Classes of kernels and continuity properties of the tangential gradient of an integral operator in Hölder spaces on a manifold

M. Lanza de Cristoforis

Key words: Continuity, classes of kernels, tangential gradient, integral operator, manifold.

AMS Mathematics Subject Classification: 31B10 (Primary); 42B20, 42B37, 47G40, 47G10 (Secondary).

Abstract. We prove multiplication and imbedding theorems for classes of kernels of integral operators in subsets of metric spaces with a measure. Then we prove a tangential differentiation theorem with respect to a semi-tangent vector for integral operators defined on an upper-Ahlfors regular subset of the Euclidean space and a continuity theorem for the corresponding integral operator in Hölder spaces in the specific case of a differentiable manifold.

1 Introduction

Volume and layer potentials are integrals on a subset Y of the Euclidean space $\mathbb{R}^n$ that depend on a variable in a subset X of $\mathbb{R}^n$. Typically, X and Y are either measurable subsets of $\mathbb{R}^n$ with the n-dimensional Lebesgue measure, or manifolds imbedded in $\mathbb{R}^n$, or boundaries of open subsets of $\mathbb{R}^n$ with the surface measure and X may well be different from Y.

For many relevant results in Hölder spaces, one can introduce a unified approach by assuming that X and Y are subsets of a metric space (M, d) and that Y is equipped with a measure ν that satisfies an upper Ahlfors growth condition that includes non-doubling measures (cf. (4.2)). With this respect we mention the work of García-Cuerva and Gatto [6], [7], Gatto [8] who have considered case $X = Y = M$ and proved $T1$ Theorems for integral operators. Then one can also consider a stronger growth condition. Namely, the strong upper Ahlfors growth condition (4.9) that has been introduced in [15] to treat the dependence of singular and weakly singular integral operators both upon the variation of the density and of the kernel, when the kernel belongs to certain classes of kernels that generalize those of Giraud [10], Gegelia [9], Kupradze, Gegelia, Basheleishvili and Burchuladze [13, Chap. IV] and the so-called standard kernels.

In this paper, we first introduce some basic multiplication and imbedding theorems for such classes of kernels (see section 3).

In section 4, we summarize and complement some results of [15].

In section section 5, we prove the tangential differentiation Theorem 5.1 with respect to a semi-tangent vector for integral operators defined on an upper-Ahlfors regular subset of the Euclidean space.

In section 6, we consider the case in which Y is a compact manifold of codimension 1 in $\mathbb{R}^n$, and we show application of the results of [15], of the above mentioned properties of the kernel classes and of Theorem 5.1 by proving Theorem 6.3 on the continuity of the tangential gradient of a weakly singular integral operator that is defined in Y upon variation both of the kernel and of the density in Hölder spaces. Here we mention that Theorem 6.3 applies to relevant integral operators such as the layer potentials. In a forthcoming paper, we plan to apply the multiplication and imbedding theorems of the classes of kernels of section 3 and of Theorem 6.3 to analyze the continuity properties of the double layer potential associated to the fundamental solution of a second order elliptic operator with constant coefficients.

2 Notation

Let X be a set. Then we set

$$B(X) \equiv \left\{ f \in \mathbb{C}^X : \, f \text{ is bounded} \right\} , \quad \|f\|_{B(X)} \equiv \sup_X |f| \qquad \forall f \in B(X) .$$

If (M, d) is a metric space, we set

$$B(\xi, r) \equiv \{\eta \in M : \, d(\xi, \eta) < r\} \tag{2.1}$$

for all $(\xi, r) \in M \times]0, +\infty[$ and

$$\text{diam}\,(X) \equiv \sup\{d(x_1, x_2) : \, x_1, x_2 \in X\}$$

for all subsets X of M. Then $C^0(M)$ denotes the set of continuous functions from M to $\mathbb{C}$ and we introduce the subspace $C^0_b(M) \equiv C^0(M) \cap B(M)$ of $B(M)$. Let ω be a function from $[0, +\infty[$ to itself such that

$$\begin{aligned} &\omega(0) = 0, \qquad \omega(r) > 0 \qquad \forall r \in]0, +\infty[\,, \\ &\omega \text{ is increasing}, \ \lim_{r \to 0^+} \omega(r) = 0 \,, \\ &\text{and} \qquad \sup_{(a,t) \in [1, +\infty[\times]0, +\infty[} \frac{\omega(at)}{a\omega(t)} < +\infty \,. \end{aligned} \tag{2.2}$$

If f is a function from a subset $\mathbb{D}$ of a metric space (M, d) to $\mathbb{C}$, then we denote by $|f : \mathbb{D}|_{\omega(\cdot)}$ the $\omega(\cdot)$-Hölder constant of f, which is delivered by the formula

$$|f : \mathbb{D}|_{\omega(\cdot)} \equiv \sup \left\{ \frac{|f(x) - f(y)|}{\omega(d(x, y))} : x, y \in \mathbb{D}, x \neq y \right\} .$$

If $|f : \mathbb{D}|_{\omega(\cdot)} < \infty$, we say that f is $\omega(\cdot)$-Hölder continuous. Sometimes, we simply write $|f|_{\omega(\cdot)}$ instead of $|f : \mathbb{D}|_{\omega(\cdot)}$. The subset of $C^0(\mathbb{D})$ whose functions are $\omega(\cdot)$-Hölder continuous is denoted by $C^{0,\omega(\cdot)}(\mathbb{D})$ and $|f : \mathbb{D}|_{\omega(\cdot)}$ is a semi-norm on $C^{0,\omega(\cdot)}(\mathbb{D})$. Then we consider the space $C^{0,\omega(\cdot)}_b(\mathbb{D}) \equiv C^{0,\omega(\cdot)}(\mathbb{D}) \cap B(\mathbb{D})$ with the norm

$$\|f\|_{C^{0,\omega(\cdot)}_b(\mathbb{D})} \equiv \sup_{x \in \mathbb{D}} |f(x)| + |f|_{\omega(\cdot)} \qquad \forall f \in C^{0,\omega(\cdot)}_b(\mathbb{D}) .$$

In the case in which $\omega(\cdot)$ is the function r^α for some fixed $\alpha \in]0, 1]$, a so-called Hölder exponent, we simply write $|\cdot : \mathbb{D}|_\alpha$ instead of $|\cdot : \mathbb{D}|_{r^\alpha}$, $C^{0,\alpha}(\mathbb{D})$ instead of $C^{0,r^\alpha}(\mathbb{D})$, $C^{0,\alpha}_b(\mathbb{D})$ instead of $C^{0,r^\alpha}_b(\mathbb{D})$, and we say that f is α-Hölder continuous provided that $|f : \mathbb{D}|_\alpha < +\infty$.

3 Special classes of potential type kernels in metric spaces

If X and Y are sets, then we denote by $\mathbb{D}_{X\times Y}$ the diagonal of $X \times Y$, i.e., we set

$$\mathbb{D}_{X\times Y} \equiv \{(x,y) \in X \times Y : x = y\} \tag{3.1}$$

and if $X = Y$, then we denote by $\mathbb{D}_X$ the diagonal of $X \times X$, i.e., we set

$$\mathbb{D}_X \equiv \mathbb{D}_{X\times X} .$$

An off-diagonal function in $X \times Y$ is a function from $(X \times Y) \setminus \mathbb{D}_{X\times Y}$ to $\mathbb{C}$. We now wish to consider a specific class of off-diagonal kernels in a metric space (M, d).

Definition 1. *Let X and Y be subsets of a metric space (M, d). Let $s \in \mathbb{R}$. We denote by $\mathcal{K}_{s,X\times Y}$, the set of continuous functions K from $(X \times Y) \setminus \mathbb{D}_{X\times Y}$ to $\mathbb{C}$ such that*

$$\|K\|_{\mathcal{K}_{s,X\times Y}} \equiv \sup_{(x,y)\in(X\times Y)\setminus\mathbb{D}_{X\times Y}} |K(x,y)|\, d(x,y)^s < +\infty .$$

The elements of $\mathcal{K}_{s,X\times Y}$ are said to be kernels of potential type s in $X \times Y$.

We plan to consider 'potential type' kernels as in the following definition. See also Dondi and the author [5], where such classes have been introduced in a form that generalizes those of Giraud [10], Gegelia [9], Kupradze, Gegelia, Basheleishvili and Burchuladze [13, Chap. IV].

Definition 2. *Let X and Y be subsets of a metric space (M, d). Let s_1, s_2, $s_3 \in \mathbb{R}$. We denote by $\mathcal{K}_{s_1,s_2,s_3}(X \times Y)$ the set of continuous functions K from $(X \times Y) \setminus \mathbb{D}_{X\times Y}$ to $\mathbb{C}$ such that*

$$\begin{aligned}\|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \equiv \sup&\Big\{ d(x,y)^{s_1}|K(x,y)| : (x,y) \in X \times Y, x \neq y \Big\}\\ &+ \sup\Big\{ \frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}}|K(x',y) - K(x'',y)| :\\ &x', x'' \in X, x' \neq x'', y \in Y \setminus B(x', 2d(x',x'')) \Big\} < +\infty .\end{aligned}$$

One can easily verify that $(\mathcal{K}_{s_1,s_2,s_3}(X \times Y), \|\cdot\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)})$ is a normed space. By our definition, if s_1, s_2, $s_3 \in \mathbb{R}$, we have

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \subseteq \mathcal{K}_{s_1,X\times Y}$$

and

$$\|K\|_{\mathcal{K}_{s_1,X\times Y}} \leq \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \qquad \forall K \in \mathcal{K}_{s_1,s_2,s_3}(X \times Y) .$$

We note that if we choose $s_2 = s_1 + s_3$ we have a so-called class of standard kernels. We now turn to prove a series of statements in a metric space setting that extend the validity of corresponding statements for the classes that had been introduced in Giraud [10], Gegelia [9], Kupradze, Gegelia, Basheleishvili and Burchuladze [13, Chap. IV]. We start with the following elementary known embedding lemma.

