

Robust fixed-lag smoothing under model perturbations

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Abstract

A robust fixed-lag smoothing approach is proposed in the case there is a mismatch between the nominal model and the actual model. The resulting robust smoother is characterized by a dynamic game between two players: one player selects the least favorable model in a prescribed ambiguity set, while the other player selects the fixed-lag smoother minimizing the smoothing error with respect to least favorable model. We propose an efficient implementation of the proposed smoother. Moreover, we characterize the corresponding least favorable model over a finite time horizon. Finally, we test the robust fixed-lag smoother in two examples. The first one regards a target tracking problem, while the second one regards a parameter estimation problem.

Keywords: Robust fixed-lag smoothing; minimax problem; reduced order smoothing; least favorable model.

1. Introduction

Fixed-lag smoothing aims to estimate the state of a dynamical system at time t using the observations in the interval $[0, t + L - 1]$ with $L > 1$. This algorithm is fundamental in various applications, e.g. tracking and navigation because it can handle online requirements, see [1, 2, 3]. Moreover, it can be used in the expectation maximization (EM) algorithm to compute the maximum likelihood (ML) estimator of the unknown parameters characterizing the matrices of a state space model, see e.g. [4, 5, 6]. Indeed, although the EM algorithm is based on the Rauch-Tung-Striebel (RTS) smoother, [7], its

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estimate can be approximated by the one given by the fixed-lag smoother provided that L is taken sufficiently large. However, in all the aforementioned applications, the actual model is typically known only imprecisely that is only the nominal model is known. In this situation, these smoothers could perform poorly.

Model uncertainty is traditionally addressed by risk sensitive filtering, see e.g. [8, 9, 10, 11, 12, 13]. Here, the robust estimator minimizes an exponential loss function which severely penalizes large errors. These filters can be also interpreted as the solution of a dynamic minimax game [14], see also [15, 16, 17, 18]. Then, [19] proposed a robust estimator which solves an incremental minimax game. At time t the actual model belongs to the ambiguity set which is a ball, in the Kullback-Leibler (KL) topology, about the nominal model. In this way, the uncertainty is “spread” along the time and not concentrated in specific time steps. Then, there are two players which operate against. One player, say nature, selects the least favorable model in this prescribed “ball”, and the other player designs the optimum estimator for the least favorable model. It is worth noting that many extensions of this paradigm have been proposed such as: the case with different ambiguity sets [20, 21, 22, 23]; the distributed case [24, 25]; the case with external input [26]; the case of degenerate densities [27, 28].

In the literature, the robust smoothing problems mainly consider two situations. In the first case, the noise distribution is known but it is not necessarily Gaussian [29, 30, 31, 32, 33, 34, 35, 36, 37, 38], for instance the noise process is assumed to have a non-Gaussian distribution in order to model outliers, temporary model uncertainties, missing observations or sensor delays. Some of these robust paradigms are adaptive because the parameters of the noises characterizing the state space model are inferred from the collected data. In the second situation, the noise distribution is not known but this process takes values in a bounded set, e.g. an ellipsoidal set [39, 40]. However, there are relatively few studies on robust smoothing problems which use the risk sensitive philosophy, see [41, 42].

In this paper, we propose a new robust fixed-lag smoothing problem where the model uncertainty is expressed incrementally as in [19, 27]. Thus, at each time step we have to solve a dynamic game between two players: the nature which selects the least favorable model in the ambiguity set and the other player which designs the optimal fixed-lag smoother according to the least favorable model. The resulting smoother is characterized by matrices whose dimension is proportional to the lag L . On the other hand, the typical value of

the lag is large. Accordingly, numerical instabilities and high computational burden characterize the algorithm. In order to avoid those issues, we propose an efficient implementation drawing inspiration from the reduced order fixed-lag smoother proposed in [43]. Then, the corresponding least favorable model over a finite simulation horizon is derived to evaluate the performance of the smoother. Finally, we consider a target tracking problem and a parameter estimation problem to test the performance of the robust fixed-lag smoother.

The outline of the paper is as follows. In Section 2, we introduce the problem formulation about robust fixed-lag smoothing. In Section 3 we derive the robust fixed-lag smoother. The corresponding algorithm is then reformulated in an efficient way to reduce the computational burden in Section 4. In Section 5, we derive the least favorable model corresponding to the robust fixed-lag smoother. The numerical examples are provided in Section 6, which is devoted to the target tracking problem, and in Section 7, which is devoted to the parameter estimation problem. Finally, in Section 8 we draw the conclusions.

2. Problem Formulation

We consider the nominal state space model:

$$\begin{aligned} x_{t+1} &= Ax_t + Bv_t \\ y_t &= Cx_t + Dv_t \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times (m+n)}$, $C \in \mathbb{R}^{m \times n}$ and $D \in \mathbb{R}^{m \times (m+n)}$, x_t is the state vector, y_t is the observation vector, and $v_t \in \mathbb{R}^{m+n}$ is normalized white Gaussian noise. Moreover, $x_0 \sim \mathcal{N}(\hat{x}_0, V_0)$, with $V_0 > 0$, which is independent from v_t . We also assume that $BD^\top = 0$, $\text{rank}(B) = n$ and $\text{rank}(D) = m$. In this way, the process noise, say $w_t := Bv_t$, and the measurement noise, say $v_t := Dv_t$, are independent and their covariance matrices are BB^\top and DD^\top , respectively. Fixed-lag smoothing aims to find an estimate of $\hat{x}_{t-L+1|t}$ of x_{t-L+1} given $Y_t = \{y_0 \cdots y_t\}$ and L denotes the lag. It is well-known that such a problem can be interpreted as a Kalman prediction problem corresponding to the following nominal augmented state space model [43]:

$$\begin{aligned} \xi_{t+1} &= \tilde{A}\xi_t + \tilde{B}v_t \\ y_t &= \tilde{C}\xi_t + \tilde{D}v_t \\ x_{t-L+1} &= \tilde{H}\xi_{t+1} \end{aligned} \tag{2}$$

where $\tilde{A} \in \mathbb{R}^{(L+1)n \times (L+1)n}$, $\tilde{B} \in \mathbb{R}^{(L+1)n \times (m+n)}$, $\tilde{C} \in \mathbb{R}^{m \times (L+1)n}$, $\tilde{D} \in \mathbb{R}^{m \times (m+n)}$ and $\tilde{H} \in \mathbb{R}^{n \times (L+1)n}$ are such that

$$\xi_t = \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-L} \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & 0 & \cdots & 0 \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\tilde{C} = [C \ 0 \ \cdots \ 0], \quad \tilde{D} = D, \quad \tilde{H} = [0 \ 0 \ \cdots \ I].$$

Then,

$$\hat{x}_{t-L+1|t} = \tilde{H} \hat{\xi}_{t+1}$$

where $\hat{\xi}_{t+1}$ is the one step-ahead predictor of ξ_{t+1} given Y_t . We define $z_t := [\hat{\xi}_{t+1}^\top \ y_t^\top]^\top$. Let $\phi(z_t|\xi_t)$ be the transition probability density function of z_t given ξ_t corresponding to the nominal model (2). Then, $\phi_t(z_t|\xi_t) \sim \mathcal{N}(m_{z_t}, K_{z_t})$ with

$$m_{z_t} = \begin{bmatrix} \tilde{A} \\ \tilde{C} \end{bmatrix} \xi_t, \quad K_{z_t} = \begin{bmatrix} \tilde{B}\tilde{B}^\top & 0 \\ 0 & \tilde{D}\tilde{D}^\top \end{bmatrix}.$$

Notice that $\phi_t(z_t|\xi_t)$ is a degenerate probability density function because $\tilde{B}\tilde{B}^\top$ is singular and thus K_{z_t} as well. More precisely, $\text{rank}(K_{z_t}) = n + m$. Accordingly, the support of $\phi_t(z_t|\xi_t)$ is the $n + m$ -dimensional affine subspace

$$\mathcal{A}_t = \{m_{z_t} + v, \quad v \in \text{Im}(K_{z_t})\}$$

which depends on ξ_t . Then,

$$\phi_t(z_t|\xi_t) = [(2\pi)^{n+m} \det^+(K_{z_t})]^{-1/2} \times \exp \left[-\frac{1}{2} (z_t - m_{z_t})^\top K_{z_t}^+ (z_t - m_{z_t}) \right]$$

where $K_{z_t}^+$ is the pseudo-inverse of K_{z_t} and $\det^+(K_{z_t})$ is the pseudo-determinant of K_{z_t} .

The nominal model in (2) in the time horizon $[0, N]$ is described by the joint probability density

$$f(\Xi_{N+1}, Y_N) = \tilde{f}_0(\xi_0) \prod_{t=0}^N \phi_t(z_t|\xi_t) \quad (3)$$

where

$$\begin{aligned}\Xi_{N+1}^\top &= [\xi_0^\top \cdots \xi_t^\top \cdots \xi_{N+1}^\top] \\ Y_N^\top &= [y_0^\top \cdots y_t^\top \cdots y_N^\top],\end{aligned}$$

$\tilde{f}_0(\xi_0) \sim \mathcal{N}(\hat{\xi}_0, \tilde{V}_0)$ with

$$\hat{\xi}_0 = \begin{bmatrix} \hat{x}_0 \\ \star \end{bmatrix}, \quad \tilde{V}_0 = \begin{bmatrix} V_0 & 0 \\ 0 & \star \end{bmatrix}. \quad (4)$$

In the above equations the star symbol means that it is an arbitrary vector or matrix. Indeed, since x_0 is independent from x_t with $-L \leq t < 0$, the smoother does not depend on those parameters for $t \geq 0$. Accordingly, without loss of generality, we assume that $\tilde{V}_0 > 0$.

In this paper, we consider the situation in which the actual model does not coincide with the nominal one in (2). In particular, we assume that the probability density of the actual model has a structure similar to the one in (3):

$$\tilde{f}(\Xi_{N+1}, Y_N) = \tilde{f}_0(\xi_0) \prod_{t=0}^N \tilde{\phi}_t(z_t|\xi_t)$$

where we assume that $\tilde{\phi}_t$ has the same support of ϕ_t . In this way, we can measure the discrepancy between f and \tilde{f} through the KL-divergence:

$$D(\tilde{f}_Z, f_Z) := \int_{\mathcal{A}_\Xi} \int_{\mathcal{A}_Y} \tilde{f}(\Xi_{N+1}, Y_N) \ln \frac{\tilde{f}(\Xi_{N+1}, Y_N)}{f(\Xi_{N+1}, Y_N)} dY_N d\Xi_{N+1}$$

where $\tilde{E}[\cdot]$ is the expected value operator with respect to the actual probability density \tilde{f} and $\mathcal{A}_\Xi \times \mathcal{A}_Y$ is the support of f and \tilde{f} . It is worth noting that the KL divergence is the natural metric to measure such a mismatch in the case the nominal model is inferred from data, see [44]. It is not difficult to see that

$$D(\tilde{f}, f) = \sum_{t=0}^N D(\tilde{\phi}_t, \phi_t) \quad (5)$$

where

$$\begin{aligned}D(\tilde{\phi}_t, \phi_t) &= \tilde{\mathbb{E}} \left[\ln \left(\frac{\tilde{\phi}_t(z_t|\xi_t)}{\phi_t(z_t|\xi_t)} \right) \right] \\ &:= \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \tilde{\phi}_t(z_t|\xi_t) \tilde{f}_t(\xi_t) \ln \left(\frac{\tilde{\phi}_t(z_t|\xi_t)}{\phi_t(z_t|\xi_t)} \right) dz_t d\xi_t\end{aligned}$$

where $\bar{\mathcal{A}}_t$ is the support of $\tilde{f}_t(\xi_t)$ which denotes the actual marginal density of ξ_t .

