BEC in Nonextensive Statistical Mechanics: Some Additional Results

Luca Salasnich

Istituto Nazionale per la Fisica della Materia, Unità di Milano, Dipartimento di Fisica, Università di Milano, Via Celoria 16, 20133 Milano, Italy

Abstract

In a recent paper [Int. J. Mod. Phys. B 14, 405 (2000)] we discussed the Bose-Einstein condensation (BEC) in the framework of Tsallis's nonextensive statistical mechanics. In particular, we studied an ideal gas of bosons in a confining harmonic potential. In this memoir we generalize our previous analysis by investigating an ideal Bose gas in a generic power-law external potential. We derive analytical formulas for the energy of the system, the BEC transition temperature and the condensed fraction.

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Recently we analyzed the consequences of weak nonextensivity on Bose-Einstein Condensation for an ideal Bose gas¹ by using Tsallis's nonextensive statistical mechanics (NSM).² From the nonextensive Bose-Einstein distribution³⁻⁵ we derived the BEC transition temperature, the condensed fraction and the energy per particle in three different cases: the homogeneous gas, the gas in a harmonic trap and the relativistic homogenous gas.

In this short contribution we generalize our previous results by considering the ideal Bose gas in a generic power-law potential. All the calculations are performed by assuming a D-dimensional space.

The NSM predicts that, for an ideal quantum gas of identical bosons in the grand canonical ensemble, the weak nonextensivity formula of the average number of particles with energy ϵ is given by

$$\langle n(\epsilon) \rangle_q = \frac{1}{e^{\beta(\epsilon-\mu)} - 1} + \frac{1}{2}(q-1)\frac{\beta^2(\epsilon-\mu)^2 e^{\beta(\epsilon-\mu)}}{(e^{\beta(\epsilon-\mu)} - 1)^2} , \qquad (1)$$

where μ is the chemical potential and $\beta = 1/(kT)$ with k the Boltzmann constant and T the temperature.³ The parameter q is a measure of the lack of extensivity: for q = 1 one recovers the familiar Bose-Einstein distribution. An interesting interpretation of q, in terms of fluctuations of the parameters which appear in the standard exponential distribution, can be found in Ref. 6.

The total number N of particle and the energy E for a system of noninteracting bosons can be written

$$N = \int_0^\infty d\epsilon \ \rho(\epsilon) \ \langle n(\epsilon) \rangle_q \quad \text{and} \quad E = \int_0^\infty d\epsilon \ \epsilon \ \rho(\epsilon) \ \langle n(\epsilon) \rangle_q , \quad (2)$$

where $\rho(\epsilon)$ is the density of states. It can be obtained from the formula

$$\rho(\epsilon) = \int \frac{d^D \mathbf{r} d^D \mathbf{p}}{(2\pi\hbar)^D} \delta(\epsilon - H(\mathbf{p}, \mathbf{r})) , \qquad (3)$$

where $H(\mathbf{p}, \mathbf{r})$ is the classical single-particle Hamiltonian of the system in a D-dimensional space. If $H(\mathbf{p}, \mathbf{r}) = \mathbf{p}^2/(2m) + U(\mathbf{r})$ then it is easy to show that

$$\rho(\epsilon) = \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{D}{2}} \frac{1}{\Gamma(\frac{D}{2})} \int d^D \mathbf{r} \ \left(\epsilon - U(\mathbf{r})\right)^{\frac{(D-2)}{2}} , \qquad (4)$$

where $\Gamma(x)$ is the factorial function.

Let us consider the power-law potential given by

$$U(\mathbf{r}) = A r^{\alpha} , \qquad (5)$$

where $r = |\mathbf{r}|$ and α is the exponent. The study of this potential is useful to analyze the effects of adiabatic changes in the confining trap.⁷ The density of states can be calculated from the two previous formulas and reads

$$\rho(\epsilon) = \left(\frac{m}{2\hbar^2}\right)^{\frac{D}{2}} \left(\frac{1}{A}\right)^{\frac{D}{\alpha}} \frac{\Gamma(\frac{D}{\alpha}+1)}{\Gamma(\frac{D}{2}+1)\Gamma(\frac{D}{2}+\frac{D}{\alpha})} \epsilon^{\frac{D}{2}+\frac{D}{\alpha}-1} .$$
(6)

