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# LIOUVILLE EQUATION AND SCHOTTKY PROBLEM<sup>¶</sup>

MARCO MATONE\*

*Department of Mathematics  
 Imperial College  
 180 Queen's Gate, London SW7 2BZ, U.K.*

*and*

*Department of Physics "G. Galilei" - Istituto Nazionale di Fisica Nucleare  
 University of Padova  
 Via Marzolo, 8 - 35131 Padova, Italy\*\**

## ABSTRACT

An Ansatz for the Poincaré metric on compact Riemann surfaces is proposed. This implies that the Liouville equation reduces to an equation resembling a non chiral analog of the higher genus relationships (KP equation) arising in the framework of Schottky's problem solution. This approach connects uniformization (Fuchsian groups) and moduli space theories with KP hierarchy. Besides its mathematical interest, the Ansatz has some applications in the framework of quantum Riemann surfaces arising in 2D gravity.

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\*e-mail: matone@padova.infn.it, mvxpd5::matone

\*\*Present address

# 1 Schottky Problem And KP Hierarchy

Let us consider a genus  $h$  compact Riemann surface  $\Sigma$ . A fundamental object defining the complex structure of  $\Sigma$  is the Riemann period matrix

$$\Omega_{ij} \equiv \oint_{\beta_i} \omega_j, \quad (1.1)$$

where the  $\omega_k$ 's denote the  $h$  holomorphic differentials with the standard normalization  $\oint_{\alpha_i} \omega_j = \delta_{ij}$ . By means of the Riemann bilinear relations it can be proved that  $\Omega_{ij}$  is symmetric and has positive definite imaginary part (see for example [1]). Let us consider the Siegel space

$$\mathcal{A}_h = \mathcal{H}_h / Sp(h, \mathbf{Z}), \quad (1.2)$$

where  $\mathcal{H}_h$  denotes the Siegel upper-half plane, that is the space of symmetric  $h \times h$  matrices with positive definite imaginary part. To recognize the locus in  $\mathcal{A}_h$  of the Riemann period matrices is the famous Schottky problem. This problem has been solved essentially by Dubrovin, Mulase and Shiota [2-4]. The solution is based on the proof of the Novikov conjecture stating that

$$u(x, y, t) = 2\partial_x^2 \log \Theta(Ux + Vy + Wt + z_0 | \Omega), \quad (1.3)$$

satisfies the KP equation if and only if  $\Omega$  is the period matrix of some  $\Sigma$ . The corresponding equations on  $\Omega$  (see eq.(2.15)) were derived in [2] where it was proved that they determine an algebraic variety with a component given by the matrices of the  $\beta$ -periods. In [4] Shiota pointed out that if  $u$  in eq.(1.3) satisfies the KP equation, then there are vectors  $U^k$ , such that the function

$$u(t_1, t_2, \dots) = 2\partial_t^2 \log \Theta \left( \sum_{k=1}^{\infty} U^k t_k | \Omega \right), \quad t_1 = x, t_2 = y, t_3 = t, \quad (1.4)$$

determines solutions of the KP hierarchy

$$\left[ \frac{\partial}{\partial t_j} - L_j, \frac{\partial}{\partial t_k} - L_k \right] = 0, \quad (1.5)$$

where the order  $k$  differential operators  $L_k$  have coefficients depending on  $\vec{t} \equiv (t_1, t_2, \dots)$  and are determined by the equation  $(\partial_{t_k} - L_k)\psi(\vec{t}, z) = 0$ ,  $\psi$  being the Baker-Akhiezer function on  $\Sigma$ . Since the space of vectors  $U^k$  is  $h$ -dimensional, there are two commuting operators of coprime order which are linear combinations of the  $L_k$ 's. Therefore one can apply the results in [5] to show that  $\Omega$  is the Riemann matrix of the surface defined by these operators.