Since the actual model is not known, we assume the latter belongs to the ambiguity set which is a ball about f formed by placing an upper bound on $D(\tilde{f}, f)$. However, this ambiguity set contains models which concentrate the uncertainty in a unique time step, i.e. a situation which is unrealistic in practice. On the other hand, in view of Equation (5), we can express such a mismatch incrementally through ϕ_t and $\tilde{\phi}_t$. Accordingly, we assume that $\tilde{\phi}_t$ given Y_{t-1} belongs to the following ambiguity set:

$$\mathcal{B}_t := \left\{ \tilde{\phi}_t \text{ s.t. } \tilde{\mathbb{E}} \left[\ln \left(\frac{\tilde{\phi}_t(z_t|\xi_t)}{\phi_t(z_t|\xi_t)} \right) \middle| Y_{t-1} \right] \leq c_t \right\}$$

where

$$\begin{aligned} & \tilde{\mathbb{E}} \left[\ln \left(\frac{\tilde{\phi}_t(z_t|\xi_t)}{\phi_t(z_t|\xi_t)} \right) \middle| Y_{t-1} \right] \\ & := \int_{\bar{\mathcal{A}}_t} \int_{\mathcal{A}_t} \tilde{\phi}_t(z_t|\xi_t) \tilde{f}_t(\xi_t|Y_{t-1}) \ln \left(\frac{\tilde{\phi}_t(z_t|\xi_t)}{\phi_t(z_t|\xi_t)} \right) dz_t d\xi_t, \end{aligned} \quad (6)$$

and $\check{\mathcal{A}}_t$ is the support of $\tilde{f}_t(\xi_t|Y_{t-1})$. It is worth noting that $c_t > 0$, hereafter called tolerance, is the mismodeling budget allowed at time step t . Our aim is to address the following problem.

Problem 1. *Design a fixed-lag smoother with respect to the ambiguity set \mathcal{B}_t for $t = 0 \dots N$.*

3. Robust smoothing

We propose a robust fixed-lag smoother of x_{t-L+1} given Y_t with respect to \mathcal{B}_t solving the following minimax problem:

$$\begin{aligned} \hat{x}_{t-L+1|t} &= \tilde{H} \hat{\xi}_{t+1} \\ \hat{\xi}_t &= \operatorname{argmin}_{g_t \in \mathcal{G}_t} \max_{\tilde{\phi}_t \in \mathcal{B}_t} J_t(\tilde{\phi}_t, g_t) \end{aligned} \quad (7)$$

where

$$\begin{aligned} J_t(\tilde{\phi}_t, g_t) &= \frac{1}{2} \tilde{\mathbb{E}} \left[\|\tilde{H}(\xi_{t+1} - g_t(y_t))\|^2 \middle| Y_{t-1} \right] \\ &= \frac{1}{2} \int_{\bar{\mathcal{A}}_t} \int_{\mathcal{A}_t} \|\tilde{H}(\xi_{t+1} - g_t(y_t))\|^2 \tilde{\phi}_t(z_t|\xi_t) \times \tilde{f}_t(\xi_t|Y_{t-1}) dz_t d\xi_t, \end{aligned}$$

\mathcal{G}_t denotes the class of estimators with finite second-order moments with respect to all the densities $\tilde{\phi}_t(z_t|\xi_t)\tilde{f}_t(\xi_t|Y_{t-1})$ such that $\tilde{\phi}_t \in \mathcal{B}_t$. Notice that $\tilde{\phi}_t$ must satisfy the constraint:

$$I_t(\tilde{\phi}_t) \triangleq \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \tilde{\phi}_t(z_t|\xi_t)\tilde{f}_t(\xi_t|Y_{t-1}) dz_t d\xi_t = 1. \quad (8)$$

It is worth noting that problems like (7) can be written as a risk-sensitive problem, i.e. as a minimization problem where the standard quadratic cost function is replaced by an exponential cost function, see [45] for more details.

Lemma 1. *For a fixed estimator $g_t \in \mathcal{G}_t$, the density $\tilde{\phi}_t(z_t|\xi_t) \in \mathcal{B}_t$ that maximizes the objective function*

$$J_t(\tilde{\phi}_t, g_t) = \tilde{\mathbb{E}} \left[\|\tilde{H}(\xi_{t+1} - g_t(y_t))\|^2 | Y_{t-1} \right]$$

under constraint $D_t(\tilde{\phi}_t, \phi_t) \leq c_t$ is given by

$$\tilde{\phi}_t^0 = \frac{1}{M_t(\lambda_t)} \exp \left(\frac{1}{2\lambda_t} \|\tilde{H}(x_{t+1} - g_t(y_t))\|^2 \right) \phi_t \quad (9)$$

where $M_t(\lambda_t)$ is the normalizing constant defined as follows:

$$M_t(\lambda_t) = \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \exp \left(\frac{1}{2\lambda_t} \|\tilde{H}(\xi_{t+1} - g_t(y_t))\|^2 \right) \phi_t \times \tilde{f}_t(\xi_t|Y_{t-1}) dz_t d\xi_t.$$

Moreover, for $c_t > 0$ sufficiently small, there exists a unique $\lambda_t > 0$ such that $D(\tilde{\phi}_t^0, \phi_t) = c_t$.

PROOF. The proof is similar to the one of [27, Lemma2]. \square

Once we get the function $\tilde{\phi}_t^0$, the estimator $g_t \in \mathcal{G}_t$ minimizing the objective function $J_t(\tilde{\phi}_t^0, g_t)$ is given by

$$\hat{\xi}_{t+1} = g_t^0(y_t) = \tilde{\mathbb{E}}[\xi_{t+1}|Y_t] = \int_{\tilde{\mathcal{A}}_{t+1}} \xi_{t+1} \tilde{f}_{t+1}(\xi_{t+1}|Y_t) d\xi_{t+1}$$

where

$$\tilde{f}_{t+1}(\xi_{t+1}|Y_t) = \frac{\int_{\tilde{\mathcal{A}}_t} \tilde{\phi}_t^0(z_t|\xi_t)\tilde{f}_t(\xi_t|Y_{t-1}) d\xi_t}{\int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t^*} \tilde{\phi}_t^0(z_t|\xi_t)\tilde{f}_t(\xi_t|Y_{t-1}) d\xi_{t+1} d\xi_t} \quad (10)$$

where \mathcal{A}_t^* is defined as follows: $\xi_{t+1} \in \mathcal{A}_t^*$ if and only if there exists at least one y_t for which $[\xi_{t+1}^\top \ y_t^\top]^\top \in \mathcal{A}_t$. The optimal estimator g_t^0 solution to (7) relies on the least-favorable density $\tilde{\phi}_t^0(z_t|\xi_t)$. On the other hand, the latter depends on the estimator g_t^0 . In order to break this deadlock problem, an additional assumption is needed. More precisely, we assume that the *a priori* conditional density $\tilde{f}_t(\xi_t|Y_{t-1})$ is Gaussian $\tilde{f}_t(\xi_t|Y_{t-1}) \sim \mathcal{N}(\hat{\xi}_t, \tilde{V}_t)$. In view of (2), the marginal density

$$\bar{f}_t(z_t|Y_{t-1}) := \int_{\mathcal{A}_t} \phi_t(z_t|\xi_t) \tilde{f}_t(\xi_t|Y_{t-1}) d\xi_t$$

is Gaussian so that

$$\bar{f}_t(z_t|Y_{t-1}) \sim \mathcal{N}(m_{z_t|Y_{t-1}}, K_{z_t|Y_{t-1}}) \quad (11)$$

where

$$\begin{aligned} m_{z_t|Y_{t-1}} &= \begin{bmatrix} \tilde{A} \\ \tilde{C} \end{bmatrix} \hat{\xi}_t, \\ K_{z_t|Y_{t-1}} &= \begin{bmatrix} \tilde{A} \\ \tilde{C} \end{bmatrix} \tilde{V}_t \begin{bmatrix} \tilde{A}^\top & \tilde{C}^\top \end{bmatrix} + \begin{bmatrix} \tilde{B} \\ \tilde{D} \end{bmatrix} \begin{bmatrix} \tilde{B}^\top & \tilde{D}^\top \end{bmatrix}. \end{aligned}$$

Then, on the basis of Lemma 1, we have that the least favorable density of z_t given Y_{t-1} is

$$\begin{aligned} \tilde{f}_t(z_t|Y_{t-1}) &:= \int_{\mathcal{A}_t} \tilde{\phi}_t^0(z_t|\xi_t) \tilde{f}_t(\xi_t|Y_{t-1}) d\xi_t \\ &= \frac{1}{M_t(\lambda_t)} \exp\left(\frac{1}{2\lambda_t} \|\tilde{H}(\xi_{t+1} - g_t(y_t))\|^2\right) \bar{f}_t(z_t|Y_{t-1}). \end{aligned} \quad (12)$$

Accordingly, $\tilde{f}_t(z_t|Y_{t-1})$ is a Gaussian probability density.

Lemma 2. *Consider the state space model (1) with $\text{rank}(B) = n$ and $\text{rank}(D) = m$. If $\tilde{V}_t > 0$ then $K_{z_t|Y_{t-1}} > 0$, i.e. $\tilde{f}_t(z_t|Y_{t-1})$ is a non-degenerate density. Moreover,*

$$\tilde{P}_{t+1} := \tilde{A}\tilde{V}_t\tilde{A}^\top - \tilde{A}\tilde{V}_t\tilde{C}^\top(\tilde{C}\tilde{V}_t\tilde{C}^\top + \tilde{D}\tilde{D}^\top)^{-1}\tilde{C}\tilde{V}_t\tilde{A}^\top + \tilde{B}\tilde{B}^\top$$

is positive definite.

PROOF. First, notice that the block in position (2,2) of $K_{z_t|Y_{t-1}}$ is $\tilde{C}\tilde{V}_t\tilde{C}^\top + \tilde{D}\tilde{D}^\top$. The latter is positive definite because $\tilde{C}\tilde{V}_t\tilde{C}^\top + \tilde{D}\tilde{D}^\top \geq \tilde{D}\tilde{D}^\top > 0$. Accordingly, in order to prove that $K_{z_t|Y_{t-1}} > 0$ it is sufficient to prove that \tilde{P}_{t+1} , which is the Schur complement of the block (2,2) of $K_{z_t|Y_{t-1}}$, is positive definite. Since \tilde{V}_t is invertible we can rewrite \tilde{P}_{t+1} by using the Woodbury formula:

$$\tilde{P}_{t+1} = \tilde{A}[\tilde{V}_t^{-1} + \tilde{C}^\top(\tilde{D}\tilde{D}^\top)^{-1}\tilde{C}]^{-1}\tilde{A}^\top + \tilde{B}\tilde{B}^\top.$$

Let $v = [v_1 \ v_2]^\top \in \mathbb{R}^{(n+1)L}$ be such that $v_1 \in \mathbb{R}^n$ and $v_2 \in \mathbb{R}^{nL}$. Notice that

$$v^\top \tilde{P}_{t+1} v = v^\top \tilde{A}[\tilde{V}_t^{-1} + \tilde{C}^\top(\tilde{D}\tilde{D}^\top)^{-1}\tilde{C}]^{-1}\tilde{A}^\top v + v_1^\top \tilde{B}\tilde{B}^\top v_1$$

and thus

$$v^\top \tilde{P}_{t+1} v \geq v_1^\top \tilde{B}\tilde{B}^\top v_1 \tag{13}$$

$$v^\top \tilde{P}_{t+1} v \geq v^\top \tilde{A}[\tilde{V}_t^{-1} + \tilde{C}^\top(\tilde{D}\tilde{D}^\top)^{-1}\tilde{C}]^{-1}\tilde{A}^\top v. \tag{14}$$

Assume that $v^\top \tilde{P}_{t+1} v = 0$. Since $\tilde{B}\tilde{B}^\top > 0$, by (13) we have that $v_1 = 0$. Accordingly, the inequality in (14) for $v_1 = 0$ becomes

$$v^\top \tilde{P}_{t+1} v \geq [0 \ v_2^\top] \tilde{A}[\tilde{V}_t^{-1} + \tilde{C}^\top(\tilde{D}\tilde{D}^\top)^{-1}\tilde{C}]^{-1}\tilde{A}^\top \begin{bmatrix} 0 \\ v_2 \end{bmatrix}.$$

Moreover, in view of the particular structure of \tilde{A} , we have that $[0 \ v_2^\top] \tilde{A} = [v_2^\top \ 0] = 0$ if and only if $v_2 = 0$. Since $[\tilde{V}_t^{-1} + \tilde{C}^\top(\tilde{D}\tilde{D}^\top)^{-1}\tilde{C}]^{-1} > 0$, because $\tilde{V}_t > 0$, it follows that $v_2 = 0$ and thus $v = 0$. We proved that if $v^\top \tilde{P}_{t+1} v = 0$ then $v = 0$, i.e. \tilde{P}_{t+1} is positive definite. \square

Finally, if $\tilde{f}_t(\xi_t|Y_{t-1})$ is Gaussian then, in view of (10), also $\tilde{f}_{t+1}(\xi_{t+1}|Y_t)$ is Gaussian. Accordingly, the assumption that $\tilde{f}_0(\xi_0)$ is Gaussian, implies that $\tilde{f}_t(\xi_t|Y_{t-1})$ is Gaussian for any t .