Lemma 3.1. *Let X and Y be subsets of a metric space (M,d). Let s_1, s_2, $s_3 \in \mathbb{R}$. If $a \in]0,+\infty[$, then $\mathcal{K}_{s_1,s_2,s_3}(X\times Y)$ is continuously embedded into $\mathcal{K}_{s_1,s_2-a,s_3-a}(X\times Y)$.*

Proof. It suffices to note that if $x', x'' \in X$, $x' \neq x''$, then

$$\frac{d(x',y)^{s_2-a}}{d(x',x'')^{s_3-a}} = \frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}}\frac{d(x',x'')^{a}}{d(x',y)^{a}} \leq \frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}}\left(\frac{\frac{1}{2}d(x',y)}{d(x',y)}\right)^{a}$$
$$= \frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}}2^{-a} \qquad \forall y \in Y\setminus B(x',2d(x',x''))\,.$$

□

Next we introduce the following known elementary lemma, which we exploit later and which can be proved by the triangular inequality.

Lemma 3.2. *Let (M,d) be a metric space. Then*

$$\frac{1}{2}d(x',y) \leq d(x'',y) \leq 2d(x',y)\,,$$

for all $x', x'' \in M$, $x' \neq x''$, $y \in M\setminus B(x',2d(x',x''))$.

Next we prove the following product rule for kernels.

Theorem 3.1. *Let X and Y be subsets of a metric space (M,d). Let s_1, s_2, s_3, t_1, t_2, $t_3 \in \mathbb{R}$.*

(i) If $K_1 \in \mathcal{K}_{s_1,s_2,s_3}(X\times Y)$ and $K_2 \in \mathcal{K}_{t_1,t_2,t_3}(X\times Y)$, then the following inequality holds

$$|K_1(x',y)K_2(x',y) - K_1(x'',y)K_2(x'',y)|$$
$$\leq \|K_1\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}\|K_2\|_{\mathcal{K}_{t_1,t_2,t_3}(X\times Y)}\left(\frac{d(x',x'')^{s_3}}{d(x',y)^{s_2+t_1}} + \frac{2^{|s_1|}d(x',x'')^{t_3}}{d(x',y)^{t_2+s_1}}\right)$$

for all $x', x'' \in X$, $x' \neq x''$, $y \in Y\setminus B(x',2d(x',x''))$.

(ii) The pointwise product is bilinear and continuous from

$$\mathcal{K}_{s_1,s_1+s_3,s_3}(X\times Y)\times \mathcal{K}_{t_1,t_1+s_3,s_3}(X\times Y) \quad \textit{to} \quad \mathcal{K}_{s_1+t_1,s_1+s_3+t_1,s_3}(X\times Y)\,.$$

Proof. (i) By the triangular inequality and by the definition of norm for kernels, we have

$$|K_1(x',y)K_2(x',y) - K_1(x'',y)K_2(x'',y)|$$
$$\leq |K_1(x',y) - K_1(x'',y)|\,|K_2(x',y)| + |K_1(x'',y)|\,|K_2(x',y) - K_2(x'',y)|$$
$$\leq \|K_1\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}\|K_2\|_{\mathcal{K}_{t_1,t_2,t_3}(X\times Y)}\left(\frac{d(x',x'')^{s_3}}{d(x',y)^{s_2+t_1}} + \frac{d(x',x'')^{t_3}}{d(x',y)^{t_2}d(x'',y)^{s_1}}\right)$$

If $s_1 \geq 0$, Lemma 3.2 implies that

$$\frac{1}{d(x'',y)^{s_1}} \leq \frac{1}{d(x',y)^{s_1}2^{-s_1}} = \frac{2^{s_1}}{d(x',y)^{s_1}}\,.$$

If instead $s_1 < 0$, Lemma 3.2 implies that

$$\frac{1}{d(x'',y)^{s_1}} \leq \frac{1}{d(x',y)^{s_1}2^{s_1}} = \frac{2^{-s_1}}{d(x',y)^{s_1}}\,.$$

Hence, the validity of the inequality of statement (i) follows.

(ii) Since

$$|K_1(x,y)K_2(x,y)| \leq \frac{\|K_1\|_{\mathcal{K}_{s_1,X\times Y}}\|K_2\|_{\mathcal{K}_{t_1,X\times Y}}}{d(x,y)^{s_1}d(x,y)^{t_1}} \qquad \forall x,y \in X\times Y, x\neq y\,,$$

statement (ii) is an immediate consequence of the inequality of statement (i) with $s_3 = t_3$, $s_2 = s_1 + s_3$, $t_2 = t_1 + s_3$. □

Then we have the following product rule of a kernel and of a function of either $x \in X$ or $y \in Y$.

Proposition 3.1. *Let X and Y be subsets of a metric space (M,d). Let s_1, s_2, $s_3 \in \mathbb{R}$, $\alpha \in]0,1]$. Then the following statements hold.*

(i) If $K \in \mathcal{K}_{s_1,s_2,s_3}(X\times Y)$ and $f \in C^{0,\alpha}_b(X)$, then

$$|K(x,y)f(x)|\,d(x,y)^{s_1} \leq \|K\|_{\mathcal{K}_{s_1,X\times Y}} \sup_X |f| \quad \forall (x,y) \in (X\times Y)\setminus \mathbb{D}_{X\times Y}$$

and

$$\begin{aligned}|K(x',y)f(x') &- K(x'',y)f(x'')| \\ &\leq \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}\|f\|_{C^{0,\alpha}_b(X)}\left\{\frac{d(x',x'')^{s_3}}{d(x',y)^{s_2}} + 2^{|s_1|}\frac{d(x',x'')^{\alpha}}{d(x',y)^{s_1}}\right\}\end{aligned}$$

for all $x', x'' \in X$, $x' \neq x''$, $y \in Y \setminus B(x', 2d(x',x''))$.

(ii) If $s_2 \geq s_1$ and X and Y are both bounded, then the map from

$$\mathcal{K}_{s_1,s_2,s_3}(X\times Y)\times C^{0,s_3}_b(X) \quad \text{to} \quad \mathcal{K}_{s_1,s_2,s_3}(X\times Y)$$

that takes (K,f) to the kernel $K(x,y)f(x)$ of the variable $(x,y) \in (X\times Y)\setminus\mathbb{D}_{X\times Y}$ is bilinear and continuous.

(iii) The map from

$$\mathcal{K}_{s_1,s_2,s_3}(X\times Y)\times C^{0}_b(Y) \quad \text{to} \quad \mathcal{K}_{s_1,s_2,s_3}(X\times Y)$$

that takes (K,f) to the kernel $K(x,y)f(y)$ in the variable $(x,y) \in (X\times Y)\setminus\mathbb{D}_{X\times Y}$ is bilinear and continuous.

Proof. (i) The first inequality is an obvious consequence of the definition of the norm in $\mathcal{K}_{s_1,X\times Y}$. To prove the second one, we note that

$$\begin{aligned}&|K(x',y)f(x')-K(x'',y)f(x'')|\\&\qquad\le |K(x',y)-K(x'',y)|\,|f(x')|+|K(x'',y)|\,|f(x')-f(x'')|\\&\qquad\le \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}\|f\|_{C_b^{0,\alpha}(X)}\left\{\frac{d(x',x'')^{s_3}}{d(x',y)^{s_2}}+\frac{d(x',x'')^{\alpha}}{d(x'',y)^{s_1}}\right\}\end{aligned}$$

If $s_1\ge 0$, Lemma 3.2 implies that

$$\frac{d(x',x'')^{\alpha}}{d(x'',y)^{s_1}}\le\frac{d(x',x'')^{\alpha}}{d(x',y)^{s_1}2^{-s_1}}=2^{s_1}\frac{d(x',x'')^{\alpha}}{d(x',y)^{s_1}}\,.$$

If instead $s_1<0$, Lemma 3.2 implies that

$$\frac{d(x',x'')^{\alpha}}{d(x'',y)^{s_1}}\le\frac{d(x',x'')^{\alpha}}{d(x',y)^{s_1}2^{s_1}}=2^{-s_1}\frac{d(x',x'')^{\alpha}}{d(x',y)^{s_1}}\,.$$

Hence, the second inequality in statement (i) holds true. To prove (ii), it suffices to note that

$$\frac{d(x',x'')^{s_3}}{d(x',y)^{s_1}}=\frac{d(x',x'')^{s_3}d(x',y)^{s_2-s_1}}{d(x',y)^{s_1}d(x',y)^{s_2-s_1}}\le(\mathrm{diam}\,(X\cup Y))^{s_2-s_1}\frac{d(x',x'')^{s_3}}{d(x',y)^{s_2}}\,,$$

to apply the second inequality of statement (i) and to invoke the first inequality of statement (i). Statement (iii) is obvious. □

We also point out the validity of the following elementary remark that holds if both X and Y are bounded.

Remark 1. Let (M,d) be a metric space. Let X, Y be bounded subsets of M. Let s_1, s_2, $s_3\in[0,+\infty[$. If $a\in]0,+\infty[$, then Lemma 3.2 implies the validity of the following inequality

$$\begin{aligned}&\sup\biggl\{\frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}}|K(x',y)-K(x'',y)|:\\&\qquad\qquad x',x''\in X, a\le d(x',x''), y\in Y\setminus B(x',2d(x',x''))\biggr\}\\&\qquad\le\frac{(\mathrm{diam}\,(X\cup Y))^{s_2}}{a^{s_3}}\|K\|_{\mathcal{K}_{s_1,X\times Y}}\left((2a)^{-s_1}+(2^{-1}2a)^{-s_1}\right)\end{aligned}$$

for all $K\in\mathcal{K}_{s_1,X\times Y}$ and accordingly the norm on $\mathcal{K}_{s_1,s_2,s_3}(X\times Y)$ defined by setting

$$\begin{aligned}\|K\|_{a;\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}&\equiv\|K\|_{\mathcal{K}_{s_1,X\times Y}}+\sup\biggl\{\frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}}|K(x',y)-K(x'',y)|:\\&\qquad x',x''\in X, 0<d(x',x'')<a, y\in Y\setminus B(x',2d(x',x''))\biggr\}\end{aligned}$$

is equivalent to the norm $\|\cdot\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}$ on $\mathcal{K}_{s_1,s_2,s_3}(X\times Y)$.

Next we prove the following imbedding statement that holds for bounded sets.

Proposition 3.2. *Let (M,d) be a metric space. Let X, Y be bounded subsets of M. Let s_1, s_2, s_3, t_1, t_2, $t_3 \in \mathbb{R}$. Then the following statements hold.*

(i) If $t_1 \geq s_1$ then $\mathcal{K}_{s_1,X\times Y}$ is continuously embedded into $\mathcal{K}_{t_1,X\times Y}$.