## 2 The Ansatz

Let  $\Sigma$  be a compact Riemann surface of genus  $h > 1$ . It is well-known that the Liouville equation on  $\Sigma$

$$\partial_z \partial_{\bar{z}} \varphi(z, \bar{z}) = \frac{1}{2} e^{\varphi(z, \bar{z})}, \quad (2.1)$$

is uniquely satisfied by the Poincaré metric (with Gaussian curvature  $-1$ ). This metric can be written in terms of the inverse map of uniformization

$$J_H^{-1} : \Sigma \longrightarrow H, \quad (2.2)$$

where  $H = \{w | \text{Im } w > 0\}$  denotes the upper half plane. The Poincaré metric on  $H$  is

$$ds^2 = \frac{|dw|^2}{(\text{Im } w)^2}, \quad (2.3)$$

so that on  $\Sigma \cong H/\Gamma$  (here  $\Gamma$  is a hyperbolic Fuchsian group)

$$e^{\varphi(z, \bar{z})} = \frac{|J_H^{-1}(z)'|^2}{(\text{Im } J_H^{-1}(z))^2}, \quad (2.4)$$

which is invariant under  $SL(2, \mathbf{R})$  fractional transformations of  $J_H^{-1}$ . Unfortunately no one has succeeded in writing down  $J_H^{-1}$  in terms of the moduli of  $\Sigma$ .

Here we consider the following Ansatz for the Poincaré metric<sup>1</sup>

$$e^\varphi = \sum_{i,j=1}^h \omega_i A_{ij} \bar{\omega}_j. \quad (2.5)$$

To get the inverse map one has to solve the Schwarzian equation

$$\{J_H^{-1}, z\} = T^F(z), \quad (2.6)$$

where

$$T^F(z) = \varphi_{zz} - \frac{1}{2} \varphi_z^2, \quad (2.7)$$

is the classical Liouville stress tensor (or Fuchsian projective connection). By (2.5) we have

$$T^F(z) = \frac{2 \sum_{i,j=1}^h \omega_i'' A_{ij} \bar{\omega}_j - 3 \left( \sum_{i,j=1}^h \omega_i' A_{ij} \bar{\omega}_j \right)^2}{2 \left( \sum_{i,j=1}^h \omega_i A_{ij} \bar{\omega}_j \right)^2}. \quad (2.8)$$

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<sup>1</sup>Notice that a possible choice for the matrix to be positive definite is to set  $A_{ij} = \Omega_{ij}^{(2)-1}$ , in this case (2.5) coincides with the Bergman metric.

Observe that eq.(2.1) implies that

$$\partial_{\bar{z}} T^F(z) = 0. \quad (2.9)$$

Eq.(2.6) can be reduced to the linear equation

$$\left(2\partial_z^2 + T(z)\right) \psi = 0. \quad (2.10)$$

Actually it turns out that, up to  $SL(2, \mathbf{C})$  linear fractional transformations,

$$J^{-1} = \psi_1/\psi_2, \quad (2.11)$$

with  $\psi_1$  and  $\psi_2$  two linearly independent solutions of (2.10) (see [7] for a discussion on this point).

Inserting (2.5) in (2.1), the Liouville equation becomes

$$\sum_{i,j,k,l=1}^h \omega_l^2 \partial_z (\omega_i/\omega_l) A_{ij} A_{lk} \bar{\omega}_k^2 \partial_{\bar{z}} (\bar{\omega}_j/\bar{\omega}_k) = \left( \sum_{i,j=1}^h \omega_i A_{ij} \bar{\omega}_j \right)^3. \quad (2.12)$$

This equation has a strict similarity with the relations between the periods of holomorphic differentials on Riemann surfaces [2]. Thus one should expect that  $A_{ij}$  depends on the moduli through the Riemann period matrix. To show this similarity, we write down the fundamental relations given in [2]. Let us introduce the following notation

$$\begin{aligned} U_k &= -\omega_k(P), \\ V_k &= -\omega'_k(P), \\ W_k &= -\frac{1}{2}\omega_k''(P) - \frac{1}{2}c(P)U_k, \end{aligned} \quad (2.13)$$

where  $c(P)$  is a projective connection [2] and  $P$  is an arbitrary point on  $\Sigma$ . In [2] Dubrovin proved that the function (1.3) is a solution of the KP equation

