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The Helmholtz legacy in color metrics: Schrödinger's color theory

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Abstract

This study is a continuation of the authors' previous work entitled "Helmholtz and the geometry of color space: gestation and development of Helmholtz's line element" (Peruzzi and Roberti in Arch Hist Exact Sci. https://doi.org/10.1007/s00407-023-00304-2, 2023), which provides an account of the first metrically significant model of color space proposed by the German polymath Hermann von Helmholtz in 1891–1892. Helmholtz's Riemannian line element for three-dimensional color space laid the foundation for all subsequent studies in the field of color metrics, although it was largely forgotten for almost three decades from the time of its first publication. The rediscovery of Helmholtz's masterful work was due to one of the founders of quantum mechanics, Erwin Schrödinger. He established his color metric in three extended papers submitted in 1920 to the *Annalen der Physik*. Two memoirs were devoted to the so-called *lower color metric,* which laid the basis for the development of his *higher color metric*, exposed in the last paper. Schrödinger's approach to the geometry of color space has been taken as a starting point for future elaborations of color metrics and allows a close examination of the current assumptions about the analysis of color-matching data. This paper presents an overall picture of Schrödinger's works on color. His color theory developed a tradition first inaugurated by Newton and Young, and which acquired strong scientific ground with Grassmann's, Maxwell's, and Helmholtz's contributions in the 1850s. Special focus will be given to Schrödinger's account of color metric, which responded directly to Helmholtz's hypothesis of a Riemannian line element for color space.

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1 Introduction

The scientific value of Erwin Schrödinger's contributions cannot be overstated. His fame resulted in particular from his contribution to quantum mechanics, for which he was awarded the Nobel Prize in 1933 jointly with Paul Dirac. In addition to this, Schrödinger explored a wide variety of research topics, and, in particular, from 1920 to 1926, he wrote extensively on color. This paper investigates Schrödinger's breakthroughs in the field of color theory, with a special focus on his implementation of Helmholtz's line element for color space. In Sect. 2, Schrödinger's Viennese background will be discussed. During the years spent at the University of Vienna, he became aware of the latest advances in colorimetry thanks to the studies of Exner and Kohlrausch. While Exner and Kohlrausch limited their research interests to the domain of lower color metrics, Schrödinger went further extending the mathematical treatment of color to the field of advanced color metric. This led him to the rediscovery of Helmholtz's contributions of 1891 and 1892, in which he found crucial insights for the development of his own color metric. Schrödinger's seminal works of 1920 will be exposed in paragraph 3. In Sect. 4, the influence of both Helmholtz's and Schrödinger's works on the subsequent developments of color metrics will be analyzed. The main improvements related to this new branch of color science will be outlined, starting with the experimental results obtained by MacAdam in 1942. In the last paragraph, Schrödinger's other papers on color—both popular and scientific publications—written between 1923 and 1926 will be discussed.

2 Schrödinger's Viennese background

In the winter semester 1906–1907, Erwin Schrödinger enrolled in the University of Vienna. During this time, he came under the influence of Friedrich Hasenhörl, professor of theoretical physics, and Franz Serafin Exner, professor of experimental physics. Both Hasenhörl and Exner can be considered as Schrödinger's "scientific fathers" since they were able to leave a strong impact on his future scientific career. Franz Exner, in particular, initiated Schrödinger into the study of color science.

When Schrödinger matriculated at the University of Vienna, Exner was the director of the "Zweites Physikalisches Institut" ("Second Physics Institute"), the new name of the "Physikalisch-chemisches Institut" ("Physico-Chemical Institute") since 1905. Exner devoted his career to a wide range of physical research topics, in which he obtained remarkable results, to the extent that he was recognized as one of the leaders of the Viennese physics community. These included electrochemistry, atmospheric electricity, meteorology, radioactivity, crystal physics, spectroscopy, and color theory. From 1902 to nearly until his death, Exner investigated the consequences of the Young–Maxwell–Helmholtz color theory: his main aim was to further establish the experimental foundation of the trichromatic theory, defending it against attacks from scholars supporting Hering's theory of color. At that time, indeed, one of the most heated controversies in the field of color was the Helmholtz–Hering dispute. Until the 1870s, the Young–Maxwell–Helmholtz trichromatic theory imposed itself as the only fulfilling theory of color vision and attracted important supporters among the scientific community. Between 1872 and 1874, the German physiologist Ewald Hering provided an alternative theory in his "Zur Lehre vom Lichtsinne" ("Theory of the Light Sense"), published as a whole in 1878 (Hering 1878). This work developed an experimental critique of Helmholtz's psychophysical theory of simultaneous contrast and contained the formulation of Hering's theory of color vision, in which black–white, blue–yellow, and green–red were interpreted as antagonistic psychophysical processes. Hering's theory was developed and strongly defended by his school against criticism from the supporters of Young–Maxwell–Helmholtz's theory for the subsequent half-century.[1]

Besides the topic just mentioned, Exner's work on color includes studies on the Purkinje effect, i.e., a shift of the peak sensitivity of the eye toward the violet/blue end of the spectrum in twilight, the fundamental colors and their sensitivity curves, and the function of rods and cones. He also provided a quantitative determination of brightness for different colors, which was a much-debated topic at the time.[2] Exner served as a mentor with strong enthusiasm, sharing his research interests with his pupils so that his research group in Vienna started to be named after him, "Exner Kreis", Exner Circle.[3] Schrödinger was also powerfully influenced by his professor. After obtaining his doctorate in 1910, he became an assistant to Exner in experimental physics. From 1911 to 1914, the year of his promotion as Privatdozent, Schrödinger contributed to various topics of both experimental and theoretical physics, such as, dielectric properties of matter, atmospheric electricity, and X-ray diffraction in lattices, whereas from 1918 to 1920, his most important research was in the field of color theory, in the footsteps of Exner.

Schrödinger shared his interest in color theory with his colleague and closest friend Karl Wilhelm Friedrich Kohlrausch. Kohlrausch, three years older than him, was assistant to Exner from 1908 to 1920, and in those years, he inevitably got acquainted with both the established laws of color mixing and the most debated topics in the field of color perception. From 1917 to 1920, indeed, he held a lectureship on subtractive color theory (pigment mixing) in Vienna at the Kunstgewerbeschule, the School of Applied Arts. In particular, he investigated the artist's color pigments in terms of the three variables, hue, shade, and tint, first introduced by the mathematician Hermann Günther Grassmannin 1853 (see Sect. 3). In 1920, Kohlrausch published three contributions, which included, *inter alia*, a demonstration of the additivity law for brightness, i.e.,

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[1] For more details on the controversy, see the contribution of Turner, "In the Eye's Mind, Vision and the Helmholtz-Hering Controversy" (Turner 1994).

[2] A collection of four of Exner's contributions to color theory can be found in the volume "Vorlesungen über die Physikalischen Grundlagen der Naturwissenschaften" ("Lectures on the Foundations of the Natural Sciences"), first published in 1919 and extended in a second edition in 1922: Chapter 82. "Gegensatz von Aussen- und Innenwelt. Korrelation derselben in Bezug auf das Licht. Farbentheorie. Grassmanns Regeln. Newtons Prinzip und das Farbendreieck" ("Contrast between the outer and inner world. Its correlation with respect to the light. Color theory. Grassmann's rules. Newton's principle and the color triangle"); Chapter 83. "Sättigung der Farben. Begriff des Weiss. Young-Helmholtzschen Theorie" ("Saturation of Colors. The concept of white. The Young-Helmholtz theory"); Chapter 84. "Bestimmung der drei Grundempfindungen. Festlegung von Farben. Die Funktion der Stäbchen und Zapfen im Auge" ("Determination of the three fundamental sensations. The Definition of colors. The function of rods and cones in the eye"); Chapter 85 "Herings und Goethes Theorie der Farben" ("Hering's and Goethe's color theory"). See Exner (1922, pp. 627–656).

