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Analysis and Models for the Dynamics of Opinions and Interpersonal Relationships

Director and Coordinator: Prof. Andrea Neviani
Advisor: Prof. Maria Elena Valcher

Ph. D. Candidate : Giulia De Pasquale

"Success is not the key to happiness.

Happiness is the key to success.

If you love what you are doing,

you will be successful."

[Albert Schweitzer]

Con affetto, ai miei genitori.

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Abstract

This doctoral thesis focuses on the study of opinion dynamics over social networks. The treatment is made from a control system perspective: psychology and social sciences underpin the modeling, analysis and control of complex networked systems. The common thread of this work is the mathematical formalization of the *micro-mechanisms* of social networks in order to understand how they affect the dynamics behaviour of the network at a *macro-level*.

This thesis develops along three main directions: in the first part we assume the topology to be fixed in a social equilibrium configurations and we study the effects of the weights given to the interpersonal and personal ties on the opinion dynamics. In particular we assume the network to be *clustering balanced* and we study, under some assumption on the interpersonal weights, how much each individual should be convinced about its own opinion so that agents' opinion cluster conformably to the clusters in the topology, thus giving rise to the "k-partite consensus" phenomenon.

In the second part we take into account the case of opinion varying network topologies and we study the reaching of structural equilibria in the network. This paradigm is the most suitable when opinions and interpersonal ties evolve on time scales that are comparable in magnitude. We take into account two mechanisms according to which the interpersonal ties evolve along time: the *influence mechanism* and the *homophily mechanism*. In this context we also study two multi-dimensional extensions of one of the pioneering models in the scientific literature related to opinion dynamics: the Hegselmann-Krause model. In the *average-based* model, agents compare the average opinion that they have on different topics while in the *uniform-affinity* model agents compare their opinions topic-wise. The first model suits better for contexts in which the topics into play are related so that, supposedly, there is not much deviation among the opinions that each agents has on the various topic. The second variation is suitable also for contexts in which the opinions are expressed on a wider variety of topics.

The third and last part of this thesis is related to the study of herdability, namely the capability of a system to be driven towards the interior of the positive orthant. The study of this property becomes of interest in contexts in which (positive) thresholds come into play, such as in marketing advertisement, or electoral contexts. We will investigate

under what structural properties some specific leader-follower network topologies lead to herdable systems.

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Chapter 1

Introduction

During the 20th century social sciences witnessed a substantial change of paradigm from individualistic approaches towards social groups and their structural properties Proskurnikov & Tempo (2017). Consequently the focus has been moved from individuals to the social ties among them.

The pioneering work Moreno (1934) was the first in which the use of graphs was promoted for the study of social relations. The works Moreno (1934, 1951) gave later rise to a new discipline known as *sociometry* starting from which the *Social Network Analysis (SNA)* was later developed in Scott (2000), thus giving to sociology a quantitative slant. The development of this new discipline led to the formalization of new concepts, such as clustering coefficients, centrality measure, cliques, etc. Proskurnikov & Tempo (2017). Despite the development of this discipline and the advancements in the control of complex network systems Murray (2003), a gap between SNA and Control Theory persisted for a long time. The reason of such gap can be explained by the lack of accurate models in the description of social phenomena and by the lack of empirical data to validate these models.

Social dynamics are oftentimes very complex and manifest behaviours that are difficult to predict. The reaching of consensus, when it comes to social systems, is a very rare occurrence. Other behaviours, like persistent disagreements and clustering, are way more likely Friedkin (2015). The design of models that are explicative enough, but still mathematically tractable, is a tough goal and an active research topic for scientists from various fields, e.g. control Altafini (2013), physics Castellano et al. (2009), computer science Seerat & Azam (2012) etc.

However, during the last decades, substantial progresses in filling the gap between mathematical models and social network analysis have been done Friedkin & Bullo (2017), Pagan et al. (2021), Acemoglu et al. (2011), Friedkin (2015). This is mainly due to the large availability of real world data from which scientists can benefit nowadays. Consequently, new disciplines such as *sociodynamics* Castellano et al. (2009) and *sociophysics* Papanikolaou et al. (2022), arised and researchers from different fields put their effort in order either to readjust the fundamental social quantitative models that mimic opinion dynamics nowadays or to validate them.

Another research topic to which this doctoral thesis dedicates attention is the one known in the literature as *herdability* Ruf et al. (2018), She & Kan (2020). Herdability refers to the capability of a system to be driven towards the interior of the positive orthant. It is strictly related to the concept of *structural controllabilty*, namely the study of the structural properties a dynamical system needs to enjoy so that its dynamics can be driven towards any point of the state space Egerstedt et al. (2012), Parlangeli & Notarstefano (2012). The study of herdability, specifically, becomes of interest in all those contexts for which investigating if the system state can be brought towards any point of the state space is not of practical interest, and may lead to overly restrictive conditions on the model into play as it happens in chemistry Bower & Bolouri (2001), biology Jacquez (1972), neuroscience Gupta et al. (2007), etc. Herdability applies to leader-follower network systems in which one may wonder where to locate the leaders in the network so that the system is herdable.

For example, in the context of marketing advertisement, it is of interest to devise strategies targeting some individuals to bring the consumption level of a certain good for a group of consumers over a certain threshold. In many electoral systems there is an election threshold that represents the minimum share of votes which a candidate or political party has to achieve to become entitled to any representation in a legislature. In these contexts, it is pointless to require that the state entries may assume any real value, including the negative ones.

It is in contexts like these, in which (positive) thresholds come into play, that the investigation of herdability becomes of interest. For the aforementioned reasons we will focus on the herdability of some leader-follower networks from a structural perspective.

In this manuscript we extend some of the results presented in the scientific literature both in the context of opinion dynamics Altafini (2013), Mei et al. (2019), Cisneros-Velarde & Bullo (2020), Hegselmann & Krause (2002), Etesami et al. (2013) and herdability Ruf et al. (2018, 2019), She et al. (2019) by trying to overcome some of the current limitations. In particular, in the context of opinion dynamics we will show how the pioneristic Altafini model Altafini (2013) con be revisited so that multi-partite consensus can be achieved. We also study a binary version of the models proposed in Cisneros-Velarde & Bullo (2020), Mei et al. (2019) and show that our simplified formulation leads to equally accurate results. Both models are able to drive and initially unbalanced network towards a socially balanced one. Finally, we look at multi-dimensional versions of the classical Hegselmann-Krause model Hegselmann & Krause (2002), by proposing two different variants suitable to two different contexts, and we analize their convergence and order preservation properties.

As far as herdability is concerned, we propose some algebraic conditions to reduce the dimensionality of the problem and we study the herdabiltiy of leader-follower network structures with special structural topologies, in doing so, beside providing new structural conditions that guarantee herdability, we also extend some of the results already presented in Ruf et al. (2018, 2019), She et al. (2019).

All the results presented in this manuscript have a social network interpretation.

In the sequel, Chapter 2 includes notation and technical background that are exploited during the rest of the manuscript. Section 3 deals with the study of k-partite consensus for clustering balanced social networks, Chapter 4 is dedicate do the study of social balance in binary social models, Chapter 5 is related with the study of networked social system with opinion varying topological structure, Chapter 6 pertains the study of herdability for networked systems with special topologies, finally Chapter 7 concludes the manuscript.

Chapter 2

Notation and Technical Background

In this chapter the notation and technical background exploited in the next part of this manuscript are introduced.

2.1 Notation

Given $k, n \in \mathbb{Z}$, with $k \leq n$, the symbol $[k, n]$ denotes the integer set $\{k, k+1, \dots, n\}$. The (i, j)-th entry of a matrix A is denoted by $[A]_{i,j}$ while the i-th entry of a vector $\mathbf{v}$ either by $[\mathbf{v}]_i$ or by v_i. Given a matrix $A \in \mathbb{R}^{n\times n}$ we use the symbols $A \geq 0$ if it is *nonnegative*, and $A > 0$ if it is *p*ositive, namely if all its entries are nonnegative and positive respectively. The same notation holds for vectors. Given a matrix $A \in \mathbb{R}^{n\times n}$, we denote by A_{i*} the i-th row of A, by A_{*j} its j-th column.

A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is *positive (semi) definite* if $\mathbf{x}^\top P\mathbf{x} > 0$ ($\mathbf{x}^\top P\mathbf{x} \geq 0$) for every $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$, and when so we use the notation $P \succ 0$ ($P \succeq 0$).
$A = A_1 \oplus \cdots \oplus A_n$ indicates a block diagonal matrix whose diagonal blocks are $A_1, \dots, A_n$, while $\mathrm{diag}(A)$ denotes the diagonal matrix whose diagonal entries are the diagonal entries of A. The symbols $\mathbf{0}_n$ and $\mathbf{1}_n$ denote the vectors in $\mathbb{R}^n$ with all entries equal to 0 and to 1, respectively. Similarly, $0_{p\times m}$ denotes the $p \times m$ matrix with all zero entries. The symbol $\mathbf{e}_i$ denotes the i-th canonical vector of the canonical basis of $\mathbb{R}^n$. By *signed canonical vectors* we will mean all canonical vectors and their opposite, i.e. the set $\{\pm\mathbf{e}_i, i \in [1, n]\}$. Every nonzero multiple of a canonical vector is called *monomial vector*.

A matrix $A \in \mathbb{R}^{n\times n}_+$ is *row stochastic* if it is a nonnegative matrix and $A\mathbf{1}_n = \mathbf{1}_n$.

The function $\text{sign}(\cdot) : \mathbb{R}^{n\times m} \to \{-1,0,1\}^{n\times m}$ is the function that maps a real matrix into a matrix taking values in $\{-1,0,1\}$ in accordance with the sign of its entries.

Given two vectors $\mathbf{v}$ and $\mathbf{w}$ of the same size n, the expression $\max\{\mathbf{v},\mathbf{w}\}$ denotes the n-dimensional vector $\mathbf{z}$ with $z_i = \max\{v_i, w_i\}$, $i \in [1,n]$.

Given a vector $\mathbf{v} \in \mathbb{R}^n$, the set $\overline{ZP} = \{i \in [1,n] : [\mathbf{v}]_i \neq 0\}$ denotes the *non-zero pattern* of $\mathbf{v}$ Valcher & Santesso (2010). Similarly, one can define the nonzero pattern of a matrix A. A nonzero vector v is said to be unisigned Ruf et al. (2019) if all its nonzero entries have the same sign. If $\mathbf{v}$ is a unisigned vector, then by $\text{sign}(\mathbf{v}) \in \{-1,1\}$ we mean the common sign of its nonzero entries. Given a vector $\mathbf{v} \in \mathbb{R}^n$, we define $\|\mathbf{v}\|_1 := \sum_{i=1}^n |v_i|$ and $\|\mathbf{v}\|_2 := \sqrt{\mathbf{v}^\top \mathbf{v}}$.A real square matrix A is *Hurwitz* if all its eigenvalues lie in the open left complex halfplane, i.e. for every λ belonging to $\sigma(A)$, the *spectrum* of A, we have $\text{Re}(\lambda) < 0$. A matrix $\Pi \in \mathbb{R}^{n\times n}$ is a *p*ermutation matrix if its columns are a permuted version of the columns of the identity matrix I_n. A matrix $A \in \mathbb{R}^{n\times n}$ is said *i*rreducible if $I_n + |\mathcal{A}| + \cdots + |\mathcal{A}|^{n-1} > 0$,where $|\mathcal{A}|$ is a matrix whose entries are the absolute values of the entries of $\mathcal{A}$. Equivalently, the matrix A can not be reduced to upper block triangular form via permutation transformations. Every reducible square matrix $A \in \mathbb{R}_+^{n\times n}$ can be brought, by means of a permutation matrix Π, to the *Frobenius form*:

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & 0 & \dots & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{k1} & \mathcal{A}_{k2} & \dots & \mathcal{A}_{kk} \end{bmatrix}, \tag{2.1}$$

with k diagonal blocks $\mathcal{A}_{ii}$ that are either scalar or irreducible matrices. If A is an $n \times n$ Metzler matrix, its a real dominant (not necessarily simple) eigenvalue, known as *Frobenius eigenvalue* will be denoted by $\lambda_F(A)$. This means that $\lambda_F(A) > \text{Re}(\lambda), \forall\ \lambda \in \sigma(A), \lambda \neq \lambda_F(A)$. If A is Metzler and irreducible, $\lambda_F(A)$ is necessarily simple (see Theorem 2.3, below). Given a set $\mathcal{S}$, the *cardinality* of $\mathcal{S}$ is denoted by $|\mathcal{S}|$.

2.2 Graph Theory

Graphs are mathematical entities that play an important role in the study of the dynamics of networked systems and, in particular, of social network systems.

A *directed, signed and weighted graph* is a triple Mohar (1991) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, (in short $\mathcal{G}(\mathcal{A})$), where $\mathcal{V} = \{1, \ldots, N\}$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs, and $\mathcal{A} \in \mathbb{R}^{N\times N}$ the *adjacency matrix* of the weighted graph $\mathcal{G}$. An arc (j, i) belongs to $\mathcal{E}$ if and only if $[\mathcal{A}]_{i,j} \neq 0$. We assume that the graph $\mathcal{G}$ has no self-loops, i.e., $[\mathcal{A}]_{i,i} = 0$ for every $i \in [1, N]$,and arcs in $\mathcal{E}$ have either positive or negative weights, namely the (nonzero) off-diagonal entries of $\mathcal{A}$ are either positive or negative. If all the nonzero weights take values in $\{-1, 1\}$, we call the graph *unweighted*. We say that two vertices i and j are *friends* (*enemies*) if there is a direct edge with positive (negative) weight connecting them.

A sequence of k consecutive arcs $(j, j_2), (j_2, j_3), \ldots, (j_k, i) \in \mathcal{E}$ is a *walk* (or *path*) of length k from j to i. A walk from j to i is said to be *positive (negative)* if the product of the weights of the edges that compose the walk is positive (negative). A closed path in which each node, except the start-end node, is distinct is called *c*ycle, and a cycle of unitary length is also known as *s*elf-loop.

A *minimum walk* from j to i is a walk of minimum length connecting the two nodes. We define the *distance* $d(j, i)$ from the node j to the node i as the length of the minimum walk from j to i. If there is no path from j to i then $d(j, i) = +\infty$. The distance $d(j, \mathcal{I})$ from the node j to the set of nodes $\mathcal{I}$ is the minimum among all the distances $d(j, i)$, $i \in \mathcal{I}$. Similarly, the distance $d(\mathcal{I}, j)$ from the set of nodes $\mathcal{I}$ to the vertex j is the minimum among all the distances $d(i, j)$, $i \in \mathcal{I}$.

Given a node $i \in \mathcal{V}$, we define the *out-neighbourhood* of node i as the set of nodes j such that $d(i, j) = 1$, namely $\text{Out}(i) = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. We define the *positive out-neighbourhood* of node i as the set of nodes j such that (i, j) is an arc of $\mathcal{G}(A)$ of positive weight, namely $\text{Out}_+(i) = \{j \in \mathcal{V} : [A]_{j,i} > 0\}$. The definition of *negative out-neighbourhood* of a node is analogous. The out-neighbourhood can be also defined for a set of nodes $\mathcal{I} \subset \mathcal{V}$ as $\text{Out}(\mathcal{I}) = \{j \in \mathcal{V} \setminus \mathcal{I} : (i, j) \in \mathcal{E}, \exists\, i \in \mathcal{I}\}$. The definitions of $\text{Out}_+(\mathcal{I})$ and $\text{Out}_-(\mathcal{I})$ are analogous.

If $\mathcal{A}$ is a symmetric matrix, namely $\mathcal{A} = \mathcal{A}^\top$, the graph $\mathcal{G}(\mathcal{A})$ is (signed, weighted and) undirected, and all previous concepts (in particular, the concepts of walk and distance) become symmetric.

The graph $\mathcal{G}$ is said to be *complete* if, for every two nodes $i, j \in \mathcal{V}$, $i \neq j$, there is an edge connecting them, namely $(i, j) \in \mathcal{E}$. Consider the complete undirected graph with N nodes $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ and $\mathcal{E} = (\mathcal{V} \times \mathcal{V}) \setminus \{(i, i) : i \in \mathcal{V}\}$. If $|\mathcal{E}| = m$, we let

$C_N \in \{-1, 0, 1\}^{N\times m}$ denote its *oriented incidence matrix* Bullo (2020), defined as follows. For every vertex $h \in \mathcal{V}$ and every edge $e = (i, j) \in \mathcal{E}$, we have

$$[C_N]_{h,e} = \begin{cases} 1, & h = i; \\ -1, & h = j; \\ 0, & \text{otherwise.} \end{cases}$$

$\mathcal{G}$ has a (nontrivial) *clustering* Davis (1957) if it has at least one negative edge and the set of vertices $\mathcal{V}$ can be partitioned into say $k \geq 2$ disjoint subsets $\mathcal{V}_1, \dots, \mathcal{V}_k$ such that for every $i, j \in \mathcal{V}_p, p \in [1, k]$, $[\mathcal{A}]_{i,j} \geq 0$, while for every $i \in \mathcal{V}_p, j \in \mathcal{V}_q$, $p, q \in [1, k]$, $p \neq q$, $[\mathcal{A}]_{i,j} \leq 0$.

If the adjacency matrix $\mathcal{A}$ is in Frobenius form (2.1), then we can partition the set of vertices $\mathcal{V} = \{1, 2, \dots, N\}$ into k *communication classes* $\mathcal{C}_i, i \in \{1, 2, \dots, k\}$, where $\mathcal{C}_i$ is the set of nodes $\{(\sum_{h=0}^{i-1} n_h) + 1, (\sum_{h=0}^{i-1} n_h) + 2, \dots, (\sum_{h=0}^{i} n_h)\}$ (with $n_0 := 0$ and $n_i := |\mathcal{C}_i|$ for $i \in \{1, 2, \dots, k\}$) corresponding to the row/column indices of the (entries of the) diagonal block A_{ii}. For $j < i$ the class $\mathcal{C}_i$ is *accessible* from the class $\mathcal{C}_j$ (for short, $\mathcal{C}_j \to \mathcal{C}_i$) if there is a walk in $\mathcal{G}(\mathcal{A})$ from some node of $\mathcal{C}_j$ to some node of $\mathcal{C}_i$. Clearly, $\mathcal{C}_i$ is accessible from itself, while for $j > i$ the class $\mathcal{C}_i$ is never accessible from $\mathcal{C}_j$.

For an undirected graph, given 3 distinct vertices i, j and $k \in \mathcal{V}$, the *triad* (i, j, k) is called *balanced* Cisneros-Velarde & Bullo (2020) if $[\mathcal{A}]_{ij}[\mathcal{A}]_{jk}[\mathcal{A}]_{ki} = 1$ and *unbalanced* if $[\mathcal{A}]_{ij}[\mathcal{A}]_{jk}[\mathcal{A}]_{ki} = -1$.

In the following we list two important notions related to graphs and their equivalent conditions in terms of adjacency matrix.

- A directed graph $\mathcal{G}(A)$ is *strongly connected* if for every pair of vertices $i, j \in \mathcal{V}$ there exists a walk from i to j. Equivalently, the adjacency matrix is *i*rreducible Minc (1988).

- A strongly connected graph $\mathcal{G}(A)$ is *aperiodic* if the greatest common divisor of the lengths of the cycles is equal to one. Equivalently, the adjacency matrix is *p*rimitive, namely there exists a positive integer k such that $\mathcal{A}^k > 0$ Minc (1988).

- A graph $\mathcal{G}(\mathcal{A})$ is said to be *structurally balanced* if all its nodes can be partitioned into two disjoint subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ in a way such that $\forall i, j \in \mathcal{V}_p$, $p \in \{1, 2\}$, $[A]_{ij} \geq 0$

and $\forall i \in \mathcal{V}_p$ and $\forall j \in \mathcal{V}_q$, $p, q \in \{1, 2\}$, $p \neq q$, it holds that $[A]_{ij} \leq 0$. If $\mathcal{V}_1 = [1, m]$, while $\mathcal{V}_2 = [m + 1, n]$, the matrix A can be block partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{m \times m}$ and $A_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$ are nonnegative matrices, while A_{12} and A_{21} are nonpositive matrices (i.e., the opposite of nonnegative matrices).

- A graph $\mathcal{G}(\mathcal{A})$ is said to be *clustering balanced* if all its nodes can be partitioned into more than two disjoint subsets $\mathcal{V}_1$, $\mathcal{V}_2, \ldots, \mathcal{V}_k$ in a way such that $\forall i, j \in \mathcal{V}_p$, $p \in \{1, 2, \ldots, k\}$, $[A]_{ij} \geq 0$ and $\forall i \in \mathcal{V}_p$ and $\forall j \in \mathcal{V}_q$, $p, q \in \{1, 2, \ldots, k\}$, $p \neq q$, it holds that $[A]_{ij} \leq 0$. If $\mathcal{V}_1 = [1, n_1]$, $\mathcal{V}_2 = [n_1 + 1, n_2], \ldots \mathcal{V}_k = [n_{k-1} + 1, N]$ the matrix A can be block partitioned as

$$A = \begin{bmatrix} A_{11} & \ldots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \ldots & A_{kk} \end{bmatrix},$$

where $A_{ii} \in \mathbb{R}^{n_i \times n_i}$ $i \in [1, k]$ are nonnegative matrices, while $A_{ij} \in \mathbb{R}^{ni \times n_j}$, $i \neq j$, $i, j \in [1, k]$ are nonpositive matrices.

Two vertices i and j are *familiar* if they belong to the same connected component of the same cluster, namely $i, j \in \mathcal{V}_h$ for some $h \in [1, k]$ and there exists a path (with all positive weights) from i to j passing only through vertices of $\mathcal{V}_h$.

2.3 Matrix Spectral Properties

This section refers to the spectral properties of nonnegative matrices, i.e. matrices whose entries are nonnegative and positive matrices, namely matrices whose entries are positive. For a real matrix $A \in \mathbb{R}^{n \times n}$ we will denote by $\sigma(A)$ its spectrum, namely the set of all its eigenvalues, by $\rho(A)$ its spectral radius, i.e., $\rho(A) = \max\{|\lambda : \lambda \in \sigma(A)|\}$ and by $\mu(A)$ its spectral abscissa, $\mu(A) = \max\{\text{Re}[\lambda], \lambda \in \sigma(A)\}$.

Theorem 2.1 (Horn & Johnson (1985))**.** *Let $A \in \mathbb{R}^{n \times n}$ be a positive matrix, Then*

(i) $\rho(A) > 0$ is eigenvalue of A with algebraic multiplicity 1,

(ii) *there exists an essentially unique vector* $v \in \mathbb{R}^n$ *such that* $Av = \rho(A)v$, *and* $v > 0$,

(iii) *there exists an essentially unique vector* $w \in \mathbb{R}^n$ *such that* $w^\top A = \rho(A)w^\top$, *and* $w > 0$,

(iv) $\rho(A) > |\lambda|$, *for all* $\lambda \in \sigma(A)$,

(v) $\lim_{k\to\infty}(\rho(A)^{-1}A)^k = vw^\top$.

Theorem 2.1 generalizes to the class of nonnegative and primitive matrices. For the class of nonnegative irreducible matrices weaker results hold as formalized by the following theorem.

Theorem 2.2 (Perron-Frobenius theorem Horn & Johnson (1985)). *Let* $A \in \mathbb{R}^{n\times n}$ *be nonnegative and irreducible, Then*

(i) $\rho(A) > 0$ *is eigenvalue of* A *with algebraic multiplicity* 1,

(ii) *there exists a unique vector* $v \in \mathbb{R}^n$ *such that* $Av = \rho(A)v$, *and* $v > 0$,

(iii) *there exists a unique vector* $w \in \mathbb{R}^n$ *such that* $w^\top A = \rho(A)w^\top$, *and* $w > 0$.

For Metzler matrices, namely square matrices that can be expressed $A = sI - B$ with $s \in \mathbb{R}$ and $B \geq 0$ the Perron-Frobenius theorem generalizes as follows.

Theorem 2.3 (Horn & Johnson (1985)). *Let* $A = B - sI \in \mathbb{R}^{n\times n}$ *with* $B \geq 0$, $s \in \mathbb{R}$ *be an irreducible Metzler matrix. Then*

(i) $\mu(A) = \rho(B) - s$ *and it is a simple eigenvalue of* A,

(ii) *there is a unique vector* $v \in \mathbb{R}^n$, $v > 0$ *such that* $Av = \mu(A)v$,

(iii) *there is a unique vector* $w \in \mathbb{R}^n$, $w > 0$ *such that* $w^\top A = w^\top \mu(A)$.

Theorem 2.4 (Horn & Johnson (1985)). *Let* $A = [A_{ij}] \in \mathbb{R}^{n\times n}$. *The eigenvalues of* A *lay in the union of the* n *Gershgorin's disks:*

$$\Big\{\lambda \in \mathbb{C} : |\lambda - [A]_{ii}| \leq \sum_{j=1}^{n} |[A]_{ij}|\Big\}, \quad i = 1, \dots, n. \tag{2.2}$$

2.3.1 Conditions for Positive (Semi)Definite Matrices

Consider a matrix $M = M^\top \in \mathbb{R}^{n\times n}$ block-partitioned as

$$M = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}, \tag{2.3}$$

where $A = A^\top \in \mathbb{R}^{k\times k}$. If $\det(A) \neq 0$, then the matrix $S = C - B^\top A^{-1} B \in \mathbb{R}^{(n-k)\times(n-k)}$ is well-defined and it is said the Schur complement of A in M Boyd & Vandenberghe (2004). Then it holds that

(i) the matrix M is positive definite, $M \succ 0$, if and only if $A \succ 0$ and $S \succ 0$,

(ii) if $A \succ 0$, the matrix M is positive semidefinite, $M \succeq 0$, if and only if $S \succeq 0$.

Lemma 2.5 (Berman & Plemmons (1979)). *Let $D \in \mathbb{R}^{n\times n}$ be a diagonal matrix and let $A \in \mathbb{R}^{n\times n}$ be a symmetric Metzler matrix, then:*

i) $D - A$ is positive definite if and only if there exists a positive vector $\mathbf{z} \in \mathbb{R}^n$ such that $(D - A)\mathbf{z} > 0$.

ii) If condition i) holds, then $(D - A)^{-1} \geq 0$ and is symmetric.

2.4 Stability and Controllability of Dynamical Systems

Theorem 2.6 (Ahmadi & Parrilo (2008)). *Consider the discrete time dynamical system*

$$x(k+1) = f(x(k)) \tag{2.4}$$

with $k \in \mathbb{N}, x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(0) = 0$. If there exists a scalar $\tau \geq 0$, and a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$, such that

$$\begin{aligned} V(x(t)) > 0, \quad \forall x \neq 0 \\ V(0) = 0 \\ \tau(V(x(k+2)) - V(x(k))) + (V(x(k+1)) - V(x(k))) < 0 \end{aligned}$$

then the origin is a globally asymptotically stable equilibrium of (2.4).

Theorem 2.7 (Kailath (1980))**.** *Consider the discrete time linear system*

$$x(k+1) = Ax(k) + Bu(k) \tag{2.5}$$

where $x \in \mathbb{R}^n$ *is the state variable and* $u \in \mathbb{R}^m$ *the control input. The system* (2.5) *is controllable if and only if the rank of* $[(A - \lambda I_n)B]$ *is equal to* n *for all* $\lambda \in \mathbb{C}$.

Chapter 3

Consensus for Clusters of Agents with Cooperative and Antagonistic Relationships

In this chapter we address the consensus problem in the context of networked agents whose communication graph can be split into a certain number of clusters in such a way that interactions between agents in the same clusters are cooperative, while interactions between agents belonging to different clusters are antagonistic. This problem set-up arises in the context of social networks and opinion dynamics, where reaching consensus means that the opinions of the agents in the same cluster converge to the same decision. Under the assumption that agents in the same cluster have the same constant and pre-fixed amount of trust (/distrust) to be distributed among their cooperators (/adversaries), we propose a modified version of the Altafini model Altafini (2013) that, by simply constraining how much agents in each group should be conservative about their own opinions, allows to achieve a nontrivial solution by means of a distributed algorithm. The result is then particularised to unweighted complete communication graphs, and subsequently extended to a class of nonlinear multi-agent systems, in which the state variable is not directly accessible.

Finally, we will relax the assumption on the amount of trust/distrust being constant and we will study the reaching of a relaxed meaning of consensus, by requiring consensus just over the signs of the opinions, thus leading to the notion of *sign consensus*. The results presented in this chapter can be found in:

- G. De Pasquale, M. E. Valcher, "Consensus for clusters of agents with cooperative and antagonistic relationships", Automatica, Vol. 135, pp. 1-9, 2022, https://doi.org/10.1016/j.automatica.2021.110002.
- G. De Pasquale, M. E. Valcher, "Consensus problems on clustered networks" Proc. of the 59th IEEE Conf. Decision and Control, 2020, pp. 3675-3680, Jeju Island, Republic of Korea, 10.1109/CDC42340.2020.9303877;
- G. De Pasquale, M. E. Valcher, "Tripartite and Sign Consensus for Clustering Balanced Social Networks" Proc. of the American Control Conference (ACC) 2021, pp. 3056-3061, New Orleans,LA, USA, 10.23919/ACC50511.2021.9483355.

3.1 Introduction

Social networks provide clear evidence that mutual relationships may not always be cooperative, and yet the dynamics of opinion forming may exhibit stable asymptotic patterns. In particular, Altafini (2013) has shown that in a multi-agent system with cooperative and antagonistic relationships, *bipartite consensus*, namely the splitting of the agents' opinions into two groups that asymptotically converge to two opposite values, is possible. This is the case if the communication network is *structurally balanced*, namely agents split into two groups such that intra-group relationships are cooperative and inter-group relationships are antagonistic, and agents update their opinions based on DeGroot's control law DeGroot (1974). This analysis has been later extended from the case of simple integrator to the case of homogeneous agents described by an arbitrary state-space model Valcher & Misra (2014) (see, also, Bauso et al. (2009), Easley & Kleinberg (2010)), and has been in turn investigated by several other authors under different working conditions. See for example Shang (2016, 2014), Han et al. (2013), Monaco & Celsi (2019). In Shang (2016) cluster consensus in a discrete time, stochastic setting is addressed. The author provides a combinatorial necessary and sufficient condition for a compact set of stochastic matrices to be a cluster consensus set. In Shang (2014) group consensus for multi-agent systems with information flow is addressed under pinning control in both fixed topology and randomly switching topology driven by a continuous-time homogeneous Markov process. In Y. Han & Chen (2015), the authors study consensus cluster of continuous time multi-agent systems with time-varying topologies to which non-identical inter-cluster inputs are applies. Similarly to what we address in this chapter, also in this case consensus is related both to the intra-cluster synchronization and inter-cluster separation. The first one is ensured through structural constraints on the network, the latter one is insured by imposing adaptive inputs that are identical within the same cluster and different between different clusters. Also, in Monaco & Celsi (2019) the authors deal with the concept of *multi-consensus* in a multi-agent setting. The distinct achieved consensuses are as many as the number of groups of agents. It is shown that multi-consensus is achieved when the underlying digraph admits a suitable almost equitable partition.

In social contexts, structural balance characterizes networks whose individuals adhere to the four Heider's rules Heider (1944): 1) the friend of my friend is my friend, 2) the enemy of my friend is my enemy, 3) the friend of my enemy is my enemy, 4) the enemy of my enemy is my friend. On the other hand, the case may occur that agents apply all Heider's rules but the fourth one, and the community splits into $k \geq 3$ groups such that intra-group relationships are cooperative and inter-group relationships are antagonistic, thus giving rise to a *weakly balanced* network Wasserman & Faust (1994). In this scenario the Altafini model does not guarantee any consensus configuration, except for the trivial one in which all the states of the agents converge to zero Altafini (2013). Therefore, it is interesting to understand if it is possible to modify DeGroot's algorithm, so that non trivial equilibrium configurations may be obtained, where all agents in the same cluster converge to the same decision. In other words, we are interested in a form of "group consensus" that represents the natural generalisation of bipartite consensus to $k \geq 3$ clusters and guarantees that the splitting of the final decisions in groups is representative of the group partitioning of the underlying communication graph. In this regard it is worth mentioning the pioneering work from Hegselmann & Krause (2002) in which a clustering of opinions is obtained in a network in which all agents are connected but only agents whose opinions is close enough, communicate. In this chapter, we will show that under suitable hypotheses a different type of group consensus, associated with a pre-fixed network structure, is possible, by introducing a minor modification of DeGroot's law, that requires agents in the same cluster to adjust the coefficients that weight their own opinions, while leaving unchanged the weights given to all the other individuals' opinions.

By assuming that interactions between agents in the same clusters are cooperative, while interactions between agents belonging to different clusters may only be antagonistic, we aim to achieve a group consensus where all individuals that cooperate (and hence necessarily belong to the same cluster) converge to the same decision/opinion. This problem set-up seems more suitable to formalise consensus problems arising in the economical, biological, sociological fields (see, e.g., Easley & Kleinberg (2010), Wasserman & Faust (1994)). Sociological models were, in fact, the primary motivation behind the set-up adopted in Altafini (2013).

Our study is first developed under the *homogeneity condition* that requires that each agent in a group distributes the same amount of "trust" to the agents in its own group and "distrust" to the agents belonging to adverse groups. This is equivalent to saying that given two arbitrary (not necessarily distinct) classes, say i and j, the sum of the weights of the incoming edges from all the agents of class j to an agent of class i depends on i and j, and not on the specific agent.
More in detail, we assume that the communication graph between agents is modeled by an undirected, signed, weighted, connected graph, and that the agents are partitioned into k clusters, such that intra-cluster interactions may only be nonnegative, while inter-cluster interactions can only be nonpositive. We investigate under what conditions a revised version of the DeGroot's distributed feedback control law that only requires to modify the weight that each agent belonging to the same class has to give to its own opinion, can lead the multi-agent system to *k-partite consensus*.

It is worthwhile remarking that the design of these coefficients cannot be obtained in a fully distributed way, since the algorithm we propose requires that each cluster is aware of the choices made by the clusters preceding it, with respect to some suitable ordering. However, once the parameters have been chosen the control algorithm is completely distributed.

The homogeneity conditions will be relaxed in the last part of this chapter in wich a weaker notion of consensus, known in the literature as *sign consensus*, will be studied.

3.2 k-Partite Consensus: Problem Statement

We consider a multi-agent system consisting of N agents, each of them described as a continuous-time integrator Altafini (2013), Olfati-Saber et al. (2007), Olfati-Saber & Murray (2004), Ren et al. (2007). The overall system dynamics is described as

$$\dot{\mathbf{x}}(t) = \mathbf{u}(t), \tag{3.1}$$

where $\mathbf{x} \in \mathbb{R}^N$ and $\mathbf{u} \in \mathbb{R}^N$ are the state and input variables, respectively.

Assumption 1 on the communication structure. [Connectedness and clustering] The communication among the N agents is described by an undirected, signed and weighted communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, with $\mathcal{V} = [1, N]$. The entries $[\mathcal{A}]_{i,j} = [\mathcal{A}]_{j,i}$, $i \neq j$, of $\mathcal{A}$ are nonzero if and only if the i-th and the j-th agents have a direct relationship, which is cooperative if $[\mathcal{A}]_{i,j} > 0$ and antagonistic if $[\mathcal{A}]_{i,j} < 0$. We also assume that the graph $\mathcal{G}$ is connected and *all the agents are grouped in* $k \geq 3$ *clusters*, $\mathcal{V}_i, i \in [1, k]$, *with* $n_i = |\mathcal{V}_i|$.

The aim of this work is to propose an extension to the case of k clusters of the results reported in Altafini (2013) for *structurally balanced graphs*, namely graphs with two clusters, by proposing conditions under which agents in the same cluster $\mathcal{V}_i, i \in [1, k]$, reach *consensus*. In other words, we investigate conditions ensuring that the state variables of the agents belonging to the same cluster asymptotically converge to the same value:

$$\lim_{t \to +\infty} x_k(t) = \gamma_i, \qquad \forall\ k \in \mathcal{V}_i,\ \forall\ i \in [1, k].$$

When dealing with multi-agent systems with cooperative and antagonistic relationships, one can use the DeGroot's type distributed feedback control law Altafini (2013), Ren et al. (2007):

$$u_i(t) = -\sum_{j:(j,i)\in\mathcal{E}} |[\mathcal{A}]_{i,j}| \cdot [x_i(t) - \text{sign}([\mathcal{A}]_{i,j}) x_j(t)],$$

$i \in [1, N]$, with $\text{sign}(\cdot)$ as the sign function, that corresponds, in aggregated form, to

$$\mathbf{u}(t) = -\mathcal{L}\mathbf{x}(t),$$

where $\mathcal{L}$ is the *Laplacian matrix* associated with the adjacency matrix $\mathcal{A}$, defined as Altafini (2013), Hou et al. (2003) $\mathcal{L} := \mathcal{C} - \mathcal{A}$, where $\mathcal{C}$ is the (diagonal) connectivity matrix, with diagonal entries $[\mathcal{C}]_{ii} = \sum_{h:(h,i)\in\mathcal{E}} |[\mathcal{A}]_{ih}|, \forall i \in [1, N]$. In other words

$$[\mathcal{L}]_{ij} = \begin{cases} \sum_{h:(h,i)\in\mathcal{E}} |[\mathcal{A}]_{i,h}|, & \text{if } i = j; \\ -[\mathcal{A}]_{i,j}, & \text{if } i \neq j. \end{cases}$$

As shown in Altafini (2013), however, this control law leads to an autonomous multi-agent system $\dot{\mathbf{x}}(t) = -\mathcal{L}\mathbf{x}(t)$, that may achieve a nontrivial consensus only if the underlying communication graph is structurally balanced. This immediately implies that if the agents can be partitioned into $k \geq 3$ clusters, but not into a smaller number of clusters, then the only possible consensus is the one to the zero value. So, in the following we investigate how to modify the distributed control law (3.2), to achieve consensus when the communication graph is connected and signed, but the agents split into $k \geq 3$ disjoint groups.
For the sake of simplicity, we will assume that the agents are ordered in such a way that the agents belonging to the cluster $\mathcal{V}_1$ are the first n_1, the agents in the cluster $\mathcal{V}_2$ are the subsequent n_2, ... and the agents in the cluster $\mathcal{V}_k$ are the last n_k. Clearly, $n_1 + n_2 + \cdots + n_k = N$. This assumption entails no loss of generality, since it is always possible to reduce ourselves to this structure by means of a relabelling of the nodes/agents. Accordingly, the adjacency matrix of the graph $\mathcal{G}$ is block-partitioned as follows

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{1,1} & \mathcal{A}_{1,2} & \dots & \mathcal{A}_{1,k} \\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2} & \dots & \mathcal{A}_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{k,1} & \mathcal{A}_{k,2} & \dots & \mathcal{A}_{k,k} \end{bmatrix} \tag{3.2}$$

with $\mathcal{A}_{i,j} \in \mathbb{R}^{n_i \times n_j}$, $\mathcal{A}_{i,i} = \mathcal{A}_{i,i}^\top \geq 0$, $\forall i \in [1, k]$, $\mathcal{A}_{i,j} \leq 0$ $\forall i \neq j$, $i, j \in [1, k]$, $[\mathcal{A}_{i,i}]_{\ell,\ell} = 0$, $\forall i \in [1, k], \ell \in [1, n_i]$. We consider a distributed control law for the system (4.3.1) of the type

$$\mathbf{u} = -\mathcal{L}_{\mathcal{D}}\mathbf{x}, \tag{3.3}$$

where $\mathcal{L}_{\mathcal{D}} \in \mathbb{R}^{N\times N}$ takes the form

$$\mathcal{L}_{\mathcal{D}} = \mathcal{D} - \mathcal{A}, \tag{3.4}$$

with $\mathcal{A}$ the adjacency matrix of $\mathcal{G}$ and $\mathcal{D} \in \mathbb{R}^{N\times N}$ ia diagonal matrix that can be partitioned according to the block-partition of $\mathcal{A}$, namely

$$\mathcal{D} = \text{diag}\{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_k\}, \quad \mathcal{D}_i \in \mathbb{R}^{n_i\times n_i},$$

where $\mathcal{D}_i = \delta_i I_{n_i}$, n_i being the cardinality of the i-th cluster, is a scalar matrix. The overall multi-agent system is hence described as

$$\dot{\mathbf{x}}(t) = -\mathcal{L}_{\mathcal{D}}\mathbf{x}(t), \tag{3.5}$$

and we aim to investigate if it is possible to choose the parameters δ_i so that all the agents reach *k-partite consensus*, by this meaning that for every initial condition $\mathbf{x}(0) \in \mathbb{R}^N$ (except for a set of zero measure in $\mathbb{R}^N$) all the state variables, associated to agents in the same cluster, converge to the same value, namely

$$\lim_{t\to+\infty} \mathbf{x}(t) = [\gamma_1\mathbf{1}_{n_1}^\top, \gamma_2\mathbf{1}_{n_2}^\top, \dots, \gamma_n\mathbf{1}_{n_k}^\top]^\top, \tag{3.6}$$

for suitable $\gamma_i = \gamma_i(\mathbf{x}(0)) \in \mathbb{R}, i \in [1, k]$, not all of them equal to zero.
The diagonal entries $\delta_i, i \in [1, k]$, of the matrix $\mathcal{D}$ are henceforth our design parameters. Each δ_i can be seen as the degree of "self-confidence" of the agents of the i-th cluster. It quantifies how much the individuals in the cluster $\mathcal{V}_i$ are conservative about their own opinions. As it will be clarified in the following, the proposed control scheme is not fully distributed, since the agents will not be able to autonomously decide the level of self-confidence they have to adopt in order to guarantee that the final target is achieved. However the proposed modification of the standard control law is minimal, since it only requires the agents to modify the weight that each of them gives to its own opinion. Note that once the diagonal entries of $\mathcal{D}$ have been set, the control algorithm is implemented in a purely distributed way.

3.3 k-Partite Consensus: Preliminary Results

In order to provide a solution to the k-partite consensus problem under certain assumptions on the communication graph, we first present a simple lemma that provides necessary and sufficient conditions for k-partite consensus. The result is elementary and extends the analogous result for consensus of cooperative multi-agent systems.

