ON THE OPTIMAL L^q-REGULARITY FOR VISCOUS HAMILTON-JACOBI EQUATIONS WITH SUB-QUADRATIC GROWTH IN THE GRADIENT

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Abstract. This paper studies a maximal L^q-regularity property for nonlinear elliptic equations of second order with a zero-th order term and gradient nonlinearities having superlinear and sub-quadratic growth, complemented with Dirichlet boundary conditions. The approach is based on the combination of linear elliptic regularity theory and interpolation inequalities, so that the analysis of the maximal regularity estimates boils down to determine lower order integral bounds. The latter are achieved via a L^p duality method, which exploits the regularity properties of solutions to stationary Fokker-Planck equations. For the latter problems, we discuss both global and local estimates. Our main novelties for the regularity properties of this class of nonlinear elliptic boundary-value problems are the treatment of equations with a zero-th order term together with the analysis of the end-point summability threshold $q = d(\gamma-1)/\gamma$, d being the dimension of the ambient space and $\gamma > 1$ the growth of the first-order term in the gradient variable.

1. Introduction

In this note we establish maximal regularity properties in Lebesgue spaces for a large class of second order nonlinear elliptic equations, whose main model is the viscous Hamilton-Jacobi equation, equipped with Dirichlet boundary conditions of the form

$$\begin{cases} -\Delta u(x) + \lambda u(x) + H(x, Du(x)) = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

with an unbounded right-hand side $f \in L^q(\Omega)$, Ω being a C^2 bounded domain of the d-dimensional Euclidean space $\mathbb{R}^d$, $\lambda \in \mathbb{R}$, $H = H(x,p)$ a nonlinearity with superlinear growth in the second entry. By maximal regularity in Lebesgue spaces we mean that an a priori information on the source term $f \in L^q$ implies bounds on the individual terms $D^2u, H(Du)$ on the left-hand side of the equation, see [20, 31] for related properties for linear equations. Here, $H \in C(\Omega \times \mathbb{R}^d)$ is convex in the second variable and satisfies for $\gamma > 1$ the following assumption

$$C_H^{-1}|p|^\gamma - C_H \le H(x,p) \le C_H(|p|^\gamma + 1) \tag{H}$$

for every $x \in \Omega$, $p \in \mathbb{R}^d$. We further assume that

$$\lambda > 0 \; . \tag{2}$$

It is well-known that such an assumption avoid to impose size conditions on the source term f of the equation, cf [34, 47].

The problem of optimal gradient regularity in Lebesgue spaces for these classes of PDEs has been proposed in a series of seminars by P.-L. Lions[1], who conjectured its validity under the general

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[1] P.-L. Lions "On Mean Field Games". Seminar at the conference "Topics in Elliptic and Parabolic PDEs", Napoli, September 11-12, 2014.
P.-L. Lions, Recorded video of Séminaire de Mathématiques appliquées at Collége de France, available at `https://www.college-de-france.fr/site/pierre-louis-lions/seminar-2014-11-14-11h15.htm`, November 14, 2014.

assumption $q > d(\gamma-1)/\gamma$, $q > 1$, d being the dimension of the ambient space, when $\lambda = 0$. Except for some specific cases discussed in his seminars, see e.g. [24, pp.1522-1523] for more details, the full conjecture was investigated in [24] for equations posed on the flat torus (i.e. the case of the ergodic control of a diffusion) without zero-th order terms in the equation, under the (sharp) assumption

$$q > \frac{d(\gamma-1)}{\gamma} \tag{3}$$

through a delicate refinement of the integral Bernstein method, which in turn forced the additional restriction $q > 2$. Nonetheless, the above coupled condition is always realized in a suitable range of the parameter γ, e.g. when $\gamma > \frac{d}{d-2}$.
Notice that the condition (3) is natural if one regards the problem as a nonlinear Poisson equation: indeed, if u solves (1) with $\lambda = 0$, then it is a solution to

$$-\Delta u = -H(x, Du) + f(x) \ ,$$

so that classical maximal L^q-elliptic regularity properties, (H) and Sobolev embeddings lead to the estimates

$$\|Du\|_{L^{\frac{dq}{d-q}}(\Omega)} \lesssim \|u\|_{W^{2,q}(\Omega)} \lesssim \||Du|\|^{\gamma}_{L^{q\gamma}(\Omega)} + \|f\|_{L^q(\Omega)}.$$

Therefore, one can expect maximal regularity properties for the nonlinear problem whenever the integrability of the gradient on the right-hand side is less than the one on the left, i.e. whenever $q^* = \frac{dq}{d-q} \geq q\gamma$, which implies $q \geq \frac{d(\gamma-1)}{\gamma}$.
In this note we continue the analysis on this regularity problem for different boundary conditions, as suggested by P.-L. Lions in his seminars. We first show that it is possible to have a control on $D^2u, |Du|^\gamma \in L^q$ removing the above restriction $q > 2$ in the *sub-quadratic* regime $\gamma < 2$ proposing a different proof with respect to [24]. Here, the approach is inspired by that employed for the parabolic problem in [23], and will be based on a perturbation argument that combines L^q-regularity results for linear problems, cf Appendix A, and interpolation inequalities [44], handling Dirichlet boundary conditions. As a consequence, the obtainment of maximal L^q-regularity estimates through interpolation methods boils down to establish a lower order bound for solutions to (1). We deduce such a level of regularity via duality arguments, following [22, 23]. The study of regularity properties for linear problems (even at the level of maximal regularity) shifting the attention to their adjoint equations is a classical idea, see e.g. [27, 40] and the references therein, and it has been recently and systematically explored in the nonlinear setting of Hamilton-Jacobi equations starting with the work by L.C. Evans [28], and later in the context of Mean Field Games, see e.g. [33, 23] for a thorough bibliography. Due to the presence of a nonlinear term, this approach is tied up with the smoothness properties, at the level of Sobolev spaces, of a (dual) stationary Fokker-Planck equation with a "singular" forcing term of the form

$$\begin{cases} -\Delta\rho + \lambda\rho + \operatorname{div}(b(x)\rho) = \delta_{x_0} & \text{in } \Omega \\ \rho = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where δ_{x_0} stands for the Dirac measure at some $x_0 \in \Omega$, when the drift b satisfies some a priori integrability conditions against the solution ρ itself.
As a second aim, we show that the presence of a zero-th order term in the equation allows to cover the maximal regularity property even at the critical level of summability $q = \frac{d(\gamma-1)}{\gamma}$, thus leading to new advances on this regularity problem with respect to [24].

The first main maximal L^q-regularity property in the *subcritical regime* $q > \frac{d(\gamma-1)}{\gamma}$ will be addressed in Theorem 2.1 through the lower-order bounds in Lebesgue spaces that will be detailed in Corollary 4.3. We emphasize that this seems, to the author's knowledge, a slightly novel viewpoint of the analysis of integral estimates in the framework of maximal regularity for these nonlinear elliptic equations with coercive gradient terms and Dirichlet boundary conditions. However, we borrow several ideas already appeared in the context of parabolic equations from [22, 23]. It is worth pointing out that a similar L^p duality argument was developed in a rather different context by M. Pierre [46, Section 3.2] to prove integral estimates for solutions to time-dependent reaction-diffusion systems, while some duality methods in the context of elliptic boundary value problems were previously used in various papers, see e.g. [6, 7, 12, 10, 15], the more recent [11] and the references therein, but at different summability scales and for different notions of solutions. We note that the integral estimates we obtain in Corollary 4.3 for (1) in the case $q > \frac{d(\gamma-1)}{\gamma}$ are not new when $\gamma < 2$ and can be found in [34] for more general equations with diffusion operators in divergence-form modeled over the p-Laplacian, see Remark 2.5. Nevertheless, they were obtained using completely different methods, yet based on variational arguments, under the restriction $\gamma < p$ when the leading operator is the p-Laplacian. Hence, our method of proof seems new with respect to the current literature of boundary-value problems. Instead, some integral bounds for distributional subsolutions when $\gamma > 2$ can be found in [26]. With respect to the L^∞ bounds, our methods unify the treatment in [34, 26] and allow to handle both the sub- and super-quadratic regimes of the nonlinearity.

Our second main result is a treatment of the maximal regularity problem at the end-point summability threshold $q = d(\gamma-1)/\gamma$ (at the expenses of assuming a finer dependence on the data), which in general fails without the presence of a zero-th order term, see [24, Remark 1]. Our proof is inspired by a stability argument recently proposed in the parabolic setting in [23, Theorem 1.3]. Our advances in the regularity theory for such second order nonlinear elliptic problems are twofold. On one hand, we notice that the treatment of the maximal regularity problem in the end-point case cannot be inferred from the known results in the literature, e.g. from [34], and hence Theorem 2.2 represents our main contribution in the theory. On the other hand, the maximal regularity properties in Theorems 2.1 and 2.2 are new in the regime $\frac{d+2}{d} < \gamma \le \frac{d}{d-2}$, which was not covered by the results in [24]. In both cases, we provide maximal regularity results for equations having zero-th order terms, which have not yet been discussed in the literature.

We emphasize that the presence of the zero-th order term in the equation is crucial not only to have a maximal regularity property in the critical case $q = d(\gamma-1)/\gamma$, but even to deduce integral bounds via duality. As already mentioned, both results are achieved through Sobolev regularity estimates for stationary Fokker-Planck equations having drifts satisfying suitable summability conditions against the solution itself. Here, we provide new global bounds for the Dirichlet problem (4) as well as local estimates for solutions to linear equations with lower-order coefficients in divergence form, when $|b| \in L^k(\rho\, dx)$, for some $k > 1$, that might be of independent interest. This goal is achieved by adapting the approach of earlier works on the subject [43, 22, 23]. In particular, the integral estimates for (1) follow by observing that an integrability information on the quantity $D_pH(x, Du)$ along the solution controls the regularity in Lebesgue scales of the solution u itself, since in turn it controls the (Sobolev) regularity of the solution of the dual problem (4) with $b = -D_pH(x, Du)$, as in [22, 23]. This crucial step is achieved through an integral representation formula, see the proofs of Propositions 4.1 and 4.2 for further details.

We remark that our variation on the Evans' scheme seems to be the first application to stationary Hamilton-Jacobi equations with L^p coefficients, and also to different boundary conditions than the periodic setting, except the work [29] on differentiability properties of solutions to boundary-value problems of equations modeled over the ∞-Laplacian. We refer to [33, 22, 23, 50] and the references therein for further developments and earlier results through this nonlinear duality method.