$$u_{yy} = (4u_t - 6uu_x - u_{xxx})_x, \quad (2.14)$$

if and only if the following relations between  $U, V, W, \Omega$  and an additional constant  $d$  are satisfied (see [2] for notation)

$$\sum_{i,j,k,l=1}^h U_i U_j U_k U_l \hat{\Theta}_{ijkl}[n] + \sum_{i,j=1}^h \left( \frac{3}{4} V_i V_j - U_i W_j \right) \hat{\Theta}_{ij}[n] + d \hat{\Theta}[n] = 0, \quad n \in \mathbf{Z}_2^h. \quad (2.15)$$

We emphasize that this result is a fundamental step to solve Schottky's problem.

Our remark is that eq.(2.12) looks like a non chiral generalization of (2.15). In the notation introduced above eq.(2.12) reads

$$\sum_{i,j,k,l=1}^h (U_l V_i - U_i V_l) A_{ij} A_{lk} (\bar{U}_k \bar{V}_j - \bar{U}_j \bar{V}_k) = \left( \sum_{i,j=1}^h U_i A_{ij} \bar{U}_j \right)^3. \quad (2.16)$$

We stress that solving this equation is equivalent to solving crucial questions arising in uniformization theory, Fuchsian groups and related subjects. In particular, Weil-Petersson's 2-form  $\omega_{WP}$  can be recovered using the fact that the classical Liouville action evaluated at the classical solution is the Kähler potential of  $\omega_{WP}$  [6].

Another aspect that should be investigated is whether eq.(2.16) furnishes conditions on the period matrix in a more manageable form than KP equation (2.14)-(2.15).

A possible approach to study eq.(2.12) is using Krichever-Novikov's differentials  $\psi_j^{(n)}$  [8]. These differentials are holomorphic on  $\Sigma \setminus \{P_+, P_-\}$  with prescribed behaviour at  $P_\pm$ . In particular, in terms of local coordinates  $z_\pm$  vanishing at  $P_\pm \in \Sigma$ , we have

$$\psi_j^{(n)}(z_\pm)(dz_\pm)^n = a_j^{(n)\pm} z_\pm^{\pm j - s(n)} (1 + \mathcal{O}(z_\pm)) (dz_\pm)^n, \quad s(n) = \frac{h}{2} - n(h-1), \quad (2.17)$$

where  $j \in \mathbf{Z} + h/2$  and  $n \in \mathbf{Z}$ . By the Riemann-Roch theorem,  $\psi_j^{(n)}$  is uniquely fixed by choosing the value of  $a_j^{(n)+}$  or  $a_j^{(n)-}$ . In the following we set  $a_j^{(n)+} = 1$ .

These differentials can be written in terms of theta functions<sup>2</sup> [9]

$$\psi_j^{(n)}(z) = C_j^{(n)} \Theta \left( I(z) + \mathcal{D}^{j;n} | \Omega \right) \frac{\sigma(z)^{2n-1} E(z, P_+)^{j-s(n)}}{E(z, P_-)^{j+s(n)}}, \quad (2.18)$$

where

$$\mathcal{D}^{j;n} = (j - s(n)) I(P_+) - (j + s(n)) I(P_-) + (1 - 2n)\Delta,$$

and constant  $C_j^{(n)}$  is fixed by the condition  $a_j^{(n)+} = 1$ .

Let  $\mathcal{C}$  be a homologically trivial contour separating  $P_+$  and  $P_-$ . The dual of  $\psi_j^{(n)}$  is defined by

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \psi_j^{(n)} \psi_{(n)}^k = \delta_j^k, \quad (2.19)$$

which implies

$$\psi_{(n)}^j = \psi_{-j}^{(1-n)}. \quad (2.20)$$

Note that (2.17) provides a basis for the  $1-2s(n) = (2n-1)(h-1)$  holomorphic  $n$ -differentials on  $\Sigma$  ( $h \geq 2$ )

$$\mathcal{H}^{(n)} = \left\{ \psi_k^{(n)} \mid s(n) \leq k \leq -s(n) \right\}, \quad n \geq 2. \quad (2.21)$$

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<sup>2</sup>In the appendix we illustrate the method to construct differentials in higher genus Riemann surfaces.