(ii) If $t_1 \geq s_1$, $t_3 \leq s_3$ and $(t_2 - t_3) \geq (s_2 - s_3)$, then $\mathcal{K}_{s_1,s_2,s_3}(X\times Y)$ is continuously embedded into $\mathcal{K}_{t_1,t_2,t_3}(X\times Y)$.

(iii) If $t_1 \geq s_1$, $t_3 \leq s_3$, then $\mathcal{K}_{s_1,s_1+s_3,s_3}(X\times Y)$ is continuously embedded into $\mathcal{K}_{t_1,t_1+t_3,t_3}(X\times Y)$.

Proof. Statement (i) is an immediate consequence of the following elementary inequality

$$
\begin{aligned}
d(x,y)^{t_1}|K(x,y)| &\leq d(x,y)^{t_1-s_1} d(x,y)^{s_1}|K(x,y)| \\
&\leq (\mathrm{diam}\,(X\cup Y))^{t_1-s_1}\|K\|_{\mathcal{K}_{s_1,X\times Y}} \qquad \forall (x,y)\in X\times Y\setminus \mathbb{D}_{X\times Y}\,,
\end{aligned}
$$

which holds for all $K \in \mathcal{K}_{s_1,X\times Y}$. To prove (ii), it suffices to invoke (i) and to note that

$$
\begin{aligned}
&\frac{d(x',y)^{t_2}}{d(x',x'')^{t_3}}|K(x',y)-K(x'',y)| \\
&\quad = \frac{d(x',y)^{t_2-s_2}}{d(x',x'')^{t_3-s_3}}\frac{d(x',y)^{s_2}}{d(x',x'')^{s_3}}|K(x',y)-K(x'',y)| \\
&\quad \leq d(x',y)^{t_2-s_2} d(x',x'')^{s_3-t_3}\|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \\
&\quad \leq d(x',y)^{t_2-s_2}(2^{-1}d(x',y))^{s_3-t_3}\|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \\
&\quad \leq d(x',y)^{(t_2-t_3)-(s_2-s_3)}2^{t_3-s_3}\|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \\
&\quad \leq (\mathrm{diam}\,(X\cup Y))^{(t_2-t_3)-(s_2-s_3)}2^{t_3-s_3}\|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}
\end{aligned}
$$

for all $x', x'' \in X$, $x' \neq x''$, $y \in Y \setminus B(x', 2d(x',x''))$ and $K \in \mathcal{K}_{s_1,s_2,s_3}(X\times Y)$. Finally, statement (iii) is an immediate consequence of statement (ii). □

We now show that we can associate a potential type kernel to all Hölder continuous functions.

Lemma 3.3. *Let X and Y be subsets of a metric space (M,d). Let $\alpha \in]0,1]$. Let $C^{0,\alpha}(X\cup Y)$ be endowed with the Hölder seminorm $|\cdot : X\cup Y|_\alpha$. Then the following statements hold.*

(i) If $\mu \in C^{0,\alpha}(X\cup Y)$, then the map $\Xi[\mu]$ defined by

$$
\Xi[\mu](x,y) \equiv \mu(x) - \mu(y) \qquad \forall (x,y)\in (X\times Y)\setminus \mathbb{D}_{X\times Y} \tag{3.2}
$$

belongs to $\mathcal{K}_{-\alpha,0,\alpha}(X\times Y)$.

(ii) The operator Ξ from $C^{0,\alpha}(X\cup Y)$ to $\mathcal{K}_{-\alpha,0,\alpha}(X\times Y)$ that takes μ to $\Xi[\mu]$ is linear and continuous.

Proof. It suffices to observe that

$$|\mu(x)-\mu(y)| \le |\mu : X\cup Y|_\alpha d(x,y)^\alpha \qquad \forall (x,y)\in (X\times Y)\setminus \mathbb{D}_{X\times Y}$$

and that

$$|(\mu(x')-\mu(y))-(\mu(x'')-\mu(y))| = |\mu(x')-\mu(x'')| \le |\mu : X\cup Y|_\alpha \frac{d(x',x'')^\alpha}{d(x',y)^0}$$

for all $x', x'' \in X$, $x'\neq x''$, $y\in Y\setminus B(x', 2d(x',x''))$. □

Sometimes the kernel has a special form which we need later on. Thus we introduce the following preliminary lemma for standard kernels.

Lemma 3.4. *Let X and Y be subsets of a metric space (M,d). Let $s_1\in\mathbb{R}$, $s_3\in]-\infty,1]$, $\theta\in]0,1]$. Let $C^{0,\theta}(X\cup Y)$ be endowed with the Hölder seminorm $|\cdot : X\cup Y|_\theta$. Then the following statements hold.*

(i) The map H from $\mathcal{K}_{s_1,X\times Y}\times C^{0,\theta}(X\cup Y)$ to $\mathcal{K}_{s_1-\theta,X\times Y}$, which takes (Z,g) to the function from $(X\times Y)\setminus\mathbb{D}_{X\times Y}$ to $\mathbb{C}$ defined by

$$H[Z,g](x,y)\equiv (g(x)-g(y))Z(x,y) \qquad \forall (x,y)\in (X\times Y)\setminus \mathbb{D}_{X\times Y} \tag{3.3}$$

is bilinear and continuous.

(ii) The map H from

$$\mathcal{K}_{s_1,s_1+s_3,s_3}(X\times Y)\times C^{0,\theta}(X\cup Y) \quad \text{to} \quad \mathcal{K}_{s_1-\theta,s_1+s_3-1,s_3-(1-\theta)}(X\times Y),$$

which takes (Z,g) to the function defined by (3.3) is bilinear and continuous.

Proof. (i) It suffices to note that the Hölder continuity of g implies that

$$|H[Z,g](x,y)| \le \frac{|g : X\cup Y|_\theta}{d(x,y)^{s_1-\theta}}\|Z\|_{\mathcal{K}_{s_1,X\times Y}} \qquad \forall (x,y)\in (X\times Y)\setminus\mathbb{D}_{X\times Y}\,. \tag{3.4}$$

(ii) By Lemma 3.3, the linear operator from

$$\mathcal{K}_{s_1,s_1+s_3,s_3}(X\times Y)\times C^{0,\theta}(X\cup Y) \quad \text{to} \quad \mathcal{K}_{s_1,s_1+s_3,s_3}(X\times Y)\times \mathcal{K}_{-\theta,0,\theta}(X\times Y)$$

that takes (Z,g) to $(Z,\Xi[g])$ is linear and continuous. By the elementary embedding Lemma 3.1, the inclusion map from

$$\mathcal{K}_{s_1,s_1+s_3,s_3}(X\times Y)\times \mathcal{K}_{-\theta,0,\theta}(X\times Y)$$

to $\mathcal{K}_{s_1,s_1+s_3-(1-\theta),s_3-(1-\theta)}(X\times Y)\times\mathcal{K}_{-\theta,-(1-s_3),\theta-(1-s_3)}(X\times Y)$ is linear and continuous. Then the product Theorem 3.1 (ii) for standard kernels implies that the product is continuous from

$$\mathcal{K}_{s_1,s_1+s_3-(1-\theta),s_3-(1-\theta)}(X\times Y)\times\mathcal{K}_{-\theta,-(1-s_3),\theta-(1-s_3)}(X\times Y)$$

to $\mathcal{K}_{s_1-\theta,s_1+s_3-1,s_3-(1-\theta)}(X\times Y)$ and thus the proof is complete. □

4 Preliminaries on upper υ_Y-Ahlfors regular sets

We plan to consider integral operators in subsets X and Y of a metric space (M,d) when Y is endowed of a measure as follows.

$$\begin{aligned}&\text{Let } \mathcal{N} \text{ be a } \sigma\text{-algebra of parts of } Y\,, \mathcal{B}_Y \subseteq \mathcal{N}\,.\\ &\text{Let } \nu \text{ be measure on } \mathcal{N}\,.\\ &\text{Let } \nu(B(x,r)\cap Y) < +\infty \qquad \forall (x,r)\in X\times]0,+\infty[\,.\end{aligned} \tag{4.1}$$

Here $\mathcal{B}_Y$ denotes the σ-algebra of the Borel subsets of Y.

Definition 3. *Let X and Y be subsets of a metric space (M,d). Let $\upsilon_Y \in]0,+\infty[$. Let ν be as in (4.1). We say that Y is upper υ_Y-Ahlfors regular with respect to X provided that the following condition holds*

$$\begin{aligned}&\textit{there exist } r_{X,Y,\upsilon_Y} \in]0,+\infty]\,,\ c_{X,Y,\upsilon_Y} \in]0,+\infty[\textit{ such that}\\ &\nu(B(x,r)\cap Y) \le c_{X,Y,\upsilon_Y} r^{\upsilon_Y}\\ &\textit{for all } x\in X \textit{ and } r \in]0, r_{X,Y,\upsilon_Y}[\,.\end{aligned} \tag{4.2}$$

In case $X = Y$, we say that Y is upper υ_Y-Ahlfors regular.

One could show that if $n \in \mathbb{N}$, $n \ge 2$ and if Y is a compact embedded differential manifold in $\mathbb{R}^n$ of codimension 1, then Y is upper $(n-1)$-Ahlfors regular with respect to $\mathbb{R}^n$. Then one can prove the following basic inequalities for the integral on an upper Ahlfors regular set Y and on the intersection of Y with balls with center at a point x of X of the powers of $d(x,y)^{-1}$ with exponent $s \in]-\infty, \upsilon_Y[$, that are variants of those proved by Gatto [8, p. 104] in case $X = Y$ (for a proof see [15, Lem. 3.2, 3.4]).

Lemma 4.1. *Let X and Y be subsets of a metric space (M,d). Let $\upsilon_Y \in]0,+\infty[$. Let ν be as in (4.1). Let Y be upper υ_Y-Ahlfors regular with respect to X. Then the following statements hold.*

(i) $\nu(\{x\}) = 0$ *for all* $x \in X\cap Y$.