Lemma 3.1. *A multi-agent system (4.3.1), whose communication graph $\mathcal{G}$ satisfies Assumption 1, adopting the distributed control law (3.3), and hence described as in (3.5), with $\mathcal{L}_{\mathcal{D}} \in \mathbb{R}^{N\times N}$ as in (3.4), $\mathcal{A}$ as in (3.2), $\mathcal{D} = \mathrm{diag}\{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_k\} \in \mathbb{R}^{N\times N}$ and $\mathcal{D}_i \in \mathbb{R}^{n_i\times n_i}$, $i \in [1, k]$, scalar matrices, reaches k-partite consensus if and only if the following conditions hold:*

(C.1) $\mathcal{L}_{\mathcal{D}}$ *is a singular positive semidefinite matrix.*

(C.2) The kernel of $\mathcal{L}_{\mathcal{D}}$ *is spanned by vectors of the type* $\mathbf{z} = [\alpha_1 \mathbf{1}_{n_1}^\top, \dots, \alpha_k \mathbf{1}_{n_k}^\top]^\top, \alpha_i \in \mathbb{R}, i \in [1, k]$.

Sufficiency. If $\mathcal{L}_{\mathcal{D}}$ is a singular positive semidefinite matrix, then the system $\dot{\mathbf{x}} = -\mathcal{L}_{\mathcal{D}}\mathbf{x}$ is stable (but not asymptotically stable), and for every $\mathbf{x}(0) \in \mathbb{R}^N$

$$\lim_{t \to +\infty} \mathbf{x}(t) = \sum_{i=1}^{m} p_i \mathbf{v}_i, \tag{3.7}$$

where m is the dimension of the eigenspace associated with the (dominant) zero eigenvalue, $p_i \in \mathbb{R}$ are coefficients that depend on the initial conditions and $\mathbf{v}_i \in \mathbb{R}^N$ are the eigenvectors associated with the zero eigenvalue. By condition (C.2), each $\mathbf{v}_i$ is block-partitioned in k blocks, conformably with the clusters' dimensions, and hence $\sum_{i=1}^m p_i \mathbf{v}_i$ takes the form $[\gamma_1 \mathbf{1}_{n_1}^\top, \dots, \gamma_k \mathbf{1}_{n_k}^\top]^\top$.

[Necessity] If condition (3.6) holds for (almost) every $\mathbf{x}(0)$, then 0 must be the dominant eigenvalue of the matrix $-\mathcal{L}_{\mathcal{D}}$, and hence, being a symmetric matrix, it follows that $\mathcal{L}_{\mathcal{D}}$ is (singular and) positive semidefinite. Moreover, as condition (3.6) has to hold for every $\mathbf{x}(0)$ that is an eigenvector of $-\mathcal{L}_{\mathcal{D}}$ corresponding to 0, this implies condition (C.2). □

We now introduce some additional assumptions on the communication graph that will be used in the following analysis, and comment on their meaning.

Assumption 2 on the communication structure. [Homogeneity of trust/mistrust] *All the agents in a class* $\mathcal{V}_i$ *have the same constant and pre-fixed amount of trust to be distributed among their cooperators and distrust, specific for each class* $\mathcal{V}_j, j \neq i$, *to be distributed among the agents in antagonistic classes. This translates into assuming that the sums of the elements of the rows belonging to the same block assume the same value, namely for every* $i, j \in [1, k]$, $\mathcal{A}_{i,j}\mathbf{1}_{n_j} = c_{ij}\mathbf{1}_{n_i}$, *where* $c_{ii} \geq 0$ *and* $c_{ij} \leq 0$, $\forall i \neq j$.
Note that even if the adjacency matrix is symmetric, c_{ij} *may differ from* c_{ji}.

Example 3.2. *Consider the undirected, signed, unweighted, connected and clustered communication graph, with* $k = 3$ *clusters of cardinality* $n_1 = 2$, $n_2 = 4$, $n_3 = 1$, *and adjacency* *matrix*

$$\mathcal{A} = \left[\begin{array}{cc|cccc|c} 0 & 1 & -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ \hline -1 & 0 & 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ \hline -1 & -1 & -1 & -1 & -1 & -1 & 0 \end{array}\right]$$

It is easy to see that this graph satisfies both Assumption 1 and Assumption 2, and the parameters c_{ij} *are* $c_{11} = 1, c_{12} = -2, c_{13} = -1, c_{21} = -1, c_{22} = 2, c_{23} = -1, c_{31} = -2, c_{32} = -4, c_{33} = 0$.

Remark 3.3. *Assumption 2 may be regarded as a generalization of the concept of* equitable partition, *originally introduced in Egerstedt et al. (2012) for undirected, unweighted and unsigned graphs. In an equitably partitioned (unweighted, unsigned and undirected) graph, in fact, all the agents in the same cluster are restricted to have the same number of neighbours in every cluster, i.e.* $\mathcal{A}_{i,j}\mathbf{1}_{n_j} = c_{ij}\mathbf{1}_{n_i}, \forall\ i,j \in [1,k]$, *and each* c_{ij} *is a nonnegative integer number, representing the number of unitary entries in each row of* $\mathcal{A}_{i,j}$. *Moreover, this assumption is similar to the one introduced in the first part of Xia & Cao (2011) dealing with cooperative multi-agent systems, where it was assumed that the blocks* $\mathcal{A}_{i,j}, i \neq j$, *have constant (and nonnegative) row sums.*

Assumption 3 on the communication structure. [Close friendship] *There exist* $k-1$ *distinct indices* $i_1, i_2, \dots, i_{k-1} \in [1,k]$ *such that every cluster* $\mathcal{V}_h, h \in \{i_2, \dots, i_{k-1}\}$, *either consists of a single node/agent or for every choice of two distinct agents* $i, j \in \mathcal{V}_h$ *either one of the following cases applies:*

i) i *and* j *are* friends *(the edge* (i,j) *belongs to* $\mathcal{E}$ *and it has positive weight);*

ii) i *and* j *are* enemies *of two (not necessarily distinct) vertices in* $\mathcal{V}_{i_1}$ *that are* familiar *to each other. This means that there exist* $r, s \in \mathcal{V}_{i_1}$, *and belonging to the same connected component in* $\mathcal{V}_{i_1}$, *such that the edges* (r,i) *and* (j,s) *belong to* $\mathcal{E}$ *(and have negative weights).*

It is worthwhile to better illustrate this graph property. Conditions i) and ii) amount to saying that either the vertices i and j of $\mathcal{V}_h$ are connected by an edge or there is a path connecting them whose intermediate vertices are all in $\mathcal{V}_{i_1}$. Figure 3.1 provides a graphical representation of this property. The property holds for $\mathcal{V}_{i_h}$ and $\mathcal{V}_{i_{k-1}}$, but not for $\mathcal{V}_{i_k}$, the remaining set.

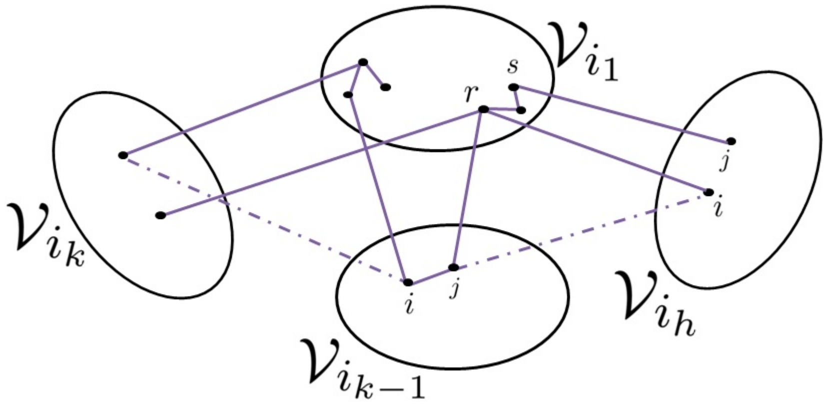

Figure 3.1: Graphical representation of Assumption 3.

The idea behind this assumption is that if two agents belong to the same clusters $\mathcal{V}_h, h \in \{i_2, i_3, \ldots, i_{k-1}\}$, they have a close relationship: they are either friends or they are enemies of agents belonging to the same group of friends in $\mathcal{V}_{i_1}$.

From an algebraic point of view, Assumption 3 states that for every $h \in \{i_2, i_3, \ldots, i_{k-1}\}$ and for every $i, j \in \mathcal{V}_h, i \neq j$, either $[\mathcal{A}_{h,h}]_{i,j} > 0$ or there exists $t \in \mathbb{Z}_+$ such that $[\mathcal{A}_{h,i_1}\mathcal{A}^t_{i_1,i_1}\mathcal{A}_{i_1,h}]_{i,j} > 0$. As a consequence, for every scalar matrix $\mathcal{D}_{i_1}$ such that $\mathcal{D}_{i_1} - \mathcal{A}_{i_1,i_1}$ is positive definite (see Lemma 2.5 in Section 2.3.1), and hence $(\mathcal{D}_{i_1} - \mathcal{A}_{i_1,i_1})^{-1} \geq 0$, we have that

$$[\mathcal{A}_{h,h} + \mathcal{A}_{h,i_1}(\mathcal{D}_{i_1} - \mathcal{A}_{i_1,i_1})^{-1}\mathcal{A}_{i_1,h}]_{i,j} > 0, \ \forall\, i \neq j.$$

By referring to the previous Example 3.2, it is easy to see that Assumption 3 trivially holds for every choice of $i_1, i_2 \in [1, 3], i_1 \neq i_2$. Note that $\mathcal{V}_3$ consists of a single node, while $\mathcal{V}_1$ and $\mathcal{V}_2$ consist of a single connected component.

3.4 k-Partite Consensus: Problem Solution Under the Homogeneity Constraint

We are now in a position to prove that under the homogeneity constraint imposed by Assumption 2 and the close friendship hypothesis formalised in Assumption 3, we can always find suitable choices of the scalar matrices $\mathcal{D}_i = \delta_i I_{n_i}, i \in [1, k]$, that lead the multi-agent system to k-partite consensus.

Theorem 3.4. *Consider the multi-agent system (4.3.1), with communication graph $\mathcal{G}$ satisfying Assumptions 1, 2 and 3. There exist $\delta_i \in \mathbb{R}, i \in [1, k]$, such that the closed-loop multi-agent system (3.5) reaches k-partite consensus, (i.e., (3.6) holds for suitable $\gamma_i = \gamma_i(\mathbf{x}(0)) \in \mathbb{R}, i \in [1, k]$), under the distributed control law (3.3), with $\mathcal{L}_\mathcal{D} \in \mathbb{R}^{N\times N}$ described as in (3.4), $\mathcal{A}$ as in (3.2), $\mathcal{D} = \text{diag}\{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_k\} \in \mathbb{R}^{N\times N}$ and $\mathcal{D}_i = \delta_i I_{n_i}$, $i \in [1, k]$.*

Proof. We assume without loss of generality that Assumption 3 holds for $i_1 = 1$ and $i_h = h + 1$ for $h = 2, 3, \dots, k - 1$. In fact, we can always relabel the clusters, and accordingly permute the blocks of $\mathcal{A}$, so that this condition is satisfied.

By Lemma 3.1, we need to prove that under the theorem assumptions it is always possible to choose the real parameters $\delta_1, \delta_2, \dots, \delta_k$ so that (C.1) the matrix $\mathcal{L}_\mathcal{D}$ is singular and positive semidefinite, and (C.2) its kernel is spanned by vectors taking the form $\mathbf{z} = [\alpha_1 \mathbf{1}_{n_1}^\top, \alpha_2 \mathbf{1}_{n_2}^\top, \dots, \alpha_k \mathbf{1}_{n_k}^\top]^\top, \alpha_i \in \mathbb{R}, i \in [1, k]$.

Condition (C.1). To impose that $\mathcal{L}_\mathcal{D}$ is singular and positive semidefinite, we make use of the result in Section 2.3.1. Specifically, we first set $\mathcal{H}_1 := \mathcal{L}_\mathcal{D}$ and block-partition it as follows

$$\mathcal{H}_1 := \mathcal{L}_\mathcal{D} = \left[\begin{array}{c|ccc} \mathcal{D}_1 - \mathcal{A}_{1,1} & -\mathcal{A}_{1,2} & \dots & -\mathcal{A}_{1,k} \\ \hline -\mathcal{A}_{2,1} & \mathcal{D}_2 - \mathcal{A}_{2,2} & \dots & -\mathcal{A}_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathcal{A}_{k,1} & -\mathcal{A}_{k,2} & \dots & \mathcal{D}_k - \mathcal{A}_{k,k} \end{array}\right] = \left[\begin{array}{c|c} \Phi_1 & S_1 \\ \hline S_1^\top & Q_1 \end{array}\right].$$

We impose that $\Phi_1 \in \mathbb{R}^{n_1 \times n_1}$ is positive definite and its Schur complement in $\mathcal{H}_1$, i.e., $\mathcal{H}_2 := Q_1 - S_1^\top \Phi_1^{-1} S_1$, is positive semidefinite and singular. This means that condition

(3.4) holds:

$$\Phi_1 := \mathcal{D}_1 - \mathcal{A}_{1,1} = \delta_1 I_{n_1} - \mathcal{A}_{1,1} \succ 0,$$

and the matrix $\mathcal{H}_2$ in (3.8) is positive semidefinite and singular.

$$\mathcal{H}_2 = \begin{bmatrix} \Phi_2 & S_2 \\ S_2^\top & Q_2 \end{bmatrix} \tag{3.8}$$

with

$$\begin{aligned}
\Phi_2 &= \left[\mathcal{D}_2 - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}\right], \\
S_2 &= \left[-\mathcal{A}_{2,3} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3} \cdots - \mathcal{A}_{2,k} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,k}\right], \\
Q_2 &= \begin{bmatrix} \mathcal{D}_3 - \mathcal{A}_{3,3} - \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3} & \dots & -\mathcal{A}_{3,k} - \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,k} \\ \vdots & \ddots & \vdots \\ -\mathcal{A}_{k,3} - \mathcal{A}_{k,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3} & \dots & \mathcal{D}_k - \mathcal{A}_{k,k} - \mathcal{A}_{k,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,k} \end{bmatrix}.
\end{aligned}$$

We note that if we assume $\delta_1 > c_{11} \geq 0$, then $\Phi_1 \mathbf{1}_{n_1} = (\delta_1 I_{n_1} - \mathcal{A}_{1,1})\mathbf{1}_{n_1} > 0$. By making use of Lemma 2.5, part i), in the Appendix for $D = \delta_1 I_{n_1}$, $A = \mathcal{A}_{1,1}$ and $\mathbf{z} = \mathbf{1}_{n_1}$, we can claim that $\Phi_1 = D - A$ is positive definite, i.e., (3.4) holds.
To ensure that $\mathcal{H}_2 \succeq 0$ (and is singular), we apply again the result from Section 2.3.1, and impose that its first block Φ_2 is positive definite, namely condition (3.4) holds:

$$\Phi_2 := \mathcal{D}_2 - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}\Phi_1^{-1}\mathcal{A}_{1,2} \succ 0,$$

while its Schur complement $\mathcal{H}_3$ (see (3.9)) is positive semidefinite and singular.

To address condition (3.4), we first observe that by Lemma 2.5, part ii), $\Phi_1^{-1} = (\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}$ is symmetric and nonnegative, and hence so is $\mathcal{A}_{2,2} + \mathcal{A}_{2,1}\Phi_1^{-1}\mathcal{A}_{1,2}$. But then we can apply Lemma 2.5, part i), again, by assuming $D = \mathcal{D}_2$ and $A = \mathcal{A}_{2,2} + \mathcal{A}_{2,1}\Phi_1^{-1}\mathcal{A}_{1,2}$. Indeed, if we impose the following constraint on δ_2:

$$\delta_2 > c_{22} + \frac{c_{12}c_{21}}{\delta_1 - c_{11}}, \tag{3.10}$$

then it is easy to verify that

$$\begin{aligned}
\Phi_2 \mathbf{1}_{n_2} &= (D - A)\mathbf{1}_{n_2} = (\delta_2 - c_{22})\mathbf{1}_{n_2} - \mathcal{A}_{2,1}\Phi_1^{-1}c_{12}\mathbf{1}_{n_1} \\
&= (\delta_2 - c_{22})\mathbf{1}_{n_2} - c_{21}(\delta_1 - c_{11})^{-1}c_{12}\mathbf{1}_{n_1} > 0,
\end{aligned}$$

$$\begin{aligned}\mathcal{H}_3 \;&:=\; \begin{bmatrix} \mathcal{D}_3-\mathcal{A}_{3,3}-\mathcal{A}_{3,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3} & \dots & -\mathcal{A}_{3,k}-\mathcal{A}_{3,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,k} \\ \vdots & \ddots & \vdots \\ -\mathcal{A}_{3,k}^\top-\mathcal{A}_{k,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3} & \dots & \mathcal{D}_k-\mathcal{A}_{k,k}-\mathcal{A}_{k,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,k}\end{bmatrix} \\ &-\begin{bmatrix} -\mathcal{A}_{3,2}-\mathcal{A}_{3,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2} \\ \vdots \\ -\mathcal{A}_{k,2}-\mathcal{A}_{k,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}\end{bmatrix}\Phi_2^{-1}\cdot \\ &\cdot\begin{bmatrix} -\mathcal{A}_{2,3}-\mathcal{A}_{2,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3} & \dots & -\mathcal{A}_{2,k}-\mathcal{A}_{2,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,k}\end{bmatrix} = \begin{bmatrix}\Phi_3 & S_3 \\ S_3^\top & Q_3\end{bmatrix}\end{aligned} \tag{3.9}$$

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where we used the fact that $\Phi_1^{-1}\mathbf{1}_{n_1} = (\mathcal{D}_1-\mathcal{A}_{11})^{-1}\mathbf{1}_{n_1} = (\delta_1-c_{11})^{-1}\mathbf{1}_{n_1}$. Therefore $D-A$ is positive definite, namely (3.4) holds.
Consider, now, the first block of $\mathcal{H}_3$ in (3.9):

$$\Phi_3 \;:=\; \mathcal{D}_3-\mathcal{A}_{3,3}-\mathcal{A}_{3,1}\Phi_1^{-1}\mathcal{A}_{1,3}-[\mathcal{A}_{3,2}+\mathcal{A}_{3,1}\Phi_1^{-1}\mathcal{A}_{1,2}]\cdot\Phi_2^{-1}[\mathcal{A}_{2,3}+\mathcal{A}_{2,1}\Phi_1^{-1}\mathcal{A}_{1,3}].$$

We want to prove that for a suitable choice of δ_3 we can ensure that Φ_3 is positive definite and impose that its Schur complement is positive semidefinite and singular. We observe that from Assumption 3 (see also (3.3)) and the properties of $(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}$ it follows that $\mathcal{A}_{\ell,\ell}+\mathcal{A}_{\ell,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,\ell}, \ell\in[3,k]$, is a nonnegative matrix whose off-diagonal entries are all positive. On the other hand, by Lemma A.1 we can always choose $\delta_2>0$ sufficiently large (something that ensures, in particular, that (3.10) is met) to guarantee that the entries of Φ_2^{-1} are arbitrarily small, and hence the entries of $[\mathcal{A}_{\ell,2}+\mathcal{A}_{\ell,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}]\Phi_2^{-1}[\mathcal{A}_{2,\ell}+\mathcal{A}_{2,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,\ell}], \ell\in[3,k]$, are arbitrarily small. This ensures, in particular, that the matrix $A=-\Phi_3+\mathcal{D}_3\approx\mathcal{A}_{3,3}+\mathcal{A}_{3,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}$ has positive off-diagonal entries, and hence $-\Phi_3$ is an irreducible Metzler matrix.

If we now choose δ_3 such that

$$\delta_3 > c_{33}+\frac{c_{31}c_{13}}{\delta_1-c_{11}}+\left(c_{32}+\frac{c_{31}c_{12}}{\delta_1-c_{11}}\right)\cdot\left(\delta_2-c_{22}-\frac{c_{21}c_{12}}{\delta_1-c_{11}}\right)^{-1}\left(c_{23}+\frac{c_{21}c_{13}}{\delta_1-c_{11}}\right) \tag{3.11}$$

we ensure that Φ_3 satisfies $\Phi_3\mathbf{1}_{n_3}>0$. This proves that Φ_3 is positive definite.
The previous reasoning, that we have commented in detail for the first three steps, namely for the matrices $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, (and that is based on result from Section 2.3.1 and Lemma 2.5 in Appendix A), needs to be applied k times, as many as the number of clusters in the

communication graph. The procedure is described in detail in Algorithm 1 and it consists of recursively imposing, for $h = 1, 2, \ldots, k-1$, that Φ_h, the first block of $\mathcal{H}_h$, is positive definite and the opposite of a Metzler matrix, while its Schur complement $\mathcal{H}_{h+1}$, is positive semidefinite and singular. Finally, Φ_k is positive semidefinite and singular, with a single eigenvalue in 0. In detail, the algorithm is initialised by identifying $\mathcal{H}_1$ with the original matrix $\mathcal{L}_{\mathcal{D}}$, each matrix $\mathcal{L}_{\mathcal{D}i,j}^{(0)}$ with the (i,j)-th block of $\mathcal{A}$ and each coefficient $m_{i,j}^{(0)}$ with the row sum $c_{i,j}$ (of $\mathcal{A}_{i,j}$). At each step h, ranging from 1 to $k-1$, we choose the parameter δ_h large enough so that

- The Metzler matrix $-\Phi_h = -\mathcal{D}_h + \mathcal{L}_{\mathcal{D}h,h}^{(h-1)} = -\delta_h I_{n_h} + \mathcal{L}_{\mathcal{D}h,h}^{(h-1)}$ satisfies $-\Phi_h \mathbf{1}_{n_h} < 0$, something that requires $\delta_h \mathbf{1}_{n_h} > \mathcal{L}_{\mathcal{D}h,h}^{(h-1)} \mathbf{1}_{n_h}$ (i.e., $\delta_h > m_{h,h}^{(h-1)}$), and that ensures that the Metzler matrix $-\Phi_h$ is Hurwitz (equivalently Φ_h is (symmetric and) positive definite).

- While $-\Phi_1$ and $-\Phi_2$ are Metzler for every choice of δ_1 and δ_2, in order to ensure that $-\Phi_h, h \in [3,k]$, are all Metzler matrices, we exploit Assumption 3 that ensures that $\mathcal{L}_{\mathcal{D}h,h}^{(1)}, h \in [3,k]$, are all Metzler matrices with strictly positive off-diagonal entries. By making use of Lemma A.1, it is immediate to show that if $\mathcal{L}_{\mathcal{D}h,h}^{(i)}$ has strictly positive off-diagonal entries, then also $\mathcal{L}_{\mathcal{D}h,h}^{(i+1)}$ has strictly positive off-diagonal entries. This allows to say that all the matrices $\mathcal{L}_{\mathcal{D}h+1,h+1}^{(h)}, h \in [2,k]$, (have strictly positive off-diagonal entries and hence) are Metlzer. This, in turn, allows to say that all matrices $-\Phi_h, h \in [3,k]$, are (irreducible) Metzler matrices.

Once we have chosen δ_h and hence uniquely identified Φ_h, we first update the matrices $\mathcal{L}_{\mathcal{D}i,j}^{(h)}$ and then define $\mathcal{H}_{h+1}$, the Schur complement of Φ_h in $\mathcal{H}_h$, that turns out to be expressed in terms of the diagonal blocks $\mathcal{D}_{h+1}, \ldots, \mathcal{D}_k$ and of the matrices $\mathcal{L}_{\mathcal{D}i,j}^{(h)}$ as described in the algorithm.
So, by proceeding in this way, we construct all positive definite matrices $\Phi_1, \ldots, \Phi_{k-1}$ and at the last step we choose $\delta_k > 0$ so that $-\Phi_k \mathbf{1}_{n_k} = 0$. Being $-\Phi_k$ an irreducible Metzler matrix, this ensures (see Theorem 2.3) that 0 is a simple dominant eigenvalue of $-\Phi_k$, and therefore Φ_k is positive semidefinite and singular, with a simple eigenvalue in 0.

Since the spectrum of $\mathcal{L}_{\mathcal{D}}$ is the union of the spectra of the positive definite matrices $\Phi_h, h \in [1, k-1]$, and of the positive semidefinite and singular matrix Φ_k, then $\mathcal{L}_{\mathcal{D}}$ is positive semidefinite with a simple eigenvalue in 0.

Algorithm 1 Selection of the $\delta_h, h = 1, 2, \ldots, k$.

for $i, j \in [1, k]$ **do** ▷ Initialization

$\mathcal{L}_{\mathcal{D}}{}_{i,j}^{(0)} := \mathcal{A}_{i,j}$

$m_{i,j}^{(0)} := c_{i,j}$

$\mathcal{H}_1 := \mathcal{L}_{\mathcal{D}}$

for $h \in [1, k-1]$ **do** ▷ Recursive Step

Choose $\delta_h > 0$ so that

$\delta_h > m_{h,h}^{(h-1)}$ and

if $h > 1$ **then**

$\forall\ \ell \in [h+1, k], \forall\ i \neq j$

$[\mathcal{L}_{\mathcal{D}}{}_{\ell,\ell}^{(h-1)} + \mathcal{L}_{\mathcal{D}}{}_{\ell,h}^{(h-1)}[\delta_h I_{n_k} - \mathcal{L}_{\mathcal{D}}{}_{h,h}^{(h-1)}]^{-1}\mathcal{L}_{\mathcal{D}}{}_{h,\ell}^{(h-1)}]_{ij} > 0$

Set

$\mathcal{D}_h := \delta_h I_{n_h}$

$\Phi_h := \mathcal{D}_h - \mathcal{L}_{\mathcal{D}}{}_{h,h}^{(h-1)}$

$\phi_h := \delta_h - m_{h,h}^{(h-1)}$

$\mathcal{L}_{\mathcal{D}}{}_{i,j}^{(h)} := \mathcal{L}_{\mathcal{D}}{}_{i,j}^{(h-1)} + \mathcal{L}_{\mathcal{D}}{}_{i,h}^{(h-1)}\Phi_h^{-1}\mathcal{L}_{\mathcal{D}}{}_{h,j}^{(h-1)}$

$m_{i,j}^{(h)} := m_{i,j}^{(h-1)} + m_{i,h}^{(h-1)}\phi_h^{-1}m_{h,j}^{(h-1)} \quad \forall i, j$

$$\mathcal{H}_{h+1} := \begin{bmatrix} \mathcal{D}_{h+1} - \mathcal{L}_{\mathcal{D}}{}_{h+1,h+1}^{(h)} & \ldots & -\mathcal{L}_{\mathcal{D}}{}_{h+1,k}^{(h)} \\ \vdots & \ddots & \vdots \\ -\mathcal{L}_{\mathcal{D}}{}_{k,h+1}^{(h)} & \ldots & \mathcal{D}_k - \mathcal{L}_{\mathcal{D}}{}_{k,k}^{(h)} \end{bmatrix}$$

(Schur complement of Φ_h in $\mathcal{H}_h$)

Set ▷ Final Step

$\delta_k := m_{k,k}^{(k-1)}$

$\mathcal{D}_k := \delta_k I_{n_k}$

$\Phi_k := \mathcal{D}_k - \mathcal{L}_{\mathcal{D}}{}_{k,k}^{(k-1)}$

$\phi_k := \delta_k - m_{k,k}^{(k-1)} = 0$

Condition (C.2). We want to prove that if we assume for the parameters $\delta_1, \delta_2, \dots, \delta_k$ the values obtained by means of the previous algorithm we ensure that $\mathcal{L}_\mathcal{D}$ has an eigenvector associated with the 0 eigenvalue with the desired block structure. We note that $\mathcal{L}_\mathcal{D}\mathbf{z} = \mathbf{0}_N$ is equivalent to the family of equations

$$\alpha_i \delta_i \mathbf{1}_{n_i} = \alpha_i c_{ii} \mathbf{1}_{n_i} + \sum_{j=1, j\neq i}^{k} \alpha_j c_{ij} \mathbf{1}_{n_i}, \quad i \in [1,k],$$

that, in turn, can be equivalently rewritten as

$$\alpha_i \delta_i = \alpha_i c_{ii} + \sum_{j=1, j\neq i}^{k} \alpha_j c_{ij}, \quad i \in [1,k],$$

and hence in matrix form as $(\mathbb{D} - \mathbb{C}) \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \end{bmatrix}^\top = \mathbf{0}_k$, where $\mathbb{D} := \text{diag}\{\delta_1, \delta_2, \dots, \delta_k\}$ and $\mathbb{C} := [c_{ij}]_{i,j\in[1,k]}$.

Now we observe that all the constraints on the $\delta_i, i \in [1,k]$, that we have derived, can be simply obtained from the (non symmetric) matrix $\mathbb{D} - \mathbb{C}$ by imposing that the $(1,1)$-entry of each of the first $k-1$ Schur complements, obtained according to the same algorithm that we used to define the matrices $\Phi_h, h \in [1, k-1]$, are positive, while the k-th one is zero. Indeed, such $(1,1)$-entries just correspond to the coefficients $\phi_1, \phi_2, \dots, \phi_k$. But this implies that if we choose $\delta_i, i \in [1,k]$, according to the previous algorithm, we ensure that $\det(\mathbb{D} - \mathbb{C}) = \phi_1\phi_2 \dots \phi_k = 0$, namely $\mathbb{D} - \mathbb{C}$ is singular. Therefore $\mathbb{D} - \mathbb{C}$ has an eigenvector $\mathbf{w} = [\alpha_1, \alpha_2, \dots, \alpha_k]^\top$, corresponding to 0, and hence $[\alpha_1 \mathbf{1}_{n_1}^\top, \alpha_2 \mathbf{1}_{n_2}^\top, \dots, \alpha_k \mathbf{1}_{n_k}^\top]^\top$ is an eigenvector of $\mathcal{L}_\mathcal{D}$ associated with the zero eigenvalue. Moreover, since we proved that 0 is a simple eigenvalue, all the eigenvectors of $\mathcal{L}_\mathcal{D}$ corresponding to 0 have the desired block structure. □

Example 3.5. *Consider, again, Example 3.2. As previously remarked, the communication graph satisfies Assumptions 1, 2 and 3 for $i_1 = 1$ and $i_2 = 3$ (as in the proof). If we apply Algorithm 1 we obtain the constraints*

$$\delta_1 > 1, \qquad \delta_2 > 2 + \frac{2}{\delta_1 - 1}, \qquad \delta_3 = \frac{2}{\delta_1 - 1} + \frac{\left(-4 + \frac{4}{\delta_1 - 1}\right)\left(-1 + \frac{1}{\delta_1 - 1}\right)}{\left[\delta_2 - 2 - \frac{2}{\delta_1 - 1}\right]^{-1}}.$$

If we assume $\delta_1 = 2$ then, independently of δ_2, one gets $\delta_3 = 2$. It turns out that for every choice of $\delta_2 > 4$ the eigenvector corresponding to the zero eigenvalue of $\mathcal{L}_\mathcal{D}$ is $\mathbf{z} =$

$[1\ 1\,|\,0\ 0\ 0\ 0\,|\,-1]^\top$.

Figure 3.2 shows the state evolution of the system described as in (3.5), with adjacency matrix as in Example 3.2, with random initial conditions $\mathbf{x}(0)$ *taken as realizations of a Gaussian vector with* 0 *mean and variance* $\sigma^2 = 4$*, i.e.* $\mathbf{x}(0) \sim \mathcal{N}(0, 4)$*. The graph shows that tripartite consensus is reached after about* 1.5 *units of time with regime values* $\gamma_1 = -1.39$*,* $\gamma_2 = 0$*,* $\gamma_3 = 1.39$*.*

Alternatively, one can choose $\delta_1 = 3, \delta_2 = 4$ *and* $\delta_3 = 2$*, and get as dominant eigenvector* $\bar{\mathbf{z}} = [0\ 0\,|\,1\,1\,1\,1\,|\,-2]^\top$.

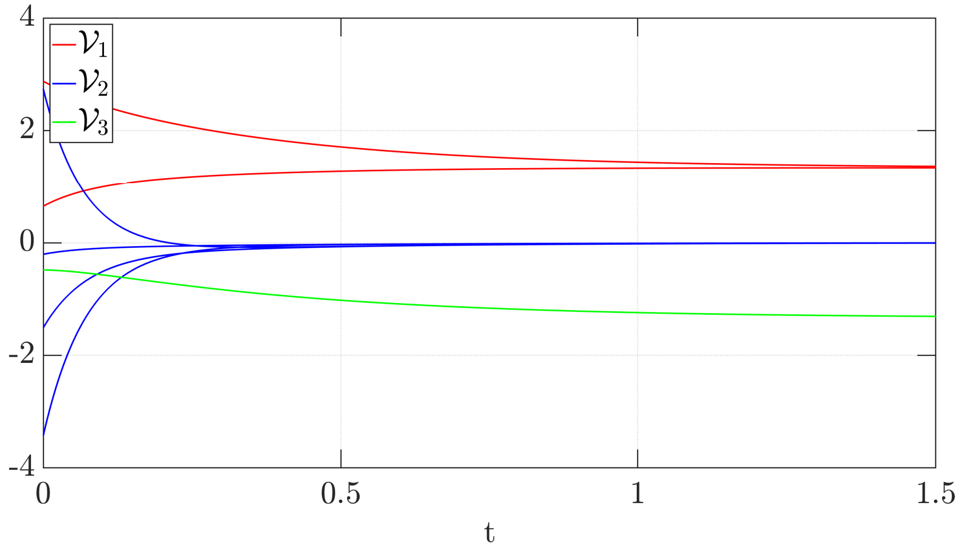

Figure 3.2: Graph corresponding to Example 3.5.

Remark 3.6. *If the number of clusters coincides with the number of agents, i.e. each cluster consists of a single node and all nodes are enemies to each other, the homogeneity constraint trivially holds* ($c_{ij} = \mathcal{A}_{ij} = [\mathcal{A}]_{ij}$) *and hence Theorem 3.4 applies under Assumption 1 alone.*

3.5 k-partite Consensus for Multi-agent Systems with Complete Unweighted Graph

In this subsection we will focus our attention on multi-agent systems with complete, unweighted and undirected communication graphs, clustered into an arbitrary number k of groups. By resorting to a suitable relabelling of the agents, we can always assume that the adjacency matrix $\mathcal{A}$ is described as in (3.2) with $\mathcal{A}_{i,i} = \mathbf{1}_{n_i}\mathbf{1}_{n_i}^\top - I_{n_i}$ and $\mathcal{A}_{i,j} = -\mathbf{1}_{n_i}\mathbf{1}_{n_j}^\top$ for $i \neq j$, i.e.,

$$\mathcal{A} = \begin{bmatrix} \mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top - I_{n_1} & -\mathbf{1}_{n_1}\mathbf{1}_{n_2}^\top & \dots & -\mathbf{1}_{n_1}\mathbf{1}_{n_k}^\top \\ -\mathbf{1}_{n_2}\mathbf{1}_{n_1}^\top & \mathbf{1}_{n_2}\mathbf{1}_{n_2}^\top - I_{n_2} & \dots & -\mathbf{1}_{n_2}\mathbf{1}_{n_k}^\top \\ \vdots & \vdots & \ddots & \vdots \\ -\mathbf{1}_{n_k}\mathbf{1}_{n_1}^\top & -\mathbf{1}_{n_k}\mathbf{1}_{n_2}^\top & \dots & \mathbf{1}_{n_k}\mathbf{1}_{n_k}^\top - I_{n_k} \end{bmatrix} \tag{3.12}$$

n_i being the cardinality of the i-th cluster. Also, in this case we plan to design a distributed control law for the system (4.3.1) of the type (3.3), with $\mathcal{L}_\mathcal{D} = \mathcal{D} - \mathcal{A}$, and $\mathcal{D} = \text{diag}\{\delta_1 I_{n_1}, \dots, \delta_k I_{n_k}\} \in \mathbb{R}^{N \times N}$.

Under the previous hypotheses on the adjacency matrix $\mathcal{A}$, Assumptions 1, 2 and 3 are trivially satisfied. So, the existence of a suitable choice of the coefficients $\delta_i, i \in [1, k]$, that ensures k-partite consensus follows from the previous Theorem 3.4. On the other hand, the particular structure of $\mathcal{A}$ allows to obtain a much simpler proof as well as an explicit expression of (a possible choice of) the δ_i's that cannot be obtained in the general homogeneous case. For this reason we provide here an independent proof of this result.

Theorem 3.7. *Consider the multi-agent system* (4.3.1)*, with unweighted and* complete *communication graph $\mathcal{G}$ split into k clusters, and adjacency matrix $\mathcal{A}$ as in* (3.12)*. If we assume*

$$\delta_i = 2n_i - 1, \quad i \in [1, k],$$

the closed-loop multi-agent system (3.5) *reaches k-partite consensus, under the distributed control law* (3.3)*, with $\mathcal{L}_\mathcal{D} \in \mathbb{R}^{N \times N}$ described as in* (3.4)*, and $\mathcal{D} = \text{diag}\{\delta_1 I_{n_1}, \dots, \delta_k I_{n_k}\} \in \mathbb{R}^{N \times N}$.*

Proof. By Lemma 3.1, we need to prove that under the theorem hypotheses and by assuming the parameters $\delta_i, i \in [1, k]$, as in (3.7), we can ensure that (C.1) the matrix $\mathcal{L}_\mathcal{D}$ is singular and positive semidefinite, and (C.2) its kernel is spanned by vectors taking the block form $\mathbf{z} = [\alpha_1 \mathbf{1}_{n_1}^\top, \alpha_2 \mathbf{1}_{n_2}^\top, \dots, \alpha_k \mathbf{1}_{n_k}^\top]^\top, \alpha_i \in \mathbb{R}, i \in [1, k]$.

Condition (C.2). By assuming $\delta_i, i \in [1, k]$, as in (3.7), and by imposing $\mathcal{L}_\mathcal{D}\mathbf{z} = \mathbf{0}_N$, for $\mathbf{z}$ described as above, we obtain the family of equations

$$\mathbb{N}_k \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \end{bmatrix}^\top = 0,$$

where $\mathbb{N}_k := \mathbf{1}_k \begin{bmatrix} n_1 & n_2 & \dots & n_k \end{bmatrix}$ is a singular matrix whose kernel coincides with

$$\mathcal{H} = \begin{bmatrix} \mathcal{H}_{2,2} & \mathcal{H}_{2,3} & \dots & \mathcal{H}_{2,k} \\ \mathcal{H}_{3,2} & \mathcal{H}_{3,3} & \dots & \mathcal{H}_{3,k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}_{k,2} & \mathcal{H}_{k,3} & \dots & \mathcal{H}_{k,k} \end{bmatrix}, \quad \begin{array}{l} \mathcal{H}_{i,i} = 2n_i I_{n_i} - \mathbf{1}_{n_i}\mathbf{1}_{n_i}^\top - \mathbf{1}_{n_i}\mathbf{1}_{n_1}^\top (2n_1 I_{n_1} - \mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top)^{-1}\mathbf{1}_{n_i}\mathbf{1}_{n_i}^\top \\ \mathcal{H}_{i,j} = \mathbf{1}_{n_i}\mathbf{1}_{n_j}^\top - \mathbf{1}_{n_i}\mathbf{1}_{n_1}^\top (2n_1 I_{n_1} - \mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top)^{-1}\mathbf{1}_{n_i}\mathbf{1}_{n_j}^\top, \quad i \neq j. \end{array}$$

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ker $\begin{bmatrix} n_1 & n_2 & \dots & n_k \end{bmatrix}$. This implies that ker $\mathcal{L}_\mathcal{D}$ includes all the vectors $\mathbf{z} = [\alpha_1 \mathbf{1}_{n_1}^\top, \alpha_2 \mathbf{1}_{n_2}^\top, \dots, \alpha_k \mathbf{1}_{n_k}^\top]^\top$, with $[\alpha_1, \alpha_2, \dots, \alpha_k] \in$ ker $\begin{bmatrix} n_1 & n_2 & \dots & n_k \end{bmatrix}$. To prove that *all* the eigenvectors of $\mathcal{L}_\mathcal{D}$ corresponding to the zero eigenvalue take the form $[\alpha_1 \mathbf{1}_{n_1}^\top, \dots, \alpha_k \mathbf{1}_{n_k}^\top]^\top, \alpha_i \in \mathbb{R}, i \in [1,k]$, let $\mathbf{w} = \begin{bmatrix} \mathbf{w}_1^\top & \mathbf{w}_2^\top & \dots & \mathbf{w}_k^\top \end{bmatrix}^\top$ be any eigenvector of $\mathcal{L}_\mathcal{D}$ corresponding to 0. Then condition $\mathcal{L}_\mathcal{D}\mathbf{w} = \mathbf{0}_N$ implies

$$2n_i \mathbf{w}_i = (\mathbf{1}_{n_i}^\top \mathbf{w}_i)\mathbf{1}_{n_i} - \sum_{j=1, j\neq i}^{k} (\mathbf{1}_{n_j}^\top \mathbf{w}_j)\mathbf{1}_{n_i}, \; i \in [1,k],$$

which ensures that every $\mathbf{w}_i$ is a scalar multiple of $\mathbf{1}_{n_i}$.

