Classes of kernels and continuity properties of the double layer potential in Hölder spaces

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Abstract: We prove the validity of regularizing properties of the boundary integral operator corresponding to the double layer potential associated to the fundamental solution of a *nonhomogeneous* second order elliptic differential operator with constant coefficients in Hölder spaces by exploiting an estimate on the maximal function of the tangential gradient with respect to the first variable of the kernel of the double layer potential and by exploiting specific imbedding and multiplication properties in certain classes of integral operators and a generalization of a result for integral operators on differentiable manifolds.

Keywords: Double layer potential, second order differential operators with constant coefficients, boundary behavior, Hölder spaces.
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1 Introduction

In this paper, we consider the double layer potential associated to the fundamental solution of a second order differential operator with constant coefficients in Hölder spaces. Unless otherwise specified, we assume throughout the paper that

$$n \in \mathbb{N} \setminus \{0, 1\},$$

where $\mathbb{N}$ denotes the set of natural numbers including 0. Let $\alpha \in [0, 1]$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ and we understand that $C^{m,0} \equiv C^m$. For the definition and properties of the classical Schauder spaces we refer for example to [7, Chap. 2], [8, §2]. We employ the same notation of reference [8] with Dondi that we now introduce.

Let ν_Ω or simply $\nu \equiv (\nu_l)_{l=1,\dots,n}$ denote the external unit normal to $\partial\Omega$. Let N_2 denote the number of multi-indexes $\gamma \in \mathbb{N}^n$ with $|\gamma| \leq 2$. For each

$$\mathbf{a} \equiv (a_\gamma)_{|\gamma|\leq 2} \in \mathbb{C}^{N_2}\,, \tag{1.1}$$

we set

$$a^{(2)} \equiv (a_{lj})_{l,j=1,\dots,n} \qquad a^{(1)} \equiv (a_j)_{j=1,\dots,n} \qquad a \equiv a_0\,.$$

with $a_{lj} \equiv 2^{-1} a_{e_l+e_j}$ for $j \neq l$, $a_{jj} \equiv a_{e_j+e_j}$, and $a_j \equiv a_{e_j}$, where $\{e_j : j = 1, \dots, n\}$ is the canonical basis of $\mathbb{R}^n$. We note that the matrix $a^{(2)}$ is symmetric. Then we assume that $\mathbf{a} \in \mathbb{C}^{N_2}$ satisfies the following ellipticity assumption

$$\inf_{\xi\in\mathbb{R}^n, |\xi|=1} \mathrm{Re}\left\{\sum_{|\gamma|=2} a_\gamma \xi^\gamma\right\} > 0\,, \tag{1.2}$$

and we consider the case in which

$$a_{lj} \in \mathbb{R} \qquad \forall l, j = 1, \dots, n\,. \tag{1.3}$$

Then we introduce the operators

$$\begin{aligned} P[\mathbf{a}, D]u &\equiv \sum_{l,j=1}^n \partial_{x_l}(a_{lj}\partial_{x_j} u) + \sum_{l=1}^n a_l \partial_{x_l} u + au\,, \\ B^*_\Omega v &\equiv \sum_{l,j=1}^n \overline{a}_{jl} \nu_l \partial_{x_j} v - \sum_{l=1}^n \nu_l \overline{a}_l v\,, \end{aligned}$$

for all $u, v \in C^2(\overline{\Omega})$, and a fundamental solution $S_{\mathbf{a}}$ of $P[\mathbf{a}, D]$, and the boundary integral operator corresponding to the double layer potential

$$\begin{aligned} W_\Omega[\mathbf{a}, S_{\mathbf{a}}, \mu](x) &\equiv \int_{\partial\Omega} \mu(y) \overline{B^*_{\Omega,y}}\left(S_{\mathbf{a}}(x-y)\right)\, d\sigma_y \qquad (1.4) \\ &= -\int_{\partial\Omega} \mu(y) \sum_{l,j=1}^n a_{jl}\nu_l(y) \frac{\partial S_{\mathbf{a}}}{\partial x_j}(x-y)\, d\sigma_y \\ &\quad - \int_{\partial\Omega} \mu(y) \sum_{l=1}^n \nu_l(y) a_l S_{\mathbf{a}}(x-y)\, d\sigma_y \qquad \forall x \in \partial\Omega\,, \end{aligned}$$

