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Ample vector bundles with zero loci of small Δ-genera

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Abstract. Let X be a smooth complex projective variety endowed with an ample vector bundle $\mathcal{E}$ admitting a global section whose zero locus is a smooth subvariety Z of the expected dimension, and let H be an ample line bundle on X, whose restriction H_Z to Z is very ample. Triplets $(X, \mathcal{E}, H)$ are studied and classified under the assumption that the delta genus of (Z, H_Z) is either small (≤ 3) or small in comparison with the corank of $\mathcal{E}$ or the degree.

Key words. Ample vector bundles, special varieties, Δ-genus, adjunction theory, Fano manifolds.

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1 Introduction

Let X be a smooth complex projective variety and let L be an ample line bundle on X. In order to study polarized manifolds (X, L) Fujita [9] introduced the Δ-*genus* of (X, L), which is a nonnegative integer defined by the formula

$$\Delta(X, L) := \dim X + L^{\dim X} - h^0(X, L).$$

This character turned out to be very useful even in the classification of projective manifolds (e.g., see [11]) combined with other numerical characters like the sectional genus. Extending classification results of smooth projective varieties in terms of hyperplane sections to the more general framework of ample vector bundles arises as a very natural problem.

The setting we consider in this paper is as follows:

1.1. Let X be a smooth complex projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank $r \geq 2$ on X such that there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z := (s)_0$ is a smooth subvariety of X of the expected dimension $n - r$.

Note that Condition 1.1 is certainly satisfied if the ample vector bundle $\mathcal{E}$ is also globally generated. Next, consider an ample line bundle H on X and suppose that its restriction H_Z to Z is very ample.

Generally speaking, our aim is to study and classify triplets $(X, \mathcal{E}, H)$ as above under the assumption that $\Delta(Z, H_Z)$ is small. As a first thing we investigate the setting above when $\Delta(Z, H_Z) \leq 3$. Cases $\Delta = 0$ and 1 do not require any further restriction. Moreover, for $\Delta = 0$ we do not even need to require the very ampleness of H_Z. However, to study the next cases $\Delta = 2$ and 3 we have to assume that $n - r \geq 2$. On the other hand we should note that for $n - r = 1$ if H_Z is very ample and non-special, then $\Delta(Z, H_Z)$ is simply the genus g of the smooth curve Z. Thus our problem overlaps that of classifying pairs $(X, \mathcal{E})$ as in 1.1 with $\mathcal{E}$ having curve genus g. As far as we know, results on this related problem are available only for $g \leq 2$, with $\mathcal{E}$ being very ample when equality holds [25], [26]. For $n - r \geq 2$, starting from the known classification of projective manifolds of small Δ and using miscellaneous results concerning ample vector bundles, we get satisfactory structure theorems for our triplets $(X, \mathcal{E}, H)$. The results are expressed by Theorems 3.2, 3.5, 3.6, and 3.12 for the values $\Delta = 0, 1, 2, 3$, respectively. They are complete except for $\Delta(Z, H_Z) = 3$ and $n - r = 2$, because the case in which (Z, H_Z) is a nongeneral quintic surface in $\mathbb{P}^3$ and X has Picard number $\rho(X) \geq 2$ is not covered.

Next we address the same problem when $\Delta(Z, H_Z)$ is small in comparison with the corank $n - r$ of $\mathcal{E}$. We assume that $\Delta(Z, H_Z) \leq \operatorname{cork}(\mathcal{E}) - 1$. Of course, if $n - r = 1$ this means $\Delta = 0$ and this situation falls in Theorem 3.2. On the other hand, if $n - r \geq 2$, our assumption is equivalent to the requirement that Z is embedded in $\mathbb{P}^N$ by $|H_Z|$ with degree $\leq N$. Relying on a nice classification result of Ionescu [12], we succeed to describe the possible structures of the triplets $(X, \mathcal{E}, H)$ for each pair (Z, H_Z) occurring in Ionescu's list. Our result in this setting is expressed by Theorem 4.1. Apart from triplets arising from a Fano manifold Z with $\operatorname{Pic}(Z) \cong \mathbb{Z}$ generated by H_Z, $(X, \mathcal{E}, H)$ is described in precise way. We would like to note that, in both investigations above, adjunction theoretic results for triplets $(X, \mathcal{E}, H)$ developed by Maeda and the first author provide a useful tool. This should be not surprising in view of the key role played by adjunction theory in classification of projective manifolds.

One more way to rephrase that $\Delta(Z, H_Z)$ is small is to compare this character with the degree $d = d(Z, H_Z)$. If $n - r \geq 2$ we show that condition $\Delta(Z, H_Z) < \frac{d}{2}$ implies the non-nefness of the adjoint line bundle $K_X + \det \mathcal{E} + (n - r - 2)H$. As a consequence, assuming $n - r \geq 3$ we conclude that triplets $(X, \mathcal{E}, H)$ satisfying this condition are those occurring in an adjunction theoretic classification result for ample vector bundles due to Maeda [24].

The paper is organized as follows. In Section 2 we provide some background material including results of interest in themselves, like Lemmas 2.4, 2.5 and 2.9. Section 3 is devoted to the study of $\Delta(Z, H_Z) \leq 3$; in Section 4 we consider the case $\Delta(Z, H_Z) \leq \operatorname{cork}(\mathcal{E}) - 1$ and in Section 5 the case $\Delta(Z, H_Z) < \frac{d}{2}$.

2 Background material

We use the standard notation from algebraic geometry. The tensor products of line bundles are denoted additively. The pullback $i^*\mathcal{E}$ of a vector bundle $\mathcal{E}$ on X by an embedding of projective varieties $i \colon Y \hookrightarrow X$ is denoted by $\mathcal{E}_Y$. We denote by K_X the canonical bundle of a smooth variety X. The blow-up of a variety X along a smooth subvariety Y is denoted by $\mathrm{Bl}_Y(X)$. We say that a vector bundle is spanned to mean that it is generated by global sections.

A smooth complex projective variety X is called a *Fano manifold* if its anticanonical bundle $-K_X$ is ample. For a Fano manifold X, the largest integer, r_X, which divides $-K_X$ in the Picard group $\mathrm{Pic}(X)$ is called the *index* of X while the integer $q_X := \dim X - r_X + 1$ is called the *coindex* of X.

A *polarized manifold* is a pair (X, L) consisting of a smooth complex projective variety X and an ample line bundle L on X. A polarized manifold (X, L) is said to be a *scroll* over a smooth variety W if there exists a surjective morphism $f\colon X \longrightarrow W$ such that $(F, L_F) \cong (\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ with $r = \dim X - \dim W$ for any fiber F of f. This condition is equivalent to saying that $(X, L) \cong (\mathbb{P}_W(\mathcal{F}), H(\mathcal{F}))$ for some ample vector bundle $\mathcal{F}$ on W, where $H(\mathcal{F})$ is the tautological line bundle on the projective space bundle $\mathbb{P}_W(\mathcal{F})$ associated to $\mathcal{F}$. A polarized manifold (X, L) is said to be a *quadric fibration* over a smooth curve W if there exists a surjective morphism $f\colon X \longrightarrow W$ and any general fiber F of f is a smooth quadric hypersurface $\mathbb{Q}^{n-1}$ in $\mathbb{P}^n$ with $n = \dim X$ such that $L_F = \mathcal{O}_{\mathbb{Q}^{n-1}}(1)$. A polarized manifold (X, L) is said to be a *del Pezzo manifold* if $K_X + (\dim X - 1)L = \mathcal{O}_X$.

The following fact is well known.

Lemma 2.1. *Let $\mathcal{E}$ be an ample vector bundle of rank r on a compact complex manifold X. For any rational curve $C \subset X$ we have*

$$\det \mathcal{E} \cdot C \geq r.$$

Moreover, if C is smooth and equality holds, then $(C, \mathcal{E}_C) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r})$.

Theorem 2.2 (Lefschetz–Sommese). *Let $X, \mathcal{E}$ and Z be as in* 1.1 *and let $\iota\colon Z \hookrightarrow X$ be the inclusion. Then:*

(1) $H^i(\iota)\colon H^i(X, \mathbb{Z}) \longrightarrow H^i(Z, \mathbb{Z})$ *is an isomorphism for $i \leq \dim Z - 1$, and injective with torsion free cokernel for $i = \dim Z$;*
(2) $\mathrm{Pic}(\iota)\colon \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Z)$ *is an isomorphism for $\dim Z \geq 3$, and injective with torsion free cokernel for $\dim Z = 2$.*

We recall the following facts that we will use in our proofs.

Proposition 2.3. *Let X, $\mathcal{E}$ and Z be as in* 1.1 *with $n - r \geq 3$. Let H be an ample line bundle on X. Assume that one of the following holds:*

(1) *(X, H) is a scroll over a smooth curve C and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(r-1)}$ for any fiber F of the projection $X \longrightarrow C$;*
(2) *(X, H) is a quadric fibration over a smooth curve C and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus r}$ for any general fiber F of the fibration $X \longrightarrow C$;*
(3) *(X, H) is a scroll over a smooth surface S and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus r}$ for any fiber F of the projection $X \longrightarrow S$.*

If (1) *or* (2) *holds, then (Z, H_Z) is a quadric fibration over C. If* (3) *holds, then (Z, H_Z) is a scroll over S.*

Proof. These assertions are shown in the proofs of [18, Theorem 2 and Theorem 3]. □

Here is a general result that we will use in the proofs of Theorems 3.6, 3.12 and 4.1.

Lemma 2.4. *Let X, $\mathcal{E}$ and Z be as in* 1.1 *with $n-r \geq 4$. Let H be an ample line bundle on X and let (X,H) be a quadric fibration over $\mathbb{P}^1$ with $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus r}$ for any general fiber F of the fibration. Then (Z,H_Z) cannot be $\mathbb{P}^1 \times \mathbb{Q}^{n-r-1}$ Segre embedded.*

Proof. We argue by contradiction. Consider the quadric fibration $q\colon X \longrightarrow \mathbb{P}^1$ and note that $q|_Z\colon Z \to \mathbb{P}^1$ is the first projection of $\mathbb{P}^1 \times \mathbb{Q}^{n-r-1}$. So, $f := F \cap Z \cong \mathbb{Q}^{n-r-1}$ for every fiber F of q. We know that $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus r}$ for the general fiber F. Since $H_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)$, we know that

$$(\mathcal{E} \otimes [-H])_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}^{\oplus r}$$

for the general fiber F of q.

Claim: $(\mathcal{E} \otimes [-H])$ restricted to every fiber of q is trivial.

Let F_0 be a singular fiber of q and set $E := (\mathcal{E} \otimes [-H])_{F_0}$. By [19, Lemma 0.3] we know that (F_0, H_{F_0}) is a quadric cone of $\mathbb{P}^n$ having a single point v as vertex, with the polarization induced by $\mathcal{O}_{\mathbb{P}^n}(1)$. Consider $f_0 := F_0 \cap Z$ and note that $(f_0, H_{f_0}) \cong (\mathbb{Q}^{n-r-1}, \mathcal{O}_{\mathbb{Q}^{n-r-1}}(1))$ can be identified with the section of the quadric cone $F_0 \subset \mathbb{P}^n$ with r general hyperplanes. Let $\ell \subset f_0$ be any line. As the lines contained in the fibers of $q|_Z$ belong to a single algebraic family we know that $\deg \mathcal{E}_\ell = r$; moreover $H_\ell \cong \mathcal{O}_\ell(1)$. Since $\mathcal{E}$ is ample, this implies that $E_\ell \cong \mathcal{O}_\ell^{\oplus r}$ for every line $\ell \subset f_0$. By applying [36, Lemma 3.6.1] we thus get

$$E_{f_0} \cong \mathcal{O}_{\mathbb{Q}^{n-r-1}}^{\oplus r}. \tag{2.4.1}$$

Let W be a smooth hyperplane section of F_0 in the linear system having f_0 as base locus. Then $W \cong \mathbb{Q}^{n-2}$. Moreover we can look at $F_0 \subset \mathbb{P}^n$ as the cone projecting W from v. Since there is a ladder of (W, H_W), all of whose elements from f_0 to W itself are smooth quadrics, by applying inductively the same argument as in [18, Proof of Lemma 2.1] (inspired by [32, Chapter I, Section 2.3]), we can infer from (2.4.1) that

$$E_W \cong \mathcal{O}_{\mathbb{Q}^{n-2}}^{\oplus r}. \tag{2.4.2}$$

Set $T = \mathcal{O}_{F_0}^{\oplus r}$ and call $\varphi\colon T_W \to E_W$ the inverse of the isomorphism in (2.4.2). Now consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{F_0}(-W) \longrightarrow \mathcal{O}_{F_0} \longrightarrow \mathcal{O}_W \longrightarrow 0. \tag{2.4.3}$$

Tensor with $E(k) := E \otimes [kH]_{F_0}$, and consider the cohomology exact sequence

$$\ldots \longrightarrow H^1(E(k-1)) \longrightarrow H^1(E(k)) \longrightarrow H^1(E_W(k)) \longrightarrow \ldots .$$

Since $W \cong \mathbb{Q}^{n-2}$ we know that $h^1(E_W(k)) = rh^1(\mathcal{O}_W(k)) = 0$ for every integer k. Thus $h^1(E(k-1)) \leq h^1(E(k))$. But the latter is zero for $k >> 0$. Hence

$$h^1(E(k)) = 0 \quad \text{for every integer } k. \tag{2.4.4}$$

Now tensor (2.4.3) with $T^\vee \otimes E$, where $^\vee$ denotes the dual, and consider the cohomology exact sequence

$$\dots \longrightarrow H^0(T^\vee \otimes E) \longrightarrow H^0((T^\vee \otimes E)_W) \longrightarrow H^1(T^\vee \otimes E(-1)) \dots .$$

Note that $H^1(T^\vee \otimes E(-1)) = (H^1(E(-1)))^{\oplus r} = 0$ by (2.4.4). Thus the section of $(T^\vee \otimes E)_W$ corresponding to the isomorphism $\varphi\colon T_W \longrightarrow E_W$ can be extended to a section of $T^\vee \otimes E$, i.e. to a homomorphism $\phi\colon T \longrightarrow E$. Note that $K_Z + (n-r-1)H_Z - q|_Z^* \mathcal{O}_{\mathbb{P}^1}(n-r-3) = \mathcal{O}_Z$. Therefore $K_X + \det \mathcal{E} + (n-r-1)H - q^*\mathcal{O}_{\mathbb{P}^1}(n-r-3) = \mathcal{O}_X$, by the Lefschetz–Sommese theorem. This says that

$$\det(\mathcal{E} \otimes [-H])_F = (\det \mathcal{E} - rH)_F = -(K_X + (n-1)H)_F \cong \mathcal{O}_F$$

for every fiber F of q. In particular, we get $\det E \cong \mathcal{O}_{F_0}$, and so

$$\det \phi \in \mathrm{Hom}(\det T, \det E) = H^0(\mathcal{H}om_{\mathcal{O}_{F_0}}(\mathcal{O}_{F_0}, \mathcal{O}_{F_0})) = H^0(\mathcal{O}_{F_0}) = \mathbb{C}.$$

Since $\det \phi$ is non-zero on W and constant on F_0, $\det \phi$ vanishes nowhere. Therefore ϕ is an isomorphism and $E = (\mathcal{E} \otimes [-H])_{F_0}$ is trivial as claimed.