(ii) *Let $\nu(Y) < +\infty$. If $s \in]0,\upsilon_Y[$, then*

$$c'_{s,X,Y} \equiv \sup_{x\in X} \int_Y \frac{d\nu(y)}{d(x,y)^s} \le \nu(Y)a^{-s} + c_{X,Y,\upsilon_Y} \frac{\upsilon_Y}{\upsilon_Y - s} a^{\upsilon_Y - s}$$

for all $a \in]0, r_{X,Y,\upsilon_Y}[$. If $s = 0$, then

$$c'_{0,X,Y} \equiv \sup_{x\in X} \int_Y \frac{d\nu(y)}{d(x,y)^0} = \nu(Y)\,.$$

(iii) *Let $\nu(Y) < +\infty$ whenever $r_{X,Y,\upsilon_Y} < +\infty$. If $s \in]-\infty, \upsilon_Y[$, then*

$$c''_{s,X,Y} \equiv \sup_{(x,t)\in X\times]0,+\infty[} t^{s-\upsilon_Y} \int_{B(x,t)\cap Y} \frac{d\nu(y)}{d(x,y)^s} < +\infty\,.$$

By the Hölder inequality one can prove the following statement of Hille-Tamarkin (see [15, Prop. 4.1]).

Proposition 4.1. *Let X and Y be subsets of a metric space (M,d). Let $\upsilon_Y \in]0,+\infty[$, $s \in [0,\upsilon_Y[$. Let ν be as in (4.1). Let $\nu(Y) < +\infty$. Let Y be upper υ_Y-Ahlfors regular with respect to X. Then the following statements hold.*

(i) If $(K,\varphi) \in \mathcal{K}_{s,X\times Y} \times L^\infty_\nu(Y)$, then the function $K(x,\cdot)\varphi(\cdot)$ is integrable in Y for all $x \in X$ and the function $A[K,\varphi]$ defined by

$$A[K,\varphi](x) \equiv \int_Y K(x,y)\varphi(y)\,d\nu(y) \qquad \forall x \in X \tag{4.3}$$

is bounded.

(ii) The bilinear map from $\mathcal{K}_{s,X\times Y} \times L^\infty_\nu(Y)$ to $B(X)$, which takes (K,φ) to $A[K,\varphi]$ is continuous and the following inequality holds

$$\sup_X |A[K,\varphi]| \leq c'_{s,X,Y} \|K\|_{\mathcal{K}_{s,X\times Y}} \|\varphi\|_{L^\infty_\nu(Y)} \tag{4.4}$$

for all $(K,\varphi) \in \mathcal{K}_{s,X\times Y} \times L^\infty_\nu(Y)$ (see Lemma 4.1 (ii) for $c'_{s,X,Y}$).

Under the assumptions of the previous proposition, one can actually prove that the function $A[K,\varphi]$ is continuous. To do so, we first introduce the following result for potential type operators.

Proposition 4.2. *Let X and Y be subsets of a metric space (M,d). Let ν be as in (4.1). Let $\nu(Y) < +\infty$. Let $s \in \mathbb{R}$. Let $K \in \mathcal{K}_{s,X\times Y}$. Let $d(x,\cdot)^{-s}$ belong to $L^1_\nu(Y\setminus\{x\})$ for all $x \in X$. Let*

$$\sup_{x\in X} \int_{Y\setminus\{x\}} d(x,y)^{-s}\,d\nu(y) < +\infty\,. \tag{4.5}$$

If $\nu(\{x\}) = 0$ for all $x \in X \cap Y$ and if for each $\epsilon \in]0,+\infty[$ there exists $\delta \in]0,+\infty[$ such that

$$\sup_{x\in X} \int_{F\setminus\{x\}} d(x,y)^{-s}\,d\nu(y) \leq \epsilon \qquad \text{if } F \in \mathcal{N},\ \nu(F) \leq \delta\,, \tag{4.6}$$

and if $\varphi \in L^\infty_\nu(Y)$, then the function $A[K,\varphi]$ from X to $\mathbb{C}$ defined by (4.3) is continuous.

Proof. Let $\tilde{x} \in X$. It suffices to show that if $\{x_j\}_{j\in\mathbb{N}}$ is a sequence in X which converges to $\tilde{x}$, then

$$\lim_{j\to\infty} \int_Y K(x_j,y)\varphi(y)\,d\nu(y) = \int_Y K(\tilde{x},y)\varphi(y)\,d\nu(y)\,.$$

We now turn to prove such a limiting relation by exploiting the Vitali Convergence Theorem. To do so, we prove the validity of the following two statements.

(j) There exists $N_{\tilde{x}} \in \mathcal{N}$ such that $\nu(N_{\tilde{x}}) = 0$ and

$$\lim_{j\to\infty} K(x_j,y)\varphi(y) = K(\tilde{x},y)\varphi(y) \qquad \forall y \in Y \setminus N_{\tilde{x}}\,.$$

(jj) For each $\epsilon \in]0,+\infty[$, there exists $\delta \in]0,+\infty[$ such that

$$\sup_{j\in\mathbb{N}} \int_F |K(x_j,y)\varphi(y)|\,d\nu(y) \leq \epsilon \qquad \text{if } F \in \mathcal{N}\,, \nu(F) \leq \delta\,.$$

Since $\nu(\{\tilde{x}\} \cap Y) = 0$, we can take $N_{\tilde{x}} \equiv \{\tilde{x}\} \cap Y$ and statement (j) follows by our continuity assumption on K that follows by the membership of K in $\mathcal{K}_{s,X\times Y}$. We now turn to prove (jj). By our assumptions on K, we have

$$\int_F |K(x_j,y)\varphi(y)|\; d\nu(y) \leq \|K\|_{\mathcal{K}_{s,X\times Y}} \int_F d(x_j,y)^{-s}\,d\nu(y) \|\varphi\|_{L^\infty_\nu(F)}$$

for all $j \in \mathbb{N}$. Thus it suffices to choose $\delta \in]0,+\infty[$ such that

$$\sup_{x\in X} \int_F d(x,y)^{-s}\,d\nu(y) \leq \epsilon(1+\|K\|_{\mathcal{K}_{s,X\times Y}}\|\varphi\|_{L^\infty_\nu(Y)})^{-1} \;\text{ if } F \in \mathcal{N},\;\nu(F) \leq \delta\,,$$

and statement (jj) holds true and the proof is complete. □

In order to apply Proposition 4.2 in case Y is upper Ahlfors regular, we need to prove the following lemma.

Lemma 4.2. *Let X and Y be subsets of a metric space (M,d). Let $\upsilon_Y \in]0,+\infty[$, $s \in [0,\upsilon_Y[$. Let ν be as in (4.1). Let $\nu(Y) < +\infty$. Let Y be upper υ_Y-Ahlfors regular with respect to X. Then for each $\epsilon \in]0,+\infty[$ there exists $\delta \in]0,+\infty[$ such that*

$$\sup_{x\in X} \int_F d(x,y)^{-s}\,d\nu(y) \leq \epsilon \qquad \textit{if } F \in \mathcal{N},\; \nu(F) \leq \delta\,, \tag{4.7}$$

Proof. We first note that if $F \in \mathcal{N}$, then F is a subset of Y. Accordingly F is also upper υ_Y-Ahlfors regular with respect to X and we can choose $r_{X,F,\upsilon_Y} = r_{X,Y,\upsilon_Y}$, $c_{X,F,\upsilon_Y} = c_{X,Y,\upsilon_Y}$. If $s > 0$, then Lemma 4.1 (ii) implies that

$$\sup_{x\in X} \int_F d(x,y)^{-s}\,d\nu(y) \leq \nu(F)a^{-s} + c_{X,Y,\upsilon_Y}\frac{\upsilon_Y}{\upsilon_Y - s}a^{\upsilon_Y - s} \qquad \forall a \in]0, r_{X,Y,\upsilon_Y}[\,.$$

Thus if $\epsilon \in]0,+\infty[$, then we choose $a_\epsilon \in]0, r_{X,Y,\upsilon_Y}[$ such that

$$c_{X,Y,\upsilon_Y}\frac{\upsilon_Y}{\upsilon_Y - s}a_\epsilon^{\upsilon_Y - s} < \frac{\epsilon}{2}$$

and we can set $\delta \equiv \frac{\epsilon}{2}a_\epsilon^s$. Then we have

$$\sup_{x\in X} \int_F d(x,y)^{-s}\,d\nu(y) \leq \delta a_\epsilon^{-s} + \frac{\epsilon}{2} = \epsilon$$

whenever $F \in \mathcal{N}$ and $\nu(F) \leq \delta$. If instead $s = 0$, then condition (4.7) holds trivially with $\delta = \epsilon$. □

Proposition 4.3. *Let X and Y be subsets of a metric space (M,d). Let ν be as in (4.1). Let ν be finite. Let $s \in [0, \upsilon_Y[$. Let Y be upper υ_Y-Ahlfors regular with respect to X. If $(K,\varphi) \in \mathcal{K}_{s,X\times Y} \times L^\infty_\nu(Y)$, then the function $A[K,\varphi]$ from X to $\mathbb{C}$ defined by (4.3) is continuous.*

Proof. We plan to deduce the continuity of $A[K,\varphi]$ by the continuity Proposition 4.2. To do so, it suffices to note that Lemma 4.1 (i), (ii) imply that $\nu(\{x\}) = 0$ for all $x \in X \cap Y$ and that condition (4.5) is satisfied. Moreover, Lemma 4.2 implies that condition (4.6) is satisfied. □

Next we plan to introduce a result on the integral operator

$$Q[Z,g,1](x) \equiv \int_Y Z(x,y)(g(x)-g(y))\,d\nu(y) \qquad \forall x \in X\,. \tag{4.8}$$

when Z belongs to a class $\mathcal{K}_{s_1,s_2,s_3}(X \times Y)$ as in Definition 2 and g is a $\mathbb{C}$-valued function in $X \cup Y$. We exploit the operator in (4.8) in the next section and we note that operators as in (4.8) appear in the applications (cf. *e.g.*, Colton and Kress [3, p. 56], and Dondi and the author [5, §8]). In order to estimate the Hölder quotient of $Q[Z,g,1]$, we need to introduce a further norm for kernels.