Theorem 3. *Consider the state space model (1) where we recall that $\text{rank}(B) = n$ and $\text{rank}(D) = m$. Let $\tilde{f}_t(\xi_t|Y_{t-1}) \sim \mathcal{N}(\hat{\xi}_t, \tilde{V}_t)$ with $\tilde{V}_t > 0$. Then, the estimator*

$$g_t^0(y_t) = \tilde{A}\hat{\xi}_t + \tilde{G}_t(y_t - \tilde{C}\hat{\xi}_t) \tag{15}$$

with

$$\tilde{G}_t = \tilde{A}\tilde{V}_t\tilde{C}^\top(\tilde{C}\tilde{V}_t\tilde{C}^\top + \tilde{D}\tilde{D}^\top)^{-1}$$

solves Problem (7). The nominal error covariance of ξ_{t+1} given y_t is

$$\tilde{P}_{t+1} = \tilde{A}\tilde{V}_t\tilde{A}^\top - \tilde{G}_t(\tilde{C}\tilde{V}_t\tilde{C}^\top + \tilde{D}\tilde{D}^\top)\tilde{G}_t^\top + \tilde{B}\tilde{B}^\top$$

and the perturbed error covariance of ξ_{t+1} given Y_t is

$$\tilde{V}_{t+1} = (\tilde{P}_{t+1}^{-1} - \lambda^{-1}\tilde{H}^\top\tilde{H})^{-1}.$$

If we denote $r(P)$ as the largest eigenvalue of P , then the Lagrange multiplier $\lambda_t > r(\tilde{H}\tilde{P}_{t+1}\tilde{H}^\top)$ is unique and such that

$$\gamma(\lambda_t) = \frac{1}{2} \left[\text{tr} \left((I - \lambda_t^{-1}\tilde{H}^\top\tilde{H}\tilde{P}_{t+1})^{-1} - I \right) + \ln \det(I - \lambda_t^{-1}\tilde{H}^\top\tilde{H}\tilde{P}_{t+1}) \right] = c_t. \quad (16)$$

Finally, the least favorable density $\tilde{f}_t^0(z_t|Y_{t-1})$ corresponding to the solution of (7) is a non-degenerate Gaussian density.

PROOF. As we already noticed, $\bar{f}_t(z_t|Y_{t-1})$ is Gaussian, and in view of Lemma 2, non-degenerate. Accordingly, in view of (12), $\tilde{f}_t(z_t|Y_{t-1})$ is Gaussian and non-degenerate. Let

$$\tilde{f}_t(z_t|Y_{t-1}) \sim \mathcal{N}(\tilde{m}_{z_t|Y_{t-1}}, \tilde{K}_{z_t|Y_{t-1}})$$

with

$$\tilde{m}_{z_t|Y_{t-1}} = \begin{bmatrix} \tilde{m}_{\xi_{t+1}|Y_{t-1}} \\ \tilde{m}_{y_t|Y_{t-1}} \end{bmatrix},$$

and

$$\tilde{K}_{z_t|Y_{t-1}} = \begin{bmatrix} \tilde{K}_{\xi_{t+1}|Y_{t-1}} & \tilde{K}_{\xi_{t+1}y_t|Y_{t-1}} \\ \tilde{K}_{y_t\xi_{t+1}|Y_{t-1}} & \tilde{K}_{y_t|Y_{t-1}} \end{bmatrix}.$$

In view of (11) and (12), the conditional KL-divergence in (6) admits the closed-form expression

$$\begin{aligned} \tilde{\mathbb{E}} \left[\ln(\tilde{\phi}_t^\circ/\phi_t) | Y_{t-1} \right] &= \tilde{\mathbb{E}} \left[\ln(\tilde{f}_t/\bar{f}_t) | Y_{t-1} \right] \\ &= \frac{1}{2} \left[\|\Delta m\|_{K_{z_t|Y_{t-1}}^{-1}}^2 + \text{tr}(K_{z_t|Y_{t-1}}^{-1}\tilde{K}_{z_t|Y_{t-1}} - I) - \ln \det(K_{z_t|Y_{t-1}}^{-1}\tilde{K}_{z_t|Y_{t-1}}) \right] \end{aligned} \quad (17)$$

where

$$\Delta m = \tilde{m}_{z_t|Y_{t-1}} - m_{z_t|Y_{t-1}}.$$

Therefore, we can rewrite the minimax game (7) with respect to $\tilde{f}_t(z_t|Y_{t-1})$ and $\bar{f}_t(z_t|Y_{t-1})$:

$$(g_t^0, \tilde{f}_t^0) = \arg \min_{g_t \in \bar{\mathcal{G}}_t} \max_{\tilde{f}_t \in \bar{\mathcal{B}}_t} J_t(\tilde{f}_t, g_t)$$

where

$$\bar{\mathcal{B}}_t = \left\{ \tilde{f}_t(z_t|Y_{t-1}) \text{ s.t. } \mathbb{E} \left[\ln(\tilde{f}_t/\bar{f}_t) | Y_{t-1} \right] \leq c_t \right\},$$

$\bar{\mathcal{G}}_t$ is the set of estimators with finite second order moments with respect to all the densities $\tilde{f}_t \in \bar{\mathcal{B}}_t$ and

$$J_t(\tilde{f}_t, g_t) := \frac{1}{2} \int_{\mathbb{R}^{(L+1)n+m}} \|\tilde{H}(\xi_{t+1} - g_t(y_t))\|^2 \times \tilde{f}_t(z_t|Y_{t-1}) dz_t.$$

Next, we prove that \tilde{f}_t^0 and g_t^0 are such that

$$J_t(\tilde{f}_t, g_t^0) \leq J_t(\tilde{f}_t^0, g_t^0) \leq J_t(\tilde{f}_t^0, g_t), \quad (18)$$

where $\tilde{f}_t^0(z_t|Y_{t-1}) \sim \mathcal{N}(\tilde{m}_{z_t|Y_{t-1}}^0, \tilde{K}_{z_t|Y_{t-1}}^0)$ with

$$\begin{aligned} \tilde{m}_{z_t|Y_{t-1}}^0 &= m_{z_t|Y_{t-1}}, \\ \tilde{K}_{z_t|Y_{t-1}}^0 &= \begin{bmatrix} \tilde{K}_{\xi_{t+1}|Y_{t-1}} & K_{\xi_{t+1}y_t|Y_{t-1}} \\ K_{y_t\xi_{t+1}|Y_{t-1}} & K_{y_t|Y_{t-1}} \end{bmatrix}. \end{aligned} \quad (19)$$

Since \tilde{f}_t^0 is Gaussian, the optimal estimator satisfying the second inequality in (18) is (15). Then, it remains to prove that the least favorable density \tilde{f}_t^0 is such that (19) holds.

It is not difficult to see that

$$J_t(\tilde{f}_t, g_t^0) = \frac{1}{2} \text{tr} \left\{ \begin{bmatrix} I \\ -\tilde{G}_t^\top \end{bmatrix} \tilde{H}^\top \tilde{H} \begin{bmatrix} I & -\tilde{G}_t \end{bmatrix} \times \left(\tilde{K}_{z_t|Y_{t-1}} + \Delta m \Delta m^\top \right) \right\}. \quad (20)$$

Then, based on the parametric structure of the KL divergence in (17) and the objective function in (20), we consider the corresponding Lagrangian as

a function of $\tilde{m}_{z_t|Y_{t-1}}$ and $\tilde{K}_{z_t|Y_{t-1}}$ as follows:

$$\begin{aligned}
& \mathcal{L}(\tilde{m}_{z_t|Y_{t-1}}, \tilde{K}_{z_t|Y_{t-1}}, \lambda_t) \\
&= J(\tilde{m}_{z_t|Y_{t-1}}, \tilde{K}_{z_t|Y_{t-1}}) + \lambda_t \left(c_t - \tilde{\mathbb{E}}[\ln(\tilde{f}_t/\bar{f}_t)|Y_{t-1}] \right) \\
&= \frac{1}{2} \text{tr} \left\{ \begin{bmatrix} I \\ -\tilde{G}_t^\top \end{bmatrix} \tilde{H}^\top \tilde{H} \begin{bmatrix} I & -\tilde{G}_t \end{bmatrix} (\tilde{K}_{z_t|Y_{t-1}} + \Delta m \Delta m^\top) \right\} \\
&\quad - \frac{\lambda_t}{2} \Delta m^\top K_{z_t|Y_{t-1}}^{-1} \Delta m - \frac{\lambda_t}{2} \text{tr}(K_{z_t|Y_{t-1}}^{-1} \tilde{K}_{z_t|Y_{t-1}}) + \frac{\lambda_t}{2} \text{tr}(I) \\
&\quad + \frac{\lambda_t}{2} \ln \det(K_{z_t|Y_{t-1}}^{-1} \tilde{K}_{z_t|Y_{t-1}}) + \lambda_t c_t \\
&= \lambda_t c_t + \frac{\lambda_t}{2} \text{tr}(I) + \frac{\lambda_t}{2} \ln \det(K_{z_t|Y_{t-1}}^{-1} \tilde{K}_{z_t|Y_{t-1}}) \\
&\quad + \frac{1}{2} \text{tr}(W(\lambda_t) \tilde{K}_{z_t|Y_{t-1}}) + \frac{1}{2} \Delta m^\top W(\lambda_t) \Delta m
\end{aligned}$$

where

$$W(\lambda_t) \triangleq \begin{bmatrix} I \\ -\tilde{G}_t^\top \end{bmatrix} \tilde{H}^\top \tilde{H} \begin{bmatrix} I & -\tilde{G}_t \end{bmatrix} - \lambda_t K_{z_t|Y_{t-1}}^{-1}.$$

