A Wasserstein norm for signed measures, with application to non local transport equation with source term

Benedetto Piccoli * Francesco Rossi † Magali Tournus ‡

Abstract

We introduce the optimal transportation interpretation of the Kantorovich norm on the space of signed Radon measures with finite mass, based on a generalized Wasserstein distance for measures with different masses.

With the formulation and the new topological properties we obtain for this norm, we prove existence and uniqueness for solutions to non-local transport equations with source terms, when the initial condition is a signed measure.

Keywords. Wasserstein distance, Transport equation, Signed measures, Kantorovich duality.

AMS subject classifications. 28A33, 35A01.

1 Introduction

The problem of optimal transportation, also called Monge-Kantorovich problem, has been intensively studied in the mathematical community. Related to this problem, Wasserstein distances in the space of probability measures have revealed to be powerful tools, in particular for dealing with dynamics of measures like the transport Partial Differential Equation (PDE in the following), see e.g. [**?**, **?**]. For a complete introduction to Wasserstein distances, see [13, 14].

The main limit of this approach, at least for its application to dynamics of measures, is that the Wasserstein distances $W_p(\mu,\nu)$ $(p \geq 1)$ are defined only if the two positive measures μ,ν have the same mass. For this reason, in [11, 12] we introduced the generalized Wasserstein distances $W_p^{a,b}(\mu,\nu)$, combining the standard Wasserstein and total variation distances. In rough words, for $W_p^{a,b}(\mu,\nu)$ an infinitesimal mass $\delta\mu$ of μ can either be removed at cost $a|\delta\mu|$, or moved from μ to ν at cost $bW_p(\delta\mu,\delta\nu)$. Further generalizations for positive measures with different masses, based on the Wasserstein distance, are introduced in [4, 8, 9].

Such generalizations still have a drawback: both measures need to be positive. The first contribution of this paper is then the definition of a norm on the space of signed Radon measures with finite mass on $\mathbb{R}^d$. Such norm, based on an optimal transport approach, induces a distance generalizing the Wasserstein distance to signed measures. We then prove that this norm corresponds to the extension of the so-called Kantorovich distance for finite signed Radon measures introduced in [6] in the dual form

$$\|\mu\| = \sup_{\|f\|_\infty \leq 1,\ \|f\|_{Lip} \leq 1} \int_{\mathbb{R}^d} f d\mu. \tag{1}$$

*Department of Mathematical Sciences, Rutgers University Camden, Camden, NJ, USA. Email: piccoli@camden.rutgers.edu.

†Dipartimento di Matematica "Tullio Levi-Civita" Università degli Studi di Padova, Via Trieste 63, 35121 Padova, Italy. Email: francesco.rossi@math.unipd.it

‡Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France. Email: mtournus@math.cnrs.fr.

The novelty then lies in the dual interpretation of this norm in the framework of optimal transportation. We also prove new topological properties and characterizations of this norm.

One of the interests of signed measures is that they can be used to model phenomena in which the density is a quantity can increase (mass source) or decrease (mass sink). This also implies that the density can eventually become negative. In this setting, the second main contribution of the paper is to use this norm to guarantee well-posedness of the following non local transport equation with a source term being a signed measure. We study the following PDE

$$\partial_t \mu_t(x) + \operatorname{div}(v[\mu_t](x)\mu_t(x)) = h[\mu_t](x), \qquad \mu_{|t=0}(x) = \mu_0(x), \tag{2}$$

for $x \in \mathbb{R}^d$ and $\mu_0 \in \mathcal{M}^s(\mathbb{R}^d)$, where $\mathcal{M}^s(\mathbb{R}^d)$ is the space of signed Radon measures with finite mass on $\mathbb{R}^d$. Equation (2) has already been studied in the framework of positive measures, where it has been used for modeling several different phenomena such as crowd motion and development in biology, se a review in [**?**]. Our main motivation to study equation (2) in the framework of signed measure is the interpretation of μ_t as the spatial derivative of the entropy solution $\rho(x,t)$ to a scalar conservation law. A link between scalar conservation laws and non local transport equation has been initiated in [2, 7], but until now, studies are restricted to convex fluxes and monotonous initial conditions, so that the spatial derivative μ_t is a positive measure for all $t > 0$. Moreover, the mass of μ_t is preserved, since $\rho(+\infty,t) - \rho(-\infty,t)$ is a constant.

The authors of [1] suggested to extend the usual Wasserstein distance W_1 to the couples of signed measures $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$ such that $|\mu^+| + |\nu^-| = |\mu^-| + |\nu^+|$ by the formula $\mathbb{W}_1(\mu,\nu) = \mathbb{W}_1(\mu^+ + \nu^-, \mu^- + \nu^+)$. This procedure fails for $p \neq 1$, since triangular inequality is lost (see a counter-example in [1]).

We use the same trick to turn the generalized Wasserstein distance into a distance for signed measure and define $\mathbb{W}_1^{a,b}(\mu,\nu)$ as $W_1^{a,b}(\mu^+ + \nu^-, \mu^- + \nu^+)$. The space of signed measures being a vector space, we also define a norm $\|\mu\|^{a,b} = \mathbb{W}^{a,b}(\mu,0)$. Notice that to define the norm $\|.\|^{a,b}$, we need to restrict ourselves to Radon measures with finite mass, since the generalized Wasserstein distance [11] may not be defined for Radon measures with infinite mass.

We then use the norm $\|\cdot\|^{a,b}$ to study existence and uniqueness of the solution to the equation (2). The regularity assumptions made in this paper on the vector field and on the source term are the following:

(H-1) There exists K such that for all $\mu,\nu \in \mathcal{M}^s(\mathbb{R}^d)$ it holds

$$\|v[\mu] - v[\nu]\|_{\mathcal{C}^0(\mathbb{R}^d)} \leq K\|\mu - \nu\|^{a,b}. \tag{3}$$

(H-2) There exist L, M such that for all $x, y \in \mathbb{R}^d$, for all $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ it holds

$$|v[\mu](x) - v[\mu](y)| \leq L|x-y|, \qquad |v[\mu](x)| \leq M. \tag{4}$$

(H-3) There exist Q, P, R such that for all $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ it holds

$$\|h[\mu] - h[\nu]\|^{a,b} \leq Q\|\mu - \nu\|^{a,b}, \qquad |h[\mu]| \leq P, \qquad \operatorname{supp}(h[\mu]) \subset B_0(R). \tag{5}$$

The main result about equation (2) is the following:

Theorem 1 (Existence and uniqueness)**.** *Let v and h satisfy (H-1)-(H-2)-(H-3) and $\mu_0 \in \mathcal{M}^s(\mathbb{R}^d)$ compactly supported be given. Then, there exists a unique distributional solution to* (2) *in the space* $\mathcal{C}^0\left([0,1], \mathcal{M}^s(\mathbb{R}^d)\right)$*. In adition, for μ_0 and ν_0 in $\mathcal{M}^s(\mathbb{R}^d)$, denoting by μ_t and ν_t the solution, we have the following property of continuous dependence with respect to initial data:*

$$\|\mu_t - \nu_t\|^{a,b} \leq \|\mu_0 - \nu_0\|^{a,b} \exp(Mt), \qquad M = 2L + 2K(P + \min\{|\mu_0|, |\nu_0|\}) + Q,\ t \geq 0.$$

Remark 1. *We emphasize that the assumptions (H-2)-(H-3) are incompatible with a direct interpretation of the solution of* (2) *as the spatial derivative of a conservation law and need to be relaxed in a future work. Indeed, to draw a parallel between conservation laws and non-local equations, discontinuous vector fields need to be considered.*

The structure of the article is the following. In Section 2, we state and prove preliminary results which are needed for the rest of the paper. In Section 3, we define the generalized Wasserstein distance for signed measures, we show that it can be used to define a norm, and prove some topological properties. Section 4 is devoted to the use of the norm defined here to guarantee existence, uniqueness, and stability to initial condition for the transport equation (2).

2 Measure theory and the Generalized Wasserstein distance

In this section, we introduce the notations and state preliminary results. Throughout the paper, $\mathcal{B}(\mathbb{R}^d)$ is the space of Borel sets on $\mathbb{R}^d$, $\mathcal{M}(\mathbb{R}^d)$ is the space of Radon measures with finite mass (i.e. Borel regular, positive, and finite on every set).

2.1 Recalls on measure theory

In this section, μ and ν are in $\mathcal{M}(\mathbb{R}^d)$.

Definition 1. *We say that*

- $\mu << \nu$ *if* $\forall A \in \mathcal{B}(\mathbb{R}^d),\ (\nu(A) = 0) \Rightarrow (\mu(A) = 0)$
- $\mu \leq \nu$ *if* $\forall A \in \mathcal{B}(\mathbb{R}^d),\ \mu(A) \leq \nu(A)$
- $\mu \perp \nu$ *if there exists* $E \in \mathcal{B}(\mathbb{R}^d)$ *such that* $\mu(\mathbb{R}^d) = \mu(E)$ *and* $\nu(E^c) = 0$

The concept of largest common measure between measures is now recalled.

Lemma 1. *We consider μ and ν two measures in $\mathcal{M}(\mathbb{R}^d)$. Then, there exists a unique measure $\mu \wedge \nu$ which satisfies*

$$\mu \wedge \nu \leq \mu, \quad \mu \wedge \nu \leq \nu, \quad (\eta \leq \mu \text{ and } \eta \leq \nu) \Rightarrow \eta \leq \mu \wedge \nu. \tag{6}$$

We refer to $\mu \wedge \nu$ as the largest common measure to μ and ν. Moreover, denoting by f the Radon Nikodym derivative of μ with respect to ν, i.e. the unique measurable function f such that $\mu = f\nu + \nu_\perp$, with $\nu_\perp \perp \nu$, we have

$$\mu \wedge \nu = \min\{f, 1\}\nu. \tag{7}$$

Proof. The uniqueness is clear using (6). Existence is given by formula (7) as follows. First, it is obvious that $\min\{f, 1\}\nu \leq \nu$ and using $\mu = f\nu + \nu_\perp$, it is also clear that $\min\{f, 1\}\nu \leq \mu$. Let us now assume by contradiction the existence of a measure η and of $A \in \mathcal{B}(\mathbb{R}^d)$ such that

$$\eta \leq \mu, \quad \eta \leq \nu, \quad \eta(A) > \int_A \min\{f, 1\} d\nu. \tag{8}$$

Since $\nu_\perp \perp \nu$, there exists $E \in \mathcal{B}(\mathbb{R}^d)$ such that $\nu(A) = \nu(A \cap E)$ and $\nu_\perp(A) = \nu_\perp(A \cap E^c)$. Since $\eta \le \nu$, we have

$$\eta(A \cap E) = \eta(A) > \int_{A\cap E} \min\{f, 1\} d\nu.$$

We define

$$B = A \cap E \cap \{f > 1\}.$$

If $\nu(B) = 0$, then $f \le 1$ ν-a.e., hence $\eta(A) \le \int_A \min\{f, 1\} d\nu$. We then assume $\nu(B) > 0$. Then

$$\eta(B) + \eta((A \cap E) \setminus B) = \eta(A \cap E) > \int_B \min\{f, 1\} d\nu(x) + \int_{(A\cap E)\setminus B} \min\{f, 1\} d\nu$$
$$= \int_B 1 d\nu + \int_{(A\cap E)\setminus B} f d\nu = \nu(B) + \mu((A \cap E) \setminus B)$$

which contradicts the fact that $\eta \le \nu$ and $\eta \le \mu$. This implies that η satisfying (8) does not exist, and then (7) holds. □

Lemma 2. *Let μ and ν be two measures in $\mathcal{M}(\mathbb{R}^d)$. Then $\eta \le \mu + \nu$ implies $\eta - (\mu \wedge \eta) \le \nu$.*

Proof. Take A a Borel set. We write $\mu = f\eta + \eta_\perp$, with $\eta_\perp \perp \eta$. Then $\eta \wedge \mu = \min\{f, 1\}\eta$, and we can write

$$\eta(A) - (\eta \wedge \mu)(A) = \int_A \Big(\max\{1 - f, 0\} \Big) d\eta.$$

Define $B = A \cap \{f < 1\}$, and E such that $\eta(A \cap E) = \eta(A)$ and $\eta_\perp(A \cap E^c) = \eta_\perp(A)$. It then holds

$$\eta(A) - (\eta \wedge \mu)(A) = \int_{B\cap E} (1 - f) d\eta(x) = \eta(B \cap E) + \eta_\perp(B \cap E) - \mu(B \cap E) \le \nu(B \cap E) \le \nu(A).$$

Since this estimate holds for any Borel set A, the statement is proved.