Condition (C.1). We now prove that by assuming $\delta_i, i \in [1,k]$, as in (3.7):
(A) the upper diagonal block of $\mathcal{L}_D$, namely $\Phi := 2n_1 I_{n_1} - \mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top$, is positive definite, and
(B) its Schur complement $\mathcal{H}$, given in (3.13), is positive semidefinite and singular.
Therefore, by the result in Section 2.3.1, $\mathcal{L}_\mathcal{D}$ is positive semidefinite and singular.
By Lemma 2.5 part i), we can claim that, since $(2n_1 I_{n_1} - \mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top)\mathbf{1}_{n_1} = n_1 \mathbf{1}_{n_1} > 0$, (A) holds.
Now, we observe that, for any vector $\mathbf{z} = [\alpha_1 \mathbf{1}_{n_1}^\top, \alpha_2 \mathbf{1}_{n_2}^\top, \dots, \alpha_k \mathbf{1}_{n_k}^\top]^\top$, with $[\alpha_1, \alpha_2, \dots, \alpha_k] \in$ ker $\begin{bmatrix} n_1 & n_2 & \dots & n_k \end{bmatrix}$, we have $0 = (2n_1 I_{n_1} - \mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top)\alpha_1 \mathbf{1}_{n_1} + \mathbf{1}_{n_1}\alpha_2 n_2 + \dots + \mathbf{1}_{n_1}\alpha_k n_k$, and hence

$$(2n_1 I_{n_1} - \mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top)^{-1}\mathbf{1}_{n_1} = -\frac{\alpha_1}{(\sum_{i=2}^k \alpha_i n_i)}\mathbf{1}_{n_1} = \frac{1}{n_1}\mathbf{1}_{n_1}.$$

This allows to verify that the matrix $\mathcal{H}$ takes the block diagonal form

$$\mathcal{H} = \text{diag}\{2n_2 I_{n_2} - 2\mathbf{1}_{n_2}\mathbf{1}_{n_2}^\top, 2n_3 I_{n_3} - 2\mathbf{1}_{n_3}\mathbf{1}_{n_3}^\top, \dots, 2n_k I_{n_k} - 2\mathbf{1}_{n_k}\mathbf{1}_{n_k}^\top\}.$$

Each diagonal block $2n_i I_{n_i} - 2\mathbf{1}_{n_i}\mathbf{1}_{n_i}^\top, i \in [2,k]$, is easily seen (by a straightforward extension of Lemma 2.5) to be positive semidefinite and singular (with 0 as a simple eigenvalue). So, we have shown that $\mathcal{L}_\mathcal{D}$ is positive semidefinite and singular and hence (B) holds. Therefore condition (C.1) holds and k-partite consensus is asymptotically achieved. □

Example 3.8. *Consider the multi-agent system* (3.5)*, with unweighted and complete communication graph and 5 clusters of size* $n_1 = 9, n_2 = 13, n_3 = 14, n_4 = 11, n_5 = 7$*. We assume* $\delta_i, i \in [1, 5]$*, as in* (3.7) *and* $\mathbf{x}(0) \sim \mathcal{N}(0, 4)$*. The system reaches* 5*-partite consensus after about* 0.2 *units of time, with regime values* $\gamma_1 = -0.1781, \gamma_2 = 0.484, \gamma_3 = -0.9866, \gamma_4 = 0.1849, \gamma_5 = 1.004$*, as illustrated in Fig. 3.3.*

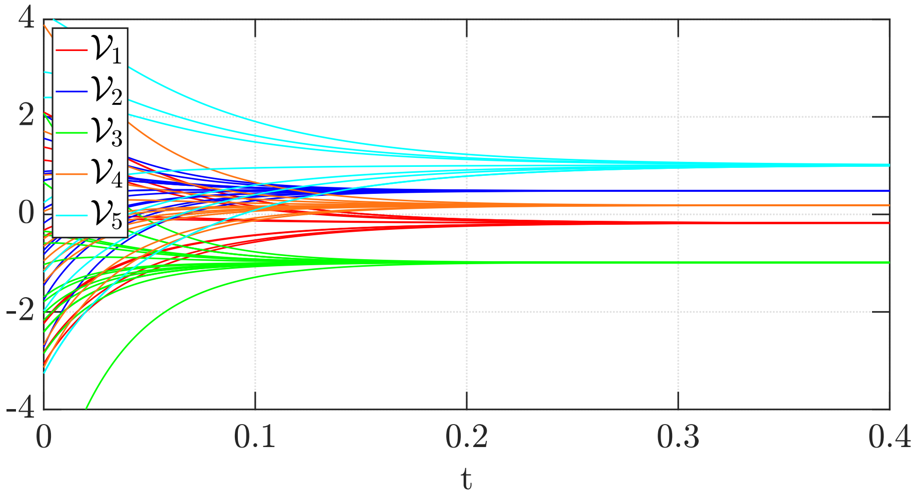

Figure 3.3: Graph corresponding to Example 3.8.

3.6 k-Partite Consensus for a Class of Nonlinear Models

In the following, an extension of the k-partite consensus analysis to nonlinear systems is proposed. To this aim, by adopting a set-up similar to the one in Altafini (2013), we consider a multi-agent system described as in (4.3.1), with communication graph $\mathcal{G}$ satisfying Assumption 1 and subjected to the feedback law

$$\mathbf{u} = \mathbf{f}(\mathbf{x}), \tag{3.13}$$

where $\mathbf{f} : \mathbb{R}^\mathbf{N} \to \mathbb{R}^\mathbf{N}$ is a Lipschitz continuous function satisfying $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.

Assumption 4 on the vector field f: *We assume for* $\mathbf{f}$ *a distributed additive expression. Specifically, we assume that each component* $f_i(\mathbf{x})$*,* $i \in [1, N]$*, of the function* $\mathbf{f}$ *depends only on the states of the neighbouring agents of the agent* i*, namely for every* $i \in [1, N]$*, the function* $f_i(\mathbf{x})$ *depends only on those entries* x_j *such that* $(j, i) \in \mathcal{E}$*, and is* *expressed as follows*

$$f_i(\mathbf{x}) = -\sum_{j:(j,i)\in\mathcal{E}} \left([\mathcal{D}]_i \tilde{h}_i(x_i(t)) - [\mathcal{A}]_{i,j}\tilde{h}_j(x_j(t))\right),$$

where $[\mathcal{D}]_i$ *is a real number, and the nonlinear function* $\tilde{h}_k(\cdot)$ *is the same for all the agents belonging to the same cluster. So, if we assume that the agents are partitioned into* k *clusters and ordered in such a way that* $\mathcal{A}$ *is described as in* (3.2)*, the vector* $\mathbf{x}$ *is accordingly partitioned as* $\mathbf{x} = \begin{bmatrix}\mathbf{x}_1^\top & \mathbf{x}_2^\top & \dots & \mathbf{x}_k^\top\end{bmatrix}^\top$, *with* $\mathbf{x}_i \in \mathbb{R}^{n_i}$ *representing the states of the agents belonging to the* i*-th cluster. The function* $\mathbf{f}$ *can be expressed as the product of the matrix* $\mathcal{L}_\mathcal{D}$, *given in* (3.4)*, and of a nonlinear function* $\mathbf{h}(\mathbf{x})$*:*

$$\dot{\mathbf{x}} = -\mathcal{L}_\mathcal{D}\mathbf{h}(\mathbf{x}), \tag{3.14}$$

with $\mathbf{h}(\mathbf{x}) = [\mathbf{h_1}(\mathbf{x_1})^\top\ \mathbf{h_2}(\mathbf{x_2})^\top\ \dots\ \mathbf{h_k}(\mathbf{x_k})^\top]^\top$, *and* $\mathbf{h_i}(\mathbf{x_i}) : \mathbb{R}^{\mathbf{n_i}} \to \mathbb{R}^{\mathbf{n_i}}$, $i \in [1, k]$, *described as follows*

$$\mathbf{h}_i(\mathbf{x}_i) = [h_i(x_{s_i+1})\ h_i(x_{s_i+2})\ \dots\ h_i(x_{s_i+n_i})]^\top, \tag{3.15}$$

$$s_i = \begin{cases} 0, & i = 1; \\ \sum_{j<i} n_j, & i = 2, \dots, k. \end{cases} \tag{3.16}$$

The scalar functions $h_i(\cdot)$ *are assumed to be monotone, bijective functions belonging to the following set:*

$$\mathcal{R} := \Big\{h : \mathbb{R} \to \mathbb{R} : (h(x_a) - h(x_b))(x_a - x_b) > 0, x_a \neq x_b,$$
$$h(0) = 0, \int_{x_b}^{x_a} (h(z) - h(x_b))dz \to \infty \text{ as } |x_a - x_b| \to \infty\Big\}.$$

The following theorem provides sufficient conditions for a networked closed-loop system described as in (3.14) to reach k-partite consensus that extend those given in Theorem 3.4. Similarly, the extension of Theorem 3.7 would be possible.

Theorem 3.9. *Consider the multi-agent system* (4.3.1)*, with communication graph* $\mathcal{G}$ *satisfying Assumptions 1, 2 and 3, and distributed control law* (3.13) *satisfying Assumption 4 and* (3.6)*. Consequently, the multi-agent system is described as in* (3.14)*, with the function* $\mathbf{h}(\mathbf{x})$ *defined as above,* $\mathcal{L}_\mathcal{D} \in \mathbb{R}^{N\times N}$ *described as in* (3.4)*,* $\mathcal{D} \in \mathbb{R}^{N\times N}$ *described as in* (3.2) *and* $\mathcal{D}_i = \delta_i I_{n_i}$*, for* $i \in [1, k]$*. There exist* $\delta_i \in \mathbb{R}, i \in [1, k]$*, such that the closed-loop multi-agent system* (3.14) *reaches* k*-partite consensus.*

Proof. Clearly, the equilibrium points of system (3.14) are all the vectors $\mathbf{x}^*$ in $\mathbb{R}^N$ such that $\mathbf{0} = \mathcal{L}_{\mathcal{D}}\mathbf{h}(\mathbf{x}^*)$. We want to show that it is possible to choose the coefficients $\delta_i, i \in [1,k]$, so that all the equilibrium points of the system are block partitioned according to the block partitioning of the matrix $\mathcal{L}_{\mathcal{D}}$, and they are globally simply stable. This ensures that the set of all such equilibrium points is the attractor of every state trajectory (there cannot be limit cycles and the trajectories cannot diverge), and hence the multi-agent system asymptotically reaches k-partite consensus.

We have proved (see Theorem 3.4) that under Assumptions 1, 2 and 3 it is possible to choose the coefficients $\delta_1, \delta_2, \dots, \delta_k \in \mathbb{R}$ so that $\mathcal{L}_{\mathcal{D}}$ is a singular positive semidefinite matrix, having 0 as a simple eigenvalue and the corresponding eigenvector takes the form $\mathbf{z} = [\alpha_1 \mathbf{1}_{n_1}^\top, \alpha_2 \mathbf{1}_{n_2}^\top, \dots \alpha_k \mathbf{1}_{n_k}^\top]^\top$, for suitable $\alpha_i \in \mathbb{R}, i \in [1,k]$. This implies that the equilibrium points of the system (3.14) are the vectors $\mathbf{x}^*$ such that $\mathbf{h}(\mathbf{x}^*) \in \langle \mathbf{z} \rangle$. As the maps h_i belong to $\mathcal{R}$, for every $c \in \mathbb{R}$ such that $c \cdot \alpha_i$ belongs to the image of the corresponding h_i for every $i \in [1,k]$, there exist $\beta_1, \beta_2 \dots \beta_k \in \mathbb{R}$ such that $c \cdot [\alpha_1 \mathbf{1}_{n_1}^\top, \dots \alpha_k \mathbf{1}_{n_k}^\top]^\top = \mathbf{h}([\beta_1 \mathbf{1}_{n_1}^\top, \dots \beta_k \mathbf{1}_{n_k}^\top]^\top)$.

Suppose, without loss of generality, that this is the case for $c = 1$, set

$\mathbf{x}^* := [\beta_1 \mathbf{1}_{n_1}^\top, \beta_2 \mathbf{1}_{n_2}^\top, \dots, \beta_k \mathbf{1}_{n_k}^\top]^\top$, and consider a suitably modified version of the Lyapunov function $V : \mathbb{R}^N \to \mathbb{R}$ adopted in Altafini (2013):

$$V(\mathbf{x}) = \sum_{i=1}^{k} \sum_{j=s_i+1}^{s_i+n_i} \int_{x_j^*}^{x_j} (h_i(z) - h_i(x_j^*))dz = \sum_{i=1}^{k} \sum_{j=s_i+1}^{s_i+n_i} \int_{\beta_i}^{x_j} (h_i(z) - \alpha_i)dz, \quad (3.17)$$

(see (3.16) for the definition of s_i) for $\mathbf{x} \neq \mathbf{x}^*$. Moreover, $V(\mathbf{x})$ is radially unbounded and its derivative is

$$\dot{V}(\mathbf{x}) = \sum_{i=1}^{k} \sum_{j=s_i+1}^{s_i+n_i} (h_i(x_j) - h_i(x_j^*))\dot{x}_j = -(\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{x}^*))^\top \mathcal{L}_{\mathcal{D}}\mathbf{h}(\mathbf{x}) = -\mathbf{h}(\mathbf{x})^\top \mathcal{L}_{\mathcal{D}}\mathbf{h}(\mathbf{x}) \leq 0,$$

where we used the fact that $\mathcal{L}_{\mathcal{D}} = \mathcal{L}_{\mathcal{D}}^\top$ and $\mathcal{L}_{\mathcal{D}}\mathbf{h}(\mathbf{x}^*) = \mathcal{L}_{\mathcal{D}}\mathbf{z} = 0$, and the last inequality holds since $\mathcal{L}_{\mathcal{D}}$ is a singular positive semidefinite matrix. This ensures that every equilibrium point $\mathbf{x}^*$ of the system is globally stable and since all such equilibrium points have the required block-structure, k-partite consensus is guaranteed. □

Example 3.10. *Consider the multi-agent system* (3.14)*, with unweighted and complete communication graph,* $\mathbf{h}(\mathbf{x}(t)) = \tanh(\mathbf{x}(t))$*, and 4 clusters of size* $n_1 = 6, n_2 = 9, n_3 = 11, n_4 = 7$*. We have assumed that* $\mathbf{x}(0) \sim \mathcal{N}(0,4)$ *and* $\delta_i = 2n_i - 1$ *for every* $i \in [1,4]$*. The* *system reaches* 4*-partite consensus after approximately* 0.5 *time units, with regime values* $\gamma_1 = 0.4967, \gamma_2 = -0.5369, \gamma_3 = -1.925, \gamma_4 = 2.418$*, as illustrated in Fig. 3.4.*

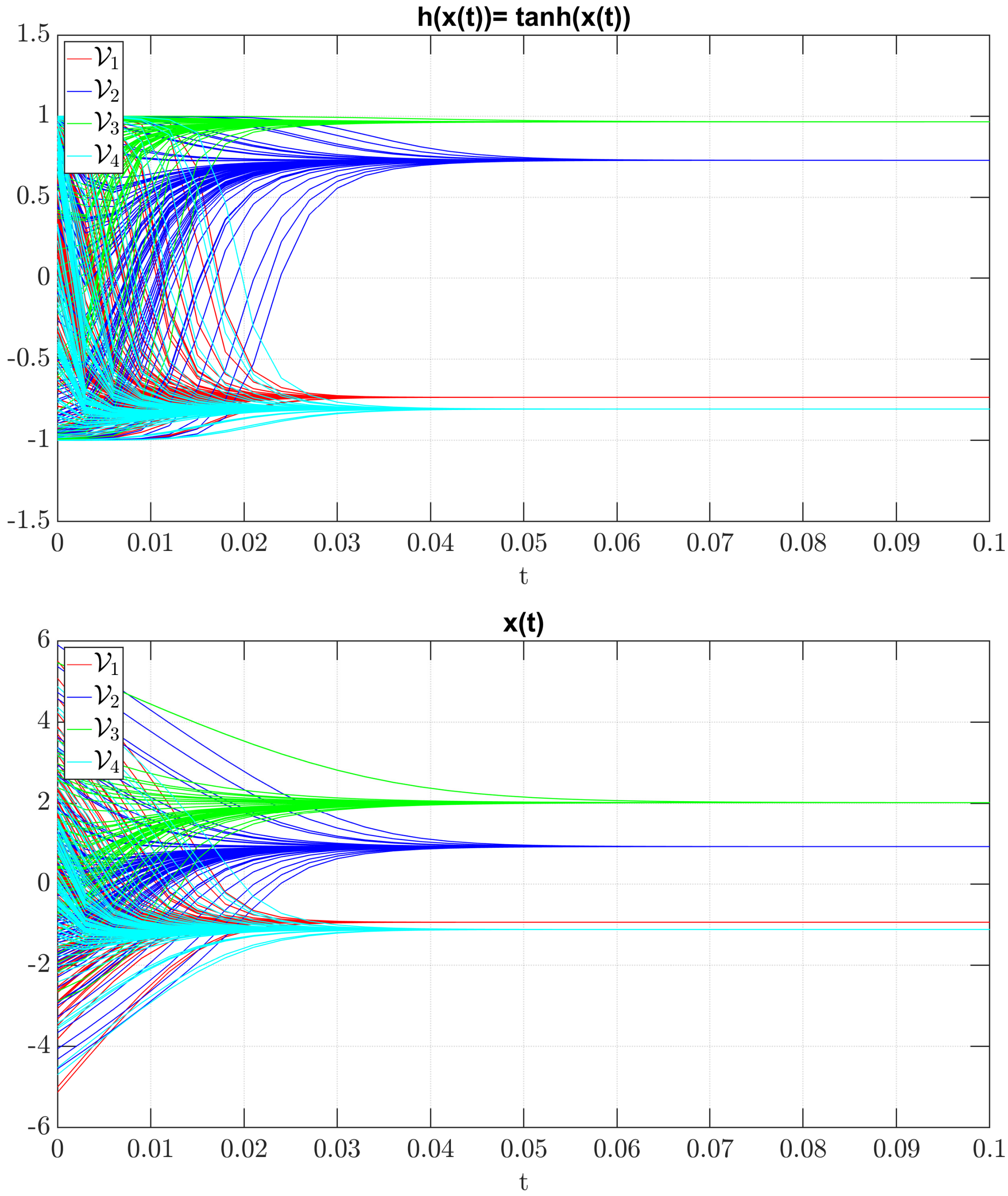

Figure 3.4: Graphs associated with Example 3.10. The upper shows the time evolution of $\mathbf{h}(\mathbf{x}(t)) = \tanh(\mathbf{x}(t))$. The one below shows the time evolution of $\mathbf{x}(t)$.

3.7 Relaxation of the Homogeneity Assumption

The remaining part of this chapter addresses networks that are partitioned into three clusters of agents that do not fulfil the homogeneity assumption (see Assumption 2). In this setting, we will focus on the reaching of tripartite consensus and sign consensus. While in the homogeneous case the desired goal was achieved by choosing a self-confidence degree common to all the agents belonging to the same class, from now onward the agents' degree of self-confidence will be individually tuned.

By focusing on Lemma 3.1 (ii), we provide the following lemma whose easy proof is omitted.

Lemma 3.11. *Given the matrix $\mathcal{L}_\mathcal{D} \in \mathbb{R}^{N\times N}$ described as in (3.4), $\mathcal{D} \in \mathbb{R}^{N\times N}$ described as in (3.2) and $\mathcal{D}_i \in \mathbb{R}^{n_i\times n_i}$, for $i \in [1,3]$, diagonal matrices, the kernel of $\mathcal{L}_\mathcal{D}$ includes a vector of the type $\mathbf{v} = [v_1\mathbf{1}_{n_1}^\top, v_2\mathbf{1}_{n_2}^\top, v_3\mathbf{1}_{n_3}^\top]^\top, v_i \in \mathbb{R}, i \in [1,3]$, if and only if*

$$\text{rank}\left(\begin{bmatrix} \mathbf{d}_1 - \mathbf{a}_{11} & -\mathbf{a}_{12} & -\mathbf{a}_{13} \\ -\mathbf{a}_{21} & \mathbf{d}_2 - \mathbf{a}_{22} & -\mathbf{a}_{23} \\ -\mathbf{a}_{31} & -\mathbf{a}_{32} & \mathbf{d}_3 - \mathbf{a}_{33} \end{bmatrix}\right) < 3, \tag{3.18}$$

where

$$\mathbf{d}_i := \mathcal{D}_i\mathbf{1}_{n_i}, \qquad \mathbf{a}_{ij} := \mathcal{A}_{i,j}\mathbf{1}_{n_j}, \qquad i,j \in [1,3].$$

Based on Lemmas 3.1 and 3.11, in the sequel we will provide conditions ensuring the existence of diagonal matrices $\mathcal{D}_i$ (equivalently, of vectors $\mathbf{d}_i = \mathcal{D}_i\mathbf{1}_{n_i}$), for $i \in [1,3]$, such that the corresponding matrix $\mathcal{L}_\mathcal{D}$ is a singular positive semi-definite matrix, with a simple eigenvalue in 0 and condition (3.18) holds. In fact, if 0 is a simple eigenvalue, in order to fulfil condition (ii) of Lemma 3.1 it is sufficient to prove that there exists a single vector in the kernel of $\mathcal{L}_D$ having the desired block structure.

Assumption 5 The clustering partition into 3 clusters is minimal.

It is worth noticing that $\mathbf{a}_{ij} \neq 0$ for every pair $i,j \in [1,3], i \neq j$. In fact $\mathbf{a}_{ij} = 0$ implies $\mathcal{A}_{i,j} = 0$ and hence also $\mathcal{A}_{j,i} = 0$ which means that $\mathcal{V}_i$ and $\mathcal{V}_j$ could be grouped together, thus contradicting the minimality of the partitioning into 3 clusters introduced in Assumption 5.

We are now in a position to introduce one of the two main results of this section.

Theorem 3.12. *Consider the multi-agent system (4.3.1), with undirected, signed, weighted and connected communication graph $\mathcal{G}$ satisfying Assumption 1, Assumption 2 and Assumption 5 for a suitable choice of $i_1, i_2 \in [1,3], i_1 \neq i_2$. Also, suppose that the following conditions hold:*

1) *every agent in $\mathcal{V}_{i_3}$ has at least one enemy in $\mathcal{V}_{i_2}$, namely $\mathcal{A}_{i_3,i_2}\mathbf{1}_{n_{i_2}} \ll 0$, and*

2) *there exists $h \in \{i_2, i_3\}$ such that every agent in $\mathcal{V}_{i_1}$ has at least one enemy in $\mathcal{V}_h$, namely $\mathcal{A}_{i_1,h}\mathbf{1}_h \ll 0$.*

Then there exist diagonal matrices $\mathcal{D}_i \in \mathbb{R}^{n_i \times n_i}, i \in [1,3]$, such that the distributed control law (3.3), with $\mathcal{L}_\mathcal{D} \in \mathbb{R}^{N\times N}$ described as in (3.4), $\mathcal{D} \in \mathbb{R}^{N\times N}$ described as in (3.2), makes the closed-loop multi-agent system (3.5) reach tripartite consensus.

Proof. We can always relabel the vertices in $\mathcal{V}$ so that $i_1 = 1, i_2 = 3$ and $i_3 = 2$. Note, also, that $\mathbf{a}_{ij} = \mathcal{A}_{i,j}\mathbf{1}_{n_j} \leq 0$, for every $i, j \in [1,3], i \neq j$ (but $\mathbf{a}_{ij} \neq 0$), and $\mathbf{a}_{ii} = \mathcal{A}_{i,i}\mathbf{1}_{n_i} \geq 0$.
Based on the previous comments, related to Lemmas 3.1 and 3.11, we prove that there exist diagonal matrices $\mathcal{D}_i$, $i \in [1,3]$, such that **(A)** $\mathcal{L}_\mathcal{D}$ is a singular positive semi-definite matrix, with a simple eigenvalue in 0, and **(B)** the matrix in (3.18) has a nontrivial vector $\tilde{\mathbf{v}} := \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^\top$ in its kernel.
We first prove that Assumption 2 ensures that **(A)** holds. To ensure that the matrix

$$\mathcal{L}_\mathcal{D} = \left[\begin{array}{c|cc} \mathcal{D}_1 - \mathcal{A}_{1,1} & -\mathcal{A}_{1,2} & -\mathcal{A}_{1,3} \\ \hline -\mathcal{A}_{2,1} & \mathcal{D}_2 - \mathcal{A}_{2,2} & -\mathcal{A}_{2,3} \\ -\mathcal{A}_{3,1} & -\mathcal{A}_{3,2} & \mathcal{D}_3 - \mathcal{A}_{3,3} \end{array}\right] \tag{3.19}$$

is positive semidefinite, we impose (see Section 2.3.1) that the upper diagonal block is positive definite and its Schur complement is positive semi-definite, i.e., that conditions (3.20):

$$\mathcal{D}_1 - \mathcal{A}_{1,1} \succ 0 \tag{3.20}$$

and (3.21) hold.

Assume that

$$\mathbf{d}_1 \gg \mathbf{a}_{11} \geq 0. \tag{3.22}$$

$$\begin{bmatrix} \mathcal{D}_2 - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2} & -\mathcal{A}_{2,3} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3} \\ -\mathcal{A}_{3,2} - \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2} & \mathcal{D}_3 - \mathcal{A}_{3,3} - \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3} \end{bmatrix} \succeq 0. \quad (3.21)$$

$$\begin{aligned} \Phi_3 \;:=\; & \mathcal{D}_3 - \mathcal{A}_{3,3} - \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3} - [\mathcal{A}_{3,2} + \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}] \\ & \cdot[\mathcal{D}_2 - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}]^{-1}[\mathcal{A}_{2,3} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}] \succeq 0 \\ & \text{and singular.} \end{aligned} \quad (3.24)$$

Then $(\mathcal{D}_1 - \mathcal{A}_{1,1})\mathbf{1}_{n_1} \gg 0$, and hence Lemma 2.5, part i), holds for $\mathbf{v} = \mathbf{1}_{n_1}$, thus ensuring that $\mathcal{D}_1 - \mathcal{A}_{1,1}$ is positive definite.

To ensure that (3.21) holds, we iterate the same procedure, and impose condition:

$$\mathcal{D}_2 - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2} \succ 0, \quad (3.23)$$

as well as condition (3.24).

To address condition (3.23), we first observe that by Lemma 2.5, part ii), $(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}$ is symmetric and nonnegative, and hence so is $\mathcal{A}_{2,2}+\mathcal{A}_{2,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}$. But then we can apply Lemma 2.5, part i), again, by assuming $D = \mathcal{D}_2$ and $A = \mathcal{A}_{2,2}+\mathcal{A}_{2,1}(\mathcal{D}_1-\mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}$. Indeed, if we impose the following constraint on $\mathbf{d}_2$:

$$\mathbf{d}_2 \gg \mathbf{a}_{22} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{12} \geq 0, \quad (3.25)$$

then it is easy to verify that

$$(D - A)\mathbf{1}_{n_2} = \mathbf{d}_2 - \mathbf{a}_{22} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{12} \gg 0.$$

Therefore $D - A$ is positive definite, namely (3.23) holds.

On the other hand, we can always choose the positive diagonal entries of the diagonal matrix $\mathcal{D}_2$, namely the vector $\mathbf{d}_2$, so that not only $\mathbf{d}_2$ fulfils condition (3.25), but it is also sufficiently large to ensure that the entries of $[\mathcal{A}_{3,2} + \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}][\mathcal{D}_2 - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}]^{-1}[\mathcal{A}_{2,3} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}]$ are small enough to guarantee that

$$-(\Phi_3 - \mathcal{D}_3) \approx \mathcal{A}_{3,3} + \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}.$$

By Assumption 2, for $i_1 = 1$ and $i_3 = 2$, the matrix $\mathcal{A}_{3,3} + \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}$ has positive off-diagonal entries, and hence the same is true for $-(\Phi_3 - \mathcal{D}_3)$. This ensures that $-\Phi_3$ is an irreducible Metzler matrix.

So, now, we are remained with proving that for a suitable choice of $\mathcal{D}_3$ we can ensure that (3.24) holds. If we apply the vector $\mathbf{1}_{n_3}$ on the right side of the matrix Φ_3, by making use of reasonings similar to those just exploited to prove (3.23), we obtain

$$\begin{aligned}\Phi_3\mathbf{1}_{n_3} &= \mathbf{d}_3 - \mathbf{a}_{33} - \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{13} - [\mathcal{A}_{3,2} + \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}] \\ &\cdot\ [\mathcal{D}_2 - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}]^{-1} \cdot [\mathbf{a}_{23} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{13}].\end{aligned}$$

Therefore, by imposing

$$\begin{aligned}\mathbf{d}_3 &= \mathbf{a}_{33} + \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{13} + [\mathcal{A}_{3,2} + \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}] \\ &\cdot\ [\mathcal{D}_2 - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}]^{-1} \cdot [\mathbf{a}_{23} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{13}],\end{aligned}$$

we ensure that $\Phi_3\mathbf{1}_{n_3} = 0$. This guarantees that the matrix Φ_3 has 0 as an eigenvalue corresponding to the eigenvector $\mathbf{1}_{n_3} \gg 0$, and therefore (see Theorem 2.3) 0 is the simple dominant eigenvalue of the irreducible Metzler matrix $-\Phi_3$. Since the eigenvalues of $\mathcal{L}_\mathcal{D}$ are the union of the eigenvalues of the matrices in (3.20) and (3.23) and of the matrix Φ_3, that have been obtained from $\mathcal{L}_\mathcal{D}$ by applying the Schur complement, then $\mathcal{L}_\mathcal{D}$ is positive semidefinite with 0 as simple eigenvalue and thus condition **(A)** holds. Now we show that under conditions 1) and 2) we can determine vectors $\mathbf{d}_i, i \in [1,3]$, so that also condition **(B)** holds.

Note that by condition 1), $\mathbf{a}_{23} \ll 0$, and by condition 2), either $\mathbf{a}_{12} \ll 0$ or $\mathbf{a}_{13} \ll 0$. In the sequel we will focus on the case $\mathbf{a}_{12} \ll 0$, the other case being completely equivalent. We want to prove that we can always find vectors $\mathbf{d}_i, i \in [1,3]$, consistent with the constraints (3.22), (3.25) and (3.26), so that **(B)** holds and hence there exist v_2, v_3 such that

$$\begin{bmatrix}\mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13}\\ \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23}\\ \mathbf{a}_{31} & \mathbf{a}_{32} & \mathbf{a}_{33}\end{bmatrix}\begin{bmatrix}1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}\mathbf{d}_1\\ v_2\mathbf{d}_2\\ v_3\mathbf{d}_3\end{bmatrix}. \tag{3.26}$$

This is equivalent to determining scalars v_2 and v_3 that make the vectors

$$\mathbf{d}_1 = \mathbf{a}_{11} + v_2\mathbf{a}_{12} + v_3\mathbf{a}_{13} \tag{3.27}$$

$$\mathbf{d}_2 = \mathbf{a}_{22} + \frac{1}{v_2}\mathbf{a}_{21} + \frac{v_3}{v_2}\mathbf{a}_{23} \tag{3.28}$$

$$\mathbf{d}_3 = \mathbf{a}_{33} + \frac{1}{v_3}\mathbf{a}_{31} + \frac{v_2}{v_3}\mathbf{a}_{32}, \tag{3.29}$$

consistent with the constraints (3.22), (3.25) and (3.26).

We first note that since $\mathbf{a}_{12} \ll 0$, we can always choose $v_2 < 0$ with large module, and $v_3 > 0$ and small, so that $v_2\mathbf{a}_{12} + v_3\mathbf{a}_{13} \gg 0$, which automatically implies that $\mathbf{d}_1$ satisfies condition (3.22). Also, we can choose the modules of v_2 and v_3 in such a way that the entries of $\mathbf{d}_1$ and hence of $\mathcal{D}_1$ are so large that $\mathbf{a}_{23} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{13} \approx \mathbf{a}_{23} \ll 0$ and hence $\mathbf{a}_{23} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{13} \ll 0$, and also

$$\frac{v_3}{v_2}\left[\mathbf{a}_{23} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{13}\right] \gg 0. \tag{3.30}$$

By making use of (3.30), we obtain that

$$\begin{aligned}
\mathbf{d}_2 &= \mathbf{a}_{22} + \frac{1}{v_2}\mathbf{a}_{21} + \frac{v_3}{v_2}\mathbf{a}_{23} \gg \mathbf{a}_{22} + \frac{1}{v_2}\mathbf{a}_{21} - \frac{v_3}{v_2}\mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{13} \\
&= \mathbf{a}_{22} + \frac{1}{v_2}\mathcal{A}_{2,1}[\mathbf{1}_{n_1} - v_3(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{13}] = \mathbf{a}_{22} + \frac{1}{v_2}\mathcal{A}_{2,1}[v_2(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{12}] \\
&= \mathbf{a}_{22} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{12},
\end{aligned}$$

where we used the fact that condition (3.27) is equivalent to

$$\mathbf{1}_{n_1} = v_2(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{12} + v_3(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{13}. \tag{3.31}$$

So, this proves that also (3.25) holds. Finally, it is possible to prove that if the identities (3.27) and (3.28) hold, then the constraints (3.29) and (3.26) are equivalent.

Hence we conclude that there exist suitable choices of $\mathbf{d}_i, i \in [1,3]$, such that both conditions **(A)** and **(B)** are fulfilled and hence the overall multi-agent system reaches tripartite consensus. □

Example 3.13. *Consider the undirected, signed, weighted, connected and clustered communication graph, with three clusters of cardinalities $n_1 = 5$, $n_2 = 4$, $n_3 = 2$, respectively, and adjacency matrix composed of the sub-matrices*

$$\mathcal{A}_{1,1} = \begin{bmatrix} 0 & 4 & 0 & 0 & 1 \\ 4 & 0 & 3 & 10 & 2 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 10 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 \end{bmatrix} \mathcal{A}_{1,2} = -\begin{bmatrix} 1.5 & 1.5 & 0 & 1.5 \\ 0.5 & 3 & 0 & 2.5 \\ 3 & 0.5 & 0 & 2.5 \\ 3 & 3 & 0 & 2 \\ 0 & 0 & 0.5 & 2.5 \end{bmatrix},$$

$$\mathcal{A}_{1,3} = -\begin{bmatrix} 7 & 2 \\ 0 & 3 \\ 4 & 2 \\ 8 & 0 \\ 7 & 3 \end{bmatrix}, \mathcal{A}_{2,2} = \begin{bmatrix} 0 & 6 & 0 & 2 \\ 6 & 0 & 4 & 4 \\ 0 & 4 & 0 & 0 \\ 2 & 4 & 0 & 0 \end{bmatrix}, \mathcal{A}_{2,3} = -\begin{bmatrix} 3 & 4 \\ 6 & 1 \\ 1 & 0 \\ 4 & 6 \end{bmatrix}, \mathcal{A}_{3,3} = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix},$$

all the others being deduced by symmetry. Assumption 2 holds for $i_1 = 1$ and $i_2 = 3$, and both assumptions 1) and 2) of Theorem 3.12 hold, since $\mathbf{a}_{23} \ll 0$, $\mathbf{a}_{12}$ and $\mathbf{a}_{13}$ are both strictly negative vectors. So, we can assume, for example, $(v_1, v_2, v_3) = (1, 5, -8)$, and hence $\mathbf{d}_1 = [54.5\ 13\ 19.5\ 36\ 74]^\top$, $\mathbf{d}_2 = [17.6\ 23.6\ 5.5\ 19.9]^\top$ and $\mathbf{d}_3 = [18\ 14.125]^\top$. The dynamics of the state vector of the system, with random initial conditions $\mathbf{x}(\mathbf{0})$ taken as realizations of a gaussian vector with 0 mean and variance $\sigma^2 = 4$, i.e. $\mathbf{x}(0) \sim \mathcal{N}(0, 4)$, is given in Fig. 3.5. The plot shows that tripartite consensus is reached after about 1.8 units of time with regime values $(c_1\ c_2\ c_3) = (0.22\ \ 1.11\ \ -1.76) = 0.22\cdot(1\ 5\ -8) = 0.22\cdot(v_1\ v_2\ v_3)$.

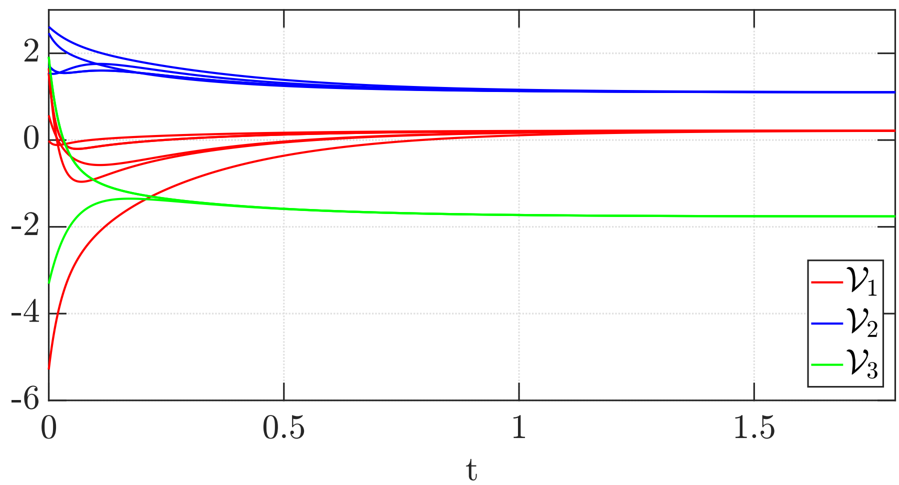

Figure 3.5: Tripartite consensus for Example 1.

3.8 Sign Consensus

In this section we introduce the concept of sign consensus for which a formal definition is given in the following.

Definition 3.14 (Sign Consensus). *The overall multi-agent system described as in (3.5), with $\mathcal{L}_{\mathcal{D}} \in \mathbb{R}^{N\times N}$ described as in (3.4), $\mathcal{D} \in \mathbb{R}^{N\times N}$ described as in (3.2) and $\mathcal{D}_i \in \mathbb{R}^{n_i\times n_i}$, for $i \in [1,3]$, diagonal matrices, whose interconnection topology is described by an undirected, signed and connected communication graph $\mathcal{G}$, having 3 clusters, reaches sign consensus if there exists a relabelling of the three clusters such that, for every index $i \in \mathcal{V}_2$, $\lim_{t\to\infty} x_i(t) = 0$, while for every $i,j \in \mathcal{V}_1 \cup \mathcal{V}_3$*

$$\lim_{t\to\infty} \text{sgn}(x_i(t)) - \text{sgn}(x_j(t)) = 0, \quad \text{if } \exists m : i,j \in \mathcal{V}_m,$$
$$\lim_{t\to\infty} \text{sgn}(x_i(t)) - \text{sgn}(x_j(t)) \neq 0, \quad \text{if } \nexists m : i,j \in \mathcal{V}_m.$$

The following lemma provides necessary and sufficient conditions for sign consensus to be reached.

Lemma 3.15. *Given an undirected, signed, weighted and connected communication graph, $\mathcal{G}$, having 3 clusters, the multi-agent system (4.3.1), with communication graph $\mathcal{G}$ and distributed control law (3.3), and hence described as in (3.5), with $\mathcal{L}_{\mathcal{D}} \in \mathbb{R}^{N\times N}$ given in (3.4), $\mathcal{A}$ in (3.2) for $k = 3$, $\mathcal{D} \in \mathbb{R}^{N\times N}$ in (3.2) and $\mathcal{D}_i \in \mathbb{R}^{n_i\times n_i}$, for $i \in [1,3]$, diagonal matrices reaches sign consensus if and only if the following conditions hold:*

i) $\mathcal{L}_{\mathcal{D}}$ is a singular positive semi-definite matrix.

ii) There exists a reordering $\{i_1, i_2, i_3\}$ of the index set $\{1,2,3\}$ such that every nonzero vector in the kernel of $\mathcal{L}_{\mathcal{D}}$ can be expressed as $\mathbf{v} = [\mathbf{v}_1^\top, \mathbf{v}_2^\top, \mathbf{v}_3^\top]^\top$ with $\mathbf{v}_{i_2} = 0$, and in the pair $(\mathbf{v}_{i_1}, \mathbf{v}_{i_3})$ one of the vectors is strictly positive and one is strictly negative.

Proof. Analogous to the proof of Lemma 3.1. □

Theorem 3.16. *Consider the multi-agent system (4.3.1), with undirected, signed, weighted and connected communication graph $\mathcal{G}$ satisfying Assumption 1 and Assumption 2 for a suitable choice of $i_1, i_2 \in [1,3], i_1 \neq i_2$. Also, suppose that the following conditions hold:*

a) every agent in $\mathcal{V}_{i_1}$ has at least one enemy in $\mathcal{V}_{i_2}$, which means that $\mathcal{A}_{i_1,i_2}\mathbf{1}_{n_{i_2}} \ll 0$, and

b) there exist vectors $\mathbf{v}_{i_1} \in \mathbb{R}^{n_{i_1}}$ and $\mathbf{v}_{i_2} \in \mathbb{R}^{n_{i_2}}$, one of them strictly positive and the other strictly negative, such that $\mathcal{A}_{i_3,i_1}\mathbf{v}_{i_1} + \mathcal{A}_{i_3,i_2}\mathbf{v}_{i_2} = 0$, where $i_3 = [1,3] \setminus \{i_1, i_2\}$.

Then there exist diagonal matrices $\mathcal{D}_i \in \mathbb{R}^{n_i \times n_i}, i \in [1,3]$, such that the distributed control law (3.3), with $\mathcal{L}_\mathcal{D} \in \mathbb{R}^{N\times N}$ described as in (3.4), $\mathcal{D} \in \mathbb{R}^{N\times N}$ described as in (3.2), makes the closed-loop multi-agent system (3.5) reach sign consensus.

Proof. We can always relabel the vertices in $\mathcal{V}$ so that assumption a) and b) hold for $i_1 = 1, i_2 = 3, i_3 = 2$.

By Lemma 3.15, it will be sufficient to prove that under assumptions a)-b), it is always possible to choose the diagonal matrices $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ so that **(A)** the matrix $\mathcal{L}_\mathcal{D}$ is singular and positive semi-definite with a simple eigenvalue in 0, and **(B)** the kernel of $\mathcal{L}_\mathcal{D}$ includes the vector $\mathbf{v} = [\mathbf{v}_1^\top, \mathbf{0}_{n_2}^\top, \mathbf{v}_3^\top]^\top$, where $\mathbf{v}_1 \in \mathbb{R}^{n_1}$ and $\mathbf{v}_3 \in \mathbb{R}^{n_3}$ are two vectors satisfying assumptions b), and we assume w.lo.g. that $\mathbf{v}_1 \gg 0$ and $\mathbf{v}_3 \ll 0$.