where the density or moment μ is a function from $\partial\Omega$ to $\mathbb{C}$. Here the subscript y of $\overline{B^*_{\Omega,y}}$ means that we are taking y as variable of the differential operator $\overline{B^*_{\Omega,y}}$. The role of the double layer potential in the solution of boundary value problems for the operator $P[\mathbf{a}, D]$ is well known (cf. *e.g.*, Günter [13], Kupradze, Gegelia, Basheleishvili and Burchuladze [19], Mikhlin [23].)

The analysis of the continuity and compactness properties of $W_\Omega[\mathbf{a}, S_\mathbf{a}, \cdot]$ is a classical topic and several results in the literature show that $W_\Omega[\mathbf{a}, S_\mathbf{a}, \cdot]$ improves the regularity of Hölder continuous functions on $\partial\Omega$. We briefly recall some references (see also [8]).

In case $n = 3$ and Ω is of class $C^{1,\alpha}$ and $S_\mathbf{a}$ is the fundamental solution of the Laplace operator, it has long been known that $W_\Omega[\mathbf{a}, S_\mathbf{a}, \cdot]$ is a linear and compact operator in $C^{1,\alpha}(\partial\Omega)$ and is linear and continuous from $C^0(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ (cf. Schauder [29], [30], Miranda [25].)

In case $n = 3$, $m \geq 2$ and Ω is of class $C^{m,\alpha}$ and if $P[\mathbf{a}, D]$ is the Laplace operator, Günter [13, Appendix, § IV, Thm. 3] has proved that $W[\partial\Omega, \mathbf{a}, S_\mathbf{a}, \cdot]$ is bounded from $C^{m-2,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha'}(\partial\Omega)$ for $\alpha' \in]0, \alpha[$.

In case $n \geq 2$, $\alpha \in]0, 1]$, O. Chkadua [2] has pointed out that one could exploit Kupradze, Gegelia, Basheleishvili and Burchuladze [19, Chap. IV, Sect. 2, Thm 2.9, Chap. IV, Sect. 3, Theorems 3.26 and 3.28] and prove that if Ω is of class $C^{m,\alpha}$, then $W[\partial\Omega, \mathbf{a}, S_\mathbf{a}, \cdot]$ is bounded from $C^{m-1,\alpha'}(\partial\Omega)$ to $C^{m,\alpha'}(\partial\Omega)$ for $\alpha' \in]0, \alpha[$.

In case $n = 3$ and Ω is of class C^2 and if $P[\mathbf{a}, D]$ is the Helmholtz operator, Colton and Kress [4] have developed previous work of Günter [13] and Mikhlin [23] and proved that the operator $W_\Omega[\mathbf{a}, S_\mathbf{a}, \cdot]$ is bounded from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ and that accordingly it is compact in $C^{1,\alpha}(\partial\Omega)$.

In case $n \geq 2$, $\alpha \in]0, 1[$ and Ω is of class C^2 and if $P[\mathbf{a}, D]$ is the Laplace operator, Hsiao and Wendland [15, Remark 1.2.1] deduce that the operator $W[\partial\Omega, \mathbf{a}, S_\mathbf{a}, \cdot]$ is bounded from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ by the work of Mikhlin and Prössdorf [24].

In case $n = 3$, $m \geq 2$ and Ω is of class $C^{m,\alpha}$ and if $P[\mathbf{a}, D]$ is the Helmholtz operator, Kirsch [17] has proved that the operator $W_\Omega[\mathbf{a}, S_\mathbf{a}, \cdot]$ is bounded from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$ and that accordingly it is compact in $C^{m,\alpha}(\partial\Omega)$.