□

2.2 Signed measures

We now introduce signed Radon measures, that are measures μ that can be written as $\mu = \mu_+ - \mu_-$ with $\mu_+, \mu_- \in \mathcal{M}(\mathbb{R}^d)$. We denote with $\mathcal{M}^s(\mathbb{R}^d)$ the space of such signed Radon measures.

For $\mu \in \mathcal{M}^s(\mathbb{R}^d)$, we define $|\mu| = |\mu_+^J| + |\mu_-^J|$ where (μ_+^J, μ_-^J) is the unique Jordan decomposition of μ, i.e. $\mu = \mu_+^J - \mu_-^J$ with $\mu_+^J \perp \mu_-^J$. Observe that $|\mu|$ is always finite, since $\mu_+^J, \mu_-^J \in \mathcal{M}(\mathbb{R}^d)$. We now recall the definition of tightness for a sequence in $\mathcal{M}^s(\mathbb{R}^d)$ and a weak compactness theorem.

Definition 2. *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures in $\mathcal{M}(\mathbb{R}^d)$ is tight if for each $\varepsilon > 0$, there is a compact set $K \subset \mathbb{R}^d$ such that for all $n \ge 0$, $\mu_n(\mathbb{R}^d \setminus K) < \varepsilon$. A sequence $(\mu_n)_{n\in\mathbb{N}}$ of signed measures of $\mathcal{M}^s(\mathbb{R}^d)$ is tight if the sequences $(\mu_n^+)_{n\in\mathbb{N}}$ and $(\mu_n^-)_{n\in\mathbb{N}}$ given by the Jordan decomposition are both tight.*

Lemma 3 (Weak compactness)**.** *Let μ_n be a sequence of measures in $\mathcal{M}(\mathbb{R}^d)$ that are uniformly bounded in mass. We can then extract a subsequence $\mu_{\phi(n)}$ such that $\mu_{\phi(n)} \underset{n\to\infty}{\rightharpoonup} \mu$ for some $\mu \in \mathcal{M}(\mathbb{R}^d)$, i.e. for all continuous and bounded function ϕ, we have $\int_{\mathbb{R}^d} \varphi d\mu_{\phi(n)} \underset{n\to\infty}{\rightarrow} \int_{\mathbb{R}^d} \varphi d\mu$.*

A proof can be found in [5].

2.3 Properties of the generalized Wasserstein distance

In this section, we recall key properties of the generalized Wasserstein distance. The usual Wasserstein distance $W_p(\mu,\nu)$ was defined between two measures μ and ν of same mass $|\mu| = |\nu|$, see more details in [13]. A transference plan between two positive measures of same mass μ and ν is a measure $\pi \in \mathcal{P}(\mathbb{R}^d, \mathbb{R}^d)$ which satisfies for all $A, B \in \mathcal{B}(\mathbb{R}^d)$

$$\pi(A \times \mathbb{R}^d) = \mu(A), \quad \pi(\mathbb{R}^d \times B) = \nu(B).$$

We denote by $\Pi(\mu,\nu)$ the set of transference plans between μ and ν. The p-Wasserstein distance for positive Radon measures of same mass is defined as

$$W_p(\mu,\nu) = \left(\min_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p d\pi(x,y) \right)^{\frac{1}{p}}.$$

It was extended to positive measures having possibly different mass in [11, 12], where the authors introduce the distance $W_p^{a,b}$ on the space $\mathcal{M}(\mathbb{R}^d)$ of Radon measures with finite mass. The formal definition is the following.

Definition 3 (Generalized Wasserstein distance [11])**.** *Let μ,ν be two positive measures in $\mathcal{M}(\mathbb{R}^d)$. The generalized Wasserstein distance between μ and ν is given by*

$$W_p^{a,b}(\mu,\nu) = \left(\inf_{\substack{\tilde\mu,\tilde\nu\in\mathcal{M}(\mathbb{R}^d)\\ |\tilde\mu|=|\tilde\nu|}} a^p(|\mu-\tilde\mu| + |\nu-\tilde\nu|)^p + b^p W_p^p(\tilde\mu,\tilde\nu) \right)^{1/p}. \tag{9}$$

We notice that

$$W_p^{a,b} = \frac{b}{b'}\, W_p^{a',b'}, \quad \text{for } \frac{a}{b} = \frac{a'}{b'}. \tag{10}$$

Notice that the infimum in (9) is always attained. Moreover, there always exists a minimizer that satisfy the additional constraint $\tilde\mu \le \mu, \tilde\nu \le \nu$.

We denote by $\mathcal{C}_c^0(\mathbb{R}^d;\mathbb{R})$ the set of continuous functions with compact support on $\mathbb{R}^d$. For $f \in \mathcal{C}_c^0(\mathbb{R}^d;\mathbb{R})$, we define

$$\|f\|_\infty = \sup_{x\in\mathbb{R}^d} |f(x)|, \qquad \|f\|_{Lip} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}.$$

We also denote by $\mathcal{C}_c^{0,Lip}(\mathbb{R}^d;\mathbb{R})$ the subset of functions $f \in \mathcal{C}_c^0(\mathbb{R}^d;\mathbb{R})$ for which it holds $\|f\|_{Lip} < +\infty$.

The following result is stated in [12, Theorem 13].

Lemma 4 (Kantorovitch Rubinstein duality)**.** *For μ, ν in $\mathcal{M}(\mathbb{R}^d)$, it holds*

$$W_1^{1,1}(\mu,\nu) = \sup\left\{ \int_{\mathbb{R}^d} \varphi\, d(\mu-\nu);\ \varphi \in \mathcal{C}_c^{0,Lip}, \|\varphi\|_\infty \le 1, \|\varphi\|_{Lip} \le 1 \right\}.$$

Lemma 5 (Properties of the generalized Wasserstein distance)**.** *Let $\mu,\nu,\eta,\mu_1,\mu_2,\nu_1,\nu_2$ be some positive measures with finite mass on $\mathbb{R}^d$. The following properties hold*

1. $W_p^{a,b}(\mu_1+\mu_2, \nu_1+\nu_2) \le W_p^{a,b}(\mu_1,\nu_1) + W_p^{a,b}(\mu_2,\nu_2)$.
2. $W_1^{a,b}(\mu+\eta, \nu+\eta) = W_1^{a,b}(\mu,\nu)$,

Proof. The first property is taken from [11, Proposition 11]. The second statement is a direct consequence of the Kantorovitch-Rubinstein duality in Lemma 4 for $W^{1,1}$. For general $a > 0,\ b > 0$, we proceed as follows. Let μ, ν be two measures. Define

$$C^{a,b}(\bar{\mu}, \bar{\nu}, \pi; \mu, \nu) := a(|\mu - \bar{\mu}| + |\nu - \bar{\nu}|) + b \int |x - y| \, d\pi(x, y),$$

where π is a transference plan in $\Pi(\bar{\mu}, \bar{\nu})$. Define $D_\lambda : x \to \lambda x$ with $\lambda > 0$ the dilation in $\mathbb{R}^n$. It holds

$$\begin{aligned} C^{a,b}(D_\lambda\#\bar{\mu}, D_\lambda\#\bar{\nu}, &(D_\lambda \times D_\lambda)\#\pi; D_\lambda\#\mu, D_\lambda\#\nu) \\ &= a(|D_\lambda\#\mu - D_\lambda\#\bar{\mu}| + |D_\lambda\#\nu - D_\lambda\#\bar{\nu}|) + b \int |x - y| \, d(D_\lambda \times D_\lambda)\pi(x, y), \\ &= a(|\mu - \bar{\mu}| + |\nu - \bar{\nu}|) + b \int |\lambda x - \lambda y| \, d\pi(x, y) = C^{a,\lambda b}(\bar{\mu}, \bar{\nu}, \pi; \mu, \nu). \end{aligned}$$

As a consequence, it holds

$$W^{a,b}(D_\lambda\#\mu, D_\lambda\#\nu) = W^{a,\lambda b}(\mu, \nu).$$

We now show that this implies $W^{a,b}(\mu+\eta, \nu+\eta) = W^{a,b}(\mu, \nu)$. Indeed, also applying Kantorovich-Rubinstein for $W^{1,1}$ and (10) with $a' = 1, b' = \lambda = \frac{b}{a}$, it holds

$$\begin{aligned} &W^{a,b}(\mu + \eta, \nu + \eta) = aW^{1,\frac{b}{a}}(\mu + \eta, \nu + \eta) = aW^{1,1}(D_\lambda\#\mu + D_\lambda\#\eta, D_\lambda\#\nu + D_\lambda\#\eta) = \\ &= aW^{1,1}(D_\lambda\#\mu, D_\lambda\#\nu) = aW^{1,\frac{b}{a}}(\mu, \nu) = W^{a,b}(\mu, \nu). \end{aligned}$$

□

Definition 4 (Image of a measure under a plan)**.** *Let μ and ν two measures in $\mathcal{M}(\mathbb{R}^d)$ of same mass and $\pi \in \Pi(\mu, \nu)$. For $\eta \leq \mu$, we denote by f the Radon-Nykodym derivative of η with respect to μ and by π_f the transference plan defined by $\pi_f(x, y) = f(x)\pi(x, y)$. Then, we define the image of η under π as the second marginal η' of π_f.*

Observe that the second marginal satisfies $\eta' \leq \nu$. Indeed, since $\eta \leq \mu$, it holds $f \leq 1$. Thus, for all Borel set B of $\mathbb{R}^d$ we have

$$\eta'(B) = \pi_f(\mathbb{R}^d \times B) \leq \pi(\mathbb{R}^d \times B) = \nu(B).$$

3 Generalized Wasserstein norm for signed measures

In this section, we define the generalized Wasserstein distance for signed measures and prove some of its properties. The idea is to follow what was already done in [1] for generalizing the classical Wasserstein distance.

Definition 5 (Generalized Wasserstein distance extended to signed measures)**.** *For μ, ν two signed measures with finite mass over $\mathbb{R}^d$, we define*

$$\mathbb{W}_1^{a,b}(\mu, \nu) = W_1^{a,b}(\mu_+ + \nu_-, \mu_- + \nu_+),$$

where μ_+, μ_-, ν_+ and ν_- are any measures in $\mathcal{M}(\mathbb{R}^d)$ such that $\mu = \mu_+ - \mu_-$ and $\nu = \nu_+ - \nu_-$.

Proposition 1. *The operator $\mathbb{W}_1^{a,b}$ is a distance on the space $\mathcal{M}^s(\mathbb{R}^d)$ of signed measures with finite mass on $\mathbb{R}^d$.*

Proof. First, we point out that the definition does not depend on the decomposition. Indeed, if we consider two distinct decompositions, $\mu = \mu_+ - \mu_- = \mu_+^J - \mu_-^J$, and $\nu = \nu_+ - \nu_- = \nu_+^J - {\nu_-}^J$, with the second one being the Jordan decomposition, then we have $(\mu_+ + \nu_-) - (\mu_+^J + \nu_-^J) = (\mu_- + \nu_+) - (\mu_-^J + \nu_+^J)$, and this is a positive measure since $\mu_+ \geq \mu_+^J$ and $\nu_+ \geq \nu_+^J$. The first property of Lemma 5 then gives

$$
\begin{aligned}
&W_1^{a,b}(\mu_+^J + \nu_-^J, \mu_-^J + \nu_+^J) = \\
&W_1^{a,b}(\mu_+^J + \nu_-^J + (\mu_+ + \nu_-) - (\mu_+^J + \nu_-^J), \mu_-^J + \nu_+^J + (\mu_- + \nu_+) - (\mu_-^J + \nu_+^J)) = \\
&W_1^{a,b}(\mu_+ + \nu_-, \mu_- + \nu_+).
\end{aligned}
$$

We now prove that $\mathbb{W}_1^{a,b}(\mu,\nu) = 0$ implies $\mu = \nu$. As explained above, we can choose the Jordan decomposition for both μ and ν. Since $W_1^{a,b}$ is a distance, we obtain $\mu_+ + \nu_- = \mu_- + \nu_+$. The orthogonality of μ_+ and μ_- and of ν_+ and ν_- implies that $\mu_+ = \nu_+$ and $\mu_- = \nu_-$, and thus $\mu = \nu$.