It is worth pointing out that some maximal regularity results have been already obtained when H is slowly increasing in the gradient variable in [8] (precisely when $\gamma \le \frac{d+2}{2}$, which prevents from the use of energy formulations), when the source f belongs to finer Lorentz classes without zero-th order terms, see also the references therein and Remark 5.2. Still, a quite general regularity theory on spaces of maximal regularity for such equations, mostly for systems of Hamilton-Jacobi-Bellman equations with bounded data and first-order terms having at most quadratic growth, can be found in [7]. Some results on second order Sobolev spaces via the Amann-Crandall approach [4] can be found in the monograph [42], and are based on Aleksandrov-Bakel'man-Pucci estimates, which in

turn require $f \in L^d$.
We conclude by saying that our underlying motive for this analysis relies on the application of the maximal regularity estimates to the existence problem of classical solutions to the systems of PDEs arising in the theory of Mean Field Games introduced by J.-M. Lasry-P.-L. Lions [39], when the coupling term of the Hamilton-Jacobi equation has power-like growth, cf [21, 19] and [23, Theorems 1.4 and 1.5]. Some recent results in this direction through maximal regularity in the case of defocusing systems [21] posed on convex domains of the Euclidean space and Neumann boundary conditions have been discussed in [32].
Rather different methods to develop interior estimates for Hamilton-Jacobi equations with superquadratic Hamiltonians can be found in [25].

Plan of the paper. Section 2 is devoted to present the statements of the main results of the manuscript along with some related remarks. In Section 3 we establish the Sobolev regularity of solutions for the adjoint Fokker-Planck equation, analyzing both global and interior estimates, while Section 4 studies the integral estimates for (1) by duality methods. Section 5 discusses the proofs of the main results. Section 6 and Appendix A conclude the paper with some remarks on the case of the quadratic growth and an auxiliary Calderón-Zygmund result for linear problems.

2. Main results

From now on, $\Omega \subset \mathbb{R}^d$ will be a C^2 bounded domain. This is only one of the possible regularity conditions one can assume on the domain to ensure the simultaneous validity of Sobolev's inequality as well as Calderón-Zygmund estimates for the Dirichlet problem. The latter will be recalled in Theorem A.1.
Our main results are the following: the first one deals with the subcritical regime $q > \frac{d(\gamma-1)}{\gamma}$, while the second one focuses on the critical threshold of summability $q = \frac{d(\gamma-1)}{\gamma}$.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a C^2 bounded domain, $d > 2$, and $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, $q > \frac{d(\gamma-1)}{\gamma}$, $1 + \frac{2}{d} < \gamma < 2$, be a strong solution to* (1) *and assume* (H)-(2)*. Suppose there exists $K > 0$ such that*

$$\|f\|_{L^q(\Omega)} \leq K \ .$$

Then there exists a constant M_1 depending on $K, d, q, \gamma, C_H, |\Omega|, \lambda$ such that

$$\|u\|_{W^{2,q}(\Omega)} + \||Du|^\gamma\|_{L^q(\Omega)} \leq M_1$$

Our approach closely follows [23] and can be described as follows: first, considering (1) as a nonlinear Poisson equation, by linear maximal regularity and (H), any strong solution to (1) satisfies

$$\|D^2 u\|_{L^q} \lesssim \||Du|\|_{L^{q\gamma}}^\gamma + \|f\|_{L^q} \ .$$

Using standard Gagliardo-Nirenberg interpolation inequalities one gets

$$\|Du\|_{L^{q\gamma}}^\gamma \lesssim \|D^2 u\|_{L^q}^{\theta\gamma} \|u\|_{L^s}^{(1-\theta)\gamma}$$

with $\theta\gamma < 1$ when s is suitably chosen. This reduces the maximal regularity estimate, through the application of the weighted Young's inequality, to a lower order estimate in Lebesgue spaces, see Corollary 4.3. The latter is accomplished by duality methods through the study of maximal regularity properties of stationary Fokker-Planck equations, cf Corollary 3.3.
The second main result deals with the critical value of summability, where the above interpolation argument leads to the condition $\theta\gamma = 1$, so that the Young's inequality does not allow to conclude the statement. It is based on a careful analysis of stability estimates in Lebesgue spaces, as stated in Proposition 4.5 below.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^d$ be as in Theorem 2.1, $d > 2$, and $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, $1 + \frac{2}{d} < \gamma < 2$, be a solution to* (1) *satisfying* (H)-(2) *and*

$$q = \frac{d(\gamma-1)}{\gamma}.$$

Then there exists a constant M_2 depending on $q, \gamma, C_H, |\Omega|, \lambda$ and f such that

$$\|u\|_{W^{2,q}(\Omega)} + \||Du|^\gamma\|_{L^q(\Omega)} \leq M_2 \ .$$

In particular, the constant M_2 does not depend only on $\|f\|_{L^q(\Omega)}$, but remains bounded when f varies in a set which is bounded and equi-integrable in $L^q(\Omega)$.

Some remarks on the above results are now in order. In the sequel, γ' will denote the Hölder conjugate exponent of γ, i.e. $\gamma' = \frac{\gamma}{\gamma-1}$.

Remark 2.3. The results of this manuscript can be extended to more general second order diffusions of the form $-\mathrm{Tr}(A(x)D^2u)$. We emphasize that some control on the derivatives of A is needed to study the regularity of the dual Fokker-Planck equation via the auxiliary problem (4), while Theorem A.1 merely requires the leading coefficients to be continuous in the whole domain, cf [31, Lemma 9.17]. Partial results with Sobolev regularity assumptions on the diffusion matrix A already appeared in [5] through the integral Bernstein method to prove Sobolev estimates for problems with first order terms having natural growth, under further integrability conditions on the source term. It is still unclear which are the minimal regularity requirements for the validity of the maximal L^q-regularity for viscous Hamilton-Jacobi equations driven by operators in non-divergence form. However, being our methods of variational nature, extensions to more general diffusions of the form $-\mathrm{Tr}(A(x)D^2u)$ would be possible provided that the coefficients are smooth enough, e.g. Lipschitz continuous, since $a_{ij}\partial_{ij}u = \partial_i(a_{ij}\partial_j u) - \partial_i a_{ij}\partial_j u$, which in turn results in the presence of an additional transport term with a bounded coefficient.

Remark 2.4. As pointed out in [47], the dependence on f in the a priori estimates, as well as in the existence problems for such nonlinear equations, changes when $\lambda = 0$ and $\lambda > 0$. The property of maximal regularity when $\lambda = 0$ has been established in [24] in the periodic setting, and one needs, in general, a control on $f \in L^q$, $q > \frac{d}{\gamma'}$. Moreover, existence and uniqueness may fail in $W_0^{1,2}$ when $\lambda = 0$, see [47], while necessary regularity conditions (i.e. $f \in L^q$, $q \geq \frac{d}{\gamma'}$), and also size hypotheses on the datum, are needed to derive a priori estimates, cf [35, 47]. In addition, maximal L^q-regularity fails when $q \leq \frac{d}{\gamma'}$ [24]. In particular, the end-point summability threshold $q = \frac{d}{\gamma'}$ is in general dealt with smallness assumptions, see e.g. [34, Theorem 1.1] for the case of integrability estimates or even [3, 8] for the case of data in Lorentz spaces. When instead $\lambda > 0$, which can be regarded as the closest regime to the parabolic framework, one can encompass even the case $q = \frac{d}{\gamma'}$ without requiring smallness assumptions, where, however, the constant of the estimate does not depend only on $\|f\|_{L^q}$, but remains bounded when f varies in a set which is bounded and equi-integrable in L^q, see Remark 4.6. This last observation agrees with the results found in [23, Theorem 1.3] for parabolic equations and also with the analysis carried out in [34]. We further refer to [47, Section 3], [48] for additional comments on this matter.

Remark 2.5. As already mentioned, our proof combines linear maximal regularity for the Dirichlet problem with interpolation inequalities, reducing the problem to finding a lower order estimate. We emphasize that the integral estimates we obtain in Corollary 4.3 have been already addressed in much more generality in [34, Theorem 1.4] (especially with respect to the diffusive term, which encompasses p-Laplacian operators and, more generally, pseudo-monotone operators satisfying Leray-Lions-type growth), but for $\gamma < 2$. However, the operators considered in [34] do not allow to use Calderón-Zygmund estimates. Hence, on one hand, our (linear) duality method provides a new approach to produce a priori regularity estimates in Lebesgue scales for nonlinear problems driven by second order operators with Dirichlet boundary conditions and, on the other hand, gives new advances on the regularity theory of two separate equations. Finally, though the integral estimates in the borderline case $q = d(\gamma-1)/\gamma$ have been already studied in [34], we emphasize that standard interpolation methods do not lead to maximal regularity. Therefore, we stress once more that the results in Theorem 2.2, obtained through a delicate stability argument, cannot be inferred from the standard literature of boundary value problems for nonlinear elliptic equations. Finally, we refer to [34, Theorem 4.9] for the analysis of integrability estimates in the case $\frac{d}{d-1} < \gamma < \frac{d+2}{d}$, which requires a different formulation of the problem.

Remark 2.6. The previous main results can be extended to non-homogeneous boundary conditions via the corresponding results for the linear problem, see e.g. [31, Theorem 9.15] or [20, Section 5]. Moreover, the techniques to prove Theorems 2.1 and 2.2 apply identically to problems with zero-th order terms posed on the flat torus $\mathbb{T}^d \equiv \mathbb{R}^d/\mathbb{Z}^d$ typical of the ergodic control setting, i.e. with periodic data, or with other boundary conditions (e.g. of Neumann and mixed type, see [34, Section 7.1]), leading to new results even in these contexts. Still, we emphasize that the same strategy presented here could also be extended with appropriate modifications to study parabolic Cauchy-Dirichlet problems in the sub-quadratic regime through the corresponding result for the linear problem. In this case, the parabolic dimension $d+2$ would replace the dimension d of the state space in the integrability conditions, see [23] for additional details.