Due to the claim there exists a vector bundle $\mathcal{G}$ of rank r on $\mathbb{P}^1$ such that

$$\mathcal{E} \cong H \otimes q^*\mathcal{G}. \tag{2.4.5}$$

Denote by $p_1\colon Z \longrightarrow \mathbb{P}^1$ and $p_2\colon Z \longrightarrow \mathbb{Q}^{n-r-1}$ the projections onto the factors and let $l \subset Z$ be a fiber of p_2. Then $p_1|_l\colon l \longrightarrow \mathbb{P}^1$ is an isomorphism. As $\mathcal{E}_Z$ is ample, $\deg \mathcal{E}_l \geq \mathrm{rk}(\mathcal{E})$ by Lemma 2.1. Since $\mathcal{E}_Z \cong (H \otimes q^*\mathcal{G})_Z \cong H_Z \otimes p_1^*\mathcal{G}$, we get

$$r = \mathrm{rk}(\mathcal{E}_l) \leq \deg \mathcal{E}_l = \deg(\mathcal{E}_Z)_l = \deg(H_Z \otimes p_1^*\mathcal{G})_l = rH_Z \cdot l + \deg \mathcal{G} = r + \deg \mathcal{G},$$

hence

$$\deg \mathcal{G} \geq 0. \tag{2.4.6}$$

Note that $H_Z^{n-r} = H^{n-r} \cdot Z = H^{n-r} \cdot c_r(\mathcal{E})$. So in view of (2.4.5) we have

$$c_r(\mathcal{E}) = \sum_{j=0}^{r} q^*c_j(\mathcal{G}) \cdot H^{r-j} = H^r + q^*c_1(\mathcal{G}) \cdot H^{r-1} = H^r + \deg \mathcal{G} H^{r-1} \cdot F.$$

Therefore, noting that $H^{n-1} \cdot F = (H_F)^{n-1} = 2$, we deduce that

$$2(n-r) = H_Z^{n-r} = H^n + 2\deg \mathcal{G}. \tag{2.4.7}$$

Now come back to the quadric fibration $q\colon X \longrightarrow \mathbb{P}^1$ and set $\mathcal{V} := q_*H$. Then $\mathcal{V}$ is a vector bundle of rank $n+1$ on $\mathbb{P}^1$. Set $P := \mathbb{P}_{\mathbb{P}^1}(\mathcal{V})$, let $\pi\colon P \longrightarrow \mathbb{P}^1$ be the projection, and let ξ and D be the tautological line bundle and a fiber respectively. Then X is embedded fiberwise in P as a divisor $X \in |2\xi - bD|$ for some integer b, and $\xi_X = H$.

Note that $H^n = (\xi_X)^n = \xi^n \cdot X = \xi^n \cdot (2\xi - bD)$. So taking into account the relation $\xi^{n+1} = \deg \mathcal{V}$ provided by the Chern–Wu formula and the fact that $\xi^n \cdot D = 1$, (2.4.7) gives

$$2(n-r) = 2(\deg \mathcal{V} + \deg \mathcal{G}) - b. \tag{2.4.8}$$

Now look at $Z = \mathbb{P}^1 \times \mathbb{Q}^{n-r-1}$ and recall that $\xi_Z = H_Z = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{Q}^{n-r+1}}(1,1)$. Moreover any fiber f of the first projection p_1 can be regarded as $Z \cap F = Z \cap D$, for some fiber D of π, in view of the commutative diagram

$$\begin{array}{ccccc} Z = \mathbb{P}^1 \times \mathbb{Q}^{n-r-1} & \hookrightarrow & X & \hookrightarrow & P := \mathbb{P}_{\mathbb{P}^1}(\mathcal{V}) \\ & {\scriptstyle p_1} \searrow & \downarrow {\scriptstyle q} & \swarrow {\scriptstyle \pi} & \\ & & \mathbb{P}^1. & & \end{array}$$

Due to the inclusion $Z \subset X$, adjunction and the canonical bundle formula for $\mathbb{P}$-bundles allow us to compute K_Z. We have

$$\begin{aligned} K_Z &= (K_X + \det \mathcal{E})_Z = ((K_P + X)_X + \det \mathcal{E})_Z = \\ &= ((-(n+1)\xi + (-2 + \deg \mathcal{V})D + 2\xi - bD)_X + rH + q^* \det \mathcal{G})_Z = \\ &= -(n-r-1)H_Z + (\deg \mathcal{V} + \deg \mathcal{G} - 2 - b)f. \end{aligned}$$

On the other hand, we know that

$$K_Z = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{Q}^{n-r-1}}(-2, -(n-r-1)) = -(n-r-1)H_Z + (n-r-3)f,$$

since $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{Q}^{n-r-1}}(1,0) = p_1^* \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_Z(f)$. Comparing the expressions obtained for K_Z we get the following relation

$$\deg \mathcal{V} + \deg \mathcal{G} = n - r - 1 + b. \tag{2.4.9}$$

Thus, (2.4.8) and (2.4.9) show that

$$b = 2 \quad \text{and} \quad \deg \mathcal{V} = (n - r + 1) - \deg \mathcal{G}. \tag{2.4.10}$$

Now, by the same argument as in [8, (3.3)] we can compute the number δ of singular fibers of q and get

$$0 \leq \delta = 2 \deg \mathcal{V} - (n+1)b.$$

Combining this with (2.4.10) gives

$$0 \leq \delta = -2(r + \deg \mathcal{G}).$$

But this is a contradiction, since we know that $r \geq 2$ and $\deg \mathcal{G} \geq 0$ by (2.4.6). □

Lemma 2.5. *Let X, $\mathcal{E}$ and Z be as in* 1.1 *with $n - r \geq 3$. Let H be an ample line bundle on X such that H_Z is very ample. Suppose that $g(Z, H_Z) = 3$ and (Z, H_Z) is a scroll over $\mathbb{P}^2$. Then $n - r = 3$.*

Proof. Assume that $n - r \geq 4$. Then, by [12, Proof of Proposition 5] and [11, Proposition 4.7], $(Z, H_Z) \cong (\mathbb{P}_{\mathbb{P}^2}(\mathcal{F}), H(\mathcal{F}))$, where $\mathcal{F}$ is one of the following vector bundles:

(i) $\mathcal{F} \cong T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$;
(ii) $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$;

(iii) $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4}$.

We have $b_2(X) = b_2(Z) = 2$ by the Lefschetz–Sommese theorem. Moreover, denoted by p the bundle projection, $K_Z + \mathrm{rk}(\mathcal{F})H_Z = p^*(K_{\mathbb{P}^2} + \det \mathcal{F})$ is nef, since in all three cases $K_{\mathbb{P}^2} + \det \mathcal{F} = K_{\mathbb{P}^2} + \mathcal{O}_{\mathbb{P}^2}(4)$ is ample; so $K_Z + (\dim Z - 1)H_Z$ is nef. Note that $K_Z + (\dim Z - 2)H_Z$ is not nef since it is negative on curves in a fiber of p. So we are in the assumption of [18, Theorem 3]. Note that condition $b_2(X) > 1$ rules out Cases (1)–(9), while the fact that $n - r > 3$ rules out Case (11). Therefore we get one of the following possibilities:

(a) there exists an effective divisor E on X such that

$$(E, \mathcal{E}_E, H_E, \mathcal{O}_E(E)) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r}, \mathcal{O}_{\mathbb{P}^{n-1}}(1), \mathcal{O}_{\mathbb{P}^{n-1}}(-1));$$

(b) (X, H) is a scroll over a smooth curve C and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(r-1)}$ for any fiber F of the projection $X \longrightarrow C$;

(c) (X, H) is a quadric fibration over a smooth curve C and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus r}$ for any general fiber F of the fibration $X \longrightarrow C$;

(d) (X, H) is a scroll over a smooth surface S and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus r}$ for every fiber F of the projection $X \longrightarrow S$.

We check this list case-by-case to rule out all possibilities.

Case (a). The restriction of the section s to E is a section $s_E \in \Gamma(\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r})$, hence its zero locus $Z_0 := (s_E)_0 = Z \cap E$ is a linear subspace of E, so $\dim Z_0 \geq n - r - 1$. Note that Z_0 cannot be equal to Z, otherwise Z would be $\mathbb{P}^{n-r}$ contradicting $b_2(Z) = 2$; moreover, Z is irreducible, hence Z_0 is a divisor in Z. Therefore $Z_0 \cong \mathbb{P}^{n-r-1}$. This implies that $p(Z_0)$ cannot have positive dimension. On the other hand the restriction of p to Z_0 cannot be constant, otherwise Z_0 would be contained in a fiber of p, which is impossible.

Cases (b) and (c). In both cases the restriction $q := f_Z$ of the morphism $f\colon X \longrightarrow C$ to Z gives to (Z, H_Z) a structure of quadric fibration, by Proposition 2.3. So Z is endowed with two morphisms

$$C \xleftarrow{\ q\ } Z \xrightarrow{\ p\ } \mathbb{P}^2$$

where p is the scroll projection over $\mathbb{P}^2$. Let $F \cong \mathbb{P}^{n-r-2}$ be any fiber of the scroll projection. Note that $q(F)$ is a point since $\dim F \geq 2$. Hence the general fiber $F \cong \mathbb{P}^{n-r-2}$ is contained in a smooth fiber $\cong \mathbb{Q}^{n-r-1}$ of q. Recalling that $\dim Z$ is either 4 or 5 we get a contradiction, since a smooth $\mathbb{Q}^{n-r-1}$ can contain a linear space of dimension at most $[\frac{n-r-1}{2}]$.

Case (d). The restriction π_Z of the projection $\pi\colon X \longrightarrow S$ to Z gives to (Z, H_Z) a structure of scroll over S, by Proposition 2.3. So Z is endowed with two morphisms

$$\begin{array}{ccc} & Z & \\ {}^{\pi_Z}\swarrow & & \searrow^{p} \\ S & & \mathbb{P}^2. \end{array}$$

We claim that $\pi_Z = p$. Assume by contradiction that $\pi_Z \neq p$. Then there is a fiber F of p such that $\pi_Z|_F : F \longrightarrow S$ is not constant. Hence $F \cong \mathbb{P}^2$, otherwise $\pi_Z|_F$ would give a fibration of $\mathbb{P}^3$ either onto S or onto a curve of S, which is a contradiction. Moreover $\pi_Z|_F : F \longrightarrow S$ is a surjective morphism onto a smooth projective surface, so $S \cong \mathbb{P}^2$. By [35, Theorem A], we obtain $Z \cong \mathbb{P}^2 \times \mathbb{P}^2$, which is not one of our cases. We have thus proved that $\pi_Z = p$, therefore the scroll structure of (Z, H_Z) comes from that of (X, H) and $S \cong \mathbb{P}^2$. So we have the following commutative diagram

$$\begin{array}{ccc} Z = \mathbb{P}_{\mathbb{P}^2}(\mathcal{F}) & \overset{\iota}{\hookrightarrow} & X := \mathbb{P}_{\mathbb{P}^2}(\mathcal{V}) \\ {}_{p=\pi_Z}\searrow & & \swarrow_{\pi} \\ & \mathbb{P}^2. & \end{array}$$

Now denote by F a fiber of the scroll projection $\pi : X \longrightarrow \mathbb{P}^2$. We observe that $H_F \cong \mathcal{O}_{\mathbb{P}^{n-2}}(1)$; so $(\mathcal{E} \otimes [-H])_F \cong \mathcal{O}_F^{\oplus r}$, hence $\mathcal{E} \otimes [-H] \cong \pi^* \mathcal{G}$ for a vector bundle $\mathcal{G}$ of rank r on $\mathbb{P}^2$. We can thus write

$$\det \mathcal{E} = \det(\pi^* \mathcal{G} \otimes [H]) = rH + \pi^* \det \mathcal{G}.$$

Let $\mathcal{V}$ be the ample vector bundle of rank $n-1$ on $\mathbb{P}^2$ such that $(X, H) = (\mathbb{P}_{\mathbb{P}^2}(\mathcal{V}), H(\mathcal{V}))$. Clearly $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ generated by H and $\pi^* \mathcal{O}_{\mathbb{P}^2}(1)$; moreover, by the canonical bundle formula,

$$K_X = -(n-1)H + \pi^*(\det \mathcal{V} + \mathcal{O}_{\mathbb{P}^2}(-3)).$$

We have the following equalities

$$\begin{aligned} -(n-r-1)H_Z + \pi_Z^*(K_{\mathbb{P}^2} + c_1(\mathcal{F})) = K_Z = \\ = (K_X + \det \mathcal{E})_Z = (-(n-r-1)H + \pi^* \mathcal{O}_{\mathbb{P}^2}(c_1(\mathcal{V}) + c_1(\mathcal{G}) - 3))_Z \end{aligned}$$

where $\mathcal{F}$ is the ample vector bundle such that $(Z, H_Z) \cong (\mathbb{P}_{\mathbb{P}^2}(\mathcal{F}), H(\mathcal{F}))$. From this we derive

$$\pi_Z^*(K_{\mathbb{P}^2} + c_1(\mathcal{F})) = \pi_Z^* \mathcal{O}_{\mathbb{P}^2}(c_1(\mathcal{V}) + c_1(\mathcal{G}) - 3),$$

which, recalling that $c_1(\mathcal{F}) = 4$ in all three Cases (i)–(iii), gives

$$c_1(\mathcal{V}) + c_1(\mathcal{G}) = 4. \tag{2.5.1}$$

Note that in each case $\mathcal{F}$ has a summand given by copies of $\mathcal{O}_{\mathbb{P}^2}(1)$. Let

$$\Sigma := \begin{cases} \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(1)) & \text{in Case (i);} \\ \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}) & \text{in Case (ii);} \\ \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4}) & \text{in Case (iii).} \end{cases}$$

Then $\Sigma \cong \mathbb{P}^2$, $\mathbb{P}^2 \times \mathbb{P}^1$ or $\mathbb{P}^2 \times \mathbb{P}^3$ respectively and we have the following diagram

$$\begin{array}{ccccc} \Sigma & \hookrightarrow & Z = \mathbb{P}_{\mathbb{P}^2}(\mathcal{F}) & \overset{\iota}{\hookrightarrow} & X := \mathbb{P}_{\mathbb{P}^2}(\mathcal{V}) \\ & {\scriptstyle \pi_\Sigma} \searrow & \downarrow {\scriptstyle \pi_Z} & \swarrow {\scriptstyle \pi} & \\ & & \mathbb{P}^2, & & \end{array}$$

where π_Σ is the restriction of π_Z to Σ. Moreover the surjection from $\mathcal{F}$ to the summand defining Σ gives an injection $\Sigma \subseteq Z$ such that $H_\Sigma = (H_Z)_\Sigma$ is either $\mathcal{O}_{\mathbb{P}^2}(1)$, $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(1,1)$, or $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^3}(1,1)$, respectively. Let $l \subset \Sigma$ be a line in Case (i) and a line contained in a fiber of the second projection in Cases (ii) and (iii). Then

$$H \cdot l = H_\Sigma \cdot l = 1.$$

Note that the restriction of π to l maps l isomorphically onto a line $\ell \subset \mathbb{P}^2$. Recalling that $\mathcal{E}$ is ample we thus get

$$r \leq \deg \mathcal{E}_l = \deg(\pi^*\mathcal{G} \otimes H)_l = \deg(\pi^*\mathcal{G})_l + rH \cdot l = \deg \mathcal{G}_\ell + rH \cdot l = c_1(\mathcal{G}) + r.$$