Then Heinemann [14] has developed the ideas of von Wahl in the frame of Schauder spaces and has proved that if Ω is of class C^{m+5} and if $S_\mathbf{a}$ is the fundamental solution of the Laplace operator, then the double layer improves the regularity of one unit on the boundary, *i.e.*, $W_\Omega[\mathbf{a}, S_\mathbf{a}, \cdot]$ is linear and continuous from $C^{m,\alpha}(\partial\Omega)$ to $C^{m+1,\alpha}(\partial\Omega)$.

Mitrea [27] has proved that the double layer of second order equations and systems is compact in $C^{0,\beta}(\partial\Omega)$ for $\beta \in]0, \alpha[$ and bounded in $C^{0,\alpha}(\partial\Omega)$ under the assumption that Ω is of class $C^{1,\alpha}$. Then by exploiting a formula for the tangential derivatives such results have been extended to compactness and boundedness results in $C^{1,\beta}(\partial\Omega)$ and $C^{1,\alpha}(\partial\Omega)$, respectively.

In [8], we have proved that if $m \geq 1$, $\beta \in]0, \alpha]$, $\alpha \in]0,1[$, then $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, \cdot]$ is linear and continuous from $C^{m,\beta}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$ and a related result if we chose $\beta = 0$.

In this paper we plan to consider the case in which Ω is of class $C^{1,1}$ and show that if the maximal function of the tangential gradient with respect to the first variable of the kernel of the double layer potential is bounded, then $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, \cdot]$ is linear and continuous from $C^{0,\beta}(\partial\Omega)$ to $C^{1,\beta}(\partial\Omega)$ for $\beta \in]0,1[$ and is linear and continuous from $C^{0,1}(\partial\Omega)$ to the generalized Schauder space $C^{1,\omega_1}(\partial\Omega)$ of functions with 1-st order tangential derivatives which satisfy a generalized ω_1-Hölder condition with

$$\omega_1(r) \sim r^1 |\ln r| \qquad \text{as } r \to 0\,,$$

see Theorem 5.11. Our proofs are based on Theorem of [21, Thm. 6.6] on integral operators, that we report here in the case in which the domain of integration is a compact differentiable manifold, see Theorem 3.12. Theorem 3.12 requires that we can estimate the maximal function associated to the tangential gradient of the kernel of the double layer potential with respect to its first variable and that the same tangential gradient belongs to a certain class of kernels. Then we prove the membership in the class of kernels by exploiting the imbedding and multiplication properties that we have highlighted and proved in [21] and that we report here in the special case we need, see Section 3. Here we note that the properties of Section 3 actually simplify a proof that would be otherwise long to explain.

2 Notation

Let $M_n(\mathbb{R})$ denote the set of $n \times n$ matrices with real entries. $\delta_{l,j}$ denotes the Kronecker symbol. Namely, $\delta_{l,j} = 1$ if $l = j$, $\delta_{l,j} = 0$ if $l \neq j$, with $l, j \in \mathbb{N}$. $|A|$ denotes the operator norm of a matrix A, A^t denotes the transpose matrix of A. We set

$$\mathbb{B}_n(\xi, r) \equiv \{\eta \in \mathbb{R}^n : \, |\xi - \eta| < r\}\ , \tag{2.1}$$

for all $(\xi, r) \in \mathbb{R}^n \times]0, +\infty[$. If $\mathbb{D}$ is a subset of $\mathbb{R}^n$, then we set

$$B(\mathbb{D}) \equiv \left\{ f \in \mathbb{C}^{\mathbb{D}} : \, f \text{ is bounded} \right\}\ , \quad \|f\|_{B(\mathbb{D})} \equiv \sup_{\mathbb{D}} |f| \qquad \forall f \in B(\mathbb{D})\,.$$