3. Preliminary results for stationary Fokker-Planck equations

3.1. **Global Sobolev regularity for the Dirichlet problem.** In this section we give some global regularity results for the (dual) Dirichlet problem

$$\begin{cases} -\Delta\rho(x) + \lambda\rho(x) + \operatorname{div}(b(x)\rho(x)) = \psi(x) & \text{in } \Omega, \\ \rho(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Here, ψ should be thought as a L^p approximation of a Dirac delta, for p to be specified later, cf [28]. We denote, as usual, by $W_0^{k,p}(\Omega)$ the closure of $C_0^k(\Omega)$ in $W^{k,p}(\Omega)$, while we consider weak solutions to (5) belonging to $W_0^{1,2}(\Omega)$ in the sense that the following identity holds

$$\int_\Omega D\rho \cdot D\varphi \, dx + \lambda \int_\Omega \rho\varphi \, dx - \int_\Omega b\rho \cdot D\varphi \, dx = \int_\Omega \psi\varphi \, dx \ , \forall \varphi \in W_0^{1,2}(\Omega) \ .$$

We start with a well-known well-posedness result for the Dirichlet problem.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^d$, $d > 2$, be a bounded domain, $b \in L^p(\Omega)$, $p \geq d$, and $\psi \in L^\infty(\Omega)$. Then, there exists a unique weak solution $\rho \in W_0^{1,2}(\Omega)$ of the Dirichlet problem* (5). *When $b \in L^p(\Omega)$, $p > d$, $\rho \in L^\infty(\Omega)$, while when $p = d$ one has $\rho \in L^r(\Omega)$ for any finite $r > 1$. Moreover, if $\psi \geq 0$ in Ω, then $\rho \geq 0$ in Ω. Finally, if $\psi \in L^1(\Omega)$ we have $\|\rho\|_{L^1(\Omega)} \leq \frac{\|\psi\|_{L^1(\Omega)}}{\lambda}$.*

Proof. The existence and uniqueness follow from [17, Proposition 2.1.4] combined with [17, Theorem 2.1.8] (applied with $\gamma = 0$ and $\beta \equiv 0$, which requires $\lambda \geq 0$ using the notation of this paper, cf [17, Remark 2.1.10]), while the positivity is a consequence of [17, Theorem 2.2.1-(i) and (iii)]. The L^∞ estimates when $b \in L^p$, $p > d$, are standard and can be obtained through the estimate of $\log(1 + |u|) \in W^{1,2}$, cf [9, Theorem 5.6] or [14, Theorem 2.1], while the estimates in $L^r(\Omega)$ when $b \in L^d$ can be obtained arguing as in [14, p. 414], see also [9, Theorem 5.5]. The last assertion

$$\int_\Omega \rho \leq \frac{\int_\Omega \psi}{\lambda} \tag{6}$$

was proved in e.g. [10, Lemma 4.1] (where it is enough to have $b \in L^2(\Omega)$).

□

We now prove the following regularity result for solutions to the adjoint problem (5) with Dirichlet boundary conditions in terms of an integrability information of the drift term against the solution itself, cf [22, 21] for deeper results on the subject and the importance in connection with Mean Field Games. The proof is inspired by some ideas already appeared in [43] (cf [16] for a different approach), see also [22, 23] for related results for the parabolic problem along with the general reference [17]. Note that here the right-hand side term ψ plays the same role of the terminal datum of the backward adjoint problem in the parabolic case analyzed in [23]. Our result extends with a different proof those in [16] and provides, in addition, an explicit control on the size of the Sobolev and Lebesgue norms.

Proposition 3.2. *Let $1 < \sigma' < d$. Let then $\psi \in L^{p'}(\Omega)$ and $|b| \in L^k(\Omega; \rho\, dx)$ with $k = 1 + \frac{d}{\sigma}$, with k, σ, p satisfying the following conditions*

- *$p = \frac{d}{k-2}$ when $\sigma' > \frac{d}{d-1}$ with $2 < k < 1 + \frac{d}{2}$;*

- *$p' = 1$ when $1 < \sigma' < \frac{d}{d-1}$;*
- *any $p' < \infty$ when $\sigma' = \frac{d}{d-1}$.*

Then, every nonnegative weak solution to the Dirichlet problem (5) *satisfies the estimate*

$$\|\rho\|_{L^{\frac{d}{d-k}}(\Omega)} + \|D\rho\|_{L^{\sigma'}(\Omega)} \leq C\left(\int_\Omega |b|^k \rho\, dx + \|\psi\|_{L^{p'}(\Omega)} + 1\right)$$

where C depends only on $d, \sigma', |\Omega|$ when $\sigma' > \frac{d}{d-1}$, while it depends also on λ when $\sigma' \leq \frac{d}{d-1}$.

Proof. We first discuss the case $\sigma' > \frac{d}{d-1}$ using a variational argument. Let $\beta = \frac{k-2}{d-k}$ and use the test function $\varphi = \rho^\beta$. However, since $\beta \in (0,1)$, it follows that $D\varphi$ may not be in L^2, and hence one has to make the argument rigorous by taking $\varphi = (\delta+\rho)^\beta - \delta^\beta \in W_0^{1,2}$, where $\delta > 0$, and then let $\delta \to 0$. For the sake of presentation, we derive some formal estimates using $\varphi = \rho^\beta$ as a test function in the weak formulation of (5). We have

$$\int_\Omega D\rho \cdot D(\rho^\beta)\, dx - \int_\Omega \rho b \cdot D(\rho^\beta)\, dx + \lambda \int_\Omega \rho^{\beta+1}\, dx = \int_\Omega \rho^\beta \psi\, dx\ ,$$

which is equivalent to

$$\int_\Omega |D\rho|^2 \rho^{\beta-1}\, dx + \frac{\lambda}{\beta}\int_\Omega \rho^{\beta+1}\, dx \leq \int_\Omega |b|\rho^\beta |D\rho|\, dx + \frac{1}{\beta}\int_\Omega \rho^\beta \psi\, dx\ .$$

We then use the generalized Young's inequality on the first term of the right-hand side to get

$$\begin{aligned}(7)\quad \frac{1}{2}\int_\Omega |D\rho|^2\rho^{\beta-1}\, dx + \frac{1}{2}\int_\Omega |D\rho|^2\rho^{\beta-1}\, dx + \frac{\lambda}{\beta}\int_\Omega \rho^{\beta+1}\, dx &\leq \frac{1}{4}\int_\Omega |D\rho|^2\rho^{\beta-1}\, dx \\ &\quad + \int_\Omega \rho^{\beta+1}|b|^2\, dx + \frac{1}{\beta}\int_\Omega \rho^\beta\psi\, dx\ .\end{aligned}$$

As for the left-hand side, we observe that by Sobolev's inequality we have

$$\int_\Omega |D\rho|^2\rho^{\beta-1}\, dx = c_\beta \int_\Omega |D\rho^{\frac{\beta+1}{2}}|^2 \geq c_{\beta,d}\left(\int_\Omega \rho^{(\beta+1)\frac{d}{d-2}}\, dx\right)^{1-\frac{2}{d}}\ .$$

Then, writing $\int_\Omega \rho^{\beta+1}|b|^2\, dx = \int_\Omega |b|^2\rho^{\frac{2}{k}}\rho^{\beta+\frac{k-2}{k}}\, dx$, we first apply Hölder's inequality with exponents $\left(\frac{k}{2}, \frac{k}{k-2}\right)$ and (p, p'), $p = \frac{d}{k-2}$, respectively to the first and second term of the right-hand side of (7), and then the generalized Young's inequality with the pairs of conjugate exponents $\left(\frac{(d-2)k}{2(d-k)}, \frac{(d-2)k}{d(k-2)}\right)$ and $\left(\frac{d-2}{d-k}, \frac{d-2}{k-2}\right)$ respectively to get

$$\begin{aligned}&\int_\Omega \rho^{\beta+1}|b|^2\, dx + \frac{1}{\beta}\int_\Omega \rho^\beta\psi\, dx \\ &\qquad \leq \left(\int_\Omega |b|^k\rho\, dx\right)^{\frac{2}{k}}\left(\int_\Omega \rho^{\beta\frac{k}{k-2}+1}\, dx\right)^{1-\frac{2}{k}} + \frac{1}{\beta}\|\psi\|_{L^{p'}(\Omega)}\left(\int_\Omega \rho^{\beta p}\right)^{\frac{1}{p}} \\ &\leq \tilde{c}_{d,\beta}\left[\left(\int_\Omega |b|^k\rho\, dx\right)^{\frac{d-2}{d-k}} + \|\psi\|_{L^{p'}(\Omega)}^{\frac{d-2}{d-k}}\right] + \frac{c_{d,\beta}}{8}\left(\int_\Omega \rho^{\beta\frac{k}{k-2}+1}\, dx\right)^{1-\frac{2}{d}} + \frac{c_{d,\beta}}{8}\left(\int_\Omega \rho^{\beta p}\right)^{1-\frac{2}{d}}\ .\end{aligned}$$

One immediately checks the validity of the following chain of identities

$$\frac{d}{d-k} = (\beta+1)\frac{d}{d-2} = \beta\frac{k}{k-2} + 1 = \beta p\ .$$

Then, for some positive constant C depending solely on d, β we get

$$\left(\int_\Omega \rho^{\frac{d}{d-k}}\, dx\right)^{\frac{d-2}{d}} + \frac{1}{4}\int_\Omega \rho^{\beta-1}|D\rho|^2\, dx \leq C\left(\int_\Omega |b|^k\rho\, dx + \|\psi\|_{L^{p'}(\Omega)}\right)^{\frac{d-2}{d-k}}$$

giving the desired estimate on $\rho \in L^{\frac{d}{d-k}}(\Omega)$. Exploiting the fact that $\beta \in (0,1)$ we conclude by the Hölder's inequality (recalling that $\sigma' = \frac{d}{d-k+1}$ and $k < 1 + \frac{d}{2}$)

$$\|D\rho\|_{L^{\sigma'}(\Omega)} = \|D\rho\|_{L^{\frac{d}{d-k+1}}(\Omega)} \leq \|\rho^{\frac{\beta-1}{2}} D\rho\|_{L^2(\Omega)} \|\rho^{\frac{1-\beta}{2}}\|_{L^{\frac{2d}{d-2k+2}}(\Omega)}$$

and using the previous estimates we conclude the assertion.
The proof in the case $\sigma' < \frac{d}{d-1}$ is based on maximal regularity arguments, and it can be obtained as follows. By [49] (or argue as in [19, Lemma 1] via [2, Theorem 8.1]), we have

$$\|D\rho\|_{L^{\sigma'}(\Omega)} \leq C\left(\|b\rho\|_{L^{\sigma'}(\Omega)} + \|\rho\|_{L^{\sigma'}(\Omega)} + \|\psi\|_{W^{-1,\sigma'}(\Omega)}\right),$$

where C depends on λ, Ω, σ. We first handle the last term on the right-hand side, observing that

$$\int_\Omega \psi\varphi\, dx \leq \|\psi\|_{L^1(\Omega)}\|\varphi\|_{L^\infty(\Omega)} \leq C\|\psi\|_{L^1(\Omega)}\|\varphi\|_{W_0^{1,\sigma}(\Omega)} \ , \sigma > d \ ,$$

so that by [1, Section 3.13] we get $\|\psi\|_{W^{-1,\sigma'}(\Omega)} \leq C\|\psi\|_{L^1(\Omega)}$ for some positive constant $C > 0$. Then, we have

$$\begin{aligned}
\|D\rho\|_{L^{\sigma'}(\Omega)} &\leq C\left(\|b\rho\|_{L^{\sigma'}(\Omega)} + \|\rho\|_{L^{\sigma'}(\Omega)} + \|\psi\|_{L^1(\Omega)}\right) \\
&= C\left(\|b\rho^{\frac{1}{k}}\rho^{\frac{1}{k'}}\|_{L^{\sigma'}(\Omega)} + \|\rho\|_{L^{\sigma'}(\Omega)} + \|\psi\|_{L^1(\Omega)}\right) \\
&\leq C\left(\left(\int_\Omega |b|^k\rho\, dx\right)^{\frac{1}{k}} \|\rho\|_{L^z(\Omega)}^{\frac{1}{k'}} + \|\rho\|_{L^{\sigma'}(\Omega)} + \|\psi\|_{L^1(\Omega)}\right) \\
&\leq C\left(\frac{1}{2\delta}\int_\Omega |b|^k\rho\, dx + \frac{\delta}{2}\|\rho\|_{L^z(\Omega)} + \|\rho\|_{L^{\sigma'}(\Omega)} + \|\psi\|_{L^1(\Omega)}\right),
\end{aligned}$$

where we applied first the Hölder's inequality for an exponent $z > \sigma'$ satisfying

$$\frac{1}{\sigma'} = \frac{1}{k} + \frac{1}{zk'} \ , \tag{8}$$

and then the generalized Young's inequality. By the interpolation inequalities we have

$$\|\rho\|_{L^{\sigma'}(\Omega)} \leq \|\rho\|_{L^1(\Omega)}^{1-\theta}\|\rho\|_{L^z(\Omega)}^{\theta} \ , \theta \in (0,1) \ , \frac{1}{\sigma'} = 1 - \theta + \frac{\theta}{z} \ .$$