Furthermore, from

$$\widetilde{\mathcal{H}}^{(m)} = \left\{ \psi_k^{(m)} \mid 1 - s(m) \leq k \leq s(m) - 1 \right\}, \quad m \leq -1, \quad (2.22)$$

one can define the space of generalized Beltrami differentials. They are vanishing everywhere on  $\Sigma$  except in a disk where coincide with [7]

$$\widetilde{\mathcal{B}}^{(m)} = \left\{ \partial_{\bar{z}} \psi_k^{(m)} \mid 1 - s(m) \leq k \leq s(m) - 1 \right\}, \quad m \leq -1, \quad (2.23)$$

(for  $m = -1$  one gets the Beltrami differentials considered in [9]). Observe that the differentials in (2.22) have poles both in  $P_+$  and  $P_-$ . In particular,  $\widetilde{\mathcal{H}}^{(1-n)}$  is the dual space of  $\mathcal{H}^{(n)}$ .

We now expand the holomorphic 3-differentials in (2.12) in terms of the basis introduced above. We have

$$\omega_i^2 \partial_z (\omega_j / \omega_i) = \sum_{p=1}^{5h-5} a_{ij}^p \psi_{p+s(3)-1}^{(3)}, \quad a_{ij}^p = \frac{1}{2\pi i} \oint_{\mathcal{C}} \psi_{-p-s(3)+1}^{(-2)} \omega_i^2 \partial_z (\omega_j / \omega_i), \quad (2.24)$$

$$\omega_i \omega_j \omega_k = \sum_{p=1}^{5h-5} b_{ijk}^p \psi_{p+s(3)-1}^{(3)}, \quad b_{ijk}^p = \frac{1}{2\pi i} \oint_{\mathcal{C}} \psi_{-p-s(3)+1}^{(-2)} \omega_i \omega_j \omega_k. \quad (2.25)$$

Inserting these expansions in (2.12) we get the ‘Liouville relations’

$$\sum_{i,j,k,l=1}^h a_{ij}^p A_{ik} A_{jl} \bar{a}_{kl}^q = \sum_{i,j,k,l,m,n=1}^h b_{ijm}^p A_{ik} A_{jl} A_{mn} \bar{b}_{klm}^q. \quad (2.26)$$

Let us notice that the coefficients  $a_{kl}^q$  and  $b_{klm}^q$  are functionals of the holomorphic differentials and their derivatives computed at  $P_+$  and coincide with the vectors of  $\beta$ -periods of second-kind differentials.

The above expansions provide relations involving the holomorphic differentials, theta functions and their derivatives. To see this it is sufficient to notice that the coefficients  $a_{ij}^p$  and  $b_{ijk}^p$  are vanishing for  $p < 1$  and  $p > 5h - 5$ . The reason is that in this range the  $\psi_{-p-s(3)+1}^{(-2)}$ ’s are holomorphic in  $P_-$  or  $P_+$ . This implies that for  $p < 1$  and  $p > 5h - 5$ , the contribution to  $a_{ij}^p$  and  $b_{ijk}^p$  coming from the poles at  $P_-$  or  $P_+$  add to zero. Notice that this ‘residue formula’ is crucial to get important relations such as Hirota’s formulation of the KP hierarchy (see for example [10]).