[3] On Franz Serafin Exner and his circle in Vienna, see the contribution of Karlik and Schmid, "Franz Serafin Exner und sein Kreis—Ein Beitrag zur Geschichte der Physik in Österreich" (Karlik and Schmidt 1982).

he proved that the quantity of brightness could be expressed as a linear combination of the fundamental colors (Kohlrausch 1920b). In the third contribution entitled "Bemerkungen zur Oswaldschen Theorie" ("Observations on the Ostwald's theory"), Kohlraush expressed strong criticism toward Ostwald's theory of color (Kohlrausch 1920c), which supported Hering's opponent-color theory. The German chemist Wilhelm Ostwald was a keen amateur painter. He used his knowledge of chemistry to study pigments and, inspired by the works of his contemporaries, to develop his own color classification system. This was extensively exposed in the monumental work "Farbenlehre" published in 1918 (Ostwald 1918). Using the Maxwellian color spinning top method (Clerk Maxwell 1855), Ostwald was able to detect hundreds of color tones that he arranged on a disk, sorting them according to the perceived equal differences and compensating each tone with the one diametrically opposed. In his color atlas, however, he introduced only twenty-four full colors, *Vollfarben*, by selecting the ones most easily available at that time, and he used filters to measure their intensity. These were arranged on a circle, which formed the boundary between an upper and a lower cone so that his final model assumed the form of a double cone. According to him, this form was naturally based on the choice of three parameters: full color, white and black. By recognizing *blackness* and *whiteness* as the attributes that contribute to human color perception, Ostwald openly manifested his support for Hering's color theory.

In 1920, Schrödinger provided a preliminary study on color theory in his paper "Theorie der Pigmente grösster Leuchtkraft" ("A Theory of Pigments of Maximum Luminous Efficiency"), published in the *Annalen der Physik* (Schrödinger 1920a). In this first contribution, he studied the maximum intensity of light that may be manifested by a pigment for a particular point of the color diagram and for an arbitrary illumination condition. Schrödinger introduced the concept of "ideal colors", i.e., colors for which the spectral reflectance has a value of either 0 or 1. Although his model was purely theoretical—in practice, indeed, the reflectance function does not change abruptly from 0 to 1—some actual pigments approximate this behavior quite closely. Schrödinger called these pigments "optimal colors" because, when mixed, the resulting color will have the maximum brightness obtainable for that particular pigment. He formalized the theory of subtractive color mixing by providing a two-dimensional manifold for optimal pigment with fixed intensity (he chose the intensity of sunlight). Although Schrödinger departed from Ostwald's color theory, his results furnished a theoretical understanding of some empirical rules obtained by Ostwald a few years earlier.

Schrödinger's first memoir testified to his profound knowledge of the subject under investigation, he cited, among others, the works of Newton, Grassmann, Helmholtz, Helmholtz's pupils, and Ostwald's experimental results. This memoir was an important contribution to color theory and the concept of "ideal colors" has since been used in the development of color models and for the analysis of color pigments.

3 Schrödinger's "Grundlinien"

« [...] a computational error by Helmholtz, the discovery of which further weakens the connection to experience, is probably the reason why his intellectually interesting idea of a Riemann geometry of color has not been appreciated, as far as I know. The relevant sections have even been deleted from the latest edition of the Helmholtz Handbook of Physiological Optics» (Schrödinger 1920b, p. 401).

After having displayed the concepts of ideal color pigments, Schrödinger turned to a more ambitious project, i.e., the definition of a metric in color space, inspired by the works published by Helmholtz in 1891 and 1892.

His studies on color metrics were published in 1920 in the *Annalen der Physik* and appeared as three separate contributions to the organic work entitled "Grundlinien einer Theorie der Farbenmetrik im Tagessehen" ("Outline of a Theory of Color Measurement for Daylight vision"). "Grundlinien" Part I, Part II, and Part III represent Schrödinger's main achievement in the field of color theory and made him be honored by the prestigious Haitinger Prize of the Austrian Academy of Sciences "for fundamentals of color theory", which he received in the same year of their publication. In the introductory paragraph of Part I, Schrödinger coined the dichotomy *lower-higher color metric* and clarified the distinction between these two theoretical domains: "It emerges through experience that all efforts to quantify color fall into two fundamental categories. Their distinction lies in the criterion used for the adjustment of two adjoining color fields on an experimental device. Either a criterion of complete identity is applied exclusively (indistinguishability), or other criteria are applied (such as just-noticeable difference, maximum similarity, or eventual maximum contrast). Measurement results of the first kind do form a unified, internally consistent system whose simple axiomatic rules have been formalized by Grassmann, which rules have been experimentally verified by König. I believe that this system of rules—frequently known as the laws of light mixture—may reasonably be called lower color metric, to distinguish it from higher color metric, whose laws are much more complicated and much less well understood" (Schrödinger 1920b, p. 399, translated from German to English by the authors[4]). According to Schrödinger's definition, the pioneering color-matching experiments carried out by James Clerk Maxwell[5] in the 1850s felt in the domain of the lower color metric, while Helmholtz's work on the line element implied the definition of *just noticeable color differences* (i.e., the smallest perceptual

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[4] In this section, all the subsequent quotations from Schrödinger's original work have been taken from the English translation provided by Niall (2017). However, in order to allow a more accurate reproduction of the original, we have translated the terms "niedere Farbenmetrik" and "höheren Farbenmetrik" as "lower color metric" and "higher color metric", instead of "basic colorimetry" and "advanced colorimetry".

[5] In his masterful papers "Experiments on Colour, as perceived by the Eye, with Remarks on Colour-Blindness" and "On the theory of compound colors, and the relations of the colors of the spectrum" published in 1855 and 1860, respectively, James Clerk Maxwell laid down the foundation for the construction of chromaticity diagrams, derived from his color matching experiments. Maxwell's color diagram is a 2-dimensional representation of color space, which assumes the shape of an equilateral triangle in the Euclidean plane with the chosen primary colors as vertices, V (vermillion), U (ultramarine), and EG (emerald green), and white (W) as the orthocenter. The two variables defining the color diagram are *hue* (wavelength) and *saturation*, whereas the *brightness* is maintained constant. For this reason, all saturated colors (wavelengths) are located on the perimeter of the triangle. Any other color inside the triangle can be expressed in terms of the "amount" of light of the three primary colors involved in the mixture and identified in terms of its distance from white. Once the position of a given color inside the triangle is detected graphically, it is possible to identify both its wavelength and the wavelength of its complementary color by extending the straight line passing through it and W (white) to a side of the triangle (Clerk Maxwell 1855, 1860).

differences) between two color stimuli and, therefore, belonged to the domain of the higher color metric. Schrödinger maintained a strong distinction between lower and higher color metrics. Indeed, as we will show in this section, the first two memoirs of the "Grundlinien" are entirely devoted to the analysis of the lower color metric, while in Part III Schrödinger entered the domain of higher color metric, proposing his implemented version of Helmholtz's line element for perceptual color space.