Therefore, exploiting the fact that $\|\rho\|_{L^1(\Omega)} \leq \frac{\|\psi\|_{L^1(\Omega)}}{\lambda}$ (see [10] or (6)) and applying once more the generalized Young's inequality to the above term, we get

$$\|D\rho\|_{L^{\sigma'}(\Omega)} \leq \tilde{C}\left(\frac{1}{\delta}\int_\Omega |b|^k\rho\, dx + \delta\|\rho\|_{L^z(\Omega)} + \|\psi\|_{L^1(\Omega)}\right) \ ,$$

where $\tilde{C}$ now depends also on λ. The conditions $k = 1 + \frac{d}{\sigma}$ and (8) lead to

$$\frac{1}{z} = \frac{1}{\sigma'} - \frac{1}{d} \ ,$$

so that the Sobolev embedding applies to obtain

$$\|\rho\|_{L^z(\Omega)} \leq C_1\|D\rho\|_{L^{\sigma'}(\Omega)} \ .$$

This implies, by choosing $\delta = \frac{1}{2\tilde{C}C_1}$, a bound on $\rho \in L^z(\Omega)$, and hence the assertion. The case $\sigma' = \frac{d}{d-1}$ can be treated similarly owing to the embedding $W_0^{1,\sigma}$ onto L^p for any finite $p > 1$. □

Corollary 3.3. *Let ρ be the nonnegative weak solution to* (5). *There exists a constant $C > 0$ depending on $d, q, |\Omega|$ and not on λ such that if $\frac{2d}{d+2} < q < \frac{d}{2}$ we have*

$$\|\rho\|_{L^{q'}(\Omega)} \leq C\left(\int_\Omega |b|^{\frac{d}{q}}\rho\, dx + \|\psi\|_{L^{p'}(\Omega)}\right)$$

with $p = \frac{dq}{d-2q}$. If $q = \frac{d}{2}$ or $q > \frac{d}{2}$ we have the same estimate for any finite $p' < \infty$ and $p' = 1$ respectively, but the constant of the estimate depends also on λ.

Proof. The proof follows from Proposition 3.2 applied with $k = \frac{d}{q}$ and the embedding of $W_0^{1,\sigma'}(\Omega)$ onto $L^{q'}(\Omega)$. $\square$

3.2. **Local Sobolev regularity.** In this section we prove a local counterpart of the regularity results of the previous section for weak solutions to

$$-\Delta\rho + \lambda\rho + \mathrm{div}(b(x)\rho) = \psi(x) \text{ in } \Omega \ , \tag{9}$$

focusing only on the case $b \in L^k(\rho)$ for $k = 1 + \frac{d}{\sigma}$, $\frac{d}{d-1} < \sigma' < d$. This gives an alternative proof of [16, Theorem 1-(ii)] without using elliptic regularity theory, and provides an explicit estimate on the size of the norm.

Proposition 3.4. *Let $\rho \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a weak solution to (9) and let $B_R = \{x \in \mathbb{R}^d : |x| < R\}$. Let $\frac{d}{d-1} < \sigma' < d$, $\psi \in L^{p'}(B_R)$, $p = \frac{d}{k-2}$, with $|b| \in L^k_{loc}(B_R; \rho\, dx)$ where $k = 1 + \frac{d}{\sigma}$ satisfies $2 < k < 1 + \frac{d}{2}$. Then $\rho \in W^{1,\sigma'}_{\mathrm{loc}}(\Omega)$, and every weak solution to (9) satisfies the interior estimate*

$$\|\rho\|_{L^{\frac{d}{d-k}}(B_{\frac{R}{2}})} + \||D\rho|\|_{L^{\sigma'}(B_{\frac{R}{2}})} \le C\left(\||b|\|_{L^k_{\mathrm{loc}}(B_R;\rho\, dx)} + \|\psi\|_{L^{p'}(B_R)} + 1\right)$$

where C depends in particular on $d, \sigma', k, R, \|\rho\|_{L^1(B_R)}$.

Proof. Let $\zeta \in C_0^\infty(B_R)$ be such that $0 \le \zeta \le 1$ satisfying $\zeta > 0$ in $B := B_R$, $\zeta = 1$ on the twice smaller ball $B_{R/2}$ and assume

$$\sup_x |D\zeta|\zeta^{-\eta} \le C_\zeta \tag{10}$$

for $\eta = \frac{\beta+1-2/k}{\beta+1} \in (0,1)$, $\beta = \frac{k-2}{d-k}$ and some positive constant C_ζ. Such conditions are verified by $\zeta(x) = \psi(|x|/R)$, $\psi \in C_0^\infty(\mathbb{R})$ with ψ such that $0 \le \psi \le 1$, $\psi(y) > 0$ for $|y| < 1$, $\psi(y) = 0$ when $|y| \ge 1$ and $\psi(y) = 1$ on $|y| \le \frac{1}{2}$, with $\psi(y) = \exp((y^2-1)^{-1})$ near the end-points -1 and 1, cf [17, Theorem 1.7.4].

We test the equation against $\varphi = \rho^\beta\zeta^2$, $\beta = \frac{k-2}{d-k}$ and set $\alpha = \frac{2(d-k)}{d-2}$. We have

$$\int_B D\rho \cdot D(\zeta^2\rho^\beta)\, dx - \int_B \rho b \cdot D(\zeta^2\rho^\beta)\, dx + \lambda \int_B \rho^{\beta+1}\zeta^2\, dx = \int_B \zeta^2\rho^\beta\psi\, dx \ .$$

We thus write

$$\begin{aligned}\int_B |D\rho|^2\rho^{\beta-1}\zeta^2\, dx + \frac{\lambda}{\beta}\int_B \rho^{\beta+1}\zeta^2\, dx &\le \frac{2}{\beta}\int_B |D\rho||D\zeta|\rho^\beta\zeta\, dx \\ &+ \int_B \zeta^2|b|\rho^\beta|D\rho|\, dx + \frac{2}{\beta}\int_B |D\zeta|\zeta\rho^{\beta+1}|b|\, dx + \frac{1}{\beta}\int_B \zeta^2\rho^\beta\psi\, dx \ .\end{aligned}$$

We then use Young's inequality to get

$$\begin{aligned}&\int_B |D\rho|^2\rho^{\beta-1}\zeta^2\, dx + \frac{\lambda}{\beta}\int_B \rho^{\beta+1}\zeta^2\, dx \le \frac{2}{\beta}\int_B |D\rho||D\zeta|\rho^\beta\zeta\, dx \\ &+ \frac{1}{8}\int_B |D\rho|^2\rho^{\beta-1}\zeta^2\, dx + 2\int_B \rho^{\beta+1}|b|^2\zeta^2\, dx + \frac{2}{\beta}\int_B |D\zeta|\zeta\rho^{\beta+1}|b|\, dx + \frac{1}{\beta}\int_B \zeta^2\rho^\beta\psi\, dx \\ &= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V}) \ .\end{aligned} \tag{11}$$

Note that (II) can be absorbed on the left-hand side. We observe that by the Sobolev's inequality [31, Theorem 7.10]

$$\begin{aligned}\frac{1}{2}\int_B |D\rho|^2\rho^{\beta-1}\zeta^2\, dx &= c_\beta \int_B |D\rho^{\frac{\beta+1}{2}}|^2\zeta^2\, dx = c_\beta\int_B |D(\rho^{\frac{\beta+1}{2}}\zeta)|^2\, dx - c_\beta\int_B \rho^{\beta+1}|D\zeta|^2\, dx \\ &\ge c_{d,\beta}\left(\int_B \rho^{(\beta+1)\frac{d}{d-2}}\zeta^{\frac{2d}{d-2}}\, dx\right)^{1-\frac{2}{d}} - c_\beta\int_B \rho^{\beta+1}|D\zeta|^2\, dx \ ,\end{aligned} \tag{12}$$

where $c_\beta = \frac{2}{(\beta+1)^2}$. We now start estimating the terms in (11) (and the one on the right-hand side of the above inequality, which has a negative sign). Applying the Hölder's inequality with exponents $\left(\frac{k}{2}, \frac{k}{k-2}\right)$ and the weighted Young's inequality with the pair $\left(\frac{(d-2)k}{2(d-k)}, \frac{(d-2)k}{d(k-2)}\right)$, we get

$$\begin{aligned}
(\mathrm{III}) = \int_B \zeta^2 \rho^{\beta+1}|b|^2\,dx &= \int_B |b|^2 \rho^{\frac{2}{k}} \zeta^{\frac{2\alpha}{k}} \rho^{\beta+1-\frac{2}{k}} \zeta^{\frac{2d(k-2)}{k(d-2)}}\,dx \\
&\leq \left(\int_B \zeta^{\frac{2(d-k)}{d-2}} |b|^k \rho\right)^{\frac{2}{k}} \left(\int_B \rho^{\beta\frac{k}{k-2}+1} \zeta^{\frac{2d}{d-2}}\,dx\right)^{1-\frac{2}{k}} \\
&\leq C_1(d,\beta,k) \left(\int_B \zeta^{\frac{2(d-k)}{d-2}} |b|^k \rho\,dx\right)^{\frac{d-2}{d-k}} + \frac{c_{d,\beta}}{24} \left(\int_B \rho^{\beta\frac{k}{k-2}+1} \zeta^{\frac{2d}{d-2}}\,dx\right)^{1-\frac{2}{d}}.
\end{aligned}$$