To prove **(B)** we note that solving the system of equations $\mathcal{L}_\mathcal{D}\mathbf{v} = 0\mathbf{v}$ is equivalent to solve the system

$$\begin{bmatrix} \mathcal{D}_1 - \mathcal{A}_{1,1} & -\mathcal{A}_{1,2} & -\mathcal{A}_{1,3} \\ -\mathcal{A}_{2,1} & \mathcal{D}_2 - \mathcal{A}_{2,2} & -\mathcal{A}_{2,3} \\ -\mathcal{A}_{3,1} & \mathcal{A}_{3,2} & \mathcal{D}_3 - \mathcal{A}_{3,3} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{0}_{n_2} \\ \mathbf{v}_3 \end{bmatrix} = \mathbf{0}_N, \tag{3.32}$$

and this in turn is equivalent to the three identities

$$\mathcal{D}_1\mathbf{v}_1 = \mathcal{A}_{1,1}\mathbf{v}_1 + \mathcal{A}_{1,3}\mathbf{v}_3 \tag{3.33}$$

$$\mathbf{0}_{n_2} = \mathcal{A}_{2,1}\mathbf{v}_1 + \mathcal{A}_{2,3}\mathbf{v}_3. \tag{3.34}$$

$$\mathcal{D}_3\mathbf{v}_3 = \mathcal{A}_{3,1}\mathbf{v}_1 + \mathcal{A}_{3,3}\mathbf{v}_3. \tag{3.35}$$

Identity (3.34) holds by assumption b) and we note that the constraint (3.33) and (3.35) allow to uniquely determine [1] the diagonal matrices $\mathcal{D}_1$ and $\mathcal{D}_3$, since they can be componentwise written as

$$[\mathcal{D}_p]_{i,i} = \frac{1}{[\mathbf{v}_p]_i}\left(\sum_{j;j\neq i} [\mathcal{A}_{p,p}]_{ij}[\mathbf{v}_p]_j + \sum_{k=1}^{n_q} [\mathcal{A}_{p,q}]_{i,k}[\mathbf{v}_q]_k \right) \tag{3.36}$$

[1] Note, however, that the values of $\mathcal{D}_1$ and $\mathcal{D}_3$ depend on the specific choice of the vectors $\mathbf{v}_1$ and $\mathbf{v}_3$ satisfying (3.34), which are not necessarily uniquely determined.

$$\begin{bmatrix} \mathcal{D}_2 - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2} & -\mathcal{A}_{2,3} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3} \\ -\mathcal{A}_{3,2} - \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2} & \mathcal{D}_3 - \mathcal{A}_{3,3} - \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3} \end{bmatrix} \succeq 0. \quad (3.39)$$

---

for $p, q \in \{1, 3\}, p \neq q$.

We are now remained with proving that, after having determined the matrices $\mathcal{D}_1$ and $\mathcal{D}_3$, it is always possible to choose $\mathcal{D}_2$ so that **(A)** is satisfied. To do so we proceed as follows (see Section 2.3.1): we first verify that the upper diagonal block of $\mathcal{L}_\mathcal{D}$:

$$\mathcal{L}_\mathcal{D} = \left[\begin{array}{c|cc} \mathcal{D}_1 - \mathcal{A}_{1,1} & -\mathcal{A}_{1,2} & -\mathcal{A}_{1,3} \\ \hline -\mathcal{A}_{2,1} & \mathcal{D}_2 - \mathcal{A}_{2,2} & -\mathcal{A}_{2,3} \\ -\mathcal{A}_{3,1} & -\mathcal{A}_{3,2} & \mathcal{D}_3 - \mathcal{A}_{3,3} \end{array}\right], \quad (3.37)$$

is positive definite, namely condition

$$\mathcal{D}_1 - \mathcal{A}_{1,1} \succ 0 \quad (3.38)$$

holds, and then impose (by means of a suitable choice of $\mathcal{D}_2$) that its Schur complement is positive semi-definite with a simple eigenvalue in 0, namely it verifies condition (3.39), and it has a simple eigenvalue in 0. Condition (3.33) ensures that

$$(\mathcal{D}_1 - \mathcal{A}_{1,1})\mathbf{v}_1 = \mathcal{A}_{1,3}\mathbf{v}_3 \gg 0, \quad (3.40)$$

where we used the fact that $\mathbf{v}_3 \ll 0$ and $\mathcal{A}_{1,3}$ has no zero rows. Then Lemma 2.5, part i), holds with $\mathbf{v} = \mathbf{v}_1$, thus ensuring that $\mathcal{D}_1 - \mathcal{A}_{1,1}$ is positive definite.

To ensure that (3.39) holds for a suitable choice of $\mathcal{D}_2$, we iterate the same procedure, and impose condition (3.41):

$$\mathcal{D}_{2,2} - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2} \succ 0, \quad (3.41)$$

as well as condition (3.43).

To address condition (3.41), we first observe that by Lemma 2.5, part ii), $(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}$ is symmetric and nonnegative, and hence so is $A := \mathcal{A}_{2,2} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}$. Let us set $\mathbf{a}_{i2} := \mathcal{A}_{i,2}\mathbf{1}_{n_2}, i \in [1, 2]$, and $\mathbf{d}_2 := \mathcal{D}_2\mathbf{1}_{n_2}$, and impose the following constraint on $\mathbf{d}_2$:

$$\mathbf{d}_2 \gg \mathbf{a}_{22} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{12}. \quad (3.43)$$

$$\begin{aligned}\Phi_3 \ := \ & \mathcal{D}_3 - \mathcal{A}_{3,3} - \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3} - [\mathcal{A}_{3,2} + \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}] \\ & \cdot[\mathcal{D}_2 - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}]^{-1}[\mathcal{A}_{2,3} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}] \succeq 0.\end{aligned} \tag{3.42}$$

Then it is easy to verify that

$$(D - A)\mathbf{1}_{n_2} = \mathbf{d}_2 - \mathbf{a}_{22} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{12} \gg 0,$$

where $D = \mathcal{D}_2$. But then we can apply Lemma 2.5, part i), again, to claim that $D - A$ is positive definite, namely (3.41) holds.

We now observe that we can always choose the positive diagonal entries of the diagonal matrix $\mathcal{D}_2$ sufficiently large to ensure that (not only (3.43) holds, but also) the entries of $[\mathcal{A}_{3,2} + \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}][\mathcal{D}_2 - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}]^{-1}[\mathcal{A}_{2,3} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}]$ are arbitrarily small and hence also the matrix $A = -\Phi_3 + \mathcal{D}_3$ has positive off-diagonal entries. This ensures that $-\Phi_3$ is an irreducible Metzler matrix.

So, we now prove that (3.43) holds. We observe that from condition a) for $i_1 = 1$ and $i_2 = 3$ it follows that $\mathcal{A}_{3,3} + \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}$ is a nonnegative matrix whose off-diagonal entries are all positive. If we apply the vector $-\mathbf{v}_3 \gg 0$ on the right side of the matrix Φ_3, appearing in (3.43), we obtain

$$\begin{aligned}-\Phi_3\mathbf{v}_3 \ = \ & -\mathcal{D}_3\mathbf{v}_3 + \mathcal{A}_{3,3}\mathbf{v}_3 + \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}\mathbf{v}_3 + [\mathcal{A}_{3,2} + \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}] \\ \cdot \ & [\mathcal{D}_2 - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,2}]^{-1} \cdot [\mathcal{A}_{2,3}\mathbf{v}_3 + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}\mathbf{v}_3].\end{aligned}$$

We first note that by (3.35) we have $-\mathcal{D}_3\mathbf{v}_3 + \mathcal{A}_{3,3}\mathbf{v}_3 = -\mathcal{A}_{3,1}\mathbf{v}_1$. On the other hand, from equation (3.33) one gets

$$\mathcal{A}_{1,3}\mathbf{v}_3 = (\mathcal{D}_1 - \mathcal{A}_{1,1})\mathbf{v}_1, \tag{3.44}$$

from which it follows

$$\mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}\mathbf{v}_3 = \mathcal{A}_{3,1}\mathbf{v}_1. \tag{3.45}$$

Therefore

$$-\mathcal{D}_3\mathbf{v}_3 + \mathcal{A}_{3,3}\mathbf{v}_3 + \mathcal{A}_{3,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}\mathbf{v}_3 = 0. \tag{3.46}$$

On the other hand, from (3.44) it also follows that $\mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}\mathbf{v}_3 = \mathcal{A}_{21}\mathbf{v}_1$, and making use of (3.34), this latter identity leads to

$$\mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,3}\mathbf{v}_3 + \mathcal{A}_{2,3}\mathbf{v}_3 = 0. \tag{3.47}$$

So, by replacing (3.46) and (3.47) in the expression of $-\Phi_3\mathbf{v}_3$ we obtain the zero vector. Since $-\Phi_3$ is an irreducible Metzler matrix, this ensures (see Theorem 2.3) that 0 is the simple dominant eigenvalue of $-\Phi_3$ and hence Φ_3 is positive semidefinite and singular with a simple eigenvalue in 0. Since the eigenvalues of $\mathcal{L}_{\mathcal{D}}$ are the union of the eigenvalues of the matrices in (3.38) and (3.41) and of the matrix Φ_3, that have been obtained from $\mathcal{L}_{\mathcal{D}}$ by applying the Schur complement, then $\mathcal{L}_{\mathcal{D}}$ is positive semidefinite with a simple eigenvalue in 0, and hence **(A)** holds.
To conclude, we have proved that by setting $\mathcal{D}_1$ and $\mathcal{D}_3$ as in (3.36), by choosing the diagonal entries of the diagonal matrix $\mathcal{D}_2$ sufficiently large, both conditions of Lemma 3.15 are fulfilled, and the overall multi-agent system reaches sign consensus. □

Example 3.17. *Consider the undirected, signed, unweighted, connected and clustered communication graph, with three clusters of cardinality $n_1 = 5$, $n_2 = 4$, $n_3 = 2$, respectively, and adjacency matrix whose submtarices $\mathcal{A}_{i,i}, i \in [1,3]$ are as in Example 1, while the remaining blocks are*

$$\mathcal{A}_{1,2} = -\begin{bmatrix} 3 & 3 & 0 & 3 \\ 6 & 6 & 0 & 12 \\ 6 & 6 & 0 & 6 \\ 6 & 6 & 0 & 6 \\ 0 & 0 & 3 & 3 \end{bmatrix}, \mathcal{A}_{1,3} = -\begin{bmatrix} 3.5 & 1 \\ 0 & 1.5 \\ 1 & 1 \\ 1.5 & 0 \\ 3.5 & 1.5 \end{bmatrix}, \mathcal{A}_{2,3} = -\begin{bmatrix} 36 & 3 \\ 24 & 6 \\ 12 & 0 \\ 12 & 15 \end{bmatrix}.$$

Condition b) of Theorem 3.16 holds for $\mathbf{v}_1 = [2,1,1,1,2]^\top \gg 0$, $\mathbf{v}_3 = -[0.5,2]^\top \ll 0$, while $\mathbf{d}_2 = \mathbf{a}_{22} + \mathcal{A}_{2,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathbf{a}_{12} + \mathbf{v}$, where $\mathbf{v}$ is a vector whose entries are the absolute value of the entries of the realization of a Gaussian vector with 0 mean and standard deviation $\sigma = 200$.

The dynamics of the state vector of system (3.5), with random initial conditions $\mathbf{x}(\mathbf{0})$ taken as realizations of a Gaussian vector with 0 mean and variance $\sigma^2 = 4$, i.e. $\mathbf{x}(0) \sim \mathcal{N}(0,4)$, is given in Fig. 3.6.

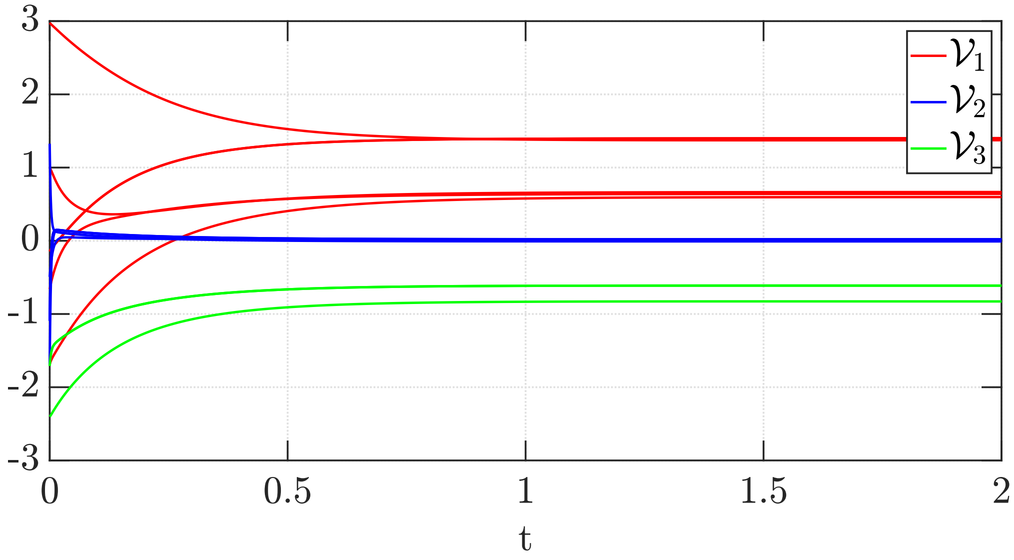

Figure 3.6: Sign consensus for Example 2.

Chapter 4

Dynamic Social Balance

In this chapter we propose two discrete time binary models, based on the homophily and influence social mechanisms, that dynamically reduce the cognitive dissonance among the agents in a social network. The first model, according to our interpretation, represents a mathematical formalization of a cognitive bias known in the literature as *b*andwagon bias and its dynamics is based on the interplay between appraisals and opinions. The dynamics of the second model is uniquely driven by the homophily mechanism and only the interpersonal appraisals are taken into account. It will be shown that both models can drive an initially structurally unbalanced network towards a socially balanced one.

The results presented in this chapter can be found in:

- G. De Pasquale, M. E. Valcher, "A Bandwagon Bias Based Model for Opinion Dynamics: Intertwining between Homophily and Influence Mechanisms", European Journal for Control, pp. 1-22, 2022, https://doi.org/10.1016/j.ejcon.2022.100675.
- G. De Pasquale, M. E. Valcher, "A binary homophily model for opinion dynamics", European Control Conference (ECC), pp. 1515-1520, Rotterdam, The Netherlands, 2022, 10.23919/ECC54610.2021.9655049.

4.1 Introduction

In several cases, sociological models represent the primary focus of the investigation, but there are numerous contexts, such as product promotion, spread of diseases, resource allocation, etc., where social dynamics represents the context in which other phenomena evolve. Consequently, understanding its behaviour is a preliminary but fundamental step in order to investigate and understand the evolution of the process of interest Aghbolagh et al. (2019), Marvel et al. (2011). As a result, it becomes of great importance to build a reliable model for the social dynamics, that allows to forecast the network evolution and thus to design strategies aimed at driving the network towards the desired configuration Proskurnikov & Tempo (2017). Dynamic social balance theory is concerned with the study and analysis of the evolution of socially unbalanced networks towards socially balanced ones, namely networks in balanced configurations in which all the agents split in (at most) two groups in such a way that all the agents in the same group have friendly relationships, while agents from different groups have not Harary et al. (1988), Heider (1944).

Even if, from a modeling perspective, the study of social balance has rather remote origins, as witnessed by the pioneering works of Heider (1944),Harary (1959), Harary et al. (1988), the dynamic social balance theory represents an active and timely research topic. In this regard, a recent interesting work in which a sociological mathematical model, including two coexisting social mechanisms, is studied, is the work from Liu et al. (2020). This work inspired the model presented in Section 4.3 that represents its mixed binary and real valued counterpart.

Both models can be interpreted as a mathematical formalization of a form of cognitive bias known in psychology as "bandwagon bias" Niesiobedzka (2017). Namely the bias according to which our opinions on topics are influenced by the opinions that other individuals have on the same topics and by the relationships we have with those individuals. Bandwagon bias results in an intertwined dynamics involving both a homophily mechanisms for the interpersonal relationships and an influence mechanism for the agents' opinions. This is in line with the fact that, in real life, interpersonal appraisals influence individual opinions and viceversa. It is worth noticing that this model applies better to online social-networks rather than real-life ones, since in the first case it is reasonable to assume that opinions and interpersonal relationships evolve according to the same time-scale.

The model presented in Section 4.4 is entirely binary and it only involves mutual interpersonal relationships among agents. Its dynamics is driven by the natural human propensity to minimize the number of unbalance triads every pairs of individuals belongs to Heider (1944).

The motivation behind the study of the dynamical evolution of unweighted signed social networks comes from the fact that there are many circumstances in which recognizing the type (friendly or hostile) of relationship between individuals is easy, while assessing its intensity is complicated and prone to model errors. In fact, while individual evaluations of certain products or their opinions on certain topics can be easily obtained, attributing numerical values to the mutual appraisals is more challenging.

4.2 Preliminaries

4.2.1 Two Social Mechanisms

We introduce two social mechanisms that drive human behaviour to minimize the cognitive dissonance of the network and that drive the dynamics of the models presented in this chapter Mei et al. (2019), Lazarsfeld & Merton (1954), Friedkin & Johnsen (2011).

(i) **Homophily mechanism**: the interpersonal appraisals of any two individuals in a social group are adjusted based on whether they agree on the appraisals of the group members,

(ii) **Influence mechanism**: each individual assigns influence to others proportionally to her/his appraisal of them.

In this chapter we consider undirected and signed graphs with unitary self loops. Therefore the adjacency matrix of the graph belongs to the set De Pasquale & Valcher (2021)

$$\mathcal{S}_1^N := \{\mathbf{M} \in \{-1,0,1\}^{N\times N} : \mathbf{M} = \mathbf{M}^\top,\ [M]_{ii} = 1,\ \forall i \in [1,N]\}. \tag{4.1}$$

Lemma 4.1 (Structural balance for complete graphs)**.** *Given a matrix $\mathbf{X} \in \mathcal{S}_1^N \cap \{-1,1\}^{N\times N}$, the following facts are equivalent:*

i) $\mathbf{X} = \mathbf{p}\mathbf{p}^\top$*, for some vector* $\mathbf{p} \in \{-1,1\}^N$*;*

ii) *rank*$(\mathbf{X}) = 1$*;*

ii) *for every* $a, b \in [1,N]$ *either* $\mathbf{e}_a^\top \mathbf{X} = \mathbf{e}_b^\top \mathbf{X}$ *or* $\mathbf{e}_a^\top \mathbf{X} = -\mathbf{e}_b^\top \mathbf{X}$ *;*

iv) the graph $\mathcal{G}(\mathbf{X})$ is structurally balanced;

v) all the triads (i, j, k) of distinct vertices in $\mathcal{G}(\mathbf{X})$ are balanced.

In the following we will say that $\mathbf{X}$ is structurally balanced if $\mathcal{G}(\mathbf{X})$ is structurally balanced. Given a group of N agents, we denote by $\mathbf{X}(t) \in \{-1, 0, 1\}^{N\times N}$ the *appraisal matrix at time* t of the agents, whose (i, j)-th entry represents agent i's appraisal of agent j at time t. $[X(t)]_{ij} = 1$ if i has positive feelings towards j and $[X(t)]_{ij} = -1$ if i has negative feelings towards j, while $[X(t)]_{ij} = 0$ if i chooses not rely on j in forming its opinion[1]. We assume that for each pair of agents (i, j) at each time instant t the appraisal is mutual, namely $[X(t)]_{ij} = X_{ji}(t)$ $\forall i, j \in [1, N]$, and hence $\mathbf{X}(t)$ is a symmetric matrix $\forall t \geq 0$. The (undirected and signed) graph $\mathcal{G}(\mathbf{X})$, having $\mathbf{X}$ as adjacency matrix, represents the *appraisal network* Mei et al. (2019).

4.3 A Bandwagon Bias Based Model for Opinion Dynamics: Intertwining between Homophily and Influence Mechanisms

4.3.1 The Model: Properties, Equilibrium Points and Periodic Solutions

In this section we assume that the agents express their opinions about a certain number, say m, of issues. This information is collected in a matrix $\mathbf{Y}(t) \in \mathbb{R}^{N\times m}$, whose (i, j)-th entry is the opinion that agent i has about the issue j at the time instant t. $\mathbf{Y}(t)$ is called the *opinion matrix at the time instant* t of the social network. We assume that the opinion matrix and the appraisal matrix evolve according to an intertwined dynamics expressed by the following equations

$$\mathbf{X}(t+1) = \text{sgn}(\mathbf{Y}(t)\mathbf{Y}(t)^\top) \tag{4.2}$$

$$\mathbf{Y}(t+1) = \frac{1}{N}\mathbf{X}(t+1)\mathbf{Y}(t) \tag{4.3}$$

[1]Since we consider small-medium size networks, this formalizes the case when agent i knows agent j but does not find correlation between its own choices and agent j's opinions, and hence chooses not to give it any weight.

that component-wise correspond to

$$[X(t+1)]_{ij} = \text{sgn}\left(\sum_{k=1}^{m}[Y(t)]_{ik}[Y(t)]_{jk}\right) \tag{4.4}$$

$$[Y(t+1)]_{ij} = \frac{1}{N}\sum_{k=1}^{N}[X(t+1)]_{ik}[Y(t)]_{kj}. \tag{4.5}$$

Equation (4.5) shows that the opinion that agent i has about issue j at the time instant $t+1$ is a (signed) weighted average of the opinions that all agents have about the topic j at the time instant t, where the weights are the appraisals that agent i has about them at the time instant t, divided by the number of agents.

On the other hand, from equation (4.4), we notice that the the (i, j)-th entry of the appraisal matrix at the time instant $t+1$, namely, the appraisal that agent i has about agent j at the time instant $t+1$, depends on the comparison between the opinions that agents i and j have about all the topics at the time instant t. In particular, if the agents agree (resp. disagree) on a specific issue k, this will give a positive (resp. negative) contribution $[Y(t)]_{ik}[Y(t)]_{jk} > 0$ (resp. $[Y(t)]_{ik}[Y(t)]_{jk} < 0$), in determining the relationship between i and j at the time instant $t+1$.

Essentially, this model captures the evolution of opinion-dependent time-varying graph structures. In this regard one can see analogies with the pioneering work form Hagselmann-Krause Hegselmann & Krause (2002), in which the closeness of opinions determines the structure topology of the (unweighted) interaction graph. On the other hand, in our model all agents potentially communicate and their opinions will rather determine the type (friendly/antagonistic) of relationship. Equations (4.2) and (4.3) can be grouped into a single equation that describes the update of the opinion matrix alone and takes the form

$$\mathbf{Y}(t+1) = \frac{1}{N}\text{sgn}(\mathbf{Y}(t)\mathbf{Y}(t)^\top)\mathbf{Y}(t). \tag{4.6}$$

Equation (4.6) shows that the mathematical abstraction of the bandwagon bias leads the intertwining between opinion dynamics and appraisal dynamics to a peculiar form of opinion dynamics model. It is immediate to notice that if $\mathbf{Y}(0)$ has a zero row (a situation that formalizes the case when one of the agents expresses no opinion on any of the m topics), then that same row remains zero in every subsequent opinion matrix $\mathbf{Y}(t), t \geq 0$. Similarly, if $\mathbf{Y}(0)$ has a zero column (none of the agents expresses any judgement on a specific topic), that same column remains zero in all the matrices $\mathbf{Y}(t), t \geq 0$. Therefore both cases are of no interest (substantially, one can always remove the agent and/or the topic and focus on the analysis of the remaining variables) and will not be considered in the following.

Remark 4.2. *Compared with the model proposed and investigated in Liu et al. (2020), we have modified the law that governs the appraisal matrix update and how it affects the opinion dynamics in two aspects. First, we have chosen to keep into account only the signs of the mutual appraisals, rather than their absolute values. This is motivated by the fact that, in a lot of practical situations, being able to assess the sign of the mutual appraisal is easier and more robust to modeling errors with respect to determining the numerical value associated to the tie strength. Moreover, the influence that agent j can have on the opinion agent i has on a certain issue does not necessarily scale with the absolute value of $[X]_{ij}$. Secondly, we have chosen to "give a weight" also to the fact that a pair of agents chooses not rely on each other's opinion, namely to the fact that $[X]_{ij} = 0$. Since we consider small-medium size networks, this formalizes the case when agent i knows agent j but does not find correlation between its own choices and agent j's opinions, and hence chooses not to give it any weight. In this perspective, the fact that the mutual appraisal is 0 is an information that should be considered and this motivates the fact that in the opinion dynamics update equation (4.5) each row is divided by the overall number of agents N, rather than by the absolute value of its entries. It is worth noticing that, however, since the appraisal matrix is obtained by comparing the (real valued) opinions of the agents on the various topics into play, and its (i, j)-th entry is zero only if the opinion vectors of agents i and j are orthogonal, a zero entry in the appraisal matrix is a very rare occurrence, as it will be confirmed by the numerical simulations at the end of the section.*
As we will see in the following, our model retains all the relevant features of the model investigated in Liu et al. (2020), and it is simpler to analyse and implement.

Assumption 1 (No zero rows/columns). *In the following, we will steadily assume that $\mathbf{Y}(0)$ is devoid of zero rows/columns.*

Lemma 4.3 (No zero rows dynamics)**.** *If $\mathbf{Y}(0) \in \mathbb{R}^{N\times m}$ has no zero rows, then for every $t \geq 0$ the matrix $\mathbf{Y}(t)$, obtained from the model (4.6) corresponding to the initial condition $\mathbf{Y}(0)$, has no zero rows.*

Proof. Suppose, by contradiction, that this is not the case, and let $t_0 \geq 0$ be the smallest time instant such that $\mathbf{Y}(t_0)$ has no zero rows, but $\mathbf{Y}(t_0 + 1)$ has (at least) one zero row.

It entails no loss of generality assuming that the first row of $\mathbf{Y}(t_0+1)$ becomes zero (if not we can always resort to a relabelling of the agents to reduce ourselves to this case). If we set $\mathbf{Y} := \mathbf{Y}(t_0)$, this means that $\mathbf{Y}$ has no zero rows, but

$$\mathbf{e}_1^\top \mathrm{sgn}(\mathbf{Y}\mathbf{Y}^\top)\mathbf{Y} = \mathbf{0}^\top.$$

Set $\mathbf{z}^\top := \mathbf{e}_1^\top \mathrm{sgn}(\mathbf{Y}\mathbf{Y}^\top) \in \{-1,0,1\}^{1\times N}$. We observe that since the first row of $\mathbf{Y}$ is not zero then the $(1,1)$-entry of $\mathbf{Y}\mathbf{Y}^\top$ is positive and hence the first entry of $\mathbf{z}$ is 1. The remaining ones belong to $\{-1,0,1\}$. We distinguish two cases: either all the other entries of $\mathbf{z}$ are zero (Case A) or there exist other nonzero entries in $\mathbf{z}$ (Case B), and in this latter case we can assume without loss of generality (if not, we can always permute the m topics, namely the m columns of $\mathbf{Y}$, to make this possible) that

$$\mathbf{z}^\top = \left[\begin{array}{c|ccc|ccc} 1 & z_2 & \dots & z_r & 0 & \dots & 0 \end{array}\right], \quad \begin{array}{c} z_i \in \{-1,1\}, \\ i \in [2,r]. \end{array}$$

Condition $\mathbf{z}^\top \mathbf{Y} = \mathbf{0}^\top$ implies that the columns of $\mathbf{Y}$ are all orthogonal to the vector $\mathbf{z}$. In Case B this implies that $\mathbf{Y}$ can be expressed as

$$\mathbf{Y} = \left[\begin{array}{c|c} \mathbf{1}_{r-1}^\top & \mathbf{0}^\top \\ \Sigma & \mathbf{0}^\top \\ \hline 0 & I_{N-r} \end{array}\right] \begin{bmatrix} C_a \\ C_b \end{bmatrix} \tag{4.7}$$

for some matrices $C_a \in \mathbb{R}^{(r-1)\times m}$ and $C_b \in \mathbb{R}^{(N-r)\times m}$, where $\Sigma := -\mathrm{diag}\{z_2,\dots,z_r\}$. On the other hand, the vector $\mathbf{z}^\top$ and the matrix $\mathbf{Y}$ are related by the identity $\mathbf{z}^\top = \mathbf{e}_1^\top \mathrm{sgn}(\mathbf{Y}\mathbf{Y}^\top) = \mathrm{sgn}(\mathbf{e}_1^\top \mathbf{Y}\mathbf{Y}^\top)$, and hence it must be

$$\left[\begin{array}{c|ccc|ccc} 1 & z_2 & \dots & z_r & 0 & \dots & 0 \end{array}\right] = \mathrm{sgn}\left(\mathbf{1}_{r-1}^\top C_a \begin{bmatrix} C_a^\top & C_b^\top \end{bmatrix} \left[\begin{array}{c|c|c} \mathbf{1}_{r-1} & \Sigma & 0 \\ 0 & 0 & I_{N-r} \end{array}\right]\right).$$

This implies, in particular, that

$$\begin{bmatrix} z_2 & \dots & z_r \end{bmatrix} = \mathrm{sgn}(\mathbf{1}_{r-1}^\top C_a C_a^\top \Sigma),$$

or, entrywise, keeping into account the definition of Σ:

$$z_i = -\mathrm{sgn}(\mathbf{1}_{r-1}^\top C_a C_a^\top z_i \mathbf{e}_{i-1}), \quad \forall\, i \in [2,r].$$

This amounts to saying that

$$\mathrm{sgn}(\mathbf{1}_{r-1}^\top C_a C_a^\top \mathbf{e}_{i-1}) = -1, \quad \forall\, i \in [2,r],$$

namely $\mathbf{1}_{r-1}^\top C_a C_a^\top \ll 0$, by this meaning that it is a vector with all negative entries. But this would imply $\|C_a^\top \mathbf{1}_{r-1}\|^2 = \mathbf{1}_{r-1}^\top C_a C_a^\top \mathbf{1}_{r-1} < 0$, which is clearly impossible.
We consider now Case A. If the only nonzero entry of $\mathbf{z}$ is the first one, then $\mathbf{Y}$ can be expressed as $\mathbf{Y} = \mathbf{W}C_0$, where $\mathbf{W} = [\mathbf{0}|I_{N-1}]^\top$ and C_0 is a real matrix of size $(N-1) \times m$. By resorting to the same reasoning as in Case B, condition $\mathbf{z}^\top = \mathbf{e}_1^\top \text{sgn}(\mathbf{Y}\mathbf{Y}^\top)$ becomes

$$\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} = \text{sgn}(\mathbf{e}_1 \mathbf{W} C_0 C_\mathbf{0}^\top \mathbf{W}^\top) = \text{sgn}(\mathbf{0}^\top),$$

which is impossible. Therefore it is not possible that there exists $t_0 \geq 0$ such that $\mathbf{Y}(t_0)$ has no zero rows, but $\mathbf{Y}(t_0+1)$ has (at least) one zero row. □

Based on the preliminary remarks and Lemma 4.3, we introduce the set Mei et al. (2019)

$$\mathcal{S}_{nz-rows} := \{\mathbf{Y} \in \mathbb{R}^{N \times m} : \mathbf{e}_i^\top \mathbf{Y} \neq \mathbf{0}^\top, \forall\, i \in [1, N]\},$$

and in the following we will steadily assume that $\mathbf{Y}(0) \in \mathcal{S}_{nz-rows}$, and hence $\mathbf{Y}(t) \in \mathcal{S}_{nz-rows}$ for every $t \geq 0$. It is worth noticing that, differently from Liu et al. (2020), we do not need to impose that $\mathbf{Y}(0) \in \mathcal{Y} := \{\mathbf{Y} : \mathbf{Y}(t) \in \mathcal{S}_{nz-rows} \forall t \geq 0\}$, since for our model it suffices to assume that $\mathbf{Y}(0) \in \mathcal{S}_{nz-rows}$ to guarantee that $\mathbf{Y}(t) \in \mathcal{S}_{nz-rows}$, $\forall t \geq 0$.
Note that, as a further consequence, for every $t \geq 0$, $\mathbf{X}(t+1) = \text{sgn}(\mathbf{Y}(t)\mathbf{Y}(t)^\top)$ is a symmetric matrix with unitary diagonal entries, and hence belongs to $\mathcal{S}_1^N$, $\forall t \geq 0$.

Remark 4.4. *The case when there exists $t > 0$ such that the matrix $\mathbf{Y}(t)$ has a zero column, even if $\mathbf{Y}(0)$ has no zero columns, may arise, but it is a rare occurrence. This happens if and only if one of the columns of $\mathbf{Y}(t)$ belongs to the kernel of the matrix $\mathbf{X}(t+1) = sgn(\mathbf{Y}(t)\mathbf{Y}(t)^\top)$. This means that at the time t the column vector describing the opinions that the agents have on some specific topic is such that* for every agent i *the sum of the opinions of the agents trusted by i equals the sum of the opinions of the agents not trusted by agent i. Since the agents'opinions are arbitrary real numbers this case arises for a set of initial conditions $\mathbf{Y}(0)$ having zero measure.*

An elementary example is represented by the case when $\mathbf{Y}(0) = \begin{bmatrix} 1 & \epsilon \\ 2 & -\epsilon \end{bmatrix}$, where ϵ is nonzero and sufficiently small. Correspondingly, we get $\mathbf{Y}(1) = \begin{bmatrix} 3/2 & 0 \\ 3/2 & 0 \end{bmatrix}$.

After having explored these preliminary aspects regarding agents that become indifferent to all issues, or issues that become irrelevant to all agents, we want to investigate the existence and structure of the equilibrium points for the model (4.2)- (4.3), when starting from initial opinion matrices $\mathbf{Y}(0)$ satisfying Assumption 1.

Definition 4.5 (Equilibrium point)**.** *A pair* $(\mathbf{Y}^*, \mathbf{X}^*)$ *is an* equilibrium point *for the model* (4.2)- (4.3) *if*

$$\mathbf{X}^* = \operatorname{sgn}(\mathbf{Y}^*(\mathbf{Y}^*)^\top) \tag{4.8}$$

$$\mathbf{Y}^* = \frac{1}{N}\mathbf{X}^*\mathbf{Y}^*. \tag{4.9}$$

It is interesting to notice that the only possible nontrivial equilibrium points for the model are those that correspond to a structurally balanced configuration of the appraisal network $\mathcal{G}(\mathbf{X}^*)$. Moreover, the appraisal network is necessarily complete, namely each agent needs to express its appraisal towards all the other agents.

Proposition 4.6 (Equilibrium equivalence conditions)**.** *A pair* $(\mathbf{Y}^*, \mathbf{X}^*) \neq (\mathbf{0}, \mathbf{0})$ *is an equilibrium point for the model* (4.2)- (4.3) *if and only if*

i) $\mathbf{X}^* = \mathbf{p}\mathbf{p}^\top$*, for some* $\mathbf{p} \in \{-1, 1\}^N$*;*

ii) $\mathbf{Y}^* = \mathbf{p}\begin{bmatrix} a_1 & a_2 & \dots & a_m \end{bmatrix}$*, for some* $a_i \in \mathbb{R}, \sum_{i=1}^m a_i^2 \neq 0$.

Proof. It is immediate to observe that if i) and ii) hold, then the identities (4.8) and (4.9) hold.
Conversely, assume that the pair $(\mathbf{Y}^*, \mathbf{X}^*)$ is an equilibrium point. Then (4.9) holds, but this means that the nonzero columns of $\mathbf{Y}^*$ are eigenvectors of $\frac{1}{N}\mathbf{X}^*$ corresponding to the unitary eigenvalue. This means that $1 \in \sigma\left(\frac{1}{N}\mathbf{X}^*\right)$ and therefore, by Lemma A.2 in the Appendix, i) holds. On the other hand, by replacing the matrix $\mathbf{X}^*$ in (4.9) with $\mathbf{p}\mathbf{p}^\top$, we obtain ii). □

We want now to show that the model we have proposed cannot exhibit periodic solutions and hence limit cycles. To prove this result we need a preliminary lemma, that will be useful also for the subsequent analysis.

Lemma 4.7 (Upper bounded opinion dynamics)**.** *For every $j \in [1,m]$ and every $t \geq 0$*

i)

$$\max_{i\in[1,N]} |[Y(t+1)]_{ij}| \leq \max_{i\in[1,N]} |[Y(t)]_{ij}|. \tag{4.10}$$

ii) Condition

$$\max_{i\in[1,N]} |[Y(t+1)]_{ij}| = \max_{i\in[1,N]} |[Y(t)]_{ij}| \neq 0$$

holds if and only if

(a) $\mathbf{Y}(t)\mathbf{e}_j = \mathbf{p} \cdot \mu_j,\ \exists\ \mathbf{p} \in \{-1,1\}^N$ *and* $\mu_j > 0$; *and*

(b) once we set $h := \text{argmax}_{i\in[1,N]}|[Y(t+1)]_{ij}|$ *then* $\mathbf{e}_h^\top \mathbf{X}(t+1)$ *has no zero entries and* $\mathbf{e}_h^\top \mathbf{X}(t+1) = p_h \cdot \mathbf{p}^\top$.

Proof. i) From equation (4.3) it follows that

$$\begin{aligned} |[Y(t+1)]_{ij}| &= |\frac{1}{N}\sum_{k=1}^{N}[X(t+1)]_{ik}[Y(t)]_{kj}| \leq \frac{1}{N}\sum_{k=1}^{N}|[X(t+1)]_{ik}||[Y(t)]_{kj}| \\ &\leq \frac{1}{N}\sum_{k=1}^{N}|[Y(t)]_{kj}| \leq \frac{1}{N} N \max_k |[Y(t)]_{kj}| = \max_k |[Y(t)]_{kj}|, \end{aligned}$$

and hence (4.10) holds.

ii) Set $h := \text{argmax}_{i\in[1,N]}|[Y(t+1)]_{ij}|$. Then $|[Y(t+1)]_{hj}| = \max_{i\in[1,N]}|[Y(t+1)]_{ij}|$ coincides with $\max_{i\in[1,N]} |[Y(t)]_{ij}|$ if and only if

$$\sum_{\ell=1}^{N} |[X(t+1)]_{h\ell}||[Y(t)]_{\ell j}| = N \cdot \max_{i\in[1,N]} |[Y(t)]_{ij}|$$

and this is possible if and only if all the entries in the j-th column of $\mathbf{Y}(t)$ have the same absolute value $\mu_j > 0$ (and this leads to (a), for some suitable vector $\mathbf{p}$) and all the terms $[X(t+1)]_{h\ell}[Y(t)]_{\ell j}$, $\ell \in [1,N]$, have the same sign. But this latter condition means that $\mathbf{e}_h^\top \mathbf{X}(t+1)$ either coincides with $\mathbf{p}^\top$ or with its opposite, and since $[X(t+1)]_{hh} = 1$ this means that condition (b) holds. □

We are now in a position to prove the following result.

Proposition 4.8 (Aperiodicity in opinion dynamics)**.** *Suppose that there exist* $\bar{t} \geq 0$, $T \geq 1$ *and nonzero matrices* $\tilde{\mathbf{Y}}_i \in \mathbb{R}^{N\times m}, i \in [1,T]$, *such that*

$$\mathbf{Y}(\bar{t}+i) = \tilde{\mathbf{Y}}_i, i \in [1,T], \quad \text{and} \quad \mathbf{Y}(\bar{t}+T+1) = \tilde{\mathbf{Y}}_1,$$

namely from $\bar{t}+1$ *onward the sequence of matrices* $\{\mathbf{Y}(t)\}_{t\geq \bar{t}+1}$ *becomes periodic of period* T, *then* $T = 1$, *namely the sequence becomes constant.*

Proof. From Lemma 4.7, part i), we can claim that for every $j \in [1, m]$ and every $t \geq 0$

$$\max_{\ell\in[1,N]} |[Y(t+T+1)]_{\ell j}| \leq \max_{\ell\in[1,N]} |Y_{\ell j}(t+T)|$$

$$\leq ... \leq \max_{\ell\in[1,N]} |[Y(t+2)]_{\ell j}| \leq \max_{\ell\in[1,N]} |[Y(t+1)]_{\ell j}|.$$

But since for $t = \bar{t}$ we have $\mathbf{Y}(\bar{t}+T+1) = \mathbf{Y}(\bar{t}+1) = \tilde{\mathbf{Y}}_1$ and hence the two extremes in the previous sequence of inequalities coincide, it follows that all the symbols $\leq$ are equalities, namely

$$\max_{\ell\in[1,N]} |[\tilde{\mathbf{Y}}_i]_{\ell j}| = \mu_j > 0, \qquad \forall\ j \in [1, m],\ \forall\ i \in [1, T].$$

This also implies, see Lemma 4.7 part ii), that, for every non zero column in $\tilde{\mathbf{Y}}_i$,

$$\tilde{\mathbf{Y}}_i \mathbf{e}_j = \mathbf{p}_i \cdot \mu_j, \qquad \exists\ \mathbf{p}_i \in \{-1, 1\}^N, \mu_j > 0, \tag{4.11}$$

and that, for every $h \in [1, N]$, one has $\mathbf{e}_h^\top \mathrm{sgn}(\tilde{\mathbf{Y}}_i \tilde{\mathbf{Y}}_i^\top) = [\mathbf{p}_i]_h \cdot \mathbf{p}_i^\top$. This implies that for every $i \in [1, T]$

$$\mathrm{sgn}([\tilde{\mathbf{Y}}_i \tilde{\mathbf{Y}}_i^\top]) = \mathbf{p}_i \mathbf{p}_i^\top, \qquad \exists\ \mathbf{p}_i \in \{-1, 1\}^{N\times N}.$$

Consequently[2]

$$\tilde{\mathbf{Y}}_{(i+1 \mod T)} = \frac{1}{N} \mathrm{sgn}([\tilde{\mathbf{Y}}_i \tilde{\mathbf{Y}}_i^\top]) \tilde{\mathbf{Y}}_i = \frac{1}{N} \mathbf{p}_i \mathbf{p}_i^\top \tilde{\mathbf{Y}}_i. \tag{4.12}$$

So, by comparing (4.11) and (4.12) one gets that every matrix $\tilde{\mathbf{Y}}_i, i \in [1, T]$, takes the form

$$\tilde{\mathbf{Y}}_i = \mathbf{p}_i \begin{bmatrix} a_1^{(i)} & \dots & a_m^{(i)} \end{bmatrix}, \qquad \exists\ \mathbf{p}_i \in \{-1, 1\}^N, a_k^{(i)} \in \mathbb{R},$$

but this also implies that

$$\tilde{\mathbf{Y}}_{(i+1 \mod T)} = \tfrac{1}{N} \mathrm{sgn}([\tilde{\mathbf{Y}}_i \tilde{\mathbf{Y}}_i^\top]) \tilde{\mathbf{Y}}_i = \tfrac{1}{N} \mathrm{sgn}(\mathbf{p}_i \mathbf{p}_i^\top \cdot \textstyle\sum_k [a_k^{(i)}]^2) \mathbf{p}_i \begin{bmatrix} a_1^{(i)} & \dots & a_m^{(i)} \end{bmatrix}$$

$$= \tfrac{1}{N} \mathbf{p}_i \mathbf{p}_i^\top \mathbf{p}_i \begin{bmatrix} a_1^{(i)} & \dots & a_m^{(i)} \end{bmatrix} = \mathbf{p}_i \begin{bmatrix} a_1^{(i)} & \dots & a_m^{(i)} \end{bmatrix} = \tilde{\mathbf{Y}}_i.$$

So, all matrices $\tilde{\mathbf{Y}}_i$ coincide. □

4.3.2 Convergence to an Equilibrium in a Finite Number of Steps

We want to explore under what conditions the equilibrium can be reached in a finite number of steps. It is easy to see that if there exists a time instant $t_0 \geq 0$ such that

[2] The expression $i + 1 \mod T$ means the remainder of $i + 1$ when divided by T.