In the "Grundlinien", Part I, Schrödinger introduced the nomenclature related to light and color, which he coherently adopted for the definition of the general laws of light mixture. In particular, he introduced the definition of the term "color", in the wavelength range from $\lambda = 400$ nm to $\lambda = 800$ nm, as follows: "For quantitative ends, let us designate the set of identically-appearing lights as a color" (Niall 2017, p. 20, original in Schrödinger 1920b, p. 420). Starting point for the whole treatment were Grassmann's laws of colorimetry, which, therefore, require further elucidation. In his seminal paper—his sole contribution to color theory—published in 1853 and entitled "Zur Theorie der Farbenmischung" ("On the theory of compound colors"), Grassmann proposed the formal equipment of affine geometry (or projective geometry) to the problem of characterizing color space by introducing his four laws of additive color mixture, later known as "Grassmann's laws". The first law concerns the dimension of color space; Grassmann considered the nature of human color vision as dependent on three variables, i.e., *tint*, *shade*, and *hue*. Grassmann's first rule was experimentally verified by Maxwell in 1855 (Clerk Maxwell 1855). Although Maxwell considered red, blue, and green as the three independent color variables, he showed that these two methods (his own and Grassmann's) of considering color could be easily deduced one from the other via coordinate transformations. The second law can be expressed by the following statement: "In the second place, we assume that if one of two mingling lights be continuously altered (while the other remains unchanged), the impression of the mixed light also is continuously changed" (Grassmann 1854, p. 256). Grassmann's second law, also known as "continuity law", reflected the property of color singled out by Riemann in his 1854 Habilitation dissertation, and later by Helmholtz in 1868: colors form a continuously extended manifoldness. The first two laws imply the existence of a complementary color for each color of the spectrum. However, Helmholtz's experimental findings of 1855 contradicted this proposition (see Peruzzi and Roberti 2023). By analyzing pairs of complementary spectral lights, indeed, Helmholtz was not able to find a simple complementary color for green, which had to be mixed with purple, i.e., a mixture of violet and red, to furnish white (von Helmholtz 1855). Grassmann's third law states that there are lights with different spectral power distributions that appear identical (the color match is said to be metameric), while Grassmann's fourth law concerns the additivity of brightness: the total intensity of an additive mixture of colors is the sum of the intensities of the mixed colors.

Schrödinger performed a synthesis of the most important achievements in the field of colorimetry in a set of axioms, which incorporated, extended, and refined Grassmann's laws. In Sect. 2, Schrödinger discussed the composition of lights and colors ("Addition von Lichtern und Farben"), which he interpreted as the superposition of colored light vectors, following Grassmann's intuition. In this paragraph, Schrödinger elucidated his first axiom by rephrasing Grassmann's third law as follows: "the unconditional and complete equivalence of lights which we have defined as the same in color, is retained when they are mixed" (Niall 2017, p. 21, original in Schrödinger 1920b, p. 411). Section 3 is devoted to the operation of scalar multiplication ("Multiplicaktion mit einer Zahl), which allowed Schrödinger to introduce his second axiom. He refined Grassmann's second law, replacing it with the following proposition: "There are no two lights that become equal and later unequal in a periodic manner as the intensities of these lights increase proportionally" (Niall 2017, p. 27, original in Schrödinger 1920b, p. 418). The third proposition is introduced in Sect. 4 "Die Dimensionstatsache", which deals with the dimensionality of color space. Schrödinger extended Grassmann's first law to the case of dichromatic and monochromatic color vision. While in trichromatic color space, there are linearly independent triplets of colors, and in dichromatic color space, there are linearly independent pairs of colors, i.e., color space for normal trichromats is three-dimensional and color space for dichromats is two-dimensional. In monochromatic color space, on the other hand, any two colors are linearly dependent, i.e., color space for subjects affected by achromatopsia is one-dimensional. It is worth pointing out, that Grassmann's fourth law of colorimetry was not considered in this first memoir. A deeper analysis was reserved to this law, known as the additivity law of brightness, in Part III of the "Grundlinien", devoted to advanced color metric.

As shown above, Schrödinger's account responded directly to Grassmann's formalization of color theory as a vector space: "it turns out that basic colorimetry provides a complete model for the affine geometry of a pencil of vectors, taken to represent the color manifold" (Niall 2017, p.13, original in Schrödinger 1920b, p. 420). He envisaged the color domain to have a bijective correspondence with a pencil of vectors in space, namely vectors that radiate from a point. Schrödinger recognized, however, that the laws of color mixture deviate from the axioms of affine geometry on one point: this concerns the application of the operation of subtraction and the lack of physical significance related to negative colors, as well as the operation of multiplication by negative numbers. For any vector representing a color, the vector of the opposite sign cannot be used, since there are no lights that produce darkness when superimposed.

After this work of axiomatization, the next aim was that of representing geometrically the color manifold. "Grundlinien", Part II, is entirely focused on this topic. To construct the tridimensional color space, Schrödinger chose three objective colors as calibration colors, between which there is no linear relation. These calibration colors are assigned to three arbitrarily chosen noncoplanar (i.e., linearly independent) vectors of the pencil. Once indicated as **A**, **B**, and **C** the calibration colors, i.e., the basis vectors, any other color **F** can be expressed by the following relation:

$$\mathbf{F} = x_1\boldsymbol{A} + x_2\boldsymbol{B} + x_3\boldsymbol{C}, \tag{1}$$

where x_i indicates the color coordinates of **F**, namely its trichromatic coefficients, relating to **A**, **B**, and **C**. Any color, therefore, can be specified by its components in the direction of each basis vector. By representing each color vector in this way, the color manifold of vectors can be easily built up: it assumes the form of a regular convex cone, which Schrödinger called the "spectral cone", embedded in a real vector space and whose apex coincides with the origin (see Fig. 1). Any given spectral distribution could be represented by a curve on the surface of the cone. The heads of

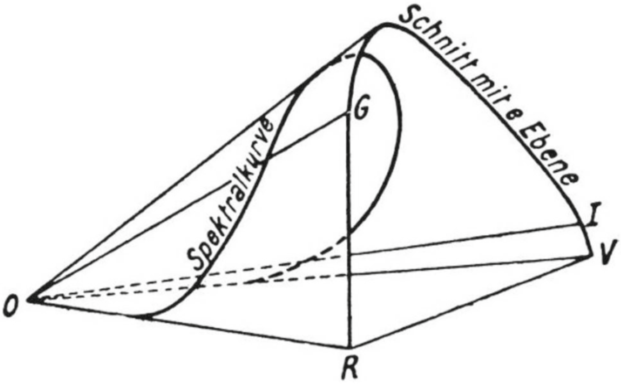

Fig. 1 Schrödinger's cone of perceived colors. A *Spektralkurve*, spectral curve, and its intersection with an arbitrarily inclined plane (*Schnitt mit einer Ebene*) are shown. The planar surfaces ROG, ROV, and VOI are also indicated (Schrödinger 1920c, p. 428)

the vectors indeed trace a specific curve—whose shape depends on the spectral energy distribution—around the surface of the spectral cone. This curve represents the spectral curve and its position in the color manifold is determined by the brightness. The higher the brightness, the farther the spectral curve will be from the origin (Schrödinger 1920c, p. 428). It is worth noticing that the intersection between the color cone and an arbitrary inclined plane (Fig. 1) gives rise to a plane section whose shape closely approximates that of Helmholtz's two-dimensional color diagram (Fig. 2).