We now prove the triangle inequality. We have $\mathbb{W}_1^{a,b}(\mu,\eta) = W_1^{a,b}(\mu_+ + \eta_-, \mu_- + \eta_+)$. Using Lemma 5, we have

$$
\begin{aligned}
\mathbb{W}_1^{a,b}(\mu,\eta) &= W_1^{a,b}(\mu_+ + \eta_- + \nu_+ + \nu_-, \mu_- + \eta_+ + \nu_+ + \nu_-) \\
&\leq W_1^{a,b}(\mu_+ + \nu_-, \mu_- + \nu_+) + W_1^{a,b}(\eta_- + \nu_+, \eta_+ + \nu_-) \\
&= \mathbb{W}_1^{a,b}(\mu,\nu) + \mathbb{W}_1^{a,b}(\nu,\eta).
\end{aligned}
$$

□

We also state the following lemma about adding and removing masses.

Lemma 6. *Let $\mu, \nu, \eta, \mu_1, \mu_2, \nu_1, \nu_2$ in $\mathcal{M}^s(\mathbb{R}^d)$ with finite mass on $\mathbb{R}^d$. The following properties hold*

- $\mathbb{W}_1^{a,b}(\mu + \eta, \nu + \eta) = \mathbb{W}_1^{a,b}(\mu,\nu)$,
- $\mathbb{W}_1^{a,b}(\mu_1 + \mu_2, \nu_1 + \nu_2) \leq \mathbb{W}_1^{a,b}(\mu_1,\nu_1) + \mathbb{W}_1^{a,b}(\mu_2,\nu_2)$.

Proof. The proof is direct. For the first item, it holds $\mathbb{W}_1^{a,b}(\mu + \eta, \nu + \eta) = W_1^{a,b}(\mu_+ + \nu_+ + \eta_+ + \eta_-, \mu_- + \nu_- + \eta_+ + \eta_-)$ which is $W_1^{a,b}(\mu_+ + \nu_+, \mu_- + \nu_-) = \mathbb{W}_1^{a,b}(\mu,\nu)$.

For the second item, it holds

$$
\begin{aligned}
\mathbb{W}_1^{a,b}(\mu_1 + \mu_2, \nu_1 + \nu_2) &= W_1^{a,b}(\mu_{1,+} + \mu_{2,+} + \nu_{1,-} + \nu_{2,-}, \nu_{1,+} + \nu_{2,+} + \mu_{1,-} + \mu_{2,-}) \\
&\leq W_1^{a,b}(\mu_{1,+} + \nu_{1,-}, \nu_{1,+} + \mu_{1,-}) + W_1^{a,b}(\mu_{2,+} + \nu_{2,-}, \nu_{2,+} + \mu_{2,-}) \\
&= \mathbb{W}_1^{a,b}(\mu_1,\nu_1) + \mathbb{W}_1^{a,b}(\mu_2,\nu_2).
\end{aligned}
$$

□

Definition 6. *For $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ and $a > 0,\ b > 0$, we define*

$$\|\mu\|^{a,b} = \mathbb{W}_1^{a,b}(\mu, 0) = W_1^{a,b}(\mu_+, \mu_-),$$

where μ_+ and μ_- are any measures of $\mathcal{M}(\mathbb{R}^d)$ such that $\mu = \mu_+ - \mu_-$.

It is clear that the definition of $\|\mu\|^{a,b}$ does not depend on the choice of μ_+, μ_- as a consequence of the corresponding property for $W_1^{a,b}$.

Proposition 2. *The space of signed measures* $(\mathcal{M}^s(\mathbb{R}^d), \|.\|^{a,b})$ *is a normed vector space.*

Proof. First, we notice that $\|\mu\|^{a,b} = 0$ implies that $W_1^{a,b}(\mu_+, \mu_-) = 0$, which is $\mu_+ = \mu_-$ so that $\mu = \mu_+ - \mu_- = 0$. For triangular inequality, using the second property of Lemma 6, we have that for $\mu, \eta \in \mathcal{M}^s(\mathbb{R}^d)$,

$$\|\mu + \eta\|^{a,b} = \mathbb{W}_1^{a,b}(\mu + \eta, 0) \leq \mathbb{W}_1^{a,b}(\mu, 0) + \mathbb{W}_1^{a,b}(\eta, 0) = \|\mu\|^{a,b} + \|\eta\|^{a,b}.$$

Homogeneity is obtained by writing for $\lambda > 0$, $\|\lambda\mu\|^{a,b} = \mathbb{W}_1^{a,b}(\lambda\mu, 0) = W_1^{a,b}(\lambda\mu_+, \lambda\mu_-)$ where $\mu = \mu_+ - \mu_-$. Using Lemma 4, we have

$$\begin{aligned} W_1^{a,b}(\lambda\mu_+, \lambda\mu_-) &= \sup\left\{\int_{\mathbb{R}^d} \varphi \, d(\lambda\mu_+ - \lambda\mu_-);\ \varphi \in \mathcal{C}_c^{0,Lip}, \|\varphi\|_\infty \leq 1, \|\varphi\|_{Lip} \leq 1\right\} \\ &= \lambda \sup\left\{\int_{\mathbb{R}^d} \varphi \, d(\mu_+ - \mu_-);\ \varphi \in \mathcal{C}_c^{0,Lip}, \|\varphi\|_\infty \leq 1, \|\varphi\|_{Lip} \leq 1\right\} = \lambda W_1^{a,b}(\mu_+, \mu_-). \end{aligned}$$

□

3.1 Topological properties

In this section, we study the topological properties of the norm introduced above. In particular, we aim to prove that it admits a duality formula that indeed coincides with (1). We first prove that the topology of $\|.\|^{a,b}$ does not depend on $a, b > 0$.

Proposition 3. *For* $a > 0,\ b > 0$*, the norm* $\|.\|^{a,b}$ *is equivalent to* $\|.\|^{1,1}$.

Proof. For $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ denote by $(m_+^{a,b}, m_-^{a,b})$ the positive measures such that

$$\|\mu\|^{a,b} = a|\mu_+ - m_+^{a,b}| + a|\mu_- - m_-^{a,b}| + bW_1(m_+^{a,b}, m_-^{a,b}),$$

and similarly define $(m_+^{1,1}, m_-^{1,1})$ By definition of the minimizers, we have

$$\begin{aligned} \|\mu\|^{a,b} &= a|\mu_+ - m_+^{a,b}| + a|\mu_- - m_-^{a,b}| + bW_1(m_+^{a,b}, m_-^{a,b}) \\ &\leq a|\mu_+ - m_+^{1,1}| + a|\mu_- - m_-^{1,1}| + bW_1(m_+^{1,1}, m_-^{1,1}). \leq \max\{a,b\}\|\mu\|^{1,1}, \end{aligned}$$

In the same way, we obtain

$$\min\{a,b\}\|\mu\|^{1,1} \leq \|\mu\|^{a,b} \leq \max\{a,b\}\|\mu\|^{1,1}.$$

□

We give now an equivalent Kantorovich-Rubinstein duality formula for the new distance. We denote by $\mathcal{C}_b^0(\mathbb{R}^d;\mathbb{R})$ the set of bounded and continuous functions on $\mathbb{R}^d$. For $f \in \mathcal{C}_b^0(\mathbb{R}^d;\mathbb{R})$, similarly to $\mathcal{C}_c^0(\mathbb{R}^d;\mathbb{R})$, we define the following

$$\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|, \qquad \|f\|_{Lip} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

We introduce

$$\mathcal{C}_b^{0,Lip} = \{f \in \mathcal{C}_b^0(\mathbb{R}^d;\mathbb{R}) \mid \|f\|_{Lip} < \infty\}.$$

In the next proposition, we express the Kantorovich duality for the norm $\mathbb{W}_1^{1,1}$. This shows that $\mathbb{W}_1^{1,1}$ coincides with the bounded Lipschitz distance introduced in [6], also called Fortet Mourier distance in [14].

Proposition 4 (Kantorovich duality)**.** *The signed generalized Wasserstein distance $\mathbb{W}_1^{1,1}$ coincides with the bounded Lipschitz distance: for μ, ν in $\mathcal{M}^s(\mathbb{R}^d)$, it holds*

$$\mathbb{W}_1^{1,1}(\mu,\nu) = \sup\left\{\int_{\mathbb{R}^d} \varphi \; d(\mu-\nu);\; \varphi \in \mathcal{C}_b^{0,Lip}, \|\varphi\|_\infty \leq 1, \|\varphi\|_{Lip} \leq 1\right\}$$

Proof. By using Lemma 4 we have

$$\begin{aligned}\mathbb{W}_1^{1,1}(\mu,\nu) &= W_1^{a,b}(\mu_+ + \nu_-, \nu_+ + \mu_-)\\ &= \sup\left\{\int_{\mathbb{R}^d} \varphi \; d(\mu_+ - \mu_- - (\nu_+ - \nu_-));\; \varphi \in \mathcal{C}_c^{0,Lip}, \|\varphi\|_\infty \leq 1, \|\varphi\|_{Lip} \leq 1\right\}\\ &= \sup\left\{\int_{\mathbb{R}^d} \varphi \; d(\mu-\nu);\; \varphi \in \mathcal{C}_c^{0,Lip}, \|\varphi\|_\infty \leq 1, \|\varphi\|_{Lip} \leq 1\right\}.\end{aligned}$$

We denote by

$$S = \sup\left\{\int_{\mathbb{R}^d} \varphi \; d(\mu-\nu);\; \varphi \in \mathcal{C}_b^{0,Lip}, \|\varphi\|_\infty \leq 1, \|\varphi\|_{Lip} \leq 1\right\}.$$

First observe that $S < +\infty$. Indeed, it holds $\int_{\mathbb{R}^d} \varphi \; d(\mu-\nu) \leq \|\varphi\|_\infty(|\mu|+|\nu|) < +\infty$. Denote with φ_n a sequence of functions of $\mathcal{C}_b^{0,Lip}$ such that $\int_{\mathbb{R}^d} \varphi_n \; d(\mu-\nu) \to S$ as $n \to \infty$. Consider a sequence of functions ρ_n in $\mathcal{C}_c^{0,Lip}$ such that $\rho_n(x) = 1$ for $x \in B_0(n)$, $\rho_n(x) = 0$ for $x \notin B_0(n+1)$ and $\|\rho_n\|_\infty \leq 1$. For the sequence $\psi_n = \varphi_n \rho_n$ of functions of $\mathcal{C}_c^{0,Lip}$, it holds

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \psi_n \; d(\mu-\nu) - S\right| &\leq \left|\int_{\mathbb{R}^d} (\psi_n - \varphi_n) \; d(\mu-\nu)\right| + \left|\int_{\mathbb{R}^d} \varphi_n \; d(\mu-\nu) - S\right|\\ &\leq 2\left|\int_{\mathbb{R}^d \setminus B_0(n)} d(\mu-\nu)\right| + \left|\int_{\mathbb{R}^d} \varphi_n \; d(\mu-\nu) - S\right|\end{aligned}$$

since $\|\varphi_n\|_\infty \leq 1$. The first term goes to zero with n, since $(\mu-\nu)$ being of finite mass is tight, and the second term goes to zero with n by definition of S and φ_n. Then

$$S = \sup\left\{\int_{\mathbb{R}^d} \varphi \; d(\mu-\nu);\; \varphi \in \mathcal{C}_c^{0,Lip}, \|\varphi\|_\infty \leq 1, \|\varphi\|_{Lip} \leq 1\right\},$$

and Proposition 4 is proved. □

Remark 2. *We observe that a sequence μ_n of $\mathcal{M}^s(\mathbb{R})$ which satisfies $\|\mu_n\|^{a,b} \underset{n\to\infty}{\to} 0$ is not necessarily tight, and its mass is not necessarily bounded. For instance, we have that*

$$\nu_n = \delta_n - \delta_{n+\frac{1}{n}}$$

is not tight, whereas it satisfies for n sufficiently large $\|\nu_n\|^{a,b} = \frac{b}{n} \underset{n\to\infty}{\to} 0$. The sequence

$$\mu_n = n\,\delta_{\frac{1}{n^2}} - n\,\delta_{-\frac{1}{n^2}}$$

satisfies $\|\mu_n\|^{a,b} = \dfrac{2bn}{n^2}$ for n sufficiently large (depending on a and b, it may be less expensive to cancel the mass than to transport it), so that $\|\mu_n\|^{a,b} \underset{n\to\infty}{\to} 0$ whereas $|\mu_n| = 2n$ is not bounded.