Then $C^0(\mathbb{D})$ denotes the set of continuous functions from $\mathbb{D}$ to $\mathbb{C}$ and we introduce the subspace $C^0_b(\mathbb{D}) \equiv C^0(\mathbb{D}) \cap B(\mathbb{D})$ of $B(\mathbb{D})$. Let ω be a function from $[0,+\infty[$ to itself such that

$$\begin{aligned} & \omega(0)=0, \qquad \omega(r)>0 \qquad \forall r\in]0,+\infty[\,, \\ & \omega \text{ is increasing}, \ \lim_{r\to 0^+}\omega(r)=0\,, \\ & \text{and} \quad \sup_{(a,t)\in[1,+\infty[\times]0,+\infty[} \frac{\omega(at)}{a\omega(t)}<+\infty\,. \end{aligned} \tag{2.2}$$

If f is a function from a subset $\mathbb{D}$ of $\mathbb{R}^n$ to $\mathbb{C}$, then we denote by $|f:\mathbb{D}|_{\omega(\cdot)}$ the $\omega(\cdot)$-Hölder constant of f, which is delivered by the formula

$$|f:\mathbb{D}|_{\omega(\cdot)} \equiv \sup\left\{\frac{|f(x)-f(y)|}{\omega(|x-y|)} : x,y\in\mathbb{D}, x\neq y\right\}.$$

If $|f:\mathbb{D}|_{\omega(\cdot)}<\infty$, we say that f is $\omega(\cdot)$-Hölder continuous. Sometimes, we simply write $|f|_{\omega(\cdot)}$ instead of $|f:\mathbb{D}|_{\omega(\cdot)}$. The subset of $C^0(\mathbb{D})$ whose functions are $\omega(\cdot)$-Hölder continuous is denoted by $C^{0,\omega(\cdot)}(\mathbb{D})$ and $|f:\mathbb{D}|_{\omega(\cdot)}$ is a semi-norm on $C^{0,\omega(\cdot)}(\mathbb{D})$. Then we consider the space $C^{0,\omega(\cdot)}_b(\mathbb{D}) \equiv C^{0,\omega(\cdot)}(\mathbb{D})\cap B(\mathbb{D})$ with the norm

$$\|f\|_{C^{0,\omega(\cdot)}_b(\mathbb{D})} \equiv \sup_{x\in\mathbb{D}}|f(x)| + |f|_{\omega(\cdot)} \qquad \forall f\in C^{0,\omega(\cdot)}_b(\mathbb{D})\,.$$

Remark 2.3 *Let ω be as in (2.2). Let $\mathbb{D}$ be a subset of $\mathbb{R}^n$. Let f be a bounded function from $\mathbb{D}$ to $\mathbb{C}$, $a\in]0,+\infty[$. Then,*

$$\sup_{x,y\in\mathbb{D},\ |x-y|\geq a}\frac{|f(x)-f(y)|}{\omega(|x-y|)} \leq \frac{2}{\omega(a)}\sup_{\mathbb{D}}|f|\,.$$

In the case in which $\omega(\cdot)$ is the function r^α for some fixed $\alpha\in]0,1]$, a so-called Hölder exponent, we simply write $|\cdot:\mathbb{D}|_\alpha$ instead of $|\cdot:\mathbb{D}|_{r^\alpha}$, $C^{0,\alpha}(\mathbb{D})$ instead of $C^{0,r^\alpha}(\mathbb{D})$, $C^{0,\alpha}_b(\mathbb{D})$ instead of $C^{0,r^\alpha}_b(\mathbb{D})$, and we say that f is α-Hölder continuous provided that $|f:\mathbb{D}|_\alpha<\infty$.