The most peculiar property of Schrödinger's cone is that it is "not so tightly curved": the envelope of the cone has, indeed, three plane sections, ROG (red, origin, green), ROV (red, origin, violet), and VOI (violet, origin, indigo), as shown in Fig. 1., which represent the spectral intervals detected by one of Helmholtz's pupils, Arthur König: from λ = 655 nm to λ = 630 nm; from λ = 630 nm to λ = 475 nm and from λ = 475 nm to λ = 430 nm, respectively (König and Dieterici 1892). According to the experimental data, the preferred basis vectors would be those which lie on these three

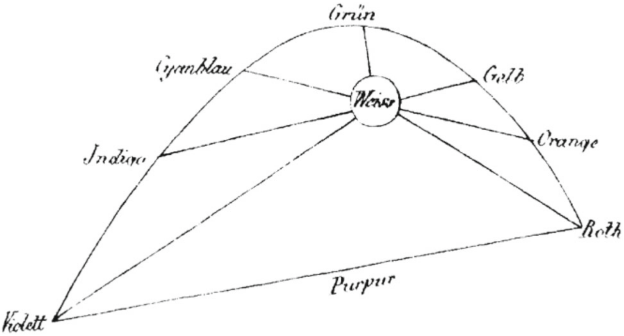

Fig. 2 Helmholtz's color diagram at constant brightness, which was experimentally derived in 1855 (von Helmholtz 1855, Tafel I, Fig. 5)

plane sections, in the directions of the three spectral primaries detected by König: far-red end of the spectrum, R; far violet, V; and green, G.[6] The two sections ROG and VOI form the borders of the envelope, and between ROG and VOI the spectral cone has a convex curvature. Moreover, in the interior of the envelope, there is a vector direction that corresponds to white. The sheaf of planes passing through the direction of white separates the totality of colors into groups: "The groups which lie on the same half-plane of the sheaf of planes are related to the same spectral color by 'dilution with white'" (Niall 2017, p. 37, original in Schrödinger 1920c, p. 430). Pairs of colors that lie on the same plane enclosing white are defined as complementary colors.

Once he displayed the construction of his color cone and elucidated its geometrical properties, Schrödinger provided in the final section of the paper the derivation of color coordinates for an arbitrary function of wavelength and explored the relation between two-dimensional and three-dimensional color space, i.e., between the color space for dichromats and trichromats.

Schrödinger's discussion of lower color metrics contained in these first papers laid the foundation for his work on higher color metrics, "Grundlinien", Part III. As discussed above, in the domain of lower color metrics, an affine space sufficed to describe the laws of color mixture. On the other hand, in the domain of higher color metrics, a Riemannian space was demonstrated to be necessary to translate color differences into color distances in the three-dimensional and two-dimensional color spaces. Therefore, in higher color metrics distances between colors are defined by a Riemannian metric. This is, as Moore noticed, the same kind of geometry used by Einstein in his general theory of relativity, although the color space is three-dimensional and not four-dimensional as space–time (Moore 2015). It is worthwhile to point out that Schrödinger had previously familiarized himself with Riemann geometry while working on problems of general relativity (see Schrödinger 1918a, b). Certainly, his consolidated knowledge of the Riemannian formalism allowed him to envisage its application to color theory.

In "Grundlinien", Part III, Schrödinger acknowledged, on the one hand, Helmholtz's intellectually interesting idea of a Riemann geometry of color, whereas, on the other hand, he recognized some critical aspects that had escaped Helmholtz's attention in the definition of the line element. Schrödinger argued that the identification of these shortcomings by Helmholtz's pupils might have led to the suppression of the key passages related to higher color metrics in the third edition of the "Handbuch" (Von Helmholtz 1909–1911), condemning his pioneering work to oblivion.

Helmholtz's original line element formulation (Helmholtz 1892) is reported below:

$$\mathrm{d}E^2 = \left(\frac{\mathrm{d}x}{x+a}\right)^2 + \left(\frac{\mathrm{d}y}{y+b}\right)^2 + \left(\frac{\mathrm{d}z}{z+c}\right)^2, \tag{2}$$

where x, y, and z refer to the *physiologische Urfarben* (physiological primary colors) and a, b, and c represent the self-light constants. The physiological primary colors were expressed as linear homogeneous equations of R (red), G (green), and$V$ (violet),

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[6] See Sect. 9 of Schrödinger's paper entitled "Irreelle Eichfarben. Die Young-Helmholtzsche Theorie" ("Virtual-Valued Calibration Colors. The Young-Helmholtz Theory").

i.e., the *Elementarfarben*, elementary colors, found by Helmholtz's assistants Arthur König and Conrad Dieterici (König and Dieterici 1886).

By analyzing Helmholtz's mathematical treatment, Schrödinger first noticed that the new primaries chosen by Helmholtz as a linear combination of König' and Dieterici's fundamentals had to be computed ad hoc to make the line element for just distinguishable color pairs approximately constant. Moreover, Schrödinger recognized that the brightness function ($V(\lambda)$ in modern terms) obtained from his model was in contradiction to experience. Two pronounced maxima resulted from the spectral sensitivity of the primary sensations: "this brightness function is quite absurd, since e.g., it produces a hideous dromedary-like curve with two pronounced maxima as a brightness distribution for the spectrum of sunlight" (Niall 2017, p. 71, original in Schrödinger 1920d, p. 493).

But for a still more general reason this brightness function, and therefore Helmholtz's line element, was unacceptable to Schrödinger. Experiments carried out by William de Wiveleslie Abney,[7] Edward Robert Festing (see Abney and Festing 1886), and subsequently by Exner and Kohlrausch (Exner 1918, 1920; Kohlrausch 1920b), proved that brightness, at least to a high degree of approximation, was an additive property of colors, confirming Grassmann's fourth law of colorimetry (Grassmann 1854). Geometrically speaking, this means that the surfaces of constant brightness should be portions of planes in the 3-dimensional Euclidean space in which the color space is embedded. The surfaces of constant brightness derived from Helmholtz's metric were not planar, i.e., brightness was not an additivity property in Helmholtz's model.

Between the nineteenth and the twentieth centuries the modern concept of luminance saw birth, thanks to more accurate and elaborate studies in the field of photometry. It has to be underlined, however, that Helmholtz had already conducted refined experiments on brightness comparison using a modified version of the color wheel used by Maxwell for mixing colors. Schrödinger was aware of this study and argued that Helmholtz had misinterpreted his results, which only apparently led to a non-additive brightness function. Since that time, Helmholtz had always doubted the additivity of brightness.

Another significant aspect related to the nature of color space was highlighted by Schrödinger. Helmholtz's model was Riemannian in the sense that any distance between two neighboring points was given by a differential expression of coordinates, but the resulting color space turned out to be isometric to Euclidean space. Indeed, by neglecting the self-light constants a, b, and c in Eq. (2), Helmholtz's color space is Euclidean on a logarithmic basis (lnx, lny, lnz):

$$\mathrm{d}E^2 = (\mathrm{d}lnx)^2 + (\mathrm{d}lny)^2 + (\mathrm{d}lnz)^2. \tag{3}$$

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[7] William Abney was a photographic chemist. He succeeded Arthur König as the leading international pioneer in precision colorimetry. Abney formulated the now-called "Abney's law", or brightness additivity law, named after him. Abney's Law of additivity states that the total brightness of a mixture of wavelengths is equal to the sum of the brightness of its monochromatic components, something already postulated by Grassmann.

If the mathematical treatment is reduced to a two-dimensional color vision system, i.e., color space for subjects suffering from dichromatism, the orthogonality requirement between lines of constant brightness (*isophotes*) and lines of constant hue (*isohues*) leads to a contradiction to Abney's additivity rule. One of the consequences of Helmholtz's line element was, thus, that *isophotes* and *isohues* were not mutually orthogonal (see Vos 2006).

The disagreement with Abney's law of additivity led Schrödinger to undertake a re-examination of the entire metrical theory of color perception, and his proposal of a new metric compatible with both Abney's and Weber-Fechner's results took shape.