Definition 4. *Let X and Y be subsets of a metric space (M,d). Let ν be as in (4.1). Let s_1, s_2, $s_3 \in \mathbb{R}$. We set*

$$\begin{aligned}\mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y) \equiv \Big\{ & K \in \mathcal{K}_{s_1,s_2,s_3}(X\times Y) : \\ & K(x,\cdot) \text{ is } \nu-\text{integrable in } Y\setminus B(x,r) \text{ for all } (x,r)\in X\times]0,+\infty[\,, \\ & \sup_{x\in X}\sup_{r\in]0,+\infty[}\left|\int_{Y\setminus B(x,r)} K(x,y)\,d\nu(y)\right| < +\infty \Big\}\end{aligned}$$

and

$$\begin{aligned}\|K\|_{\mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y)} &\equiv \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \\ &+\sup_{x\in X}\sup_{r\in]0,+\infty[}\left|\int_{Y\setminus B(x,r)} K(x,y)\,d\nu(y)\right| \quad \forall K \in \mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y)\,.\end{aligned}$$

Clearly, $(\mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y), \|\cdot\|_{\mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y)})$ is a normed space. By definition, the space $\mathcal{K}^\sharp_{s_1,s_2,s_3}(X\times Y)$ is continuously embedded into the space $\mathcal{K}_{s_1,s_2,s_3}(X\times Y)$. Then we consider a stronger version of the upper Ahlfors regularity. Namely, we assume that Y is strongly upper υ_Y-Ahlfors regular with respect to X, *i.e.*, that

$$\begin{aligned}&\text{there exist } r_{X,Y,\upsilon_Y} \in]0,+\infty]\,,\ c_{X,Y,\upsilon_Y} \in]0,+\infty[\text{ such that} \\ &\nu((B(x,r_2)\setminus B(x,r_1))\cap Y) \le c_{X,Y,\upsilon_Y}(r_2^{\upsilon_Y} - r_1^{\upsilon_Y}) \\ &\text{for all } x\in X \text{ and } r_1, r_2 \in [0, r_{X,Y,\upsilon_Y}[\text{ with } r_1 < r_2\,,\end{aligned} \tag{4.9}$$

where we understand that $B(x,0) \equiv \emptyset$ (in case $X = Y$, we just say that Y is strongly upper υ_Y-Ahlfors regular). So for example if Y a compact manifold of class C^1 that is imbedded in $M = \mathbb{R}^n$, then Y can be proved to be strongly upper $(n-1)$-Ahlfors regular with respect to Y. Next we introduce a function that we need for a generalized Hölder norm. For each $\theta \in]0,1]$, we define the function $\omega_\theta(\cdot)$ from $[0,+\infty[$ to itself by setting

$$\omega_\theta(r) \equiv \begin{cases} 0 & r = 0\,, \\ r^\theta |\ln r| & r \in]0, r_\theta]\,, \\ r_\theta^\theta |\ln r_\theta| & r \in]r_\theta, +\infty[\,, \end{cases}$$

where $r_\theta \equiv e^{-1/\theta}$ for all $\theta \in]0,1]$. Obviously, $\omega_\theta(\cdot)$ is concave and satisfies condition (2.2). We also note that if $\mathbb{D} \subseteq M$, then the continuous embedding

$$C_b^{0,\theta}(\mathbb{D}) \subseteq C_b^{0,\omega_\theta(\cdot)}(\mathbb{D}) \subseteq C_b^{0,\theta'}(\mathbb{D})$$

holds for all $\theta' \in]0,\theta[$ (cf. Section 2). We are now ready to state the following statement of [15, Prop. 6.3] on the Hölder continuity of $Q[Z,g,1]$, where we understand that $C^{0,\beta}(X \cup Y)$ is endowed with the semi-norm $|\cdot : X \cup Y|_\beta$.

Proposition 4.4. *Let X and Y be subsets of a metric space (M,d). Let*

$$\upsilon_Y \in]0,+\infty[\,,\ \beta \in]0,1]\,,\ s_1 \in [\beta, \upsilon_Y + \beta[\,,\ s_2 \in [\beta, +\infty[\,,\ s_3 \in]0,1]\,.$$

Let ν be as in (4.1), $\nu(Y) < +\infty$. Then the following statements hold.

(i) If $s_1 < \upsilon_Y$, then the following statements hold.

(a) If $s_2 - \beta > \upsilon_Y$, $s_2 < \upsilon_Y + \beta + s_3$ and Y is upper υ_Y-Ahlfors regular with respect to X, then the bilinear map from

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\min\{\beta,\upsilon_Y+s_3+\beta-s_2\}}(X)\,,$$

which takes (Z,g) to $Q[Z,g,1]$ is continuous.

(aa) If $s_2 - \beta = \upsilon_Y$ and Y is strongly upper υ_Y-Ahlfors regular with respect to X, then the bilinear map from

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\max\{r^\beta,\omega_{s_3}(r)\}}(X)\,,$$

which takes (Z,g) to $Q[Z,g,1]$ is continuous.

(ii) If $s_1 = \upsilon_Y$, then the following statements hold.

(b) If $s_2 - \beta > \upsilon_Y$, $s_2 < \upsilon_Y + \beta + s_3$ and Y is upper υ_Y-Ahlfors regular with respect to X, then the bilinear map from

$$\mathcal{K}^\sharp_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\min\{\beta,\upsilon_Y+s_3+\beta-s_2\}}(X)\,,$$

which takes (Z,g) to $Q[Z,g,1]$ is continuous.

(bb) If $s_2 - \beta = \upsilon_Y$ and Y is strongly upper υ_Y-Ahlfors regular with respect to X, then the bilinear map from

$$\mathcal{K}^\sharp_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad \text{to} \quad C_b^{0,\max\{r^\beta,\omega_{s_3}(r)\}}(X)\,,$$

which takes (Z,g) to $Q[Z,g,1]$ is continuous.

(iii) If $s_1 > \upsilon_Y$, then the following statements hold.

(c) If $s_2 - \beta > \upsilon_Y$, $s_2 < \upsilon_Y + \beta + s_3$ and Y is upper υ_Y-Ahlfors regular with respect to X, then the bilinear map from

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad to \quad C_b^{0,\min\{\upsilon_Y+\beta-s_1,\upsilon_Y+s_3+\beta-s_2\}}(X)\,,$$

which takes (Z, g) to $Q[Z, g, 1]$ is continuous.

(cc) If $s_2 - \beta = \upsilon_Y$ and Y is strongly upper υ_Y-Ahlfors regular with respect to X, then the bilinear map from

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C^{0,\beta}(X \cup Y) \quad to \quad C_b^{0,\max\{r^{\upsilon_Y+\beta-s_1},\omega_{s_3}(r)\}}(X)\,,$$

which takes (Z, g) to $Q[Z, g, 1]$ is continuous.

5 A differentiation theorem for integral operators on upper Ahlfors regular subsets of $\mathbb{R}^n$

We first introduce some preliminaries.

Definition 5. *Let $n \in \mathbb{N} \setminus \{0\}$. Let X be a subset of $\mathbb{R}^n$, $p \in X$. We say that the vector $w \in \mathbb{R}^n$ is semi-tangent to X at the point p provided that either $w = 0$ or there exists a sequence $\{x_j\}_{j\in\mathbb{N}}$ in $X \setminus \{p\}$ which converges to p and such that*

$$\frac{w}{|w|} = \lim_{j\to\infty} \frac{x_j - p}{|x_j - p|}\,.$$

We say that the vector $w \in \mathbb{R}^n$ is tangent to X at the point p provided that both w and $-w$ are semi-tangent to X at the point p.

We denote by T_pX the set of all semi-tangent vectors to X at p. One can easily check that T_pX is a cone of $\mathbb{R}^n$, i.e., that

$$\lambda w \in T_pX \qquad \text{whenever} \quad (\lambda, w) \in]0, +\infty[\times T_pX\,.$$

We say that T_pX is the cone of semi-tangent vectors to X at p. If T_pX is also a subspace of $\mathbb{R}^n$, then we say that X has a tangent space at p, that T_pX is the tangent space to X at p and that $p + T_pX$ is the affine tangent space to X at p. Next we state the definition of directional derivative for a function defined on an arbitrary subset of $\mathbb{R}^n$.

Definition 6. *Let Z be a real or complex normed space. Let X be a subset of $\mathbb{R}^n$. Let ϕ be a function from X to Z. Let $p \in X$, $v \in T_pX$, $|v| = 1$.*

We say that ϕ has a derivative in p with respect to the direction v provided that there exists an element $D_{X,v}\varphi(p) \in Z$ such that

$$D_{X,v}\varphi(p) = \lim_{j\to\infty} \frac{\phi(x_j) - \phi(p)}{|x_j - p|} \qquad in\ Z$$

for all sequences $\{x_j\}_{j\in\mathbb{N}}$ *in* $X\setminus\{p\}$ *which converge to* p *and such that*

$$v=\lim_{j\to\infty}\frac{x_j-p}{|x_j-p|}\,.$$

Then we say that $D_{X,v}\varphi(p)$ *is the derivative of* ϕ *in* p *with respect to the direction* v.

We note that if there exist an open neighborhood W of p in $\mathbb{R}^n$ and if $\tilde{\phi}$ is a (real) continuously differentiable function from W to Z and satisfies the equality $\tilde{\phi}_{|X\cap W}=\phi_{|X\cap W}$, then ϕ has a derivative in p with respect to the direction v and

$$D_{X,v}\varphi(p)=D_v\tilde{\phi}(p)=d\tilde{\phi}(p)[v]\,.$$

Indeed, $d\tilde{\phi}(p)[v]=\lim_{j\to\infty}d\tilde{\phi}(p)\left[\frac{x_j-p}{|x_j-p|}\right]$ in Z and

$$\begin{aligned}0=\lim_{j\to\infty}&\frac{\|\tilde{\phi}(x_j)-\tilde{\phi}(p)-d\tilde{\phi}(p)[x_j-p]\|_Z}{|x_j-p|}\\&=\lim_{j\to\infty}\left\|\frac{\tilde{\phi}(x_j)-\tilde{\phi}(p)}{|x_j-p|}-d\tilde{\phi}(p)\left[\frac{x_j-p}{|x_j-p|}\right]\right\|_Z\\&=\lim_{j\to\infty}\left\|\frac{\phi(x_j)-\phi(p)}{|x_j-p|}-d\tilde{\phi}(p)\left[\frac{x_j-p}{|x_j-p|}\right]\right\|_Z\end{aligned}$$

and accordingly

$$\lim_{j\to\infty}\frac{\phi(x_j)-\phi(p)}{|x_j-p|}=\lim_{j\to\infty}d\tilde{\phi}(p)\left[\frac{x_j-p}{|x_j-p|}\right]=d\tilde{\phi}(p)[v]\qquad\text{in }Z$$

for all sequences $\{x_j\}_{j\in\mathbb{N}}$ as in Definition 5 of semi-tangent vector. Then we can prove the following differentiation theorem for integral operators that are defined on upper Ahlfors regular subsets of $\mathbb{R}^n$. To do so, we set

$$\mathbb{B}_n(x,\rho)\equiv\{y\in\mathbb{R}^n:\,|x-y|<\rho\}$$

For all $\rho>0$, $x\in\mathbb{R}^n$.