Remark 3. *Norm* $\|.\|^{1,1}$ *does not metrize tight convergence, contrarily to what is stated in [6]. Indeed, take* $\mu_n = \delta_{\sqrt{2\pi n+\frac{\pi}{2}}} - \delta_{\sqrt{2\pi n+\frac{3\pi}{2}}}$. *We have*

$$\|\mu_n\|^{1,1} \leq \left|\sqrt{2\pi n + \frac{\pi}{2}} - \sqrt{2\pi n + \frac{3\pi}{2}}\right| \underset{n\to\infty}{\to} 0,$$

even though for $\varphi(x) = \sin(x^2)$ *in* $\mathcal{C}_b^0(\mathbb{R})$, *we have*

$$\int_{\mathbb{R}} \varphi d\mu_n = 2, \quad n \in \mathbb{N}.$$

Remark 4. *We have as a direct consequence of Proposition 4 that*

$$\|\mu_n - \mu\|^{a,b} \underset{n\to\infty}{\to} 0 \qquad \Rightarrow \quad \forall \varphi \in \mathcal{C}_b^{0,Lip}(\mathbb{R}^d), \int_{\mathbb{R}^d} \varphi d\mu_n \underset{n\to\infty}{\to} \int_{\mathbb{R}^d} \varphi d\mu. \tag{11}$$

However, the reciprocal statement of (11) *is false: define*

$$\mu_n := n\cos(nx)\chi_{[0,\pi]}.$$

For

$$\varphi_n := \frac{1}{n}\cos(nx),$$

it is clear that

$$\int_{\mathbb{R}} \varphi_n \, d\mu_n = \int_0^\pi \cos^2(nx)\, dx = \frac{\pi}{2} \not\to 0.$$

We now prove that, for each φ *in* $\mathcal{C}_b^{0,Lip}(\mathbb{R})$, *it holds* $\int_{\mathbb{R}} \varphi \, d\mu_n \to 0$. *Given* $\varphi \in \mathcal{C}_b^{0,Lip}(\mathbb{R})$, *define*

$$f(x) := \begin{cases} \varphi(-x), & \text{when } x \in [-\pi, \pi], \\ \varphi(x - 2\pi), & \text{when } x > \pi, \\ \varphi(x + 2\pi), & \text{when } x < -\pi. \end{cases}$$

We have

$$\int_{\mathbb{R}} \varphi \, d\mu_n = \int_{\mathbb{R}} f \, d\mu_n.$$

Since f *is a* 2π*-periodic function, it also holds* $\int f \, d\mu_n = na_n$, *where* a_n *is the* n*-th cosine coefficient in the Fourier series expansion of* f. *We then prove* $na_n \to 0$ *for any* 2π*-periodic Lipschitz function* f, *following the ideas of [15, p. 46, last line]. Since* f *is Lipschitz, then its distributional derivative is in* $L^\infty[-\pi, \pi]$ *and thus in* $L^1[-\pi, \pi]$. *Then*

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx = -\frac{1}{2n\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\, dx = -\frac{b'_n}{n},$$

where b'_n *is the* n*-th sine coefficient of* f'. *As a consequence of the Riemann-Lebesgue lemma,* $b'_n \to 0$, *and this implies* $na_n \to 0$.

Proposition 5. *Assume that* $\|\mu_n\|^{a,b} \underset{n\to\infty}{\to} 0$, *then* $\Delta m_n := |\mu_n^+| - |\mu_n^-| \underset{n\to\infty}{\to} 0$.

Proof. We have by definition $\|\mu_n\|^{a,b} = W_1^{a,b}(\mu_n^+, \mu_n^-)$. We denote by $\bar{\mu}_n^+, \bar{\mu}_n^-$ the minimizers in the right hand side of (9) realizing the distance $W_1^{a,b}(\mu_n^+, \mu_n^-)$. We have

$$\|\mu_n\|^{a,b} = a\left(|\mu_n^+ - \bar{\mu}_n^+| + |\mu_n^- - \bar{\mu}_n^-|\right) + bW_1(\bar{\mu}_n^+, \bar{\mu}_n^-), \qquad |\bar{\mu}_n^+| = |\bar{\mu}_n^-|.$$

Since $\|\mu_n\|^{a,b} \underset{n\to\infty}{\to} 0$, each of the three terms converges to zero as well. Thus,

$$\begin{aligned}\big||\mu_n^+| - |\mu_n^-|\big| &= \big||\mu_n^+ - \bar{\mu}_n^+ + \bar{\mu}_n^+| - |\mu_n^- - \bar{\mu}_n^- + \bar{\mu}_n^-|\big| \\ &= \big||\mu_n^+ - \bar{\mu}_n^+| + |\bar{\mu}_n^+| - |\mu_n^- - \bar{\mu}_n^-| - |\bar{\mu}_n^-|\big| \\ &= \big||\mu_n^+ - \bar{\mu}_n^+| - |\mu_n^- - \bar{\mu}_n^-|\big| \underset{n\to\infty}{\to} 0.\end{aligned}$$

□

Theorem 2. *The two following statements are equivalent:*

$$(i) \quad \|\mu_n - \mu\|^{a,b} \underset{n\to\infty}{\to} 0.$$

(ii) There exists z_n^+, z_n^-, m_n^+, $m_n^- \in \mathcal{M}(\mathbb{R}^d)$ such that

$$\begin{aligned}\mu_n^+ &= z_n^+ + m_n^+,\\ \mu_n^- &= z_n^- + m_n^-,\end{aligned} \qquad \text{with} \qquad \begin{aligned}&W_1^{a,b}(z_n^+, z_n^-) \underset{n\to\infty}{\to} 0,\\ &W_1^{a,b}(m_n^+, \mu^+) \underset{n\to\infty}{\to} 0,\\ &W_1^{a,b}(m_n^-, \mu^-) \underset{n\to\infty}{\to} 0,\\ &\{m_n^+\}_n \text{ and } \{m_n^-\}_n \text{ are tight and bounded in mass,}\end{aligned}$$

where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition, and $\mu_n = \mu_n^+ - \mu_n^-$ is any decomposition.

Proof. We start by proving $(i) \Rightarrow (ii)$. We have $\|\mu_n - \mu\|^{a,b} = \mathbb{W}_1^{a,b}(\mu_n, \mu) = W_1^{a,b}(\mu_n^+ + \mu^-, \mu_n^- + \mu^+) \underset{n\to\infty}{=\to} 0$. We denote by $a_n \le (\mu_n^+ + \mu^-)$ and $b_n \le (\mu_n^- + \mu^+)$ a choice of minimizers realizing $W_1^{a,b}(\mu_n^+ + \mu^-, \mu_n^- + \mu^+)$, as well as π_n being a minimizing transference plan from a_n to b_n.

Step 1. The removed mass. We define by a_n^+ and b_n^- the largest transported mass which is respectively below μ_n^+ and μ_n^-

$$\begin{aligned}a_n^+ &= \mu_n^+ \wedge a_n, \qquad & b_n^- &= \mu_n^- \wedge b_n,\\ a_n^- &= a_n - a_n^+, \qquad & b_n^+ &= b_n - b_n^-.\end{aligned}$$

The mass which is removed is then $r_n = r_n^+ + r_n^- := (\mu_n^+ - a_n^+) + (\mu^- - a_n^-)$ and $r_n^* = r_n^{*,-} + r_n^{*,+} := (\mu_n^- - b_n^-) + (\mu^+ - b_n^+)$. The removed mass r_n and r_n^* are expressed here as the sum of two positive measures. Indeed, it is clear by definition that $a_n^+ \le \mu_n^+$, and since $a_n \le \mu_n^+ + \mu^-$, Lemma 2, gives that $a_n^- = a_n - a_n \wedge \mu_n^+ \le \mu^-$. We reason the same way for r_n^*. Then, we have $W_1^{a,b}(\mu_n^+ + \mu^-, \mu_n^- + \mu^+) = a\,(|\mu_n^+ - a_n^+| + |\mu^- - a_n^-| + |\mu_n^- - b_n^-| + |\mu^+ - b_n^+|) + bW_1(a_n, b_n)$. Since $W_1^{a,b}(\mu_n^+ + \mu^-, \mu_n^- + \mu^+)$ goes to zero, each of the five terms of the above decomposition goes to zero, and in particular, $|\mu_n^+ - a_n^+| \underset{n\to\infty}{\to} 0$ and $|\mu_n^- - b_n^-| \underset{n\to\infty}{\to} 0$ which implies that that

$$W_1^{a,b}(\mu_n^+ - a_n^+, 0) \underset{n\to\infty}{\to} 0, \qquad W_1^{a,b}(\mu_n^- - b_n^-, 0) \underset{n\to\infty}{\to} 0. \tag{12}$$

Step 2. The transported mass. The mass a_n^+ is split into two pieces: ν_n is sent to μ_n^-, and ξ_n is sent to μ^+. Denote by $\bar{a}_n^+$ the image of a_n^+ under π_n (using Definition 4), then we define

$\nu_n^* = \bar{a}_n^+ \wedge \mu_n^-$. (Still using Definition 4), we denote by ν_n the image of ν_n^* under π_n. Then, we define ξ_n such that $a_n^+ = \nu_n + \xi_n$, and we denote by ξ_n^* the image of ξ_n under π_n. By definition, we have

$$W_1(a_n, b_n) = W_1(\nu_n, \nu_n^*) + W_1(\xi_n, \xi_n^*) + W_1(w_n, w_n^*) + W_1(\alpha_n, \alpha_n^*), \tag{13}$$

with $a_n^+ = \nu_n + \xi_n$, w_n^* is defined so that $b_n^- = \nu_n^* + w_n^*$, w_n is the image of w_n^* under π_n, α_n is defined so that $\mu^- = w_n + \alpha_n$, α_n^* is the image of α_n under π_n, and it can be checked that $\mu^+ = \xi_n^* + \alpha_n^*$. Since $W_1(a_n, b_n) \underset{n\to\infty}{\rightarrow} 0$, each of the four term of the sum (13) is going to zero.

Step 3. Conclusion.

Let us write

$$z_n^+ = \nu_n + (\mu_n^+ - a_n^+), \quad z_n^- = \nu_n^* + (\mu_n^- - b_n^-), \quad m_n^+ = \xi_n, \quad m_n^- = w_n^*.$$

We show here that the sequences defined hereinabove satisfy the conditions stated in (ii). First, we have $z_n^+ + m_n^+ = \nu_n^+ + (\mu_n^+ - a_n^+) + \xi_n = \mu_n^+$ and similarly, $z_n^- + m_n^- = \nu_n^* + (\mu_n^- - b_n^-) + w_n^* = \mu_n^-$.