Schrödinger used the additivity law of brightness as a guide for his mathematical treatment:

$$h = a_1 x_1 + a_2 x_2 + a_3 x_3, \tag{4}$$

where h stands for "Helligkeit" (brightness), a_i indicate constants, derived experimentally by Exner ($a_1 = 43, 33$; $a_2 = 32, 76$; $a_3 = 1$) and corresponding approximately to the brightness of the three fundamental processes x_1, x_2, x_3, as experimentally determined by König and Dieterici in 1892 (König and Dieterici 1892). To agree with the existing experimental data, therefore, the line element must produce a family of planes of constant brightness:

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = \text{const.} \tag{5}$$

Planes of this family should be everywhere perpendicular to those of constant hue. By performing a coordinate transformation, Schrödinger obtained the Euclidean line element, which turned out to satisfy the orthogonality requirement but, on the other hand, did not satisfy the Weber–Fechner law. The incompatibility with the law of psychophysics was first pointed out by Wolfgang Pauli, whose key contribution was openly acknowledged by Schrödinger in the original memoir. Thus, by selecting a factor for reciprocal luminance of the following form:

$$\frac{1}{a_1 x_1 + a_2 x_2 + a_3 x_3}. \tag{6}$$

Schrödinger obtained the final expression for his line element, which satisfied both Abney and Weber-Fechner laws:

$$\mathrm{d}s^2 = \frac{1}{a_1 x_1 + a_2 x_2 + a_3 x_3} \left(\frac{a_1 \mathrm{d}x_1^2}{x_1} + \frac{a_2 \mathrm{d}x_2^2}{x_2} + \frac{a_3 \mathrm{d}x_3^2}{x_3} \right). \tag{7}$$

In Schrödinger's metric, furthermore, color space has a curvature different from zero, i.e., it is not isometric to Euclidean space.

A highly significant application of the line element reported in this paper was the determination of the lines of constant hue (wavelength) inside the triangular color diagram. Indeed, using his line element, Schrödinger was able to depict in the color diagram containing all colors with equal brightness the geodesic lines connecting the

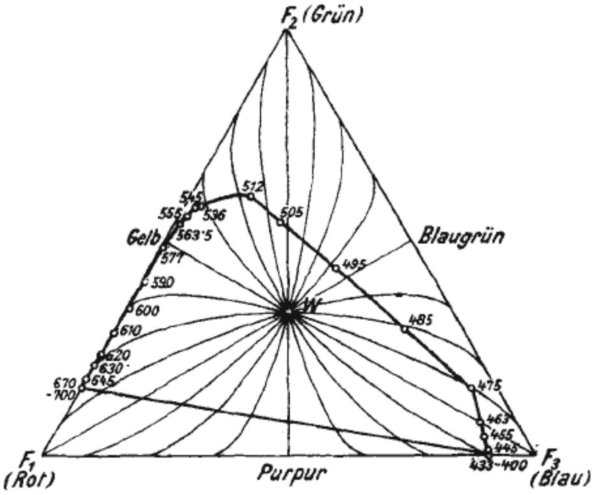

Fig. 3 Theoretical constant-hue lines in Maxwell's color triangle (Rot–Blau–Grün, Red–Blue–Green) showing the spectrum locus determined by König and Dieterici (thicker black line with the indication of wavelength in nm) (Schrödinger 1920a, b, c, d, p. 515)

white point with the spectral color points on the perimetry. These were in general curved lines, as shown in Fig. 3, except for the shortest lines to the three fundamental colors, which were straight. Colors of constant hue were those that lay on the geodesics between the fully saturated color and white. Such geodesics were loci of *constant hue*, or *isochromes*. The *isochromes* diverged from yellow, magenta, and cyan and converged toward the primaries red, green, and blue. The derivation of geodesic lines of constant hue provided an important clue for the non-Euclidean interpretation of the geometry of color space and constituted a pivotal contribution to the advancement of color science.

4 The reception of Schrödinger's line element

Since Schrödinger's theoretical work, other mathematicians and physicists have developed line element theory further, so that higher color metric started constituting a proper branch of color theory, as Turner pointed out (Turner 1994), almost inaccessible to other specialists working in the field.

Whereas Helmholtz's line element had a firm structural basis in the Weber–Fechner law, Schrödinger's line element was obtained by a purely mathematical construction. It gave, however, a satisfactory description of the experimental color discrimination data available at the time.

The first significant implementation of line element theory was provided by the American physicist and color scientist David MacAdam, who experimentally derived his ellipses for color stimuli in 1942 (MacAdam 1942). He was well aware of Schrödinger's work, as testified by the publication of an abridged translation of

Schrödinger's papers of 1920 in his "Sources of Color Science" (MacAdam 1970). MacAdam had the opportunity to perform precise measurements at the Kodak firm in Rochester, New York. These aimed to create a connection between the perception of the chromatic difference between two colors and a geometric chromatic distance in the CIE 1931 color space (the first color space created by the International Commission on Illumination) of constant luminance. He carried out color-matching experiments and determined the spread of color matches for a set of 25 test colors. MacAdam used a bipartite comparison field of view, exploiting a method first developed by Maxwell in the 1850s; one color served as a test, and the other color was subjected to small variations in chromaticity by the observer until it matched the test color. He found out that colors, within a certain small region in the diagram, which assumed the shape of an ellipse, appeared identical to a human eye. Figure 4 shows MacAdam's discrimination ellipses, containing all the colors that are indistinguishable to the human eye from the color located at the center of the ellipses, and the perimeter of each ellipse represents the *just noticeable difference in chromaticity*; therefore, the smaller the perimeter, the better the discrimination. MacAdam's ellipses rapidly became a key set of data, used as a reference for line elements and color difference formulae.

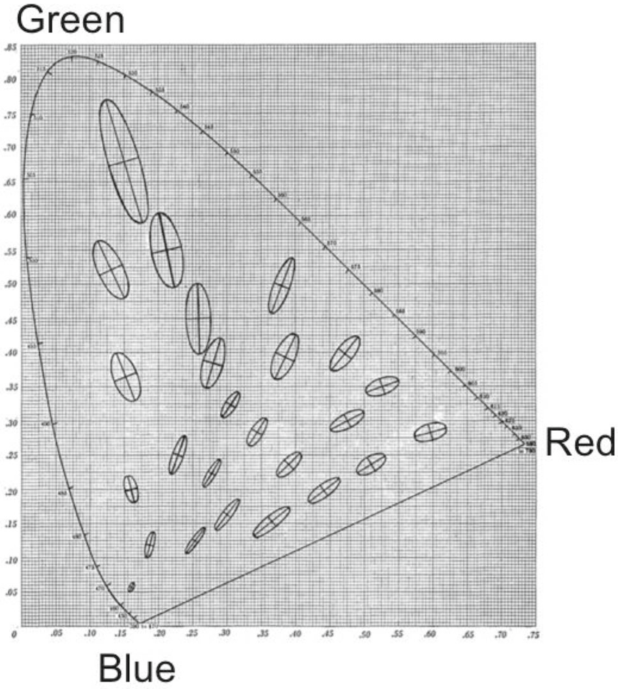

Fig. 4 MacAdam's ellipses, represented ten times their true size on the CIE (x, y) chromaticity diagram. Each ellipse on the diagram contains all the colors that are indistinguishable to the human eye from the color at the center of the ellipse. The perimeter of the ellipses represents the just noticeable difference in chromaticity. Therefore, the smaller the perimeter, the better the discrimination of chromaticity. The reason for the large variability in size, shape, and orientation of the ellipses is expressed in Helmholtz's papers of 1891 and 1892 (MacAdam 1942, p. 271)

Another metric for color space was provided by the English physicist and mathematician Walter S. Stiles in 1946. Stiles, relying on his skills as a mathematical physicist, proposed a version of the line element based on his experimental investigations. He found some weaknesses in both Helmholtz's and Schrödinger's treatments. In the intermediate luminance range, both scientists assumed that the Weber fractions were equal for all three response mechanisms. By conducting his sensitivity experiments, however, Stiles recognized that different Weber fractions had to be applied to the three cone systems rather than the common fraction used by Helmholtz and Schrödinger. Stiles suggested, thus, a modification of the line element introducing the different Weber fractions for each fundamental process:

$$\mathrm{d}E^2 = \left(\frac{\mathrm{d}x}{0.78x}\right)^2 + \left(\frac{\mathrm{d}y}{1.0y}\right)^2 + \left(\frac{\mathrm{d}z}{4.46z}\right)^2, \tag{8}$$

where x, y, and z represent the three fundamental processes and 0.78, 1.0, and 4.46 indicate the quantities, obtained experimentally, that are proportional to the limiting Fechner fractions for red, green, and blue, respectively. By rewriting Eq. (8) as a logarithmic expression, the color space can be converted to a uniform Euclidean space, in line with Helmholtz's proposal, but with different scales along the three axes.