### 3 The Accessory Parameters

Here we consider some aspects concerning the Fuchsian accessory parameters. First of all we introduce the projective connection

$$T^S(z) = \{J_\Omega^{-1}, z\}, \quad (3.1)$$

where  $J_\Omega : \Omega \rightarrow \Sigma$  denote the Schottkian uniformization map. Here  $\Omega$  denotes the region of discontinuity in  $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$  of the Schottky group  $\mathcal{S}$  and  $\Sigma \cong \Omega/\mathcal{S}$ . Let us introduce the following notation for the Krichever-Novikov vector fields and quadratic differentials

$$e_k \equiv \psi_k^{(-1)}, \quad \Omega^k \equiv \psi_{-k}^{(2)}. \quad (3.2)$$

Let  $\mathcal{T}_\Sigma$  be the holomorphic projective connection on  $\Sigma$  obtained from the symmetric bidifferential of the second-kind with bi-residue 1 and zero  $\alpha$ -periods. The Fuchsian accessory parameters  $\lambda_1^{(F)}, \dots, \lambda_{3h-3}^{(F)}$  and the Schottkian accessory parameters  $\lambda_1^{(S)}, \dots, \lambda_{3h-3}^{(S)}$  are defined by

$$T^F = \mathcal{T}_\Sigma + \sum_{k=1}^{3h-3} \lambda_k^{(F)} \Omega^{k+1-h_0}, \quad T^S = \mathcal{T}_\Sigma + \sum_{k=1}^{3h-3} \lambda_k^{(S)} \Omega^{k+1-h_0}, \quad h_0 \equiv \frac{3}{2}h. \quad (3.3)$$

In order to write  $\mathcal{T}_\Sigma$  explicitly we consider an arbitrary nonsingular point  $f$  of the theta divisor, that is  $\Theta(f) = 0$  and  $\text{grad } \Theta(f) \neq 0$ . We define

$$H_f(z) = \sum_{k=1}^h \Theta_k(f) \omega_k(z), \quad (3.4)$$

$$Q_f(z) = \sum_{j,k=1}^h \Theta_{jk}(f) \omega_j(z) \omega_k(z), \quad (3.5)$$

$$T_f(z) = \sum_{i,j,k=1}^h \Theta_{ijk}(f) \omega_i(z) \omega_j(z) \omega_k(z). \quad (3.6)$$

The holomorphic projective connection is [11]

$$\mathcal{T}_\Sigma(z) = \left\{ \int_{P_0}^z H_f, z \right\} + \frac{3}{2} \left( \frac{Q_f(z)}{H_f(z)} \right)^2 - 2 \frac{T_f(z)}{H_f(z)}. \quad (3.7)$$

At a zero of  $H_f$  we have

$$Q_f(z_0) = \pm H'_f(z_0), \quad T_f(z_0) = -H''_f(z_0) \pm \frac{3}{2} Q'_f(z_0), \quad (3.8)$$

with the sign  $\pm$  chosen accordingly as  $\Theta(z - z_0 \mp f) \equiv 0, \forall z \in \Sigma$ .

Besides  $T^F$  and  $T^S$ , also  $\mathcal{T}_\Sigma$  can be expressed as a Schwarzian derivative. To do this we simply note that according to the general rule described above the equation

$$\left(\frac{\partial^2}{\partial z^2} + \frac{1}{2}\mathcal{T}_\Sigma(z)\right)\phi(z) = 0, \quad (3.9)$$

has as solutions two linearly independent  $-\frac{1}{2}$ -differentials  $\phi_1, \phi_2$ , satisfying the equation

$$\mathcal{T}_\Sigma(z) = \{\phi_2/\phi_1, z\}. \quad (3.10)$$

Note that the Fuchsian accessory parameters are given by

$$\lambda_k^{(F)} = \frac{1}{2\pi i} \oint_{\mathcal{C}} \left( \{J_H^{-1}(z), z\} - \{J_\Sigma^{-1}(z), z\} \right) e^{k+1-h_0}, \quad (3.11)$$

where

$$J_\Sigma^{-1}(z) = \phi_2/\phi_1. \quad (3.12)$$

It is interesting to note that the integrand resembles the chain rule for the Schwarzian derivative

$$\{w(t(z)), z\}(dz)^2 - \{t(z), z\}(dz)^2 = \{w(t), t\}(dt)^2, \quad (3.13)$$

in particular

$$\{J_H^{-1}(J_\Sigma^{-1}(z)), z\} - \{J_\Sigma^{-1}(z), z\} = \{J_H^{-1}(J_\Sigma^{-1}), J_\Sigma^{-1}\} (\partial_z J_\Sigma^{-1}(z))^2. \quad (3.14)$$