Owing to the above estimate, we then obtain via the Young's inequality

$$\begin{aligned}
(\mathrm{IV}) = \frac{2}{\beta} \int_B |D\zeta| \zeta \rho^{\beta+1}|b|\,dx &\leq \frac{1}{\beta} \int_B \rho^{\beta+1}|D\zeta|^2\,dx + \frac{1}{\beta} \int_B |b|^2 \zeta^2 \rho^{\beta+1}\,dx \\
&\leq \frac{1}{\beta} \int_B \rho^{\beta+1}|D\zeta|^2\,dx + \frac{1}{\beta} \left(\int_B \zeta^{\frac{2(d-k)}{d-2}} |b|^k \rho\right)^{\frac{2}{k}} \left(\int_B \rho^{\beta\frac{k}{k-2}+1} \zeta^{\frac{2d}{d-2}}\,dx\right)^{1-\frac{2}{k}} \\
\leq \frac{1}{\beta} \int_B \rho^{\beta+1}|D\zeta|^2\,dx + C_2(d,k,\beta) &\left(\int_B \zeta^{\frac{2(d-k)}{d-2}} |b|^k \rho\,dx\right)^{\frac{d-2}{d-k}} + \frac{c_{d,\beta}}{24} \left(\int_B \rho^{\beta\frac{k}{k-2}+1} \zeta^{\frac{2d}{d-2}}\,dx\right)^{1-\frac{2}{d}}.
\end{aligned}$$

We now estimate $\int_B \rho^{\beta+1}|D\zeta|^2\,dx$ (which appeared in (12) and from the above inequality) using (10) for $\eta := \frac{\beta+1-2/k}{\beta+1} \in (0,1)$, and Hölder's inequality applied with the exponents $\xi = \frac{d}{(d-2)\eta}$ and ξ'. In particular, using the definition of β and η one first checks that $2\xi'/k = 1$. Indeed,

$$\frac{1}{\xi} = \frac{d-2}{d}\eta$$

hence

$$\frac{1}{\xi'} = 1 - \frac{d-2}{d}\eta.$$

Therefore

$$\frac{k}{2\xi'} = \frac{k}{2} - \frac{k}{2}\frac{d-2}{d}\frac{\beta+1-\frac{2}{k}}{\beta+1} = \frac{k}{2} - \frac{k}{2}\frac{d-2}{d}\frac{d(k-2)}{d-2}\frac{d-k}{k(d-k)} = \frac{k}{2} - \frac{k-2}{2} = 1.$$

We use once more the Hölder's inequality first and then the Young's inequality, together with (10), to conclude

$$\begin{aligned}
\int_B \rho^{\beta+1}|D\zeta|^2\,dx &\leq \frac{C_\zeta}{\beta} \int_B \rho^{\beta+1-\frac{2}{k}} \zeta^{2\eta} \rho^{\frac{2}{k}}\,dx \\
&\leq \frac{C_\zeta}{\beta} \left(\int_B \rho^{(\beta+1)\frac{d}{d-2}} \zeta^{\frac{2d}{d-2}}\,dx\right)^{(1-\frac{2}{d})\eta} \left(\int_B \rho^{\frac{2\xi'}{k}}\,dx\right)^{\frac{1}{\xi'}} \\
&\leq \frac{c_{d,\beta}}{24} \left(\int_B \rho^{(\beta+1)\frac{d}{d-2}} \zeta^{\frac{2d}{d-2}}\,dx\right)^{1-\frac{2}{d}} + C(d,\beta,\zeta,\|\rho\|_{L^1(B_1)}),
\end{aligned}$$

where we also used that $\frac{k}{2\xi'} = 1$. Moreover, using that $\frac{2d}{d-2} < \frac{2d}{k-2}$ (note that $k < 1 + \frac{d}{2}$, hence $k < d$ for $d > 2$), we write, applying the Hölder and Young's inequalities

$$\begin{aligned}
(\mathrm{V}) = \frac{1}{\beta} \int_B \zeta^2 \rho^\beta \psi\,dx &\leq \frac{1}{\beta}\|\psi\|_{L^{p'}(B)} \left(\int_B \rho^{\beta p} \zeta^{\frac{2d}{k-2}}\,dx\right)^{\frac{1}{p}} \leq \frac{1}{\beta}\|\psi\|_{L^{p'}(B)} \left(\int_B \rho^{\beta p} \zeta^{\frac{2d}{d-2}}\,dx\right)^{\frac{1}{p}} \\
&\leq \frac{c_{d,\beta}}{24} \left(\int_B \rho^{\beta p} \zeta^{\frac{2d}{d-2}}\,dx\right)^{1-\frac{2}{d}} + C(\beta,p,c_{d,\beta})\|\psi\|_{L^{p'}(B)}^{\frac{d-2}{d-k}}.
\end{aligned}$$

We now consider

$$
\begin{aligned}
\text{(I)} = \frac{2}{\beta}\int_B |D\rho||D\zeta|\rho^\beta\zeta\,dx &\le \frac{1}{8}\int_B |D\rho|^2\rho^{\beta-1}\zeta^2\,dx + C(\beta)\int_B \rho^{\beta+1}|D\zeta|^2\,dx\\
&\le \frac{1}{8}\int_B |D\rho|^2\rho^{\beta-1}\zeta^2\,dx + \frac{c_{d,\beta}}{24}\left(\int_B \rho^{(\beta+1)\frac{d}{d-2}}\zeta^{\frac{2d}{d-2}}\,dx\right)^{1-\frac{2}{d}} + C(d,\beta,\zeta,\|\rho\|_{L^1(B_1)})\ .
\end{aligned}
$$

We plug all the estimates together in (11) noting that

$$
\frac{d}{d-k} = (\beta+1)\frac{d}{d-2} = \beta\frac{k}{k-2} + 1 = \beta p\ ,
$$

to obtain

$$
\begin{aligned}
\frac{c_{d,\beta}}{4}\left(\int_B \rho^{(\beta+1)\frac{d}{d-2}}\zeta^{\frac{2d}{d-2}}\,dx\right)^{1-2/d} &+ \frac{1}{4}\int_B \rho^{\beta-1}|D\rho|^2\zeta^2\,dx\\
&\le C_3\left[\left(\int_B \zeta^{\frac{2(d-k)}{d-2}}|b|^k\rho\,dx\right)^{\frac{d-2}{d-k}} + \|\psi\|_{L^{q'}(B)}^{\frac{d-2}{d-k}} + 1\right]\\
&\le C_4\left[\int_B \zeta^{\frac{2(d-k)}{d-2}}|b|^k\rho\,dx + \|\psi\|_{L^{p'}(B)} + 1\right]^{\frac{d-2}{d-k}}\ ,
\end{aligned}
$$

where C_3, C_4 depends on d,β,C_ζ,k together with $\|\rho\|_{L^1(B_R)}$. This in turn allows to conclude for some positive constant $C_5 > 0$

$$
\left(\int_B \rho^{\frac{d}{d-k}}\zeta^{\frac{2d}{d-2}}\,dx\right)^{1-2/d} \le C_5\left[\left(\int_B \zeta^{\frac{2(d-k)}{d-2}}|b|^k\rho\,dx\right) + \|\psi\|_{L^{p'}(B)} + 1\right]^{\frac{d-2}{d-k}}\ .
$$

This in particular yields the estimate $\|\rho\zeta^{\frac{2(d-k)}{d-2}}\|_{L^{\frac{d}{d-k}}(B_R)} \ge \|\rho\|_{L^{\frac{d}{d-k}}(B_{\frac{R}{2}})}$. Finally, by the Hölder's inequality, using that $2 < k < 1 + \frac{d}{2}$, we have

$$
\begin{aligned}
\|\zeta^{\frac{2(d-k)}{d-2}}D\rho\|_{L^{\frac{d}{d-k+1}}(B)} &\le \|\zeta\rho^{\frac{\beta-1}{2}}D\rho\|_{L^2(B)}\|\rho^{\frac{1-\beta}{2}}\zeta^{\frac{d-2k+2}{d-2}}\|_{L^{\frac{2d}{d-2k+2}}(B)}\\
&= \left(\int_B \rho^{\beta-1}\zeta^2|D\rho|^2\,dx\right)^{\frac{1}{2}}\left(\int_B \rho^{\frac{d}{d-k}}\zeta^{\frac{2d}{d-2}}\,dx\right)^{\frac{d-2k+2}{2d}}\ ,
\end{aligned}
$$

and we conclude the local gradient regularity using the previous estimates. □

4. Integral estimates for viscous Hamilton-Jacobi equations

In this section we focus on strong solutions in the standard sense of [31, Chapter 9] of (1) such that $u \in W^{2,q}(\Omega)\cap W^{1,q}_0(\Omega)$, $q > \frac{d}{\gamma'}$, dealing with the case $\gamma > \frac{d+2}{d}$. On one hand, we observe that under the restriction $q \ge \frac{d}{\gamma'}$ we have the embedding

$$
W^{2,q} \hookrightarrow W^{1,q\gamma}
$$

and hence $b \sim |Du|^{\gamma-1} \in L^p$, $p \ge d$. Therefore, when the velocity filed $b = -D_pH(Du)$, under the standing growth conditions on the Hamiltonian term H, the adjoint equation (5) turns out to be well-posed by Proposition 3.1. The above restriction on γ is imposed because we use the energy formulation of (1) via the embedding

$$
W^{2,q} \hookrightarrow W^{1,2}\ ,
$$

which occurs when $q > \frac{2d}{d+2}$. Then, $\frac{d}{\gamma'} > \frac{2d}{d+2}$ precisely when $\gamma > \frac{d+2}{d}$, which is indeed the critical threshold guaranteeing the validity of the energy formulation of the problem, see also [34]. One expects to address the case below $\gamma = \frac{d+2}{d}$ using different techniques and notion of solutions, cf [8, 34] and the references therein.

In this section, we denote by L the Lagrangian of H, defined as its Legendre transform, i.e. $L(x,\nu)=\sup_{p\in\mathbb{R}^d}\{p\cdot\nu-H(x,p)\}$. Moreover, by the convexity property of H we have

$$H(x,p)=\sup_{\nu\in\mathbb{R}^d}\{p\cdot\nu-L(x,\nu)\}$$

and

$$H(x,p)=p\cdot\nu-L(x,\nu) \text{ if and only if } \nu=D_pH(x,p)\ .$$

Under the standing assumptions we will henceforth use that for some $C_L>0$ depending on C_H

$$C_L^{-1}|\nu|^{\gamma'}-C_L\le L(x,\nu)\le C_L|\nu|^{\gamma'}+C_L\ ,\gamma'=\frac{\gamma}{\gamma-1}\ . \tag{L}$$

We will focus here on the case $\frac{d}{\gamma'}<q<\frac{d}{2}$, i.e. when $\gamma<2$, and deduce L^p integral estimates in the next sections.

4.1. **The subcritical case $q>\frac{d}{\gamma'}$.** We start with the following bound on the positive part of u, $u^+=\max\{u,0\}$. It holds for any $q>\frac{2d}{d+2}$. Below, $u^-=(-u)^+$, while $T_k(s)=\max\{-s,\min\{s,k\}\}$ will denote the standard truncation operator at level $k>0$.