An exhaustive collection of the main achievements in the field of color metrics up to 1971 can be found in the Proceeding of the Memorial Symposium on Color Metrics entitled to Helmholtz on the occasion of the 150th anniversary of his birth. Among the 64 participants, the names of MacAdam and Stiles stood out. On that occasion, a whole range of line elements and color difference formulas passed the review. One of the most advanced attempts to define a line element in color space was that proposed by J. J. Vos and P. L. Walraven of the Institute for Perception TNO of Soesterberg. It consists of a theoretical model, the so-called "zone-theory" compromise, which contemplates the validity of both Young-Maxwell-Helmholtz's theory and Hering's opponent process theory, as we will see in the following section. Between 1955 and 1965, important breakthroughs in the study of vision came from various direct techniques, which established the existence of three photopigments in the retinal cones. Those were demonstrated to provide neural inputs to cells located higher in the visual tract, which in turn displayed chromatic opponency in a manner very similar to that predicted by Hering.[8]

Up to the present time, several research studies have been performed aimed at fulfilling what Helmholtz and Schrödinger first began, i.e., an ever more accurate mathematical description of color by the application of differential geometry to the perceptual color space, in step with the modern achievements in the fields of both geometry, psychophysics, and visual neuroscience.

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[8] In particular, the 1950s-1960s contributions of the American psychologists Dorothea Jameson, Leo M. Hurvich, Russell L. De Valois and Arthur E. Jones made the scientific community begin reconsidering Hering's opponent process theory (see, e.g., Jameson and Hurvich 1955, De Valois and Jones 1961).

5 Schrödinger's other papers on color

After his masterly work on color metrics, Schrödinger continued publishing occasional papers on color theory between 1923 and 1926. In this paragraph, we present some of these largely unexplored contributions, which testify to the breadth and depth of Schrödinger's research interests.

On February 3, 1923, Schrödinger's short popular article entitled "Ton und Farbe" ("Sound and Color") appeared in the Swiss daily newspaper *Neue Zürcher Zeitung* (Fig. 5). Schrödinger offered a comparison between the two sensations of sound and color. He stated that sounds and colors were both periodic phenomena, but their esthetic effects through the respective senses of hearing and sight were quite different in modality. According to him, music depends upon the *temporal* sequence of sounds, whereas painting depends on the *spatial* arrangement of colors: "wahrgenommene Farben bilden räumliche Formen; die Klänge bilden nur zeitliche Formen, im besonderen Melodien" ("perceived colors form spatial forms; sounds form only temporal forms, especially melodies"). This *Fuilletton* reveals Schrödinger's inclination for philosophical investigations and also testifies to his interest in both music and art. Concerning the latter, he wondered about the qualities that made a painting satisfactory. In particular, he did not believe that an abstract arrangement of colors could constitute an art piece because it did not furnish an adequate depiction of reality.

In the following year, Schrödinger submitted to *Die Naturwissenschaften* a scientific paper entitled "Über den Ursprung der Empfindlichkeitskurven des Auges" ("On the Origin of the Eye's Sensitivity Curves"). This memoir contains key arguments in favor of a phylogenetic evolution of the human visual system and testifies to Schrödinger's awareness of the latest achievements in the field of biology.[9] He considered the spectral brightness sensitivity function of the ordinary human eye in terms of the availability of natural sources of illumination over time. Schrödinger speculated that the peak brightness sensitivities of cones and rods arose in phylogenetic development, with the cone system being relatively recent in appearance compared to the rod system. In that period, the comparative physiology of vision offered support to these conjectures, based on differences in illumination for terrestrial and aquatic animals. For the development of this study, he embraced the duplicity theory formulated in 1894 by the German physiological psychologist Johannes von Kries. This theory concerns the comparisons and interactions between the cone and rod systems in the visual pathways, with the assumption that twilight vision is a function of the retinal rods and daylight vision is a function of the retinal cones. In 1866, the German anatomist Max Schultze supposed the rods to be more primitive organs than the cones, i.e., achromatic twilight vision to be more primitive than chromatic daylight vision (Schultze 1866). Most of the subsequent intervening discussions of rod vision

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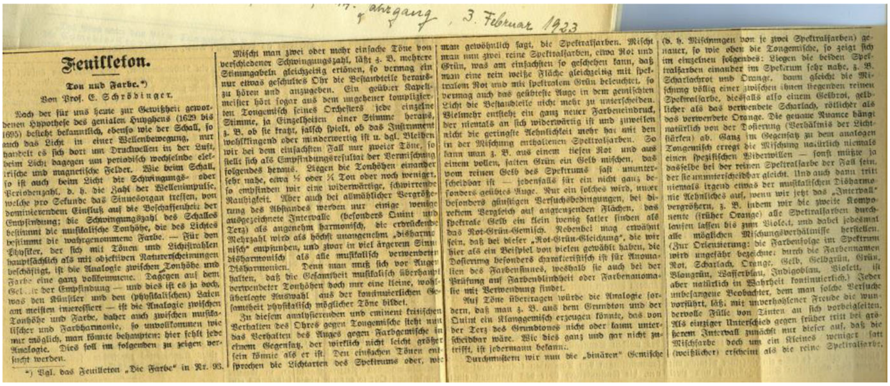
[9] Schrödinger extended his interest toward biology while he was a Senior Professor at the School of Theoretical Physics of the Dublin Institute for Advanced Studies. In 1943, Schrödinger gave a series of lectures at Trinity College in which he explored areas of biology. These lectures were focused on two main topics: the nature of heredity and the thermodynamics of living systems. In particular, Schrödinger investigated to what extent quantum physics could be used to explain the genetic structure of living beings. These lectures were collected in a volume entitled "What is life? The Physical Aspect of the Living Cell" (Schrödinger 1944), published in 1944. "What is life?" was widely read and became one of the most influential science books of the twentieth century.



7. Jahrgang, 3. Februar 1923

Feuilleton.

Ton und Farbe.*)

Von Prof. E. Schrödinger.