3 Special classes of potential type kernels in $\mathbb{R}^n$

In this section we collect some basic properties of the classes of kernel that we need. For the proofs, we refer to [21, §3]. If X and Y are subsets of $\mathbb{R}^n$, then we denote by $\mathbb{D}_{X\times Y}$ the diagonal of $X\times Y$, i.e., we set

$$\mathbb{D}_{X\times Y} \equiv \{(x,y)\in X\times Y : x=y\} \tag{3.1}$$

and if $X = Y$, then we denote by $\mathbb{D}_X$ the diagonal of $X \times X$, i.e., we set

$$\mathbb{D}_X \equiv \mathbb{D}_{X \times X} \,.$$

An off-diagonal function in $X \times Y$ is a function from $(X \times Y) \setminus \mathbb{D}_{X \times Y}$ to $\mathbb{C}$. We now wish to consider a specific class of off-diagonal kernels.

Definition 3.2 *Let X and Y be subsets of $\mathbb{R}^n$. Let $s \in \mathbb{R}$. We denote by $\mathcal{K}_{s,X \times Y}$ (or more simply by $\mathcal{K}_s$), the set of continuous functions K from $(X \times Y) \setminus \mathbb{D}_{X \times Y}$ to $\mathbb{C}$ such that*

$$\|K\|_{\mathcal{K}_{s,X\times Y}} \equiv \sup_{(x,y)\in (X\times Y)\setminus \mathbb{D}_{X\times Y}} |K(x,y)|\,|x-y|^s < +\infty \,.$$

The elements of $\mathcal{K}_{s,X \times Y}$ are said to be kernels of potential type s in $X \times Y$.

We plan to consider 'potential type' kernels as in the following definition (see also paper [8] with Dondi, where such classes have been introduced in a form that generalizes those of Gegelia [10], [19, Chap. IV] and Giraud [12]).

Definition 3.3 *Let X, $Y \subseteq \mathbb{R}^n$. Let s_1, s_2, $s_3 \in \mathbb{R}$. We denote by $\mathcal{K}_{s_1,s_2,s_3}(X \times Y)$ the set of continuous functions K from $(X \times Y) \setminus \mathbb{D}_{X \times Y}$ to $\mathbb{C}$ such that*

$$\begin{aligned}
\|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \equiv \sup&\Big\{|x-y|^{s_1}|K(x,y)| : \, (x,y) \in X \times Y, x \neq y\Big\} \\
+\sup&\Big\{\frac{|x'-y|^{s_2}}{|x'-x''|^{s_3}}|K(x',y) - K(x'',y)| : \\
&x', x'' \in X, x' \neq x'', y \in Y \setminus \mathbb{B}_n(x', 2|x'-x''|)\Big\} < +\infty \,.
\end{aligned}$$

One can easily verify that $(\mathcal{K}_{s_1,s_2,s_3}(X \times Y), \|\cdot\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)})$ is a normed space. By our definition, if s_1, s_2, $s_3 \in \mathbb{R}$, we have

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \subseteq \mathcal{K}_{s_1,X\times Y}$$

and

$$\|K\|_{\mathcal{K}_{s_1,X\times Y}} \leq \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)} \qquad \forall K \in \mathcal{K}_{s_1,s_2,s_3}(X \times Y) \,.$$

We note that if we choose $s_2 = s_1 + s_3$ we have a so-called class of standard kernels. Then we have the following elementary known embedding lemma (cf. *e.g.*, [21, §3]).

Lemma 3.4 *Let X, $Y \subseteq \mathbb{R}^n$. Let s_1, s_2, $s_3 \in \mathbb{R}$. If $a \in]0,+\infty[$, then $\mathcal{K}_{s_1,s_2,s_3}(X \times Y)$ is continuously embedded into $\mathcal{K}_{s_1,s_2-a,s_3-a}(X \times Y)$.*

Next we introduce the following known elementary lemma, which we exploit later and which can be proved by the triangular inequality.

Lemma 3.5

$$\frac{1}{2}|x'-y| \leq |x''-y| \leq 2|x'-y|\,,$$

for all $x', x'' \in \mathbb{R}^n$, $x' \neq x''$, $y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x'-x''|)$.

Next we state the following two product rule statements (cf. [21, §3]).