Proposition 4.1. *Assume that H is nonnegative and let $u\in W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega)$, $q>\frac{2d}{d+2}$, be a strong solution to the Dirichlet problem (1). There exists a positive constant C_0 such that*

$$\|u^+\|_{L^p(\Omega)}\le C_0\|f^+\|_{L^q(\Omega)}$$

with $p=\frac{dq}{d-2q}$ if $q<\frac{d}{2}$, $p=\infty$ if $q>\frac{d}{2}$. Here, C_0 depends on $d,q,|\Omega|$ but not on λ when $q<\frac{d}{2}$, while depends on λ whenever $q\ge\frac{d}{2}$.

Proof. We first consider the case $q<\frac{d}{2}$. For $k>0$ consider the weak nonnegative solution $\mu=\mu_k$ to the Dirichlet problem

$$\begin{cases}-\Delta\mu+\lambda\mu=\psi_1(x) & \text{in } \Omega\ ,\\ \mu=0 & \text{on } \partial\Omega\ .\end{cases} \tag{13}$$

where

$$\psi_1(x)=\frac{[T_k(u^+(x))]^{p-1}}{\|u^+\|^{p-1}_{L^p(\Omega)}}.$$

Note that $\|\psi_1\|_{p'}\le 1$. By Corollary 3.3 with $b\equiv 0$ we deduce

$$\|\mu\|_{L^{q'}(\Omega)}\le C$$

with C independent of k. By Kato's inequality [18], u^+ is a weak subsolution to

$$-\Delta u^++\lambda u^+\le[f(x)-H(x,Du(x))]\mathbb{1}_{\{u>0\}} \text{ in } \Omega\ .$$

Then, using μ as a test function in the previous equation and u^+ as a test function in problem (13), we deduce

$$\begin{aligned}\int_\Omega\psi_1(x)u^+(x)\,dx &\le\int_{\Omega\cap\{u>0\}}f\mu\,dx-\int_\Omega H(x,Du)\mathbb{1}_{\{u>0\}}\mu\,dx\\ &\le\|f^+\|_{L^q(\Omega)}\|\mu\|_{L^{q'}(\Omega)}\le C\|f^+\|_{L^q(\Omega)}\ .\end{aligned}$$

The estimate follows by duality using the definition of ψ_1, applying Hölder's inequality on the right-hand side of the above inequality and sending $k\to\infty$. We now briefly discuss the case $q>\frac{d}{2}$. First, one considers (13) with a right-hand side $\psi_1\ge 0$ such that $\|\psi_1\|_{L^1(\Omega)}=1$. Using the same strategy, without need to use a truncation argument, using that μ is nonnegative and $H\ge 0$, one obtains $u^+\in L^\infty(\Omega)$ by applying Corollary 3.3 in the right regime of integrability, i.e. with $p'=1$ and $b\equiv 0$. We refer to [22, Proposition 3.7] for further details, being the proof similar. □

We now consider the more delicate bound on the negative part u^-, which needs the extra integrability requirement $q>\frac{d}{\gamma'}$.

Proposition 4.2. *Assume that* (H) *holds. Let $u \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ be a strong solution to* (1) *with $q > d/\gamma'$. There exists a positive constant C depending on d, q, C_H, γ such that*

$$\|u^-\|_{L^p(\Omega)} \leq C\|f^-\|^{\frac{q\gamma'}{q\gamma'-d}}_{L^q(\Omega)} + CC_L\frac{|\Omega|^{\frac{1}{p}}}{\lambda} + C\|f^-\|_{L^q(\Omega)}$$

where $p = \frac{dq}{d-2q}$ when $q < \frac{d}{2}$, any $p < \infty$ when $q = \frac{d}{2}$ and $p = \infty$ when $q > \frac{d}{2}$.

Proof. For $q < \frac{d}{2}$, $k > 0$, we define as $\rho = \rho_k$ the weak nonnegative solution to the Dirichlet problem

$$\begin{cases} -\Delta\rho(x) + \lambda\rho(x) + \operatorname{div}(D_pH(x, Du(x))\mathbb{1}_{\{u<0\}}\rho(x)) = \psi_2(x) & \text{in } \Omega\ , \\ \rho = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{14}$$

where $\psi_2(x) = \frac{[T_k(u^-(x))]^{p-1}}{\|u^-\|^{p-1}_{L^p(\Omega)}}$. By duality as in the case of Proposition 4.1, using again Kato's inequality as in [34, p. 148], [23], we get

$$\int_\Omega u^-(x)\psi_2(x)\,dx + \int_\Omega \big[-D_pH(x, Du)\cdot Du^- - H(x, Du)\big]\,\mathbb{1}_{\{u<0\}}\rho\,dx \leq \int_{\Omega\cap\{u<0\}} f\rho\,dx\ .$$

By the properties of the Lagrangian (L), we estimate

$$\begin{aligned} \big[-D_pH(x, Du)\cdot Du^- - H(x, Du)\big]\,\mathbb{1}_{\{u<0\}} &= L(x, D_pH(x, -Du^-))\mathbb{1}_{\{u<0\}} \\ &\geq [C_L^{-1}|D_pH(x, Du)|^{\gamma'} - C_L]\mathbb{1}_{\{u<0\}}\ . \end{aligned}$$

Therefore, we have by Corollary 3.3 (note that it can be safely applied since $\gamma > \frac{d+2}{d}$) and the fact that the condition $\|\psi_2\|_{p'} \leq 1$ implies that $\|\rho\|_1 \leq \frac{|\Omega|^{\frac{1}{p}}}{\lambda}$ through (6)

$$\begin{aligned} &\int_\Omega u^-(x)\psi_2(x)\,dx + C_L^{-1}\int_\Omega |D_pH(x, Du)|^{\gamma'}\mathbb{1}_{\{u<0\}}\rho\,dx \leq C_L\int_\Omega \rho\mathbb{1}_{\{u<0\}} + \int_\Omega f^-\rho\,dx \\ &\leq C_L\frac{|\Omega|^{\frac{1}{p}}}{\lambda} + \|f^-\|_{L^q(\Omega)}\|\rho\|_{L^{q'}(\Omega)} \leq C_L\frac{|\Omega|^{\frac{1}{p}}}{\lambda} + C_1\|f^-\|_{L^q(\Omega)}\left(\int_\Omega |D_pH(x, Du)|^{\frac{d}{q}}\mathbb{1}_{\{u<0\}}\rho\,dx + 1\right). \end{aligned}$$

Since $q > d/\gamma'$, we can absorb the second term in the above right-hand side via Young's inequality to obtain

$$\int_\Omega u^-(x)\psi_2(x)\,dx \leq C_L\frac{|\Omega|^{\frac{1}{p}}}{\lambda} + C_2\left(\|f^-\|^{\frac{q\gamma'}{q\gamma'-d}}_{L^q(\Omega)} + \frac{|\Omega|^{\frac{1}{p}}}{\lambda} + \|f^-\|_{L^q(\Omega)}\right)$$

and then let $k \to \infty$ to conclude the estimate. The case $q = \frac{d}{2}$ is similar, while the case $q > \frac{d}{2}$ requires to take a nonnegative $\psi = \psi_2 \in L^1(\Omega)$ with $\|\psi_2\|_{L^1(\Omega)} = 1$ to conclude that $u^- \in L^\infty(\Omega)$ via the same duality argument and exploiting that

$$\int_\Omega \rho \leq \frac{\|\psi_2\|_{L^1(\Omega)}}{\lambda} = \frac{1}{\lambda}\ .$$

Indeed, fix $q > \frac{d}{\gamma'}$ and start again with

$$\begin{aligned} \int_\Omega u^-(x)\psi_2(x)\,dx + C_L^{-1}\int_\Omega |D_pH(x, Du)|^{\gamma'}\mathbb{1}_{\{u<0\}}\rho\,dx &\leq C_L\int_\Omega \rho\mathbb{1}_{\{u<0\}} + \int_\Omega f^-\rho\,dx \\ &\leq \frac{C_L}{\lambda} + \|f^-\|_{L^q(\Omega)}\|\rho\|_{L^{q'}(\Omega)}. \end{aligned}$$

One then chooses σ such that

$$\frac{1}{q'} = \frac{1}{\sigma} - \frac{1}{d}$$

so that taking $q > \frac{d}{2}$ we get the condition $\sigma < \frac{d}{d-1}$. We can then apply Corollary 3.3 to get

$$C_L^{-1}\int_\Omega |D_pH(x, Du)|^{\gamma'}\mathbb{1}_{\{u<0\}}\rho\,dx \leq \frac{C_L}{\lambda} + C\|f^-\|_{L^q(\Omega)}\left(\int_\Omega |D_pH(x, Du)|^k\mathbb{1}_{\{u<0\}}\rho\,dx + 1\right)$$

with $k = 1 + \frac{d}{\sigma} = \frac{d}{q} < \gamma'$, and then apply the Young's inequality to conclude the assertion. $\square$

Combining Propositions 4.1 and 4.2 we get the following result under the assumption $q > \frac{d}{\gamma'}$ (recall that $\frac{d}{\gamma'} > \frac{2d}{d+2}$ under the standing restrictions on γ).

Corollary 4.3. *Assume that* (H) *holds. Let $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ be a strong solution to* (1) *with $q > d/\gamma'$. Assume there exists $K > 0$ such that $\|f\|_{L^q(\Omega)} \le K$. There exists a positive constant C depending on $K, d, q, C_H, |\Omega|, \lambda$ such that*

$$\|u\|_{L^p(\Omega)} \le C$$

where

- *$p = \infty$ when $q > \frac{d}{2}$ and any $\gamma > \frac{d+2}{d}$;*
- *any $p < \infty$ when $q = \frac{d}{2} > \frac{d}{\gamma'}$ and $\gamma > \frac{d+2}{d}$;*
- *$p = \frac{dq}{d-2q}$ when $\frac{d}{\gamma'} < q < \frac{d}{2}$ (which implies $\gamma < 2$) and $\gamma > \frac{d+2}{d}$.*

Proof. It readily follows from Propositions 4.1 and 4.2. $\square$

Remark 4.4. L^∞ bounds for distributional subsolutions in $W_0^{1,\gamma}$ in the case $\gamma > 2$ already appeared in [26, Theorem 5.2] through a Stampacchia-type argument under the restriction $f \in L^q$, $q > \frac{d}{\gamma}$ (note that when $\gamma > 2$ we have $\frac{d}{\gamma} < \frac{d}{2}$). When $\gamma = 2$ the L^∞ bounds for solutions belonging a priori to $W_0^{1,2} \cap L^\infty$ were proved in [13, Theorem 2.1] again via a Stampacchia's method. Instead, our L^∞ bound from Corollary 4.3 holds for any $\gamma > 1 + \frac{2}{d}$, and thus provide a unified proof for this end-point integral estimate. We further remark that, in general, the L^∞ estimate is false when $\gamma \le 2$ if u does not belong to a suitable class of solutions, cf [26, Remark 5.1] or [13, Remark 2.1]. Indeed, Example 1.3 in [34] shows that there exists an unbounded weak solution for problems with sub-quadratic growth in the gradient if u does not satisfy the condition

$$|u|^\tau \in W_0^{1,2}(\Omega)\ , \tau = \frac{(d-2)(\gamma-1)}{2(2-\gamma)}\ .$$

We observe that in the limiting case $q = \frac{d}{\gamma'}$, for $\gamma < 2$, the membership $u \in W^{2,q}$ leads to the above condition when $\gamma < 2$. Indeed, if $u \in W^{2,q}$ for $q = \frac{d}{\gamma'}$, it follows by Sobolev embeddings that $u \in W^{1,\frac{dq}{d-q}}$, i.e. $u \in W^{1,d(\gamma-1)}$. This in particular implies, applying again the Sobolev inequality, that $u \in L^{\frac{d(\gamma-1)}{2-\gamma}}$, i.e. $|u|^\tau \in L^{2^*} \hookrightarrow L^2$. Finally, $D|u|^\tau \in L^2$ when $|u|^{\tau-1}|Du| \in L^2$, which is true by the Hölder's inequality using that $u \in L^{\frac{d(\gamma-1)}{2-\gamma}}$ together with $Du \in L^{d(\gamma-1)}$. Finally, for the case of natural gradient growth $\gamma = 2$, the Sobolev embedding for $q > \frac{d}{2}$ implies that $u \in L^\infty$, and hence we are in the same situation of [13, Remark 2.1], where solutions belong a priori to L^∞.