Then, we have

$$\begin{aligned}
W_1^{a,b}(z_n^+, z_n^-) &= W_1^{a,b}(\nu_n + (\mu_n^+ - a_n^+), \nu_n^* + (\mu_n^- - b_n^-)) \\
&\leq W_1^{a,b}(\nu_n, \nu_n^*) + W_1^{a,b}(\mu_n^+ - a_n^+, \mu_n^- - b_n^-) \quad \text{using Lemma 5} \\
&\leq bW_1(\nu_n, \nu_n^*) + W_1^{a,b}(\mu_n^+ - a_n^+, 0) + W_1^{a,b}(0, \mu_n^- - b_n^-) \\
&\underset{n\to\infty}{\rightarrow} 0, \text{ using (12) and (13).}
\end{aligned}$$

Here, we also used that for $|\mu| = |\nu|$, $W_1^{a,b}(\mu, \nu) \leq bW_1(\mu, \nu)$. This is trivial with the definition of $W_1^{a,b}$. Now, we also have

$$\begin{aligned}
W_1^{a,b}(m_n^+, \mu^+) = W_1^{a,b}(\xi_n, \mu^+) &\leq W_1^{a,b}(\xi_n, \xi_n^*) + W_1^{a,b}(\xi_n^*, b_n^+) + W_1^{a,b}(b_n^+, \mu^+) \quad \text{(triangular inequality)} \\
&= W_1^{a,b}(\xi_n, \xi_n^*) + W_1^{a,b}(\alpha_n^*, 0) + W_1^{a,b}(\mu^+ - b_n^+, 0)
\end{aligned}$$

since $\alpha_n^* + \xi_n^* = b_n^+$. We know that $W_1^{a,b}(\xi_n, \xi_n^*) \leq bW_1(\xi_n, \xi_n^*) \underset{n\to\infty}{\rightarrow} 0$ using (13), and that $W_1^{a,b}(\mu^+ - b_n^+, 0) \underset{n\to\infty}{\rightarrow} 0$ using (12). Let us explain now why $W_1^{a,b}(\alpha_n^*, 0) \underset{n\to\infty}{\rightarrow} 0$. We recall that $W_1(\alpha_n, \alpha_n^*) \underset{n\to\infty}{\rightarrow} 0$, $\alpha_n \leq a_n^- \leq \mu^-$, $\alpha_n^* \leq b_n^+ \leq \mu^+$. Since $(\alpha_n)_n$ is uniformly bounded in mass, then there exists $\alpha \in \mathcal{M}(\mathbb{R}^d)$ such that $\alpha_{\varphi(n)} \underset{n\to\infty}{\rightharpoonup} \alpha$ (weak compactness of uniformly bounded in mass Radon measures, see [5]). We have also that $(\alpha_{\varphi(n)})_n$ is tight, since $\alpha_{\varphi(n)} \leq \mu^-$ which has a finite mass. Using Theorem 13 of [10], we deduce that $W_1^{a,b}(\alpha_{\varphi(n)}, \alpha) \underset{n\to\infty}{\rightarrow} 0$. Then, $W_1^{a,b}(\alpha_{\varphi(n)}^*, \alpha) \leq W_1^{a,b}(\alpha_{\varphi(n)}^*, \alpha_{\varphi(n)}) + W_1^{a,b}(\alpha_{\varphi(n)}, \alpha) \leq W_1(\alpha_{\varphi(n)}^*, \alpha_{\varphi(n)}) + W_1^{a,b}(\alpha_{\varphi(n)}, \alpha) \underset{n\to\infty}{\rightarrow} 0$. Then, using again Theorem 13 of [10], we deduce that $\alpha_{\varphi(n)}^* \underset{n\to\infty}{\rightharpoonup} \alpha$ Since $\alpha_n \leq \mu^-$, we have $\alpha \leq \mu^-$. Likewise, $\alpha_n^* \leq \mu^+$ implies $\alpha \leq \mu^+$. Since $\mu^- \perp \mu^+$, we have $\alpha = 0$. We have $W_1^{a,b}(\alpha_{\varphi(n)}, 0) \underset{n\to\infty}{\rightarrow} 0$ and $W_1^{a,b}(\alpha_{\varphi(n)}, 0) \underset{n\to\infty}{\rightarrow} 0$. The sequence $(\alpha_n)_n$ satisfies the following property: each of its subsequences admits a subsequence converging to zero. Thus, we have that the whole sequence is converging to zero, i.e. $W_1^{a,b}(\alpha_n, 0) \underset{n\to\infty}{\rightarrow} 0$ and $W_1^{a,b}(\alpha_n^*, 0) \underset{n\to\infty}{\rightarrow} 0$. Lastly, the tightness of $(m_n^+)_n$ and $(m_n^-)_n$ is given again by Theorem 13 of [10], since $W_1^{a,b}(m_n^\pm, \mu^\pm) \underset{n\to\infty}{\rightarrow} 0$.

We prove now that $(ii) \Rightarrow (i)$. Let us assume (ii). We have

$$\|\mu_n - \mu\|^{a,b} = W_1^{a,b}(\mu_n^+ + \mu^-, \mu_n^- + \mu^+) = W_1^{a,b}(z_n^+ + m_n^+ + \mu^-, z_n^- + m_n^- + \mu^+)$$

$$\leq W_1^{a,b}(z_n^+, z_n^-) + W_1^{a,b}(m_n^+, \mu^+) + W_1^{a,b}(\mu^-, m_n^-) \underset{n\to\infty}{\to} 0,$$

which is (i). □

We recall from [12] that the space $(\mathcal{M}(\mathbb{R}^d), W_p^{a,b})$ is a Banach space. The proof is based on the fact that a Cauchy sequence of positive measures is both uniformly bounded in mass and tight. This is not true anymore for a Cauchy sequence of signed measures.

Remark 5. *Observe that $(\mathcal{M}^s(\mathbb{R}^d), \|.\|^{a,b})$ is not a Banach space. Indeed, take the sequence*

$$\mu_n = \sum_{i=1}^{n} \left(\delta_{i+\frac{1}{2^i}} - \delta_{i-\frac{1}{2^i}}\right).$$

It is a Cauchy sequence in $(\mathcal{M}^s(\mathbb{R}^d), \|.\|^{a,b})$, since it holds

$$\mathbb{W}_1^{a,b}(\mu_n, \mu_{n+k}) \leq 2b \sum_{i=n+1}^{n+k} \frac{1}{2^i} \leq 2b \sum_{i=n+1}^{+\infty} \frac{1}{2^i} \underset{n\to\infty}{\to} 0.$$

However, such sequence does not converge in $(\mathcal{M}^s(\mathbb{R}^d), \|.\|^{a,b})$. As seen in Remark (3), the convergence for the norm $\|.\|^{a,b}$ implies the convergence in the sense of distributions. In the sense of distributions we have

$$\mu_n \rightharpoonup \mu^* := \sum_{i=1}^{+\infty} \left(\delta_{i+\frac{1}{2^i}} - \delta_{i-\frac{1}{2^i}}\right) \notin \mathcal{M}^s(\mathbb{R}).$$

Indeed, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$, since φ is compactly supported, it holds

$$\langle \mu_n - \mu, \varphi \rangle = \sum_{i=n+1}^{+\infty} \left(\varphi\left(i + \frac{1}{2^i}\right) - \varphi\left(i - \frac{1}{2^i}\right)\right) \underset{n\to\infty}{\to} 0.$$

Nevertheless, we have the following convergence result.

Theorem 3. *A Cauchy sequence in $(\mathcal{M}^s(\mathbb{R}^d), \|.\|^{a,b})$ uniformly bounded in mass and tight converges in $(\mathcal{M}^s(\mathbb{R}^d), \|.\|^{a,b})$.*

Proof. Take a tight Cauchy sequence $(\mu_n)_n \in \mathcal{M}^s(\mathbb{R}^d)$ such that the sequences given by the Jordan decomposition $|\mu_n^+|$ and $|\mu_n^-|$ are uniformly bounded. Then, by Lemma 3, there exists μ^+ and μ^- in $\mathcal{M}(\mathbb{R}^d)$ and φ non decreasing such that, $\mu_{\varphi(n)}^+ \underset{n\to\infty}{\rightharpoonup} \mu^+$ and $\mu_{\varphi(n)}^- \underset{n\to\infty}{\rightharpoonup} \mu^-$. Since μ_n^+ and μ_n^- are assumed to be tight, it holds $W_1^{a,b}(\mu_{\varphi(n)}^+, \mu^+) \underset{n\to\infty}{\to} 0$ and $W_1^{a,b}(\mu_{\varphi(n)}^-, \mu^-) \underset{n\to\infty}{\to} 0$ (see [11, Theorem 13]). Then, we have

$$\begin{aligned}\|\mu_n - (\mu^+ - \mu^-)\|^{a,b} \leq & \|\mu_n - \mu_{\varphi(n)}\|^{a,b} + \|\mu_{\varphi(n)} - (\mu^+ - \mu^-)\|^{a,b} \\ & \leq \|\mu_n - \mu_{\varphi(n)}\|^{a,b} + W_1^{a,b}(\mu_{\varphi(n)}^+ + \mu^-, \mu_{\varphi(n)}^- + \mu^+) \\ & \leq \|\mu_n - \mu_{\varphi(n)}\|^{a,b} + W_1^{a,b}(\mu_{\varphi(n)}^+, \mu^+) + W_1^{a,b}(\mu_{\varphi(n)}^-, \mu^-) \underset{n\to\infty}{\to} 0\end{aligned}$$

since $(\mu_n)_n$ is a Cauchy sequence. □

4 Application to the transport equation with source term

This section is devoted to the use of the norm defined in Definition 6 to guarantee existence, uniqueness, and stability with respect to initial condition for the transport equation (2).

4.1 Estimates of the norm under flow action

In this section, we extend the action of flows on probability measures to signed measures, and state some estimates about the variation of $\|\mu-\nu\|^{a,b}$ after the action of a flow on μ and ν. Notice that for $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ and T a map, we have $T\#\mu = T\#\mu^+ - T\#\mu^-$, where $\mu = \mu^+ - \mu^-$ is any decomposition of μ. Observe that in general, given $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ and $T : \mathbb{R}^d \mapsto \mathbb{R}^d$ a Borel map, it only holds $|T\#\mu| \leq |\mu|$, even by choosing the Jordan decomposition for (μ^+, μ^-), since it may hold that $T\#\mu^+$ and $T\#\mu^-$ are not orthogonal. However, if T is injective (as it will be in the rest of the paper), it holds $T\#\mu^+ \perp T\#\mu^-$, hence $|T\#\mu| = |\mu|$.

Lemma 7. *For $v(t,x)$ measurable in time, uniformly Lipschitz in space, and uniformly bounded, we denote by Φ_t^v the flow it generates, i.e. the unique solution to*

$$\frac{d}{dt}\Phi_t^v = v(\Phi_t^v), \qquad \Phi_0^v = I_d.$$

Given $\mu_0 \in \mathcal{M}^s(\mathbb{R}^d)$, then, $\mu_t = \Phi_t^v\#\mu_0$ is the unique solution of the linear transport equation

$$\begin{cases} \dfrac{\partial}{\partial t}\mu_t + \nabla.(v(t,x)\mu_t) = 0, \\ \mu_{|t=0} = \mu_0 \end{cases}$$

in $\mathcal{C}([0,T], \mathcal{M}^s(\mathbb{R}^d))$.

Proof. The proof is a direct consequence of [13, Theorem 5.34] combined with [3, Theorem 2.1.1]. □

Lemma 8. *Let v and w be two vector fields, both satisfying for all $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$, the following properties:*

$$|v(t,x) - v(t,y)| \leq L|x-y|, \qquad |v(t,x)| \leq M.$$

Let μ and ν be two measures of $\mathcal{M}^s(\mathbb{R}^d)$. Then

- $\|\phi_t^v\#\mu - \phi_t^v\#\nu\|^{a,b} \leq e^{Lt}\|\mu-\nu\|^{a,b}$
- $\|\mu - \phi_t^v\#\mu\|^{a,b} \leq b\, t\, M|\mu|$,
- $\|\phi_t^v\#\mu - \phi_t^w\#\mu\|^{a,b} \leq b|\mu|\frac{(e^{Lt}-1)}{L}\|v-w\|_{\mathcal{C}^0}$
- $\|\phi_t^v\#\mu - \phi_t^w\#\nu\|^{a,b} \leq e^{Lt}\|\mu-\nu\|^{a,b} + b\min\{|\mu|,|\nu|\}\frac{(e^{Lt}-1)}{L}\|v-w\|_{\mathcal{C}^0}$

Proof. The first three inequalities follow from [12, Proposition 10]. For the first inequality, we write

$$\begin{aligned}
\|\phi_t^v\#\mu - \phi_t^v\#\nu\|^{a,b} = \mathbb{W}_1^{a,b}(\phi_t^v\#\mu, \phi_t^v\#\nu) &= \mathbb{W}_1^{a,b}(\phi_t^v\#\mu^+ - \phi_t^v\#\mu^-, \phi_t^v\#\nu^+ - \phi_t^v\#\nu^-) \\
&= W_1^{a,b}(\phi_t^v\#(\mu^+ + \nu^-), \phi_t^v\#(\mu^- + \nu^+)) \\
&\leq e^{Lt}W_1^{a,b}(\mu^+ + \nu^-, \mu^- + \nu^+) \quad \text{by [12, Prop. 10]} \\
&= e^{Lt}\|\mu-\nu\|^{a,b}.
\end{aligned}$$