Hence $c_1(\mathcal{G}) \geq 0$. Recalling (2.5.1), the ampleness of $\mathcal{V}$ implies

$$4 = c_1(\mathcal{V}) + c_1(\mathcal{G}) \geq c_1(\mathcal{V}) \geq \mathrm{rk}(\mathcal{V}) = n - 1.$$

So, $n \leq 5$. On the other hand, $n = r + (n - r) \geq 2 + 4 = 6$, a contradiction. □

Lemma 2.6. *Let $S \subset \mathbb{P}^3$ be a smooth quintic surface and let $\mathcal{G}$ be an ample and spanned vector bundle of rank $r \geq 2$ on S. Then $c_1(\mathcal{G})^2 \geq 6$.*

Proof. Put $P := \mathbb{P}_S(\mathcal{G})$ and let $\pi\colon P \longrightarrow S$ be the bundle projection. Denote by ξ the tautological line bundle associated to $\mathcal{G}$ on P. Note that ξ is spanned, since $\mathcal{G}$ is so. Then there exists a smooth surface $\tilde{S} = \bigcap_{i=1}^{r-1} D_i$, where $D_i \in |\xi|$ (i.e. $\tilde{S} = (\tau)_0$ with τ a regular section of $\Gamma(\xi^{\oplus(r-1)})$). Note that $(\tilde{S}, \xi_{\tilde{S}})$ has $(S, \det \mathcal{G})$ as adjunction theoretic reduction. Indeed, the canonical bundle of $\tilde{S}$ is $K_{\tilde{S}} = (K_P + (r-1)\xi)_{\tilde{S}} = (-r\xi + \pi^*(K_S + \det \mathcal{G}) + (r-1)\xi)_{\tilde{S}} = -\xi_{\tilde{S}} + \pi_{\tilde{S}}^*(K_S + \det \mathcal{G})$, where $\pi_{\tilde{S}}$ is the restriction of π to $\tilde{S}$. Therefore $d(P, \xi) = d(\tilde{S}, \xi_{\tilde{S}}) = \xi_{\tilde{S}}^2 = \xi^2 \cdot \xi^{r-1} = \xi^{r+1}$. Since $\xi^r - \xi^{r-1} \cdot \pi^* c_1(\mathcal{G}) + \xi^{r-2} \cdot \pi^* c_2(\mathcal{G}) = 0$ by the Chern–Wu relation, we derive $\xi^{r+1} = \xi^r \cdot \pi^* c_1(\mathcal{G}) - \xi^{r-1} \cdot \pi^* c_2(\mathcal{G}) = \xi^{r-1} \cdot \pi^*(c_1(\mathcal{G})^2 - c_2(\mathcal{G})) = c_1(\mathcal{G})^2 - c_2(\mathcal{G})$. So

$$c_1(\mathcal{G})^2 = d(\tilde{S}, \xi_{\tilde{S}}) + c_2(\mathcal{G}). \tag{2.6.1}$$

Note that the morphism $\pi_{\tilde{S}}$ is birational and contracts exactly $c_2(\mathcal{G})$ (-1)-curves of $(\tilde{S}, \xi_{\tilde{S}})$. In particular, $\tilde{S}$ is not minimal, since $c_2(\mathcal{G}) > 0$.

Now we show that both summands on the right-hand side of (2.6.1) are ≥ 3.

If $c_2(\mathcal{G}) = 1$, then $(S, \mathcal{G}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus r})$, by [3, Theorem 11.1.3], but this is a contradiction, since the Kodaira dimension of S is $\kappa(S) = 2$. If $c_2(\mathcal{G}) = 2$, by [23, Corollary] and [28], we have the following possibilities for $(S, \mathcal{G})$:

(i) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$;
(ii) $(\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2})$, where $\mathbb{Q}^2$ is a smooth quadric in $\mathbb{P}^3$;
(iii) $S = \mathbb{P}_B(\mathcal{F})$ is a $\mathbb{P}^1$-bundle over an elliptic curve B and $\mathcal{G} = \xi_{\mathcal{F}} \otimes \pi^*\mathcal{V}$, where $\pi\colon S \longrightarrow B$ is the ruling projection, $\mathcal{F}$ and $\mathcal{V}$ are normalized rank-2 vector bundles of degree 1 on B and $\xi_{\mathcal{F}}$ is the tautological line bundle of $\mathcal{F}$;
(iv) there exists a finite morphism $f\colon S \longrightarrow \mathbb{P}^2$ of degree 2 and $\mathcal{G} \cong f^*\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$.

The first three cases are ruled out since in our assumption $\kappa(S) = 2$. To exclude the last one, let $\Delta \in |\mathcal{O}_{\mathbb{P}^2}(2b)|$, with $b > 0$, be the branch divisor of f. We can compute $K_S = f^*\mathcal{O}_{\mathbb{P}^2}(b-3)$ to derive that $b \geq 4$, since $\kappa(S) = 2$. Therefore $5 = K_S^2 = 2(b-3)^2$, which is a contradiction. So $c_2(\mathcal{G}) \geq 3$.

Now look at the summand $d(\tilde{S}, \xi_{\tilde{S}})$. Since $\xi_{\tilde{S}}$ is ample and spanned, if $d(\tilde{S}, \xi_{\tilde{S}}) = 1$, then $(\tilde{S}, \xi_{\tilde{S}}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, contradicting $\kappa(\tilde{S}) = \kappa(S) = 2$. On the other hand, if $d(\tilde{S}, \xi_{\tilde{S}}) = 2$, then either $(\tilde{S}, \xi_{\tilde{S}}) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))$, or there exists a finite morphism $f\colon \tilde{S} \longrightarrow \mathbb{P}^2$ of degree 2 and $\xi_{\tilde{S}} \cong f^*\mathcal{O}_{\mathbb{P}^2}(1)$. The former case gives again a contradiction with the Kodaira dimension of $\tilde{S}$. In the latter case, denote again by $\Delta \in |\mathcal{O}_{\mathbb{P}^2}(2b)|$, with $b > 0$, the branch divisor of f. The canonical bundle of $\tilde{S}$ is $K_{\tilde{S}} = f^*((b-3)\mathcal{O}_{\mathbb{P}^2}(1)) = (b-3)\xi_{\tilde{S}}$; hence $b \geq 4$, as $\kappa(\tilde{S}) = \kappa(S) = 2$. But then $K_{\tilde{S}}$ is ample, whence $\tilde{S}$ is minimal, a contradiction. □

Let X be a smooth complex projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank $r = 2$ on X. If $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold, we say that $(\mathbb{P}_X(\mathcal{E}), X, \mathcal{E})$ is a ruled Fano manifold, according to [29, Definition 3.1]. We need the following result from [29] (see [29, Theorem 1.1, Propositions 5.1 and 5.2, Corollary 5.3]), which we restate for our use in Lemma 2.9.

Theorem 2.7. *Let X and $\mathcal{E}$ be as above and assume $n = 4$. Suppose that $(\mathbb{P}_X(\mathcal{E}), X, \mathcal{E})$ is a ruled Fano manifold such that $K_X + \det\mathcal{E} = \mathcal{O}_X$. If X has Picard number $\rho(X) \geq 2$, then either*

(1) *X is a Fano 4-fold of index 2 (for the classification of these manifolds see [37]) and $\mathcal{E} = \mathcal{L}^{\oplus 2}$, where $\mathcal{L}$ is an ample line bundle on X, or*
(2) *$(X, \mathcal{E})$ is one of the following pairs:*
  (2a) $(\mathrm{Bl}_p(\mathbb{P}^4), [2h + E] \oplus [3h + E])$;
  (2b) $(\mathrm{Bl}_l(\mathbb{P}^4), [2h - E] \oplus [3h - E])$;
  (2c) $(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,2) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(2,1))$;
  (2d) $(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(2,2))$;
  (2e) $X = \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}) \subset \mathbb{P}^2 \times \mathbb{P}^3$ *and* $\mathcal{E} = (\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,1) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,2))_X$.
  In (2a) *and* (2b) *E stands for the exceptional divisor and h for the pullback of $\mathcal{O}_{\mathbb{P}^4}(1)$ on X.*

Remark 2.8. Note that the list in Table 0.3 of [37] is not affected by the missed case in the Mori–Mukai classification of Fano 3-folds with $b_2 \geq 2$. Actually for such a 3-fold Y (Case (13) in Table 4 of [27]) we know that $\rho(Y) = 4$ and $(-K_Y)^3 = 26$. So, if X is a Fano 4-fold such that $-K_X = 2H$ with $Y \in |H|$, we have $\rho(X) = 4$ by the Lefschetz theorem. But then the validity of the generalized Mukai conjecture in dimension 4 [4] would imply that $X = (\mathbb{P}^1)^4$. Therefore $Y \in |\mathcal{O}_{(\mathbb{P}^1)^4}(1,1,1,1)|$, which gives $(-K_Y)^3 = (H_Y)^3 = H^4 = 24$, a contradiction.

Lemma 2.9. *Let $X, \mathcal{E}$ and Z be as in* 1.1*, with $n - r = 2$. Let H be an ample line bundle on X such that H_Z is very ample, and suppose that (Z, H_Z) is a K3 surface of degree ≤ 8. Then $\mathrm{Pic}(X) \cong \mathbb{Z}$, generated by H.*

Proof. If $\mathrm{Pic}(X) \cong \mathbb{Z}$, then $\mathrm{Pic}(X)$ is generated by H; otherwise $H = a\mathcal{L}$, where $\mathcal{L}$ is an ample line bundle and $a \geq 2$. Note that $\mathcal{L}_Z^2$ is even, by the genus formula, since K_Z is trivial. Hence $H_Z^2 = a^2\mathcal{L}_Z^2 \geq 2a^2 \geq 8$. Since $H_Z^2 \leq 8$, this implies $H = 2\mathcal{L}$ with $\mathcal{L}_Z^2 = 2$. We show that this is impossible. Note that $|\mathcal{L}_Z|$ has no fixed part, otherwise we could contradict the ampleness of $\mathcal{L}_Z$. It then follows from [34, Corollary 3.2] that $\mathcal{L}_Z$ is spanned, so, since $h^0(\mathcal{L}_Z) = 3$, we have that the map $\psi\colon Z \longrightarrow \mathbb{P}^2$ defined by $|\mathcal{L}_Z|$ is a morphism of degree 2. As $H_Z = 2\mathcal{L}_Z = \psi^*\mathcal{O}_{\mathbb{P}^2}(2)$, we see that

$$\psi^*|\mathcal{O}_{\mathbb{P}^2}(2)| \subseteq |H_Z|. \tag{2.9.1}$$

On the other hand, by the Riemann–Roch theorem and the Kodaira vanishing theorem, we get $h^0(H_Z) = 6$. This shows that (2.9.1) is in fact an equality. As a consequence, the morphism $\varphi\colon Z \longrightarrow \mathbb{P}^5$ defined by $|H_Z|$ factors via ψ and the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ as follows:

$$\begin{array}{ccc} Z & \xrightarrow{\quad\varphi\quad} & \mathbb{P}^5 \\ & {\scriptstyle\psi}\searrow \quad \nearrow & \\ & \mathbb{P}^2. & \end{array}$$

But then φ cannot be an embedding, since $\deg\psi = 2$. This contradicts the very ampleness of H_Z.

Next, note that $K_X + \det\mathcal{E} = \mathcal{O}_X$ by the Lefschetz–Sommese theorem, hence X is Fano. Now, suppose that X has Picard number $\rho(X) \geq 2$. We claim that

$$4 \leq n \leq 5. \tag{2.9.2}$$

This follows from [21, Proposition 5], for $H_Z^2 = 4$, but note that the same argument works for $H_Z^2 \leq 8$. The remaining part of the proof is devoted to show that (2.9.2) does not occur. The procedure, which has to be adapted to many cases, is the following. We find a suitable basis $\{L_i\}$, $i = 1, \dots, \rho := \rho(X)$, of $\mathrm{Pic}(X)$. Using an explicit expression of $\mathcal{E}$ (which in most cases turns out to be decomposable) and writing $H = \sum_{i=1}^{\rho} a_i L_i$, where the integers a_i have to satisfy some conditions reflecting the ampleness of H, we get

$$H_Z^2 = H^2 \cdot c_2(\mathcal{E}) = f(a_1, \dots, a_\rho),$$

where f is a polynomial of degree 2 in the a_i's. Moreover, computing the intersection indexes $L_i^k \cdot L_j^{(n-k)}$ we succeed to show that $f(a_1, \dots, a_\rho) > 8$ for all admissible values of the a_i's.

First assume that $n = 5$. Then $r = 3$. As $K_X + \det\mathcal{E} = \mathcal{O}_X$, by [30, Theorem 1], $(X, \mathcal{E})$ is one of the following:

(i) $(\mathbb{P}^2 \times \mathbb{Q}^3, \mathcal{O}_{\mathbb{P}^2\times\mathbb{Q}^3}(1,1)^{\oplus 3})$;
(ii) $(\mathbb{P}^2 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,2) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,1)^{\oplus 2}$ or $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,0) \otimes p_2^*T_{\mathbb{P}^3})$, where p_2 is the projection onto the second factor;
(iii) $X = \mathbb{P}_{\mathbb{P}^3}(\mathcal{V})$, where $\mathcal{V} = \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$, and $\mathcal{E} = [\xi + \pi^*\mathcal{O}_{\mathbb{P}^3}(1)]^{\oplus 3}$, where ξ is the tautological line bundle of $\mathcal{V}(-1)$ and $\pi\colon X \longrightarrow \mathbb{P}^3$ is the projection;
(iv) $X = \mathbb{P}_{\mathbb{P}^3}(T_{\mathbb{P}^3}) \subset \mathbb{P}^3 \times \mathbb{P}^3$ and $\mathcal{E} = (\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^3}(1,1)^{\oplus 3})_X$.

A direct computation shows that none of these cases can occur under the assumption $H_Z^2 \leq 8$. To give an example let us discuss Case (iv) in detail.

Case (iv). Put $A_1 := \mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^3}(1,0)$ and $A_2 := \mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^3}(0,1)$, and note that $X \in |A_1 + A_2|$. Moreover $A_1^4 = A_2^4 = 0$, while $A_1^3 \cdot A_2^3 = 1$. Set $L_i = (A_i)_X$; clearly L_1 and L_2 generate $\mathrm{Pic}(X)$. Since H is ample, we can write $H = a_1L_1 + a_2L_2$ for some positive integers a_1 and a_2. To see this, note that H_{F_i} is an ample divisor for any fiber fiber F_i of the map $X \longrightarrow \mathbb{P}^3$ induced by the i-th projection of $\mathbb{P}^3\times\mathbb{P}^3$, $i = 1, 2$. As $L_1 + L_2$ is ample and $F_i = A_i^3 \cdot X$, we have $0 < H_{F_i} \cdot (L_1 + L_2)_{F_i} = (a_1A_1 + a_2A_2) \cdot (A_1 + A_2)^2 \cdot (A_i)^3 = a_j$, $j \neq i$. We get $H_Z^2 = H^2 \cdot Z = H^2 \cdot c_3(\mathcal{E}) = (a_1L_1 + a_2L_2)^2 \cdot (L_1 + L_2)^3 = (a_1A_1 + a_2A_2)^2 \cdot (A_1 + A_2)^4 = 4(a_1^2 + 3a_1a_2 + a_2^2)A_1^3 \cdot A_2^3 \geq 20$, which is a contradiction.