Nach der für uns heute zur Gewißheit gewordenen Hypotheſe des genialen Huyghens (1629 bis 1695) beſteht bekanntlich, ebenſo wie der Schall, ſo auch das Licht in einer Wellenbewegung, nur handelt es ſich dort um Druckwellen in der Luft, beim Licht dagegen um periodiſch wechſelnde elektriſche und magnetiſche Felder. Wie beim Schall, ſo iſt auch beim Licht die Schwingungs- oder Periodenzahl, d. h. die Zahl der Wellenimpulſe, welche pro Sekunde das Sinnesorgan treffen, von dominierendem Einfluß auf die Beſchaffenheit der Empfindung; die Schwingungszahl des Schalles beſtimmt die muſikaliſche Tonhöhe, die des Lichtes beſtimmt die wahrgenommene Farbe. — Für den Phyſiker, der ſich mit Tönen und Lichtſtrahlen hauptſächlich als mit objektiven Naturerſcheinungen beſchäftigt, iſt die Analogie zwiſchen Tonhöhe und Farbe eine ganz vollkommene. Dagegen auf dem Gebiete der Empfindung — und dies iſt es ja doch, was den Künſtler und den (phyſikaliſchen) Laien am meiſten intereſſiert — iſt die Analogie zwiſchen Tonhöhe und Farbe, daher auch zwiſchen muſikaliſcher und Farbharmonie, ſo unvollkommen wie nur möglich, man könnte behaupten: hier fehlt jede Analogie. Dies ſoll im folgenden zu zeigen verſucht werden.

Miſcht man zwei oder mehr einfache Töne von verſchiedener Schwingungszahl, läßt z. B. mehrere Stimmgabeln gleichzeitig ertönen, ſo vermag ein nur etwas geſchultes Ohr die Beſtandteile herauszuhören und anzugeben. Ein geübter Kapellmeiſter hört ſogar aus dem ungeheuer komplizierten Tongemiſch ſeines Orcheſters jede einzelne Stimme, ja Einzelheiten einer Stimme heraus, z. B. ob ſie kratzt, falſch ſpielt, ob das Inſtrument wohlklingend oder minderwertig iſt u. dgl. Bleiben wir bei dem einfachſten Fall nur zweier Töne, ſo ſtellt ſich als Empfindungsreſultat der Vermiſchung folgendes heraus. Liegen die Tonhöhen einander ſehr nahe, etwa ½ oder ¼ Ton oder noch weniger, ſo empfinden wir eine widerwärtige, ſchwirrende Rauhigkeit. Aber auch bei allmählicher Vergrößerung des Abſtandes werden nur einige wenige ausgezeichnete Intervalle (beſonders Quint und Terz) als angenehm harmoniſch, die erdrückende Mehrzahl wird als höchſt unangenehm „disharmoniſch" empfunden, und zwar in viel ärgerem Sinn disharmoniſch als alle muſikaliſch verwendeten Disharmonien. Denn man muß ſich vor Augen halten, daß die Geſamtheit muſikaliſch überhaupt verwendeter Tonhöhen doch nur eine kleine, wohlüberlegte Auswahl aus der kontinuierlichen Geſamtheit phyſikaliſch möglicher Töne bildet.

Zu dieſem analyſierenden und eminent kritiſchen Verhalten des Ohres gegen Tongemiſche ſteht nun das Verhalten des Auges gegen Farbgemiſche in einem Gegenſatz, der wirklich nicht leicht größer ſein könnte als er iſt. Den einfachen Tönen entſprechen die Lichtarten des Spektrums oder, wie man gewöhnlich ſagt, die Spektralfarben. Miſcht man nun zwei reine Spektralfarben, etwa Rot und Grün, was am einfachſten ſo geſchehen kann, daß man eine rein weiße Fläche gleichzeitig mit ſpektralem Rot und mit ſpektralem Grün beleuchtet, ſo vermag auch das geübteſte Auge in dem gemiſchten Licht die Beſtandteile nicht mehr zu unterſcheiden. Vielmehr entſteht ein ganz neuer Farbeneindruck, der niemals an ſich widerwärtig iſt und zuweilen nicht die geringſte Aehnlichkeit mehr hat mit den in der Miſchung enthaltenen Spektralfarben. So kann man z. B. aus einem tiefen Rot und aus einem vollen, ſatten Grün ein Gelb miſchen, das vom reinen Gelb des Spektrums faſt ununterſcheidbar iſt — jedenfalls für ein nicht ganz beſonders geübtes Auge. Nur ein ſolches wird, unter beſonders günſtigen Verſuchsbedingungen, bei direktem Vergleich auf angrenzenden Flächen, das ſpektrale Gelb ein klein wenig ſatter finden als das Rot-Grün-Gemiſch. Nebenbei mag erwähnt ſein, daß bei dieſer „Rot-Grün-Gleichung", die wir hier als ein Beiſpiel von vielen gewählt haben, die Doſierung beſonders charakteriſtiſch iſt für Anomalien des Farbenſinnes, weshalb ſie auch bei der Prüfung auf Farbenblindheit oder Farbenanomalie mit Verwendung findet.

Auf Töne übertragen würde die Analogie fordern, daß man z. B. aus dem Grundton und der Quint ein Klanggemiſch erzeugen könnte, das von der Terz des Grundtones nicht oder kaum unterſcheidbar wäre. Wie dies ganz und gar nicht zutrifft, iſt jedermann bekannt.

Durchmuſtern wir nun die „binären" Gemiſche (d. h. Miſchungen von je zwei Spektralfarben) genauer, ſo wie oben die Tongemiſche, ſo zeigt ſich im einzelnen folgendes: Liegen die beiden Spektralfarben einander im Spektrum ſehr nahe, z. B. Scharlachrot und Orange, dann gleicht die Miſchung völlig einer zwiſchen ihnen liegenden reinen Spektralfarbe, diesfalls alſo einem Gelbrot, gelblicher als das verwendete Scharlach, rötlicher als das verwendete Orange. Die genaue Nuance hängt natürlich von der Doſierung (Verhältnis der Lichtſtärken) ab. Ganz im Gegenſatz zu dem analogen Tongemiſch erregt die Miſchung natürlich niemals einen ſpezifiſchen Widerwillen — ſonſt müßte ja dasſelbe bei der reinen Spektralfarbe der Fall ſein, der ſie ununterſcheidbar gleicht. Und auch dann tritt niemals irgend etwas der muſikaliſchen Disharmonie Aehnliches auf, wenn wir jetzt das „Intervall" vergrößern, z. B. indem wir die zweite Komponente (früher Orange) alle Spektralfarben durchlaufen laſſen bis zum Violett, und dabei jedesmal alle möglichen Miſchungsverhältniſſe herſtellen. (Zur Orientierung: die Farbenfolge im Spektrum wird ungefähr bezeichnet durch die Farbennamen Rot, Scharlach, Orange, Gelb, Gelbgrün, Grün, Blaugrün, Waſſerblau, Indigoblau, Violett, iſt aber natürlich in Wahrheit kontinuierlich.) Jeder unbefangene Beobachter, dem man ſolche Verſuche vorführt, läßt mit unverhohlener Freude die wundervolle Fülle von Tinten an ſich vorbeigleiten. Als einziger Unterſchied gegen früher tritt bei größerem Intervall zunächſt nur dieſer auf, daß die Miſchfarbe doch um ein Kleines weniger ſatt (weißlicher) erſcheint als die reine Spektralfarbe.

*) Vgl. das Feuilleton „Die Farbe" in Nr. 93.