At the BEC transition temperature, the chemical potential μ is zero and at $\mu = 0$ the number N of particles and energy E can be analytically determined. The number of particles is given by

$$N = (kT)^{\frac{D}{2} + \frac{D}{\alpha}} \left(\frac{m}{2\hbar^2}\right)^{\frac{D}{2}} \left(\frac{1}{A}\right)^{\frac{D}{\alpha}} \frac{\Gamma(\frac{D}{\alpha} + 1)\zeta(\frac{D}{2} + \frac{D}{\alpha})}{\Gamma(\frac{D}{2} + 1)} \times \left[1 + \frac{1}{2}(q-1)\frac{\Gamma(\frac{D}{2} + \frac{D}{\alpha} + 2)\zeta(\frac{D}{2} + \frac{D}{\alpha} + 1)}{\Gamma(\frac{D}{2} + \frac{D}{\alpha})\zeta(\frac{D}{2} + \frac{D}{\alpha})}\right]$$
(7)

and the energy satisfies the following relation

$$\frac{E}{kT} = (kT)^{\frac{D}{2} + \frac{D}{\alpha}} \left(\frac{m}{2\hbar^2}\right)^{\frac{D}{2}} \left(\frac{1}{A}\right)^{\frac{D}{\alpha}} \frac{\left(\frac{D}{2} + \frac{D}{\alpha}\right)\Gamma\left(\frac{D}{\alpha} + 1\right)\zeta\left(\frac{D}{2} + \frac{D}{\alpha} + 1\right)}{\Gamma\left(\frac{D}{2} + 1\right)} \times$$

$$\times \left[1 + \frac{1}{2} (q-1) \frac{\Gamma(\frac{D}{2} + \frac{D}{\alpha} + 3)\zeta(\frac{D}{2} + \frac{D}{\alpha} + 2)}{\Gamma(\frac{D}{2} + \frac{D}{\alpha} + 1)\zeta(\frac{D}{2} + \frac{D}{\alpha} + 1)} \right] , \qquad (8)$$

where $\zeta(x)$ is the Riemann ζ -function. Note that our formula of the energy can be easily generalized above the critical temperature by substituting the Riemann function $\zeta(D)$ with the polylogarithm function $Li_D(z) = \sum_{k=1}^{\infty} z^k / k^D$, that depends on the fugacity $z = e^{\beta\mu}$. From the energy one can easily obtain the specific heat and the other thermodynamical quantities.

By inverting the function N = N(T) one finds the transition temperature T_q . It is given by

$$kT_q = \left[\left(\frac{2\hbar^2}{m}\right)^{\frac{D}{2}} A^{\frac{D}{\alpha}} \frac{\Gamma(\frac{D}{2}+1)}{\Gamma(\frac{D}{\alpha}+1)\zeta(\frac{D}{2}+\frac{D}{\alpha})} N \right]^{\frac{1}{\frac{D}{2}+\frac{D}{\alpha}}} \times \left[1 + \frac{1}{2}(q-1)\frac{\Gamma(\frac{D}{2}+\frac{D}{\alpha}+2)\zeta(\frac{D}{2}+\frac{D}{\alpha}+1)}{\Gamma(\frac{D}{2}+\frac{D}{\alpha})\zeta(\frac{D}{2}+\frac{D}{\alpha})} \right]^{\frac{D}{2}+\frac{D}{\alpha}}.$$
 (9)

The Eqs. (6)–(9) generalize the results obtained in our previous paper.¹ In fact, by setting $\alpha = 2$ and $A = m\omega^2 r^2/2$ one recovers the formulas for the Bose gas in a harmonic trap. The results for a rigid box are instead obtained by letting $\frac{D}{\alpha} \to 0$, where the density of particles per unit length is given by N/Ω_D and $\Omega_D = D\pi^{\frac{D}{2}}/\Gamma(\frac{D}{2}+1)$ is the volume of the D-dimensional sphere. Obviously, for q = 1 one gets the equations of the extensive thermodynamics.⁸ In particular, with q = 1 and D = 3, the Eq. (9) gives the extensive BEC transition temperature obtained by Bagnato, Pritchard and Kleppner.⁹

The Eq. (9) shows that the critical temperature T_q grows by increasing the nonextensive parameter q. Moreover, one observes that, because $\zeta(1) = \infty$, BEC is possible if and only if the following condition is satisfied

$$\frac{D}{2} > 1 - \frac{D}{\alpha} . \tag{10}$$

Thus, we have found a remarkable relation between the space dimension Dand the exponent α of the confining power-law potential. For example, for D = 2 there is no BEC in a homogenous gas $(\frac{D}{\alpha} \rightarrow 0)$ but it is possible in a harmonic trap ($\alpha = 2$). Instead, for D = 1 BEC is possible with $1 < \alpha < 2$ (see also Ref. 8).

Below T_q , a macroscopic number N_0 of particle occupies the single-particle ground-state of the system. It follows that Eq. (7) gives the number $N - N_0$ of non-condensed particles and the condensed fraction reads

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_q}\right)^{\frac{D}{2} + \frac{D}{\alpha}} .$$
 (11)

In conclusion, we have analyzed the consequences of Tsallis's nonextensive statistical mechanics for an ideal Bose gas confined in a power-law potential. We have obtained analytical formulas for the energy of the system, the BEC transition temperature and the condensed fraction. In the appropriate limits, such formulas reduce to results found in previous papers. Moreover, we have shown that BEC is possible if and only if $\frac{D}{2} > 1 - \frac{D}{\alpha}$, where D is the space dimension and α is the exponent of the confining power-law potential.

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