4.2. **The end-point threshold $q = \frac{d}{\gamma'}$.** The results of the previous section do not encompass the critical integrability value $q = \frac{d}{\gamma'}$. In this case one expects integrability estimates to depend on finer properties than the sole control of $f \in L^q$, see e.g. [34]. Aim of this section is to derive integrability estimates in this end-point situation via duality methods, together with establishing new stability properties in Lebesgue spaces for strong solutions. The approach of this section is inspired by that appeared in the context of parabolic problems in [23, Corollary 3.4].
Consider for $\lambda > 0$ the Dirichlet problem with truncated right-hand side datum

$$\begin{cases} -\Delta u_k(x) + H(x, Du_k(x)) + \lambda u_k(x) = T_k(f(x)) & \text{in } \Omega \\ u_k(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

Note that now the right-hand side of the equation belongs to $L^\infty(\Omega)$, and existence and uniqueness of strong solutions in $W^{2,q}$ for the Dirichlet problem is guaranteed by the results proved in e.g. [4, Lemma 3], see also [42, 36].

Proposition 4.5. *Assume that* (H) *holds. Let u and u_k be strong solutions to* (1) *and* (15) *respectively, $\gamma < 2$ and $q = \frac{d}{\gamma'}$. Then, there exists a constant C depending on $f, C_H, q, d, |\Omega|, \lambda$ such that*

$$\|u - u_k\|_{L^p(\Omega)} \le C\|f - T_k(f)\|_{L^q(\Omega)} \tag{16}$$

where $p = \frac{dq}{d-2q} = \frac{d(\gamma-1)}{2-\gamma}$.

Proof. Let $w = u - u_k$. Here, w depends on k, but we will drop the subscript k. Let ρ be the weak solution to the Dirichlet problem

$$\begin{cases} -\Delta\rho(x) + \lambda\rho(x) - \operatorname{div}(D_pH(x, Du_k(x))\rho(x)) = \psi_3(x) & \text{in } \Omega\ , \\ \rho = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{17}$$

where

$$\psi_3(x) = \frac{[w^+(x)]^{p-1}}{\|w^+\|^{p-1}_{L^p(\Omega)}}\ .$$

As in Propositions 4.1 and 4.2, one should truncate the right-hand side term ψ_3 to ensure the existence of solutions on energy spaces, and then pass to the limit. We will drop this step for simplicity. We first proceed by proving a bound on $\int_\Omega |D_pH(x, Du_k))|^{\gamma'}\rho\, dx$. By duality arguments, testing (15) with ρ and (17) with u_k we have

$$\int_\Omega L(x, D_pH(x, Du_k))\rho\, dx = -\int_\Omega T_k(f(x))\rho(x)\, dx + \int_\Omega u_k(x)\psi_3(x)\ . \tag{18}$$

For $h > 0$ to be chosen we write by (6)

$$\begin{aligned} -\int_\Omega T_k(f(x))\rho(x)\, dx \le \int_\Omega f^-(x)\rho(x)\, dx \le \int_{\Omega\cap\{f^-\ge h\}} f^-\rho\, dx + h\frac{|\Omega|^{\frac1p}}{\lambda} \\ \le \|f^-\mathbb{1}_{\{f^-\ge h\}}\|_{L^q(\Omega)}\|\rho\|_{L^{q'}(\Omega)} + h\frac{|\Omega|^{\frac1p}}{\lambda}\ . \end{aligned}$$

We then use Proposition 4.1 applied to u_k to deduce

$$\int_\Omega u_k(x)\psi_3(x) \le \|u_k\|_{L^p(\Omega)}\|\psi_3\|_{L^{p'}(\Omega)} \le C_0\|T_k(f^+)\|_{L^q(\Omega)} \le C_0\|f^+\|_{L^q(\Omega)}\ .$$

We then conclude

$$\begin{aligned} C_L^{-1}\int_\Omega |D_pH(x, Du_k))|^{\gamma'}\rho\, dx - C_L\int_\Omega \rho\, dx \le \frac{h|\Omega|^{\frac1p}}{\lambda} + C_0\|f^+\|_{L^q(\Omega)} \\ + \|f^-\mathbb{1}_{\{f^-\ge h\}}\|_{L^q(\Omega)}\|\rho\|_{L^{q'}(\Omega)}\ . \end{aligned}$$

Plugging the previous estimates on (18) it follows that

$$\begin{aligned} C_L^{-1}\int_\Omega |D_pH(x, Du_k))|^{\gamma'}\rho\, dx \le \frac{|\Omega|^{\frac1p}}{\lambda}(h + C_L) + C_0\|f^+\|_{L^q(\Omega)} \\ + C_1\|f^-\mathbb{1}_{\{f^-\ge h\}}\|_{L^q(\Omega)}\left(\int_\Omega |D_pH(x, Du_k))|^{\gamma'}\rho\, dx + 1\right) \end{aligned}$$

with C_1 depending on d, γ, Ω. We finally choose h large enough to ensure

$$\|f^-\mathbb{1}_{\{f^-\ge h\}}\|_{L^q(\Omega)} \le \frac{1}{2C_1C_L}$$

and absorb the term on the right-hand side on the left-hand one, concluding the bound

$$\int_\Omega |D_pH(x, Du_k))|^{\gamma'}\rho\, dx \le 2C_L\left[\frac{|\Omega|^{\frac1p}}{\lambda}(h + C_L) + C_0\|f^+\|_{L^q(\Omega)}\right] + 1 =: \widetilde{C}\ .$$

We are now in position to obtain the estimate on w^+. It is immediate to observe that by convexity of $H(x,\cdot)$, w^+ is a weak subsolution to

$$-\Delta w^+ + \lambda w^+ + D_pH(x, Du_k(x)) \cdot Dw^+ \le [f - T_k(f)]\mathbb{1}_{\{w>0\}} \ .$$

By duality we conclude

$$\|w^+\|_{L^p(\Omega)} = \int_\Omega w^+(x)\psi_3(x)\, dx \le \int_\Omega [f - T_k(f)]\mathbb{1}_{\{w>0\}}\rho\, dx$$

and thus by Corollary 3.3

$$\|w^+\|_{L^p(\Omega)} \le C_1\|f - T_k(f)\|_{L^q(\Omega)} \left(\int_\Omega |D_pH(x, Du_k))|^{\gamma'}\rho\, dx + 1\right) \\ \le C_1(\widetilde{C} + 1)\|f - T_k(f)\|_{L^q(\Omega)} \ .$$

Finally, to obtain the integral estimates on w^- (and hence conclude the desired estimate on w) one can proceed using the same scheme considering the dual problem

$$\begin{cases} -\Delta\hat\rho(x) + \lambda\hat\rho(x) - \operatorname{div}(D_pH(x, Du_k(x))\hat\rho(x)) = \psi_4(x) & \text{in } \Omega \ , \\ \hat\rho = 0 & \text{on } \partial\Omega \ , \end{cases}$$

where now

$$\psi_4(x) = \frac{[w^-(x)]^{p-1}}{\|w^-\|^{p-1}_{L^p(\Omega)}} \ .$$

□

Remark 4.6. First, we observe that (16) readily implies

$$\|u\|_{L^p(\Omega)} \le C$$

when $p = \frac{d(\gamma-1)}{2-\gamma}$, $q = \frac{d}{\gamma'}$. However, the dependence of the constant C in the above estimate is different with respect to the L^p bound obtained in Corollary 4.3. Here, C depends on $\|f^+\|_{L^q(\Omega)}, h$ where h has been chosen so that

$$\|f^-\mathbb{1}_{\{f^-\ge h\}}\|_{L^q(\Omega)} \le \frac{1}{2C_1C_L} \ .$$

It is worth observing that these constants remain bounded whenever f varies in bounded and equi-integrable sets in L^q, cf Definition 2.23 and Theorem 2.29 in [30].

Remark 4.7. The estimate $u \in L^p$ with $p = \frac{d(\gamma-1)}{2-\gamma}$, $q = \frac{d}{\gamma'}$ agrees with the one already found in [34, Theorem 1.1]. Indeed, the authors in [34] proved that $|u|^\tau \in W_0^{1,2}(\Omega)$ with $\tau = \frac{(d-2)(\gamma-1)}{2(2-\gamma)}$ and the same type of dependence in the constant of the estimate. By Sobolev embedding this yields $|u|^\tau \in L^{\frac{2d}{d-2}}$, which, by the definition of τ, yields $u \in L^p$ with p as above.

5. Maximal L^q-regularity for the Dirichlet problem

Proof of Theorem 2.1. We first use Theorem A.1 to conclude that the strong solutions to (1) satisfy the estimate

$$\|u\|_{W^{2,q}(\Omega)} \le C_1(\||Du|\|^\gamma_{L^{q\gamma}(\Omega)} + 1)$$

where C_1 depends on $C_H, q, d, \Omega, \lambda$. We then proceed by interpolation as in [23] (see also the references therein for other regularity results obtained through a similar scheme). First, we recall that from Corollary 4.3 we have

$$\|u\|_{L^s(\Omega)} \le C \tag{19}$$

for any $s \le p = \frac{dq}{d-2q}$ if $q < \frac{d}{2}$, $s \le \infty$ when $q > \frac{d}{2}$, with C depending on the previous quantities. The Gagliardo-Nirenberg inequality, cf [44], gives the existence of a positive constant C_{GN} such that

$$\||Du|\|_{L^{q\gamma}(\Omega)} \le C_{\mathrm{GN}}\|D^2u\|^\theta_{L^q(\Omega)}\|u\|^{1-\theta}_{L^s(\Omega)} + \|u\|_{L^s(\Omega)}, \tag{20}$$

for $s \in [1,\infty]$ and $\theta \in [1/2,1)$ such that

$$\frac{1}{\gamma q} = \frac{1}{d} + \theta\left(\frac{1}{q} - \frac{2}{d}\right) + (1-\theta)\frac{1}{s}.$$

Since $q > \frac{d}{\gamma'}$, it follows that $p > \frac{d(\gamma-1)}{2-\gamma}$, and we are then free to choose s close to $\frac{d(\gamma-1)}{2-\gamma}$ to ensure $\theta \in [1/2, 1/\gamma)$. We finally conclude by plugging (20) into (19) and get

$$\|u\|_{W^{2,q}(\Omega)} \le C_2(\|D^2u\|^{\theta\gamma}_{L^q(\Omega)} + \|f\|_{L^q(\Omega)}) \ ,$$

which gives the estimate by applying the Young's inequality since $\theta\gamma < 1$.