Theorem 3.6 *Let X, $Y \subseteq \mathbb{R}^n$. Let s_1, s_2, s_3, t_1, t_2, $t_3 \in \mathbb{R}$.*

(i) If $K_1 \in \mathcal{K}_{s_1,s_2,s_3}(X \times Y)$ and $K_2 \in \mathcal{K}_{t_1,t_2,t_3}(X \times Y)$, then the following inequality holds

$$\begin{aligned} |K_1(x',y)K_2(x',y) - K_1(x'',y)K_2(x'',y)| \\ \leq \|K_1\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}\|K_2\|_{\mathcal{K}_{t_1,t_2,t_3}(X\times Y)} \\ \times \left(\frac{|x'-x''|^{s_3}}{|x'-y|^{s_2+t_1}} + \frac{2^{|s_1|}|x'-x''|^{t_3}}{|x'-y|^{t_2+s_1}} \right) \end{aligned}$$

for all $x', x'' \in X$, $x' \neq x''$, $y \in Y \setminus \mathbb{B}_n(x', 2|x'-x''|)$.

(ii) The pointwise product is bilinear and continuous from

$$\mathcal{K}_{s_1,s_1+s_3,s_3}(X\times Y)\times\mathcal{K}_{t_1,t_1+s_3,s_3}(X\times Y) \quad \text{to} \quad \mathcal{K}_{s_1+t_1,s_1+s_3+t_1,s_3}(X\times Y)\,.$$

Proposition 3.7 *Let X, $Y \subseteq \mathbb{R}^n$. Let s_1, s_2, $s_3 \in \mathbb{R}$, $\alpha \in]0,1]$. Then the following statements hold.*

(i) If $K \in \mathcal{K}_{s_1,s_2,s_3}(X \times Y)$ and $f \in C^{0,\alpha}_b(X)$, then

$$|K(x,y)f(x)|\,|x-y|^{s_1} \leq \|K\|_{\mathcal{K}_{s_1,X\times Y}} \sup_X |f| \qquad \forall (x,y) \in X\times Y \setminus \mathbb{D}_{X\times Y}\,.$$

and

$$\begin{aligned} |K(x',y)f(x') - K(x'',y)f(x'')| \\ \leq \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(X\times Y)}\|f\|_{C^{0,\alpha}_b(X)} \left\{ \frac{|x'-x''|^{s_3}}{|x'-y|^{s_2}} + 2^{|s_1|}\frac{|x'-x''|^{\alpha}}{|x'-y|^{s_1}} \right\} \end{aligned}$$

for all $x', x'' \in X$, $x' \neq x''$, $y \in Y \setminus \mathbb{B}_n(x', 2|x'-x''|)$.

(ii) *If $s_2 \geq s_1$ and X and Y are both bounded, then the map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C_b^{0,s_3}(X) \quad to \quad \mathcal{K}_{s_1,s_2,s_3}(X \times Y)$$

that takes the pair (K, f) to the kernel $K(x, y)f(x)$ of the variable $(x, y) \in (X \times Y) \setminus \mathbb{D}_{X\times Y}$ is bilinear and continuous.

(iii) *The map from*

$$\mathcal{K}_{s_1,s_2,s_3}(X \times Y) \times C_b^{0}(Y) \quad to \quad \mathcal{K}_{s_1,s_2,s_3}(X \times Y)$$

that takes the pair (K, f) to the kernel $K(x, y)f(y)$ of the variable $(x, y) \in (X \times Y) \setminus \mathbb{D}_{X\times Y}$ is bilinear and continuous.

Next we have the following imbedding statement that holds for bounded sets (cf. [21, §3]).