$\mathbf{Y}(t_0+1) = \mathbf{Y}(t_0) \neq 0$ then $\mathbf{Y}(t) = \mathbf{Y}(t_0) =: \mathbf{Y}^*$ for every $t \geq t_0$. Consequently, also $\mathbf{X}(t)$ becomes constant starting at $t = t_0 + 1$, and it coincides with $\mathbf{X}^* := \text{sgn}(\mathbf{Y}(t_0)\mathbf{Y}(t_0)^\top)$. However, the converse is not true: if the appraisal matrix becomes constant at some time $t_0 \geq 0$, the opinion matrix $\mathbf{Y}(t)$ can still keep evolving for $t \geq t_0$. This situation is illustrated in Example 4.9, below.

As a matter of fact, if there exists a time instant $t_0 \geq 0$ such that $\mathbf{X}(t) = \mathbf{X}^*, \forall t \geq t_0$, we can only claim that $\mathbf{Y}(t+1) = \frac{1}{N}\mathbf{X}^*\mathbf{Y}(t)$. Equivalently, if we denote by $\mathbf{y}_j(t)$, the j-th column of the matrix $\mathbf{Y}(t)$, then the dynamics expressed by equation (4.2) decomposes into m linear time invariant systems of the form

$$\mathbf{y}_j(t+1) = \frac{1}{N}\mathbf{X}^*\mathbf{y}_j(t),\ \forall\ j \in [1, m]. \tag{4.13}$$

As $\mathbf{X}^* \in S_1^N$, the matrix $\frac{\mathbf{X}^*}{N}$ is symmetric and hence diagonalizable. Moreover, by Gershgorin Circle Theorem 2.4 , all its (real) eigenvalues $\lambda_i, i \in [1, N]$, satisfy

$$|\lambda_i - \frac{1}{N}| \leq \frac{N-1}{N} \iff -\frac{N-2}{N} \leq \lambda_i \leq 1,\ \forall i \in [1, N].$$

As a consequence, two cases may arise. The first case is the one depicted in Example 4.9, namely the case when the systems in (4.13) are asymptotically stable, which means that $\mathbf{Y}(t)$ asymptotically converges to 0 (and hence $\lim_{t\to+\infty}\mathbf{X}(t) = 0 \neq \mathbf{X}^*$).

Example 4.9. *Let us consider the case $N = m = 3$, with*

$$\mathbf{Y}(0) = \begin{bmatrix} 1.41 & -1.21 & 0.49 \\ 1.42 & 0.72 & 1.03 \\ 0.67 & 1.63 & 0.73 \end{bmatrix}.$$

It turns out that $\forall t \geq 1$

$$\mathbf{X}(t) = \mathbf{X}^* = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

and $\sigma(1/3 \cdot \mathbf{X}^*) = (-1/3, 2/3, 2/3)$, and indeed for $t \geq 14$ we have $Y_{ij}(t) = o(10^{-2}), \forall i, j \in [1, 3]$.

The second possible situation is when $\frac{\mathbf{X}^*}{N}$ is simply (but not asymptotically) stable. This amounts to saying that 1 is a (simple) eigenvalue of $\frac{\mathbf{X}^*}{N}$, and hence by Lemma A.2,

$\mathbf{X}^*$ takes the form $\mathbf{X}^* = \mathbf{p}\mathbf{p}^\top$, $\exists$ $\mathbf{p} \in \{-1,1\}^N$. In this case, the convergence is not asymptotic but instantaneous. In fact, it is sufficient that $\frac{1}{N}\mathbf{X}(t_0)$ becomes simply (but not asymptotically) stable at a single time instant, to ensure the instantaneous convergence of $\mathbf{Y}(t)$ to an equilibrium condition.

Proposition 4.10 (Equilibrium points characterization)**.** *If there exists $t_0 > 0$ such that $\frac{1}{N}\mathbf{X}(t_0)$, with $\mathbf{X}(t_0) \in \mathcal{S}_1^N$, is simply (but not asymptotically) stable, then $(\mathbf{X}^*, \mathbf{Y}^*) := (\mathbf{X}(t_0), \frac{1}{N}\mathbf{X}(t_0)\mathbf{Y}(t_0 - 1))$ is an equilibrium point.*

Proof. By Lemma A.2 in the Appendix, we know that if $\frac{1}{N}\mathbf{X}(t_0) \in \mathcal{S}_1^N$ is simply stable or, equivalently, $1 \in \sigma(\frac{1}{N}\mathbf{X}(t_0))$, then there exists a vector $\mathbf{p} \in \{-1,1\}^N$ such that $\mathbf{X}(t_0) = \mathbf{p}\mathbf{p}^\top$. On the other hand, if $\mathbf{X}(t_0) = \mathbf{p}\mathbf{p}^\top$, then

$$\mathbf{Y}(t_0) = \frac{1}{N}\mathbf{p}\mathbf{p}^\top\mathbf{Y}(t_0 - 1) = \mathbf{p}[a_1, \dots, a_m],$$

where

$$[a_1, \dots, a_m] := \frac{1}{N}\mathbf{p}^\top\mathbf{Y}(t_0 - 1).$$

Therefore $(\mathbf{X}^*, \mathbf{Y}^*) := (\mathbf{X}(t_0), \frac{1}{N}\mathbf{X}(t_0)\mathbf{Y}(t_0 - 1))$ is an equilibrium point. □

Remark 4.11. *If $m = 1$ the model reaches the equilibrium in one step. When so, in fact $\mathbf{X}(1) = \operatorname{sgn}(\mathbf{Y}(0)\mathbf{Y}^\top(0)) = \mathbf{p}\mathbf{p}^\top$, where $\mathbf{p} := \operatorname{sgn}(\mathbf{Y}(0))$. Numerical simulations at the end of the section will show that, when $m > 1$, namely multiple topics are considered, convergence to structural balance is almost surely guaranteed, and it occurs in a rather small number of steps even for medium size networks (e.g. $N = 100$).*

Remark 4.12. *Gershgorin Circle theorem (see Theorem 2.4) also allows to say that if $\mathbf{X}^*$ is the adjacency matrix of a disconnected graph, all the eigenvalues of the matrix $\frac{\mathbf{X}^*}{N}$ lie in the circle of the complex plane of center the origin and radius $\frac{N-1}{N}$ (or smaller), and hence $\frac{\mathbf{X}^*}{N}$ is necessarily an asymptotically stable matrix.*

Theorem 4.13 summarizes the main results of this section.

Theorem 4.13 (Main theorem)**.** *The following conditions are equivalent*

i) there exists a time instant $t_0 \geq 0$ such that $1 \in \sigma(\frac{1}{N}\mathbf{X}(t_0))$;

ii) there exists a time instant $t_0 \geq 0$ *such that* $\mathbf{Y}(t_0) = \mathbf{Y}(t_0+1)$*;*

iii) the opinion-appraisal dynamic model (4.2)- (4.3) *converges in finite time to an equilibrium* $(\mathbf{X}^*, \mathbf{Y}^*)$*;*

iv) the opinion-appraisal dynamic model (4.2)- (4.3) *converges in finite time to an equilibrium* $(\mathbf{X}^*, \mathbf{Y}^*)$*, with* $\mathbf{X}^* = \mathbf{p}\mathbf{p}^\top$ *and* $\mathbf{Y}^* = \mathbf{p}[a_1, \dots, a_m]$*,* $\exists\ \mathbf{p} \in \{-1, 1\}^N$*, and* $a_i \in \mathbb{R}, i \in [1, m]$*, with* $\sum_{i=1}^m a_i^2 \neq 0$*.*

Proof. iv) ⇔ iii) follows from Proposition 4.6. iii) ⇒ ii) is obvious, while the converse has been commented upon at the beginning of the section.
i) ⇒ iv) follows from Proposition 4.10, while iv) ⇒ i) is obvious. □

4.3.3 Long Term Behaviour

In the previous section, we have investigated what happens if either $\mathbf{Y}(t)$ or $\mathbf{X}(t)$ become constant starting at some time instant. In the former case the overall system (4.2)-(4.3) reaches the equilibrium in a finite number of steps. In the latter case a nontrivial equilibrium is reached if and only if $\mathbf{X}(t)$ at some point becomes structurally balanced. Differently the opinion matrix asymptotically converges to zero. We want to investigate now if a nontrivial equilibrium can be reached asymptotically, but not in a finite number of steps.
An immediate consequence of the analysis of the previous section is that if the sequence of appraisal matrices $\{\mathbf{X}(t)\}_{t\geq 1}$ does not converge in a finite number of steps then $\frac{\mathbf{X}(t)}{N}$ is an asymptotically stable matrix for every $t \geq 1$. This means that if we define the set

$$\mathcal{S}_{stable} := \mathcal{S}_1^N \setminus \{\mathbf{X} \in \mathcal{S}_1^N : \mathbf{X} = \mathbf{p}\mathbf{p}^\top, \exists\, \mathbf{p} \in \{-1,1\}^N\}, \tag{4.14}$$

then $\mathbf{X}(t) \in \mathcal{S}_{stable}$ for every $t \geq 1$.

Proposition 4.14 (Zero vanishing condition)**.** *If for every* $t \geq 0$*,* $\mathbf{X}(t) \in \mathcal{S}_{stable}$*, then* $\lim_{t\to+\infty} \mathbf{Y}(t) = 0$.

Proof. For every $j \in [1, m]$, let us define $\mu_j(t) := \max_{i\in[1,N]} |[Y(t)]_{ij}|$, and let us introduce the (generalized) Lyapunov function for the system in equation (4.6), $V : \mathbb{R}^{N\times m} \to \mathbb{R}$, defined as

$$V(\mathbf{Y}(t)) := \sum_{j=1}^{m} \mu_j(t).$$

We notice that $V(\mathbf{Y}) \geq 0$, $\forall \mathbf{Y} \in \mathbb{R}^{N\times m}$ and that $V(\mathbf{Y}) = 0$ if and only if $\mathbf{Y} = 0$. Define $\Delta_2 V(\mathbf{Y}(t)) := V(\mathbf{Y}(t+2)) - V(\mathbf{Y}(t))$. We want to prove that $\Delta_2 V(\mathbf{Y}(t)) < 0$, $\forall t \geq 0$.

By Lemma 4.7 it immediately follows that $\Delta_2 V(\mathbf{Y}(t)) = \sum_{j=1}^{m} \mu_j(t+2) - \mu_j(t) \leq 0$. We show now that there is not a time instant $t_0 \geq 0$ such that $\Delta_2 V(\mathbf{Y}(t)) = 0$. If this were the case, in fact, this would mean that $\forall j \in [1, m]$, $\mu_j(t_0+2) = \mu_j(t_0)$ and therefore $\mu_j(t_0+2) = \mu_j(t_0+1) = \mu_j(t_0) =: \mu_j$. As a consequence of Lemma 4.7 we deduce that

a) $\mathbf{Y}(t_0)\mathbf{e}_j = \mathbf{p}_j(t_0)\mu_j$, $\exists\ \mathbf{p}_j(t_0) \in \{-1, 1\}^N$,

$\mathbf{Y}(t_0+1)\mathbf{e}_j = \mathbf{p}_j(t_0+1)\mu_j$, $\exists\ \mathbf{p}_j(t_0+1) \in \{-1, 1\}^N$,

b) $\forall h \in [1, N]$, $\mathbf{e}_h^T \mathbf{X}(t_0+1) = p_h \mathbf{p}_j(t_0)^\top$,

from which it follows that $\mathbf{X}(t_0+1) = \mathbf{p}_j(t_0)\mathbf{p}_j(t_0)^\top$ for every $j \in [1, m]$. But then $\mathbf{X}(t_0+1) = \mathbf{p}(t_0)\mathbf{p}(t_0)^\top$ with $\mathbf{p}(t_0) = \pm\mathbf{p}_j(t_0)$, $\forall j \in [1, m]$, that means that $\mathbf{X}(t_0+1)$ is structurally balanced and hence it does not belong to $\mathcal{S}_{stable}$, thus contradicting the hypotheses. Consequently, it must be $\Delta_2 V(\mathbf{Y}(t)) < 0$, $\forall t \geq 0$. Finally, by defining $\Delta_1 V(\mathbf{Y}(t)) := V(\mathbf{Y}(t+1)) - V(\mathbf{Y}(t))$ we get that

$$\Delta_2 V(\mathbf{Y}(t)) + \Delta_1 V(\mathbf{Y}(t)) < 0, \ \ \forall t \geq 0,$$

so the thesis follows as a direct consequence of Theorem 2.6. □

Summarizing, Theorem 4.13 and Proposition 4.14 show that either there exists a time instant t_0 such that $\forall t \geq t_0$, $\mathbf{X}(t) = \mathbf{p}\mathbf{p}^\top$ and consequently $\mathbf{Y}(t) = \mathbf{p}[a_1, a_2, \ldots, a_m]$, $a_i \in \mathbb{R}$, $\sum_i a_i^2 \neq 0$, otherwise, if a time instant t such that $\mathbf{X}(t)$ reaches the structural balance does not exist, then $\mathbf{Y}(t)$ converges to zero as time goes to infinity.

4.3.4 Simulations

In this section we show the outcome of Monte Carlo simulations in order to validate the convergence properties of the model. Figure 4.1 shows how the average number of iterations needed in order to reach a structural balanced configuration over the total number of 30000 simulations varies as a function of the number of topics $m \in [1, 10]$, for networks involving $N = 9, 20, 100$ agents. Simulations are based on initial conditions $\mathbf{Y}(0)$ with entries independently drawn from a Gaussian random variable with zero mean and standard deviation $\sigma = 10$, namely $Y_{ij}(0) \sim \mathcal{N}(0, 100)$. It turns out that, in accordance with the Chernoff bound, by running 30000 simulations, the estimated probability $\hat{p}$ to reach a structurally balanced configuration is equal to 1 with accuracy $\epsilon = 0.01$ and confidence level $1 - \delta = 0.99$, namely $P(|\hat{p} - p| \leq \epsilon) \geq 1 - \delta$, for the case of $N = 20, 100$ agents, regardless of the number of topics taken into account while $\hat{p}$ is greater than or equal to 0.98 for all $m \in [1, 10]$, for $N = 9$, with the same accuracy and confidence interval.

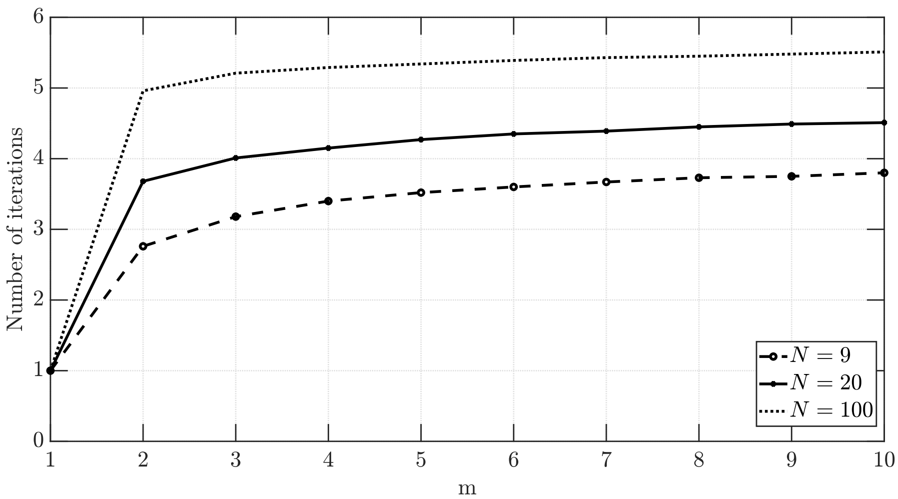

Figure 4.1: Average number of iterations, over 30000 simulations, needed in order to reach a structural balance configurations for the cases $N = 9, 20, 100$ and $m \in [1, 10]$.

4.4 A Binary Homophily Model for Opinion Dynamics

Given a group of $N \geq 3$ agents, We assume that for every $i, j \in [1, N]$, $[X(t)]_{ij} \in \{-1, 1\}$. For every pair $(i, j), i \neq j$, we introduce the sets

$$\begin{aligned}\mathcal{A}_{ij}(t) &:= \{k \in [1, N], k \neq i, j : [X(t)]_{ik}[X(t)]_{jk} = 1\},\\ \mathcal{D}_{ij}(t) &:= \{k \in [1, N], k \neq i, j : [X(t)]_{ik}[X(t)]_{jk} = -1\},\end{aligned}$$

representing the sets of agents, distinct from i and j, on which i and j agree or disagree, respectively, at time t. We assume that the relations between pairs of agents are updated according to the following *binary homophily model* (i), that is:

$$[X(t+1)]_{ij} = \begin{cases} 1, & \text{if } |\mathcal{A}_{ij}(t)| > |\mathcal{D}_{ij}(t)|; \\ -1, & \text{if } |\mathcal{A}_{ij}(t)| < |\mathcal{D}_{ij}(t)|; \\ [X(t)]_{ij}, & \text{otherwise.} \end{cases} \tag{4.15}$$

This amounts to assuming that i and j at time $t+1$ will have a good opinion of each other if at time t they agree in their evaluations of most of the other agents. They will have a bad opinion of each other if, on the contrary, they disagree on most of the other elements of the group. If, finally, their opinions coincide on exactly half of the other agents (something that is possible only if the overall number of agents, N, is even) they will keep their mutual evaluations unchanged. The binary homophily model can be equivalently described as:

$$[X(t+1)]_{ij} = \begin{cases} \text{sign}\Big(\sum_{k\neq i,j}[X(t)]_{ik}[X(t)]_{jk}\Big), & \text{if } \sum_{k\neq i,j}[X(t)]_{ik}[X(t)]_{jk} \neq 0; \\ [X(t)]_{ij}, & \text{if } \sum_{k\neq i,j}[X(t)]_{ik}[X(t)]_{jk} = 0. \end{cases} \tag{4.16}$$

So, if $\mathbf{X}(t)$ denotes the $N \times N$ symmetric matrix with entries in {-1,1} whose (i, j)-th entry is $[X(t)]_{ij}$, the binary homophily model can be expressed in matrix form as

$$\mathbf{X}(t+1) = \text{sign}\Big[\Big(\mathbf{X}(t) - \text{diag}(\mathbf{X}(t))\Big)\Big(\mathbf{X}(t) - \text{diag}(\mathbf{X}(t))\Big)^{\top} + \alpha\mathbf{X}(t)\Big] \tag{4.17}$$

with α arbitrary in $(0, 1)$. The term $\alpha\mathbf{X}(t)$ is meant to enforce the rule that if $[(\mathbf{X}(t) - \text{diag}(\mathbf{X}(t)))(\mathbf{X}(t) - \text{diag}(\mathbf{X}(t)))^{\top}]_{ij}$ is nonzero then $\alpha[X(t)]_{ij}$ is irrelevant in determining $[X(t+1)]_{ij}$, otherwise $[X(t+1)]_{ij} = [X(t)]_{ij}$.

Remark 4.15. *It is worth noticing that, after the first iteration of the binary homophily model, the matrix* $\mathbf{X}(t)$ *is not only symmetric but also with unitary diagonal elements, i.e.*

$\mathbf{X}(1) = \mathbf{X}(1)^\top$ *and* $X_{ii}(1) = 1$, $\forall i \in [1, N]$. *So, if we define*

$$S_{1,c}^N := \{\mathbf{M} = \mathbf{M}^\top \in \{-1, 1\}^{N\times N} \colon [\mathbf{M}]_{ii} = 1, \forall i \in [1, N]\} = S_1^N \cap \{-1, 1\}^{N\times N},$$

then $\forall\ \mathbf{X}(0) = \mathbf{X}(0)^\top \in \{-1, 1\}^{N\times N}$, $\mathbf{X}(t) \in S_{1,c}^N$, $\forall t \geq 1$.

If we assume $\mathbf{X}(0) \in S_{1,c}^N$ then, $\forall t \geq 0$, $\mathrm{diag}(\mathbf{X}(t)) = I_N$ and $[\mathbf{X}(t) - \mathrm{diag}(\mathbf{X}(t))]^\top = \mathbf{X}(t) - I_N$. Therefore, under this assumption, the binary homophily model can be equivalently rewritten as

$$\mathbf{X}(t+1) = \mathrm{sign}\Big((\mathbf{X}(t) - I_N)^2 + \alpha \mathbf{X}(t)\Big) = \mathrm{sign}\Big((\mathbf{X}(t))^2 + \beta \mathbf{X}(t) + I_N\Big),$$

where $\beta := -2 + \alpha$ is any real number in $(-2, -1)$. Moreover, by noticing that $[\mathbf{X}(t)^2 + \beta \mathbf{X}(t) + I_N]_{ii} = N + \beta + 1$, $\forall t \geq 0$, and $N \geq 3$, it directly follows that

$$\Big[\mathrm{sign}\Big((\mathbf{X}(t))^2 + \beta \mathbf{X}(t) + I_N\Big)\Big]_{ii} = \Big[\mathrm{sign}\Big((\mathbf{X}(t))^2 + \beta \mathbf{X}(t)\Big)\Big]_{ii}.$$

Since the identity matrix does not play any role in the calculation of the off-diagonal entries of the matrix $\mathbf{X}(t)$, the binary homophily model (under the assumption that $\mathbf{X}(0) \in S_1^N$) can be rewritten as:

$$\mathbf{X}(t+1) = \mathrm{sign}\Big((\mathbf{X}(t))^2 + \beta \mathbf{X}(t)\Big), \tag{4.18}$$

$-2 < \beta < -1$.

In the rest of the section we will steadily assume $\mathbf{X}(0) \in S_{1,c}^N$ and hence we will make use, equivalently, of the update equations (4.16) and (4.18).

4.4.1 Equilibrium Points Characterization and Structurally Balanced Equilibrium Points

A matrix $\mathbf{X}^* \in S_1^N$ is an *equilibrium point* for the binary homophily model if

$$\mathbf{X}(0) = \mathbf{X}^* \Rightarrow \mathbf{X}(t) = \mathbf{X}^*,\ \forall t \geq 0.$$

From (4.16) we deduce that $\mathbf{X}^* \in S_1^N$ is an equilibrium point if and only if the following condition holds.

Proposition 4.16. *A matrix* $\mathbf{X}^* \in S_1^N$ *is an equilibrium point if and only if* $\mathbf{X}^* = \mathrm{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)$.

Proof. We preliminary notice that, as a result of (4.16) and of the previous remark, $\mathbf{X}^*$ is an equilibrium point for the binary homophily model if and only if $\forall i, j \in [1, N]$, one has:

$$i \neq j \text{ and } \sum_{k \neq i,j} [X^*]_{ik}[X^*]_{jk} \neq 0 \;\;\Rightarrow\;\; [X^*]_{ij} = \text{sign}\Big(\sum_{k \neq i,j} [X^*]_{ik}[X^*]_{jk}\Big),$$
$$i = j \;\;\Rightarrow\;\; [X^*]_{ii} = 1. \tag{4.19}$$

On the other hand, for every $\mathbf{X}^* \in S_1^N$ condition $\mathbf{X}^* = \text{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)$ can be equivalently expressed, for every $i, j, i \neq j$, as

$$[(\mathbf{X}^*)^2 - \mathbf{X}^*]_{ij} = \sum_k [X^*]_{ik}[X^*]_{jk} - [X^*]_{ij} = [X^*]_{ii}[X^*]_{ij} + [X^*]_{ij}[X^*]_{jj} + \sum_{k \neq i,j} [X^*]_{ik}[X^*]_{jk} - [X^*]_{ij} = [X^*]_{ij} + \sum_{k \neq i,j} [X^*]_{ik}[X^*]_{jk}.$$

By making use of these preliminary remarks we can prove the result.

Necessity: Let us assume that $\mathbf{X}^*$ is an equilibrium point for the binary homophily model. Then, by making use of the characterization (4.19), we get $[(\mathbf{X}^*)^2 - \mathbf{X}^*]_{ii} = N - 1$, therefore $\text{sign}([(\mathbf{X}^*)^2 - \mathbf{X}^*]_{ii}) = 1 = [X^*]_{ii}$. For the off-diagonal entries we distinguish the following two cases:

$$a) \sum_{k \neq i,j} [X^*]_{ik}[X^*]_{jk} \neq 0 \qquad b) \sum_{k \neq i,j} [X^*]_{ik}[X^*]_{jk} = 0.$$

a) If $\sum_{k \neq i,j} [X^*]_{ik}[X^*]_{jk} > 0$ then $[X^*]_{ij} = 1$. On the other hand,

$$[(\mathbf{X}^*)^2 - \mathbf{X}^*]_{ij} = [X^*]_{ij} + \sum_{k \neq i,j} [X^*]_{ik}[X^*]_{jk} > 0$$

from which it follows that $[\text{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)]_{ij} = 1 = [X^*]_{ij}$. Analogous calculations can be done to verify the case $\sum_{k \neq i,j} [X^*]_{ik}[X^*]_{jk} < 0$.

b) If $\sum_{k \neq i,j} [X^*]_{ik}[X^*]_{jk} = 0$ then

$$[(\mathbf{X}^*)^2 - \mathbf{X}^*]_{ij} = [X^*]_{ij} + \sum_{k \neq i,j} [X^*]_{ik}[X^*]_{jk} = [X^*]_{ij}.$$

Thus, $[\text{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)]_{ij} = [X^*]_{ij}$.
Therefore, if $\mathbf{X}^*$ is an equilibrium point, then $\mathbf{X}^* = \text{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)$.

Sufficiency: Let us suppose that the matrix $\mathbf{X}^* \in S_1^N$ satisfies $\mathbf{X}^* = \text{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)$. We will show that $\mathbf{X}^*$ is an equilibrium point for the binary homophily model, namely its entries satisfy the characterization (4.19). Consider the identity

$$[(\mathbf{X}^*)^2 - \mathbf{X}^*]_{ij} = [X^*]_{ij} + \sum_{k \neq i,j} [X^*]_{ik}[X^*]_{jk}.$$

We distinguish the following cases:

- If $\sum_{k\neq i,j}[X^*]_{ik}[X^*]_{jk} \leq -2$, since

$$[X^*]_{ij} = \text{sign}\Big([X^*]_{ij} + \sum_{k\neq i,j}[X^*]_{ik}[X^*]_{jk}\Big) \tag{4.20}$$

we get $[X^*]_{ij} = -1$.

- If $\sum_{k\neq i,j}[X^*]_{ik}[X^*]_{jk} \geq 2$ from (4.20), we get $[X^*]_{ij} = 1$.

- If $\sum_{k\neq i,j}[X^*]_{ik}[X^*]_{jk} = -1$, it could not happen that $[X^*]_{ij} = 1$, otherwise one would get $[X^*]_{ij} + \sum_{k\neq i,j}[X^*]_{ik}[X^*]_{jk} = 0$, and so (4.20) could not hold, against the hypothesis. Therefore it must be $[X^*]_{ij} = -1$. An analogous reasoning applies to $\sum_{k\neq i,j}[X^*]_{ik}[X^*]_{jk} = 1$, in which case $[X^*]_{ij} = 1$ is obtained.

So, every time $\sum_{k\neq i,j}[X^*]_{ik}[X^*]_{jk} \neq 0$ we have $[X^*]_{ij} = \text{sign}\Big(\sum_{k\neq i,j}[X^*]_{ik}[X^*]_{jk}\Big)$, and this proves that $\mathbf{X}^*$ is an equilibrium point. □

Remark 4.17. *If $\mathbf{X}^* \in S^N_{1,c}$ is such that $\mathcal{G}(\mathbf{X}^*)$ is structurally balanced, i.e. (see Proposition 4.1) there exists $\mathbf{x} \in \{-1,1\}^N$ such that $\mathbf{X}^* = \mathbf{x}\mathbf{x}^\top$, then the condition expressed in Proposition 4.16 is trivially satisfied.*

For certain values of N, we can show that networks of N agents admit only structurally balanced equilibrium points.

Proposition 4.18. *Given a network of N agents, with $N \in \{3,4,5,7\}$, if $\mathbf{X}^*$ is an equilibrium point for the binary homophily model, then $\mathcal{G}(\mathbf{X}^*)$ is structurally balanced.*

Proof. We will prove this statement for $N = 2p+1$ and $p \in [1,3]$ by contrapositive. The proof of the case $N = 4$ is similar and hence omitted. If $\mathcal{G}(\mathbf{X}^*)$ is not structurally balanced then (see Proposition 4.1) there exists at least an unbalanced triad in $\mathcal{G}(\mathbf{X}^*)$. Without loss of generality we will assume that $(1,2,3)$ is such a triad. Two cases may occur as Fig. 4.2 illustrates:

a) There is only one negative edge in $(1,2,3)$. Without loss of generality we will suppose that $[X^*]_{12} = -1, [X^*]_{13} = 1, [X^*]_{23} = 1$. Since $[X^*]_{12} = -1$, it must hold that $\sum_{k\neq 1,2}[X^*]_{1k}[X^*]_{2k} \leq 0$. Moreover, as $[X^*]_{13}[X^*]_{23} = 1$, it must hold that $\sum_{k\geq 4}[X^*]_{1k}[X^*]_{2k} \leq -1$. Let us define $\mathcal{D}_{12}$, the set of agents on which agents 1 and 2 disagree, namely:

$$\mathcal{D}_{12} := \{k \neq 1,2 : [X^*]_{1k}[X^*]_{2k} = -1\}.$$

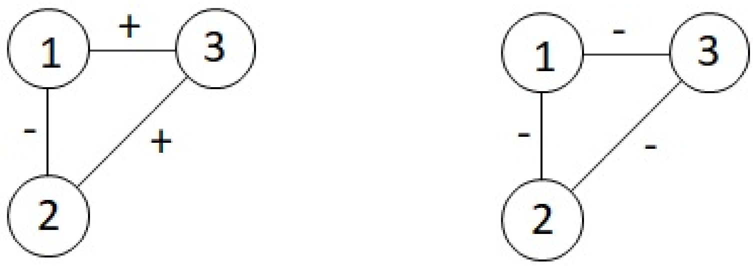

Figure 4.2: Unbalanced triads. On the left: a triad with a single negative arc (case a)). On the right: a triad with three negative arcs (case b)).

It holds that $\mathcal{D}_{12} \subseteq [4, N] = [4, 2p+1]$ and $|\mathcal{D}_{12}| \geq \lceil \frac{2p-1}{2} \rceil = p$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. On the other hand, since $[X^*]_{13} = 1, [X^*]_{23} = 1$ and $[X^*]_{12} = -1$ (and so $[X^*]_{23}[X^*]_{12} = -1$), then the set of agents on which agents 1 and 3 agree, namely

$$\mathcal{A}_{13} := \{k \neq 1, 3 : [X^*]_{1k}[X^*]_{3k} = 1\},$$

is such that $\mathcal{A}_{13} \subseteq [4, 2p+1]$ and $|\mathcal{A}_{13}| \geq p$. Keeping in mind that $\mathcal{D}_{12} \subseteq [4, 2p+1], \mathcal{A}_{13} \subseteq [4, 2p+1]$, $|\mathcal{D}_{12}| \geq p, |\mathcal{A}_{13}| \geq p$ and that $|[4, 2p+1]| = 2p-2$, it holds that $|\mathcal{D}_{12} \cap \mathcal{A}_{13}| \geq 2$. Moreover, $\forall k \in \mathcal{D}_{12} \cap \mathcal{A}_{13}$ we have that $[X^*]_{1k}[X^*]_{2k} = -1, [X^*]_{1k}[X^*]_{3k} = 1$, from which it follows that $\forall k \in \mathcal{D}_{12} \cap \mathcal{A}_{13}$, $[X^*]_{2k}[X^*]_{3k} = -1$.
Finally, since $[X^*]_{23} = 1$, the set of the agents on which agents 2 and 3 share the same opinion, $\mathcal{A}_{23}$, is such that $\mathcal{A}_{23} \subseteq [4, N] \setminus (\mathcal{D}_{12} \cap \mathcal{A}_{13})$ and $|\mathcal{A}_{23}| \geq p$, but $|[4, 2p+1] \setminus (\mathcal{D}_{12} \cap \mathcal{A}_{13})| \leq (2p-2) - 2 = 2p-4$ and for $p \in [1, 3]$, the condition $p \leq |\mathcal{A}_{23}| \leq 2p-4$ is impossible. This contradicts the fact that $[X^*]_{23} = 1$ satisfies the equilibrium condition.

b) All the edges in the triad $(1, 2, 3)$ are negative. Following a reasoning analogous to the one in a), we can show that $\mathcal{D}_{12} \subseteq [4, 2p+1]$ and $|\mathcal{D}_{12}| \geq p$. Similarly, the set $\mathcal{D}_{13}$ of the agents about which agents 1 and 3 disagree must be such that $\mathcal{D}_{13} \subseteq [4, 2p+1]$ and $|\mathcal{D}_{13}| \geq p$. But this leads to conclude that the set $\mathcal{D}_{23}$ of the agents on which agents 2 and 3 disagree satisfies the condition $p \leq |\mathcal{D}_{23}| \leq 2p-4$ that cannot be true for $p \in [1, 3]$, thus contradicting the fact that $\mathbf{X}^*$ is an equilibrium point. □

4.4.2 (V, Σ)-Factorization

In this section we propose an equivalent representation for the matrices in $S^N_{1,c}$ that allows us to derive additional conditions for the analysis of the equilibrium points of the binary homophily model.

Lemma 4.19. *Given* $\mathbf{X} \in S^N_{1,c}$*, there exist a permutation matrix* $\mathbf{P} \in \{0,1\}^{N\times N}$*, positive integers* k *and* n_i*, vectors* $\mathbf{v}_i \in \{-1,1\}^{n_i}, i \in [1,k]$*, with* $\sum_{i=1}^k n_i = N$*, and* $\mathbf{\Sigma} \in S^k_{1,c}$*, such that*

$$\mathbf{P}^\top \mathbf{X}\mathbf{P} = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top, \text{ where } \mathbf{V} := \mathbf{v}_1 \oplus \cdots \oplus \mathbf{v}_k. \tag{4.21}$$

Proof. We preliminarily notice that, given $\mathbf{X} \in S^N_{1,c}$, it is always possible to select distinct rows of $\mathbf{X}$ such that all the other rows are either identical to or the opposite of one of them. This means that $\forall\ \mathbf{X} \in S^N_{1,c}, \exists k \leq N$, $\mathbf{B} \in \{-1,1\}^{k\times N}$, whose rows are pairwise linearly independent, and a matrix $\mathbf{A} \in \{-1,0,1\}^{N\times k}$, of rank k, whose rows are signed canonical vectors, such that $\mathbf{X} = \mathbf{A}\mathbf{B}$.

Without loss of generality we can choose a permutation matrix $\mathbf{P}$ such that

$$\mathbf{P}^\top \mathbf{X}\mathbf{P} = [\mathbf{v}_1 \oplus \cdots \oplus \mathbf{v}_k] \left[\begin{array}{c} \mathbf{b}_1^\top \\ \hline \dots \\ \hline \mathbf{b}_k^\top \end{array}\right], \tag{4.22}$$

where $\mathbf{v}_i \in \{-1,1\}^{n_i}, i \in [1,k]$, $n_i \geq 1$, $\sum_{i=1}^k n_i = N$, and $\mathbf{b}_i \in \{-1,1\}^N, i \in [1,k]$. Partitioning the vectors $\mathbf{b}_i$ in blocks, according to the block partitioning of the matrix on the left in the above factorisation, we get

$$\left[\begin{array}{c} \mathbf{b}_1^\top \\ \hline \dots \\ \hline \mathbf{b}_k^\top \end{array}\right] = \left[\begin{array}{c|c|c|c} \mathbf{v}_{11}^\top & \mathbf{v}_{12}^\top & \dots & \mathbf{v}_{1k}^\top \\ \hline \dots & \dots & \dots & \dots \\ \hline \mathbf{v}_{k1}^\top & \mathbf{v}_{k2}^\top & \dots & \mathbf{v}_{kk}^\top \end{array}\right], \tag{4.23}$$

with $\mathbf{v}_{ij} \in \{-1,1\}^{n_j}$. We notice that, since $\mathbf{X} \in S^N_{1,c}$, $\mathbf{P}^\top\mathbf{X}\mathbf{P} \in S^N_{1,c}$ as well. Therefore the diagonal blocks of dimension $n_i, i \in [1,k]$, of the matrix $\mathbf{P}^\top\mathbf{X}\mathbf{P}$ belong to $S^{n_i}_{1,c}$, and this implies $\mathbf{v}_i = \mathbf{v}_{ii}$ for every $i \in [1,k]$. Putting together (4.22) and (4.23) and making use of the symmetry of $\mathbf{P}^\top\mathbf{X}\mathbf{P}$, we also obtain $\mathbf{v}_i\mathbf{v}_{ij}^\top = (\mathbf{v}_j\mathbf{v}_{ji}^\top)^\top, \forall i,j \in [1,k], i \neq j$, and due to the fact that the vector components are either 1 or -1, it must be that either (a) $\mathbf{v}_{ji} = \mathbf{v}_i$ and $\mathbf{v}_{ij} = \mathbf{v}_j$ or (b) $\mathbf{v}_{ji} = -\mathbf{v}_i$ and $\mathbf{v}_{ij} = -\mathbf{v}_j$. In light of these considerations we assume $\mathbf{v}_{ij} = \sigma_{ij}\mathbf{v}_j$, with $\sigma_{ij} \in \{-1,1\}$ and it must be $\sigma_{ij} = \sigma_{ji}$. Therefore we have

$\mathbf{P}^\top \mathbf{X} \mathbf{P} = \mathbf{V} \boldsymbol{\Sigma} \mathbf{V}^T$, where $\mathbf{V} := \mathbf{v}_1 \oplus \cdots \oplus \mathbf{v}_k \in \{-1, 0, 1\}^{N \times k}$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \sigma_{12} & \dots & \sigma_{1k} \\ \sigma_{21} & 1 & \dots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \vdots & 1 \end{bmatrix} \in S_1^k.$$

□

In the following, we will refer to the factorization (4.21) as a (V, Σ)-*factorization.* As shown in Lemma 4.19, every matrix $\mathbf{X} \in S_{1,c}^N$ admits a (V, Σ)-factorization, modulo a suitable permutation of its rows and columns, and we will provide characterisations of the matrices $\mathbf{X}^* \in S_{1,c}^N$ that represent equilibrium points of the binary homophily model in terms of the matrix $\boldsymbol{\Sigma}$ and of the sizes n_i of the vectors $\mathbf{v}_i$ appearing in $\mathbf{V}$ involved in any such factorisation. On the contrary, the specific entries of the vectors $\mathbf{v}_i$ will play no role. Finally, note that the permutation matrix $\mathbf{P}$ such that $\mathbf{P}^\top \mathbf{X}^* \mathbf{P} = \mathbf{V} \boldsymbol{\Sigma} \mathbf{V}^\top$ is not relevant when providing such a characterisation, since $\mathbf{X}^*$ is an equilibrium point if and only if $\mathbf{P}^\top \mathbf{X}^* \mathbf{P}$ is an equilibrium point. Therefore in the following we will assume, for the sake of simplicity, $\mathbf{X}^* = \mathbf{V} \boldsymbol{\Sigma} \mathbf{V}^\top$.

We now propose a graph interpretation of a (V, Σ)-factorization. To this aim it is worth noticing that the identity $\mathbf{X}^* = \mathbf{V} \boldsymbol{\Sigma} \mathbf{V}^\top$ can be equivalently expressed as

$$\mathbf{X}^* = \begin{bmatrix} \mathbf{v}_1 \mathbf{v}_1^\top & \sigma_{12} \mathbf{v}_1 \mathbf{v}_2^\top & \dots & \sigma_{1k} \mathbf{v}_1 \mathbf{v}_k^\top \\ \sigma_{12} \mathbf{v}_2 \mathbf{v}_1^\top & \mathbf{v}_2 \mathbf{v}_2^\top & \dots & \sigma_{2k} \mathbf{v}_2 \mathbf{v}_k^\top \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1k} \mathbf{v}_k \mathbf{v}_1^\top & \sigma_{2k} \mathbf{v}_k \mathbf{v}_2^\top & \dots & \mathbf{v}_k \mathbf{v}_k^\top \end{bmatrix}. \tag{4.24}$$

From this expression one deduces that the product $\mathbf{v}_i \mathbf{v}_i^\top$, $i \in [1, k]$, corresponds to a structurally balanced subclass, let us call it $\mathcal{C}_i$, of cardinality $n_i = \dim(\mathbf{v}_i)$, in the graph $\mathcal{G}(\mathbf{X}^*)$. Based on the sign of the entries of $\mathbf{v}_i$, the class $\mathcal{C}_i$ splits into two adverse factions, $\mathcal{C}_{iA}$ and $\mathcal{C}_{iB}$, each of them consisting of agents that are friends. On the other hand, σ_{ij} can be interpreted as the relation between agents in class $\mathcal{C}_i$ and agents in class $\mathcal{C}_j$. Specifically, all agents in a faction $\mathcal{C}_{iA}$ are friends *[enemies]* of all agents of $\mathcal{C}_{jA}$ and enemies *[friends]* of all agents of $\mathcal{C}_{jB}$ provided that σ_{ij} is positive *[negative]*, and the same statement holds true if the suffixes A and B are swapped. As a result, in the partition of $\mathcal{G}(\boldsymbol{\Sigma})$ thus obtained, $\forall i, j \in [1, k]$, $\mathcal{G}(\mathcal{C}_i \cup \mathcal{C}_j)$ is structurally balanced, in turn. Figure 4.3 is a graphical representation of what has just been stated above. As the graph $\mathcal{G}(\mathbf{X}^*)$ is complete and unweighted we will draw only the positive edges within the vertices of each class $\mathcal{C}_i$, while negative edges will be omitted. Self-loops will always have weight 1 and will be omitted, in turn. Arcs between two distinct classes $\mathcal{C}_i$ and $\mathcal{C}_j$ will be represented by means of the parameter σ_{ij}, as a result of the previous interpretation.