For the second inequality,

$$\begin{aligned}
\mathbb{W}_1^{a,b}(\mu, \phi_t^v \# \mu) &= W_1^{a,b}(\mu^+ + \phi_t^v \# \mu^-, \mu^- + \phi_t^v \# \mu^+) \\
&\leq W_1^{a,b}(\mu^+, \phi_t^v \# \mu^+) + W_1^{a,b}(\mu^-, \phi_t^v \# \mu^-) \quad \text{(Lemma 5)} \\
&\leq b\, t\, \|v\|_{\mathcal{C}^0}(|\mu^+| + |\mu^-|) \quad \text{ by [12, Prop. 10]} \\
&= b\, t\, \|v\|_{\mathcal{C}^0}|\mu| \quad \text{since } \mu = \mu^+ - \mu^- \text{ is the Jordan decomposition.}
\end{aligned}$$

The third inequality is given by

$$\begin{aligned}
\|\phi_t^v \# \mu - \phi_t^w \# \mu\|^{a,b} &= \mathbb{W}_1^{a,b}(\phi_t^v \# \mu^+ + \phi_t^w \# \mu^-, \phi_t^w \# \mu^+ + \phi_t^v \# \mu^-) \\
&\leq W_1^{a,b}(\phi_t^v \# \mu^+, \phi_t^w \# \mu^+) + W_1^{a,b}(\phi_t^w \# \mu^-, \phi_t^v \# \mu^-) \\
&\leq b W_1(\phi_t^v \# \mu^+, \phi_t^w \# \mu^+) + W_1(\phi_t^w \# \mu^-, \phi_t^v \# \mu^-) \\
&\leq (|\mu^+| + |\mu^-|) \frac{(e^{Lt} - 1)}{L} \|v - w\||_{\mathcal{C}^0(\mathbb{R}^d)} \quad \text{using [12, Prop. 10] with } \mu = \nu.
\end{aligned}$$

The last inequality is deduced from the first and the third one using triangular inequality. □

4.2 A scheme for computing solutions of the transport equation

In this section, we build a solution to (2) as the limit of a sequence of approximated solutions defined in the following scheme. We then prove that (2) admits a unique solution.

Consider $\mu_0 \in \mathcal{M}^s(\mathbb{R}^d)$ such that $\mathrm{supp}(\mu_0) \subset \mathcal{K}$, with $\mathcal{K}$ compact. Let $v \in \mathcal{C}^{0,Lip}(\mathcal{M}^s(\mathbb{R}^d), \mathcal{C}^{0,Lip}(\mathbb{R}^d))$ and $h \in \mathcal{C}^{0,Lip}(\mathcal{M}^s(\mathbb{R}^d), \mathcal{M}^s(\mathbb{R}^d))$ satisfying (H-1)-(H-2)-(H-3). We now define a sequence $(\mu_t^k)_k$ of approximated solutions for (2) through the following Euler-explicit-type iteration scheme. For simplicity of notations, we define a solution on the time interval $[0, 1]$.

SCHEME

Initialization. Fix $k \in \mathbb{N}$. Define $\Delta t = \dfrac{1}{2^k}$. Set $\mu_0^k = \mu_0$.

Induction. Given $\mu_{i\Delta t}$ for $i \in \{0, 1, \ldots, 2^k - 1\}$, define $v_{i\Delta t}^k := v[\mu_{i\Delta t}^k]$ and

$$\mu_t^k = \Phi_{t-i\Delta t}^{v_{i\Delta t}} \# \mu_{i\Delta t}^k + (t - i\Delta t) h[\mu_{i\Delta t}^k], \qquad t \in [i\Delta t, (i+1)\Delta t]. \tag{14}$$

Proposition 6. *The sequence $(\mu_t^k)_k$ defined in the scheme above is a Cauchy sequence in the space $\mathcal{C}^0([0,1], \mathcal{M}^s(\mathbb{R}^d), \|.\|)$ with*

$$\|\mu_t\| = \sup_{t \in [0,1]} \|\mu_t\|^{a,b}.$$

Moreover, it is uniformly bounded in mass, i.e.

$$\sup_{t \in [0,1]} |\mu_t^k| < \infty. \tag{15}$$

Proof. Let L be the Lipschitz constant in (H-2). We assume to have k sufficiently large to have $e^{Lt} \leq 1 + 2Lt$ for all $t \leq [0, \Delta t]$. This holds e.g. for $L\Delta t \leq 1$, hence $k \geq \log_2(L)$.

We also notice that the sequence built by the scheme satisfies

$$|\mu_t^k| \leq P + |\mu_0|, \quad t \in [0, 1], \tag{16}$$

where P is such that $|h[\mu]| \leq P$ by (H-3). Indeed, it holds for $t \in [i\Delta t, (i+1)\Delta t]$

$$|\mu_t^k| \leq |\Phi_t^{v_{i\Delta t}} \# \mu_{i\Delta t}^k| + \Delta t |h[\mu_{i\Delta t}^k]| \leq |\mu_{i\Delta t}^k| + \Delta t P,$$

and the result follows by induction. This proves (15). The sequence $(\mu_t^k)_{k\in\mathbb{N}}$ also has uniformly bounded support. Indeed, use (14) and (H-2)-(H-3) to write

$$\text{supp}\{\mu_t^k\} \subseteq \mathcal{K}_{t,M,R},$$

with $\text{supp}\{\mu\} = \text{supp}\{\mu^+\} \cup \text{supp}\{\mu^-\}$ where (μ^+, μ^-) is the Jordan decomposition of μ, and

$$\mathcal{K}_{t,M,R} := \{x \in \mathbb{R}^d,\ x = x_{\mathcal{K},R} + x',\ x_{\mathcal{K},R} \in \mathcal{K} \cup B_0(R),\ \|x'\| \leq tM\}.$$

Take now R' such that $\mathcal{K} \cup B_0(R) \subset B_0(R')$. Then, it holds $\mathcal{K}_{t,M,R} \subset B(0, R'+M)$. Since such set does not depend on t, while M, R are fixed, then μ_t^k have uniformly bounded support.

We now follow the notations of [10] and define $m_j^k := \mu^k_{\frac{j}{2^k}}$, $v_j^k := v[m_j^k]$ and the corresponding flow $f_t^{j,k} := \phi_t^{v_j^k}$. Fix $k \in \mathbb{N}$ and $t \in [0,1]$. Define $j \in \{0, 1, \ldots, 2^k\}$ such that $t \in \left]\frac{j}{2^k}, \frac{j+1}{2^k}\right]$

First case. If $t \in \left]\frac{j}{2^k}, \frac{2j+1}{2^{k+1}}\right]$, we call $t' = t - \frac{j}{2^k} \leq \frac{1}{2^{k+1}}$ and we obtain

$$\begin{aligned}
\mathbb{W}_1^{a,b}(\mu_t^k, \mu_t^{k+1}) &= \mathbb{W}_1^{a,b}(f_{t'}^{j,k} \# m_j^k + t' h[m_j^k], f_{t'}^{2j,k+1} \# m_{2j}^{k+1} + t' h[m_{2j}^{k+1}]) \\
&\leq \mathbb{W}_1^{a,b}(f_{t'}^{j,k} \# m_j^k, f_{t'}^{2j,k+1} \# m_{2j}^{k+1}) + \mathbb{W}_1^{a,b}(t' h[m_j^k], t' h[m_{2j}^{k+1}]) \\
&\leq e^{Lt'} \mathbb{W}_1^{a,b}(m_j^k, m_{2j}^{k+1}) + |m_j^k| \frac{(e^{Lt'} - 1)}{L} \|v_j^k - v_{2j}^{k+1}\|_{\mathcal{C}^0(\mathbb{R}^d)} + t' Q \mathbb{W}_1^{a,b}(m_j^k, m_{2j}^{k+1}) \\
&\leq \mathbb{W}_1^{a,b}(m_j^k, m_{2j}^{k+1}) \left(e^{Lt'} + (P + |\mu^0|) \frac{1}{L} (e^{Lt'} - 1) + t' Q \right)
\end{aligned}$$

Since it holds

$$e^{Lt'} \leq 1 + 2Lt' \leq 1 + 2L 2^{-(k+1)}, \qquad \frac{(e^{Lt'} - 1)}{L} \leq 2 \cdot 2^{-(k+1)},$$

we have

$$\|\mu_t^k - \mu_t^{k+1}\|^{a,b} \leq \|\mu^k_{\frac{j}{2^k}} - \mu^{k+1}_{\frac{2j}{k+1}}\|^{a,b} \left(1 + 2^{-(k+1)} \left(2L + 2(P + |\mu^0|) + Q\right)\right), \quad t \in \left[\frac{j}{2^k}, \frac{2j+1}{2^{k+1}}\right]. \tag{17}$$

Second case. If $t \in \left]\frac{2j+1}{2^{k+1}}, \frac{j+1}{2^k}\right]$, we call $t' = t - \frac{2j+1}{2^{k+1}} \leq \frac{1}{2^{k+1}}$ and we obtain

$$\begin{aligned}
\mu_t^k &= f^{j,k}_{t' + \frac{1}{2^{k+1}}} \# m_j^k + \left(t' + \frac{1}{2^{k+1}}\right) h[m_j^k] = f_{t'}^{j,k} \# f^{j,k}_{\frac{1}{2^{k+1}}} \# m_j^k + t' h[m_j^k] + \frac{1}{2^{k+1}} h[m_j^k], \\
\mu_t^{k+1} &= f_{t'}^{2j+1,k+1} \# \left(f^{2j,k+1}_{\frac{1}{2^{k+1}}} \# m_{2j}^{k+1} + \frac{1}{2^{k+1}} h[m_{2j}^{k+1}] \right) + t' h \left[f^{2j,k+1}_{\frac{1}{2^{k+1}}} \# m_{2j}^{k+1} + \frac{1}{2^{k+1}} h[m_{2j}^{k+1}] \right] \\
&= f_{t'}^{2j+1,k+1} \# f^{2j,k+1}_{\frac{1}{2^{k+1}}} \# m_{2j}^{k+1} + \frac{1}{2^{k+1}} f_{t'}^{2j+1,k+1} \# h[m_{2j}^{k+1}] + t' h \left[f^{2j,k+1}_{\frac{1}{2^{k+1}}} \# m_{2j}^{k+1} + \frac{1}{2^{k+1}} h[m_{2j}^{k+1}] \right].
\end{aligned}$$

It then holds

$$\begin{aligned}
\|\mu_t^k - \mu_t^{k+1}\|^{a,b} \leq\ & \mathbb{W}_1^{a,b} \left(f_{t'}^{j,k} \# f^{j,k}_{\frac{1}{2^{k+1}}} \# m_j^k, f_{t'}^{2j+1,k+1} \# f^{2j,k+1}_{\frac{1}{2^{k+1}}} \# m_{2j}^{k+1} \right) \\
&+ \frac{1}{2^{k+1}} \mathbb{W}_1^{a,b} \left(h[m_j^k], f_{t'}^{2j+1,k+1} \# h[m_{2j}^{k+1}] \right) \\
&+ t' \mathbb{W}_1^{a,b} \left(h[m_j^k], h \left[f^{2j,k+1}_{\frac{1}{2^{k+1}}} \# m_{2j}^{k+1} + \frac{1}{2^{k+1}} h[m_{2j}^{k+1}] \right] \right).
\end{aligned} \tag{18}$$

Use now Lemma 8 to prove the following estimate

$$\begin{aligned}&\mathbb{W}_1^{a,b}\left(f_{t'}^{j,k}\#f^{j,k}_{\frac{1}{2^{k+1}}}\#m_j^k, f_{t'}^{2j+1,k+1}\#f^{2j,k+1}_{\frac{1}{2^{k+1}}}\#m_{2j}^{k+1}\right)\\&\qquad\le (1+2L2^{-(k+1)})\mathbb{W}_1^{a,b}\left(f^{j,k}_{\frac{1}{2^{k+1}}}\#m_j^k, f^{2j,k+1}_{\frac{1}{2^{k+1}}}\#m_{2j}^{k+1}\right)+2^{-(k+1)}2P\|v_j^k-v_{2j+1}^{k+1}\|_{\mathcal{C}^0(\mathbb{R}^d)}.\end{aligned}$$