Theorem 5.1. *Let* X, $Y\subseteq\mathbb{R}^n$. *Let* $\upsilon_Y\in]0,+\infty[$. *Let* $(Y,\mathcal{N},\nu)$ *be a measured space such that* $\mathcal{B}_Y\subseteq\mathcal{N}$. *Let* ν *be finite. Let* Y *be upper* υ_Y*-Ahlfors regular with respect to* X. *Let* $s_1\in[0,\upsilon_Y[$. *Let* $x\in X$, $v\in T_xX$, $|v|=1$. *Let the kernel* $K\in\mathcal{K}_{s_1,s_1+1,1}(X\times Y)$ *satisfy the following assumptions*

$$\begin{aligned}&D_{X,v}K(x,y)\ \text{exists in }\mathbb{C}\qquad\forall y\in Y\setminus\{x\}\,,\\&D_{X,v}\int_Y K(x,y)\,d\nu(y)\ \text{exists in }\mathbb{C}\,.\end{aligned}$$

Let $\mu\in C^1_b(\mathbb{R}^n)$. *Then the function* $\int_Y K(\cdot,y)\mu(y)\,d\nu(y)$ *admits a derivative with respect to* v *at the point* x, *the function* $D_{X,v}K(x,y)(\mu(y)-\mu(x))$ *is* ν*-integrable in the variable* $y\in Y$ *and the following formula holds*

$$\begin{aligned}D_{X,v}&\int_Y K(x,y)\mu(y)\,d\nu(y)\\&=\int_Y[D_{X,v}K(x,y)](\mu(y)-\mu(x))\,d\nu(y)+\mu(x)D_{X,v}\int_Y K(x,y)\,d\nu(y)\end{aligned}\tag{5.1}$$

(see Definition 6 for $D_{X,v}$*).*

Proof. By the existence of $D_{X,v}\int_Y K(x,y)\,d\nu(y)$ and by the elementary equality

$$\int_Y K(x,y)\mu(y)\,d\nu(y) = \int_Y K(x,y)(\mu(y)-\mu(x))\,d\nu(y) + \mu(x)\int_Y K(x,y)\,d\nu(y)$$

(cf. Proposition 4.1), the existence of $D_{X,v}\int_Y K(x,y)\mu(y)\,d\nu(y)$ is equivalent to the existence of $D_{X,v}\int_Y K(x,y)(\mu(y)-\mu(x))\,d\nu(y)$ and in case of existence, we have

$$D_{X,v}\int_Y K(x,y)\mu(y)\,d\nu(y) = D_{X,v}\int_Y K(x,y)(\mu(y)-\mu(x))\,d\nu(y) + D_{X,v}\mu(x)\int_Y K(x,y)\,d\nu(y) + \mu(x)D_{X,v}\int_Y K(x,y)\,d\nu(y)\,. \tag{5.2}$$

We now turn to show the existence of

$$D_{X,v}\int_Y K(x,y)(\mu(y)-\mu(x))\,d\nu(y) \tag{5.3}$$

and to compute it. Let $\{x_j\}_{j\in\mathbb{N}}$ be a sequence in $X\setminus\{x\}$ such that

$$\lim_{j\to\infty} x_j = x\,, \qquad v = \lim_{j\to\infty}\frac{x_j - x}{|x_j - x|}\,.$$

By the existence of $D_{X,v}K(x,y)$, $D_{X,v}\mu(x)$ and by the continuity of $K(\cdot,y)$ and μ at x, we have

$$\lim_{j\to\infty}\frac{1}{|x_j-x|}\left[K(x_j,y)(\mu(y)-\mu(x_j)) - K(x,y)(\mu(y)-\mu(x))\right] = D_{X,v}K(x,y)(\mu(y)-\mu(x)) - K(x,y)D_{X,v}\mu(x) \qquad \forall y\in Y\setminus\{x\}\,. \tag{5.4}$$

We now turn to show the existence of the limit associated to the directional derivative of the integral in (5.3) by applying the Vitali Convergence Theorem. If $E\in\mathcal{N}$, the

Lipschitz continuity of μ, Lemma 3.4 with $\theta = 1$ and Lemma 4.1 imply that

$$
\begin{aligned}
&\int_E \frac{1}{|x_j - x|} \left| K(x_j, y)(\mu(y) - \mu(x_j)) - K(x,y)(\mu(y) - \mu(x)) \right| d\nu(y) \\
&\quad \leq \int_{E \cap \mathbb{B}_n(x, 2|x_j - x|)} \frac{1}{|x_j - x|} \left| K(x_j, y)(\mu(y) - \mu(x_j)) \right| \\
&\qquad + \int_{E \cap \mathbb{B}_n(x, 2|x_j - x|)} \frac{1}{|x_j - x|} \left| K(x, y)(\mu(y) - \mu(x)) \right| \\
&\qquad + \int_{E \setminus \mathbb{B}_n(x, 2|x_j - x|)} \frac{1}{|x_j - x|} \Big| K(x_j, y)(\mu(y) - \mu(x_j)) \\
&\qquad\quad - K(x,y)(\mu(y) - \mu(x)) \Big| \, d\nu(y) \\
&\quad \leq \|H[K,\mu]\|_{\mathcal{K}_{s_1 - 1, s_1 + 1 - 1, 1 - (1-1)}} \Bigg\{ \int_{E \cap \mathbb{B}_n(x_j, 3|x_j - x|)} \frac{1}{|x_j - x|} \frac{d\nu(y)}{|x_j - y|^{s_1 - 1}} \\
&\qquad + \int_{E \cap \mathbb{B}_n(x, 2|x_j - x|)} \frac{1}{|x_j - x|} \frac{d\nu(y)}{|x - y|^{s_1 - 1}} \\
&\qquad + \int_{E \setminus \mathbb{B}_n(x, 2|x_j - x|)} \frac{1}{|x_j - x|} \frac{|x_j - x|^1}{|x - y|^{s_1}} \, d\nu(y) \Bigg\} \\
&\quad \leq \|H[K,\mu]\|_{\mathcal{K}_{s_1 - 1, s_1, 1}} \Bigg\{ 3^{\upsilon_Y - (s_1 - 1)} c''_{s_1 - 1, X, Y} |x_j - x|^{\upsilon_Y - s_1} \\
&\qquad + 2^{\upsilon_Y - (s_1 - 1)} c''_{s_1 - 1, X, Y} |x_j - x|^{\upsilon_Y - s_1} \\
&\qquad + \nu(E) a^{-s_1} + c_{X,Y,\upsilon_Y} \frac{\upsilon_Y}{\upsilon_Y - s_1} a^{\upsilon_Y - s_1} \Bigg\}
\end{aligned}
$$

for all $a \in]0, r_{X,Y,\upsilon_Y}[$ and $j \in \mathbb{N}$, where the last addendum in the braces is absent if $s_1 = 0$. Now let $\epsilon \in]0, +\infty[$. Then we choose $a \in]0, r_{X,Y,\upsilon_Y}[$ such that

$$
\|H[K,\mu]\|_{\mathcal{K}_{s_1 - 1, s_1, 1}} c_{X,Y,\upsilon_Y} \frac{\upsilon_Y}{\upsilon_Y - s_1} a^{\upsilon_Y - s_1} \leq \epsilon/3
$$

and $j_\epsilon \in \mathbb{N}$ such that

$$
\begin{aligned}
\|H[K,\mu]\|_{\mathcal{K}_{s_1 - 1, s_1, 1}} \Bigg\{ & 3^{\upsilon_Y - (s_1 - 1)} c''_{s_1 - 1, X, Y} |x_j - x|^{\upsilon_Y - s_1} \\
& + 2^{\upsilon_Y - (s_1 - 1)} c''_{s_1 - 1, X, Y} |x_j - x|^{\upsilon_Y - s_1} \Bigg\} \leq \epsilon/3
\end{aligned}
$$

for all $j \in \mathbb{N}$ such that $j \geq j_\epsilon$. Thus if $E \in \mathcal{N}$ satisfies the inequality

$$
\|H[K,\mu]\|_{\mathcal{K}_{s_1 - 1, s_1, 1}} \nu(E) a^{-s_1} \leq \epsilon/3
$$

we have

$$
\int_E \frac{1}{|x_j - x|} \left| K(x_j, y)(\mu(y) - \mu(x_j)) - K(x,y)(\mu(y) - \mu(x)) \right| d\nu(y) \leq \epsilon
$$

forall $j \in \mathbb{N}$ such that $j \geq j_\epsilon$. Then the pointwise convergence of (5.4) and the Vitali Convergence Theorem implies that the pointwise limit of (5.4) is integrable in $y \in Y \setminus \{x\}$ and that

$$\lim_{j\to\infty} \int_Y \frac{1}{|x_j - x|} \left[K(x_j, y)(\mu(y) - \mu(x_j)) - K(x, y)(\mu(y) - \mu(x))\right] d\nu(y)$$
$$= \int_Y D_{X,v}K(x, y)(\mu(y) - \mu(x)) - K(x, y)D_{X,v}\mu(x)\, d\nu(y)\,. \qquad (5.5)$$

By our assumptions, $K(x, y)$ is integrable in $y \in Y \setminus \{x\}$ and $D_{X,v}\mu$ is bounded. Hence, $D_{X,v}K(x, y)(\mu(y) - \mu(x))$ is integrable in $y \in Y \setminus \{x\}$ and the right hand side of (5.5) equals

$$\int_Y D_{X,v}K(x, y)(\mu(y) - \mu(x))\, d\nu(y) - \int_Y K(x, y)D_{X,v}\mu(x)\, d\nu(y)\,.$$

Hence,

$$D_{X,v} \int_Y K(x, y)(\mu(y) - \mu(x))\, d\nu(y)$$
$$= \int_Y D_{X,v}K(x, y)(\mu(y) - \mu(x))\, d\nu(y) - \int_Y K(x, y)\, d\nu(y)D_{X,v}\mu(x)$$

and formula (5.2) implies the validity of the formula of the statement. □

6 Tangential derivatives of weakly singular integral operators on imbedded manifolds of $\mathbb{R}^n$ whose kernels have singular derivatives

Since a compact manifold Y of class C^1 that is imbedded in $\mathbb{R}^n$ can be proved to be $(n-1)$-upper Ahlfors regular and each C^1 function on Y can be extended to a C^1 function in $\mathbb{R}^n$ with compact support (cf. *e.g.*, proof of Theorem 2.85 of Dalla Riva, the author and Musolino [4]), the differentiation Theorem 5.1 implies the validity of the following theorem, which is a variant of a known result. For the definition of tangential gradient grad_Y, we refer *e.g.*, to Kirsch and Hettlich [11, A.5], Chavel [2, Chap. 1].