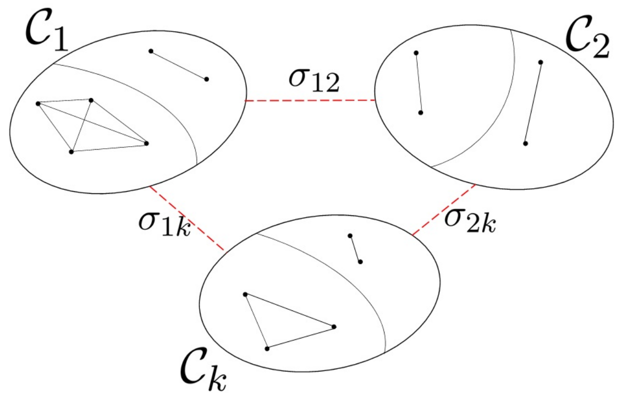

Figure 4.3: Graphic representation of $\mathcal{G}(\mathbf{V\Sigma V}^\top)$.

We now introduce a technical lemma that will be used in the following.

Lemma 4.20. *Let $\mathbf{v}_i \in \{-1,1\}^{n_i}$ $i = 1,\dots,k$ and set $\mathbf{V} = \mathbf{v}_1 \oplus \cdots \otimes \mathbf{v}_k$. Then for every $\mathbf{\Phi} \in S_1^k$ and every matrix $\mathbf{\Psi} \in \mathbb{R}^{k\times k}$, $\mathbf{V\Phi V}^\top = \operatorname{sign}(\mathbf{V\Psi V}^\top) \iff \mathbf{\Phi} = \operatorname{sign}(\mathbf{\Psi})$.*

Proof. For every $i,j \in [1,N]$ we have $\mathbf{e}_i^\top \mathbf{V\Phi V}^\top \mathbf{e}_j = \operatorname{sign}(\mathbf{e}_i^\top \mathbf{V\Psi V}^\top \mathbf{e}_j)$ if and only if $\forall \ell, s \in [1,k]$, $\forall r \in [1,n_\ell], p \in [1,n_s]$ condition $[\mathbf{v}_\ell]_r \mathbf{e}_\ell^\top \mathbf{\Phi e}_s [\mathbf{v}_s]_p = \operatorname{sign}([\mathbf{v}_\ell]_r \mathbf{e}_\ell^\top \mathbf{\Psi e}_s [\mathbf{v}_s]_p)$ holds, which means that $\forall \ell, s \in [1,k]$ we have $\mathbf{e}_\ell^\top \mathbf{\Phi e}_s = \operatorname{sign}(\mathbf{e}_\ell^\top \mathbf{\Psi e}_s)$. □

The following proposition provides a condition on a matrix $\mathbf{X}_0 \in S_{1,c}^N$ that guarantees that the binary homophily model starting from $\mathbf{X}_0$ converges to a structurally balanced equilibrium point in one step. Such a condition relies on the matrix $\mathbf{\Sigma}$ involved in a (V,Σ)-factorization of $\mathbf{X}_0$.

Proposition 4.21. *Consider a matrix $\mathbf{X}_0 \in S_{1,c}^N$ and a (V,Σ)-factorization of $\mathbf{X}_0$, i.e., $\mathbf{X}_0 = \mathbf{V\Sigma V}^\top$, where $\mathbf{V} := \mathbf{v}_1 \oplus \mathbf{v}_2 \cdots \oplus \mathbf{v}_k$, $\mathbf{v}_i \in \{-1,1\}^{n_i}$, $i \in [1,k]$, $\sum_{i=1}^k n_i = N$, and $\mathbf{\Sigma} \in S_{1,c}^k$. Set*

$$\mathbb{N} := \mathbf{V}^\top \mathbf{V} = n_1 \oplus n_2 \oplus \cdots \oplus n_k. \tag{4.25}$$

If $\mathbf{\Sigma}$ satisfies

$$\operatorname{sign}\left(\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma} - \frac{3}{2}\mathbf{\Sigma}\right) = \mathbf{w}\mathbf{w}^\top \tag{4.26}$$

for some $\mathbf{w} \in \{-1,1\}^k$, *then* $\operatorname{sign}\Big(\mathbf{X}_0^2 - \frac{3}{2}\mathbf{X}_0\Big) = \mathbf{V}\mathbf{w}\mathbf{w}^\top\mathbf{V}^\top$. *Therefore the binary homophily model starting from* $\mathbf{X}(0) = \mathbf{X}_0$ *converges in one step to the structurally balanced equilibrium point* $\mathbf{X}^* = \mathbf{v}\mathbf{v}^\top$, *where* $\mathbf{v} := \mathbf{V}\mathbf{w}$. *In particular, if all entries of* $\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma} - \frac{3}{2}\mathbf{\Sigma}$ *are positive then the equilibrium point is* $\mathbf{X}^* = \mathbf{v}\mathbf{v}^\top$, *where* $\mathbf{v} := \mathbf{V}\mathbf{1}_k$.

Proof. By Lemma 4.20, if identity (4.26) holds then

$$\begin{aligned}\mathbf{V}\mathbf{w}\mathbf{w}^\top\mathbf{V}^\top &= \operatorname{sign}\Big(\mathbf{V}\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma}\mathbf{V}^\top - \frac{3}{2}\mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top\Big) = \operatorname{sign}\Big(\mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top\mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top - \frac{3}{2}\mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top\Big)\\ &= \operatorname{sign}\Big(\mathbf{X}_0^2 - \frac{3}{2}\mathbf{X}_0\Big).\end{aligned}$$

On the other hand, the binary homophily model (4.18) for $\beta = -3/2$ leads to saying that $\mathbf{X}(1) = \operatorname{sign}\Big(\mathbf{X}(0)^2 - \frac{3}{2}\mathbf{X}(0)\Big)$. Therefore if $\mathbf{X}(0) = \mathbf{X}_0$ then $\mathbf{X}(1) = (\mathbf{V}\mathbf{w})\ (\mathbf{V}\mathbf{w})^\top$, and this concludes the proof. □

The following proposition states that when dealing with binary homophily models of size $N = 4$, every $\mathbf{X}_0 \in S^4_{1,c}$ is either a (structurally balanced) equilibrium point or it converges in one step to a structurally balanced equilibrium point (see Proposition 4.18).

Proposition 4.22. *For every* $\mathbf{X}_0 \in S^4_{1,c}$, *the matrix* $\mathbf{X}^* := \operatorname{sign}\left(\mathbf{X}_0^2 - \frac{3}{2}\mathbf{X}_0\right)$ *is an equilibrium point for the binary homophily model, and hence it is structurally balanced.*

Proof. If $\mathbf{X}_0$ is structurally balanced, then $\mathbf{X}_0$ is already an equilibrium point, and hence if we assume $\mathbf{X}(0) = \mathbf{X}_0$ then, by adopting model (4.18) with $\beta = -3/2$, we get $\mathbf{X}_0 = \mathbf{X}(1) = \operatorname{sign}\left(\mathbf{X}(0)^2 - \frac{3}{2}\mathbf{X}(0)\right)$.
Suppose, now, that $\mathbf{X}_0$ is not structurally balanced, and hence it admits a (V,Σ)-factorization $\mathbf{X}_0 = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top$, with $\mathbf{\Sigma} \in S^k_{1,c}$ for some $k \in \{2,3,4\}$ not structurally balanced. We first note that all matrices in $S^2_{1,c}$ are structurally balanced, and hence we have to rule out the case $k = 2$ because it would correspond to a structurally balanced $\mathbf{\Sigma}$ and hence to a structurally balanced $\mathbf{X}_0$. For $k = 3$ assume w.l.o.g. that

$$\mathbb{N} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}.$$

It is straightforward to prove that if $\mathbf{\Sigma} \in S_1^3$ is not structurally balanced, then

$$\text{sign}\left(\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma} - \frac{3}{2}\mathbf{\Sigma}\right) = \begin{bmatrix} 1 & -\sigma_{12} & \sigma_{13} \\ -\sigma_{12} & 1 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & 1 \end{bmatrix}$$

and this is a structurally balanced matrix. Therefore $\text{sign}\left(\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma} - \frac{3}{2}\mathbf{\Sigma}\right) = \mathbf{w}\mathbf{w}^\top$ for some $\mathbf{w} \in \{-1, 1\}^3$. So, by applying Proposition 4.21 we obtain the result.

Finally, if $k = 4$ (and hence $\mathbb{N} = I_4$), it can be proved that if $\mathbf{\Sigma}$ is not structurally balanced then there exists $\mathbf{w} \in \{-1, 1\}^4$ such that $\mathbf{\Sigma} = 2I_4 - \mathbf{w}\mathbf{w}^\top$. But then

$$\text{sign}\left(\mathbf{\Sigma}^2 - \frac{3}{2}\mathbf{\Sigma}\right) = \text{sign}[(2I_4 - \mathbf{w}\mathbf{w}^\top)^2 - \frac{3}{2}(2I_4 - \mathbf{w}\mathbf{w}^\top)] = \text{sign}(I_4 + \frac{3}{2}\mathbf{w}\mathbf{w}^\top) = \mathbf{w}\mathbf{w}^\top.$$

Therefore, by applying again Proposition 4.21, we obtain the result. □

4.4.3 Not Structurally Balanced Equilibria

For $N \geq 6$ there exist equilibrium points $\mathbf{X}^* \in S_{1,c}^N$ associated to a not structurally balanced graph $\mathcal{G}(\mathbf{X}^*)$. In the following we provide an example for the binary homophily model of dimension $N = 6$.

Example 4.23. *It is easy to verify that*

$$\mathbf{X}^* = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix} \in S_{1,c}^6$$

satisfies the condition given in Proposition 4.16, and therefore it is an equilibrium point. It is worth noticing that $\text{rank}(\mathbf{X}^*) = 3$ *(and hence* $\mathbf{X}^*$ *is not structurally balanced, see Proposition 4.1) and* $\mathbf{P}^\top\mathbf{X}^*\mathbf{P} = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top$*, where* $\mathbf{V} := \mathbf{v}_1 \oplus \mathbf{v}_2 \oplus \mathbf{v}_3$*, with*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Proposition 4.24 below states a necessary and sufficient condition for $\mathbf{X}^*$ to be an equilibrium point in terms of any matrix $\mathbf{\Sigma}$ involved in a (V, Σ)-factorization of $\mathbf{X}^*$.

Proposition 4.24. *A matrix $\mathbf{X}^* \in S^N_{1,c}$ is an equilibrium point for the binary homophily model if and only if*

$$\mathbf{\Sigma} = \mathrm{sign}(\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma} - \mathbf{\Sigma}), \tag{4.27}$$

where $\mathbf{\Sigma} \in S^k_1$ is the matrix $\mathbf{\Sigma}$ involved in a (V, Σ)-factorization of $\mathbf{X}^*$, $\mathbb{N}$ is defined as in (4.25) and $n_1, \dots, n_k$ are the sizes of the vectors $\mathbf{v}_i$ appearing in $\mathbf{V}$.

Proof. By Proposition 4.16, $\mathbf{X}^*$ is an equilibrium point if and only if $\mathbf{X}^* = \mathrm{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)$. From the identity $\mathbf{X}^* = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top$, the previous equilibrium condition can be equivalently written as $\mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top = \mathrm{sign}(\mathbf{V}(\mathbf{\Sigma}\mathbf{V}^\top\mathbf{V}\mathbf{\Sigma} - \mathbf{\Sigma})\mathbf{V}^\top)$. By Lemma 4.20, this identity is true if and only if (4.27) holds. Therefore $\mathbf{X}^*$ is an equilibrium point if and only if (4.27) holds. □

As a consequence of Proposition 4.24, we can show that if $\mathbf{\Sigma} \in S^3_{1,c}$ fulfils (4.27) it is always possible to find an equilibrium network $\mathbf{X}^*$ and a matrix $\mathbf{V}$ such that $\mathbf{X}^* = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top$. The proof is omitted due to page constraints.

Proposition 4.25. *If $\mathbf{\Sigma} \in S^3_1$, then one can always find positive integers n_1, n_2, n_3 such that*

$$\mathbf{\Sigma} = \mathrm{sign}(\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma} - \mathbf{\Sigma}), \tag{4.28}$$

where $\mathbb{N} := n_1 \oplus n_2 \oplus n_3$.

Chapter 5

Opinion Dynamics over Opinion Varying Network Topology

In this chapter we consider two multi-dimensional Hagselmann-Krause (HK) models for opinion dynamics. The two models describe how individuals adjust their opinions on multiple topics, based on the influence of their peers. The models differ in the criterion according to which individuals decide whom they want to be influenced from. In the average-based model individuals compare their average opinions on the various topics with those of the other individuals, and interact only with those individuals whose average opinions lie within a confidence interval. For this model we provide a new proof for the contractivity of the range of opinions, based on semicontraction theory, and show that the agents' opinions reach consensus/clustering if and only if their average opinions do so. In the uniform affinity model agents compare their opinions on each single topic and influence each other only if, topic-wise, such opinions do not differ more than a given tolerance. We identify conditions under which the uniform affinity model enjoys the order-preservation property topic-wise and we prove that the global range of opinions (and hence the range of opinions on each single topic) are non-increasing. The results presented in this chapter can be found in:

- G. De Pasquale, M. E. Valcher, "Multi-dimensional extensions of the Hegselmann-Krause model" accepted for presentation at the 61st Conference on Decision and Control (CDC 2022), Cancun, Mexico.

5.1 Introduction

A problem of interest when dealing with social networks is the modelling and analysis of the spread of information in the network. Different works that address this problem and that focus on different diffusion mechanisms have been proposed, see Hegselmann & Krause (2002), Mei et al. (2019). A common objective, in this context, is to understand when reaching a *consensus*, as a consequence of complex interactions among the agents in the network, is possible Etesami & Basar (2015). Consensus is an active research topic in many fields Bernardo et al. (2015), Zuo et al. (2020). It is about the achievement of an agreement or of a common goal by agents in a network. However, there are contexts in which the reaching of a consensus is either not desirable or does not represent a realistic scenario. This is the case also when dealing with social contexts, e.g. political elections, surveys. It is in these contexts that the *disagreement* phenomenon, along with consensus, becomes of interest Etesami & Basar (2015), Parsegov et al. (2017).

A sociological model that considers both consensus and disagreement is the one known in the literature as Hegselmann-Krause (HK) model Hegselmann & Krause (2002). The HK dynamics evolves under a bounded-confidence mechanism (another sociological models based on the bounded-confidence mechanism is Lorenz (2003)). Confidence intervals are expressed as a function of the gap between pairs of agents' opinions. Since only agents whose opinions are close enough interact, the model represents a mathematical abstraction of *confirmation bias* Del Vicario et al. (2017). Confirmation bias is based on the natural human propensity to search for and welcome information that supports prior beliefs Nickerson (1998). In this paper we focus our attention on the Hegselmann-Krause model, assuming that agents are asked to express their opinion on a pre-fixed and finite number of topics. This represents an extension of the classical scalar version Hegselmann & Krause (2002). A multi-dimensional version of the model has been already studied in the literature. The characterization of the dynamics in the multi-dimensional case is not trivial and some open questions still remain. In addition, we believe that the proposed multi-dimensional extension is not the only possible one. This is what motivates the work here presented. We consider two multi-dimensional HK models for opinion dynamics: the average-based model and the uniform affinity model. In the average-based mode that, to the best of our knowledge, has not been considered before in the literature, individuals compare their average opinions on the various topics with those of the other individuals, and interact only with those individuals whose average opinions lie within a confidence interval. For this model we provide an alternative proof for the contractivity of the range of opinions, and show that the agents' opinions reach consensus/clustering if and only if their average opinions do so. The uniform affinity model is a special instance of the multi-dimensional HK model investigated in Bhattacharyya et al. (2013), Etesami & Basar (2015), Etesami et al. (2013), Nedić & Touri (2012), where we specifically adopt the ℓ_∞-norm. In other words, agents compare their opinions on each single topic and influence each other only if, topic-wise, such opinions do not differ more than a given tolerance. We identify conditions under which the uniform affinity model enjoys the order-preservation property topic-wise and we prove that the global range of opinions (and hence the range of opinions on each single topic) are non-increasing.

5.2 Preliminaries

Definition 5.1 (Vector ℓ_∞-norm). *Given $x \in \mathbb{R}^N$, the* ℓ_∞-norm *of x is* $\|x\|_\infty = \max_i |x_i|$.

Definition 5.2 (Seminorms). *A function $\|\cdot\| : \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ is a* seminorm *on $\mathbb{R}^N$ if it satisfies the following properties:*

$$\text{(homogeneity): } \|ax\| = |a|\|x\|, \forall\ x \in \mathbb{R}^N \text{ and } a \in \mathbb{R};$$

$$\text{(subadditivity): } \|x + y\| \leq \|x\| + \|y\|, \forall\ x, y \in \mathbb{R}^N.$$

Definition 5.3. (ℓ_∞ weighted seminorm Jafarpour et al. (2021)) *Let $\|\cdot\|_\infty : \mathbb{R}^k \to \mathbb{R}_{\geq 0}$ be the ℓ_∞-norm on $\mathbb{R}^k$ and let $R \in \mathbb{R}^{k\times N}$. The R-*weighted seminorm on $\mathbb{R}^N$ associated with the ℓ_∞-norm on $\mathbb{R}^k$ is

$$\|x\|_{\infty,R} := \|Rx\|_\infty, \quad \forall x \in \mathbb{R}^N.$$

Example 5.4 ($C_N^\top$-weighted[1] seminorm De-Pasquale et al. (2021)). *Given a vector $x \in \mathbb{R}^N$ and the oriented incidence matrix $C_N \in \mathbb{R}^{N\times m}$, the $C_N^\top$-weighted seminorm of x associated with the ℓ_∞-norm is*

$$\|x\|_{\infty,C_N^\top} = \max_{i,j\in\{1,\dots,N\}} |x_i - x_j|.$$

[1]Note that C_N is considered as defined in Section 2.2.

Lemma 5.5 (Preliminary lemma)**.** *Given a vector $x \in \mathbb{R}^N$ and a row stochastic matrix $A \in \mathbb{R}^{N\times N}$,*

$$\|Ax\|_{\infty,C_N^\top} \leq \|A\|_{\infty,C_N^\top} \|x\|_{\infty,C_N^\top},$$

where

$$\|A\|_{\infty,C_N^\top} := \max_{\substack{\|x\|_{\infty,C_N^\top}=1 \\ x\perp \ker(C_N^\top)}} \|Ax\|_{\infty,C_N^\top}$$

is the $C_N^\top$-weighted, ℓ_∞ induced seminorm of A.

Proof. Upon noticing that $A(\ker C_N^\top) \subseteq \ker C_N^\top$, the result follows from Lemma A.3 in the Appendix and from the conditional sub-multiplicativity property of the semi norms according to which for every $x \perp \mathcal{K}$, $\|Ax\| \leq \|A\|\|x\|$, Kolpakov (1983).

□

Theorem 5.6. *(*Expression for the $C_N^\top$-weighted, ℓ_∞ induced seminorm De-Pasquale et al. (2021)*) For a row stochastic matrix $A \in \mathbb{R}^{N\times N}$,*

$$\|A\|_{\infty,C_N^\top} = 1 - \min_{ij} \sum_{k=1}^{N} \min\{[A]_{ik}, [A]_{jk}\}. \tag{5.1}$$

Proof. From Proposition 7 in Charron-Bost (2013) one gets the upper bound

$$\|A\|_{\infty,C_n^\top} \leq 1 - \min_{i,j} \sum_{k=1}^{n} \min\{[A]_{i,k}, [A]_{j,k}\}. \tag{5.2}$$

To prove that the inequality is in fact an equality we will show that there always exists a vector x^* for which the inequality in (5.2) holds as an equality. In particular, by exploiting the result from Theorem 3.7 in Ipsen & Selee (2011), according to which, for a row stochastic matrix

$$\frac{1}{2} \max_{i,j} \sum_{k=1}^{n} |[A]_{i,k} - [A]_{j,k}| = 1 - \min_{i,j} \sum_{k=1}^{n} \min\{[A]_{i,k}, [A]_{j,k}\}. \tag{5.3}$$

we will show that

$$\max_{\|x\|_{\infty,C_n^\top}\leq 1} \|Ax\|_{\infty,C_n^\top} = \frac{1}{2} \max_{i,j} \sum_{k=1}^{n} |[A]_{i,k} - [A]_{j,k}|.$$

In fact, by Definition 5.3 together with Example 5.4, it follows that

$$\max_{\|x\|_{\infty,C_n^\top}\leq 1}\|Ax\|_{\infty,C_n^\top} = \max_{\|x\|_{\infty,C_n^\top}\leq 1}\max_{i,j}|(\mathbb{e}_i^\top - \mathbb{e}_j^\top)Ax| = \max_{\|x\|_{\infty,C_n^\top}\leq 1}\max_{i,j}\left|\sum_{k=1}^{n}([A]_{i,k} - [A]_{j,k})x_k\right|$$

$$\leq \frac{1}{2}\max_{i,j}\sum_{k=1}^{n}|[A]_{i,k} - [A]_{j,k}|, \tag{5.4}$$

and the inequality in (5.4) holds as an equality for x^* such that $x_k^* = \frac{1}{2}\mathrm{sign}([A]_{\tilde{i},k} - [A]_{\tilde{j},k})$, $k = \{1,\ldots,N\}$, with $(\tilde{i},\tilde{j}) = \arg\max_{i,j}\sum_k|[A]_{i,k} - [A]_{j,k}|$ and this concludes the proof. □

5.3 The Average-Based (Multi-Dimensional) HK Model

In this section we introduce a multi-dimensional extension of the HK model in which agents compare their (scalar) average opinions on a set of topics, rather than (the vectors representing) their specific opinions topic by topic. This model is suitable to describe the situation when the opinions that an agent has on the different topics are not too far apart, as it happens, for instance, when the topics are related and homogeneous.

Given a group of $N \geq 2$ agents and $m \geq 2$ (related) topics, we let $[X(t)]_{ij}$ denote the *opinion* that agent i has about the topic j at the time instant t. The *average opinion* that the agent i has about the m topics at the time instant t is given by

$$\bar{x}_i(t) = \frac{1}{m}\sum_{j=1}^{m}[X(t)]_{ij}$$

and in vector form

$$\bar{x}(t) = \frac{1}{m}X(t)\mathbb{1}_m. \tag{5.5}$$

We assume that the opinion that the i-th agent has on topic j at time $t+1$ is influenced only by the opinions at time t on that same topic of agents whose average opinion about the m topics is not too far from agent i's average opinion at time t. Specifically, given a certain *confidence threshold* $\varepsilon > 0$, we define the set of *neighbours* (or influencers) of the agent i at the time instant t as a function of the average opinions of the agents, namely as:

$$\mathcal{N}_i^{\mathrm{ave}}(\bar{x}(t)) = \{k \in \{1,\ldots,N\} : |\bar{x}_k(t) - \bar{x}_i(t)| \leq \varepsilon\}. \tag{5.6}$$

Accordingly, by adopting a notation similar to the one in Parasnis et al. (2018), the *influence matrix* $\Phi^{\text{ave}} \in \{0,1\}^{N\times N}$ of this *average-based HK model* is defined as

$$[\Phi^{\text{ave}}(\bar{x}(t))]_{ik} := \begin{cases} 1, & \text{if } k \in \mathcal{N}_i^{\text{ave}}(\bar{x}(t)); \\ 0, & \text{otherwise.} \end{cases} \tag{5.7}$$

Upon defining the matrix

$$D^{\text{ave}}(\bar{x}(t)) := \begin{bmatrix} |\mathcal{N}_1^{\text{ave}}(\bar{x}(t))| & & \\ & \ddots & \\ & & |\mathcal{N}_N^{\text{ave}}(\bar{x}(t))| \end{bmatrix}, \tag{5.8}$$

the opinion matrix $X(t)$ evolves over time as

$$X(t+1) = A^{\text{ave}}(\bar{x}(t))X(t), \tag{5.9}$$

where

$$A^{\text{ave}}(\bar{x}(t)) := D^{\text{ave}}(\bar{x}(t))^{-1}\Phi^{\text{ave}}(\bar{x}(t))$$

is well-posed since $D^{\text{ave}}(\bar{x}(t))$ is nonsingular as a consequence of the fact that $i \in \mathcal{N}_i^{\text{ave}}(\bar{x}(t))$ (and hence $|\mathcal{N}_i^{\text{ave}}(\bar{x}(t))| \geq 1$) $\forall i \in \{1,\dots,N\}$, $\forall t \geq 0$. Equation (5.9) component-wise reads as

$$[X(t+1)]_{ij} = \frac{1}{|\mathcal{N}_i^{\text{ave}}(\bar{x}(t))|}\sum_{k=1}^{n}[\Phi^{\text{ave}}(\bar{x}(t))]_{ik}[X(t)]_{kj}. \tag{5.10}$$

5.4 Average-Based HK Model: Main Definitions

In this section we introduce some fundamental definitions for the average-based HK model that will be used in the following.

Definition 5.7 (Consensus for average-based HK model)**.** *The average-based HK model*

$$X(t+1) = A^{\text{ave}}(\bar{x}(t))X(t), \tag{5.11}$$

$$\bar{x}(t) = \frac{1}{m}X(t)\mathbb{1}_m, \tag{5.12}$$

with $X(0) \in \mathbb{R}^{N\times m}$*, is said to reach* consensus *if*

$$\lim_{t\to\infty} X(t) = \mathbf{1}_N c^\top, \quad \exists\, c \in \mathbb{R}^m. \tag{5.13}$$

Definition 5.8 (Clustering for average-based HK model)**.** *The average-based HK model* (5.11)-(5.12) *reaches* clustering *if there exists a partitioning of the agents* $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_d$ *(*$\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ *for* $i \neq j$*, and* $\cup_{i=1}^{d} \mathcal{V}_i = \{1, \ldots, N\}$*) such that* $\forall i, k \in \mathcal{V}_\ell$*,* $\ell \in \{1, \ldots, d\}$*,*

$$\lim_{t\to\infty} X_{i*}(t) = \lim_{t\to\infty} X_{k*}(t) \tag{5.14}$$

and $\forall i \in \mathcal{V}_\ell$*,* $\forall k \in \mathcal{V}_p$*,* $\ell \neq p$*,*

$$\lim_{t\to\infty} X_{i*}(t) \neq \lim_{t\to\infty} X_{k*}(t). \tag{5.15}$$

Definition 5.9 (Range of opinions on a specific topic)**.** *Given the average-based HK model* (5.11)-(5.12)*, the* range of opinions on topic j*, at the time instant* t*, is defined as*

$$\nu_j(X(t)) = \max_{i,k\in\{1,\ldots,N\}} |[X(t)]_{ij} - [X(t)]_{kj}|. \tag{5.16}$$

Remark 5.10. *Note that* $\nu_j(X(t)) = \|X_{*j}(t)\|_{\infty, C_N^\top}$*.*

5.5 Average-Based HK Model: Opinion Ranges

In this section we explore the monotonicity properties of the range of opinions defined in the previous section. As we will see, the average-based HK model preserves several nice properties of the scalar HK model Hegselmann & Krause (2002).

Proposition 5.11 (Range of opinions on topic)**.** *Given the average-based HK model* (5.11)-(5.12)*, for every choice of* $X(0) \in \mathbb{R}^{N\times m}$*, the range of opinions on a specific topic* $\{\nu_j(X(t)\}_{t\geq 0}, j \in \{1, \ldots, m\}$*, is a non-increasing sequence.*

Proof. The proof follows from the fact that each column of $X(t)$ in (5.11) updates according to the equation

$$X_{*j}(t+1) = A(\bar{x}(t))X_{*j}(t). \tag{5.17}$$

where $A(\bar{x}(t))$ is row stochastic. □

Remark 5.12. *By the same reasoning adopted to prove the previous result we can claim that* $\forall i \in \{1, \ldots, N\}$*,* $j \in \{1, \ldots, m\}$ *and* $t \geq 0$*, one has* $[X(t)]_{ij} \in [\min_k [X(0)]_{kj}, \max_k [X(0)]_{kj}]$*. Consequently, if consensus is reached and we assume* $c = [c_1 \ \ldots \ c_m]^\top$*, then*

$$\min_k [X(0)]_{ki} \leq c_i \leq \max_k [X(0)]_{ki}. \tag{5.18}$$

In the following proposition we provide an alternative proof for the rate of contractivity of the range of opinions in the average-based HK model.

Proposition 5.13 (Range of opinions)**.** *Given the average-based HK model* (5.11)-(5.12)*, for every choice of* $X(0) \in \mathbb{R}^{N\times m}$*, the range of opinions on a specific topic* $\{\nu_j(X(t)\}_{t\geq 0}, j \in \{1,\dots,m\}$*, satisfies*

$$\nu_j(X(t+1)) \leq \gamma(\bar{x}(t))\nu_j(X(t)), \tag{5.19}$$

where

$$\gamma(\bar{x}(t)) := 1 - \min_{i\ell} \sum_{k=1}^{N} \min\{[A(\bar{x}(t))]_{ik}, [A(\bar{x}(t))]_{\ell k}\}.$$

Proof. Consider (5.17), where $A(\bar{x}(t))$ is row stochastic. From Remark 5.10 and the sub-multiplicativity property of the induced matrix seminorms, we get

$$\begin{aligned}
&\nu_j(X(t+1)) = \|X_{*j}(t+1)\|_{\infty,C_N^\top} \\
&= \|A(\bar{x}(t))X_{*j}(t)\|_{\infty,C_N^\top} \leq \|A(\bar{x}(t))\|_{\infty,C_N^\top}\|X_{*j}(t)\|_{\infty,C_N^\top} \\
&= \Big(1 - \min_{i\ell}\sum_{k=1}^{N}\min\{[A(\bar{x}(t))]_{ik}, [A(\bar{x}(t))]_{\ell k}\}\Big)\nu_j(X(t))
\end{aligned}$$

where the inequality follows from Lemma 5.5, while the last identity from Theorem 5.6. □

Opinions of the agents on each topic do not enjoy any order preservation property. So, even if $\bar{x}_i(t) \leq \bar{x}_j(t)$, nothing can be said about $[X(t)]_{ik}$ and $[X(t)]_{jk}$ for specific values of $k \in \{1,\dots,m\}$. For this reason, agents whose opinions on a specific topic are very close may not influence each other. Also, differently from the scalar case, there is no guarantee for order preservation among opinions.

5.6 Average-Based HK Model: Steady State Behaviour

We first note that the vector of the average opinions $\bar{x}(t)$ in (5.5) obeys the dynamics

$$\begin{aligned}
\bar{x}(t+1) &= \frac{1}{m}X(t+1)\mathbb{1}_m = \frac{1}{m}A^{\text{ave}}(\bar{x}(t))X(t)\mathbb{1}_m \\
&= A^{\text{ave}}(\bar{x}(t))\bar{x}(t),
\end{aligned} \tag{5.20}$$

and hence it follows a scalar HK model. Upon a reordering of the agents, so that $\bar{x}_1(0) \leq \bar{x}_2(0) \leq \cdots \leq \bar{x}_N(0)$ then, see Proposition 1 in Blondel et al. (2009), we can guarantee that $\bar{x}_1(t) \leq \bar{x}_2(t) \leq \cdots \leq \bar{x}_N(t)$, $\forall t \geq 0$. Also, by Proposition 2 in Blondel et al. (2009), each sequence $\{\bar{x}_i(t)\}_{t\geq 0}$ is monotone and non-increasing and limited by $\bar{x}_N(0)$, therefore $\lim_{t\to\infty} \bar{x}_i(t)$ exists and is finite, for every $i \in \{1,\dots,m\}$. Moreover, if $\bar{x}^* := \lim_{t\to\infty} \bar{x}(t)$, $\forall i \in \{1,\dots,N-1\}$ either $\bar{x}_i^* = \bar{x}_{i+1}^*$ or $|\bar{x}_{i+1}^* - \bar{x}_i^*| > \varepsilon$, namely the steady state average opinions reach either consensus or clustering. Finally, according to Theorem 1 in Blondel et al. (2009), the limit configuration is reached in a finite number of steps, i.e., $\exists\, t^* \geq 0$ such that $\bar{x}(t^*) = \bar{x}^* \in \mathbb{R}^N$.

Remark 5.14. *As proved in Dittmer (2001) consensus is reached if and only if the sequence $\bar{x}(t)$ is an ε-chain for all $t \geq 0$, by this meaning that, assuming the initial ordering $\bar{x}_1(0) \leq \bar{x}_2(0) \leq \cdots \leq \bar{x}_N(0)$, then we have $|\bar{x}_{i+1}(t) - \bar{x}_i(t)| \leq \varepsilon$, for every $i \in \{1,\dots,N-1\}$ and $t \geq 0$.*

Let us suppose that from $t^* \geq 0$ on-wards,

$$\bar{x}(t) = \frac{1}{m} X(t)\mathbb{1}_m = \bar{x}^* \in \mathbb{R}^N \tag{5.21}$$

and hence

$$X(t+1) = A^{\text{ave}}(\bar{x}^*)X(t). \tag{5.22}$$

Let us consider first, the case when $\bar{x}^* = c^*\mathbf{1}_N$, that is, $D^{\text{ave}}(\bar{x}^*) = NI_N$ and $\Phi^{\text{ave}}(\bar{x}^*) = \mathbf{1}_N\mathbf{1}_N^\top$. Therefore, $\forall t \geq t^*$ we have $A^{\text{ave}}(\bar{x}(t)) = A^{\text{ave}}(\bar{x}^*) = \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top$ which is a constant doubly-stochastic symmetric matrix. Consequently, $\forall t \geq t^*$

$$X(t+1) = \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top X(t) \tag{5.23}$$

which implies

$$X(t^*+1) = \mathbf{1}_N[\bar{m}_1(t^*),\dots,\bar{m}_m(t^*)] \tag{5.24}$$

where

$$\bar{m}_j(t^*) := \frac{1}{N}\sum_{i=1}^{N} X_{ij}(t^*) \tag{5.25}$$

represents the average opinion of the agents on the j-th topic at the time instant t^*. Consequently,

$$X(t^*+2) = \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top X(t^*+1) \tag{5.26}$$

$$= \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top(\mathbf{1}_N[\bar{m}_1(t^*),\dots,\bar{m}_m(t^*)]) \tag{5.27}$$

$$= \mathbf{1}_N[\bar{m}_1(t^*),\dots,\bar{m}_m(t^*)], \tag{5.28}$$

which means that if the HK model that describes the evolution of the average opinions of the agents on the m topics reaches consensus at the time instant t^*, then the punctual opinions of the agents on the topics reach consensus at the next time-step

$$X(t) = X(t^*+1) = \mathbf{1}_N[m_1(t^*),\dots,m_m(t^*)],\ \forall t \geq t^*+1.$$

Let us consider now the case when there exists t^* such that $\bar{x}(t^*) = [c_1^*\mathbb{1}_{n_1}^\top|c_2^*\mathbb{1}_{n_2}^\top|\dots|c_d^*\mathbb{1}_{n_d}^\top]^\top$, namely the mean values of the agents' opinions on the m topics clusterize into d disjoint clusters: $\mathcal{V}_1,\dots,\mathcal{V}_d$, $|\mathcal{V}_i| = n_i$, in each of which the average opinion takes value c_i^* and $|c_i^* - c_{i+1}^*| > \varepsilon, \forall\, i \in \{1,\dots,N-1\}$. In this case, the matrices $D^{\text{ave}}(\bar{x}^*)$ and $\Phi^{\text{ave}}(\bar{x}^*)$ in (5.11)-(5.12) take the structure

$$D^{\text{ave}}(\bar{x}^*) = \begin{bmatrix} n_1 I_{n_1} & & \\ & \ddots & \\ & & n_d I_{n_d}\end{bmatrix}, \qquad \Phi^{\text{ave}}(\bar{x}^*) = \begin{bmatrix} \mathbb{1}_{n_1}\mathbb{1}_{n_1}^\top & & \\ & \ddots & \\ & & \mathbb{1}_{n_d}\mathbb{1}_{n_d}^\top\end{bmatrix} \tag{5.29}$$

and for all $t \geq t^*$

$$X(t+1) = \begin{bmatrix} \frac{1}{n_1}\mathbb{1}_{n_1}\mathbb{1}_{n_1}^\top & & \\ & \ddots & \\ & & \frac{1}{n_d}\mathbb{1}_{n_d}\mathbb{1}_{n_d}^\top\end{bmatrix} X(t). \tag{5.30}$$

Consequently,

$$X(t^*+1) = \begin{bmatrix} \mathbb{1}_{n_1} & & \\ & \ddots & \\ & & \mathbb{1}_{n_d}\end{bmatrix} M(t^*) \tag{5.31}$$

with $M(t^*) \in \mathbb{R}^{d\times m}$ and

$$\mathbf{e}_i^\top M(t^*) = \frac{1}{n_i}[\mathbb{0}^\top|\mathbb{1}_{n_i}^\top|\mathbb{0}^\top]X(t^*) = \frac{1}{n_i}\sum_{k\in I_i}\mathbf{e}_k^\top X(t^*),$$

where $I_i = \{n_1+\dots+n_{i-1}+1,\dots,n_1+\dots+n_{i-1}+n_i\}$ is the set of agents in the i-th cluster. The j-th entry of the row vector $\mathbf{e}_i^\top M(t^*) \in \mathbb{R}^{1\times m}$ represents the average opinion on the j-th topic of the agents in the i-th cluster. Moreover,

$$\begin{aligned} X(t^*+2) &= \begin{bmatrix} \frac{1}{n_1}\mathbb{1}_{n_1}\mathbb{1}_{n_1}^\top & & \\ & \ddots & \\ & & \frac{1}{n_d}\mathbb{1}_{n_d}\mathbb{1}_{n_d}^\top\end{bmatrix}\begin{bmatrix} \mathbb{1}_{n_1} & & \\ & \ddots & \\ & & \mathbb{1}_{n_d}\end{bmatrix} M(t^*) = \begin{bmatrix} \mathbb{1}_{n_1} & & \\ & \ddots & \\ & & \mathbb{1}_{n_d}\end{bmatrix} M(t^*) \\ &= X(t^*+1). \end{aligned}$$

Therefore, if the average opinions clusterize at the time instant t^* then, from t^*+1 onward, the punctual opinions clusterize as well by maintaining the same partition, in d clusters, as the average opinions of the agents over the m topics. Note that

$$M(t^*)\mathbb{1}_m = \begin{bmatrix} c_1^* & c_2^* & \dots & c_d^* \end{bmatrix}^\top.$$

Remark 5.15. *If the average opinion vector $\bar{x}(t)$ clusterizes in d clusters $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_d$ then the opinions on each single topic j clusterize in $d_j \leq d$ clusters and each cluster, say $\tilde{\mathcal{V}}_i$, is the union of one or more clusters $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_d$.*

To summarize the results of this section we propose the following theorem.

Theorem 5.16 (Steady state of average-based HK model)**.** *Given the average-based HK model (5.11)-(5.12), for every choice of $X(0) \in \mathbb{R}^{N\times m}$ the systems dynamics reaches a steady state configuration in a finite number of steps. Moreover, the average-based HK model reaches consensus (clustering) if and only if the HK model describing the evolution of the average opinions reaches consensus (clustering).*

The following result shows that if the maximum gap between the average opinions does not change when moving from time t to time $t+1$, then the same maximum gap remains at all subsequent times, thus showing that if such gap is nonzero then consensus is not reached.

Proposition 5.17. *Consider the average-based HK model (5.11)-(5.12). If at some time $t \geq 0$ one gets*

$$\max_{ij\in\{1,\dots,N\}} |\bar{x}_i(t) - \bar{x}_j(t)| = \max_{ij\in\{1,\dots,N\}} |\bar{x}_i(t+1) - \bar{x}_j(t+1)| \tag{5.32}$$

then

$$\max_{ij\in\{1,\dots,N\}} |\bar{x}_i(t+1) - \bar{x}_j(t+1)| = \max_{ij\in\{1,\dots,N\}} |\bar{x}_i(t+2) - \bar{x}_j(t+2)|. \tag{5.33}$$

Therefore, if the quantity in (5.32) is positive, then the average-based HK model (5.11)-(5.12) does not achieve consensus.

Proof. Assume, without loss of generality, that $\bar{x}_1(t) \leq \bar{x}_2(t) \leq \cdots \leq \bar{x}_N(t)$, then one has $\max_{ij\in\{1,\dots,N\}} |\bar{x}_i(t) - \bar{x}_j(t)| = \bar{x}_N(t) - \bar{x}_1(t)$. Since $\bar{x}_1(t+1) \geq \bar{x}_1(t)$ and $\bar{x}_N(t+1) \leq \bar{x}_N(t)$, then (5.32) implies $\bar{x}_1(t+1) = \bar{x}_1(t)$ and $\bar{x}_N(t+1) = \bar{x}_N(t)$, that easily leads to (5.33). Since the sequence of average opinions does not reach consensus, neither does the sequence $\{X(t)\}_{t\geq 0}$. $\square$

5.7 The Uniform Affinity Model

The multi-dimensional HK model investigated in Etesami & Basar (2015), Etesami et al. (2013), Nedić & Touri (2012) has a structure similar to the one we explored in the previous sections, however it adopts as a criterion to define the opinion proximity the distance (induced by the norm) between the opinion vectors of the agents. Specifically, it is assumed that the neighbours of agent i at time t are[2]

$$\mathcal{N}_i(X(t)) = \{k \in \{1, \dots, N\} : \|X_{i*}(t)^\top - X_{k*}(t)^\top\| \leq \varepsilon\},$$

where $\varepsilon > 0$ is the confidence threshold and $\|\cdot\|$ denotes an arbitrary norm. Accordingly, the influence matrix $\Phi \in \{0,1\}^{N \times N}$ at time t is the one whose (i,k)-th entry is

$$[\Phi(X(t))]_{ik} = \begin{cases} 1, & \text{if } k \in \mathcal{N}_i(X(t)); \\ 0, & \text{otherwise.} \end{cases} \tag{5.34}$$

Upon defining the matrix[3]

$$D(X(t)) := \begin{bmatrix} |\mathcal{N}_1(X(t))| & & \\ & \ddots & \\ & & |\mathcal{N}_N(X(t))| \end{bmatrix} \tag{5.35}$$

the opinion matrix $X(t)$ evolves over time as

$$X(t+1) = A(X(t))X(t), \tag{5.36}$$

where

$$A(X(t)) := D(X(t))^{-1}\Phi(X(t)) \tag{5.37}$$

is well-posed ($D(X(t))$ is nonsingular) and row stochastic.