We stress that the accessory parameters can be written as a line integral of a one-form written in terms of theta functions and holomorphic differentials. In particular for the Fuchsian accessory parameters we have

$$\begin{aligned} \lambda_k^{(F)} = \frac{1}{2\pi i} \oint_{\mathcal{C}} \left( \frac{2 \sum_{i,j=1}^h \omega_i'' A_{ij} \bar{\omega}_j - 3 \left( \sum_{i,j=1}^h \omega_i' A_{ij} \bar{\omega}_j \right)^2}{2 \left( \sum_{i,j=1}^h \omega_i A_{ij} \bar{\omega}_j \right)^2} - \left\{ \int_{P_0}^z H_f, z \right\} \right. \\ \left. - \frac{3}{2} \left( \frac{Q_f(z)}{H_f(z)} \right)^2 + 2 \frac{T_f(z)}{H_f(z)} \right) e^{k+1-h_0}. \end{aligned} \quad (3.15)$$

In the second reference in [6], where the results for the punctured Riemann sphere are generalized to higher genus Riemann surfaces, a relationship has been established between  $c_k^{(h)} = \lambda_k^{(F)} - \lambda_k^{(S)}$ , the Liouville action evaluated on the classical solution and the Weil-Petersson metric. In particular it turns out that

$$\frac{1}{2} \frac{\partial S_{cl}^{(h)}}{\partial z_i} = c_i^{(h)}, \quad \frac{\partial c_i^{(h)}}{\partial \bar{z}_j} = -\frac{1}{2} \left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right\rangle_{WP}, \quad (3.16)$$



where the brackets denote the Weil-Petersson metric on the Teichmüller space  $T_h$  projected onto the Schottky space whose coordinate are  $z_1, \dots, z_{3h-3}$ . Since the difference

$$\Theta(z) = T^F(z) - T^S(z) = \sum_{k=1}^{3h-3} c_k^{(h)} \Omega^{k+1-h_0}(z), \quad (3.17)$$

is a holomorphic quadratic differential (i.e. a section of  $T^*T_h$ ), the formulas in eq.(3.16) are equivalent to

$$\partial S_{cl}^{(h)} = 2\Theta, \quad \bar{\partial} \partial S_{cl}^{(h)} = -2i\omega_{WP}, \quad (3.18)$$

where  $d = \partial + \bar{\partial}$  is the exterior differentiation on the Schottky space and  $\omega_{WP}$  is the Weil-Petersson 2-form on this space. Because the Schottky projective connection depends holomorphically on the moduli we have

$$\bar{\partial} T^F = -i\omega_{WP}, \quad (3.19)$$

that by (2.8) gives

$$\omega_{WP} = i\bar{\partial} \frac{2 \sum_{i,j=1}^h \omega_i'' A_{ij} \bar{\omega}_j - 3 \left( \sum_{i,j=1}^h \omega_i' A_{ij} \bar{\omega}_j \right)^2}{2 \left( \sum_{i,j=1}^h \omega_i A_{ij} \bar{\omega}_j \right)^2}. \quad (3.20)$$

Similar results have been derived by Fay [12]. In particular it turns out that

$$\{J_H^{-1}, z\} = \mathcal{T}_\Sigma - 24\pi i \sum_{j,k=1}^h \left( \frac{\partial}{\partial \Omega_{jk}} \log c_0 \right) \omega_j(z) \omega_k(z), \quad (3.21)$$

where

$$c_0 = \left[ \frac{8\pi^2 \det' \Delta}{\det \text{Im } \Omega} \right]^{-1/2}, \quad (3.22)$$

is the anomaly in the spin-1/2 bosonization formula computed with respect to the Poincaré metric  $e^\varphi$ .