Since, according to the first case, it holds both

$$\mathbb{W}_1^{a,b}\left(f^{j,k}_{\frac{1}{2^{k+1}}}\#m_j^k, f^{2j,k+1}_{\frac{1}{2^{k+1}}}\#m_{2j}^{k+1}\right)\le \|m_j^k-m_{2j}^{k+1}\|^{a,b}\left(1+2^{-(k+1)}\left(2L+2(P+|\mu^0|)\right)\right)$$

and

$$\begin{aligned}\|v_j^k-v_{2j+1}^{k+1}\|_{\mathcal{C}^0(\mathbb{R}^d)}\le K\mathbb{W}_1^{a,b}(m_j^k,m_{2j+1}^{k+1}) &\le K\mathbb{W}_1^{a,b}(m_j^k,m_{2j}^{k+1})+K\mathbb{W}_1^{a,b}(m_{2j}^{k+1},m_{2j+1}^{k+1})\\&\le K\mathbb{W}_1^{a,b}(m_j^k,m_{2j}^{k+1})+K\mathbb{W}_1^{a,b}(m_{2j}^{k+1},m_{2j+1}^{k+1})\\&= K\mathbb{W}_1^{a,b}(m_j^k,m_{2j}^{k+1})+K\mathbb{W}_1^{a,b}(m_{2j}^{k+1},f^{2j,k+1}_{\frac{1}{2^{k+1}}}\#m_{2j}^{k+1})\\&= K\mathbb{W}_1^{a,b}(m_j^k,m_{2j}^{k+1})+KM2^{-(k+1)},\end{aligned}$$

we have

$$\begin{aligned}&\mathbb{W}_1^{a,b}\left(f_{t'}^{j,k}\#f^{j,k}_{\frac{1}{2^{k+1}}}\#m_j^k, f_{t'}^{2j+1,k+1}\#f^{2j,k+1}_{\frac{1}{2^{k+1}}}\#m_{2j}^{k+1}\right)\\&\quad\le \|m_j^k-m_{2j}^{k+1}\|^{a,b}\left(1+2^{-(k+1)}\left(4L+2(P+|\mu^0|)(1+L)+2KP\right)\right)+2^{-2(k+1)}2PKM.\end{aligned}\tag{19}$$

Moreover, it also holds both

$$\begin{aligned}&\mathbb{W}_1^{a,b}\left(h[m_j^k], f_{t'}^{2j+1,k+1}\#h[m_{2j}^{k+1}]\right)\\&\quad\le \mathbb{W}_1^{a,b}\left(h[m_j^k], f_{t'}^{2j+1,k+1}\#h[m_j^k]\right)+\mathbb{W}_1^{a,b}\left(f_{t'}^{2j+1,k+1}\#h[m_j^k], f_{t'}^{2j+1,k+1}\#h[m_{2j}^{k+1}]\right)\\&\quad\le t'MP+e^{Lt'}Q\|m_j^k-m_{2j}^{k+1}\|^{a,b}\le +MP2^{-(k+1)}+(1+2L2^{-(k+1)})\|m_j^k-m_{2j}^{k+1}\|^{a,b},\end{aligned}\tag{20}$$

and

$$\begin{aligned}&\mathbb{W}_1^{a,b}\left(m_j^k, f^{2j,k+1}_{\frac{1}{2^{k+1}}}\#m_{2j}^{k+1}+\frac{1}{2^{k+1}}h[m_{2j}^{k+1}]\right)\\&\quad\le \mathbb{W}_1^{a,b}\left(m_j^k, f^{2j,k+1}_{\frac{1}{2^{k+1}}}\#m_{2j}^{k+1}\right)+2^{-(k+1)}\mathbb{W}_1^{a,b}\left(0,h[m_{2j}^{k+1}]\right)\\&\quad\le \mathbb{W}_1^{a,b}\left(m_j^k,m_{2j}^{k+1}\right)+\mathbb{W}_1^{a,b}\left(m_{2j}^{k+1}, f^{2j,k+1}_{\frac{1}{2^{k+1}}}\#m_{2j}^{k+1}\right)+2^{-(k+1)}aP\\&\quad\le \|m_j^k-m_{2j}^{k+1}\|^{a,b}+2^{-(k+1)}(|\mu^0|+P(1+a)).\end{aligned}\tag{21}$$

Plugging (19), (20) and (21) into (18), and combining it with (17) we find

$$\|\mu_t^k-\mu_t^{k+1}\|^{a,b}\le (1+2^{-k}C_1)\|m_j^k-m_{2j}^{k+1}\|^{a,b}+C_2 2^{-2k},\quad t\in\left]\frac{j}{2^k},\frac{j+1}{2^k}\right],$$

with

$$C_1=\left(1+3L+(P+|\mu^0|)(1+L)+KP+Q\right),\quad C_2=\frac{1}{4}(MP(1+2K)+|\mu^0|+P(1+a)).$$

By induction on j, we obtain

$$\|\mu_t^k - \mu_t^{k+1}\| \leq \|m_{2^k}^k - m_{2^{k+1}}^{k+1}\|^{a,b} \leq \sum_{j=0}^{2^k-1} (1+2^{-k}C_1)^j 2^{-2k} C_2 \leq \frac{C_2}{C_1}(e^{C_1}-1)2^{-k}.$$

Since the right hand side is the term of a convergent series, then $(\mu_t^k)_k$ is a Cauchy sequence. $\square$

4.3 Proof of Theorem 1

In this section, we prove Theorem 1, stating existence and uniqueness of the solution to the Cauchy problem associated to (2). The proof is based on the proof of the same result for positive measures written in [12]. We first focus on existence.

Step 1. Existence. Observe that the sequence given by the scheme $(\mu_t^k)_k$ is a Cauchy sequence (Proposition 6) in the space $\big(\mathcal{C}^0[0,1],\ \mathcal{M}^s(\mathbb{R}^d)\big)$ and is uniformly bounded in mass. Then, by using Theorem 3, we define

$$\mu_t := \lim_{k\to\infty} \mu_t^k, \qquad \mathcal{C}^0\left([0,1], \mathcal{M}^s(\mathbb{R}^d)\right).$$

Denote the following:

$$\langle \mu, \varphi \rangle := \int_{\mathbb{R}^d} \varphi(t,x) d\mu_t(x).$$

The goal is to prove that for all $\varphi \in \mathcal{D}((0,1)\times\mathbb{R}^d)$ it holds

$$\int_0^1 dt\, (\langle \mu_t, \partial_t\varphi(t,x) + v[\mu_t].\nabla\varphi(t,x)\rangle + \langle h[\mu_t], \varphi(t,x)\rangle) = 0. \tag{22}$$

We first notice that

$$\sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt \left(\langle \mu_t^k, \partial_t\varphi(t,x) + v[\mu_{j\Delta t}^k].\nabla\varphi(t,x)\rangle + \langle h[\mu_{j\Delta t}^k], \varphi(t,x)\rangle\right) \underset{k\to\infty}{\longrightarrow} 0$$

Indeed, $\nu_t := \phi_t^v \# \nu_0$ is a weak solution of $\frac{\partial}{\partial t}\nu_t + \nabla.(v(x)\nu_t) = 0$ with v a fixed vector field, and $\eta_t = \eta_0 + th$ is a weak solution of $\frac{\partial}{\partial t}\eta_t = h$, with h a fixed measure. It then holds

$$\begin{aligned}
&\Big|\sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt \left(\langle \mu_t^k, \partial_t\varphi(t,x) + v[\mu_{j\Delta t}^k].\nabla\varphi(t,x)\rangle + \langle h[\mu_{j\Delta t}^k], \varphi(t,x)\rangle\right)\Big| \\
&\qquad = \left|\sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt\ \langle (t-j\Delta t) h[\mu_{j\Delta t}^k], v[\mu_{j\Delta t}^k].\nabla\varphi(t,x)\rangle\right| \\
&\qquad \leq MP\|\nabla\varphi\|_\infty 2^{-(k+1)} \underset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Now, to guarantee (22), it is enough to prove that

$$\begin{aligned}
\lim_{k\to\infty} \Big| &\int_0^1 dt\, (\langle \mu_t, \partial_t\varphi(t,x) + v[\mu_t].\nabla\varphi(t,x)\rangle + \langle h[\mu_t], \varphi(t,x)\rangle) \\
&- \sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt \left(\langle \mu_t^k, \partial_t\varphi(t,x) + v[\mu_{j\Delta t}^k].\nabla\varphi(t,x)\rangle + \langle h[\mu_t^k], \varphi(t,x)\rangle\right)\Big| = 0
\end{aligned}$$

We have

$$\Big|\int_0^1 dt\,(\langle \mu_t, \partial_t\varphi(t,x)\rangle) - \sum_{j=0}^{2^k-1}\int_{j\Delta t}^{(j+1)\Delta t} dt\left(\langle \mu_t^k, \partial_t\varphi(t,x)\rangle\right)\Big| \le \|\partial_t\varphi\|_\infty \|\mu_t-\mu_t^k\| \underset{k\to\infty}{\longrightarrow} 0,$$

$$\Big|\int_0^1 dt\langle h[\mu_t], \varphi(t,x)\rangle - \sum_{j=0}^{2^k-1}\int_{j\Delta t}^{(j+1)\Delta t} dt\langle h[\mu_t^k], \varphi(t,x)\rangle\Big| \le Q\|\varphi\|_\infty \|\mu_t-\mu_t^k\| \underset{k\to\infty}{\longrightarrow} 0,$$

and

$$\begin{aligned}
&\Big|\int_0^1 dt\langle \mu_t, v[\mu_t].\nabla\varphi(t,x)\rangle - \sum_{j=0}^{2^k-1}\int_{j\Delta t}^{(j+1)\Delta t} dt\langle \mu_t^k, v[\mu_{j\Delta t}^k].\nabla\varphi(t,x)\rangle\Big| \\
&\le \Big|\sum_{j=0}^{2^k-1}\int_{j\Delta t}^{(j+1)\Delta t} dt\langle \mu_t^k-\mu_t, v[\mu_{j\Delta t}^k].\nabla\varphi(t,x)\rangle\Big| + \Big|\sum_{j=0}^{2^k-1}\int_{j\Delta t}^{(j+1)\Delta t} dt\langle \mu_t^k, (v[\mu_{j\Delta t}^k]-v[\mu_t^k]).\nabla\varphi(t,x)\rangle\Big| \\
&+ \Big|\sum_{j=0}^{2^k-1}\int_{j\Delta t}^{(j+1)\Delta t} dt\langle \mu_t^k, (v[\mu_t]-v[\mu_t^k]).\nabla\varphi(t,x)\rangle\Big| \\
&\le \|\nabla\varphi\|_\infty \left(M\|\mu_t-\mu_t^k\| + LM(P+|\mu_0|)2^{-(k+1)} + (P+|\mu_0|)L\|\mu_t-\mu_t^k\|\right) \underset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

This proves (22).