Let us discuss also the second possibility in (ii), the only case in which $\mathcal{E}$ is not decomposable.

Case (ii) *with* $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,0) \otimes p_2^*T_{\mathbb{P}^3}$. Put $L_1 := \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,0)$ and $L_2 := \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(0,1)$, and note that $L_1^2 \cdot L_2^3 = 1$ while $L_1^3 = 0 = L_2^4$. Since H is ample, we can write again $H = a_1L_1 + a_2L_2$ for some positive integers a_1 and a_2. Moreover, we have $c_3(\mathcal{E}) = p_2^*c_3(T_{\mathbb{P}^3}) + p_2^*c_2(T_{\mathbb{P}^3}) \cdot L_1 + p_2^*c_1(T_{\mathbb{P}^3}) \cdot L_1^2 + L_1^3 = 4L_2^3 + 6L_2^2 \cdot L_1 + 4L_2 \cdot L_1^2 + L_1^3$. We get $H_Z^2 = H^2 \cdot Z = H^2 \cdot c_3(\mathcal{E}) = 4(a_1^2 + 3a_1a_2 + a_2^2)L_1^2 \cdot L_2^3 \geq 20$, which is a contradiction.

Assume now that $n = 4$. Then $r = 2$ and $(\mathbb{P}_X(\mathcal{E}), X, \mathcal{E})$ is a ruled Fano 5-fold in the sense of [29, Definition 3.1]. Since $\rho(X) \geq 2$, pairs $(X, \mathcal{E})$ are classified in Theorem 2.7.

We claim that Case (1) of Theorem 2.7 cannot occur under the assumption $H_Z^2 \leq 8$. Note that, in this case, $\mathcal{E} = \mathcal{L}^{\oplus 2}$ where $-K_X = \det \mathcal{E} = 2\mathcal{L}$. So $c_2(\mathcal{E}) = \mathcal{L}^2$. According to [37, Table 0.3] we have 18 possibilities. The last one (Case (18) in [37]) can be easily ruled out as in [21, Proof of Proposition 6, Case (d)]. In all remaining cases $\mathcal{L} = t_1L_1 + t_2L_2$ for suitable integers t_1, t_2 and then

$$f(a_1, \dots, a_\rho) = (a_1L_1 + \cdots + a_\rho L_\rho)^2 \cdot (t_1L_1 + t_2L_2)^2.$$

To provide details, let us discuss some specific cases: Cases (27), (30), (31), (32) and (41) in [29, Table] (corresponding to Cases (4), (7), (11), (12), and (16) in [37, Table 0.3], respectively). All remaining X's in the list can be ruled out in similar ways.

Case (27). X is a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ branched along a divisor in $|\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(2,2)|$. Let $\pi\colon X \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ be the morphism giving the double cover. By [6], $\mathrm{Pic}(X) \cong \mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2) \cong \mathbb{Z}^2$, generated by $L_1 = \pi^*\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,0)$ and $L_2 = \pi^*\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(0,1)$. We have $K_X = \pi^*\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-2,-2)$, so $\mathcal{L} = \pi^*\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1) = L_1 + L_2$. Note that $H = a_1L_1 + a_2L_2$, for some positive integers $a_1, a_2 \geq 1$, as H is ample. Moreover $L_1^3 = L_2^3 = 0$, while $L_1^2 \cdot L_2^2 = 2$. Therefore we can compute $H_Z^2 = H^2 \cdot c_2(\mathcal{E}) = (a_1^2 + 4a_1a_2 + a_2^2)L_1^2 \cdot L_2^2 \geq 12$, which is a contradiction with $H_Z^2 \leq 8$.

Case (30). X is the intersection of two divisors in $\mathbb{P}^3 \times \mathbb{P}^3$ of bidegree $(1,1)$. Let $A_1 = \mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^3}(1,0), A_2 = \mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^3}(0,1)$ and let $Y \in |A_1 + A_2|$ be a smooth element containing X. The Picard group of X is $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y) \cong \mathrm{Pic}(\mathbb{P}^3 \times \mathbb{P}^3) \cong \mathbb{Z}^2$, generated by $L_1 = (A_1)_X$ and $L_2 = (A_2)_X$. Clearly, the canonical bundle of X is $K_X = -2(L_1 + L_2)$, hence $\mathcal{L} = L_1 + L_2$. By the ampleness of H, we can write $H = a_1L_1 + a_2L_2$ for some positive integers a_1, a_2. Indeed, let $\pi_1 \colon X \longrightarrow \mathbb{P}^3$ be the restriction of the first projection of $\mathbb{P}^3 \times \mathbb{P}^3$ and let γ be a fiber of π_1. Then $\gamma = A_1^3$. Since H is ample, we have $H \cdot \gamma > 0$. As $A_1^4 = A_2^4 = 0$ and $A_1^3 \cdot A_2^3 = 1$, we can derive $H \cdot \gamma = (a_1A_1 + a_2A_2) \cdot (A_1 + A_2)^2 \cdot A_1^3 = a_2$, hence $a_2 > 0$. Similarly we get $a_1 > 0$. Therefore we can compute $H_Z^2 = H^2 \cdot c_2(\mathcal{E}) = (a_1A_1 + a_2A_2)^2 \cdot (A_1 + A_2)^4 = 4(a_1^2 + 3a_1a_2 + a_2^2)A_1^3 \cdot A_2^3 \geq 20$; so we have a contradiction with $H_Z^2 \leq 8$.

Case (31). $X = \mathbb{P}_{\mathbb{P}^3}(\mathcal{N})$ is the projectivization of a null-correlation bundle $\mathcal{N}$ on $\mathbb{P}^3$. Recall that $c_1(\mathcal{N}) = 0$ and $c_2(\mathcal{N}) = 1$ (see [32, p. 80]). Here $\mathrm{Pic}(X) \cong \mathbb{Z}^2$, generated by ξ and $M := \pi^*\mathcal{O}_{\mathbb{P}^3}(1)$, where π is the bundle projection and ξ is the tautological line bundle. So any line bundle on X can be written as $D = a_1\xi + a_2M$ for suitable integers a_1, a_2. Note that $a_1 > 0$ if D is ample, since it must intersect lines living in the fibers of π. Notice that $\mathcal{N}(2)$ is ample, while $\mathcal{N}(1)$ is spanned but not ample. Moreover M is spanned. Thus, writing $D = \xi + 2M + (a_1 - 1)(\xi + M) + (a_2 - a_1 - 1)M$, we see that D is ample if $a_2 \geq a_1 + 1$, while it is not if $a_1 = a_2$. It follows that the ample cone of X, being convex, cannot contain D if $a_2 \leq a_1$. Therefore D is ample if and only if $a_1 \geq 1$, $a_2 \geq a_1 + 1$. In particular, we can write $H = a_1\xi + a_2M$, where the integers a_1 and a_2 satisfy these conditions. We have $M^4 = 0$ and $\xi \cdot M^3 = 1$. Moreover, $\xi^2 - \xi \cdot \pi^*c_1(\mathcal{N}) + \pi^*c_2(\mathcal{N}) = 0$, hence $\xi^2 = -\pi^*c_2(\mathcal{N})$. We can thus derive $\xi^2 \cdot M^2 = 0$, $\xi^3 \cdot M = -1$ and $\xi^4 = 0$. By the canonical bundle formula, we get $K_X = -2(\xi + 2M)^2$. So $c_2(\mathcal{E}) = (\xi + 2M)^2$. Therefore we can compute $H_Z^2 = H^2 \cdot c_2(\mathcal{E}) = (a_1^2\xi^2 + 2a_1a_2\xi \cdot M + a_2^2M^2) \cdot (\xi^2 + 4\xi \cdot M + 4M^2) = 2(a_2 + 2a_1)(2a_2 - a_1) \geq 24$ in view of the ampleness conditions above. So we have a contradiction with $H_Z^2 \leq 8$.

Case (32). $X = \mathrm{Bl}_l(\mathbb{Q}^4)$, l a line in $\mathbb{Q}^4$. Let $\sigma \colon \mathrm{Bl}_l(\mathbb{Q}^4) \longrightarrow \mathbb{Q}^4$ be the blow-up and denote by E the exceptional divisor; let $\pi \colon E \longrightarrow l$ be the projection of the $\mathbb{P}^2$-bundle E. The Picard group of X is generated by $\sigma^*\mathcal{O}_{\mathbb{Q}^4}(1)$ and E; moreover the canonical bundle of X is $K_X = \sigma^*K_{\mathbb{Q}^4} + 2E = \sigma^*\mathcal{O}_{\mathbb{Q}^4}(-4) + 2E$, whence $\mathcal{L} = \sigma^*\mathcal{O}_{\mathbb{Q}^4}(2) - E$. Now, we can write $H = \sigma^*\mathcal{O}_{\mathbb{Q}^4}(a_1) + a_2E$. We claim that $a_1 > 0$ and $a_2 < 0$, due to the ampleness of H. Indeed, to see the first inequality it is enough to take a line in $\mathbb{Q}^4$ not meeting l and consider its proper transform, γ; then $0 < H \cdot \gamma = (\sigma^*\mathcal{O}_{\mathbb{Q}^4}(a_1) + a_2E) \cdot \gamma = a_1$. As to the second one, let λ be a line in a fiber of π. Since $\lambda \subset E$ and E induces $\mathcal{O}_{\mathbb{P}^2}(-1)$ on the fibers of π, we get $0 < H \cdot \lambda = (\sigma^*\mathcal{O}_{\mathbb{Q}^4}(a_1) + a_2E) \cdot \lambda = a_2E_\lambda = -a_2(-E_E \cdot \lambda) = -a_2$. Moreover, if we take a line in $\mathbb{Q}^4$ meeting l at one point, for its proper transform δ we find that $0 < H \cdot \delta = (\sigma^*\mathcal{O}_{\mathbb{Q}^4}(a_1) + a_2E) \cdot \delta = a_1 + a_2$. We can therefore compute $H_Z^2 = H^2 \cdot c_2(\mathcal{E}) = (\sigma^*\mathcal{O}_{\mathbb{Q}^4}(a_1) + a_2E)^2 \cdot (\sigma^*\mathcal{O}_{\mathbb{Q}^4}(2) - E)^2 = 4a_1^2(\sigma^*\mathcal{O}_{\mathbb{Q}^4}(1))^4 + a_2^2E^4$. Recalling that $E = \mathbb{P}(N_{l/\mathbb{Q}^4})$, that $-E$ induces the tautological line bundle on E and the Chern–Wu relation, we get $E^4 = -\deg N_{l/\mathbb{Q}^4}$, so the inequalities above imply that $H_Z^2 = 8a_1^2 - 2a_2^2 = 2(2a_1 - a_2)(2a_1 + a_2) > 2(2a_1 - a_2)(a_1 + 1) \geq 12$, contradicting $H_Z^2 \leq 8$.

Case (41). $X = \mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$. Let p and q be the projections onto $\mathbb{P}^1$ and $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$, respectively, and let $\pi \colon \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}) \longrightarrow \mathbb{P}^2$. Clearly $\mathrm{Pic}(\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})) \cong \mathbb{Z}^2$ and it is generated by the ample tautological line bundle ξ associated to $T_{\mathbb{P}^2}$ and the pull-back $\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. Therefore $\mathrm{Pic}(X) \cong \mathbb{Z}^3$ and it is generated by $L_1 = p^*\mathcal{O}_{\mathbb{P}^1}(1)$, $L_2 = q^*\xi$ and

$L_3 = q^*\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. Since $K_{\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})} = -2\xi + \pi^*(\mathcal{O}_{\mathbb{P}^2}(-3) + \det T_{\mathbb{P}^2}) = -2\xi$, we derive that $K_X = p^*\mathcal{O}_{\mathbb{P}^1}(-2) + q^*(-2\xi)$, so $\mathcal{L} = L_1 + L_2$. Now we write $H = a_1L_1 + q^*D$, for some positive integer a_1 and an ample line bundle $D \in \mathrm{Pic}(\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}))$. Put $D = a_2\xi + a_3\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, where a_2, a_3 are integers. We claim that $a_2 > 0$ and $a_2 + a_3 > 0$. Indeed, if f is any fiber of π, then $D \cdot f = (a_2\xi + a_3\pi^*\mathcal{O}_{\mathbb{P}^2}(1)) \cdot f = a_2\xi \cdot f = a_2$, whence $a_2 > 0$, as D is ample. On the other hand $T_{\mathbb{P}^2}(-1)$ is spanned but not ample and so is its tautological line bundle $\xi - \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. Therefore there exists an irreducible curve $\gamma \subset \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$ such that $\gamma \cdot (\xi - \pi^*\mathcal{O}_{\mathbb{P}^2}(1)) = 0$. Then $D \cdot \gamma = (a_2 + a_3)\xi \cdot \gamma - a_3(\xi - \pi^*\mathcal{O}_{\mathbb{P}^2}(1)) \cdot \gamma = (a_2 + a_3)\xi \cdot \gamma$, hence $a_2 + a_3 > 0$, since D and ξ are ample. Therefore we can write $H = a_1L_1 + a_2L_2 + a_3L_3$, with $a_1 > 0$, $a_2 > 0$ and $a_2 + a_3 > 0$. Now, by the Chern–Wu formula, we have $\xi^2 - \xi \cdot \pi^*c_1(T_{\mathbb{P}^2}) + \pi^*c_2(T_{\mathbb{P}^2}) = 0$. So $\xi^2 = \xi \cdot \pi^*c_1(T_{\mathbb{P}^2}) - 3f$, and we obtain $\xi \cdot \pi^*\mathcal{O}_{\mathbb{P}^2}(1)^2 = \xi \cdot f = 1$, $\xi^2 \cdot \pi^*\mathcal{O}_{\mathbb{P}^2}(1) = 3$ and $\xi^3 = 6$. We can thus derive $L_1^2 = 0$, $L_2^{4-i} \cdot L_3^i = 0$ for $i = 0, \dots, 3$, $L_1 \cdot L_2 \cdot L_3^2 = 1$, $L_1 \cdot L_2^2 \cdot L_3 = 3$ and $L_1 \cdot L_2^3 = 6$. Finally we compute $H_Z^2 = H^2 \cdot \mathcal{L}^2 = (a_1L_1 + a_2L_2 + a_3L_3)^2 \cdot (L_1 + L_2)^2 = 6(2a_2^2 + 2a_1a_2) + (2a_3^2) + 3(2a_1a_3 + 4a_2a_3)$. Recalling that $a_3 \geq -a_2 + 1$, we get $H_Z^2 \geq 6a_1a_2 + 2a_3^2 + 6(a_1 + 2a_2) \geq 24$, which is a contradiction with $H_Z^2 \leq 8$.

The five possibilities in Case (2) of Theorem 2.7 can also be excluded. We provide details case-by-case.