In the references Etesami & Basar (2015), Etesami et al. (2013), Nedić & Touri (2012) the main focus has been on proving that the multi-dimensional HK model (5.36), with the row stochastic matrix $A(X(t))$ defined as above, (for any choice of the norm $\|\cdot\|$) converges to a steady-state solution in a finite number of steps, and on providing an upper bound on the

[2] Since the norm is formally defined for column vectors, while $X_{i*}(t)$ and $X_{k*}(t)$ are row vectors, we moved to their transposed versions.

[3] In the following we will replace $\mathcal{N}_i(X(t))$ with the more compact notation $\mathcal{N}_i(t)$. If we assume that $X(0)$ is assigned, the notation makes perfect sense.

termination time (see, in particular, Etesami & Basar (2015)). The interesting aspect is that the termination time is independent of the number m of topics. See Figure 5.1 for an example of an uniform affinity model with $N = 10$ agents and $m = 2$ topics that reaches consensus.

In this section we want to explore some monotonicity properties of the previous model by considering specifically the case when the norm is the ℓ_∞-norm. This means that

$$\mathcal{N}_i(X(t)) = \{j : \max_{k\in\{1,\dots,m\}} |[X(t)]_{ik} - [X(t)]_{jk}| \leq \varepsilon\}$$

so, in order for two agents to influence each other, their opinions must be close topic-wise. This model is in line with the spirit of bounded-confidence even in contexts in which agents take different positions about the various topics. We will refer to the multi-dimensional HK model with ℓ_∞-norm (5.36) as the uniform affinity model.

We first prove that if we consider the range of opinions on a specific topic k at time t and we consider the largest of such values over all the possible topics, then such a quantity is non increasing over time.

Proposition 5.18. *(*Range of opinions in uniform affinity HK model*) For the uniform affinity model, the quantity*

$$\nu(X(t)) := \max_{\substack{i,j\in\{1,\dots,N\}\\ k\in\{1,\dots,m\}}} |[X(t)]_{ik} - [X(t)]_{jk}| = \max_{k\in\{1,\dots,m\}} \nu_k(X(t))$$

is non increasing over time, namely $\nu(X(0)) \geq \nu(X(1)) \geq \nu(X(2)) \geq \dots$.

Proof. We first observe that $\forall k \in \{1,\dots,m\}$

$$\nu(X(t)) \geq \max_{ij} |[X(t)]_{ik} - [X(t)]_{jk}| = \max_i [X(t)]_{ik} - \min_j [X(t)]_{jk} = [X(t)]_{uk} - [X(t)]_{lk} \tag{5.38}$$

for some specific u, l.

For all $i, j \in \{1,\dots,N\}$ and $k \in \{1,\dots,m\}$

$$|[X(t+1)]_{ik} - [X(t+1)]_{jk}| = \Big| \sum_{d\in\mathcal{N}_i(t)} \frac{1}{|\mathcal{N}_i(t)|}[X(t)]_{dk} - \sum_{d\in\mathcal{N}_j(t)} \frac{1}{|\mathcal{N}_j(t)|}[X(t)]_{dk} \Big| \leq$$
$$\Big| \max_\ell [X(t)]_{\ell k} - \min_\ell [X(t)]_{\ell k} \Big| = \Big| [X(t)]_{uk} - [X(t)]_{lk} \Big| = [X(t)]_{uk} - [X(t)]_{lk} \leq \nu(X(t)).$$

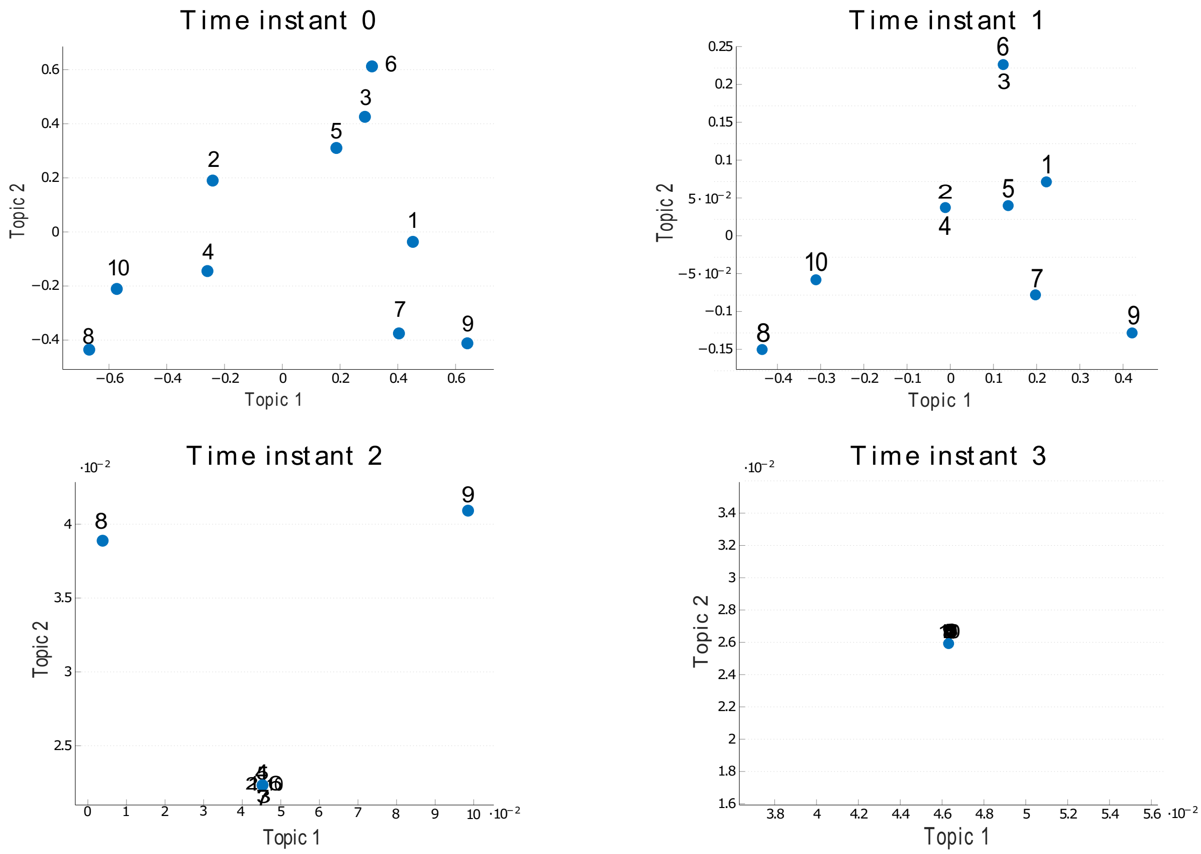

Figure 5.1: Convergence to consensus of the uniform affinity model with $N = 10$ agents, $m = 2$ topics, confidence threshold $\varepsilon = 0.8$. Initial conditions are uniformly generated in the interval $[-1, 1]^m$. Convergence occurs after 3 iterations.

Since this is true for all $i, j \in \{1, \ldots, N\}$ and for all $k \in \{1, \ldots, m\}$, then it is also true that

$$\nu(X(t+1)) = \max_{\substack{ij \in \{1,\ldots,N\} \\ k \in \{1,\ldots,m\}}} |[X(t+1)]_{ik} - [X(t+1)]_{jk}| \leq \nu(X(t)).$$

□

A consequence of Proposition 5.18 is that the opinion gap on each single topic (see Definition 5.9) is non-increasing too.

Differently from what happens with the standard HK model (and partly with the average-based HK model), there is no way to introduce a meaningful total ordering in $\mathbb{R}^m$ and hence in the set of all agents' opinions. In fact, in general, the ordering is different on each topic, and condition $|[X(t)]_{ik} - [X(t)]_{jk}| \leq \varepsilon$ for some specific k does not ensure that i and j are neighbours. So, it may happen that $[X(t)]_{ik} < [X(t)]_{jk}$, but at the subsequent time step $[X(t+1)]_{ik} > [X(t+1)]_{jk}$. See, for instance, Figure 5.1, where moving from $t = 0$ to $t = 1$ the opinions of agents 1 and 5 on topic 2 swap (even if eventually they all converge to a consensus).

However, if at some time t every pair of agents who do not influence each other have opinions about *all* the topics that differ by more than ε, then the opinions' ordering on each topic remains unaltered when moving from time t to $t+1$.

Proposition 5.19 (One-step order preservation)**.** *Consider the uniform affinity model and suppose that at some time $t \geq 0$ one has that for every $i, j \in \{1, \dots, N\}$ condition $j \notin \mathcal{N}_i(t)$ implies*

$$|[X(t)]_{ik} - [X(t)]_{jk}| > \varepsilon, \quad \forall\, k \in \{1, \dots, m\}. \tag{5.39}$$

If for every $k \in \{1, \dots, m\}$ we sort the agents' opinions (the order specifically depending on k) so that $[X(t)]_{i_1 k} \leq [X(t)]_{i_2 k} \leq \cdots \leq [X(t)]_{i_N k}$, then the same opinion ordering on that topic is preserved at $t+1$, i.e., $[X(t+1)]_{i_1 k} \leq [X(t+1)]_{i_2 k} \leq \cdots \leq [X(t+1)]_{i_N k}$.

Proof. Let k be arbitrary in $\{1, \dots, m\}$ and consider $i_h \in \{i_1, \dots, i_{N-1}\}$.
We preliminarily observe that if $i_{h+1} \notin \mathcal{N}_{i_h}(t)$, assumption (5.39) implies that

$$\mathcal{N}_{i_h}(t) \cap \mathcal{N}_{i_{h+1}}(t) = \emptyset; \quad \mathcal{N}_{i_h}(t) \setminus \mathcal{N}_{i_{h+1}}(t) \subseteq \{i_1, \dots, i_h\}; \quad \mathcal{N}_{i_{h+1}}(t) \setminus \mathcal{N}_{i_h}(t) \subseteq \{i_{h+1}, \dots, i_N\}$$

On the other hand, if $i_{h+1} \in \mathcal{N}_{i_h}(t)$, then

$$\mathcal{N}_{i_h}(t) \setminus \mathcal{N}_{i_{h+1}}(t) \subseteq \{i_1, \dots, i_{h-1}\}; \qquad \mathcal{N}_{i_{h+1}}(t) \setminus \mathcal{N}_{i_h}(t) \subseteq \{i_{h+2}, \dots, i_N\}.$$

So, we can define

$$\begin{aligned}
[\Delta(t)]_{hk} &:= \frac{1}{|\mathcal{N}_{i_h}(t) \cap \mathcal{N}_{i_{h+1}}(t)|} \sum_{\ell \in \mathcal{N}_{i_h}(t) \cap \mathcal{N}_{i_{h+1}}(t)} [X(t)]_{\ell k}, \\
[\tilde{X}(t)]_{i_h k} &:= \frac{\sum_{\ell \in \mathcal{N}_{i_h}(t) \setminus \mathcal{N}_{i_{h+1}}(t)} [X(t)]_{\ell k}}{|\mathcal{N}_{i_h}(t) \setminus \mathcal{N}_{i_{h+1}}(t)|}, \\
[\tilde{X}(t)]_{i_{h+1} k} &:= \frac{\sum_{\ell \in \mathcal{N}_{i_{h+1}}(t) \setminus \mathcal{N}_i(t)} [X(t)]_{\ell k}}{|\mathcal{N}_{i_{h+1}}(t) \setminus \mathcal{N}_i(t)|},
\end{aligned}$$

and get

$$[X(t+1)]_{i_h k} = \frac{|\mathcal{N}_{i_h}(t) \cap \mathcal{N}_{i_{h+1}}(t)|}{|\mathcal{N}_{i_h}(t)|} [\Delta(t)]_{hk} + \frac{|\mathcal{N}_{i_h}(t) \setminus \mathcal{N}_{i_{h+1}}(t)|}{|\mathcal{N}_{i_h}(t)|} [\tilde{X}(t)]_{i_h k}$$

and similarly

$$[X(t+1)]_{i_{h+1}k} = \frac{|\mathcal{N}_{i_h}(t) \cap \mathcal{N}_{i_{h+1}}(t)|}{|\mathcal{N}_{i_{h+1}}(t)|}[\Delta(t)]_{hk} + \frac{|\mathcal{N}_{i_{h+1}}(t) \setminus \mathcal{N}_{i_h}(t)|}{|\mathcal{N}_{i_{h+1}}(t)|}[\tilde{X}(t)]_{jk}.$$

Since $[\tilde{X}(t)]_{i_{h+1}k} \geq [\Delta(t)]_{hk} \geq [\tilde{X}(t)]_{i_hk}$ if i_h and i_{h+1} are neighbours, while $[\Delta]_{hk}(t) = 0$ and $[\tilde{X}(t)]_{i_{h+1}k} > [\tilde{X}(t)]_{i_hk}$ if i_h and i_{h+1} are not neighbours, it follows that $[\tilde{X}(t+1)]_{i_{h+1}k} \geq [\tilde{X}(t+1)]_{i_hk}$.

□

The reasoning behind the previous result can be extended to a different situation when the agents' opinions at some time t are ordered so that for every $k \in \{1, \dots, m\}$

$$[X(t)]_{1k} \leq [X(t)]_{2k} \leq \cdots \leq [X(t)]_{Nk}.$$

When so, such ordering is preserved at all subsequent time instants. This is based on the fact that if $i < j$ and i and j are not neighbours, then there exists $\bar{k}$ such that $[X(t)]_{j\bar{k}} - [X(t)]_{i\bar{k}} > \varepsilon$. But this implies that for every $p < i$ one has $[X(t)]_{j\bar{k}} - [X(t)]_{p\bar{k}} > \varepsilon$, and for every $q > j$ one has $[X(t)]_{q\bar{k}} - [X(t)]_{i\bar{k}} > \varepsilon$. Consequently, $\mathcal{N}_i(t) \cap \{j, j+1, \dots, N\} = \emptyset$ and similarly $\mathcal{N}_j(t) \cap \{1, 2, \dots, i\} = \emptyset$. Conversely, if $i < j$ and i and j are neighbours, then $\mathcal{N}_i(t) \cap \mathcal{N}_j(t) \supseteq \{i, i+1, \dots, j\}$. Based on these comments, the proof of the following result can be easily obtained by mimicking the proof of Proposition 5.19.

Proposition 5.20 (Sufficient condition for order preservation)**.** *Consider the uniform affinity model. If at some time $t \geq 0$ one has that for every topic $k \in \{1, \dots, m\}$*

$$[X(t)]_{1k} \leq [X(t)]_{2k} \leq \cdots \leq [X(t)]_{Nk},$$

then it is also true that for every $\tau \geq 0$ and every topic $k \in \{1, \dots, m\}$

$$[X(t+\tau)]_{1k} \leq [X(t+\tau)]_{2k} \leq \cdots \leq [X(t+\tau)]_{Nk}.$$

Chapter 6

On the Herdability of Linear Time-Invariant Systems with Special Algebraic or Topological Structure

In this chapter we investigate the herdability property, namely the capability of a system to be driven towards the (interior of the) positive orthant, for the class of linear time-invariant state space models. Some conditions that allow to reduce the problem size, as well as some sufficient conditions for the problem solvability, are first derived. Herdability of matrix pairs (A, B), with A in Jordan form, is subsequently investigated. Finally, herdability of certain matrix pairs (A, B) is explored, by analysing the corresponding leader-follower network, under the assumption that the graph $\mathcal{G}(A)$ is clustering balanced (in particular, structurally balanced), or it has a tree topology and a single leader.

The results presented in this chapter can be found in:

- G. De Pasquale, M. E. Valcher, "On the herdability of linear time-invariant systems with special algebraic or topological structure" provisionally accepted in Automatica 2022

- G. De Pasquale, M. E. Valcher. “Algebraic and graph-theoretic conditions for the herdability of linear time invariant systems", Proceedings of the 60th IEEE Conference on Decision and Control (CDC 2021), pp. 5826-5831, Austin, Texas, USA, 10.1109/CDC45484.2021.9683703

6.1 Introduction

Networked multi-agent systems have been the subject of an impressive number of contributions in the last two decades, due to their wide range of applications Antsaklis & Baillieul (2007), Baillieul & Antsaklis (2007), Olfati-Saber et al. (2007), Zhang et al. (2012). As a result, the controllability of this class of systems, namely the property of the system state to be driven towards any point of the state space, has attracted a lot of interest, mainly aimed at deriving conditions that rely on the communication graph structure, rather than on the specific weights attributed to the graph edges Boothby (1982), Egerstedt et al. (2012), Johnson et al. (1993), Mousavi et al. (2019), Parlangeli & Notarstefano (2012), Rahmani et al. (2009), Tsatsomeros (1998). However, there are many research fields, such as biology Jacquez (1972), chemistry Bower & Bolouri (2001), sociology Scott (1988), neuroscience Gupta et al. (2007), etc. for which, due to the nature of the applications involved, investigating if the system state can be brought towards any point of the state space may not be of practical interest, and may lead to overly restrictive conditions on the model into play. In particular, when dealing with social networks, in contexts in which some individuals have leading roles over the network, and hence can be identified as *leaders*, different problems with respect to the ones of consensus or polarization may be taken into account. In these situations one may wonder under what conditions leader nodes in the network can influence the state of the other individuals (*followers*) so that they can be brought above a certain threshold. For example, in the context of marketing advertisement, it is of interest to devise strategies targeting some individuals to bring the consumption level of a certain good for a group of consumers over a certain threshold. In this circumstance, it is pointless to require that the state entries may assume any real value, including the negative ones. It is in contexts like this one that the investigation of a weaker concept with respect to the one of controllability, known in the literature as herdability Ruf et al. (2018, 2019), becomes of interest. Herdability refers to the possibility of driving the state variable towards the interior of the positive orthant. More precisely, a system is herdable if, for every choice of the initial conditions, there exists a control input that drives all the state variables over a positive threshold. Generally speaking, herdability applies to leader-follower network systems in which one may wonder where to locate the leaders in the network so that the system is herdable. The problem considered in this chaper shares some similarities with the one known in the literature as *unanimity of opinions* studied in Altafini & Lini (2015) where it is shown that herdabilty can be ensured when the graph adjacency matrix is *eventually positive*.

In this chapter we study the herdability of leader-follower networked linear time invariant systems described by a matrix pair (A, B), both from an algebraic and from a topological perspective, this latter being of great relevance, as mentioned before, when the pair (A, B) represents a multi-agent system. In particular, we focus on special topologies of the graph $\mathcal{G}(A)$, as the tree topology, or the case when the graph is structurally/clustering balanced. On the other hand, we present a condition that allows to reduce the study of the herdability of a pair (A, B) to the one of a related pair $(\tilde{A}, \tilde{B})$ of lower dimensions, and we provide a complete analysis of herdability for pairs whose matrix A is in Jordan form.

6.2 Algebraic Conditions for Herdability of General Pairs (A, B)

The concept of herdability of linear and time-invariant state space models described by a matrix pair (A, B), with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, has been defined in various ways Ruf et al. (2018, 2019), She et al. (2019). In this paper we are interested in the behavior of all state variables, rather than in the behavior of a subset of them. Consequently, we assume the following definition (which is equivalent to Definition 3 in Ruf et al. (2019)).

Definition 6.1. *Given a (continuous-time or discrete-time) (linear and time-invariant) state space model of dimension n with m inputs, described by a pair (A, B), $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, the system (the pair) is said to be* herdable *if for every $\mathbf{x}(0)$ and every $h > 0$, there exists a time $t_f > 0$ and an input $\mathbf{u}(t), t \in [0, t_f)$, that drives the state of the system from $\mathbf{x}(0)$ to $\mathbf{x}(t_f) \geq h\mathbf{1}_n$.*

Both in the continuous-time case and in the discrete-time case, herdability reduces to a condition on the controllability matrix associated with the pair (A, B).

Proposition 6.2 (Corollary 1, Ruf et al. (2019))**.** *A pair $(A, B), A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, is herdable if and only if $\mathrm{Im}(\mathcal{R}(A, B))$ includes a strictly positive vector, where*

$$\mathcal{R}(A, B) := \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \tag{6.1}$$

is the controllability matrix *of the pair* (A, B).

Clearly, every reachable pair (A, B) is herdable, but the converse is not true. Also, if $\mathcal{R}(A, B)$ has zero rows then the problem is clearly not solvable. So, in the following we will investigate herdability by assuming that $\mathcal{R}(A, B)$ is devoid of zero rows and $\mathrm{Im}(\mathcal{R}(A, B))$ is a proper subset of $\mathbb{R}^n$.

In this section we present some sufficient conditions for the herdability of a generic matrix pair (A, B). We will later focus on pairs (A, B) that are endowed with specific structural properties.

Lemma 6.3. *Given a pair* $(A, B), A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times m}$*, assume that* $\mathcal{R} := \mathcal{R}(A, B) \in \mathbb{R}^{n\times nm}$ *satisfies the following conditions:*

i) $\mathcal{R}$ *has no zero rows;*

ii) the set $J := \{j \in [1, nm] : \mathcal{R}\mathbf{e}_j \text{ is unisigned}\}$ *is such that* $|\cup_{j\in J}\overline{\mathrm{ZP}}(\mathcal{R}\mathbf{e}_j)|\geq n - 1$.

Then the pair (A, B) *is herdable.*

Proof. Let us first suppose that $|\cup_{j\in J}\overline{\mathrm{ZP}}(\mathcal{R}\mathbf{e}_j)|= n$, which means that $\forall i \in [1, n]$, there exists $j \in J$ such that the i-th entry of the unisigned vector $\mathcal{R}\mathbf{e}_j$ is nonzero. By choosing the vector $\mathbf{u}$ with entries

$$[\mathbf{u}]_j = \begin{cases} 0, & \text{if } j \notin J; \\ \mathrm{sign}(\mathcal{R}\mathbf{e}_\mathrm{j}), & \text{if } j \in J; \end{cases} \tag{6.2}$$

it is immediate to see that $\mathcal{R}\mathbf{u} \gg 0$, and hence the pair (A, B) is herdable.

Let us assume now that $|\cup_{j\in J}\overline{\mathrm{ZP}}(\mathcal{R}\mathbf{e}_j)|= n - 1$, and set $J = \{j_1, j_2, \ldots, j_k\}$. This implies that there exists a unique index $i \in [1, n]$ such that $\mathbf{e}_i^\top\mathcal{R}[\mathbf{e}_{j_1}|\mathbf{e}_{j_2}|\ldots|\mathbf{e}_{j_k}] = \mathbf{0}_k^\top$. On the other hand, by hypothesis i), there exists $h \in [1, nm], h \notin J$, such that $\mathbf{e}_i^\top\mathcal{R}\mathbf{e}_h \neq 0$. Therefore, by choosing the vector $\mathbf{u}$ with entries

$$[\mathbf{u}]_j = \begin{cases} \mathrm{sign}(\mathbf{e}_i^\top\mathcal{R}\mathbf{e}_h), & \text{if } j = h; \\ 0, & \text{if } j \notin J \cup \{h\}; \\ k \cdot \mathrm{sign}(\mathcal{R}\mathbf{e}_j), & \text{if } j \in J; \end{cases} \tag{6.3}$$

there always exists $k \in \mathbb{R}, k > 0$, sufficiently large such that $\mathcal{R}\mathbf{u} \gg 0$. □

We now introduce a technical lemma, whose proof is elementary and hence omitted.

Lemma 6.4. *Given a matrix $\Phi \in \mathbb{R}^{n\times k}$, assume that there exist two permutation matrices $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{k\times k}$ such that*

$$P_1 \Phi P_2 = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix}, \tag{6.4}$$

and that both $\mathrm{Im}(\Phi_{11})$ and $\mathrm{Im}(\Phi_{22})$ include a strictly positive vector. Then $\exists\, \mathbf{u} \in \mathbb{R}^k$ such that $\Phi\mathbf{u} > 0$.

Remark 6.5. *The result of Lemma 6.4 easily extends to the case when the matrix $\Phi \in \mathbb{R}^{n\times k}$ can be reduced (by means of row and column permutations) to the more general block-triangular form:*

$$P_1 \Phi P_2 = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \dots & \Phi_{1k} \\ 0 & \Phi_{22} & \dots & \Phi_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Phi_{kk} \end{bmatrix}, \tag{6.5}$$

where the diagonal blocks Φ_{ii} are not necessarily square matrices. Indeed, if the image of each diagonal block $\Phi_{ii}, i \in [1,k]$, includes a strictly positive vector then it is straightforward to see that there exists $\mathbf{u} \in \mathbb{R}^k$ such that $\Phi\mathbf{u} > 0$. This is true, in particular, if for every $i \in [1,k]$ we can select a subset of the columns of Φ_{ii} with the following properties: 1) each of them is unsigned; 2) for each row index j, at least one of these unsigned columns has the j-th entry which is nonzero.

Based on Lemma 6.4, we can derive the following sufficient condition for heardability.

Lemma 6.6. *Given a pair (A,B), $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, assume that $\mathcal{R} := \mathcal{R}(A,B) \in \mathbb{R}^{n\times nm}$ has no zero rows. Define the sets*

$$J := \{j \in [1,nm] : \mathcal{R}\mathbf{e}_j \text{ is unsigned}\} \tag{6.6}$$

$$\mathcal{H} := \cup_{j\in J} \overline{\mathrm{ZP}}(\mathcal{R}\mathbf{e}_j), \tag{6.7}$$

and suppose that $\forall h \in [1,n] \setminus \mathcal{H}$ there exists $j \in [1,nm] \setminus J$ such that

i) $[\mathcal{R}]_{hj} = \mathbf{e}_h^\top \mathcal{R}\mathbf{e}_j \neq 0$, *and* *ii)* $\forall k \in [1,n] \setminus \mathcal{H}$, *condition* $[\mathcal{R}]_{kj} = \mathbf{e}_k^\top \mathcal{R}\mathbf{e}_j \neq 0$ *implies* $\operatorname{sign}([\mathcal{R}]_{kj}) = \operatorname{sign}([\mathcal{R}]_{hj})$,

namely for every index h that does not belong to $\mathcal{H}$ there exists a column of $\mathcal{R}$, say $\mathcal{R}\mathbf{e}_j$, where the h-th entry and all the nonzero entries corresponding to indices that do not belong to $\mathcal{H}$ are of the same sign. Then the pair (A, B) is herdable.

Proof. Under the lemma assumptions there exists a set of indices $T \subseteq [1, nm] \setminus J$ such that
a) $\left(\cup_{j\in J}\overline{\mathrm{ZP}}(\mathcal{R}\mathbf{e}_j)\right) \cup \left(\cup_{j\in T}\overline{\mathrm{ZP}}(\mathcal{R}\mathbf{e}_j)\right) = [1,n]$;
b) if we denote by $S \in \mathbb{R}^{(n-|\mathcal{H}|)\times n}$ the (selection) matrix whose rows are the n-dimensional canonical vectors indexed in $[1,n] \setminus \mathcal{H}$, then $S\mathcal{R}\mathbf{e}_j$ is unisigned for every $j \in T$.
This implies that there exist two permutation matrices $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{nm\times nm}$ such that

$$P_1 \mathcal{R} P_2 = \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ 0 & \mathcal{R}_{22} \end{bmatrix},$$

where $\mathcal{R}_{11}$ has all unisigned columns, while $\mathcal{R}_{22}$ has a subset of its columns that are unisigned and therefore $\mathrm{Im}(\mathcal{R}_{22})$ includes a strictly positive vector. So, the result follows from Lemma 6.4. □

The idea behind Lemma 6.4 and Lemma 6.6 can be recursively iterated, thus leading to an algorithm that checks a sufficient condition for herdability. The algorithm receives as input the controllability matrix and returns, if the sufficient condition is verified, a confirmation that the pair (A, B) is herdable. In detail, it proceeds as follows: at each step the algorithm detects a column vector that is unisigned, then sets to zero all the rows of $\mathcal{R}$ that correspond to the nonzero entries (the non-zero pattern) of such a column vector. Subsequently, the algorithm repeats the same step on the modified matrix $\mathcal{R}$, until either $\mathcal{R}$ becomes the zero matrix or the matrix $\mathcal{R}$ has no unisigned columns, thus iteratively applying the same strategy as in Lemma 6.4. In the former case the pair (A, B) is herdable, in the second case the algorithm stops.

Algorithm 2, below, makes use of the following notation. Given a matrix $\mathcal{R} \in \mathbb{R}^{n\times nm}$ and a set $\mathcal{I} \subseteq [1,n]$, we denote by $\mathcal{R}_{\mathcal{I}}$ the matrix obtained from $\mathcal{R}$ by (leaving unchanged all rows indexed in $\mathcal{I}$ and) replacing every row indexed in $[1,n] \setminus \mathcal{I}$ with the zero row.

Lemma 6.7. *Consider a pair (A, B), where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. For every non-singular matrix $T \in \mathbb{R}^{m\times m}$, the pair (A, B) is herdable if and only if the pair (A, BT) is herdable.*

Algorithm 2 Greedy algorithm to check herdability

$\mathcal{R} := [B|AB|\dots|A^{n-1}B]$ ▷ Initialization
$\mathcal{I} := [1, n]$
$\mathcal{J} := [1, nm]$
while $\mathcal{I} \neq \emptyset$ **do** ▷ Recursive check
  for $j \in \mathcal{J}$ **do**
    if $\mathcal{R}\mathbf{e}_j$ is unisigned **then**
      $\mathcal{J} = \mathcal{J} \setminus \{j\}$
      $\mathcal{I} = \mathcal{I} \setminus \overline{\mathrm{ZP}}(\mathcal{R}\mathbf{e}_j)$
      $\mathcal{R} = \mathcal{R}_{\mathcal{I}}$
      if $\mathcal{I} = \emptyset$ **then**
        (A,B) is herdable
  if there are no unisigned column vectors in $\mathcal{R}$ **then**
    stop

Proof. Follows from $\mathrm{Im}(\mathcal{R}(A, B)) = \mathrm{Im}(\mathcal{R}(A, BT))$. □

Proposition 6.8, below, provides a method for the dimensionality reduction of the herdability problem for matrix pairs (A, B), with A and B conformably partitioned as:

$$A = \begin{bmatrix} A_{11} & A_{21} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}. \tag{6.8}$$

It states that, when there is a set of leaders among the n nodes of the graph $\mathcal{G}(A)$, the herdability of the system depends only on the way followers interact and leaders exert their influence on their followers. How followers, in turn, “evaluate/weight" the leaders has no influence on the herdability of the system. This result will be largely exploited in the rest of the paper.

Proposition 6.8. *Consider a pair (A, B), where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are described as in (6.8), where $B_1 \in \mathbb{R}^{r\times m}$ is of full row rank and $A_{11} \in \mathbb{R}^{r\times r}$. The pair (A, B) is herdable if and only if the pair (A_{22}, A_{21}) is herdable.*

Proof. Let $\mathcal{R}(A, B)$ be the controllability matrix of (A, B) and $\mathcal{R}(A_{22}, A_{21})$ the controllability matrix of (A_{22}, A_{21}). We preliminarily note that, since B_1 is of full row rank, there exists a nonsingular matrix $T \in \mathbb{R}^{m\times m}$ such that $B_1 T = \begin{bmatrix} I_r & 0 \end{bmatrix}$. Since the zero columns of BT are irrelevant, in the following by making use of Lemma 6.7 we will assume $r = m$ and $B_1 = I_m$. Since

$$\mathcal{R}(A,B) = \begin{bmatrix} I_m & \Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix}$$

where

$$\begin{aligned}\Phi_{22} &:= \begin{bmatrix} A_{21} & A_{21}A_{11} + A_{22}A_{21} & A_{21}(A_{11}^2 + A_{12}A_{21}) + A_{22}(A_{21}A_{11} + A_{22}A_{21}) & ... \end{bmatrix} \\ &= \begin{bmatrix} 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} AB & A^2B & \dots & A^{n-1}B \end{bmatrix},\end{aligned}$$

it is immediate to see that for every $\mathbf{v}_1 \in \mathbb{R}^m$

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \in \operatorname{Im}(\mathcal{R}(A,B)) \qquad \Leftrightarrow \qquad \mathbf{v}_2 \in \operatorname{Im}(\Phi_{22}).$$

We now prove that

$$\operatorname{Im}(\Phi_{22}) = \operatorname{Im}\left(\begin{bmatrix} 0 & I_{n-m} \end{bmatrix}\begin{bmatrix} AB & A^2B & \dots & A^{n-1}B \end{bmatrix}\right) = \operatorname{Im}\left(\mathcal{R}(A_{22}, A_{21})\right).$$

To prove this result we show that for every $k \in [1, n-1]$

$$\begin{bmatrix} 0 & I_{n-m} \end{bmatrix} A^k B = \begin{bmatrix} A_{21} & A_{22}A_{21} & \dots & A_{22}^{k-1}A_{21} \end{bmatrix} \begin{bmatrix} * \\ * \\ \vdots \\ I_m \end{bmatrix} \tag{6.9}$$

where $*$ denotes a real matrix (whose value is not relevant). We proceed by induction on k. If $k = 1$ the result is true since $\begin{bmatrix} 0 & I_{n-m} \end{bmatrix} AB = A_{21} = \begin{bmatrix} A_{21} \end{bmatrix} I_m$.

We assume now that the result is true for $k < \bar{k}$ and then show that the result is true for $k = \bar{k}$. Indeed, there exists some matrix Ξ such that

$$\begin{bmatrix} 0 & I_{n-m} \end{bmatrix} A^{\bar{k}} B = \begin{bmatrix} 0 & I_{n-m} \end{bmatrix} A A^{\bar{k}-1} B = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} A^{\bar{k}-1} B =$$

$$\begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Xi \\ \begin{bmatrix} 0 & I_{n-m} \end{bmatrix} A^{\bar{k}-1} B \end{bmatrix} = A_{21}\Xi + A_{22} \begin{bmatrix} A_{21} & A_{22}A_{21} & \dots & A_{22}^{\bar{k}-2}A_{21} \end{bmatrix} \begin{bmatrix} * \\ * \\ \vdots \\ I_m \end{bmatrix} =$$

$$\begin{bmatrix} A_{21} & A_{22}A_{21} & \dots & A_{22}^{\bar{k}-1}A_{21} \end{bmatrix} \begin{bmatrix} \Xi \\ * \\ \vdots \\ I_m \end{bmatrix}.$$

From (6.9), applied for every $k \in [1, n-1]$, it follows that

$$\begin{bmatrix} 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} AB & A^2B & \dots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} A_{21} & A_{22}A_{21} & \dots & A_{22}^{n-2}A_{21} \end{bmatrix} \begin{bmatrix} I_m & * & \dots & * \\ & I_m & \dots & * \\ & & \ddots & \vdots \\ & & & I_m \end{bmatrix}$$

and hence (by Cayley-Hamilton's theorem)

$$\begin{aligned} &\mathrm{Im}(\begin{bmatrix} 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} AB & A^2B & \dots & A^{n-1}B \end{bmatrix}) = \\ &\mathrm{Im}(\begin{bmatrix} A_{21} & A_{22}A_{21} & \dots & A_{22}^{n-2}A_{21} \end{bmatrix}) = \mathrm{Im}(\mathcal{R}(A_{22}, A_{21})). \end{aligned}$$

Consequently, the pair (A, B) is herdable if and only if the pair (A_{22}, A_{21}) is herdable. □

The following result can be obtained by recursively applying Proposition 6.8.

Corollary 6.9. *Consider a pair (A, B), where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are block-partitioned as follows:*

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots & \dots & A_{1k} \\ A_{21} & A_{22} & A_{23} & \dots & \dots & A_{2k} \\ 0 & A_{32} & A_{33} & \dots & \dots & A_{3k} \\ 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & A_{k,k-1} & A_{kk} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix},$$

and assume that the matrices $B_1 \in \mathbb{R}^{n_1\times m}$, $A_{i,i-1} \in \mathbb{R}^{n_i\times n_{i-1}}$, $i \in [2, k-1]$, are all of full row rank. Then the pair (A, B) is herdable if and only if the pair $(A_{kk}, A_{k,k-1})$ is herdable.

Proposition 6.8 allows to easily obtain two results that are already available in the literature. As we will see in the next section, however, the consequences of Proposition 6.8 can be further exploited.

Corollary 6.10 (Proposition 1 Meng et al. (2020))**.** *Consider a pair (A, B), where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are described as in (6.8). If the directed graph $\mathcal{G}(A)$ is strongly connected and structurally balanced, and the classes in which the agents split are $\mathcal{V}_1 = [1, m]$ and $\mathcal{V}_2 = [m+1, n]$, then the pair (A, B) is herdable.*

Proof. We first note that as $\mathcal{G}(A)$ is strongly connected then $\mathcal{R}(A,B)$ cannot have zero rows, therefore (see Proposition 6.8) also $\mathcal{R}(A_{22},A_{21})$ has no zero rows. If $\mathcal{V}_1=[1,m]$, then A_{21} is a nonpositive matrix, while A_{22} is a nonnegative matrix, therefore the controllability matrix of the pair (A_{22},A_{21}) has all negative columns and no zero rows. This ensures that (A_{22},A_{21}) is herdable. □

Remark 6.11. *It is easily seen that the result of Corollary 6.10 would still be true if the set of leaders would include $\mathcal{V}_1$ rather than coincide with it.*

Corollary 6.12 (Theorem 1 in She et al. (2019))**.** *Consider a pair (A,B), where $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$ are described as in (6.8). If every follower is reached by at least one of the leaders in a single step, namely through a walk of length 1, and for each leader the walks of length 1 to its followers have the same sign, then the pair (A,B) is herdable.*

Proof. By the corollary assumptions the matrix A_{21} is devoid of zero rows and all its columns are either zero vectors or unisigned vectors, therefore $\mathrm{Im}(A_{21})$ includes a strictly positive vector and, since $\mathrm{Im}(A_{21})\subseteq\mathrm{Im}(\mathcal{R}(A_{22},A_{21}))$, also $\mathrm{Im}(\mathcal{R}(A_{22},A_{21}))$ does. On the other hand, by Proposition 6.8, the pair (A,B) is herdable if and only if the pair (A_{22},A_{21}) is herdable, and this completes the proof. □

6.3 Herdability of Pairs (A,B) with A in Jordan Form

In this section we show that in the special case of pairs (A,B), with A a real matrix in Jordan form, herdability can be easily checked by making use of a criterion that represents the natural extension of the reachability criterion (derived from the PBH reachability test) for such matrix pairs, see Theorem 2.7.

Proposition 6.13. *Consider a pair (A,B), where $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$. Assume that A is a real matrix in Jordan form with r distinct (real) eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_r$, namely*

$$A=\begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix} \text{ with } J_i=\begin{bmatrix} J_{i,1} & & & \\ & J_{i,2} & & \\ & & \ddots & \\ & & & J_{i,s_i} \end{bmatrix}$$

where $J_i, i \in [1, r]$, is the Jordan block and $J_{i,\ell}, \ell \in [1, s_i]$, the ℓ-th elementary Jordan block, both of them associated with the eigenvalue λ_i. We let n_i be the dimension of J_i and $n_{i,\ell}$ the dimension of $J_{i,\ell}$. Accordingly, we can partition the matrix B as follows:

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_r \end{bmatrix}, \quad \text{with} \quad B_i = \begin{bmatrix} B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,s_i} \end{bmatrix}, \ i \in [1, r]. \tag{6.10}$$

Let $\tilde{B}_i \in \mathbb{R}^{s_i \times m}$, $i \in [1, r]$, be the matrix obtained by piling up the last rows of each of the blocks $B_{i,1}, B_{i,2}, \ldots B_{i,s_i}$. Then the pair (A, B) is herdable if and only if $\forall i \in [1, r]$ there is a strictly positive vector in the image of $\tilde{B}_i$.
In particular, if $m = 1$, namely B is a column vector, then (A, B) is herdable if and only if for every $i \in [1, r]$ the vector $\tilde{B}_i$ has all nonzero entries and is unisigned.

Proof. We prove the statement for the case $m = 1$ and adopt the notation $\mathbf{b}_i$, $\mathbf{b}_{i,\ell}$ and $\tilde{\mathbf{b}}_i$ for B_i, $B_{i,\ell}$ and $\tilde{B}_i$, respectively. The case $m > 1$ is a straightforward generalization of what follows, but the notation is much more cumbersome. First of all, we define the shift function $\sigma(\cdot) : \mathbb{R}^k \to \mathbb{R}^k$ as the function that takes as input a vector, shifts up by one position all its components and puts a zero as final entry, namely

$$\sigma(\mathbf{v}) = \begin{bmatrix} v_2 \\ \vdots \\ v_k \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} =: N_k \mathbf{v}, \tag{6.11}$$

where $N_k \in \mathbb{R}^{k \times k}$ is the elementary Jordan block of size k associated with the zero eigenvalue. Clearly, $\sigma^t(\mathbf{v})$ is recursively defined as $\sigma(\sigma^{t-1}(\mathbf{v}))$ for every $t \in \{2, 3, \ldots\}$. Similarly, given a block partitioned vector $\mathbf{v} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s]$, we define the function $\tilde{\sigma}(\cdot)$ as

$$\tilde{\sigma}(\mathbf{v}) = \tilde{\sigma}\left(\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_s \end{bmatrix} \right) = \begin{bmatrix} \sigma(\mathbf{v}_1) \\ \sigma(\mathbf{v}_2) \\ \vdots \\ \sigma(\mathbf{v}_s) \end{bmatrix}. \tag{6.12}$$

Since the matrix A is in Jordan form we notice that the controllability matrix associated with the pair (A, B) takes the following structure:

$$\mathcal{R}(A,B) = \begin{bmatrix} \mathbf{b}_1 & J_1\mathbf{b}_1 & \dots & J_1^{n-1}\mathbf{b}_1 \\ \mathbf{b}_2 & J_2\mathbf{b}_2 & \dots & J_2^{n-1}\mathbf{b}_2 \\ \vdots & \vdots & & \vdots \\ \mathbf{b}_r & J_r\mathbf{b}_r & \dots & J_r^{n-1}\mathbf{b}_r \end{bmatrix}.$$

We want to show that every block $J_i^k\mathbf{b}_i, i \in [1, r], k \in [1, n-1]$, of the controllability matrix can be expressed in terms of the vectors $\tilde{\sigma}^k(\mathbf{b}_i)$, and therefore an appropriate factorization of the controllability matrix can be derived.