Step 2. Any weak solution to (2) **is Lipschitz in time.** In this step, we prove that any weak solution to the transport equation (2) is Lipschitz with respect to time, since it satisfies

$$\|\mu_{t+\tau}-\mu_t\|^{a,b} \le L_1\tau, \qquad t\ge 0,\ \tau\ge 0, \tag{23}$$

with $L_1 = P + bM(P+|\mu_0|)$. To do so, we consider a solution μ_t to (2). We define the vector field $w(t,x) := v[\mu_t](x)$ and the signed measure $b_t = h[\mu_t]$. The vector field w is uniformly Lipschitz and uniformly bounded with respect to x, since v is so. The field w is also measurable in time, since by definition, μ_t is continuous in time. Then, μ_t is the unique solution of

$$\partial_t\mu_t(x) + \mathrm{div}\,.(w(t,x)\mu_t(x)) = b_t(x), \qquad \mu_{|t=0}(x) = \mu_0(x). \tag{24}$$

Uniqueness of the solution of the linear equation (24) is a direct consequence of Lemma 7. Moreover, the scheme presented in Section 4.2 can be rewritten for the vector field w in which dependence with respect to time is added and dependence with respect to the measure is dropped. Thus, the unique solution μ_t to (24) can be obtained as the limit of this scheme. We have for $k\ge 0$ the following estimate

$$\|\mu_{t+\tau}-\mu_t\|^{a,b} \le \|\mu_t-\mu_t^k\|^{a,b} + \|\mu_t^k-\mu_{t+\tau}^k\|^{a,b} + \|\mu_{t+\tau}^k-\mu_{t+\tau}\|^{a,b},$$

where μ_t^k is given by the scheme. The first and third terms can be rendered as small as desired for $k\ge k_0$ large enough, independent on t,τ. For $\ell := \min\{i\in\{1,\ldots,2^k\},\ t\le \frac{i}{2^k}\}$, $j := \min\{i\in\{1,\ldots,2^k\},\ ,t+\tau\le\frac{i}{2^k}\}$ with the notations of the scheme, it holds

$$\begin{aligned}
\|\mu_{t+\tau}^k-\mu_t^k\|^{a,b} = \|m_j^k - m_\ell^k\|^{a,b} = \|\sum_{i=\ell}^{j-1}(m_{i+1}^k-m_i^k)\|^{a,b} &= \|\sum_{i=\ell}^{j-1}(\phi_{\Delta t}^{v[m_i^k]}\# m_i^k + \Delta t h[m_i^k] - m_i^k)\|^{a,b} \\
&\le \sum_{i=\ell}^{j-1}\|\phi_{\Delta t}^{v[m_i^k]}\# m_i^k - m_i^k\|^{a,b} + \Delta t\|\sum_{i=\ell}^{j-1} h[m_i^k]\|^{a,b}.
\end{aligned}$$

Using Lemma 8 and (16), it holds

$$\sum_{i=\ell}^{j-1} \|\phi_{\Delta t}^{v[m_i^k]} \# m_i^k - m_i^k\|^{a,b} \leq \frac{j-\ell}{2^k} bM(P+|\mu_0|) \leq bM(P+|\mu_0|)\tau + \frac{bM(P+|\mu_0|)}{2^k}. \quad (25)$$

Using (H-3), we have

$$\Delta t \| \sum_{i=\ell}^{j-1} h[m_i^k]\|^{a,b} \leq \frac{j-\ell}{2^k} P \leq P\tau + \frac{P}{2^k}, \quad (26)$$

Merging (25)-(26) and letting $k \to \infty$, we recover (23).

Step 3. Any weak solution to (2) **satisfies** the operator splitting estimate:

$$\|\mu_{t+\tau} - (\phi_\tau^{v[\mu_t]} \# \mu_t + \tau h[\mu_t])\|^{a,b} \leq K_1 \tau^2, \quad (27)$$

for some $K_1 > 0$. Indeed, let us consider a solution μ_t to (2). As in the previous step, μ_t is the unique solution to (24), and thus it can be obtained as the limit of the sequence provided by the scheme. With the notations used in Step 2 and using Lemma 8

$$\begin{aligned}\|\mu_{t+\tau} - (\phi_\tau^{v[\mu_t]} \# \mu_t + \tau h[\mu_t])\|^{a,b} &\leq \|\mu_{t+\tau} - \mu_{t+\tau}^k\|^{a,b} + \|\mu_{t+\tau}^k - (\phi_\tau^{v[\mu_t^k]} \# \mu_t^k + \tau h[\mu_t^k])\|^{a,b} \\ &+ \tau \|h[\mu_t^k] - h[\mu_t]\|^{a,b} + \|\phi_\tau^{v[\mu_t]} \# \mu_t - \phi_\tau^{v[\mu_t^k]} \# \mu_t^k\|^{a,b}.\end{aligned}$$

The first, third and fourth terms can be rendered as small as needed for k sufficiently large, independently on τ. We focus then on the second term. Assume for simplicity that $t = \ell \Delta t$ and $t + \tau = (\ell + n)\Delta t$, we have

$$\|\mu_{t+\tau}^k - (\phi_\tau^{v[\mu_t]} \# \mu_t^k + \tau h[\mu_t])\|^{a,b} = \|m_{\ell+n}^k - (\phi_{n\Delta t}^{v[m_\ell^k]} \# m_\ell^k + n\Delta t\ h[m_\ell^k])\|^{a,b}.$$

For $n = 2$, we have

$$\begin{aligned}&\|m_{\ell+2}^k - (\phi_{2\Delta t}^{v[m_\ell^k]} \# m_\ell^k + 2\Delta t\ h[m_\ell^k])\|^{a,b} = \|\phi_{\Delta t}^{v[m_{\ell+1}^k]} \# m_{\ell+1}^k + \Delta t\ h[m_{\ell+1}^k] - \phi_{\Delta t}^{v[m_\ell^k]} \# \phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k - 2\Delta t\ h[m_\ell^k]\|^{a,b} \\ &= \|\phi_{\Delta t}^{v[m_{\ell+1}^k]} \# \left(\phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k + \Delta t\ h[m_\ell^k]\right) + \Delta t\ h[m_{\ell+1}^k] - \phi_{\Delta t}^{v[m_\ell^k]} \# \phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k - 2\Delta t\ h[m_\ell^k]\|^{a,b} \\ &= \|\phi_{\Delta t}^{v[m_{\ell+1}^k]} \# \phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k + \Delta t\ \phi_{\Delta t}^{v[m_{\ell+1}^k]} \# h[m_\ell^k] + \Delta t\ h[m_{\ell+1}^k] - \phi_{\Delta t}^{v[m_\ell^k]} \# \phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k - 2\Delta t\ h[m_\ell^k]\|^{a,b} \\ &\leq \|\phi_{\Delta t}^{v[m_{\ell+1}^k]} \# \phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k - \phi_{\Delta t}^{v[m_\ell^k]} \# \phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k\|^{a,b} + \Delta t \|\phi_{\Delta t}^{v[m_{\ell+1}^k]} \# h[m_\ell^k] + h[m_{\ell+1}^k] - 2h[m_\ell^k]\|^{a,b}\end{aligned}$$

Using Step 3, we have $\|m_{\ell+n}^k - m_\ell^k\| \leq L_1 n \Delta t$. Then, using Lemma 8

$$\|m_{\ell+2}^k - (\phi_{2\Delta t}^{v[m_\ell^k]} \# m_\ell^k + 2\Delta t\ h[m_\ell^k])\|^{a,b} \leq C\Delta t^2$$

By induction on $i = 1 \ldots n$, it then holds

$$\|m_{\ell+n}^k - (\phi_{n\Delta t}^{v[m_\ell^k]} \# m_\ell^k + n\Delta t\ h[m_\ell^k])\|^{a,b} \leq C(n\Delta t)^2,$$

and (27) follows.

Step 4. Uniqueness of the solution to (2) **and continuous dependence.** Assume that μ_t and ν_t are two solutions to (2) with initial condition μ_0, ν_0, respectively. Define $\varepsilon(t) := \|\mu_t - \nu_t\|^{a,b}$. We denote

$$R_\mu(t,\tau)=\mu_{t+\tau}-(\phi_\tau^{v[\mu_t]}\#\mu_t+\tau h[\mu_t]),\quad R_\nu(t,\tau)=\nu_{t+\tau}-(\phi_\tau^{v[\nu_t]}\#\nu_t+\tau h[\nu_t]).$$

Using Lemma 8 and Step 3, and $e^{L\tau}\le 1+2L\tau$ for $0\le L\tau\le\ln(2)$, we have that $\varepsilon(t)$ is Lipschitz and it satisfies

$$\begin{aligned}
\varepsilon(t+\tau) &= \|\mu_{t+\tau}-\nu_{t+\tau}\|^{a,b}=\|\phi_\tau^{v[\mu_t]}\#\mu_t+\tau h[\mu_t]+R_\mu(t,\tau)-\phi_\tau^{v[\nu_t]}\#\nu_t-\tau h[\nu_t]-R_\nu(t,\tau)\|^{a,b}\\
&\le \|\phi_\tau^{v[\mu_t]}\#\mu_t-\phi_\tau^{v[\nu_t]}\#\nu_t\|^{a,b}+\tau\|h[\mu_t]-h[\nu_t]\|^{a,b}+\|R_\mu(t,\tau)\|^{a,b}+\|R_\nu(t,\tau)\|^{a,b}\\
&\le e^{L\tau}\|\mu_t-\nu_t\|^{a,b}+b(P+|\mu_0|)\frac{e^{L\tau}-1}{L}\|v[\mu_t]-v[\nu_t]\|_{\mathcal{C}^0}+\tau Q\|\mu_t-\nu_t\|^{a,b}+2K_1\tau^2\\
&\le \left(e^{L\tau}+b(P+\min\{|\mu_0|,|\nu_0|\})2\tau K+\tau Q\right)\|\mu_t-\nu_t\|^{a,b}+2K_1\tau^2\\
&\le (1+\tau(2L+2bK(P+\min\{|\mu_0|,|\nu_0|\})+Q))\,\|\mu_t-\nu_t\|^{a,b}+2K_1\tau^2,
\end{aligned}$$

which is

$$\frac{\varepsilon(t+\tau)-\varepsilon(t)}{\tau}\le M\varepsilon(t)+2K_1\tau,\qquad t>0,\ \tau\le\frac{\ln(2)}{L},\quad M=2L+2K(P+\min\{|\mu_0|,|\nu_0|\})+Q. \tag{28}$$

Letting τ go to zero, we deduce $\varepsilon'(t)\le M\varepsilon(t)$ almost everywhere. Then, $\varepsilon(t)\le\varepsilon(0)\exp(Mt)$, that is continuous dependence with respect to the initial data.

Moreover, if $\mu_0=\nu_0$, then $\varepsilon(0)=0$, thus $\varepsilon(t)=0$ for all t. Since $\|.\|^{a,b}$ is a norm, this implies $\mu_t=\nu_t$ for all t, that is uniqueness of the solution.

References

[1] Luigi Ambrosio, Edoardo Mainini, and Sylvia Serfaty. Gradient flow of the Chapman-Rubinstein-Schatzman model for signed vortices. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(2):217–246, 2011.

[2] Giovanni A. Bonaschi, José A. Carrillo, Marco Di Francesco, and Mark A. Peletier. Equivalence of gradient flows and entropy solutions for singular nonlocal interaction equations in 1D. *ESAIM Control Optim. Calc. Var.*, 21(2):414–441, 2015.

[3] Alberto Bressan and Benedetto Piccoli. *Introduction to the mathematical theory of control*, volume 2 of *AIMS Series on Applied Mathematics*. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2007.

[4] Lénaïc Chizat, Gabriel Peyré, Bernhard Schmitzer, and François-Xavier Vialard. An interpolating distance between optimal transport and fisher–rao metrics. *Foundations of Computational Mathematics*, Oct 2016.

[5] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[6] Leonid G. Hanin. An extension of the Kantorovich norm. In *Monge Ampère equation: applications to geometry and optimization (Deerfield Beach, FL, 1997)*, volume 226 of *Contemp. Math.*, pages 113–130. Amer. Math. Soc., Providence, RI, 1999.

[7] François James and Nicolas Vauchelet. Equivalence between duality and gradient flow solutions for one-dimensional aggregation equations. *Discrete Contin. Dyn. Syst.*, 36(3):1355–1382, 2016.

[8] Stanislav Kondratyev, Léonard Monsaingeon, and Dmitry Vorotnikov. A new optimal transport distance on the space of finite Radon measures. *Adv. Differential Equations*, 21(11-12):1117–1164, 2016.

[9] Matthias Liero, Alexander Mielke, and Giuseppe Savaré. Optimal transport in competition with reaction: the Hellinger-Kantorovich distance and geodesic curves. *SIAM J. Math. Anal.*, 48(4):2869–2911, 2016.

[10] Benedetto Piccoli and Francesco Rossi. Transport equation with nonlocal velocity in Wasserstein spaces: convergence of numerical schemes. *Acta Appl. Math.*, 124:73–105, 2013.

[11] Benedetto Piccoli and Francesco Rossi. Generalized Wasserstein distance and its application to transport equations with source. *Arch. Ration. Mech. Anal.*, 211(1):335–358, 2014.

[12] Benedetto Piccoli and Francesco Rossi. On properties of the generalized Wasserstein distance. *Arch. Ration. Mech. Anal.*, 222(3):1339–1365, 2016.

[13] Cédric Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[14] Cédric Villani. *Optimal transport*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.

[15] A. Zygmund. *Trigonometric series. Vol. I, II.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition, 2002. With a foreword by Robert A. Fefferman.