Case (2a). $X = \mathrm{Bl}_p(\mathbb{P}^4)$ and $\mathcal{E} = [2h + E] \oplus [3h + E]$, E exceptional divisor and h pullback of $\mathcal{O}_{\mathbb{P}^4}(1)$ on X. Let $\sigma\colon \mathrm{Bl}_p(\mathbb{P}^4) \longrightarrow \mathbb{P}^4$ be the blow-up and consider the $\mathbb{P}^1$-bundle $\pi\colon \mathrm{Bl}_p(\mathbb{P}^4) \longrightarrow \mathbb{P}^3$ induced by σ. If we call f a fiber of π, we derive $E \cdot f = 1$ and $h \cdot f = 1$. We note that $\mathrm{Pic}(X) \cong \mathbb{Z}^2$, generated by h and E. Therefore we can write $H = a_1h + a_2E$ for some integers a_1, a_2. Recall that H is ample. So $0 < H \cdot f = a_1 + a_2$; moreover, for a line $l \subset E$, $0 < H \cdot l = a_1h \cdot l + a_2E \cdot l = -a_2$. Combining these inequalities gives $a_1 \geq -a_2 + 1 \geq 2$. Note that $h^4 = 1$, $E^4 = (E_E)^3 = (\mathcal{O}_E(-1))^3 = -1$ and $h^i \cdot E^{4-i} = 0$ for $i = 1, 2, 3$. Now we can compute $H_Z^2 = (2h + E) \cdot (3h + E) \cdot (a_1^2h^2 + 2a_1a_2h \cdot E + a_2^2E^2) = 6a_1^2 - a_2^2 = 5a_1^2 + (a_1 + a_2)(a_1 - a_2) \geq 23$, contradicting $H_Z^2 \leq 8$.

Case (2b). $X = \mathrm{Bl}_l(\mathbb{P}^4)$ and $\mathcal{E} = [2h - E] \oplus [3h - E]$, E exceptional divisor and h pullback of $\mathcal{O}_{\mathbb{P}^4}(1)$ on X. Let $\sigma\colon \mathrm{Bl}_l(\mathbb{P}^4) \longrightarrow \mathbb{P}^4$ be the blow-up and denote by π the $\mathbb{P}^2$-bundle $\pi\colon E \longrightarrow l$. If we call f a general fiber of π, we have $E_f = \mathcal{O}_f(-1)$ and $h_f = (\sigma^*\mathcal{O}_{\mathbb{P}^4}(1))_f = \mathcal{O}_f$. Now, we note that $\mathrm{Pic}(X) \cong \mathbb{Z}^2$, generated by h and E. Therefore we can write $H = a_1h + a_2E$ for some integers a_1, a_2. Let λ and $\gamma \subset \mathrm{Bl}_l(\mathbb{P}^4)$ be the proper transforms of a line in $\mathbb{P}^4$ not meeting l and meeting l at one point, respectively; let c a line in a fiber f. Recall that H is ample. Thus we find

$$0 < H \cdot \lambda = a_1, \quad 0 < H \cdot c = -a_2 \quad \text{and} \quad 0 < H \cdot \gamma = a_1 + a_2.$$

In particular, $a_1 \geq -a_2 + 1 \geq 2$. Now, recall that $E = \mathbb{P}(N_{l/\mathbb{P}^4})$, with $-E$ inducing the tautological line bundle on E. From the Chern–Wu relation we thus get $E^4 = -\deg N_{l/\mathbb{P}^4} = -3$. Moreover we have $h^4 = 1$, $h^3 \cdot E = h^2 \cdot E^2 = 0$, $h \cdot E^3 = h \cdot s = \mathcal{O}_{\mathbb{P}^4}(1) \cdot l = 1$, where s is a section of E. So we can compute $H_Z^2 = H^2 \cdot c_2(\mathcal{E}) = (a_1^2h^2 + 2a_1a_2h \cdot E + a_2^2E^2) \cdot (2h - E) \cdot (3h - E) = 6a_1^2 + 2a_1a_2 - 8a_2^2 = 2(a_1 - a_2)(3a_1 + 4a_2)$. We thus see that

$$(a_1 - a_2)(3a_1 + 4a_2) = \frac{1}{2}H_Z^2, \tag{2.9.3}$$

where the first factor on the left-hand side is ≥ 3. Note that $H_Z^2 = 4, 6$ or 8, since H_Z is very ample and $H_Z^2 \leq 8$. In the first Case (2.9.3) is clearly impossible. In the remaining cases factoring the right term in (2.9.3) leads to systems of linear equations in a_1, a_2 not admitting integral solutions. This is a contradiction.

Cases (2c) *and* (2d). $X = \mathbb{P}^2 \times \mathbb{P}^2$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,i) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(2,j)$, with $\{i,j\} = \{1,2\}$. Clearly $H = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(a_1,a_2)$ for some positive integers a_1, a_2. Let $L_1 = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,0)$ and $L_2 = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(0,1)$; therefore $L_1^3 = L_2^3 = 0$ and $L_1^2 \cdot L_2^2 = 1$. So $H_Z^2 = (a_1L_1+a_2L_2)^2\cdot(L_1+iL_2)\cdot(2L_1+jL_2) = 2(a_1^2+a_1a_2(2i+j)+a_2^2)L_1^2\cdot L_2^2 \geq 12$, which contradicts $H_Z^2 \leq 8$.

Case (2e). $X = \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}) \subset \mathbb{P}^2 \times \mathbb{P}^3$, the inclusion deriving from the Euler sequence of $\mathbb{P}^2$, and $\mathcal{E} = (\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,1) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,2))_X$. Let $\pi\colon X \longrightarrow \mathbb{P}^2$ be the projection; denote by f a fiber of π and by ξ the tautological line bundle associated to $T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}$ on X. Note that $\xi = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(0,1)_X$ and $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,0)_X = M$, where $M = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. Then $\xi^3 - \xi^2 \cdot \pi^*c_1 + \xi \cdot \pi^*c_2 = 0$, where $c_i = c_i(T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2})$. Hence $\xi^3 = \xi^2 \cdot \pi^*\mathcal{O}_{\mathbb{P}^2}(1) - \mathcal{O}_f(1)$. This gives $\xi^4 = 0$. Since the Picard group of X is generated by ξ and M, we can write $H = a_1\xi + a_2M$, for some integers a_1, a_2. We claim that $a_1, a_2 > 0$. To see the former inequality, take a line $l \subset f$; then $0 < l \cdot H = a_1$, as $l \cdot \xi_f = 1$. To see the latter, consider a section Σ of π corresponding to the surjection $T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}$. Note that $\xi_\Sigma = \mathcal{O}_\Sigma$. If λ is a line in Σ, we get $0 < H \cdot \lambda = a_1\xi_\Sigma \cdot \lambda + a_2M_\Sigma \cdot \lambda = a_2$. We have $M^3 = 0$ and $\xi^3 \cdot M = \xi^2 \cdot M^2 = 1$. Therefore we can compute $H_Z^2 = (a_1\xi + a_2M)^2 \cdot (\xi + M) \cdot (2\xi + M) = 4a_1^2 + 10a_1a_2 + 2a_2^2 \geq 16$, contradicting $H_Z^2 \leq 8$. □

3 Ample vector bundles with zero loci of small Δ-genera

In this section we deal with triplets $(X, \mathcal{E}, H)$ where $(X, \mathcal{E})$ is as in 1.1 and H is an ample line bundle on X such that H_Z is very ample and $\Delta(Z, H_Z) \leq 3$.

Remark 3.1. Let C be a smooth curve and let $\mathcal{L} \in \mathrm{Pic}(C)$ be an ample line bundle on C. Then $\Delta(C, \mathcal{L}) = 0$ if and only if $C \cong \mathbb{P}^1$.

Theorem 3.2. *Let $X, \mathcal{E}$ and Z be as in* 1.1 *with* $n-r \geq 1$. *Let H be an ample line bundle on X and assume that*

$$\Delta(Z, H_Z) = 0.$$

Then $(X, \mathcal{E} \oplus H^{\oplus(n-r-1)})$ is one of the following:

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-1)})$;
(2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-2)})$;
(3) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-1)})$;
(4) $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{V})$, *where $\mathcal{V}$ is an ample vector bundle of rank n over $\mathbb{P}^1$, and* $(\mathcal{E} \oplus H^{\oplus(n-r-1)})_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-1)}$ *for every fiber F of the projection $X \longrightarrow \mathbb{P}^1$.*

If $n - r = 1$, then H is any ample line bundle on X.

Proof. Assume first that $n-r=1$. Then, by Remark (3.1), $Z \cong \mathbb{P}^1$. We can thus apply [14, Theorem A] to the pair $(X, \mathcal{E})$ to conclude. In all cases H is any ample line bundle on X; moreover, H is actually very ample (for Case (4) use [3, Lemma 3.2.4]).

Assume now that $n-r \geq 2$ and put $\mathcal{F} := \mathcal{E} \oplus H^{\oplus(n-r-1)}$. Then $\mathcal{F}$ is an ample vector bundle of rank $n-1$ on X. Note that $g(Z, H_Z) = 0$ because $\Delta(Z, H_Z) = 0$ by [9, Theorem 5.10]. Moreover, $g(Z, H_Z) = 1 + \frac{1}{2}(K_Z + (n-r-1)H_Z) \cdot H_Z^{n-r-1} = 1 + \frac{1}{2}(K_X + \det \mathcal{F}) \cdot c_{n-1}(\mathcal{F}) = g(X, \mathcal{F})$, so the assertion follows by [25, Theorem 1]. □

Remark 3.3. Let C be a smooth curve and let $\mathcal{L} \in \mathrm{Pic}(C)$ be an ample line bundle on C of degree $d = \deg \mathcal{L}$. The following assertion is an easy consequence of the Riemann–Roch theorem, combined with Clifford's theorem when $g(C) \geq 2$. If $\Delta(C, \mathcal{L}) = 1$, then one of the following holds:

(1) $g(C) = 1$;
(2) $g(C) \geq 2$ and C is a hyperelliptic curve (of genus $g(C) \geq 2$) and $|\mathcal{L}| = g_2^1$, hence $d = 2$;
(3) $g(C) \geq 2$ and $d = 1$.

In particular, if $\mathcal{L}$ is spanned then only (1) and (2) hold, while if $\mathcal{L}$ is very ample then Case (1) is the only possibility.

Remark 3.4. In order to investigate the case $\Delta(Z, H_Z) = 1$, it is necessary to assume H_Z very ample. Indeed, even with the assumption H_Z spanned, in Case (2) of Remark 3.3 what is known at present on the adjunction map would not allow us to classify the pairs $(X, \mathcal{E})$, even when $\mathcal{E}$ is very ample of corank 1. On the other hand, note that H_Z is very ample in all the cases of Theorem 3.2.

Therefore we give the classification in case $\Delta(Z, H_Z) = 1$ under the assumption H_Z is very ample.

Theorem 3.5. *Let $X, \mathcal{E}$ and Z be as in* 1.1 *with $n-r \geq 1$. Let H be an ample line bundle on X such that H_Z is very ample and assume that*

$$\Delta(Z, H_Z) = 1.$$

Then $(X, \mathcal{E} \oplus H^{\oplus(n-r-1)})$ is one of the following:

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-3)})$*;*
(2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(3) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-2)})$*;*
(3) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(2) \oplus \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-2)})$*;*
(4) $(X, \mathcal{L})$ *is a del Pezzo manifold, $\mathcal{E} \oplus H^{\oplus(n-r-1)} = \mathcal{L}^{\oplus(n-1)}$ and $d(X, \mathcal{L}) \geq 3$. Such pairs are classified in* [7] (*see also* [9, Chapter I, Section 8])*;*
(5) $(\mathbb{Q}^4, \mathcal{S}(2) \oplus \mathcal{O}_{\mathbb{Q}^4}(1))$*, where $\mathcal{S}$ is a spinor bundle over $\mathbb{Q}^4$;*
(6) $(\mathbb{P}^3, \mathcal{N}(2))$*, where $\mathcal{N}$ is a null-correlation bundle over $\mathbb{P}^3$;*
(7) $(\mathbb{Q}^3, \mathcal{S}(2))$*, where $\mathcal{S}$ is a spinor bundle over $\mathbb{Q}^3$;*
(8) $(\mathbb{P}^2 \times \mathbb{P}^1, \pi^* T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(1,1))$*, where π denotes the first projection;*
(9) $(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(2,1) \oplus \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(1,1))$*;*

(10) $X \cong \mathbb{P}_B(\mathcal{V})$, *where* $\mathcal{V}$ *is an ample vector bundle of rank* n *over an elliptic curve* $B \cong Z$ *and* $(\mathcal{E} \oplus H^{\oplus(n-r-1)})_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-1)}$ *for every fiber* F *of the projection* $\pi\colon X \longrightarrow B$.

If $n - r = 1$, *then* H *is any ample line bundle on* X *except in Case* (10). *In that case, let* ξ *be the tautological line bundle of* $\mathcal{V}$, *let* δ *be a line bundle on* B *and let* $\mathcal{G}$ *be a rank-*$(n-1)$ *vector bundle on* B *such that* $\mathcal{E} \cong \xi \otimes \pi^*\mathcal{G}$; *then* $H = \alpha\xi + \pi^*\delta$ *where the integer* α *satisfies the condition* $\alpha \deg \mathcal{V} + \alpha \deg \mathcal{G} + \deg \delta \geq 3$. *Moreover, if* $n - r \geq 2$ *Cases* (6)–(9) *do not occur.*

Proof. Assume first that $n-r = 1$. Then, by Remark 3.3, the assumption on the Δ-genus is equivalent to the condition $g(Z, H_Z) = 1$. Therefore $(X, \mathcal{E})$ is one of the pairs in the list of [17, Theorem 1], where the last case is ruled out by [31, Theorem 1]. Moreover H is any ample line bundle on X satisfying $\deg H_Z \geq 3$.

An easy check in Cases (1)–(9) of the statement shows that H_Z is very ample for any ample line bundle H on X. In Case (10) we can write $H = \alpha\xi + \pi^*\delta$, where ξ the tautological line bundle of $\mathcal{V}$ and δ a line bundle on B. The degree of H_Z is given by

$$\begin{aligned}\deg H_Z = (\alpha\xi + \pi^*\delta) \cdot c_{n-1}(\mathcal{E}) = \alpha\xi \cdot c_{n-1}(\mathcal{E}) + (\deg \delta) F \cdot c_{n-1}(\mathcal{E}) = \\ = \alpha\xi \cdot c_{n-1}(\mathcal{E}) + \deg \delta,\end{aligned} \tag{3.5.1}$$

as $F \cdot c_{n-1}(\mathcal{E}) = c_{n-1}(\mathcal{E}_F) = 1$. Now, since $(\mathcal{E} \otimes (-\xi))_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-1)}$ for every fiber F of the projection, then $\mathcal{E} \cong \xi \otimes \pi^*\mathcal{G}$, for a vector bundle $\mathcal{G}$ of rank $(n-1)$ on B. Moreover

$$c_{n-1}(\mathcal{E}) = \sum_{j=0}^{n-1} c_j(\pi^*\mathcal{G}) \cdot \xi^{n-1-j} = \xi^{n-1} + \pi^* c_1(\mathcal{G}) \cdot \xi^{n-2} = \xi^{n-1} + \deg \mathcal{G}(\xi^{n-2} \cdot F).$$

Recall that $\xi^n = \deg \mathcal{V}$ and $\xi^{n-1} \cdot F = 1$; substituting in (3.5.1) and recalling that $\deg H_Z \geq 3$, we get the condition in the statement.