□

Proof of Theorem 2.2. Let $w = u - u_k$, k to be chosen later, where u_k solves the Dirichlet problem with truncated right-hand side (15). By Proposition 4.5 we have the existence of a constant $C > 0$ such that

$$\|w\|_{L^p(\Omega)} \le C\|f - T_k(f)\|_{L^q(\Omega)} \ , p = \frac{d(\gamma-1)}{2-\gamma} \ .$$

In view of the Gagliardo-Nirenberg inequality we get

$$\|Dw\|_{L^{q\gamma}(\Omega)} \le C_2\|w\|^{\frac{1}{\gamma}}_{W^{2,q}(\Omega)}\|w\|^{1-\frac{1}{\gamma}}_{L^p(\Omega)} \ ,$$

where $C_2 = C_2(d,p,q,\Omega)$. Thus, we write

$$\|Dw\|^{\gamma}_{L^{q\gamma}(\Omega)} \le C_2^{\gamma}C^{\gamma-1}\|f - T_k(f)\|^{\gamma-1}_{L^q(\Omega)}\|w\|_{W^{2,q}(\Omega)} \ . \tag{21}$$

We further observe that w solves the Dirichlet problem

$$\begin{cases} -\Delta w + \lambda w = H(x, Du_k(x)) - H(x, Du(x)) + f(x) - T_k(f(x)) & \text{in } \Omega \ , \\ w = 0 & \text{on } \partial\Omega \ , \end{cases} \tag{22}$$

and, in view of the growth assumptions (H), we have $|D_pH(x,p)| \le \widetilde{C}_H(|p|^{\gamma-1} + 1)$ for some $\widetilde{C}_H > 0$, and therefore we deduce by the Young's inequality

$$\begin{aligned} |H(x, Du_k(x)) - H(x, Du(x))| &\le |Dw(x)| \cdot \max\{|D_pH(x, Du_k(x))|, |D_pH(x, Du(x))|\} \\ &\le C_3(|Du_k(x)|^{\gamma} + |Du(x)|^{\gamma} + |Dw(x)|^{\gamma} + 1) \le C_4(|Du_k(x)|^{\gamma} + |Dw(x)|^{\gamma} + 1) \ , \end{aligned}$$

where C_3, C_4 depend only on C_H. We then apply Theorem A.1 to the Dirichlet problem (22) to find

$$\begin{aligned} \|w\|_{W^{2,q}(\Omega)} &\le C_5(\|H(x, Du_k(x)) - H(x, Du(x))\|_{L^q(\Omega)} + \|f - T_k(f)\|_{L^q(\Omega)}) \\ &\le C_6\|Dw\|^{\gamma}_{L^{q\gamma}(\Omega)} + C_6(\|Du_k\|^{\gamma}_{L^q(\Omega)} + \|f - T_k(f)\|_{L^q(\Omega)} + 1) \ . \end{aligned}$$

We plug the latter inequality into (21) to obtain

$$\begin{aligned} \|Dw\|^{\gamma}_{L^{q\gamma}(\Omega)} &\le C_6C_2^{\gamma}C^{\gamma-1}\|f - T_k(f)\|^{\gamma-1}_{L^q(\Omega)}\|Dw\|^{\gamma}_{L^{q\gamma}(\Omega)} \\ &\quad + C_6(\|Du_k\|^{\gamma}_{L^q(\Omega)} + \|f - T_k(f)\|_{L^q(\Omega)} + 1)C_2^{\gamma}C^{\gamma-1}\|f - T_k(f)\|^{\gamma-1}_{L^q(\Omega)} \ , \end{aligned}$$

for a constant $C_6 = C_6(d,q,C_H)$. We then pick a certain $\bar{k}$ large enough to have

$$C_6C_2^{\gamma}C^{\gamma-1}\|f - T_{\bar{k}}(f)\|^{\gamma-1}_{L^q(\Omega)} \le \frac{1}{2} \tag{23}$$

and conclude

$$\|Dw\|^{\gamma}_{L^{q\gamma}(\Omega)} \le \|Du_{\bar{k}}\|^{\gamma}_{L^{q\gamma}(\Omega)} + \|f - T_{\bar{k}}(f)\|_{L^q(\Omega)} + 1 \le \|Du_{\bar{k}}\|^{\gamma}_{L^{q\gamma}(\Omega)} + 2\|f\|_{L^q(\Omega)} + 1 \ .$$

We can then apply Theorem 2.1 to estimate $\|Du_{\bar{k}}\|_{q\gamma}$ since $T_{\bar{k}}(f) \in L^{\infty}$ (actually, Du turns out to be even bounded via the results in [41]) and get for any $\bar{q} > q$ the bound

$$\|Du_{\bar{k}}\|_{L^{\bar{q}\gamma}(\Omega)} \le C_{\bar{k}} \ ,$$

where $C_{\bar{k}}$ depends on $\bar{k}, q, d, C_H, |\Omega|$ (indeed, $\|T_{\bar{k}}(f)\|_{L^{\bar{q}}(\Omega)} \leq \bar{k}|\Omega|^{\frac{1}{\bar{q}}}$). This allows to conclude the desired estimate, since by the definition of w we have

$$\|Du\|_{L^{q\gamma}(\Omega)} \leq \|Dw\|_{L^{q\gamma}(\Omega)} + \|Du_{\bar{k}}\|_{L^{q\gamma}(\Omega)} .$$

□

Remark 5.1. The constant of the estimate in Theorem 2.2 remains bounded when f varies in a bounded and equi-integrable set $\mathcal{F} \subset L^q(\Omega)$, see the previous Remark 4.6. Indeed, the constant of the estimate depends on $\bar{k}$ that appears in (23), where $\bar{c} = (2C_6C_2^{\gamma}C^{\gamma-1})^{-1}$ is independent of $f \in \mathcal{F}$.

Remark 5.2. In view of the recent works for the Hamilton-Jacobi equation in [45, 8], one might expect the validity of maximal regularity estimates for the Dirichlet problem when the source term f belongs to the end-point Lorentz scale $L(\frac{d}{\gamma'}, \infty)$. Some results in this direction when $\gamma < \frac{d+2}{d}$ without zero-th order terms have been obtained in [8]. Still, when $f \in L(d, 1)$ one should expect boundedness of the gradient, at least in the context of interior estimates and problems with periodic data, as analyzed in [37] for different classes of nonlinear elliptic equations. The general case at these critical integrability scales will be the matter of a future research.

6. Some remarks for the case of the natural gradient growth

As already discussed in [23] (see also [7] for a similar approach), to address the problem of maximal regularity when $\gamma = 2$ in (H), one can no longer apply the classical Gagliardo-Nirenberg interpolation inequality on Lebesgue spaces with $s = \infty$, since $\theta\gamma = 2\theta = 1$, which in turn prevents from the absorption of the second order derivative term on the left-hand side of the estimate. However, this can be circumvented by exploiting the Miranda-Nirenberg interpolation inequalities, that allow to interpolate the first order Sobolev space with a Hölder class, cf e.g. [7, Section 1.1 p. 9]. Indeed, for $\theta \in \left[\frac{1-\alpha}{2-\alpha}, 1\right]$, $\alpha \in (0, 1)$, and r, q, θ satisfying the compatibility conditions

$$\frac{1}{2q} = \frac{1}{d} + \theta\left(\frac{1}{r} - \frac{2}{d}\right) - (1-\theta)\frac{\alpha}{d},$$

any function $u \in W^{2,r}(\Omega) \cap C^{\alpha}(\overline{\Omega})$ belongs to $W^{1,2q}(\Omega)$. Then, we have, choosing $\theta = \frac{1-\alpha}{2-\alpha}$, the strict inequality $2\theta < 1$ and the estimate

$$\|Du\|_{L^{2q}(\Omega)} \leq C\|D^2u\|_{L^r(\Omega)}^{1-\theta}[u]_{C^{\alpha}(\Omega)}^{\theta} + [u]_{C^{\alpha}(\Omega)} \ , r = q\frac{2-2\alpha}{2-\alpha} \ ,$$

where $[\cdot]_{C^{\alpha}}$ is the Hölder seminorm. Then, since $r < q$ for any $\alpha > 0$, one can conclude the statement once solutions to the Dirichlet problem (1) satisfy a Hölder estimate. Since $u \in L^{\infty}(\Omega)$ by Corollary 4.3 when $q > \frac{d}{2}$, we can regard (1) as

$$-\Delta u + H(x, Du) = -\lambda u + f =: g \in L^q \ , q > \frac{d}{2}$$

and conclude invoking the Hölder estimates for the Dirichlet problem in [38, Theorem 2.2 p.441] for the case of the quadratic growth.

Appendix A. Auxiliary results

We recall the following $W^{2,q}$ a priori estimate for strong solutions to linear elliptic problems.

Theorem A.1. *Let $\Omega \subset \mathbb{R}^d$ be a C^2 bounded domain. If $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, $g \in L^q(\Omega)$ satisfies $-\Delta u + \lambda u = g$ a.e. in Ω, $\lambda \in \mathbb{R}$, $\lambda > 0$, then*

$$\|u\|_{W^{2,q}(\Omega)} \leq C_1\|g\|_{L^q(\Omega)}$$

where C_1 depends only on d, q, λ, Ω. As a consequence, any strong solution $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ to (1) satisfies

$$\|u\|_{W^{2,q}(\Omega)} \leq C_2(\||Du|\|_{L^{q\gamma}(\Omega)}^{\gamma} + \|f\|_{L^q(\Omega)} + 1)$$

where C_2 depends on $d, q, \lambda, \Omega, C_H$.

Proof. The first statement is a classical Calderón-Zygmund estimate that can be found in [31, Lemma 9.17] and [20, Theorem 6.3]. The second result follows readily from the first by observing that if u is a strong solution to (1), then it solves

$$-\Delta u + \lambda u = -H(x, Du) + g(x)$$

with the same boundary conditions. Then, the estimate is a consequence of the assumptions (H) and the properties of Lebesgue spaces. □

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