As the generic elementary Jordan block $J_{i,\ell}$ can be expressed as $J_{i,\ell} = \lambda_i I_{n_{i,\ell}} + N_{n_{i,\ell}}$, its k-th power takes the following form $(J_{i,\ell})^k = (\lambda_i I_{n_{i,\ell}} + N_{n_{i,\ell}})^k = \sum_{t=0}^k \binom{k}{t}\lambda_i^{k-t}(N_{n_{i,\ell}})^t$. Moreover, since $N_{n_{i,\ell}}\mathbf{b}_{i,\ell} = \sigma(\mathbf{b}_{i,\ell})$, we get that $(J_{i,\ell})^k\mathbf{b}_{i,\ell} = \sum_{t=0}^k \binom{k}{t}\lambda_i^{k-t}\sigma^t(\mathbf{b}_{i,\ell})$, that is:

$$(J_{i,\ell})^k\mathbf{b}_{i,\ell} = \begin{bmatrix} \mathbf{b}_{i,\ell} & \sigma(\mathbf{b}_{i,\ell}) & \dots & \sigma^k(\mathbf{b}_{i,\ell}) \end{bmatrix} \begin{bmatrix} \lambda_i^k \\ \binom{k}{1}\lambda_i^{k-1} \\ \vdots \\ \binom{k}{k}\lambda_i^0 \end{bmatrix}.$$

If we now set $c_i := \max_\ell n_{i,\ell}$ and we keep into account that $\sigma^j(\mathbf{b}_{i,\ell}) = 0$ for $j \geq c_i$ and that that $\binom{k}{i} = 0$ for $k < i$, we obtain

$$(J_{i,\ell})^k\mathbf{b}_{i,\ell} = \begin{bmatrix} \mathbf{b}_{i,\ell} & \sigma(\mathbf{b}_{i,\ell}) & \dots & \sigma^{c_i-1}(\mathbf{b}_{i,\ell}) \end{bmatrix} \begin{bmatrix} \lambda_i^k \\ \binom{k}{1}\lambda_i^{k-1} \\ \vdots \\ \binom{k}{c_i-1}\lambda_i^{k-c_i+1} \end{bmatrix}$$

Accordingly, $\begin{bmatrix} \mathbf{b}_{i,\ell} & J_{i,\ell}\mathbf{b}_{i,\ell} & \dots & J_{i,\ell}^{n-1}\mathbf{b}_{i,\ell} \end{bmatrix} =$ $\begin{bmatrix} \mathbf{b}_{i,\ell} & \sigma(\mathbf{b}_{i,\ell}) & \dots & \sigma^{c_i-1}(\mathbf{b}_{i,\ell}) \end{bmatrix} \mathbf{A}_{c_i}(\lambda_i)$, where

$$\mathbf{A}_{c_i}(\lambda_i) := \begin{bmatrix} 1 & \lambda_i & \lambda_i^2 & \lambda_i^3 & \dots & \lambda_i^{n-1} \\ & 1 & \binom{2}{1}\lambda_i & \binom{3}{1}\lambda_i^2 & \dots & \binom{n-1}{1}\lambda_i^{n-2} \\ & & \ddots & \vdots & & \vdots \\ & & & 1 & \dots & \binom{n-1}{c_i-1}\lambda_i^{n-c_i} \end{bmatrix}$$

is a matrix in $\mathbb{R}^{c_i \times n}$. Keeping into account the block partitioning of the vectors $\mathbf{b}_i$ and of the Jordan blocks J_i, we can claim that, for every $i \in [1, r]$, $\begin{bmatrix} \mathbf{b}_i & J_i\mathbf{b}_i & \dots & J_i^{n-1}\mathbf{b}_i \end{bmatrix} =$ $\begin{bmatrix} \mathbf{b}_i & \tilde{\sigma}(\mathbf{b}_i) & \dots & \tilde{\sigma}^{c_i-1}(\mathbf{b}_i) \end{bmatrix} \mathbf{A}_{c_i}(\lambda_i)$. Therefore the matrix $\mathcal{R}(A, \mathbf{b})$ factorizes as in (6.13).

$$\mathcal{R}(A,B)=\begin{bmatrix} \mathbf{b}_1 \ \tilde\sigma(\mathbf{b}_1)\ \dots\ \tilde\sigma^{c_1-1}(\mathbf{b}_1) & & & \\ & \mathbf{b}_2 \ \tilde\sigma(\mathbf{b}_2)\ \dots\ \tilde\sigma^{c_2-1}(\mathbf{b}_2) & & \\ & & \ddots & \\ & & & \mathbf{b}_r \ \tilde\sigma(\mathbf{b}_r)\ \dots\ \tilde\sigma^{c_r-1}(\mathbf{b}_r) \end{bmatrix}\begin{bmatrix} A_{c_1}(\lambda_1) \\ A_{c_2}(\lambda_2) \\ \vdots \\ A_{c_r}(\lambda_r)\end{bmatrix} \triangleq \mathcal{B}\mathcal{A} \tag{6.13}$$

We want to prove that the matrix $\mathcal{A}$ in (6.13) is of full row rank. To this end we first note that $\mathcal{A} \in \mathbb{R}^{(c_1+c_2+\dots+c_r)\times n}$ and that $\delta := c_1 + c_2 + \dots + c_r \leq n$. Therefore it will be sufficient to prove that the $\delta \times \delta$ submatrix obtained by selecting the first δ columns, i.e.,

$$\Psi := \mathcal{A}\begin{bmatrix} I_\delta \\ 0_{(n-\delta)\times\delta}\end{bmatrix} = \begin{bmatrix} A_{c_1}(\lambda_1) \\ A_{c_2}(\lambda_2) \\ \vdots \\ A_{c_r}(\lambda_r)\end{bmatrix}\begin{bmatrix} I_\delta \\ 0_{(n-\delta)\times\delta}\end{bmatrix}$$

is nonsingular. We note that a nonzero vector $\mathbf{p} \in \mathbb{R}^\delta, \mathbf{p} := \begin{bmatrix} p_0 & p_1 & \dots & p_{\delta-1}\end{bmatrix}^\top$, belongs to the kernel of Ψ if and only if

$$0 = p(\lambda_i) = \left.\frac{dp(s)}{ds}\right|_{s=\lambda_i} = \dots = \left.\frac{d^{c_i-1}p(s)}{ds^{c_i-1}}\right|_{s=\lambda_i}, i \in [1,r],$$

where $p(s) := \sum_{i=0}^{\delta-1} p_i s^i$. But this amounts to saying that $p(s)$ has λ_i as a zero of multiplicity (at least) c_i, for every $i \in [1,r]$, namely $p(s)$ is a multiple of $q(s) := (s-\lambda_1)^{c_1}(s-\lambda_2)^{c_2}\dots(s-\lambda_r)^{c_r}$, which is impossible since $\deg q(s) = \delta > \deg p(s)$. Consequently, Ψ is of full row rank. This implies that the image of $\mathcal{R}(A,B)$ coincides with the image of the matrix $\mathcal{B}$ in (6.13).

We consider, now, each i-th block of $\mathcal{B}$, i.e., $[\mathbf{b}_i \ \ \tilde\sigma(\mathbf{b}_i) \ \ \dots \ \ \tilde\sigma^{c_i-1}(\mathbf{b}_i)]$, and we note that there exist permutation matrices P_{1i} and P_{2i} such that

$$P_{1i}\begin{bmatrix}\mathbf{b}_i & \tilde\sigma(\mathbf{b}_i) & \dots & \tilde\sigma^{c_i-1}(\mathbf{b}_i)\end{bmatrix} P_{2i} = \begin{bmatrix} \Phi_{ii}^{(c_i-1)} & * & \dots & * & * \\ 0 & \Phi_{ii}^{(c_i-2)} & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Phi_{ii}^{(1)} & * \\ 0 & 0 & \dots & 0 & \Phi_{ii}^{(0)} \end{bmatrix},$$

where $\Phi_{ii}^{(0)} = \tilde{\mathbf{b}}_i$, while $\Phi_{ii}^{(k)}$ includes only the entries of the vector $\tilde{\mathbf{b}}_i$ that correspond to elementary Jordan blocks $J_{i,\ell}$ whose dimension $n_{i,\ell}$ is greater than k. It is immediate to realise that there exists a strictly positive vector in the image of $\begin{bmatrix} \mathbf{b}_i & \tilde{\sigma}(\mathbf{b}_i) & \dots & \tilde{\sigma}^{c_i-1}(\mathbf{b}_i) \end{bmatrix}$ if and only if $\Phi_{ii}^{(0)} = \tilde{\mathbf{b}}_i$ is a vector devoid of zero entries and unisigned. Therefore there exists a strictly positive vector in the image of $\mathcal{B}$ (and hence in the image of $\mathcal{R}(A, B)$) if and only if the previous condition holds for every vector $\tilde{\mathbf{b}}_i, i \in [1, r]$. This concludes the proof. □

Remark 6.14. *For pairs (A, B) with A in Jordan form and cyclic (i.e., it has one elementary Jordan block for each eigenvalue) and $B = \mathbf{b}$ a column vector, the vectors $\tilde{\mathbf{b}}_i$ reduce to scalars, thus the pair is herdable if and only if such scalars are all nonzero and this is equivalent to the reachability of the pair, as it follows from the PBH reachability test (see Theorem 2.7).*

If A is a diagonal matrix and B a column vector, Proposition 6.13 leads to the following corollary.

Corollary 6.15. *Given a matrix pair (A, B), with $A = \lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_n \in \mathbb{R}^{n \times n}$ a diagonal matrix, and $B = \mathbf{b} \in \mathbb{R}^n$, the pair is herdable if and only $\mathbf{b}$ is devoid of zero entries and condition $\lambda_i = \lambda_j$ implies $[\mathbf{b}]_i \cdot [\mathbf{b}]_j > 0$.*

6.4 Herdability of Pairs (A, B) Corresponding to a Directed Graph $\mathcal{G}(A)$ with m Leaders

We now investigate the herdability of the pairs (A, B), described as in (6.8), with $B_1 = I_m$ and $A_{11} \in \mathbb{R}^{m \times m}$. Based on Lemma 6.7, we can always reduce ourselves to this case by resorting to a change of basis in the input space and to a permutation of the state entries, every time the matrix B has a subset of its rows that is linearly independent and the remaining rows are all zero. One of the advantages of this set-up is that it allows to investigate the herdability of the pair (A, B) by resorting to the signed and weighted directed graph $\mathcal{G}(A)$ whose nodes are partitioned into leaders and followers, depending on whether the state variable associated to the node is endowed with an external and independent control input (leader) or not (follower). Specifically we introduce the following:

Assumption 1: We assume that in the signed and weighted directed graph $\mathcal{G}(A)$ the first m vertices, associated with the m canonical vectors in B, represent the set $\mathcal{L} = [1, m]$

of leaders and the remaining vertices are the set of followers, i.e., $\mathcal{F} = [m+1, n]$. We let $\mathcal{F}_d$ be the set of followers whose distance from the leaders is d, $d \in [1, k]$, by this meaning $\mathcal{F}_d := \{j \in \mathcal{F} : d(\mathcal{L}, j) = d\}$. We assume $\mathcal{F}_k \neq \emptyset$, $\mathcal{F}_d = \emptyset$, $d > k$. This means that k is the maximum distance from the set of leaders to a follower. It entails no loss of generality assuming that $\mathcal{F}_1 = [m+1, m+m_1], \dots, \mathcal{F}_d = [m+m_1+\dots+m_{d-1}+1, m+m_1+\dots+m_d]$, so that $|\mathcal{L}| = m$ and $|\mathcal{F}_d| = m_d$. We also assume that $m + m_1 + \dots + m_k = n$.

Under Assumption 1, the controllability matrix in $k+1$ steps $\mathcal{R}_{k+1} = [B|AB|A^2B|\dots|A^kB]$ has a structure as in (6.5), with $\Phi_{11} = I_m$, and all the matrices $\Phi_{dd} \in \mathbb{R}^{m_d \times m}, d \in [2, k]$, have no zero rows.

We consider the case when the leaders split into the two classes of the structurally balanced graph $\mathcal{G}(A)$.

Proposition 6.16. *Consider a pair (A, B), where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Assume that the directed graph $\mathcal{G}(A)$ is structurally balanced, and the classes in which the agents split are $\mathcal{V}_1 = [1, n_1]$ and $\mathcal{V}_2 = [n_1 + 1, n]$, so that A is described as in (6.8), with $A_{11} \in \mathbb{R}^{n_1\times n_1}$. We note that $A_{ii} \geq 0$, while $A_{ij} \leq 0$ for $i \neq j$. Assume, now, that the set of m leaders splits in the two classes as follows: $\mathcal{L} = [1, m_1] \cup [n_1 + 1, n_1 + m_2]$, with $1 \leq m_i \leq n_i$ for $i \in [1, 2]$, and hence B is described as*

$$B = \left[\begin{array}{cc} I_{m_1} & 0 \\ 0 & 0 \\ \hline 0 & I_{m_2} \\ 0 & 0 \end{array}\right].$$

If

a) $\forall i \in \mathcal{V}_1 \setminus \mathcal{L}$ there exists $\ell \in \mathcal{L} \cap \mathcal{V}_1 = [1, m_1]$ such that $d(\ell, i) < d(\ell, j), \forall j \in \mathcal{V}_2 \setminus \mathcal{L}$;

b) $\forall i \in \mathcal{V}_2 \setminus \mathcal{L}$ there exists $\ell \in \mathcal{L} \cap \mathcal{V}_2 = [n_1 + 1, n_1 + m_2]$ such that $d(\ell, i) < d(\ell, j), \forall j \in \mathcal{V}_1 \setminus \mathcal{L}$;

then the pair (A, B) is herdable.

Proof. We first observe that if two nodes (leader or follower) belong to the same class, every path that connects them has a positive weight. As a result, if $i, j \in \mathcal{V}_p$ for some

$p \in [1,2]$ and $[A^k B]_{ij} \neq 0$ for some $k > 0$, then $[A^k B]_{ij} > 0$.
Condition a) ensures that for every $i \in \mathcal{V}_1 \setminus \mathcal{L}$ there exists $\ell \in \mathcal{L} \cap \mathcal{V}_1 = [1, m_1]$ and $k_i > 0$ such that $[A^{k_i} B]_{i\ell} \neq 0$ and hence $[A^{k_i} B]_{i\ell} > 0$. On the other hand, $[A^{k_i} B]_{j\ell} = 0$ for every $j \in \mathcal{V}_2 \setminus \mathcal{L}$. Therefore if $[A^{k_i} B]_{j\ell} \neq 0$ and $j \notin \mathcal{L}$ then $[A^{k_i} B]_{j\ell} > 0$. Consequently, for every $i \in \mathcal{V}_1 \setminus \mathcal{L}$ there exists $\ell \in \mathcal{V}_1 \setminus \mathcal{L}$ and $k_i > 0$ such that the vector $A^{k_i} B \mathbf{e}_\ell$ has the i-the entry which is nonzero and its restriction to the entries that correspond to the followers is a unisigned vector. By exploiting b), we can claim the same result for all indices $i \in \mathcal{V}_2 \setminus \mathcal{L}$. So, keeping in mind the structure of B, we can claim that there exists a permutation matrix P and a selection matrix S such that

$$P\mathcal{R}(A,B)S = \begin{bmatrix} I_m & \Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix},$$

where all columns of Φ_{22} are unisigned (in fact, nonnegative) and Φ_{22} has no zero rows. By Lemma 6.4, we can claim the herdability of the pair (A, B). □

The previous result allows to extend the herdability analysis first to graphs with 3 clusters and then to graphs with an arbitrary number of clusters.

Proposition 6.17. *Consider a pair (A, B), with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Assume that $\mathcal{G}(A)$ is a clustering balanced directed graph with three clusters, and hence up to a relabelling of the nodes, A can be described as follows:*

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

with $A_{ii} \in \mathbb{R}^{n_i\times n_i}, i \in [1,3]$, $A_{ii} \geq 0$, and $A_{ij} \leq 0$ for $i \neq j$. This means that the three clusters correspond to the sets of nodes $\mathcal{V}_1 = [1, n_1]$, $\mathcal{V}_2 = [n_1+1, n_1+n_2]$ and $\mathcal{V}_3 = [n_1+n_2+1, n]$.
Let us assume, then, that the set of leaders coincides with one of the clusters, without loss of generality $\mathcal{L} = \mathcal{V}_1 = [1, n_1]$, and hence $m = n_1$ and B is described as $B = \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}$. If

a) $\forall i \in \mathcal{V}_2$ there exists $\ell \in \mathcal{L} = \mathcal{V}_1$ such that $d(\ell, i) < d(\ell, j), \forall j \in \mathcal{V}_3$;

b) $\forall i \in \mathcal{V}_3$ there exists $\ell \in \mathcal{L} = \mathcal{V}_1$ such that $d(\ell, i) < d(\ell, j), \forall j \in \mathcal{V}_2$;

then the pair (A, B) is herdable.

Proof. Under the statement assumptions, Proposition 6.8 holds for $m = n_1$, and hence (A, B) is herdable if and only if the pair

$$(\tilde{A}, \tilde{B}) := \left(\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}, \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} \right)$$

is herdable. We now set $\tilde{\mathcal{V}}_2 := [1, n_2]$ and $\tilde{\mathcal{V}}_3 := [n_2 + 1, n_2 + n_3]$ and we observe that $\mathcal{G}(\tilde{A})$ is structurally balanced.

In order to show that $(\tilde{A}, \tilde{B})$ is herdable, we first notice that the matrix $\tilde{B}$ has all non positive column vectors. Therefore, if we denote by $\mathcal{I}_1$ the index set corresponding to the non null rows of $\tilde{B}$, namely $\mathcal{I}_1 := \{i : \mathbf{e}_i^\top \tilde{B} \neq \mathbf{0}^\top\}$, then there exists a positive vector $\mathbf{v} \in \text{Im}(\tilde{B}) \subseteq \text{Im}(\mathcal{R}(\tilde{A}, \tilde{B}))$ such that $[\mathbf{v}]_i > 0$, $\forall i \in \mathcal{I}_1$, and $[\mathbf{v}]_i = 0$, $\forall i \in [1, n_2 + n_3] \setminus \mathcal{I}_1 = (\tilde{\mathcal{V}}_2 \cup \tilde{\mathcal{V}}_3) \setminus \mathcal{I}_1$.

We now observe that if we regard the set $\mathcal{I}_1$ as the set of leaders then conditions a) and b) lead to

a') $\forall i \in \tilde{\mathcal{V}}_2 \setminus \mathcal{I}_1$ there exists $\ell \in \mathcal{I}_1 \cap \tilde{\mathcal{V}}_2$ such that $d(\ell, i) < d(\ell, j), \forall j \in \tilde{\mathcal{V}}_3$;

b') $\forall i \in \tilde{\mathcal{V}}_3 \setminus \mathcal{I}_1$ there exists $\ell \in \mathcal{I}_1 \cap \tilde{\mathcal{V}}_3$ such that $d(\ell, i) < d(\ell, j), \forall j \in \tilde{\mathcal{V}}_2$.

So, by Proposition 6.16, the pair $(\tilde{A}, \tilde{B})$ is herdable and hence (A, B) is herdable.

□

The previous result generalises to any clustering balanced graph with k clusters.

Proposition 6.18. *Consider a pair (A, B), with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Assume that $\mathcal{G}(A)$ is a clustering balanced directed graph with k disjoint clusters, say $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k$, and that the set of leaders $\mathcal{L}$ coincides with $\mathcal{V}_1$. If for every $p \in [2, k]$ and $\forall i \in \mathcal{V}_p$ there exists $\ell \in \mathcal{L} = \mathcal{V}_1$ such that $d(\ell, i) < d(\ell, j), \forall j \in \cup_{\ell \notin \{1,p\}} \mathcal{V}_\ell$, then the pair (A, B) is herdable.*

6.5 Herdability of Pairs (A, B) with $\mathcal{G}(A)$ an Undirected Tree with a Single Leader

Let us now consider the case when B is a canonical vector and the matrix A is a symmetric real matrix whose associated undirected graph $\mathcal{G}(A)$ is acyclic, namely $\mathcal{G}(A)$ is a tree. This corresponds to the case of a tree with a single leader and $n-1$ followers. This case has been investigated in She et al. (2019), where a sufficient condition for the herdability of the pair (A, B) has been provided. In this section we provide a sufficient condition for herdability that is less restrictive, and in the case of trees whose followers have distance at most 2 from the leader we provide necessary and sufficient conditions.

To investigate the problem we adopt the following

Assumption 2: The graph $\mathcal{G}(A)$ is a signed, weighted, connected and acyclic undirected graph, namely a tree. The leader is $\mathcal{L} = \{1\}$ (and hence $B = \mathbf{e}_1$), while the followers split into classes, based on their distance from the leader. The followers at distance 1 from the leader are $\mathcal{F}_1 = [2, m_1 + 1]$, the followers at distance 2 from the leader are $\mathcal{F}_2 = [m_1 + 2, m_1 + m_2 + 1]$, and so on till the last class $\mathcal{F}_k = [m_1 + \cdots + m_{k-1} + 2, n]$, where k is the maximum distance between the leader and one of its followers.

Proposition 6.19. *Consider a pair (A, B), with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$ satisfying the previous Assumption 2.*
If, for every $d \in [0, k-1]$, all the edges from the vertices in $\mathcal{F}_d$ to the vertices in $\mathcal{F}_{d+1}$ have the same sign, then the pair (A, B) is herdable.

Proof. Under the previous assumption, it is easy to see that every vertex in $\mathcal{F}_d$ is reached for the first time by the leader in d steps, $d \in [0, k]$, and subsequently it is reached after $d + 2h$ steps for every $h \in \{1, 2, 3, \dots\}$ (since each undirected edge of the graph can be crossed back and forth, and hence condition $[A^d B]_i \neq 0$ implies $[A^{d+2}B]_i \neq 0$). Therefore the controllability matrix of the pair (A, B) takes the form

$$\mathcal{R} = \begin{bmatrix} 1 & 0 & * & 0 & * & \dots \\ 0 & \mathbf{v}_1 & 0 & * & 0 & \dots \\ 0 & 0 & \mathbf{v}_2 & 0 & * & \dots \\ 0 & 0 & 0 & \mathbf{v}_3 & 0 & \dots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \mathbf{v}_k & \dots \end{bmatrix}, \tag{6.14}$$

where $\mathbf{v}_d \in \mathbb{R}^{m_d}, d \in [1, k]$, are, by assumption, unisigned, while $*$ denotes (nonzero) vectors/entries whose values are not relevant. So, by making use of Remark 6.5, we immediately deduce that there exists a strictly positive vector in the image of $\mathcal{R}$, and hence (A, B) is herdable. □

Remark 6.20. *Theorem 3 in She et al. (2019) follows as a corollary of the previous proposition, since it imposes that all paths from the leader to the followers in* $\mathcal{V}_o := \cup_{h \in \mathbb{Z}_+} \mathcal{F}_{1+2h}$ *have the same sign and, at the same time, all paths from the leader to the followers in* $\mathcal{V}_e := \cup_{h \in \mathbb{Z}_+} \mathcal{F}_{2+2h}$ *have the same sign. This means that not only all the edges from vertices in* $\mathcal{F}_d$ *to vertices in* $\mathcal{F}_{d+1}, d \in [0, k-1]$*, (where* $\mathcal{F}_0 := \mathcal{L}$*) have the same signs, but such signs are uniquely determined for* $d \geq 1$ *once we choose the signs of the edges from* $\mathcal{F}_1$ *to* $\mathcal{F}_2$*.*

Example 6.21. *Consider a pair* (A, B)*, with* $A = A^\top \in \mathbb{R}^{9 \times 9}$ *and* $B = \mathbf{e}_1$*, and assume that the undirected graph* $\mathcal{G}(A)$ *associated with the matrix* A *is a tree whose structure and edge signs are described in Figure 6.1. The nodes* $i = 2$ *and* $j = 9$ *both belong to* $\mathcal{V}_o$*, since*

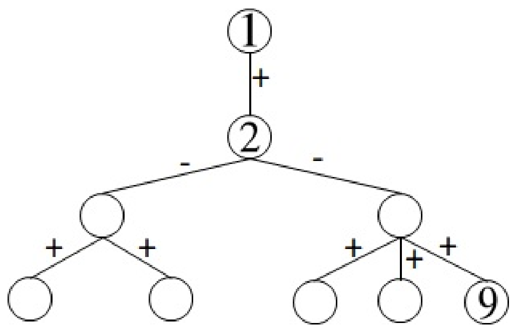

Figure 6.1: Tree structure of the herdable system of Example 6.21.

both of them are reached from the leader (node 1 *in Fig. 6.1) in an odd number of steps*

($1+2h$ and $3+2h$, $h \in \{0,1,2,\dots\}$, respectively). The node i is reached by the leader with positive walks, while j with negative ones, so the hypotheses of Theorem 3 in She et al. (2019) are violated. However, the controllability matrix of the pair takes the structure in (6.14) for $k=3$, with unisigned vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, the first one with a positive entry, while the other two with negative entries, thus the pair is herdable by Proposition 6.19.

Given a matrix A and hence a graph $\mathcal{G}(A)$ with a tree structure, we propose now Algorithm 3 for the selection of a (unique) leader i in order to ensure, if possible, that the pair $(A, \mathbf{e}_i)$ is herdable. The algorithm searches for a single node, if it exists, for which the sufficient condition given in Proposition 6.19 is satisfied. For the meaning of the symbols $\text{Out}_+(\mathcal{F}), \text{Out}_-(\mathcal{F})$ etc., we refer the reader to the Notation in Section 2.1.

Algorithm 3 Algorithm for the selection of a single leader to ensure herdability of a pair (A, B) when $\mathcal{G}(A)$ is a tree

```
for i ∈ V do
    if Out+(i) = Out(i) ≠ ∅ or Out−(i) = Out(i) ≠ ∅ then
        L := {i}
        F := {j : (i, j) ∈ E}
        H := L ∪ F
        if |H| = n then
            (A, B) is herdable
        else
            while Out+(F) = Out(F) ≠ ∅ or
                    Out−(F) = Out(F) ≠ ∅ do
                F = F ∪ Out(F)
                H = H ∪ F
                if |H| = n then
                    (A, B) is herdable
```

Propositions 6.22 and 6.23, below, provide complete characterizations of herdability for trees in which followers have all distance 1 from the leader or distance at most 2, respectively.

Proposition 6.22. *Consider a pair (A, B), with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$ satisfying Assumption 2, and suppose that all the followers have distance one from the leader.*

Then the pair (A, B) is herdable if and only if all the edges have the same sign.

Proof. If all the followers have distance 1 from the leader, namely $k = 1$, then

$$A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & \mathbf{0}_{(n-1)\times(n-1)} \end{bmatrix},$$

where $A_{21} = A_{12}^\top \in \mathbb{R}^{n-1}$ is devoid of zero entries. By Proposition 6.8, (A, B) is herdable if and only if the pair $(\mathbf{0}_{(n-1)\times(n-1)}, A_{21})$ is herdable, and this is the case if and only if A_{21} is unisigned. □

Proposition 6.23. *Consider a pair (A, B), with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$ satisfying Assumption 2, and suppose that all the followers have distance at most 2 from the leader, and hence*

$$A = \begin{bmatrix} 0 & A_{12} & \mathbf{0}_{1\times m_2} \\ A_{21} & \mathbf{0}_{m_1\times m_1} & A_{23} \\ \mathbf{0}_{m_2\times 1} & A_{32} & \mathbf{0}_{m_2\times m_2} \end{bmatrix},$$

where $A_{21} = A_{12}^\top \in \mathbb{R}^{m_1}$ and $A_{32} = A_{23}^\top \in \mathbb{R}^{m_2\times m_1}$. Then the pair (A, B) is herdable if and only if for every $i, j \in \mathcal{F}_1 = [2, m_1 + 1]$ (including $i = j$)[1] such that

$$[A_{23}A_{32}]_{ii} = [A_{23}A_{32}]_{jj}, \tag{6.15}$$

we have:

i) $[A_{21}]_i \cdot [A_{21}]_j > 0$ (namely the two edges from the leader $\mathcal{L}$ to i and j have the same sign);

ii) $A_{32}(\mathbf{e}_i+\mathbf{e}_j)$ is either zero or unisigned (namely all edges from i and j to their followers in $\mathcal{F}_2$ have the same sign).

Proof. First of all, we highlight that, by Assumption 2, A_{21} is devoid of zero entries, and for every $i \in [1, m_2]$ the row vector $\mathbf{e}_i^\top A_{32}$ is a monomial vector (namely it has a single nonzero entry). Consequently, $A_{23}A_{32} = A_{32}^\top A_{32}$ is a diagonal matrix (with nonnegative diagonal entries). By Proposition 6.8, (A, B) is herdable if and only if the pair

$$\left(\begin{bmatrix} \mathbf{0}_{m_1\times m_1} & A_{23} \\ A_{32} & \mathbf{0}_{m_2\times m_2} \end{bmatrix}, \begin{bmatrix} A_{21} \\ \mathbf{0}_{m_2} \end{bmatrix}\right)$$

[1]Note that for $i = j$ condition i) becomes trivial, while condition ii) becomes "$A_{32}\mathbf{e}_i$ is either zero or unisigned".

$$\hat{\mathcal{R}} := \begin{bmatrix} A_{21} & 0 & (A_{23}A_{32})A_{21} & 0 & (A_{23}A_{32})^2A_{21} & 0 & \dots \\ 0 & A_{32}A_{21} & 0 & A_{32}(A_{23}A_{32})A_{21} & 0 & A_{32}(A_{23}A_{32})^2A_{21} & \dots \end{bmatrix} \tag{6.16}$$

is herdable, and this is the case if and only if the image of the controllability matrix $\hat{\mathcal{R}}$ of the previous pair, given in (6.16) includes a strictly positive vector. This is the case if and only if the following conditions simultaneously hold:

a) the image of the controllability matrix $\mathcal{R}_1 := \begin{bmatrix} A_{21} & (A_{23}A_{32})A_{21} & (A_{23}A_{32})^2A_{21} & \dots \end{bmatrix}$ includes a strictly positive vector, namely the pair $(\Lambda, \Gamma) := (A_{23}A_{32}, A_{21})$ is herdable;

b) the image of the matrix $A_{32}\mathcal{R}_1$ includes a strictly positive vector.

As the matrix $\Lambda = A_{23}A_{32}$ is diagonal, while the column vector $\Gamma = A_{21}$ has no zero entries, by Corollary 6.15, the pair $(\Lambda, \Gamma) = (A_{23}A_{32}, A_{21})$ is herdable if and only if condition (6.15) implies $[A_{21}]_i \cdot [A_{21}]_j > 0$. This means that a) is equivalent to condition i).

Note, also, that by suitably relabelling the nodes in $\mathcal{F}_1$, we can always reduce ourselves to the case

$$\Lambda = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_r \end{bmatrix},$$

where $J_i = \lambda_i I_{n_i}$ is a scalar matrix (a Jordan block whose elementary blocks have all unitary size), $\lambda_i \neq \lambda_j$ for $i \neq j$, and Γ has been accordingly block-partitioned. In this set-up condition a), or equivalently i), amounts to requiring that all vectors γ_i are unisigned vectors devoid of zero entries. Moreover, by mimicking the proof of Proposition 6.13 for the special case when the Jordan form is a diagonal matrix, we deduce that $\mathrm{Im}(\mathcal{R}_1) = \mathrm{Im}\,(\gamma_1 \oplus \cdots \oplus \gamma_r)$.

So, we are now remained with proving that if i) (equivalently, a)) holds, then b) and ii) are equivalent. If i) holds, then $\mathrm{Im}\,(A_{32}\mathcal{R}_1) = \mathrm{Im}\,(A_{32} \cdot (\gamma_1 \oplus \cdots \oplus \gamma_r))$. Set $W = \begin{bmatrix} \mathbf{w}_1 \mid \dots \mid \mathbf{w}_r \end{bmatrix} := A_{32} \cdot (\gamma_1 \oplus \cdots \oplus \gamma_r)$, where each vector $\mathbf{w}_i$ is obtained by combining with the coefficients of the vector γ_i (having all the same sign) the columns of A_{32} of indices $[h_i + 1, h_i + n_i]$, where by definition $h_1 := 0$, while $h_i := n_1 + n_2 + \cdots + n_{i-1}$ for $i \in [2, r]$.

We observe that all columns of A_{32} are either zero (if a vertex in $\mathcal{F}_1 = [2, m_1 + 1]$ has no followers) or have disjoint nonzero patterns, meaning that for every $\ell, m \in [h_i + 1, h_i + n_i], \ell \neq m$, $\overline{\mathrm{ZP}}(A_{32}\mathbf{e}_\ell) \cap \overline{\mathrm{ZP}}(A_{32}\mathbf{e}_m) = \emptyset$. As a result also the columns $\mathbf{w}_i$ of W are either zero or have disjoint nonzero patterns.

We can conclude that condition b) holds if and only if $\mathrm{Im}\,(A_{32}\mathcal{R}_1) = \mathrm{Im}(W)$ contains a strictly positive vector, but this is possible if and only if all vectors $\mathbf{w}_h$ are unisigned. By the way the vectors $\mathbf{w}_h$ have been obtained, this is possible if and only if ii) holds. □

Example 6.24. *Consider the pair* (A, B)*, with*

$$A = \begin{bmatrix} 0 & A_{12} & 0 \\ A_{21} & 0 & A_{23} \\ 0 & A_{32} & 0 \end{bmatrix} = \left[\begin{array}{c|ccc|cc} 0 & 1 & a & 2 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & b & c \\ 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \end{array}\right], \quad B = \mathbf{e}_1,$$

whose graph is given in Fig. 6.2, where a, b and c are nonzero real values. Note that $\mathcal{L} = \{1\}, \mathcal{F}_1 = [2, 4]$ and $\mathcal{F}_2 = [5, 6]$, so that $m_1 = 3$ and $m_2 = 2$. We first check

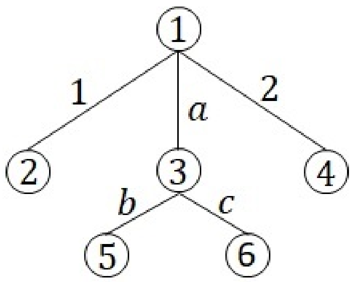

Figure 6.2: Graph structure related to the pair (A, B) in Example 6.24.

for all indices $i, j \in [1, 3], i \neq j$, whether condition (6.15) holds. It is easily seen that $[A_{23}A_{32}]_{11} = [A_{23}A_{32}]_{33} = 0$. In fact, for the pair of indices $(1, 3)$ both condition i) and condition ii) of Proposition 6.23 are satisfied, since $[A_{21}]_1 \cdot [A_{21}]_3 = 2 > 0$ and both column 1 and column 3 of A_{32} are zero. On the other hand, if we assume $i = j \in [1, 3]$, for which condition (6.15) trivially holds, condition i) is straightforward, while condition ii) holds provided that $bc > 0$ (namely b and c have the same sign), because all columns of A_{32} are zero or unisigned if and only if $bc > 0$. Therefore the pair (A, B) is herdable for every $a \neq 0$ and for $bc > 0$.

Chapter 7

Conclusions and Future Directions

In this manuscript we dealt with the study of opinion dynamics from a control system perspective by following three main directions. We first assumed the network topology of agents interaction to be fixed and in a clustering balance configuration. In this case we studied under what conditions on the weights distribution over the graph, agent's opinion reach a steady state configuration that mirrors the partition of the agents in the network, thus ensuring the reaching of k-partite consensus, where k is the number of clusters in the network. In the second part of this thesis we assumed the topological network to be opinion-varying. The network dynamics is driven by two fundamental mechanisms: the homophily mechanism and the influence mechanism. In this circumstance we investigated the reaching of socially balanced configurations in the network. The last part of this thesis pertains with the study of herdability. We focused on leader-follower networked systems with special topologies end we deduced the herdability of the systems from a structural perspective.

Future directions include the investigation of k-partite consensus under milder assumptions on the weights distribution in the network and possibly for opinion-varying network topologies. Also, we are interested in investigating other possible clustering configurations of the opinions that are of interest in social contexts, e.g. polarization and that are compatible with the clustering configuration in the network.

In the context of opinion varying network topology it would be interesting to better investigate the scientific literature from the social sciences and understand what other fundamental mechanisms characterize opinion dynamics in a social network, mathematically formalize them and possibly intertwine them with the already investigated ones and see what the consequences in terms of reaching of social equilibria are.

As far as herdabiliy is concerned, future directions involve the investigation of this property for other network topologies different from clustering, structural balanced ones and tree topology. We are also interested in the herdability of time-varying networked systems and nonlinear systems.

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APPENDICES

Appendix A

Technical Lemmas

Lemma A.1. *Given a scalar $\varepsilon > 0$ and matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$, with A Metzler and symmetric, it is always possible to choose a scalar matrix $D = \delta I_n \in \mathbb{R}^{n\times n}$, $\delta > 0$, such that $C(D-A)^{-1}B$ is a matrix whose entries satisfy $|[C(D-A)^{-1}B]_{i,j}| < \varepsilon$, $\forall\, i,j \in [1,m]$.*

Proof. As A is symmetric, there exists an orthonormal matrix $T \in \mathbb{R}^{n\times n}$ such that $T^\top AT = \mathrm{diag}\{\lambda_1,\dots,\lambda_n\} =: \Lambda$. Note also that, as D is scalar, then $T^\top DT = D$. Therefore if we denote by $\mathbf{c}_\ell$ the ℓ-th column of CT and by $\mathbf{b}_\ell^\top$ the ℓ-th row of $T^\top B$, then

$$\begin{aligned} C(D-A)^{-1}B &= (CT)[T^\top(D-A)T]^{-1}(T^\top B) \\ &= (CT)[D-\Lambda]^{-1}(T^\top B) = \sum_{\ell=1}^{n} \mathbf{c}_\ell \mathbf{b}_\ell^\top \frac{1}{\delta-\lambda_\ell}. \end{aligned}$$

Therefore $|[C(D-A)^{-1}B]_{i,j}| \le \frac{n\cdot\psi}{\delta-\max_\ell \lambda_\ell}, \forall\, i,j$, where $\psi := \max_{\substack{\ell\in[1,n] \\ h,k\in[1,m]}} |[\mathbf{c}_\ell\mathbf{b}_\ell^\top]_{h,k}|$. Therefore by imposing that $\delta > \max_\ell \lambda_\ell + \frac{n\cdot\psi}{\varepsilon}$, we ensure that all entries of $C(D-A)^{-1}B$ have modulus smaller than ε. □

Lemma A.2 (Rank-one matrices with special structures)**.** *Given a matrix $\mathbf{M} \in S_1^N$, if $1 \in \sigma\left(\frac{1}{N}\mathbf{M}\right)$, then $\mathbf{M} = \mathbf{p}\mathbf{p}^\top$ for some $\mathbf{p} \in \{-1,1\}^N$, and hence $\mathbf{M}$ has no zero entries and $\sigma(\mathbf{M}) = (0,\dots,0,1)$.*

Proof. Let $\mathbf{v} := \begin{bmatrix} v_1 & v_2 & \dots & v_N \end{bmatrix}^\top \in \mathbb{R}^N, \mathbf{v} \neq 0$, be an eigenvector of $\frac{1}{N}\mathbf{M}$ corresponding to the unitary eigenvalue, or equivalently of $\mathbf{M}$ corresponding to N. Then $\mathbf{M}\mathbf{v} = N\mathbf{v}$. Let

$h := \text{argmax}_{i\in[1,N]}|v_i|$. Then condition

$$Nv_h = \sum_{i=1}^{N} M_{hi}v_i = v_h + \sum_{\substack{i=1\\ i\neq h}}^{N} M_{hi}v_i$$

holds if and only if (a) $|v_i| = |v_h|$ for every $i \in [1, N]$; (b) $M_{hi} \neq 0$ for every $i \in [1, N]$, and $\text{sgn}(M_{hi})\text{sgn}(v_i) = \text{sgn}(v_h)$.

This implies that $\mathbf{v} = \mathbf{p}m$ for some $\mathbf{p} \in \{-1, 1\}^N$ and some $m > 0$ and $\mathbf{e}_h^\top \mathbf{M} = \text{sgn}(v_h)\mathbf{p}^\top = p_h\mathbf{p}^\top$.

On the other hand, since condition (a) holds, this means that every index $j \in [1, N]$ is $\text{argmax}_{i\in[1,N]}|v_i|$, and hence all the rows of $\mathbf{M}$ satisfy $\mathbf{e}_i^\top \mathbf{M} = p_i\mathbf{p}^\top$. This implies that $\mathbf{M} = \mathbf{p}\mathbf{p}^\top$, and the rest immediately follows. □

Lemma A.3 (De-Pasquale et al. (2021))**.** *Given a seminorm* $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *with kernel* $\mathcal{K} \subseteq \mathbb{R}^n$*, any matrix* $A \in \mathbb{R}^{n\times n}$ *such that* $A\mathcal{K} \subseteq \mathcal{K}$ *satisfies*

$$\|A\| = \max_{\|x\|\leq 1} \|Ax\|. \tag{A.1}$$

Proof. Clearly,

$$\max_{\|x\|\leq 1} \|Ax\| \geq \max_{\substack{\|x\|\leq 1\\ x\perp\mathcal{K}}} \|Ax\| = \|A\|.$$

On the other hand, every $x \in \mathbb{R}^n$ can be expressed as $x = x_1 + x_2$, for some $x_1 \in \mathcal{K}$, $x_2 \in \mathcal{K}^\perp$, and condition $\|x\| \leq 1$ implies $\|x_2\| \leq 1$. Therefore

$$\|Ax\| = \|A(x_1 + x_2)\| \leq \|Ax_1\| + \|Ax_2\| = \|Ax_2\| \leq \max_{\substack{\|x_2\|\leq 1\\ x_2\perp\mathcal{K}}} \|Ax_2\| = \|A\|,$$

where we exploited the subadditivity property of the seminorm and the fact that $Ax_1 \in \mathcal{K}$ and hence $\|Ax_1\| = 0$. Therefore (A.1) holds. □