Theorem 6.1. *Let $n \in \mathbb{N}$, $n \geq 2$. Let Y be a compact manifold of class C^1 that is imbedded in $\mathbb{R}^n$. Let $s_1 \in [0, (n-1)[$. Let the kernel $K \in \mathcal{K}_{s_1,s_1+1,1}(Y \times Y)$ satisfy the following assumptions*

$$K(\cdot, y) \in C^1(Y \setminus \{y\}) \qquad \forall y \in Y\,, \qquad \int_Y K(\cdot, y)\, d\sigma_y \in C^1(Y)\,.$$

Let $\mathrm{grad}_{Y,x}K(\cdot, \cdot)$ denote the tangential gradient of $K(\cdot, \cdot)$ with respect to the first variable. Let $\mu \in C^1(Y)$. Then the function $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ is of class $C^1(Y)$, the *function* $[\mathrm{grad}_{Y,x}K(x,y)](\mu(y)-\mu(x))$ *is integrable in the variable* $y \in Y$ *for all* $x \in Y$ *and the following formula holds for the tangential gradient of* $\int_Y K(\cdot,y)\mu(y)\,d\nu(y)$

$$\mathrm{grad}_Y \int_Y K(x,y)\mu(y)\,d\sigma_y \tag{6.1}$$
$$= \int_Y [\mathrm{grad}_{Y,x}K(x,y)](\mu(y)-\mu(x))\,d\sigma_y + \mu(x)\mathrm{grad}_Y \int_Y K(x,y)\,d\sigma_y\,,$$

for all $x \in Y$.

Next we prove formula (6.1) for the tangential gradient under weaker assumptions for μ. To do so, however we must strenghten our assumptions on the kernel.

Theorem 6.2. *Let* $n \in \mathbb{N}$, $n \geq 2$. *Let* Y *be a compact manifold of class* C^1 *that is imbedded in* $\mathbb{R}^n$. *Let* $s_1 \in [0,(n-1)[$. *Let* $\beta \in]0,1]$, $t_1 \in]0,(n-1)+\beta[$. *Let the kernel* $K \in \mathcal{K}_{s_1,s_1+1,1}(Y \times Y)$ *satisfy the following assumptions*

$$K(\cdot,y) \in C^1(Y \setminus \{y\}) \qquad \forall y \in Y\,, \qquad \int_Y K(\cdot,y)\,d\sigma_y \in C^1(Y)\,,$$
$$\mathrm{grad}_{Y,x}K(\cdot,\cdot) \in (\mathcal{K}_{t_1,Y\times Y})^n\,,$$

where $\mathrm{grad}_{Y,x}K(\cdot,\cdot)$ *denotes the tangential gradient of* $K(\cdot,\cdot)$ *with respect to the first variable. Let* $\mu \in C^{0,\beta}_b(Y)$. *Then the function* $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ *is of class* $C^1(Y)$, *the function* $[\mathrm{grad}_{Y,x}K(x,y)](\mu(y)-\mu(x))$ *is integrable in* $y \in Y$ *for all* $x \in Y$ *and formula (6.1) for the tangential gradient of* $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ *holds true.*

Proof. We plan to prove the statement by approximating μ by functions of class $C^1(Y)$ for which we know that the statement is true by Theorem 6.1. By the Mc Shane extension Theorem, there exists $\tilde{\mu} \in C^{0,\beta}_b(\mathbb{R}^n)$ that extends μ (cf. *e.g.*, Mc Shane [16], Björk [1, Prop. 1] Kufner, John and Fučik [12, Thm. 1.8.3]). Possibly multiplying $\tilde{\mu}$ by a function of class $C^\infty_c(\mathbb{R}^n)$, we can assume that $\tilde{\mu}$ has a compact support. Next we wish to approximate $\tilde{\mu}$ by functions of class $C^\infty_c(\mathbb{R}^n)$ by means of a standard family of mollifiers $\{\eta_\epsilon\}_{\epsilon \in]0,+\infty[}$ with

$$\mathrm{supp}\,\eta_\epsilon \subseteq \overline{\mathbb{B}_n(0,\epsilon)}\,, \qquad \eta_\epsilon \geq 0\,, \qquad \int_{\mathbb{R}^n} \eta_\epsilon\,dx = 1 \qquad \forall \epsilon \in]0,+\infty[$$

(cf. *e.g.*, Dalla Riva, the author and Musolino [4, A. 11]). Thus we set

$$\mu_l(x) \equiv \tilde{\mu} * \eta_{2^{-l}}(x) \qquad \forall x \in \mathbb{R}^n\,,$$

for all $l \in \mathbb{N}$. By known properties of the convolution, we have $\mu_l \in C^\infty_c(\mathbb{R}^n)$ for each $l \in \mathbb{N}$. Moreover,

$$\lim_{l\to\infty} \mu_l = \tilde{\mu} \qquad \text{uniformly in } \mathbb{R}^n\,.$$

We also observe that the Young inequality for the convolution implies that

$$\sup_{\mathbb{R}^n} |\mu_l| \leq \sup_{\mathbb{R}^n} |\tilde{\mu}| \int_{\mathbb{R}^n} |\eta_{2^{-l}}(y)|\,dy = \sup_{\mathbb{R}^n} |\tilde{\mu}| \qquad \forall l \in \mathbb{N}\,.$$

Then we note that $|\mu_l : \mathbb{R}^n|_\beta \leq |\tilde{\mu} : \mathbb{R}^n|_\beta$ for all $l \in \mathbb{N}$. Indeed, if x', $x'' \in \mathbb{R}^n$, then

$$|\mu_l(x') - \mu_l(x'')| \leq \int_{\mathbb{R}^n} |\tilde{\mu}(x'-y) - \tilde{\mu}(x''-y)|\eta_{2^{-l}}(y)\,dy$$
$$\leq |\tilde{\mu} : \mathbb{R}^n|_\beta |x'-x''|^\beta \int_{\mathbb{R}^n} \eta_{2^{-l}}(y)\,dy = |\tilde{\mu} : \mathbb{R}^n|_\beta |x'-x''|^\beta \,.$$

Then the sequence $\{\mu_{l|Y}\}_{l\in\mathbb{N}}$ is bounded in $C^{0,\beta}(Y)$ and converges uniformly to μ in Y. Now let $\beta' \in]0,\beta[$, $0 < t_1 - \beta' < n-1$. By the compactness of the imbedding of $C^{0,\beta}(Y)$ into $C^{0,\beta'}(Y)$, possibly selecting a subsequence, we can assume that

$$\lim_{l\to\infty} \mu_{l|Y} = \mu \qquad \text{in } C^{0,\beta'}(Y)\,.$$

By Lemma 3.4 (i), we have

$$\lim_{l\to\infty} \mathrm{grad}_{Y,x} K(x,y)(\mu_l(y) - \mu(x)) = \mathrm{grad}_{Y,x} K(x,y)(\mu(y) - \mu(x))$$

in $\mathcal{K}_{t_1-\beta',Y\times Y}$. Since $0 < t_1 - \beta' < n-1$, the Hille-Tamarkin Proposition 4.1 and Proposition 4.3 imply that

$$\lim_{l\to\infty} \int_Y \mathrm{grad}_{Y,x} K(x,y)(\mu_l(y) - \mu_l(x))\,d\sigma_y$$
$$= \int_Y \mathrm{grad}_{Y,x} K(x,y)(\mu(y) - \mu(x))\,d\sigma_y \qquad \text{uniformly in } x \in Y$$

and that $\int_Y \mathrm{grad}_{Y,x} K(\cdot,y)(\mu_l(y) - \mu_l(\cdot))\,d\sigma_y$ is continuous in Y for each $l \in \mathbb{N}$. Then the validity of formula (6.1) for μ_l implies that

$$\lim_{l\to\infty} \mathrm{grad}_{Y,x} \int_Y K(x,y)\mu_l(y)\,d\sigma_y \tag{6.2}$$
$$= \int_Y [\mathrm{grad}_{Y,x} K(x,y)](\mu(y) - \mu(x))\,d\nu(y) + \mu(x)\mathrm{grad}_Y \int_Y K(x,y)\,d\sigma_y$$

uniformly in $x \in Y$. Since $K \in \mathcal{K}_{s_1,Y\times Y}$ and $s_1 < n-1$, again Proposition 4.1 and Proposition 4.3 imply that

$$\lim_{l\to\infty} \int_Y K(x,y)\mu_l(y)\,d\sigma_y = \int_Y K(x,y)\mu(y)\,d\sigma_y \tag{6.3}$$

uniformly in $x \in Y$ and that $\int_Y K(\cdot,y)\mu_l(y)\,d\sigma_y$ is continuous in Y for each $l \in \mathbb{N}$. By (6.2) and (6.3), we deduce that $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ belongs to $C^1(Y)$ and that the formula (6.1) for its tangential gradient holds true. □

By combining Proposition 4.4 and the previous theorem, we can now prove a continuity theorem for the integral operator with kernel K and with values into a Schauder space on a compact manifold Y of class C^1. For the definition of the Schauder space $C^{1,\beta}(Y)$ of functions μ of class C^1 on Y such that the tangential gradient of μ is β-Hölder continuous or for an equivalent definition based on a finite family of parametrizations of Y, we refer for example to Dondi and the author [5, §2], Dalla Riva, the author and Musolino [4, §2.20].