The first variation and the second variation of \mathcal{L} with respect to $\tilde{m}_{z_t|Y_{t-1}}$ are, respectively,

$$\begin{aligned}
& \delta \mathcal{L}(\tilde{m}_{z_t|Y_{t-1}}, \tilde{K}_{z_t|Y_{t-1}}, \lambda_t; \delta \tilde{m}_{z_t|Y_{t-1}}) \\
&= \frac{1}{2} \delta \tilde{m}_{z_t|Y_{t-1}}^\top W(\lambda_t) \Delta m + \Delta m^\top W(\lambda_t) \delta \tilde{m}_{z_t|Y_{t-1}},
\end{aligned}$$

$$\delta^2 \mathcal{L}(\tilde{m}_{z_t|Y_{t-1}}, \tilde{K}_{z_t|Y_{t-1}}, \lambda_t; \delta \tilde{m}_{z_t|Y_{t-1}}, \delta \tilde{m}_{z_t|Y_{t-1}}) = \delta \tilde{m}_{z_t|Y_{t-1}}^\top W(\lambda_t) \delta \tilde{m}_{z_t|Y_{t-1}}$$

so that $\delta^2 \mathcal{L} < 0$, for any $\tilde{m}_{z_t|Y_{t-1}} \neq 0$ if and only if $W(\lambda_t) < 0$, which means \mathcal{L} is strictly concave if and only if $W(\lambda_t)$ is negative definite. Next, we find the condition on λ_t for which $W(\lambda_t) < 0$. We denote $M = \tilde{H} \begin{bmatrix} I & -\tilde{G}_t \end{bmatrix} O_{z_t}$, where $K_{z_t|Y_{t-1}} = O_{z_t} O_{z_t}^\top$, so that

$$\begin{aligned}
MM^\top &= \tilde{H} \begin{bmatrix} I & -\tilde{G}_t \end{bmatrix} O_{z_t} O_{z_t}^\top \begin{bmatrix} I \\ -\tilde{G}_t^\top \end{bmatrix} \tilde{H}^\top \\
&= \tilde{H} \left(K_{\xi_{t+1}|Y_{t-1}} - K_{\xi_{t+1}y_t|Y_{t-1}} K_{y_t|Y_{t-1}}^{-1} K_{y_t \xi_{t+1}|Y_{t-1}} \right) \tilde{H}^\top = \tilde{H} \tilde{P}_{t+1} \tilde{H}^\top.
\end{aligned}$$

Therefore,

$$\begin{aligned}
W(\lambda_t) &= O_{z_t}^{-\top} \left(O_{z_t}^\top \begin{bmatrix} I \\ -\tilde{G}_t^\top \end{bmatrix} \tilde{H}^\top \tilde{H} \begin{bmatrix} I & -\tilde{G}_t \end{bmatrix} O_{z_t} - \lambda_t I \right) O_{z_t}^{-1} \\
&= O_{z_t}^{-\top} (M^\top M - \lambda_t I) O_{z_t}^{-1}
\end{aligned}$$

and it is congruent to $\bar{W}(\lambda_t) = M^\top M - \lambda_t I$, which means $\bar{W}(\lambda_t) < 0$ if and only if $W(\lambda_t) < 0$, meanwhile, it is not difficult to see that $r(M^\top M) = r(MM^\top) = r(\tilde{H}\tilde{P}_{t+1}\tilde{H}^\top)$. Hence, $W(\lambda_t) < 0$ as long as the Lagrange multiplier $\lambda_t > r(\tilde{H}\tilde{P}_{t+1}\tilde{H}^\top)$. In such a situation the minimum is such that $\Delta m = 0$, which implies

$$\tilde{m}_{z_t|Y_{t-1}}^0 = m_{z_t|Y_{t-1}}. \quad (21)$$

The first variation and the second variation of \mathcal{L} with respect to $\tilde{K}_{z_t|Y_{t-1}}$ are, respectively,

$$\begin{aligned} & \delta L(\tilde{m}_{z_t|Y_{t-1}}, \tilde{K}_{z_t|Y_{t-1}}, \lambda_t; \delta \tilde{K}_{z_t|Y_{t-1}}) \\ &= \frac{1}{2} \text{tr} \left\{ \lambda_t \tilde{K}_{z_t|Y_{t-1}}^{-1} \delta \tilde{K}_{z_t|Y_{t-1}} + W(\lambda_t) \delta \tilde{K}_{z_t|Y_{t-1}} \right\} \end{aligned}$$

and

$$\begin{aligned} & \delta^2 L(\tilde{m}_{z_t|Y_{t-1}}, \tilde{K}_{z_t|Y_{t-1}}, \lambda_t; \delta \tilde{K}_{z_t|Y_{t-1}}, \delta \tilde{K}_{z_t|Y_{t-1}}) \\ &= -\frac{1}{2} \lambda_t \text{tr} \left\{ (\tilde{K}_{z_t|Y_{t-1}}^{-1} \delta \tilde{K}_{z_t|Y_{t-1}})^2 \right\} < 0. \end{aligned}$$

Accordingly, \mathcal{L} is strictly concave in $\tilde{K}_{z_t|Y_{t-1}}$. Thus, the minimum point $\tilde{K}_{z_t|Y_{t-1}}^0$ is given by imposing the stationarity condition

$$\delta L(\tilde{m}_{z_t|Y_{t-1}}, \tilde{K}_{z_t|Y_{t-1}}^0, \lambda_t; \delta \tilde{K}_{z_t|Y_{t-1}}) = 0.$$

The latter implies that such a point is

$$\tilde{K}_{z_t|Y_{t-1}}^{0,-1} = K_{z_t|Y_{t-1}}^{-1} - \frac{1}{\lambda_t} \begin{bmatrix} I \\ -\tilde{G}_t^\top \end{bmatrix} \tilde{H}^\top \tilde{H} \begin{bmatrix} I & -\tilde{G}_t \end{bmatrix}. \quad (22)$$

Notice that the block upper diagonal lower (UDL) form of $K_{z_t|Y_{t-1}}$ is

$$K_{z_t|Y_{t-1}} = \begin{bmatrix} I & \tilde{G}_t \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{P}_{t+1} & 0 \\ 0 & K_{y_t|Y_{t-1}} \end{bmatrix} \begin{bmatrix} I & 0 \\ \tilde{G}_t^\top & I \end{bmatrix} \quad (23)$$

and its inverse admits the following UDL decomposition

$$K_{z_t|Y_{t-1}}^{-1} = \begin{bmatrix} I & 0 \\ -\tilde{G}_t^\top & I \end{bmatrix} \begin{bmatrix} \tilde{P}_{t+1}^{-1} & 0 \\ 0 & K_{y_t|Y_{t-1}}^{-1} \end{bmatrix} \begin{bmatrix} I & -\tilde{G}_t \\ 0 & I \end{bmatrix}. \quad (24)$$

Therefore, substituting Equation (24) in Equation (22), we have

$$\tilde{K}_{z_t|Y_{t-1}}^{0,-1} = \begin{bmatrix} I & 0 \\ -\tilde{G}_t^\top & I \end{bmatrix} \begin{bmatrix} \tilde{P}_{t+1}^{-1} - \lambda_t^{-1} \tilde{H}^\top \tilde{H} & 0 \\ 0 & K_{y_t|Y_{t-1}}^{-1} \end{bmatrix} \begin{bmatrix} I & -\tilde{G}_t \\ 0 & I \end{bmatrix},$$

so that

$$\tilde{K}_{z_t|Y_{t-1}}^0 = \begin{bmatrix} I & \tilde{G}_t \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{V}_{t+1} & 0 \\ 0 & K_{y_t|Y_{t-1}} \end{bmatrix} \begin{bmatrix} I & 0 \\ \tilde{G}_t^\top & I \end{bmatrix} \quad (25)$$

where

$$\tilde{V}_{t+1} = (\tilde{P}_{t+1}^{-1} - \lambda_t^{-1} \tilde{H}^\top \tilde{H})^{-1}.$$

Then, let $\gamma(\lambda_t) := \tilde{\mathbb{E}}[\ln(\tilde{f}_t^0/\tilde{f}_t)|Y_{t-1}]$. By taking into account (17) and using Equations (21), (23) and (25), we obtain (16). The first derivative of $\gamma(\lambda_t)$ is

$$\begin{aligned} \frac{\partial \gamma(\lambda_t; \delta \lambda_t)}{\partial \lambda_t} &= \frac{\lambda_t^{-2}}{2} \operatorname{tr} \left[(I - \lambda_t^{-1} \tilde{H}^\top \tilde{H} \tilde{P}_{t+1})^{-1} \right. \\ &\quad \left. \times \tilde{H}^\top \tilde{H} \tilde{P}_{t+1} (I - (I - \lambda_t^{-1} \tilde{H}^\top \tilde{H} \tilde{P}_{t+1})^{-1}) \right] \\ &= \frac{\lambda_t^{-2}}{2} \operatorname{tr} \left[(I - \lambda_t^{-1} \tilde{H}^\top \tilde{H} \tilde{P}_{t+1})^{-1} \tilde{H}^\top \tilde{H} \tilde{P}_{t+1} \right. \\ &\quad \left. \times (I - \lambda_t^{-1} \tilde{H}^\top \tilde{H} \tilde{P}_{t+1})^{-1} ((I - \lambda_t^{-1} \tilde{H}^\top \tilde{H} \tilde{P}_{t+1}) - I) \right] \\ &= -\frac{\lambda_t^{-3}}{2} \operatorname{tr} \left[((I - \lambda_t^{-1} \tilde{H}^\top \tilde{H} \tilde{P}_{t+1})^{-1} \tilde{H}^\top \tilde{H} \tilde{P}_{t+1})^2 \right] < 0. \end{aligned}$$

Therefore, $\gamma(\lambda_t)$ is strictly monotone decreasing. Moreover, it is not difficult to see that

$$\lim_{\lambda_t \rightarrow \infty} \gamma(\lambda_t) = 0, \quad \lim_{\lambda_t \rightarrow r(\tilde{Q})} \gamma(\lambda_t) = +\infty \quad (26)$$

where $r(\tilde{Q}) = r(\tilde{H} \tilde{P}_{t+1} \tilde{H}^\top)$. As a consequence, there exists a unique Lagrangian multiplier $\lambda_t > r(\tilde{Q}) > 0$ such that $D(\tilde{f}, f) = c_t$. The fact that $\tilde{f}_t^0(z_t|Y_{t-1})$ is non-degenerate follows from the fact that $\tilde{K}_{z_t|Y_{t-1}}^0 \geq K_{z_t|Y_{t-1}} > 0$. \square

Corollary 1. *Consider the state space model (1) where we recall: $\operatorname{rank}(B) = n$, $\operatorname{rank}(D) = m$ and $\xi_0 \sim \mathcal{N}(\hat{\xi}_0, \tilde{V}_0)$ with $\tilde{V}_0 > 0$. Then, $\tilde{f}_t^0(z_t|Y_{t-1})$ is non-degenerate for any $t \geq 0$.*

PROOF. We prove the claim by induction. Let $\tilde{f}_t(\xi_t|Y_{t-1}) \sim \mathcal{N}(\hat{\xi}_t, \tilde{V}_t)$ with $\tilde{V}_t > 0$. By Theorem 3 we have that $\tilde{f}_t^0(z_t|Y_{t-1})$ is Gaussian non-degenerate and $V_{t+1} > 0$. From, $\tilde{f}_t(z_t|Y_{t-1})$ we have that $\tilde{f}_{t+1}(\xi_{t+1}|Y_t) \sim \mathcal{N}(\hat{\xi}_{t+1}, \tilde{V}_{t+1})$ which is non-degenerate. Finally, at the initial time $t = 0$, we have $f_0(\xi_0|Y_{-1}) := \tilde{f}_0(\xi_0) \sim \mathcal{N}(\hat{\xi}_0, \tilde{V}_0)$ and \tilde{V}_0 is positive definite by assumption. \square

The resulting robust fixed-lag smoother is outlined in Algorithm 1 where $\theta_t := \lambda_t^{-1}$ is the risk sensitivity parameter and

$$\gamma(\tilde{P}_{t+1}, \theta_t) := \frac{1}{2} \left[\text{tr} \left((I - \theta_t \tilde{H}^\top \tilde{H} \tilde{P}_{t+1})^{-1} - I \right) + \ln \det(I - \theta_t \tilde{H}^\top \tilde{H} \tilde{P}_{t+1}) \right]. \quad (27)$$

Algorithm 1 Robust fixed-lag smoother with lag L

Input: $y_0 \dots y_N, \hat{\xi}_0, \tilde{V}_0, c_t$

Output: $\hat{x}_{t-L+1|t}, t = L-1 \dots N$

- 1: **for** $t = 0 : N$ **do**
 - 2: $\tilde{G}_t = \tilde{A} \tilde{V}_t \tilde{C}^\top (\tilde{C} \tilde{V}_t \tilde{C}^\top + \tilde{D} \tilde{D}^\top)^{-1}$
 - 3: $\hat{\xi}_{t+1} = \tilde{A} \hat{\xi}_t + \tilde{G}_t (y_t - \tilde{C} \hat{\xi}_t)$
 - 4: $\hat{x}_{t-L+1|t} = \tilde{H} \hat{\xi}_{t+1}$
 - 5: $\tilde{P}_{t+1} = \tilde{A} \tilde{V}_t \tilde{A}^\top - \tilde{G}_t (\tilde{C} \tilde{V}_t \tilde{C}^\top + \tilde{D} \tilde{D}^\top) \tilde{G}_t^\top + \tilde{B} \tilde{B}^\top$
 - 6: Find θ_t s.t. $\gamma(\tilde{P}_{t+1}, \theta_t) = c_t$
 - 7: $\tilde{V}_{t+1} = (\tilde{P}_{t+1}^{-1} - \theta_t \tilde{H}^\top \tilde{H})^{-1}$
 - 8: **end for**
-

Finally, in the case that $c_t = 0$, i.e. the nominal model coincides with the actual one, it is not difficult to see that $\theta_t = 0$ that is we obtain the standard fixed-lag smoother.