The connection with the Weil-Petersson metric on  $T_h$  arises if we consider the quasi-conformal mapping

$$\partial_{\bar{z}} f^\rho = \rho \partial_z f^\rho, \quad \rho = t_1 \nu_1 + t_2 \nu_2. \quad (3.23)$$

It turns out that

$$-24\pi \partial \bar{\partial} \log c_0 = \langle \nu_1, \nu_2 \rangle_{WP}, \quad (3.24)$$

where  $\langle \nu_1, \nu_2 \rangle_{WP} = \int_\Sigma e^\varphi \nu_1 \bar{\nu}_2$  and

$$\partial = \partial_{t(p)} = \sum_{j,k=1}^h \frac{\partial}{\partial \Omega_{jk}} \delta \Omega_{jk}, \quad (3.25)$$

is the Schiffer variation (see [12] for details).

Another possible way to investigate eq.(2.5) is by noticing that both the first and second variations vanish for the deformation of the complex structure induced by the harmonic Beltrami differentials [13, 14]. Applying this condition to (2.5) should give further informations on the form of matrix  $A_{ij}$ .

As a final remark, we observe that, besides any mathematical interest, the solution for the Poincaré metric is crucial to get explicit expressions for correlators in string theory. In particular in the ‘uniformization approach’ to 2D quantum gravity [7, 15] one needs the explicit expression for the Liouville action evaluated at the classical solution to compute the ‘VEV of quantum Riemann surfaces’  $\langle \Sigma \rangle$  (see [16, 17]).

## A Appendix

Let us introduce the theta function with characteristic

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (z|\Omega) = \sum_{k \in \mathbf{Z}^h} e^{\pi i(k+a) \cdot \Omega \cdot (k+a) + 2\pi i(k+a) \cdot (z+b)}, \quad \Theta(z|\Omega) \equiv \Theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z|\Omega), \quad (\text{A.1})$$

where  $z \in \mathbf{C}^h$ ,  $a, b \in \mathbf{R}^h$ . When  $a_k, b_k \in \{0, 1/2\}$ ,  $\Theta \begin{bmatrix} a \\ b \end{bmatrix} (z|\Omega)$  is even or odd depending on the parity of  $4a \cdot b$ . The  $\Theta$ -function is multivalued under a lattice shift in the  $z$ -variable

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (z + n + \Omega \cdot m|\Omega) = e^{-\pi i m \cdot \Omega \cdot m - 2\pi i m \cdot z + 2\pi i(a \cdot n - b \cdot m)} \Theta \begin{bmatrix} a \\ b \end{bmatrix} (z|\Omega). \quad (\text{A.2})$$

An important object to construct differentials in higher genus is the prime form  $E(z, w)$ . It is a holomorphic  $-1/2$ -differential both in  $z$  and  $w$ , vanishing for  $z = w$  only

$$E(z, w) = \frac{\Theta \begin{bmatrix} a \\ b \end{bmatrix} (I(z) - I(w)|\Omega)}{h(z)h(w)}. \quad (\text{A.3})$$

Here  $h(z)$  denotes the square root of  $\sum_{k=1}^h \omega_k(z) \partial_{u_k} \Theta \begin{bmatrix} a \\ b \end{bmatrix} (u|\Omega)|_{u_k=0}$ ; it is the holomorphic  $1/2$ -differential with non singular (i.e.  $\partial_{u_k} \Theta \begin{bmatrix} a \\ b \end{bmatrix} (u|\Omega)|_{u_k=0} \neq 0$ ) odd spin structure  $\begin{bmatrix} a \\ b \end{bmatrix}$ . The function  $I(z)$  in (A.3) denotes the Jacobi map

$$I_k(z) = \int_{P_0}^z \omega_k, \quad z \in \Sigma, \quad (\text{A.4})$$

with  $P_0 \in \Sigma$  an arbitrary base point. This map is an embedding of  $\Sigma$  into the Jacobian

$$J(\Sigma) = \mathbf{C}^h / L_\Omega, \quad L_\Omega = \mathbf{Z}^h + \Omega \mathbf{Z}^h. \quad (\text{A.5})$$

By (A.2) it follows that the multivaluedness of  $E(z, w)$  is

$$E(z + n \cdot \alpha + m \cdot \beta, z) = e^{-\pi i m \cdot \Omega \cdot m - 2\pi i m \cdot (I(z) - I(w))} E(z, w). \quad (\text{A.6})$$