Theorem 6.3. *Let $n \in \mathbb{N}$, $n \geq 2$. Let Y be a compact manifold of class C^1 that is imbedded in $\mathbb{R}^n$. Let $s_1 \in [0,(n-1)[$. Let $\beta \in]0,1]$, $t_1 \in [\beta,(n-1)+\beta[$, $t_2 \in [\beta,+\infty[$, $t_3 \in]0,1]$. Let the kernel $K \in \mathcal{K}_{s_1,s_1+1,1}(Y \times Y)$ satisfy the following assumption*

$$K(\cdot,y) \in C^1(Y \setminus \{y\}) \quad \forall y \in Y\,.$$

Let $\mathrm{grad}_{Y,x} K(\cdot,\cdot)$ denote the tangential gradient of $K(\cdot,\cdot)$ with respect to the first variable. Then the following statements hold.

(i) If $t_1 < (n-1)$ and $\mathrm{grad}_{Y,x} K(\cdot,\cdot) \in (\mathcal{K}_{t_1,t_2,t_3}(Y \times Y))^n$, then the following statements hold.

(a) If $t_2 - \beta > (n-1)$, $t_2 < (n-1) + \beta + t_3$ and

$$\int_Y K(\cdot,y)\,d\sigma_y \in C^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ to $C^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)$ that takes μ to the function $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ is linear and continuous.

(aa) If $t_2 - \beta = (n-1)$ and

$$\int_Y K(\cdot,y)\,d\sigma_y \in C^{1,\max\{r^\beta,\omega_{t_3}(\cdot)\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ to $C^{1,\max\{r^\beta,\omega_{t_3}(\cdot)\}}(Y)$ that takes μ to the function $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ is linear and continuous.

(ii) If $t_1 = (n-1)$ and $\mathrm{grad}_{Y,x} K(\cdot,\cdot) \in \left(\mathcal{K}^\sharp_{t_1,t_2,t_3}(Y \times Y)\right)^n$, then the following statements hold.

(b) If $t_2 - \beta > (n-1)$, $t_2 < (n-1) + \beta + t_3$ and

$$\int_Y K(\cdot,y)\,d\sigma_y \in C^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ to $C^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)$ that takes μ to the function $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ is linear and continuous.

(bb) If $t_2 - \beta = (n-1)$ and

$$\int_Y K(\cdot,y)\,d\sigma_y \in C^{1,\max\{r^\beta,\omega_{t_3}(\cdot)\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ to $C^{1,\max\{r^\beta,\omega_{t_3}(\cdot)\}}(Y)$ that takes μ to the function $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ is linear and continuous.

(iii) If $t_1 > (n-1)$ and $\mathrm{grad}_{Y,x} K(\cdot,\cdot) \in (\mathcal{K}_{t_1,t_2,t_3}(Y \times Y))^n$, then the following statements hold.

(c) If $t_2 - \beta > (n-1)$, $t_2 < (n-1) + \beta + t_3$ *and*

$$\int_Y K(\cdot, y)\, d\sigma_y \in C^{1,\min\{\beta,(n-1)+\beta-t_1,(n-1)+t_3+\beta-t_2\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ *to* $C^{1,\min\{\beta,(n-1)+\beta-t_1,(n-1)+t_3+\beta-t_2\}}(Y)$ *that takes* μ *to the function* $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ *is linear and continuous.*

(cc) If $t_2 - \beta = (n-1)$ *and*

$$\int_Y K(\cdot, y)\, d\sigma_y \in C^{1,\max\{r^\beta,r^{(n-1)+\beta-t_1},\omega_{t_3}(\cdot)\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ *to* $C^{1,\max\{r^\beta,r^{(n-1)+\beta-t_1},\omega_{t_3}(\cdot)\}}(Y)$ *that takes* μ *to the function* $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ *is linear and continuous.*

Proof. Since Y is a compact manifold of class C^1 that is imbedded in $\mathbb{R}^n$, Y can be proved to be strongly upper $(n-1)$-Ahlfors regular with respect to Y. By Theorem 6.2, the function $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ is of class $C^1(Y)$ for all $\mu \in C^{0,\beta}(Y)$ and formula (6.1) for the tangential gradient of $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ holds true under any of the assumptions of (i)–(iii). Next, we consider statement (i). Under the assumptions of (a), Proposition 4.4 (i) implies that map from

$$C^{0,\beta}(Y) \qquad \text{to} \qquad C^{0,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)\,,$$

which takes μ to the function $\int_Y [\mathrm{grad}_{Y,x} K(x, y)](\mu(y) - \mu(x))\, d\sigma_y$ is linear and continuous. By our assumption on $\int_Y K(\cdot, y)\, d\sigma_y$, the map from

$$C^{0,\beta}(Y) \qquad \text{to} \qquad C^{0,\min\{\beta,\beta,(n-1)+t_3+\beta-t_2\}}(Y)\,,$$

which takes μ to $\mu(\cdot)\mathrm{grad}_Y \int_Y K(\cdot, y)\, d\sigma_y$ is linear and continuous. Then formula (6.1) implies that the map from $C^{0,\beta}(Y)$ to

$$C^{0,\min\{\beta,\beta,(n-1)+t_3+\beta-t_2\}}(Y)$$

that takes μ to $\mathrm{grad}_{Y,x} \int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ is linear and continuous. By Propositions 4.1 and 4.3, the map from $C^{0,\beta}(Y)$ to $C^0(Y)$ that takes μ to $\int_Y K(\cdot, y)\mu(y)\, d\sigma_y$ is linear and continuous. Hence, we deduce the validity of (a) of statement (i). The proof of (aa) follows the lines of that of statement (a) by invoking statement (aa) instead of statement (a) of (i) of Proposition 4.4.

The proofs of statements (ii) and (iii) follow the lines of that of statement (i) by invoking statements (ii) and (iii) instead of statement (i) of Proposition 4.4. In case of statement (iii) (c) we also observe that the pointwise product is bilinear and continuous from

$$C^{0,\beta}(Y) \times C^{0,\min\{\beta,(n-1)+\beta-t_1,(n-1)+t_3+\beta-t_2\}}(Y)$$

to $C^{0,\min\{\beta,(n-1)+\beta-t_1,(n-1)+t_3+\beta-t_2\}}(Y)$. In case of statement (iii) (cc) we also observe that the pointwise product is bilinear and continuous from

$$C^{0,\beta}(Y) \times C^{0,\max\{r^\beta,r^{(n-1)+\beta-t_1},\omega_{t_3}(\cdot)\}}(Y)$$

to $C^{0,\max\{r^\beta,r^{(n-1)+\beta-t_1},\omega_{t_3}(\cdot)\}}(Y)$ (cf. *e.g.*, Dondi and the author [5, §2]). □

Acknowledgment

The author acknowledges the support of the Research Project GNAMPA-INdAM CUP_E53C22001930001 'Operatori differenziali e integrali in geometria spettrale'.

References

[1] J.E. Björk, *On extensions of Lipschitz functions.* Ark. Mat. 7 (1969), 513–515.

[2] I. Chavel, *Eigenvalues in Riemannian geometry.* Including a chapter by Burton Randol. With an appendix by Jozef Dodziuk. Pure and Applied Mathematics, 115. Academic Press, Inc., Orlando, FL, 1984.

[3] D. Colton and R. Kress, *Integral equation methods in scattering theory*, Wiley, New York, 1983.

[4] M. Dalla Riva, M. Lanza de Cristoforis and P. Musolino, *Singularly Perturbed Boundary Value Problems. A Functional Analytic Approach*, Springer, Cham, 2021.

[5] F. Dondi and M. Lanza de Cristoforis, *Regularizing properties of the double layer potential of second order elliptic differential operators*, Mem. Differ. Equ. Math. Phys. 71 (2017), 69–110.

[6] J. García-Cuerva and A.E. Gatto, *Boundedness properties of fractional integral operators associated to non-doubling measures.* Studia Math. 162(3) (2004), 245–261.

[7] J. García-Cuerva and A.E. Gatto, *Lipschitz spaces and Calderón-Zygmund operators associated to non-doubling measures.* Publ. Mat. 49 (2005), no. 2, 285–296.

[8] A.E. Gatto, *Boundedness on inhomogeneous Lipschitz spaces of fractional integrals singular integrals and hypersingular integrals associated to non-doubling measures.* Collect. Math. 60, 1 (2009), 101–114.

[9] T.G. Gegelia, *Certain special classes of functions and their properties.* (Russian), Sakharth. SSR Mecn. Akad. Math. Inst. Šrom. **32** (1967), 94–139.

[10] G. Giraud, *Équations Γ intégrales principales; étude suivie d'une application.* (French) Ann. Sci. École Norm. Sup. **51** (1934), 251–372.

[11] A. Kirsch and F. Hettlich, *The Mathematical Theory of Time-Harmonic Maxwell's Equations; Expansion-, Integral-, and Variational Methods*, Springer, 2015.

[12] A. Kufner, O. John and S. Fučik, *Function Spaces*, Noordhoff International Publishing 1977.

[13] V.D. Kupradze, T.G. Gegelia, M.O. Basheleishvili and T.V. Burchuladze, *Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland Publ. Co., Amsterdam, 1979.

[14] M. Lanza de Cristoforis, *An inequality for Hölder continuous functions generalizing a result of Carlo Miranda.* In *Computational and Analytic Methods in Science and Engineering, C. Constanda (ed.)*, pages 185–206. Birkhäuser/Springer, New York, 2020.

[15] M. Lanza de Cristoforis, *Integral operators in Hölder spaces on upper Ahlfors regular sets*, to appear in Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., **34** (2023).

[16] E.J. Mc Shane, *Extensions of the range of functions.* Bull. Am. Math. Soc., **40** (1934), 837–842.

M. Lanza de Cristoforis
Dipartimento di Matematica 'Tullio Levi-Civita',
Università degli Studi di Padova,
Via Trieste 63,
Padova I-35121,Italy
E-mail: mldc@math.unipd.it

Received: ??.08.2022