Remark 1. In the presence of a deterministic input u_t , then it is possible to derive the corresponding robust fixed-lag smoother by using arguments similar to the ones in [26]. For instance, if the input acts only in the state equation, i.e. we have $x_{t+1} = Ax_t + Bv_t + u_t$, then Step 3 in Algorithm 1 is substituted with $\hat{\xi}_{t+1} = \tilde{A} \hat{\xi}_t + \tilde{G}_t (y_t - \tilde{C} \hat{\xi}_t) + w_t$, where $w_t := [u_t^\top u_{t-1}^\top \dots u_{t-L+1}^\top]^\top$.

4. Efficient implementation

Algorithm 1 is not numerically robust and efficient in terms of computational burden. Since the dimension of \tilde{P}_t and \tilde{V}_t is proportional to L , which is typically large, their inversion is time consuming and not accurate. Accordingly, there is the need to develop an efficient strategy which avoids those matrix inversions as it has been done in [43] for the standard fixed-lag smoother. The efficient procedure for our robust smoother is outlined in Algorithm 2. Next, we explain how to derive the salient steps. In what follows we always refer to the steps of Algorithm 2 if not specified.

First, we rewrite the risk-sensitive Riccati iteration in Step 5 of Algorithm 1 as:

$$\tilde{L}_t = \tilde{V}_t \tilde{C}^\top (\tilde{C} \tilde{V}_t \tilde{C}^\top + \tilde{D} \tilde{D}^\top)^{-1} \quad (28)$$

$$\tilde{P}_{t|t} = (I - \tilde{L}_t \tilde{C}) \tilde{V}_t \quad (29)$$

$$\tilde{P}_{t+1} = \tilde{A} \tilde{P}_{t|t} \tilde{A}^\top + \tilde{B} \tilde{B}^\top \quad (30)$$

where $\tilde{P}_{t|t} := \tilde{\mathbb{E}}[(\xi_t - \hat{\xi}_{t|t})(\xi_t - \hat{\xi}_{t|t})^\top]$ and $\hat{\xi}_{t|t}$ is the estimator of ξ_t given Y_t . Then, we parameterize \tilde{V}_t and \tilde{L}_t in blocks of $n \times n$ matrices as follows:

$$\tilde{V}_t = \begin{bmatrix} V_t & (V_t^1)^\top & \cdots & (V_t^j)^\top & \cdots & (V_t^L)^\top \\ V_t^1 & V_t^{1,1} & \cdots & (V_t^{j,1})^\top & \cdots & (V_t^{L,1})^\top \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ V_t^j & V_t^{j,1} & \cdots & V_t^{j,k} & \cdots & (V_t^{L,k})^\top \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ V_t^L & V_t^{L,1} & \cdots & V_t^{L,k} & \cdots & V_t^{L,L} \end{bmatrix}$$

$$\tilde{L}_t = [(L_t)^\top \quad (L_t^1)^\top \quad \cdots \quad (L_t^j)^\top \quad \cdots \quad (L_t^L)^\top]^\top.$$

With some abuse of notation: V_t is also denoted by $V_t^{0,0}$; V_t^j is also denoted by $V_t^{j,0}$. Substituting the above parametrizations in (28), we obtain Steps 3, 7. In regard to the initial conditions: we only need \hat{x}_0 and V_0 , we set $V_0^j = 0$ and $V_0^{j,k} > 0$, with $j, k > 1$, are set arbitrary such that $\tilde{V}_0 > 0$, see (4). Using a parametrization for $\tilde{P}_{t|t}$ and \tilde{P}_{t+1} as the one for \tilde{V}_t : from (29) and (30) we obtain Steps 4, 6, 9 and Steps 13, 15, 17, respectively. Regarding Step 12, recall that from Step 3 in Algorithm 1 we have

$$\hat{\xi}_{t+1} = \tilde{A} \hat{\xi}_t + \tilde{G}_t (y_t - \tilde{C} \hat{\xi}_t), \quad (31)$$

where $\tilde{G}_t = \tilde{A} \tilde{L}_t$. Notice that $\hat{\xi}_t$ is the predictor of ξ_t given Y_{t-1} , and it can be partitioned as $\hat{\xi}_t := [(\hat{x}_{t|t-1}^0)^\top \quad (\hat{x}_{t|t-1}^1)^\top \quad \cdots \quad (\hat{x}_{t|t-1}^j)^\top \quad \cdots \quad (\hat{x}_{t|t-1}^L)^\top]^\top$ where $\hat{x}_{t|t-1}^j = \hat{x}_{t-j|t-1}$, $j \geq 0$. Substituting the definitions of \tilde{A} , \tilde{C} , \tilde{L}_t in (31) we obtain Step 12. It remains to find an efficient way to find the risk sensitivity parameter θ_t . Indeed, in order to evaluate $\gamma(\tilde{P}_{t+1}, \cdot)$ for a specific θ in Step 6 of Algorithm 1, we have to perform the inversion and the eigenvalue decomposition of a matrix whose dimension is proportional to L . The next result shows that it is possible to find θ_t by considering a function

which requires to perform the inversion and the eigenvalue decomposition of matrices of dimension $n \times n$, see Step 20. In this way, θ_t can be computed in a numerically robust way in the case that L is large.

Proposition 1. *Consider γ defined in (27). Then, θ_t is the unique solution to $\gamma(\tilde{P}_{t+1}, \theta_t) = c_t$ if and only if θ_t is the unique solution to $\bar{\gamma}(P_{t+1}^{L,L}, \theta_t) = c_t$ where*

$$\begin{aligned} \bar{\gamma}(P_{t+1}^{L,L}, \theta_t) = & -\frac{1}{2} \left\{ \text{tr} \left[P_{t+1}^{L,L} (P_{t+1}^{L,L} - \theta_t I)^{-1} \right] \right. \\ & \left. + \ln \det \left[I - P_{t+1}^{L,L} (P_{t+1}^{L,L} - \theta_t I)^{-1} \right] \right\} = c_t. \end{aligned}$$

PROOF. First, notice that $\tilde{H}\tilde{P}_{t+1}\tilde{H}^\top = P_{t+1}^{L,L}$. Thus, condition

$$\theta_t < r(\tilde{H}\tilde{P}_{t+1}\tilde{H}^\top)^{-1}$$

is equivalent to $\theta_t < r(P_{t+1}^{L,L})^{-1}$. By (27), we know

$$\gamma(\tilde{P}_{t+1}, \theta_t) = \frac{1}{2} [\text{tr}(K_1) + \ln \det(K_2)] \quad (32)$$

where

$$K_1 = (I - \theta_t \tilde{H}^\top \tilde{H} \tilde{P}_{t+1})^{-1} - I, \quad K_2 = I - \theta_t \tilde{H}^\top \tilde{H} \tilde{P}_{t+1}.$$

Then,

$$\begin{aligned} \text{tr}(K_1) &= \text{tr} \left\{ \left[(\tilde{P}_{t+1}^{-1} - \theta_t \tilde{H}^\top \tilde{H}) \tilde{P}_{t+1} \right]^{-1} \right\} - (L+1)n \\ &= \text{tr} \left[\tilde{P}_{t+1}^{-1} (\tilde{P}_{t+1}^{-1} - \theta_t \tilde{H}^\top \tilde{H})^{-1} \right] - (L+1)n \\ &= \text{tr} \left\{ \tilde{P}_{t+1}^{-1} \left[\tilde{P}_{t+1} - \tilde{P}_{t+1} \tilde{H}^\top \times (\tilde{H} \tilde{P}_{t+1} \tilde{H}^\top - \theta_t^{-1} I)^{-1} \tilde{H} \tilde{P}_{t+1} \right] \right\} - (L+1)n \\ &= \text{tr} \left\{ \left[I - (\tilde{H} \tilde{P}_{t+1} \tilde{H}^\top - \theta_t^{-1} I)^{-1} \tilde{H} \tilde{P}_{t+1} \tilde{H}^\top \right] \right\} - n \\ &= -\text{tr}(\Gamma_{t+1}^{-1} P_{t+1}^{L,L}) \end{aligned} \quad (33)$$

where $\Gamma_{t+1} := P_{t+1|t}^{L,L} - \theta_t^{-1} I_n$ and we exploited the Woodbury matrix identity.

Then,

$$\begin{aligned}
\ln \det(K_2) &= \ln \det \left[(\tilde{P}_{t+1}^{-1} - \theta_t \tilde{H}^\top \tilde{H}) \tilde{P}_{t+1} \right] \\
&= -\ln \det \left[\tilde{P}_{t+1}^{-1} (\tilde{P}_{t+1}^{-1} - \theta_t \tilde{H}^\top \tilde{H})^{-1} \right] \\
&= -\ln \det \left[\tilde{P}_{t+1}^{-1} (\tilde{P}_{t+1} - \tilde{P}_{t+1} \tilde{H}^\top \Gamma_{t+1}^{-1} \tilde{H} \tilde{P}_{t+1}) \right] \\
&= -\ln \det(I - \tilde{H}^\top \Gamma_{t+1}^{-1} \tilde{H} \tilde{P}_{t+1}) \\
&= -\ln \det(I - \tilde{H} \tilde{P}_{t+1} \tilde{H}^\top \Gamma_{t+1}^{-1}) \\
&= -\ln \det(I - P_{t+1}^{L,L} \Gamma_{t+1}^{-1}) \tag{34}
\end{aligned}$$

where we exploited the fact that $\det(I_m + AB) = \det(I_n + BA)$ and $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times m}$. Substituting (33) and (34) in (32), we get the claim. \square

Using the Woodbury formula, we have

$$\begin{aligned}
\tilde{V}_{t+1} &= (\tilde{P}_{t+1}^{-1} - \theta_t \tilde{H}^\top \tilde{H})^{-1} \\
&= \tilde{P}_{t+1} - \tilde{P}_{t+1} \tilde{H}^\top \Gamma_{t+1}^{-1} \tilde{H} \tilde{P}_{t+1}
\end{aligned}$$

and using the parametrization of \tilde{V}_{t+1} and \tilde{P}_{t+1} in terms $V_{t+1}^{j,k}$ and $P_{t+1}^{j,k}$, we obtain Steps 21, 22, 24, 26. Finally, Step 31 is derived as follows. The estimate of ξ_{t+1} given Y_{t+1} is obtained by the update step of the standard Kalman filter:

$$\hat{\xi}_{t+1|t+1} = \tilde{A} \hat{\xi}_{t|t} + \tilde{L}_{t+1} (y_{t+1} - \tilde{C} \tilde{A} \hat{\xi}_{t|t}). \tag{35}$$

Notice that $\hat{\xi}_{t|t}$ can be partitioned as

$$\hat{\xi}_{t|t} := [(\hat{x}_{t|t}^0)^\top (\hat{x}_{t|t}^1)^\top \cdots (\hat{x}_{t|t}^j)^\top \cdots (\hat{x}_{t|t}^L)^\top]^\top$$

where $\hat{x}_{t|t}^j = \hat{x}_{t-j|t}$, $j \geq 0$. Substituting the definitions of \tilde{A} , \tilde{C} , \tilde{L}_t in (35), we obtain

$$\hat{x}_{t+1|t+1}^j = \hat{x}_{t|t}^{j-1} + L_{t+1}^j (y_{t+1} - CA \hat{x}_{t|t}), \quad j \geq 0. \tag{36}$$

Note that when $j = 0$, $\hat{x}_{t|t}^{j-1} = \hat{x}_{t|t}^{-1} = \hat{x}_{t+1|t} = A \hat{x}_{t|t}$. In addition, from Equation (29), it is not difficult to see (the derivation is the same of the one in [43])

$$L_{t+1}^j = V_{t+1}^j C^\top (C V_t C^\top + D D^\top)^{-1} = V_{t+1}^j V_{t+1}^{-1} L_{t+1}, \quad j \geq 0$$

where $L_{t+1}^0 = L_{t+1}$ and while $V_{t+1}^0 = V_{t+1}$. Substituting the latter in (36) we have

$$\hat{x}_{t+1|t+1}^j = \hat{x}_{t|t}^{j-1} + V_{t+1}^j V_{t+1}^{-1} L_{t+1} (y_{t+1} - C A \hat{x}_{t|t}). \quad (37)$$

from which Step 31 can be established.