Assume now that $n - r \geq 2$. By [9, (6.3)], $\Delta(Z, H_Z) = 1$ implies that (Z, H_Z) is a del Pezzo manifold; then $g(X, \mathcal{F}) = 1$, where $\mathcal{F} = \mathcal{E} \oplus H^{\oplus(n-r-1)}$. So we are in the assumption of [25, Theorem 2] which gives the following possibilities for $(X, \mathcal{F})$:

- (i) $X \cong \mathbb{P}_B(\mathcal{V})$, where $\mathcal{V}$ is an ample vector bundle of rank n over an elliptic curve $B \cong Z$ and $(\mathcal{F})_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-1)}$ for every fiber F of the projection $\pi\colon X \longrightarrow B$, which gives Case (10) of the statement;
- (ii) $K_X + \det \mathcal{F} = 0$. Pairs $(X, \mathcal{F})$ satisfying this condition are classified in [33, Theorem 0.3 and Proposition 7.4]. Therefore the statement follows recalling again that the doubtful case in [33, Proposition 7.4] is ruled out by [31, Theorem 1]. □

Theorem 3.6. *Let* $X, \mathcal{E}$ *and* Z *be as in* 1.1 *with* $n-r \geq 2$. *Let* H *be an ample line bundle on* X *such that* H_Z *is very ample and assume that*

$$\Delta(Z, H_Z) = 2.$$

Then $(X, \mathcal{E}, H)$ *is one of the following:*

(1) *X is a Fano manifold with* $\mathrm{Pic}(X) \cong \mathbb{Z}$ *generated by H, of index $r_X = n-r+e-2$, where* $\det \mathcal{E} = eH$*, and* $K_X + \det \mathcal{E} + (n-r-2)H = \mathcal{O}_X$*;*
(2) $(X, \mathcal{E} \oplus H) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1,2) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1,1)^{\oplus 2})$*;*
(3) $X \cong \mathbb{P}_B(\mathcal{V})$*, where $\mathcal{V}$ is an ample vector bundle of rank n over a smooth curve B of genus* 1*, and* $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-2)}$*,* $H_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ *for every fiber F of the projection $X \to B$;*
(4) $n - r \leq 3$*,* (X, H) *is a quadric fibration over* $\mathbb{P}^1$ *and* $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus r}$ *for any general fiber F of the fibration.*

Proof. By [11, Theorem 3.12 and Corollary 3.3] (Z, H_Z) is one of the following:
(i) a hypersurface of degree 4;
(ii) an elliptic scroll of dimension $n - r = 2$;
(iii) a quadric fibration over $\mathbb{P}^1$ of sectional genus 2 (for the explicit list see [11, Theorem 3.4]).

Consider Case (i). If $n - r \geq 3$, then X is a Fano manifold with $\mathrm{Pic}(X) \cong \mathbb{Z}$ generated by H, and $K_X + \det \mathcal{E} + (n-r-2)H = \mathcal{O}_X$, by [20, Theorem, Case (ix); see also Section 3]. We can write $-K_X = r_X H$ and $\det \mathcal{E} = eH$ for some positive integer e. We get $K_Z = (-r_X + e)H_Z = (2-(n-r))H_Z$, so Z, which is a Fano manifold too, has index $r_Z = r_X - e$. On the other hand, since (Z, H_Z) is a smooth quartic hypersurface we know that $r_Z = n - r - 2$. Hence $r_X = n - r + e - 2$ and therefore $(X, \mathcal{E}, H)$ is as in (1). Now assume that $n - r = 2$. Then (Z, H_Z) is a K3 surface of degree 4 and the conclusion follows from Lemma 2.9.

Case (ii) can be easily settled by using [16, Theorem] and leads to Case (3) of the statement.

The last Case (iii) falls in [10, Theorem 0.1], Cases (ii) and (iii). Case (ii) is Case (2) of the statement. As to the further specification provided in [11, Theorem 3.4], note that $n - r \leq 3$ in Case (iii), except when (Z, H_Z) is $\mathbb{P}^1 \times \mathbb{Q}^3$ Segre embedded. However this situation is ruled out by Lemma 2.4. Therefore we obtain Case (4). □

In order to study the case $\Delta(Z, H_Z) = 3$, we recall first the following classification result of Ionescu.

Theorem 3.7. [11, Theorem 4.8] *Let Y be a smooth complex projective variety of dimension ≥ 2 polarized by a very ample line bundle L, and assume that $\Delta(Y, L) = 3$. Then (Y, L) is one of the following:*
(I) *a quintic hypersurface;*
(II) *a complete intersection of type* $(2, 3)$*;*
(III) *a* 3*-dimensional scroll over a smooth curve of genus* 1*;*
(IV) *a pair with sectional genus $g = 3$, irregularity $q = 0$ and degree $d \geq 6$.*

Remark 3.8. Note that, in Case (III) of Theorem 3.7, (Y, L) has degree $d \geq 9$ (e.g. [13, Proposition 1(i)]); so $d = 5$ can occur only in Case (I).

Remark 3.9. Looking more closely at Case (IV) of Theorem 3.7 and combining results from [11, Theorems 4.1 and 4.2] and [3, Proposition 10.2.2 and Theorem 10.2.7, Case (e)], we see that (IV) gives rise to the following possibilities:

(IV-i) a quadric fibration over $\mathbb{P}^1$;

(IV-ii) $\dim Y \geq 3$ and (Y, L) is a scroll over $\mathbb{P}^2$; in particular, if $\dim Y \geq 4$, then $Y = \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ or $Y = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2})$, with L being the tautological bundle in each case, or Y is the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^3$;

(IV-iii) $\dim Y = 2$ and (Y, L) is a Bordiga surface (i.e. (Y, L) has $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$ as adjunction theoretic reduction), with $6 \leq d \leq 16$;

(IV-iv) $\dim Y = 2$ and, either Y is a del Pezzo surface with $K_Y^2 = 2$ and $H_Y = -2K_Y$, or (Y, L) admits such a pair as simple adjunction theoretic reduction.

Remark 3.10. Note that, if (Y, L) is as in Theorem 3.7 with $\dim Y \geq 2$, $\mathrm{Pic}(X) \cong \mathbb{Z}$ and degree 6, then (Y, L) can only be as in Case (II) of Theorem 3.7.

Moreover note that K_Y is not ample except in Case (I) of Theorem 3.7, when $\dim Y = 2$. So we have

Remark 3.11. Let $X, \mathcal{E}$ and Z be as in 1.1 with $n - r \geq 2$. Let H be an ample line bundle on X such that H_Z is very ample and assume that $\Delta(Z, H_Z) = 3$. Then $(K_X + \det \mathcal{E})_Z$ is not ample except for (Z, H_Z) a quintic surface in $\mathbb{P}^3$.

Theorem 3.12. *Let $X, \mathcal{E}$ and Z be as in* 1.1 *with $n - r \geq 2$. Let H be an ample line bundle on X such that H_Z is very ample and assume that*

$$\Delta(Z, H_Z) = 3.$$

If $(K_X + \det \mathcal{E})_Z$ is not ample, then $(X, \mathcal{E}, H)$ is one of the following:

(1) *$n - r \geq 3$, X is a Fano manifold of coindex $q_X \leq 4$ with $\mathrm{Pic}(X) \cong \mathbb{Z}$ generated by H, such that $K_X + \det \mathcal{E} + (n - r - 3)H = \mathcal{O}_X$. If $n - r \geq 4$, then Z is a Fano manifold of coindex $q_Z = 4$;*

(2) *X is a Fano manifold of coindex $q_X \leq 3$ with $\mathrm{Pic}(X) \cong \mathbb{Z}$ generated by H, such that $K_X + \det \mathcal{E} + (n - r - 2)H = \mathcal{O}_X$. If $n - r \geq 3$, then Z is a Fano manifold of coindex $q_Z = 3$;*

(3) *(X, H) is a scroll over a smooth curve of genus 1 and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-3)}$ for any fiber F of the scroll projection;*

(4) *(X, H) is a scroll over $\mathbb{P}^1$, and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(r-1)}$ for any fiber F of the projection $X \longrightarrow \mathbb{P}^1$ (see* [21, Proposition 1] *for the complete description);*

(5) *$n - r \leq 4$, (X, H) is a quadric fibration over $\mathbb{P}^1$ and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus r}$ for any general fiber of the fibration;*

(6) *$n - r = 2$, X is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$, $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-2)}$, $H_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(2)$ for any fiber F of the projection $X \longrightarrow \mathbb{P}^1$, and $H^2 \cdot c_{n-2}(\mathcal{E}) = 16$ (see* [21, Proposition 2] *for the complete description);*

(7) *$(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-2)}, \mathcal{O}_{\mathbb{P}^n}(4))$;*

(8) *$n - r \leq 3$, there exists an ample vector bundle $\mathcal{W}$ of rank $n - 1$ on $\mathbb{P}^2$ such that $X = \mathbb{P}_{\mathbb{P}^2}(\mathcal{W})$, H is the tautological line bundle on X, and $\mathcal{E} \cong H \otimes p^*\mathcal{G}$ for some vector bundle $\mathcal{G}$ of rank r on $\mathbb{P}^2$, where $p\colon X \longrightarrow \mathbb{P}^2$ is the bundle projection; moreover $\det \mathcal{W} + \det \mathcal{G} = \mathcal{O}_{\mathbb{P}^2}(4)$ and $H^{n-r} \cdot c_r(\mathcal{E}) = 4c_1(\mathcal{W}) - c_2(\mathcal{W}) + c_2(\mathcal{G})$;*

(9) *X is a Fano manifold of index $n-1$ with $\mathrm{Pic}(X) \cong \mathbb{Z}$, generated by an ample line bundle $\mathcal{L}$ with $\mathcal{L}^n = 2$, and $(\mathcal{E}, H) \cong (\mathcal{L}^{\oplus(n-2)}, 2\mathcal{L})$.*

Proof. We can apply Theorem 3.7 to the pair (Z, H_Z), hence we investigate the four possibilities case-by-case.

In Case (I), we recall first that in our assumption $K_X + \det \mathcal{E}$ cannot be ample, so we infer that $n - r \geq 3$ by Remark 3.11. It follows that $\mathrm{Pic}(X) \cong \mathrm{Pic}(Z) \cong \mathbb{Z}$ by Theorem 2.2 and the Lefschetz theorem. We can compute $(K_X + \det \mathcal{E})_Z = K_Z = \mathcal{O}_{\mathbb{P}^{n-r+1}}(-(n-r)+3)_Z$. Arguing as in the proof of point (1) of Theorem 3.6, we deduce that $\mathrm{Pic}(X)$ is generated by H and $K_X + \det \mathcal{E} + (n-r-3)H = \mathcal{O}_X$; in particular X is a Fano manifold. Now, let l be a line in Z. As is known, such a line exists, e.g. see [5] or [1]. Put $\det \mathcal{E} = eH$ for some positive integer e. Then $e \geq r$ by Lemma 2.1, since $l \cdot H = l \cdot H_Z = 1$. It thus follows that X has index $r_X = e + n - r - 3 \geq n - 3$, so the coindex of X is $q_X \leq 4$. Moreover, if $n - r \geq 4$, then $-K_Z = (n-r-3)H_Z$, so Z itself is a Fano manifold of coindex $q_Z = 4$. This leads to Case (1) of the statement.

In Case (II), if $n - r \geq 3$, argue as in Case (I). So X is a Fano manifold with Picard group generated by H and $K_X + \det \mathcal{E} + (n-r-2)H = \mathcal{O}_X$. Note that also Z is a Fano manifold; moreover, by [5], it contains a line l. So, with the same computations as in Case (I), we find that the coindexes of X and Z are, respectively, $q_X \leq 3$ and $q_Z = 3$. If $n - r = 2$, then (Z, H_Z) is a K3 surface of degree 6 and we can apply Lemma 2.9. This leads to Case (2) of the statement.

Case (III) easily leads, e.g. by using [15, Theorem B], to Case (3) of the statement.

Finally we consider Case (IV). By Remark 3.9, we have four possibilities. Cases (IV-i)–(IV-iv) are listed as Cases (2), (3), (5) and (6) respectively in [21, Section 1]. Notice that the extra assumption that $|H|$ embeds Z used in [21, Section 1] to study our Case (IV-ii) has been recently removed [22, Section 1]. Relying on this result, these four possibilities lead to the triplets $(X, \mathcal{E}, H)$ as in (4), (6)–(9) of the statement or to the following situation: $n - r \leq 5$, and there exists a surjective morphism $\varphi\colon X \longrightarrow \mathbb{P}^1$ whose general fiber F is a smooth quadric hypersurface $\mathbb{Q}^{n-1}$ in $\mathbb{P}^n$ with $H_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)$ and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus r}$. In this last case, note that $n - r = 5$ occurs only when $(Z, H_Z) = (\mathbb{Q}^4 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{Q}^4 \times \mathbb{P}^1}(1,1))$. But this possibility is ruled out by Lemma 2.4, so we obtain Case (5) of the statement.

More specifically, (4), (5) and (6) come from (IV-i); Case (IV-iii) leads to (7) and to (8) with $n - r = 2$; the remaining part of (8) comes from (IV-ii) taking into account Lemma 2.5; finally, only the first subcase of (IV-iv) lifts to the vector bundle setting, leading to (9). □

Remark 3.13. All cases listed in Theorem 3.12 occur. This is obvious for Cases (7) and (9) and we already said about Cases (4) and (6). As to Cases (5) and (8), examples can be found in [21, (2.3)] and [20, Section 2], respectively. Here we produce examples for the remaining cases.
Case (1):

(1a) $X = \mathbb{P}^n$, $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(5) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(r-1)}$, $H = \mathcal{O}_{\mathbb{P}^n}(1)$;
(1b) $X = V_5 \subset \mathbb{P}^{n+1}$, a smooth quintic hypersurface, $\mathcal{E} = H^{\oplus r}$, $H = \mathcal{O}_V(1)$.

Note that $(K_X + \det \mathcal{E})_Z = H_Z$ is ample for $n - r = 2$.

Case (2):

(2a) $X = \mathbb{P}^n$, $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(3) \oplus \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(r-2)}$, $H = \mathcal{O}_{\mathbb{P}^n}(1)$;

(2b) $X = \mathbb{Q}^n \subset \mathbb{P}^{n+1}$, $\mathcal{E} = \mathcal{O}_{\mathbb{Q}^n}(3) \oplus \mathcal{O}_{\mathbb{Q}^n}(1)^{(r-1)}$, $H = \mathcal{O}_{\mathbb{Q}^n}(1)$;

(2c) $X = V_3 \subset \mathbb{P}^{n+1}$, a smooth cubic hypersurface, $\mathcal{E} = \mathcal{O}_V(2) \oplus \mathcal{O}_V(1)^{(r-1)}$, $H = \mathcal{O}_V(1)$;

(2d) $X = V_{2,3} \subset \mathbb{P}^{n+2}$, a smooth complete intersection of type $(2,3)$, $\mathcal{E} = H^{\oplus r}$, $H = \mathcal{O}_V(1)$.

Case (3): $X = \mathbb{P}_B(\mathcal{V})$, with $\mathcal{V}$ a very ample vector bundle of rank n over a curve B of genus 1, $\mathcal{E} = H^{\oplus(n-3)}$, H the tautological line bundle of $\mathcal{V}$.

Unfortunately, when $(K_X + \det \mathcal{E})_Z$ is ample we cannot determine the structure of $(X, \mathcal{E}, H)$ in general. We confine to add some remarks under the assumption that $\mathrm{Pic}(X) \cong \mathbb{Z}$.

Proposition 3.14. *Let $X, \mathcal{E}$ and Z be as in* 1.1 *with $n - r = 2$. Let H be an ample line bundle on X such that H_Z is very ample and assume that (Z, H_Z) is a quintic surface in $\mathbb{P}^3$. If $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $\mathcal{E}_Z$ is spanned, then X is a Fano manifold and H is the ample generator of* $\mathrm{Pic}(X)$.