Fig. 5 The original text of Schrödinger's "Ton und Farbe". Courtesy of the Österreichische Zentralbibliothek für Physik, University of Wien

and cone vision implied an evolutionary distinction between them. In 1892, e.g., the American psychologist, logician, and mathematician Christine Ladd-Franklin made this evolutionary distinction focal.[10]

Whereas the sensitivity curves for the perception of the fundamental colors seemed to be easily explained according to the Young–Maxwell–Helmholtz theory by referring to the spectral distribution of the solar light, Schrödinger raised questions about the sensitivity curve of the rod mechanism. This appeared to shift toward the blue end of the spectrum. He investigated the possible reasons for this displacement of the rod curve toward shorter wavelengths. One plausible explanation could reside in the fact that the rod apparatus originated under the influence of another type of illumination light with a different energy distribution curve from that of the cone apparatus. However, as Schrödinger pointed out, this light had more or less the same energy distribution as daylight. The most probable explanation laid, according to him, in the green–blue color exhibited by water in thicker layers: "for an aquatic animal that lives under the surface at some depth, the composition of sunlight must be transformed in just about the way we need for our explanation. Such an animal would have particular need of robust adaptation to different levels of brightness, too, if it explored changes in depth under the water's surface. Under this hypothesis, the rod system would be an older system of vision which emerged during the age of aquatic life" (Niall 2017, p. 137, original in Schrödinger 1924, p. 927). According to this explanation, color-blind vision could be regarded as a pure return to an atavistic state. Although many of the details of rod and cone evolution are still uncertain, recent studies have demonstrated that our distant aquatic ancestor, living in the ocean more than 500 million years ago, had already evolved two classes of cones, which were sensitive to short and long wavelengths. In line with Schrödinger's intuition, the ancestral cones provided

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[10] In 1892, Ladd-Franklin presented her first findings in London at the Second International Congress of Psychology with her contribution "A new theory of Light Sensation" (Ladd-Franklin 1892). It is worth highlighting that Ladd-Franklin traveled to Europe between 1891 and 1892. In particular, while in Germany, she devoted herself to research in color vision, working, *inter alia*, at Helmholtz's laboratory in Berlin under the supervision of Arthur König.

dichromatic color vision in daylight. From these two classes of cones, new photoreceptors had developed, resembling modern-day photoreceptors, which endowed the organism with a major survival advantage: the ancestral rods, which enabled vision at very low light levels, at night and in the deep ocean, and three new classes of cones, which provided trichromatic vision in daylight lighting levels.[11]

In 1925, another contribution, entitled "Über Farbenmessung" ("On Color Measurement"), was published in the *Physikalische Zeitschrift*. Here Schrödinger expressed strong criticism toward Ostwald's theory of color, in line with Kohlrausch's arguments on the subject (Kohlrausch 1920c).

In the same year, he presented before the Vienna Academy of Science his last paper on color theory: "Über das Verhältnis der Vierfarben- zur Dreifarbentheorie" ("On the Relation of the Four Color to the Three-Color Theories"). In this work, Schrödinger revealed the formal relation between Young–Maxwell–Helmholtz's theory and Hering's four-color theory by a simple geometric transformation on color space, i.e., by a projective transformation, and proved that both theories are equally capable of representing color mixtures at the level of simple sensations, something that Helmholtz had already tried to demonstrate in the second version of the Handbuch (von Helmholtz 1885–1896, pp. 376–382).[12] Schrödinger also reported the pioneering attempts to combine the two theories in the so-called "zone theory". The zone theory compromise was first postulated in 1881 by the Dutch ophthalmologist Franciscus Cornelis Donders and later by von Kries (von Kries and Adolf 1882). It contemplated the validity of both Young–Maxwell–Helmholtz's theory and Hering's opponent process theory, although applied to different levels of the nervous system. For almost a century, the two theories were considered mutually exclusive for the explanation of human visual perception. As already mentioned, the debate between supporters of Helmholtz's theory, on the one hand, and Hering's theory, on the other hand, constituted one of the most persistent controversies of modern science relating to human color vision. At the end of the nineteenth century, hybrid theories, "zone theories", saw birth to unify the two theories in a common interpretative framework. The name "zone-theory" suggests that the mechanism of trichromatic color mixture is manifest at an "early stage" (first zone) of the visual system, while the mechanism of the opponent process color mixture comes into play at a "higher level" (second zone) of the visual system. Although Schrödinger sympathized with von Kries' proposal, his intent while writing this contribution was not to provide support for the "zone theory", but rather to demonstrate the mathematical equivalence between the two theories of color vision: "This variant theory provided by von Kries is highly credible, in my opinion. However, what I want to show in the following is completely independent of the profound conception of the physiological substrate of the visual process. It is simply a matter of establishing that, from a formal point of view, the relationship between the two theories—the three- and the four-color theory—is to be understood in a very simple manner, namely as a mere transformation of the variables") (Schrödinger 1925a, b, p. 472, translated from German to English by the authors).

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[11] On the evolution of cone and rod photoreceptors, see the contributions of Lamb, "Why rods and cones?" (Lamb 2016), and Morshedian and Fain, "The evolution of rod photoreceptors" (Morshedian et al. 2017).

[12] The first English translation, enriched with commentaries, of the paragraph in question was provided by Joseph David Everett in 1893 (Everett 1893).

Schrödinger's last contribution to color was a paragraph entitled "Die Gesichtsempfindungen" ("The visual sensations"), written for the eleventh edition of *Müller-Pouillets Lehrbuch der Physik*[13] and published in 1926 (Schrödinger 1926). In 1921, Schrödinger moved to the University of Breslau (now Wrocław) where he was appointed full professor. Here he met the German physicists Otto Lummer, professor of experimental physics and a former assistant to Helmholtz in Berlin, and Ernst Pringsheim, professor of theoretical physics. In Breslau, Schrödinger could work in an institute, where research on optical phenomena and optical physics played a pivotal role. After the publication of his monumental work of 1920, "Grundlinien einer Theorie der Farbenmetrik im Tagessehen", he rapidly became recognized as one of the world's leading experts on color theory. Indeed, when Otto Lummer, together with his colleagues Arnold Eucken and Erich Waetzmann, was in charge of planning the eleventh edition of the "Lehrbuch", he proposed to Schrödinger the writing of the chapter related to visual sensations. Schrödinger masterfully fulfilled the assignment: in more than a hundred pages, he exposed standard color theory, color sensations in daylight and twilight, and higher color metrics.

6 Conclusion

During an interval of a few years, Erwin Schrödinger succeeded in laying the seeds for future advancements in the fields of colorimetry and color metric, further developing the 19th-century contributions of Maxwell, Grassmann, and Helmholtz. Thanks to the flourishing scientific environment he encountered at the University of Vienna in his first academic years, Schrödinger was able to address open problems in color theory, inspired by the research carried out by his contemporaries, particularly by his mentor Franz Exner and his colleague Karl Kohlrausch. His research led him to rediscover Hermann von Helmholtz's pioneering discussion on the line element, which was deleted from the latest edition of the "Handbuch" falling into oblivion for almost three decades. Although Helmholtz deserves credit for the first determination of Riemannian metrics for color space, we have tried to emphasize in this paper the importance of the work conducted by Schrödinger, who not only brought back to light Helmholtz's key insights on the nature of color space but also sought to carry Helmholtz's legacy forward.

Since Schrödinger's formal development of the color metric, considerable effort has been spent on the derivation of more refined line elements for color space, both from theoretical considerations and experimental data. Nowadays, indeed, the definition of an appropriate metric for color spaces has become increasingly important also for technological applications.

Schrödinger's papers on color show us the continuing importance of his color theory, in that an analysis of his contributions permits a closer examination and a better understanding of the main progress achieved in the field, and, in parallel, sheds light on some critical aspects yet to be clarified today.

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[13] Some translated fragments can be found in MacAdam's work "Sources of Color Science", paragraph "Thresholds of color differences" (MacAdam 1970, pp. 183–193).

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Data availability statement The authors confirm that all data generated or analyzed during this study are included in this article.

Declarations

Conflict of interest This work was supported by the DOR Peruzzi (Dotazione Ordinaria di Ricerca) funds of the University of Padova. The authors have no competing interests to declare that are relevant to the content of this article.

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