Algorithm 2 Robust efficient version

Input: $y_0 \dots y_N, \hat{x}_0, V_0, c_t$

Output: $\hat{x}_{t-L+1|t}, t = L-1 \dots N$

```

1: Set  $V_0^{j,k} = 0$  for  $j \neq k$ ,  $V_0^{j,j}$  positive definite for  $j > 0$ 
2: for  $t = 0 : N$  do
3:    $L_t = V_t C^\top (C V_t C^\top + D D^\top)^{-1}$ 
4:    $P_{t|t} = V_t (I - C^\top L_t^\top)$ ;
5:   for  $j = 1 : L$  do
6:      $P_{t|t}^j = V_t^j (I - C^\top L_t^\top)$ ;
7:      $L_t^j = V_t^j C^\top (C V_t C^\top + R)^{-1}$ ;
8:     for  $k = 1 : L$  do
9:        $P_{t|t}^{j,k} = V_t^{j,k} - V_t^j C^\top (L_t^k)^\top$ ;
10:    end for
11:  end for
12:   $\hat{x}_{t+1} = A \hat{x}_t + A L_t (y_t - C \hat{x}_t)$ ;
13:   $P_{t+1} = A P_{t|t} A^\top + B B^\top$ ;
14:  for  $j = 1 : L$  do
15:     $P_{t+1}^j = P_{t|t}^{j-1} A^\top$ ;
16:    for  $k = 1 : L$  do
17:       $P_{t+1}^{j,k} = P_{t|t}^{j-1,k-1}$ ;
18:    end for
19:  end for
20:  Find  $\theta_t$  s.t.  $\bar{\gamma}(P_{t+1}^{L,L}, \theta_t) = c_t$ ;
21:   $\Gamma_{t+1} = P_{t+1}^{L,L} - \theta_t^{-1} I_n$ ;
22:   $V_{t+1} = P_{t+1} - (P_{t+1}^L)^\top \Gamma_{t+1}^{-1} P_{t+1}^L$ ;
23:  for  $j = 1 : L$  do
24:     $V_{t+1}^j = P_{t+1}^j - (P_{t+1}^{L,j})^\top \Gamma_{t+1}^{-1} P_{t+1}^L$ ;
25:    for  $k = 1 : L$  do
26:       $V_{t+1}^{j,k} = P_{t+1}^{j,k} - (P_{t+1}^{L,j})^\top \Gamma_{t+1}^{-1} P_{t+1}^{L,k}$ ;
27:    end for
28:  end for
29: end for
30: for  $t = L-1 : N$  do

```

$$31: \quad \hat{x}_{t-L+1|t} = \hat{x}_{t-L+1} + \sum_{j=t-L+1}^t V_j^{j+L-t-1} V_j^{-1} L_j (y_j - C \hat{x}_j)$$

```

32: end for

```

Computational complexity. We perform the asymptotic analysis of the computational complexity, understood as the number of floating point

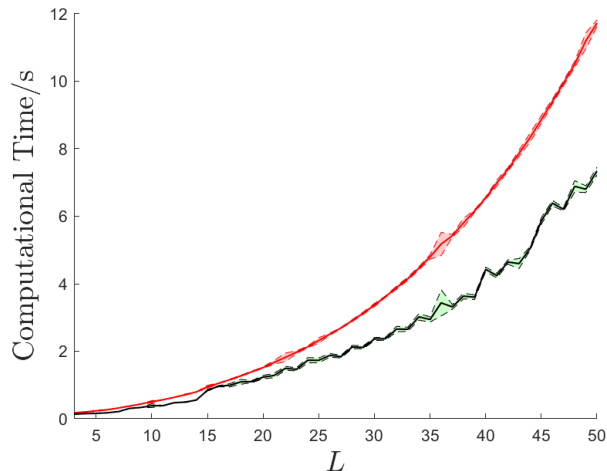


Figure 1: Computational time of ARFLS (corresponding to the red part) and RFLS (corresponding to the green part).

operations (flops) by using big O notation, of the efficient version of the robust fixed-lag smoother (RFLS), i.e. Algorithm 2, versus the one of the augmented robust fixed-lag smoother (ARFLS), i.e. Algorithm 1.

First, referring to ARFLS, Steps 2-5 have the same complexity of the standard Kalman predictor, that is:

$$O(n^3(L+1)^3) + O(mn^2(L+1)^2) + Q(m^2n(L+1)) + O(m^3).$$

Here, it is worth noting that $\tilde{Q} = \tilde{B}\tilde{B}^\top$ and $\tilde{R} = \tilde{D}\tilde{D}^\top$ are computed offline. Then, in regard to Step 6, the complexity to evaluate $\gamma(\tilde{P}_{t+1}, \theta_t)$ is $O(n^3(L+1)^3)$, see [46, Section 13.1 and 13.4]. Then, the computation of $\theta_t \in (0, r(\tilde{P}_{t+1})^{-1})$ is accomplished by a bisection method, see Algorithm 2 in [47]. Since at each step we spend constant time to reduce the problem to an instance half its size [46, Section 4.10.2], the complexity of Step 6 is

$$O(n^3(L+1)^3 \log_2(r(\tilde{P}_{t+1})^{-1}/\varepsilon))$$

where $\varepsilon > 0$ is the selected accuracy, i.e. the solution found satisfies the condition $|\gamma(\tilde{P}_{t+1}, \theta_t) - c_t| \leq \varepsilon$. Step 7 has complexity $O(n^3(L+1)^3)$. Thus, the computational complexity of Algorithm 1 is:

$$O(n^3(L+1)^3) + O(mn^2(L+1)^2) + Q(m^2n(L+1)) + O(m^3) \\ + O(n^3(L+1)^3 \log_2(r(\tilde{P}_{t+1})^{-1}/\varepsilon)).$$

Accordingly, the complexity of Algorithm 1 with respect to the instance L is $O(L^3)$.

Referring to RFLS, the total complexity of Steps 3-4, 12-13 and 21-22 is $O(n^3) + O(mn^2) + Q(m^2n) + O(m^3)$. Then, the complexity of Steps 5-11 is $O(n^3L) + O(mn^2L^2) + Q(m^2nL^2) + O(Lm^3)$; the complexity of Steps 14-19 is $O(n^3L)$; the complexity of Steps 23-28 is $O(n^3L^2)$. Next, Step 20 has the complexity of $O(n^3 \log_2(r(P_{t+1}^{L,L})^{-1}/\varepsilon))$; it is worth noting that in Step 20 the computation of θ_t is done by using the same bisection method of Step 6 in Algorithm 1: the difference is the dimension of matrices \tilde{P}_{t+1} , of dimension $n(L+1)$, and $P_{t+1}^{L,L}$, of dimension n . Finally, Step 31 has the complexity of $O(mn^2L) + O(n^3L)$. Hence, the computational complexity of Algorithm 2 is:

$$O(n^3L^2) + O(mn^2L^2) + Q(m^2nL^2) + O(m^3L) + O(n^3 \log_2(r(P_{t+1}^{L,L})^{-1}/\varepsilon)).$$

Thus, the complexity of Algorithm 2 with respect to the instance L is $O(L^2)$. We conclude that Algorithm 2 is computationally more efficient than Algorithm 1 and this advantage will become more pronounced as L grows.

Finally, we also analyze the computational time with respect to the lag L through a Monte Carlo study. In the latter, the lag ranges from $L = 3$ up to $L = 50$. Each case is composed by 100 trials. In each trial, the matrices A, B, C, D of Model (1) with $n = 2$ and $m = 1$ are randomly generated as follows. Each entry is drawn according to a uniform distribution in the interval $[0, 1]$. Then, matrix A is rescaled in such a way that its maximum eigenvalue (in modulus) is equal to 0.95. Then, an output sequence Y_N with $N = 500$ is generated. Fig. 1 shows the average value of the computational time over 100 trials required by RFLS and ARFLS with $c_t = 10^{-3}$ to estimate the state trajectory from Y_N . The results were obtained using a Huawei MateBook X Pro Laptop with Intel Core I5-8250U CPU and 8GB RAM. The dashed lines defines the corresponding confidence intervals (with level 0.95). It is possible to note that the computational time of these two algorithms grows polynomially. As expected, the growth rate of RFLS is much smaller than the one of ARFLS, i.e. RFLS drastically reduces the computational time.

5. Least-Favorable Model

In order to evaluate the performance of the robust fixed-lag smoother, we need to construct its least favorable model solution to (7). The latter can be

characterized over a finite time interval $[0, T]$ by using arguments similar to the ones in [27, Section V]. More precisely, the least favorable model takes the form

$$\begin{aligned}\eta_{t+1} &= \bar{A}_t \eta_t + \bar{B}_t \epsilon_t \\ y_t &= \bar{C}_t \eta_t + \bar{D}_t \epsilon_t\end{aligned}\tag{38}$$

where $\eta_t \triangleq [\xi_t^\top \tilde{e}_t^\top]^\top$. Moreover,

$$\begin{aligned}\bar{A}_t &:= \begin{bmatrix} \tilde{A} & \tilde{B}\tilde{S}_t \\ 0 & \tilde{A} - \tilde{G}_t\tilde{C} + (\tilde{B} - \tilde{G}_t\tilde{D})\tilde{S}_t \end{bmatrix} \\ \bar{B}_t &:= \begin{bmatrix} \tilde{B} \\ \tilde{B} - \tilde{G}_t\tilde{D} \end{bmatrix} L_t \\ \bar{C}_t &:= [\tilde{C} \quad \tilde{D}\tilde{S}_t], \quad \bar{D}_t := \tilde{D}L_t.\end{aligned}\tag{39}$$

The matrices above are computed through the backward recursion illustrated in Algorithm 3.