In terms of  $E(z, w)$  one can construct the following  $h/2$ -differential with empty divisor

$$\sigma(z) = \exp \left( - \sum_{k=1}^h \oint_{\alpha_k} \omega_k(w) \log E(z, w) \right), \quad (\text{A.7})$$

whose multivaluedness is

$$\sigma(z + n \cdot \alpha + m \cdot \beta) = e^{\pi i (h-1) m \cdot \Omega \cdot m - 2\pi i m \cdot (\Delta - (h-1)I(z))} \sigma(z), \quad (\text{A.8})$$

where  $\Delta$  is (essentially) the *vector of Riemann constants* [11]. Finally we quote two theorems:

**a. Abel Theorem** [1]. *A necessary and sufficient condition for  $\mathcal{D}$  to be the divisor of a meromorphic function is that*

$$I(\mathcal{D}) = 0 \pmod{(L_\Omega)} \text{ and } \deg \mathcal{D} = 0. \quad (\text{A.9})$$

**b. Riemann vanishing theorem** [11]. *The function*

$$\Theta \left( I(z) - \sum_{k=1}^h I(P_k) + \Delta \middle| \Omega \right), \quad z, P_k \in \Sigma, \quad (\text{A.10})$$

*either vanishes identically or else it has  $h$  zeroes at  $z = P_1, \dots, P_h$ .*

We are now ready to explicitly construct the differential  $f^{(n)}$  defined above. First of all note that

$$\tilde{f}^{(n)} = \sigma(z)^{2n-1} \frac{\prod_{k=h+1}^p E(z, P_k)}{\prod_{j=1}^{p-2n(h-1)} E(z, Q_j)}, \quad (\text{A.11})$$

is a multivalued  $n$ -differential with  $\text{Div } \tilde{f}^{(n)} = \sum_{k=h+1}^p P_k - \sum_{k=1}^{p-2n(h-1)} Q_k$ . Therefore we set

$$f^{(n)}(z) = g(z) \tilde{f}^{(n)}, \quad (\text{A.12})$$

where, up to a multiplicative constant,  $g$  is fixed by the requirement that  $f^{(n)}$  be singlevalued. From the multivaluedness of the  $E(z, w)$  and  $\sigma(z)$  it follows that, up to a multiplicative constant

$$g(z) = \Theta(I(z) + \mathcal{D} | \Omega), \quad (\text{A.13})$$

with

$$\mathcal{D} = \sum_{k=h+1}^p I(P_k) - \sum_{k=1}^{p-2n(h-1)} I(Q_k) + (1-2n)\Delta. \quad (\text{A.14})$$

By Riemann vanishing theorem  $g(z)$  has just  $h$ -zeroes  $P_1, \dots, P_h$  fixed by  $\mathcal{D}$ . Thus the requirement of singlevaluedness also fixes the position of the remainder  $h$  zeroes. To make manifest the divisor in the RHS of (A.12) we first recall that the image of the canonical line bundle  $K$  on the Jacobian of  $\Sigma$  coincides with  $2\Delta$  [11]. On the other hand, since

$$[K^n] = \left[ \sum_{k=1}^p P_k - \sum_{k=1}^{p-2n(h-1)} Q_k \right], \quad (\text{A.15})$$

by Abel theorem we have<sup>3</sup>

$$\text{Div } \Theta (I(z) + \mathcal{D}|\Omega) = \text{Div } \Theta \left( I(z) - \sum_{k=1}^h I(P_k) + \Delta \Big| \Omega \right), \quad (\text{A.16})$$

and by Riemann vanishing theorem

$$\text{Div } \Theta (I(z) + \mathcal{D}|\Omega) = \sum_{k=1}^h I(P_k). \quad (\text{A.17})$$

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<sup>3</sup>The square brackets in (A.15) denote the divisor class associated to the line bundle  $K^n$ . Two divisors belong to the same class if they differ by a divisor of a meromorphic function.

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