∞ bounds on m_k. Now, by Proposition 4.3, we get that $\bar{\lambda}_k$ are equibounded in k and that

$$|\nabla u_k(x)| \leq C(1 + |x|^{\frac{b}{\gamma}}) \qquad |\Delta u_k| \leq C(1 + |x|^b)$$

where C is independent of k. □

Proof of Theorem 1.2. Since for any $k \in \mathbb{N}$, u_k is a classical solution to the HJB equation

$$-\Delta u + \frac{1}{\gamma}|\nabla u|^\gamma + \bar{\lambda}_k = V(x) - K_\alpha * \varphi_k * m_k$$

by Lemma 4.8 and elliptic estimates (refer to [15, Theorem 8.32]) applied to $v_k(x) := u_k(x) - u_k(0)$, we obtain that for every $K \subset\subset \mathbb{R}^N$ and $\theta \in (0, 1]$

$$\|v_k\|_{C^{1,\theta}_{loc}(K)} \le C \qquad \text{uniformly with respect to } k$$

hence up to extracting a subsequence

$$v_k \to \bar{u} \qquad \text{locally uniformly in } C^1 \text{ on compact sets.}$$

Similarly, since m_k weak solution to $-\Delta m - \operatorname{div}(m|\nabla u_k|^{\gamma-2}\nabla u_k) = 0$, for every $\phi \in C_0^\infty(K)$ it holds

$$\left|\int_K m_k \Delta\phi \, dx\right| \le \|\nabla\phi\|_{L^1(K)} \|m_k|\nabla u_k|^{\gamma-1}\|_{L^\infty(K)}.$$

Using the uniform L^∞ estimates on m_k and the estimates (79), by Proposition 2.2 and Sobolev embedding, we get that for every $\beta \in (0, 1)$

$$\|m_k\|_{C^{0,\beta}(K)} \le C \qquad \text{uniformly with respect to } k$$

so up to extracting a subsequence

$$m_k \to \bar{m} \qquad \text{locally uniformly.}$$

Since the values of $\bar{\lambda}_k$ are equibounded with respect to k, we have that $\bar{\lambda}_k \to \bar{\lambda}$ up to a subsequence. Again recalling that $\int V(x)m_k \le C$ uniformly in k, we conclude by Vitali Convergence Theorem that $m_k \to \bar{m}$ in $L^1(\mathbb{R}^N)$ and hence $\int_{\mathbb{R}^N} \bar{m} = M$. From the strong convergence in $L^1(\mathbb{R}^N)$ and the uniform L^∞ estimates, we obtain also that

$$m_k \to \bar{m} \quad \text{in } L^p(\mathbb{R}^N)$$

for every $p \in [1, +\infty)$. We finally have that

$$K_\alpha * \varphi_k * m_k \to K_\alpha * m \quad \text{locally uniformly.}$$

We can pass to the limit and obtain that $(\bar{u}, \bar{m}, \bar{\lambda})$ is a solution to the MFG system (1). □

Data availability

No data was used for the research described in the article.

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