Algorithm 3 Backward recursion

Input: $\tilde{G}_0 \dots \tilde{G}_N, \theta_0 \dots \theta_N, \tilde{\Omega}_{N+1}^{-1}$

Output: $\bar{A}_t, \bar{B}_t, \bar{C}_t, \bar{D}_t, t = 0 \dots N$

- 1: $\tilde{\Omega}_{N+1}^{-1} = 0$
 - 2: **for** $t = N : 0$ **do**
 - 3: $W_{t+1}^{-1} = \tilde{\Omega}_{t+1}^{-1} + \theta_t \tilde{H}^\top \tilde{H}$
 - 4: $\tilde{K}_{v_t} = (I - (\tilde{B} - \tilde{G}_t\tilde{D})^\top W_{t+1}^{-1} (\tilde{B} - \tilde{G}_t\tilde{D}))^{-1}$
 - 5: $\tilde{S}_t = \tilde{K}_{v_t} (\tilde{B} - \tilde{G}_t\tilde{D})^\top W_{t+1}^{-1} (\tilde{A} - \tilde{G}_t\tilde{C})$
 - 6: Compute L_t such that $\tilde{K}_{v_t} = L_t L_t^\top$
 - 7: Compute $\bar{A}_t, \bar{B}_t, \bar{C}_t, \bar{D}_t$ as in (39)
 - 8: $\tilde{\Omega}_t^{-1} = (\tilde{A} - \tilde{G}_t\tilde{C})^\top W_{t+1}^{-1} (\tilde{A} - \tilde{G}_t\tilde{C}) + \tilde{S}_t^\top \tilde{K}_{v_t}^{-1} \tilde{S}_t$
 - 9: **end for**
-

It is worth noting that (38) is the least favorable model corresponding to the augmented state ξ_t . It is then natural to wonder whether such a least favorable model reduces to a least favorable model corresponding to the state x_t . The answer is affirmative. This justifies why in the minimax problem (7) we did not need to impose that $\tilde{\phi}_t(z_t|x_t)$ preserve the same structure of the augmented state space model in (2). Substituting $\bar{A}_t, \bar{B}_t, \bar{C}_t, \bar{D}_t$ in (38), we

obtain

$$\begin{aligned}\xi_{t+1} &= \tilde{A}\xi_t + \tilde{B}(\tilde{S}_t\tilde{e}_t + L_t\epsilon_t) \\ y_t &= \tilde{C}\xi_t + \tilde{D}(\tilde{S}_t\tilde{e}_t + L_t\epsilon_t)\end{aligned}$$

which is the augmented state space of the least favorable model

$$\begin{aligned}x_{t+1} &= Ax_t + B(\tilde{S}_t\tilde{e}_t + L_t\epsilon_t) \\ y_t &= Cx_t + D(\tilde{S}_t\tilde{e}_t + L_t\epsilon_t).\end{aligned}\tag{40}$$

Finally, consider a fixed-lag smoother of the form

$$\hat{x}'_{t-L+1|t} = \hat{x}'_{t-L+1} + \sum_{j=t-L+1}^t G_j'^{L-t+j} (y_j - C\hat{x}'_j)$$

where $G_t'^j$, with $0 \leq j \leq L$, are the arbitrary gains of a fixed-lag smoothing algorithm. It is not difficult to see that we can rewrite such a smoother as $\hat{x}'_{t-L+1|t} = \tilde{H}\hat{\xi}'_{t+1}$ where

$$\begin{aligned}\hat{\xi}'_{t+1} &= \tilde{A}\hat{\xi}'_t + \tilde{G}'_t(y_t - \tilde{C}\hat{\xi}'_t) \\ \tilde{G}'_t &= [(G_t'^0)^\top \quad \dots \quad (G_t'^L)^\top]^\top.\end{aligned}$$

To evaluate its performance under the least favorable model in (40), we define the corresponding smoothing error $e'_{t-L+1} = x_{t-L+1} - \hat{x}'_{t-L+1|t}$. Then it is not difficult to see that e'_{t-L+1} is zero mean and with covariance matrix

$$\bar{\Pi}_{t+1} = \tilde{H}\tilde{\Pi}_{t+1}\tilde{H}^\top$$

where: $\tilde{\Pi}_{t+1}$ is the $(L+1)n \times (L+1)n$ submatrix of Π_{t+1} in position (1, 1); Π_{t+1} is the solution to the Lyapunov equation

$$\begin{aligned}\Pi_{t+1} &= \left(\bar{A}_t - \begin{bmatrix} \tilde{G}'_t \\ 0 \end{bmatrix} \bar{C}_t \right) \Pi_t \left(\bar{A}_t - \begin{bmatrix} \tilde{G}'_t \\ 0 \end{bmatrix} \bar{C}_t \right)^\top \\ &\quad + \left(\bar{B}_t - \begin{bmatrix} \tilde{G}'_t \\ 0 \end{bmatrix} \bar{D}_t \right) \left(\bar{B}_t - \begin{bmatrix} \tilde{G}'_t \\ 0 \end{bmatrix} \bar{D}_t \right)^\top\end{aligned}$$

with initial condition $\Pi_0 = \mathbb{1}_2 \otimes \tilde{V}_0$ and $\mathbb{1}_2$ is the 2×2 matrix whose entries are equal to one.

6. Robust Target Tracking

We compare the performance of the robust and standard fixed-lag smoothers in a maneuvering target tracking problem where model uncertainty is present. More precisely, we consider as nominal model the second-order Singer model with an exponentially autocorrelated noise, see [31, 48, 49]. The state vector is defined as $x := [p^{lat}, v^{lat}, p^{lon}, v^{lon}]^\top$ where p^{lat} and v^{lat} denote the target position and velocity along the latitudinal direction, respectively; p^{lon} and v^{lon} denote the target position and velocity along the longitudinal direction, respectively. This model can be written as (1) with

$$A = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $T = 0.01$ denotes the sampling period; B is such that $BB^\top = Q$ and

$$Q = 2\alpha\sigma_m^2 \begin{bmatrix} T^3/3 & T^2/2 & 0 & 0 \\ T^2/2 & T & 0 & 0 \\ 0 & 0 & T^3/3 & T^2/2 \\ 0 & 0 & T^2/2 & T \end{bmatrix}.$$

where we assume that $\sigma_m^2 = 5$ and $1/\alpha = 0.5$ which are the instantaneous variance of the velocity and the time constant of the target velocity autocorrelation, respectively. Moreover,

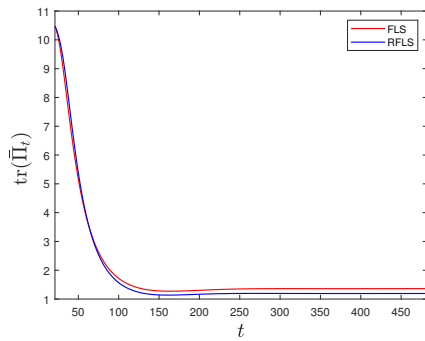
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad DD^\top = I_2$$

and the output y_t denotes the noisy position measurements along the two directions. Finally, x_0 is Gaussian distributed with zero mean and covariance matrix $V_0 = \text{diag}(50, 5, 50, 5)$.

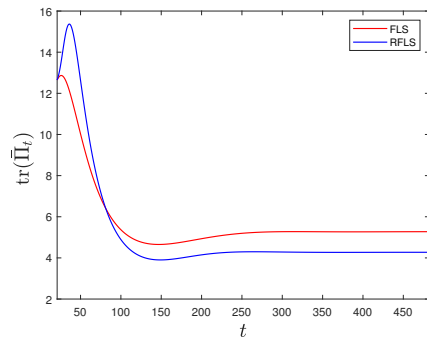
In practice, the nominal model above does not coincide with the actual one (e.g. the nominal parameters α and σ_m^2 are typically imprecise). In what follows, we assume that the actual model belongs to the ambiguity set \mathcal{B}_t . More precisely, we consider two scenarios: the first one considers the ambiguity set with $c_t = 10^{-3}$, while the second one with $c_t = 5 \cdot 10^{-3}$, i.e. the latter is larger than the former. We compare the proposed robust fixed-lag

smoother, denoted by RFLS, and the standard fixed-lag smoother, denoted by FLS, both with $L = 20$.

Fig. 2(a) shows the variances of the smoothing error under the least favorable model in the first ambiguity set with $c_t = 10^{-3}$, while Fig. 2(b) shows the variances of the smoothing error under the least favorable model in the second ambiguity set with $c_t = 5 \cdot 10^{-3}$. It is possible to see that RFLS outperforms FLS. It is also worth noting that the higher c_t is, the more RFLS outperforms FLS.

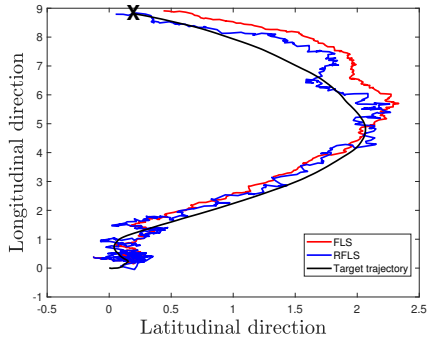


(a) Ambiguity set with $c = 10^{-3}$

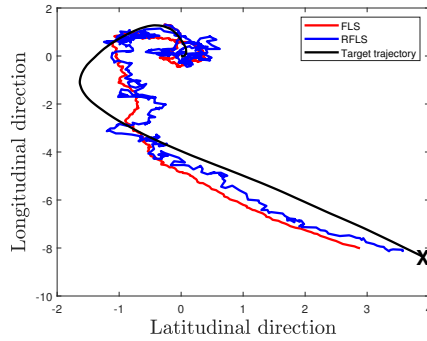


(b) Ambiguity set with $c = 5 \cdot 10^{-3}$

Figure 2: Scalar variances of the smoothing error for RFLS and FLS under the least favorable models in the different ambiguity sets.



(a) Ambiguity set with $c = 10^{-3}$



(b) Ambiguity set with $c = 5 \cdot 10^{-3}$

Figure 3: Trajectories generated by the least favorable models in the different ambiguity sets (black line) and the estimated trajectories with RFLS (blue line) and FLS (red line). The cross denotes the endpoint of the target trajectory.

	RFLS $c_t = 10^{-3}$	FLS $c_t = 10^{-3}$	RFLS $c_t = 5 \cdot 10^{-3}$	FLS $c_t = 5 \cdot 10^{-3}$
RMSE _{lat}	0.2918	0.4093	0.4962	0.6112
RMSE _{lon}	0.2891	0.3280	0.4804	0.6115

Table 1: RMSE along the latitudinal direction and longitudinal direction.

In what follows, we consider a sample $Y_N = \{y_0, y_1 \dots y_N\}$ with $N = 500$ generated by the least favorable model (40) in the first ambiguity set with $c_t = 10^{-3}$ and $\xi_0 = 0$. Fig. 3(a) shows the target trajectory (black line) and the ones estimated with RFLS (blue line) and FLS (red line). We also perform the same experiment using the least favorable model in the second ambiguity set with $c_t = 5 \times 10^{-3}$. Moreover, we quantify the performance of the smoothers in these two experiments through the root mean-square error (RMSE) along the latitudinal direction and longitudinal direction

$$\text{RMSE}_{lat} = \sqrt{\frac{1}{N} \sum_{t=1}^N (p_t^{lat} - \hat{p}_t^{lat})^2},$$

$$\text{RMSE}_{lon} = \sqrt{\frac{1}{N} \sum_{t=1}^N (p_t^{lon} - \hat{p}_t^{lon})^2}$$

whose values are displayed in Table 1. As expected, RFLS better reduces the influence of the modeling error on the estimation accuracy than others.