Proof. Note first that the Picard group of X is generated by H. Indeed, let $\mathcal{L}$ be the ample generator of $\mathrm{Pic}(X)$. Then $H = a\mathcal{L}$ for some positive integer a. So we have $5 = H_Z^2 = a^2 \mathcal{L}_Z^2$, hence $a = 1$. Moreover $K_X + \det \mathcal{E} = H$. We can write $K_X = kH$ for some integer k and $\det \mathcal{E} = eH$ for some positive integer e, hence we obtain $k+e = 1$. Therefore either $K_X = \mathcal{O}_X$ and $\det \mathcal{E} = H$, or X is a Fano manifold.

We claim that the first case cannot happen. Recall that (Z, H_Z) is a quintic surface of $\mathbb{P}^3$. Arguing as in the proof of Lemma 2.6 with $S = Z$ and $\mathcal{G} = \mathcal{E}_Z$, we get $c_1(\mathcal{E}_Z)^2 \geq 6$. But $\det \mathcal{E}_Z = H_Z$ under our assumption, hence $c_1(\mathcal{E}_Z)^2 = H_Z^2 = 5$, a contradiction. □

This allows to characterize the examples we produced for Case (1) of Theorem 3.12 in the following way.

Proposition 3.15. *Suppose that either $(X, \mathcal{E}, H)$ and Z are as in Case* (1) *of Theorem* 3.12*, or $(K_X + \det \mathcal{E})_Z$ is ample and* $\mathrm{Pic}(X) \cong Z$*. If $\mathcal{E}$ is decomposable and H is very ample, then $(X, \mathcal{E}, H)$ is as in* (1a) *or* (1b) *of Examples* 3.13.

Proof. X is Fano, $\mathrm{Pic}(X)$ is generated by H and $-K_X = r_X H$ in both cases. Since $\mathcal{E}$ is decomposable, we can write $\mathcal{E} = \bigoplus_{i=1}^r a_i H$, with $a_1 \geq \cdots \geq a_r \geq 1$. We have $5 = H_Z^{n-r} = H^{n-r} \cdot Z = (\prod_{i=1}^r a_i) H^n$, so either

(a) $H^n = 1$ and $a_1 = 5$, $a_2 = \cdots = a_r = 1$, or

(b) $H^n = 5$ and $a_1 = \cdots = a_r = 1$.

On the other hand, from $K_X + \det \mathcal{E} + (n-r-3)H = \mathcal{O}_X$ we get $r_X = \sum_{i=1}^r a_i + (n-r-3)$. So $r_X = n+1$ in Case (a) and $r_X = n-3$ in Case (b). Case (a) immediately leads to (1a) by the Kobayashi–Ochiai theorem. In Case (b) $\mathcal{E} = H^{\oplus r}$; since $h^1(\mathcal{O}_X) = 0$, by Theorem 2.2, $|H|$ has a regular ladder [9, p. 28], hence $\Delta(X, H) = \Delta(Z, H_Z) = 3$. As H is very ample, by Theorem 3.7 and Remark 3.8 (X, H) turns out to be a smooth quintic hypersurface of $\mathbb{P}^{n+1}$. This gives (1b). □

Similarly for Case (2) we have

Proposition 3.16. *Let $(X, \mathcal{E}, H)$ and Z be as in Case* (2) *of Theorem* 3.12*. If $\mathcal{E}$ is decomposable and H very ample, then $(X, \mathcal{E}, H)$ is as in* (2a)–(2d) *of Examples* 3.13.

Proof. X is Fano, $\mathrm{Pic}(X)$ is generated by H and $-K_X = r_X H$. Since $\mathcal{E}$ is decomposable, we can write $\mathcal{E} = \bigoplus_{i=1}^r a_i H$, with $a_1 \geq \cdots \geq a_r \geq 1$. We have $6 = H_Z^{n-r} = H^{n-r} \cdot Z = (\prod_{i=1}^r a_i) H^n$, so we have the following possibilities:

(a) $H^n = 1$ and $a_1 = 6$, $a_2 = \cdots = a_r = 1$, or $a_1 = 3$, $a_2 = 2$, $a_3 = \cdots = a_r = 1$;
(b) $H^n = 2$ and $a_1 = 3$, $a_2 = \cdots = a_r = 1$;
(c) $H^n = 3$ and $a_1 = 2$, $a_2 = \cdots = a_r = 1$;
(d) $H^n = 6$ and $a_1 = \cdots = a_r = 1$.

On the other hand, from $K_X + \det \mathcal{E} + (n - r - 2)H = \mathcal{O}_X$ we get $r_X = \sum_{i=1}^r a_i + (n - r - 2)$.

Recall that the index of a Fano manifold X is bounded by $1 \leq r_X \leq \dim X + 1$. So, in Case (a), $r_X = n + 1$ and then, necessarily, $a_1 = 3$, $a_2 = 2$, $a_3 = \cdots = a_r = 1$; hence we get (2a). In Case (b), X turns out to be a smooth Fano manifold of index $r_X = n$, therefore we obtain (2b). Let now X be as in (c); we see that (X, H) is a del Pezzo manifold with $H^n = 3$, so we derive (2c). In the last Case (d), we deduce that (X, H) is a Mukai manifold with $H^n = 6$ and $\mathcal{E} = H^{\oplus r}$; since $h^1(\mathcal{O}_X) = 0$, by Theorem 2.2, $|H|$ has a regular ladder [9, p. 28], hence $\Delta(X, H) = \Delta(Z, H_Z) = 3$. As H is very ample, by Theorem 3.7 and Remark 3.10 (X, H) turns out to be a complete intersection of type $(2, 3)$. This gives (2d). □

4 Δ-genera smaller than $\mathrm{cork}(\mathcal{E})$

Here we classify triplets $(X, \mathcal{E}, H)$ such that $\Delta(Z, H_Z) \leq \mathrm{cork}(\mathcal{E}) - 1$. First of all note that, if $n - r = 1$, then this means $\dim Z = 1$ and $\Delta(Z, H_Z) = 0$, which implies that $Z \cong \mathbb{P}^1$. By Section 3, this implies that X and $\mathcal{E}$ are as in Theorem 3.2 and H is any ample line bundle. So we can confine to the case $n - r \geq 2$. As in Section 3 assume that H_Z is very ample. Then we have the following

Theorem 4.1. *Let $X, \mathcal{E}$ and Z be as in* 1.1 *with $n - r \geq 2$. Let H be an ample line bundle on X such that H_Z is very ample and assume that*

$$\Delta(Z, H_Z) \leq \mathrm{cork}(\mathcal{E}) - 1.$$

Then $(X, \mathcal{E}, H)$ is one of the following:

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-2)}, \mathcal{O}_{\mathbb{P}^n}(m))$*, with* $1 \leq m \leq 3$*;*
(2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-3)}, \mathcal{O}_{\mathbb{P}^n}(1))$*;*
(3) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-3)}, \mathcal{O}_{\mathbb{P}^n}(2))$*;*
(4) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-3)}, \mathcal{O}_{\mathbb{P}^n}(2))$*;*
(5) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-4)}, \mathcal{O}_{\mathbb{P}^n}(1))$*;*
(6) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(3) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-3)}, \mathcal{O}_{\mathbb{P}^n}(1))$*;*

(7) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-2)}, \mathcal{O}_{\mathbb{Q}^n}(1))$;
(8) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-2)}, \mathcal{O}_{\mathbb{Q}^n}(2))$;
(9) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(2) \oplus \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-3)}, \mathcal{O}_{\mathbb{Q}^n}(1))$;
(10) *X is a Fano manifold of index $n-1$ with* $\mathrm{Pic}(X) \cong \mathbb{Z}$ *generated by H and $\mathcal{E} = H^{\oplus(n-2)}$;*
(11) *$n - r \geq 3$ and X is a Fano manifold with* $\mathrm{Pic}(X) \cong \mathbb{Z}$ *generated by H* (*as well as* Z) *of index $r_X > r_Z$;*
(12) $(\mathbb{Q}^4, \mathcal{S} \otimes \mathcal{O}_{\mathbb{Q}^4}(2), \mathcal{O}_{\mathbb{Q}^4}(1))$*, where $\mathcal{S}$ is a spinor bundle on $\mathbb{Q}^4$;*
(13) *(X, H) is a scroll over $\mathbb{P}^1$ and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r}$ for every fiber F of the projection $X \longrightarrow \mathbb{P}^1$;*
(14) *$n - r \geq 3$, (X, H) is a quadric fibration over $\mathbb{P}^1$ and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus r}$ for any general fiber F of the fibration $X \longrightarrow \mathbb{P}^1$;*
(15) $(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,1)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,1))$.

Proof. We note first that, H_Z being very ample, there is an embedding of Z in $\mathbb{P}^N$ with $N = h^0(H_Z) - 1$; moreover, denoted by $d = d(Z, H_Z)$ the degree of Z, the assumption on the Δ-genus is equivalent to the condition $d \leq N$.

This allows us to apply the main theorem of [12], obtaining the following possibilities for Z:

(I) Z is a Fano manifold with $b_2(Z) = 1$;
(II) (Z, H_Z) is a del Pezzo manifold with $b_2(Z) \geq 2$, $2 \leq \dim Z \leq 4$ and $3 \leq d \leq 8$;
(III) Z is the Segre embedding of $\mathbb{P}^1 \times \mathbb{F}_1$, where $\mathbb{F}_1$ is the blowing-up of $\mathbb{P}^2$ in a point, embedded in $\mathbb{P}^4$ as a rational scroll of degree 3;
(IV) (Z, H_Z) is a scroll over $\mathbb{P}^2$; more precisely, $Z = \mathbb{P}_{\mathbb{P}^2}(\mathcal{F})$ where $\mathcal{F}$ is either $T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$, or $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4}$, and H_Z stands for the tautological line bundle.
(V) (Z, H_Z) is a scroll over $\mathbb{P}^1$ with $d \geq \dim Z$ (i.e. a linear section of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^m$);
(VI) there is a vector bundle $\mathcal{G}$ over $\mathbb{P}^1$ of rank $\dim Z + 1 \geq 4$ and of splitting type $\eta = (\eta_0, \ldots, \eta_{n-r})$ such that, if L is the tautological line bundle on $\mathbb{P}_{\mathbb{P}^1}(\mathcal{G})$ and G denotes a fiber of the projection $\mathbb{P}_{\mathbb{P}^1}(\mathcal{G}) \longrightarrow \mathbb{P}^1$, Z embeds in $\mathbb{P}_{\mathbb{P}^1}(\mathcal{G})$ as $Z \in |2L + \beta G|$, $L_Z = H_Z$ and one of the following holds:
 (VI-i) $N = d = 2\dim Z - 1$, $\eta = (1, \ldots, 1, 0, 0)$, $\beta = 1$;
 (VI-ii) $N = d = 2\dim Z$, $\eta = (1, \ldots, 1, 0)$, $\beta = 0$;
 (VI-iii) $N = d = 2\dim Z + 1$, $\eta = (1, \ldots, 1)$, $\beta = -1$;
 (VI-iv) $\dim Z \geq 4$, $N = d + 1 = 2\dim Z + 1$, $\eta = (1, \ldots, 1)$, $\beta = -2$ or, equivalently, $Z \cong \mathbb{P}^1 \times \mathbb{Q}^{n-r-1}$ Segre embedded;
 (VI-v) $N = d = 2\dim Z + 2$, $\eta = (1, \ldots, 1, 2)$, $\beta = -2$.

Moreover the proof of the main theorem in [12] shows that, in Case (I), either (Z, H_Z) is $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m))$, $m = 2, 3$, $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$, or $\mathrm{Pic}(Z)$ is generated by H_Z, due to the Barth–Larsen theorem.

We proceed with a case-by-case analysis.

Case (I). Assume first that $\dim Z = 2$. In this case $Z \cong \mathbb{P}^2$, we have $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-2)})$ by [14, Theorem A]. Now, denoted by $\mathcal{H} := \mathcal{O}_{\mathbb{P}^n}(1)$ the ample generator of $\mathrm{Pic}(X)$, we have $H_Z = \mathcal{O}_{\mathbb{P}^2}(m) = m\mathcal{H}_Z$ for some positive integer m, hence $H = m\mathcal{H} = \mathcal{O}_{\mathbb{P}^n}(m)$. Moreover the condition $H_Z^2 \le N$ gives $m^2 \le \binom{m+2}{2} - 1$, so $m \le 3$. Therefore we get Case (1) of the statement.

Assume now that $\dim Z \ge 3$. Let $\mathcal{L}$ be the ample generator of $\mathrm{Pic}(Z)$. As we noted above, either $(Z, H_Z) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$, in which case $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-3})$ and $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^3}(2)$, so this leads to Case (3) of the statement, or $\mathcal{L} \cong H_Z$. In this last case we have $r_Z H_Z + K_Z = \mathcal{O}_Z$. Hence, by adjunction and the Lefschetz–Sommese theorem, $r_Z H + (K_X + \det \mathcal{E}) = \mathcal{O}_X$ and H generates $\mathrm{Pic}(X)$. Writing $\det \mathcal{E} = eH$ with e a positive integer, we get $-K_X = (r_Z + e)H$. Therefore X is a Fano manifold of index $r_X = r_Z + e$. So we are in Case (11) of the statement.

Case (II). A complete description of the triplets $(X, \mathcal{E}, H)$ and of the corresponding pairs (Z, H_Z), with (Z, H_Z) a del Pezzo manifold, is given in [18, Theorem 4 and Remark]. Note that for $\dim Z \ge 3$, condition $b_2(Z) \ge 2$ rules out all cases listed in that theorem. So $\dim Z = 2$ and the remark leads to Cases (4)–(6), (8)–(10), (12) and (15) of the statement.

Cases (III) *and* (VI). In Case (VI), let G be any fiber of $\tilde{q} \colon \mathbb{P}_{\mathbb{P}^1}(\mathcal{G}) \longrightarrow \mathbb{P}^1$; then $G \cong \mathbb{P}^{n-r}$. Let $G_0 := Z \cap G$ be the fiber of $\tilde{q}$ restricted to Z. Since $Z \in |2L + \beta F|$, we get $G_0 \cong \mathbb{Q}^{n-r-1}$, a smooth quadric hypersurface in $\mathbb{P}^{n-r}$ provided that G is a general fiber of $\tilde{q}$. Moreover $(L_Z)_{G_0} = (H_Z)_{G_0} = \mathcal{O}_{\mathbb{Q}^{n-r-1}}(1)$, hence $q := \tilde{q}|_Z$ is a quadric fibration over $\mathbb{P}^1$. In Case (III), every fiber of the morphism $q \colon \mathbb{P}^1 \times \mathbb{F}_1 \longrightarrow \mathbb{P}^1$ induced by the ruling projection $\mathbb{F}_1 \longrightarrow \mathbb{P}^1$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, denoted by F the general fiber of q, we get $(H_Z)_F \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$, hence q is a quadric fibration over $\mathbb{P}^1$. In other words (Z, H_Z) has the same structure as in (VI), so we treat Cases (III) and (VI) at the same time. Moreover $n - r \ge 3$ and, by [19, Theorem 0.4], $(X, \mathcal{E}, H)$ is one of the following:

(a) (X, H) is a scroll over $\mathbb{P}^1$ and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(r-1)}$ for every fiber F of the projection $\pi \colon X \longrightarrow \mathbb{P}^1$;

(b) (X, H) is a quadric fibration over $\mathbb{P}^1$ and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus r}$ for any general fiber F of the fibration $X \longrightarrow \mathbb{P}^1$.