7. Robust parameter estimation

Consider the following state space model

$$\begin{aligned} \tilde{\mathcal{M}}(\alpha) : x_{t+1} &= A(\alpha)x_t + B(\alpha)\tilde{v}_t \\ y_t &= C(\alpha)x_t + D(\alpha)\tilde{v}_t \end{aligned} \tag{41}$$

where $x_0 \sim \mathcal{N}(\hat{x}_0, V_0)$, $\tilde{v}_t \sim \mathcal{N}(0, \tilde{R}_t)$, i.e. \tilde{v}_t is a nonstationary process, and the matrices $A(\alpha)$, $B(\alpha)$, $C(\alpha)$ and $D(\alpha)$ are parameterized by α . In many practical applications, α is not known and needs to be estimated from the observed data $Y_N = \{y_0, y_1 \dots y_N\}$. In plain words, the latter is a system

identification problem where the model class is $\tilde{\mathcal{M}} = \{\tilde{\mathcal{M}}(\alpha), \alpha \in \Theta\}$ and Θ is the parameter space. A well established paradigm to find α is the maximum likelihood (ML) principle. However, it is usually difficult to find an explicit expression of the ML function under Model (41). Such a problem is typically addressed by using the expectation-maximization (EM) algorithm, see [6, Algorithm 12.3], which computes a lower bound of the ML function through the iterative scheme:

- Set up an initial guess α^0 ;
- For $n = 0, 1, \dots$:
 - **E-step**: compute

$$\mathcal{Q}(\alpha, \alpha^n) := \int \tilde{f}_{\alpha^n}(X_{N+1}|Y_N) \log \tilde{f}_{\alpha}(Y_N, X_{N+1}) dX_{N+1}$$

- **M-step**: compute $\alpha^{(n+1)} = \arg \max_{\alpha} \mathcal{Q}(\alpha, \alpha^n)$

where $\tilde{f}_{\alpha}(Y_N, X_{N+1})$ is the joint density of $Y_N = \{y_0, y_1, \dots, y_N\}$ and $X_{N+1} = \{x_0, x_1, \dots, x_{N+1}\}$ under $\tilde{\mathcal{M}}(\alpha)$. In most cases, however, the covariance matrix \tilde{R}_t of the noise process \tilde{v}_t is not known and it is also time-varying. Such a matrix is typically designed empirically. However, this would require to have the possibility to make more experiments on the system, i.e. a requirement that is not always met. Alternatively, we can select a nominal covariance matrix for the noise process using some a priori knowledge. However, this causes the nominal model to be inconsistent with the actual one and thus the reliability of the estimate of α will be compromised.

A possible way to address this model uncertainty is to understand (41) as the least favorable model (40) where \tilde{v}_t is equal to $\tilde{S}_t \tilde{e}_t + L_t \epsilon_t$. More precisely, assume that we want to estimate α only knowing the nominal state space model \mathcal{M} , i.e.

$$\begin{aligned} \mathcal{M}(\alpha) : x_{t+1} &= A(\alpha)x_t + B(\alpha)v_t \\ y_t &= C(\alpha)x_t + D(\alpha)v_t \end{aligned} \tag{42}$$

where v_t is normalized WGN, in particular it is a stationary process. It is worth noting that the least favorable model in (41) does not belong to \mathcal{M} . Notice that, it is not restrictive to assume that the covariance matrix of v_t is equal to the identity. Indeed, in the case its covariance matrix is R , then we

can always take $\tilde{B}(\alpha) = B(\alpha)R^{1/2}$ and $\tilde{D}(\alpha) = D(\alpha)R^{1/2}$. Then, the least favorable model, solution to (7), is (41) with $\tilde{v}_t = \tilde{S}_t\tilde{\epsilon}_t + L_t\epsilon_t$. Hence,

$$\tilde{R}_t = \tilde{S}_t \begin{bmatrix} 0 & I_n \end{bmatrix} \Pi_t \begin{bmatrix} 0 \\ I_n \end{bmatrix} \tilde{S}_t^T + L_t L_t^T.$$

At this point we can use the density, say $\tilde{f}_{\alpha^n}^0(X_{N+1}, Y_N)$, of the least favorable model in order to compute the lower bound $\mathcal{Q}(\alpha, \alpha^n)$. On the other hand, an approximation of the moments of $\tilde{f}_{\alpha^n}^0$ required in $\mathcal{Q}(\alpha, \alpha^n)$ can be constructed by the robust fixed-lag smoother of Algorithms 2 and 3 leading to

$$\begin{aligned} \mathcal{Q}(\alpha, \alpha^n) &\propto -\frac{1}{2} \sum_{t=0}^{N-1} \log |2\pi B(\alpha) \tilde{R}_t B^T(\alpha)| \\ &\quad - \frac{1}{2} \text{tr} \left\{ \sum_{t=0}^{N-1} \left[\left(B(\alpha) \tilde{R}_t B^T(\alpha) \right)^{-1} (\Phi_{1,t} \right. \right. \\ &\quad \left. \left. - \Phi_{2,t} A^T(\alpha) - A(\alpha) \Phi_{2,t}^T + A(\alpha) \Phi_{3,t} A^T(\alpha) \right) \right] \right\} \\ &\quad - \frac{1}{2} \sum_{t=0}^{N-1} \log |2\pi D(\alpha) \tilde{R}_{t+1} D^T(\alpha)| \\ &\quad - \frac{1}{2} \text{tr} \left\{ \sum_{t=0}^{N-1} \left[\left(D(\alpha) \tilde{R}_{t+1} D^T(\alpha) \right)^{-1} (\Phi_{4,t} \right. \right. \\ &\quad \left. \left. - C(\alpha) \Phi_{5,t}^T - \Phi_{5,t} C^T(\alpha) + C(\alpha) \Phi_{1,t} C^T(\alpha) \right) \right] \right\} \end{aligned}$$

where

$$\begin{aligned} \Phi_{1,t} &= \bar{\Pi}_{t+L-1}^{L,L} + \hat{x}_{t+1|t+L-1} \hat{x}_{t+1|t+L-1}^T, \\ \Phi_{2,t} &= (\bar{\Pi}_{t+L-1}^{L,L-1})^T + \hat{x}_{t+1|t+L-1} \hat{x}_{t|t+L-1}^T, \\ \Phi_{3,t} &= \bar{\Pi}_{t+L-1}^{L,L} + \hat{x}_{t|t+L-1} \hat{x}_{t|t+L-1}^T, \\ \Phi_{4,t} &= y_{t+1} y_{t+1}^T, \quad \Phi_{5,t} = y_{t+1} \hat{x}_{t+1|t+L-1}^T, \end{aligned}$$

and $\bar{\Pi}_t^{j,k}$ is the $n \times n$ block in position (j, k) of $\tilde{\Pi}_t$. Then, $\hat{x}_{t|t+L-1}$, $\hat{x}_{t-1|t+L-1}$ and y_t are given by Algorithm 2. Clearly, such an approximation is legitimate if the lag L is chosen big enough.

Next, we show a numerical example. We consider the problem to estimate

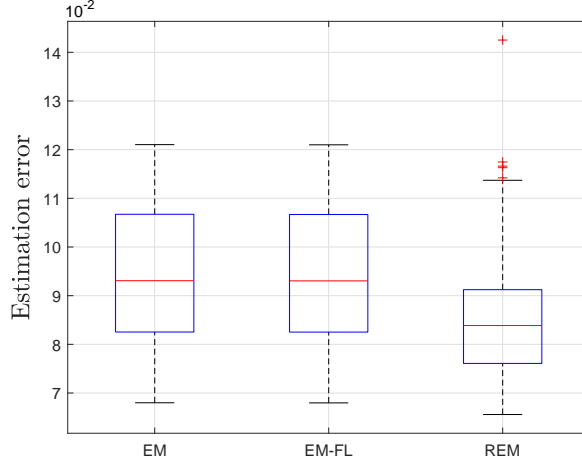


Figure 4: Estimation error for EM, EM-FL and REM in the Monte Carlo experiment.

the parameter $\alpha = [\alpha_1 \ \alpha_2]$ using the nominal model class (42) with

$$\begin{aligned}
 A(\alpha) &= \begin{bmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{bmatrix}, & B &= \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix} \\
 C &= \begin{bmatrix} 1 & -1 \end{bmatrix}, & D &= \begin{bmatrix} 0 & 0 & 0.01 \end{bmatrix},
 \end{aligned}$$

and the collected data $Y_N = \{y_0, y_1 \dots y_N\}$. We assume $x_0 \sim \mathcal{N}(0, V_0)$ with $V_0 = 0.0001I_2$. We assume that the actual model has the same structure of (42) with $\alpha^* = [0.1 \ 0.9]$, but the actual noise process, say \tilde{v}_t , is not stationary and not known. We solve the aforementioned system identification problem by means of the REM method introduced in Section 7 with $L = 50$. Moreover, we compare it with: the standard EM method where the state estimation task is performed by the RTS smoother; the “fixed-lag” EM (EM-FL) method where the state estimation task is performed by the standard fixed-lag smoother. To estimate the effectiveness of the REM method we assume that the actual model is the least favorable one belonging to the ambiguity set with $c = 2 \cdot 10^{-2}$. Moreover, we consider a Monte Carlo experiment with 100 trials. More precisely, in each trial, we generate the data set Y_N according to (40) with $\alpha = \alpha^*$ and $N = 1000$. The initial parameter estimates α_1^0 and α_2^0 , with $\alpha^0 := [\alpha_1^0 \ \alpha_2^0]$, are drawn from a uniform distribution with interval $[0.4, 0.9]$ and $[0.07, 0.13]$, respectively. Then, the termination condition is $\|\alpha^{n+1} - \alpha^n\| \leq \epsilon$ where $\epsilon = 10^{-3}$.

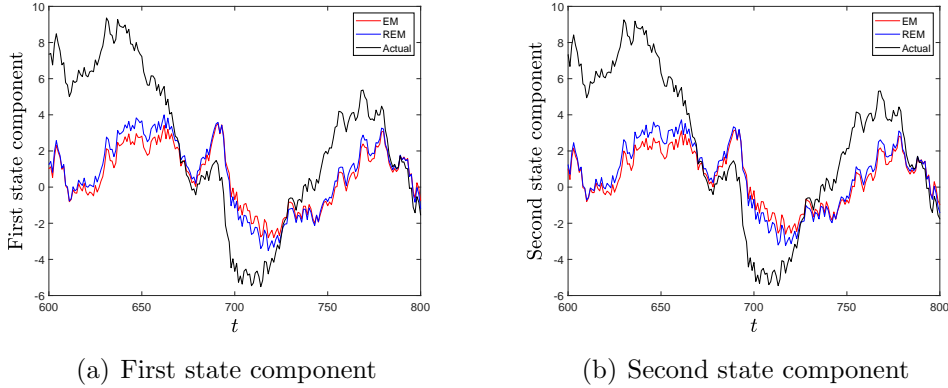


Figure 5: Comparison among the actual state components and the estimated ones obtained by EM and REM over the interval [600, 800].

Then, in order to compare the performance of these algorithms, we consider the estimation error $\|\hat{\alpha} - \alpha^*\|$ where $\hat{\alpha}$ is the parameter estimate obtained at the last stage by EM/EM-FL and REM. Fig. 4 shows the boxplot of the estimation error for EM, EM-FL and REM. We see that REM outperforms EM and EM-FL. It is worth noting that EM and EM-FL perform in the same way. This means that the value of the lag L has been chosen large enough in EM-FL and thus the fixed-lag smoother represents a good approximation of the RTS smoother.

Finally, Fig. 5 shows the estimated state trajectory in the last stage by EM and REM, with $t \in [600, 800]$, in a trial of the Monte Carlo experiment. As we can see, the one obtained with REM, and thus using RFLS, is slightly better than the one with EM, and thus using RTS. Although this advantage is not prominent, it made a dramatic improvement in the performance of the parameter estimator.

8. Conclusion

In this paper, we have proposed a robust fixed-lag smoother in the case that the actual model is different from the nominal one. More precisely, this paradigm solves a minimax game with two players: one selects the least favorable model in a prescribed ambiguity set, the other designs the optimal estimator based on the least favorable model. We also proposed an efficient implementation of the robust fixed-lag smoother in order to reduce the computational burden and avoid numerical instabilities. Then, we have characterized the least favorable model for the robust smoother over a finite

time horizon. Finally, we have presented some numerical examples showing the effectiveness of the proposed robust fixed-lag smoother.

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