Claim: Case (a) cannot occur.

To prove this fact we need to know the genus $g = g(Z, H_Z)$. In Cases (III) and (VI-i)–(VI-v), let $q \colon Z \longrightarrow \mathbb{P}^1$ be the quadric fibration.

In Case (III) it is immediate to compute g. Denote by p_1 and p_2 the projections of Z onto the first and the second factor, respectively, and consider the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{p_1} & \downarrow^{p_2} & \searrow^{q} & \\ \mathbb{P}^1 & & \mathbb{F}_1 & \longrightarrow & \mathbb{P}^1. \end{array}$$

The Picard group of Z is $\mathrm{Pic}(Z) \cong \mathbb{Z}^3$, generated by S, Σ and F, where $S = p_1^{-1}(t)$ for a point $t \in \mathbb{P}^1$, $\Sigma = p_2^{-1}(\sigma)$ with σ the (-1)-section of $\mathbb{F}_1$, and $F = p_2^{-1}(f)$ for a fiber f of the projection $\mathbb{F}_1 \longrightarrow \mathbb{P}^1$. We have

$$K_Z = p_1^* K_{\mathbb{P}^1} + p_2^* K_{\mathbb{F}_1} = [-2S - 2\Sigma - 3F]$$

and

$$H_Z = p_1^*\mathcal{O}_{\mathbb{P}^1}(1) + p_2^*[\sigma + 2f] = [S + \Sigma + 2F].$$

By the genus formula, noting that $F^2 = 0$, $S^2 = 0$ and $\Sigma^2 \cdot F = 0$, we get

$$2g - 2 = (K_Z + 2H_Z) \cdot H_Z^2 = F \cdot (S + \Sigma + 2F)^2 = 2F \cdot S \cdot \Sigma = 2,$$

whence $g = 2 = n - r - 1$.

Next we compute $g = g(Z, H_Z)$ in all subcases of (VI). Consider the following diagram

$$\begin{array}{ccc} Z & \hookrightarrow & P := \mathbb{P}_{\mathbb{P}^1}(\mathcal{G}) \\ {\scriptstyle q}\downarrow & \swarrow {\scriptstyle \tilde{q}} & \\ \mathbb{P}^1 & & \end{array}$$

and let L be the tautological line bundle of $\mathcal{G}$ on P. Denoted by $G = \tilde{q}^*\mathcal{O}_{\mathbb{P}^1}(1)$, the canonical bundle of P is

$$K_P = -(n - r + 1)L + (\deg \mathcal{G} - 2)G,$$

whence the canonical bundle of Z is

$$K_Z = (K_P + Z)_Z = (-(n - r - 1)L + (\deg \mathcal{G} - 2 + \beta)G)_Z.$$

Again by the genus formula, we derive

$$\begin{aligned} 2g - 2 = (K_Z + (n - r - 1)H_Z) \cdot H_Z^{n-r-1} &= (\deg \mathcal{G} - 2 + \beta)G_Z \cdot H_Z^{n-r-1} = \\ &= 2(\deg \mathcal{G} - 2 + \beta), \end{aligned}$$

since $G_Z \cdot H_Z^{n-r-1} = G_0 \cdot H_Z^{n-r-1} = (H_{G_0})^{n-r-1} = (\mathcal{O}_{\mathbb{Q}^{n-r-1}}(1))^{n-r-1} = 2$. Therefore $g = \deg \mathcal{G} + \beta - 1$ and we have the following table, where $\mathrm{rk}(\mathcal{G}) = n - r - 1$:

Subcases	$\deg \mathcal{G}$	β	$\deg \mathcal{G} + \beta$
(VI-i)	$\mathrm{rk}(\mathcal{G}) - 2$	1	$n - r$
(VI-ii)	$\mathrm{rk}(\mathcal{G}) - 1$	0	$n - r$
(VI-iii)	$\mathrm{rk}(\mathcal{G})$	-1	$n - r$
(VI-iv)	$\mathrm{rk}(\mathcal{G})$	-2	$n - r - 1$
(VI-v)	$\mathrm{rk}(\mathcal{G}) + 1$	-2	$n - r$

Hence

$$\begin{cases} g(Z, H_Z) = n - r - 2 & \text{in Subcase (VI-iv);} \\ g(Z, H_Z) = n - r - 1 & \text{in all the other subcases.} \end{cases}$$

Now we assume by contradiction that $(X, \mathcal{E}, H)$ is as in (a). The following argument is inspired by [10]. We can write $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{V})$, where $\mathcal{V} \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$, with $a_1 \geq a_2 \geq \cdots \geq a_n = 0$. Let ξ denote the tautological line bundle of $\mathcal{V}$ on X and identify any fiber F of $\pi\colon X \longrightarrow \mathbb{P}^1$ with $\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$. We know that $H_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1) \cong \xi_F$, hence $H = \xi + bF$, with $b \geq 1$ because of the ampleness of H by [3, Lemma 3.2.4].

Moreover we have $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(r-1)}$. Therefore $(\mathcal{E} \otimes [-2\xi])_F \cong \mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{\oplus(r-1)}$ and $h^0((\mathcal{E}\otimes[-2\xi])_F) = 1$ for any fiber F, so $\pi_*(\mathcal{E}\otimes[-2\xi]) = \mathcal{O}_{\mathbb{P}^1}(c) \in \mathrm{Pic}(\mathbb{P}^1)$. Pulling back via π, we have an injection $0 \longrightarrow \pi^*\mathcal{O}_{\mathbb{P}^1}(c) \longrightarrow \mathcal{E} \otimes [-2\xi]$. Now, twisting by $[2\xi]$, we obtain an exact sequence

$$0 \longrightarrow [2\xi + cF] \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{4.1.1}$$

where the rank-$(r-1)$ vector bundle $\mathcal{Q}$ is ample, being a quotient of an ample vector bundle. Restricting (4.1.1) to any fiber F, we get $c_1(\mathcal{Q}_F) = c_1(\mathcal{E}_F) - 2\xi \cdot F = r+1-2 = r-1 = \mathrm{rk}(\mathcal{Q}_F)$. Therefore $\mathcal{Q}_F$ is uniform of splitting type $(1,\ldots,1)$, hence $\mathcal{Q}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(r-1)}$ [32, Theorem 3.2.1, p. 51]. Moreover, being $(\mathcal{Q}\otimes[-\xi])_F \cong \mathcal{O}_F^{\oplus(r-1)}$, there exists a rank-$(r-1)$ vector bundle $\mathcal{W} \cong \bigoplus_{j=1}^{r-1}\mathcal{O}_{\mathbb{P}^1}(b_j)$ such that $\mathcal{Q}\otimes[-\xi] \cong \pi^*\mathcal{W}$, whence $\mathcal{Q} \cong \bigoplus_{j=1}^{r-1}[\xi + b_jF]$. Recalling that $\mathcal{Q}$ is ample, we have $b_j \geq 1$ for all $j = 1\ldots, r-1$. The sequence (4.1.1) gives

$$\det\mathcal{E} = (2\xi+cF)+\det\mathcal{Q} = (2\xi+cF)+(r-1)\xi+(\deg\mathcal{W})F = (r+1)\xi+(\deg\mathcal{W}+c)F.$$

Since $\det\mathcal{E}$ is ample, we obtain $\deg\mathcal{W} + c \geq 1$. Now put $\mathcal{F} := \mathcal{E} \oplus H^{\oplus(n-r-1)}$. Note that $\mathcal{F}$ is ample and of rank $(n-1)$. Moreover $g = g(Z, H_Z) = g(X, \mathcal{F})$, the curve genus of $(X, \mathcal{F})$, so we can compute g with the genus formula

$$2g - 2 = (K_Z + (n-r-1)H_Z) \cdot H_Z^{n-r-1}.$$

The canonical bundle of Z is given by

$$\begin{aligned} K_Z = (K_X + \det\mathcal{E})_Z &= ((-n\xi + (\deg\mathcal{V} - 2)F + (r+1)\xi + (\deg\mathcal{W} + c))_Z = \\ &= (-(n-r-1)\xi + (\deg\mathcal{V} - 2 + \deg\mathcal{W} + c)F)_Z, \end{aligned}$$

therefore

$$\begin{aligned} &2g-2 = \\ &= (-(n-r-1)\xi + (\deg\mathcal{V} - 2 + \deg\mathcal{W} + c)F + (n-r-1)(\xi + bF))_Z \cdot H_Z^{n-r-1} \\ &= 2(\deg\mathcal{V} - 2 + \deg\mathcal{W} + c + (n-r-1)b), \end{aligned}$$

since $F \cdot H_Z^{n-r-1} = (\mathcal{O}_{\mathbb{Q}^{n-r-1}}(1))^{n-r-1} = 2$. We have thus proved that

$$g = \deg\mathcal{V} - 1 + \deg\mathcal{W} + c + (n-r-1)b, \tag{4.1.2}$$

from which we derive

$$g \geq n - r - 1. \tag{4.1.3}$$

This is clearly a contradiction in Subcase (VI-iv). In Case (III) and Subcases (VI-i)–(VI-iii) and (VI-v) the Inequality (4.1.3) is actually an equality, so all the following condition hold: $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$, $\deg\mathcal{W} + c = 1$ and $b = 1$. Therefore $X = \mathbb{P}^1 \times \mathbb{P}^{n-1}$ and $H = \xi + F$. Let l be any fiber of $X \longrightarrow \mathbb{P}^{n-1}$. We note that $\xi^n = \deg\mathcal{V} = 0$ and $\xi^{n-1} = l$. Therefore we have

$$r \leq \deg\mathcal{E}_l = (\det\mathcal{E}) \cdot \xi^{n-1} = ((r+1)\xi + (\deg\mathcal{W} + c)F) \cdot \xi^{n-1} = \deg\mathcal{W} + c = 1,$$

which is a contradiction. This finally proves the claim.

So only Case (b) can occur. Hence both Cases (III) and (VI) lead to (14). Note that, by Lemma 2.4, (Z, H_Z) cannot be as in Subcase (VI-iv).

Case (IV). This is ruled out by Lemma 2.5.

Case (V). We claim that $K_Z + (\dim Z - 1)H_Z$ is not nef and that $K_Z + (\dim Z)H_Z$ is nef. Indeed, let F be a fiber of the scroll projection; to show the first assertion it is enough to note that $(K_Z + (\dim Z - 1)H_Z)_F = \mathcal{O}_{\mathbb{P}^{n-r-1}}(-1)$, since $K_F = (K_Z + F)_F = (K_Z)_F$ and $(H_Z)_F = \mathcal{O}_{\mathbb{P}^{n-r-1}}(1)$. As to the second, we note that if $K_Z + (\dim Z)H_Z$ is not nef, then $(Z, H_Z) \cong (\mathbb{P}^{n-r}, \mathcal{O}_{\mathbb{P}^{n-r}}(1))$ (e.g. see [3, Theorem 7.2.1]), which is a contradiction. Therefore recalling that (Z, H_Z) is a scroll over $\mathbb{P}^1$, we get the following possibilities for $(X, \mathcal{E}, H)$ by [18, Theorem 2], Cases (2), (3) and (4), respectively:

(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-3)}, \mathcal{O}_{\mathbb{P}^n}(1))$;
(b) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-2)}, \mathcal{O}_{\mathbb{Q}^n}(1))$;
(c) (X, H) is a scroll over $\mathbb{P}^1$ and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r}$ for every fiber F of the projection $X \longrightarrow \mathbb{P}^1$.

They give Cases (2), (7) and (13) of the statement, respectively. □

5 Final remarks

Another way to say that (Z, H_Z) has small Δ-genus is to mean that $\Delta(Z, H_Z)$ is small in comparison with the degree. In this section we prove a result of the same type as Theorem 4.1 in this direction.

We first need a lemma. Let S be a smooth surface and let L be a very ample line bundle on S. Set $d = L^2$. By ruled surface we mean a birationally ruled surface.

Lemma 5.1. *If $\Delta(S, L) < \frac{d}{2}$, then S is a ruled surface.*

Proof. Recall that $\Delta(S, L) = 2 + d - h^0(L)$. Let C be a smooth element in $|L|$. From the exact cohomology sequence of

$$0 \to \mathcal{O}_S \to L \to L_C \to 0,$$

we get

$$h^0(L_C) \geq h^0(L) - 1 = d + 1 - \Delta(S, L). \tag{5.1.1}$$

Suppose that S is not ruled. Then $L \cdot K_S \geq 0$, otherwise all plurigenera of S would be zero, a contradiction with the Enriques ruledness criterion [2, Theorem VI.17]. Therefore $d \leq 2g(C) - 2$, by genus formula. By applying Clifford's theorem to L_C we thus get $h^0(L_C) \leq \frac{1}{2} \deg L_C + 1 = \frac{d}{2} + 1$. Combining this with (5.1.1) we get $\Delta(S, L) \geq \frac{d}{2}$, which contradicts our assumption. □

Proposition 5.2. *Let X, $\mathcal{E}$ and Z be as in* 1.1 *with $n - r \geq 2$. Let H be an ample line bundle on X such that H_Z is very ample and assume that*

$$\Delta(Z, H_Z) < \frac{1}{2}\, c_r(\mathcal{E}) \cdot H^{n-r}. \tag{5.2.1}$$

Then $K_X + \det \mathcal{E} + (n - r - 2)H$ is not nef.

Proof. By [9, p. 28], we have $\Delta(S, H_S) \leq \Delta(Z, H_Z)$, where S is the smooth surface cut out by $n-r-2$ general elements of $|H_Z|$. Note that $c_r(\mathcal{E}) \cdot H^{n-r}$ is the degree of Z embedded by $|H_Z|$, since $c_r(\mathcal{E})$ is represented by Z. But this is also the degree d of its surface section (S, H_S). Thus $\Delta(S, H_S) < \frac{d}{2}$ by (5.2.1), and therefore S is ruled, by Lemma 5.1. By adjunction, $K_S = (K_Z + (n-r-2)H_Z)_S = (K_X + \det \mathcal{E} + (n-r-2)H)_S = (K_X + \det \mathcal{F})_S$, where $\mathcal{F} = \mathcal{E} \oplus H^{\oplus(n-r-2)}$ is an ample vector bundle of rank $n-2$ on X. Since S is ruled we conclude that $K_X + \det \mathcal{F}$ is not nef. □

In particular, for $n - r \geq 3$, the vector bundle $\mathcal{F} = \mathcal{E} \oplus H^{\oplus(n-r-2)}$ in the Proposition 5.2 has at least a direct summand H, so we obtain the following

Corollary 5.3. *Let* X, $\mathcal{E}$ *and* Z *be as in* 1.1 *with* $n-r \geq 3$. *Let* H *be an ample line bundle on* X *such that* H_Z *is very ample and assume that*

$$\Delta(Z, H_Z) < \frac{1}{2}\, c_r(\mathcal{E}) \cdot H^{n-r}.$$

Then $(X, \mathcal{E} \oplus H^{n-r-2})$ *is one of the pairs listed in* [24, Theorem]*, except Case* 7.

Proof. Simply note that Case (7) of [24, Theorem] cannot happen under our assumption, since it would require that $\mathcal{F}$ has no direct summands. □

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