Proposition 3.8 *Let X, Y be bounded subsets of $\mathbb{R}^n$. Let s_1, s_2, s_3, t_1, t_2, $t_3 \in \mathbb{R}$. Then the following statements hold.*

(i) *If $t_1 \geq s_1$ then $\mathcal{K}_{s_1,X\times Y}$ is continuously embedded into $\mathcal{K}_{t_1,X\times Y}$.*

(ii) *If $t_1 \geq s_1$, $t_3 \leq s_3$ and $(t_2 - t_3) \geq (s_2 - s_3)$, then $\mathcal{K}_{s_1,s_2,s_3}(X \times Y)$ is continuously embedded into $\mathcal{K}_{t_1,t_2,t_3}(X \times Y)$.*

(iii) *If $t_1 \geq s_1$, $t_3 \leq s_3$, then $\mathcal{K}_{s_1,s_1+s_3,s_3}(X \times Y)$ is continuously embedded into the space $\mathcal{K}_{t_1,t_1+t_3,t_3}(X \times Y)$.*

We now show that we can associate a potential type kernel to all Hölder continuous functions (cf. [21, §3]).

Lemma 3.9 *Let X, Y be subsets of $\mathbb{R}^n$. Let $\alpha \in]0, 1]$. Then the following statements hold.*

(i) *If $\mu \in C^{0,\alpha}(X \cup Y)$, then the map $\Xi[\mu]$ defined by*

$$\Xi[\mu](x, y) \equiv \mu(x) - \mu(y) \qquad \forall (x, y) \in (X \times Y) \setminus \mathbb{D}_{X\times Y} \tag{3.10}$$

belongs to $\mathcal{K}_{-\alpha,0,\alpha}(X \times Y)$.

(ii) *The operator Ξ from $C^{0,\alpha}(X \cup Y)$ to $\mathcal{K}_{-\alpha,0,\alpha}(X \times Y)$ that takes μ to $\Xi[\mu]$ is linear and continuous.*

In order to introduce a result of [21, Thm. 6.6], we need to introduce a further norm for kernels in the case in which Y is a compact manifold of class C^1 that is imbedded in $M = \mathbb{R}^n$ and $X = Y$.

Definition 3.11 *Let Y be a compact manifold of class C^1 that is imbedded in $\mathbb{R}^n$. Let s_1, s_2, $s_3 \in \mathbb{R}$. We set*

$$\mathcal{K}^\sharp_{s_1,s_2,s_3}(Y \times Y) \equiv \Big\{ K \in \mathcal{K}_{s_1,s_2,s_3}(Y \times Y) : \\ \sup_{x\in Y} \sup_{r\in]0,+\infty[} \left| \int_{Y\setminus \mathbb{B}_n(x,r)} K(x,y)\, d\nu(y) \right| < +\infty \Big\}$$

and

$$\|K\|_{\mathcal{K}^\sharp_{s_1,s_2,s_3}(Y\times Y)} \equiv \|K\|_{\mathcal{K}_{s_1,s_2,s_3}(Y\times Y)} \\ + \sup_{x\in Y} \sup_{r\in]0,+\infty[} \left| \int_{Y\setminus \mathbb{B}_n(x,r)} K(x,y)\, d\nu(y) \right| \quad \forall K \in \mathcal{K}^\sharp_{s_1,s_2,s_3}(Y \times Y) .$$

Clearly, $(\mathcal{K}^\sharp_{s_1,s_2,s_3}(Y\times Y), \|\cdot\|_{\mathcal{K}^\sharp_{s_1,s_2,s_3}(Y\times Y)})$ is a normed space. By definition, $\mathcal{K}^\sharp_{s_1,s_2,s_3}(Y \times Y)$ is continuously embedded into $\mathcal{K}_{s_1,s_2,s_3}(Y \times Y)$. Next we introduce a function that we need for a generalized Hölder norm. For each $\theta \in]0, 1]$, we define the function $\omega_\theta(\cdot)$ from $[0, +\infty[$ to itself by setting

$$\omega_\theta(r) \equiv \begin{cases} 0 & r = 0 , \\ r^\theta |\ln r| & r \in]0, r_\theta] , \\ r_\theta^\theta |\ln r_\theta| & r \in]r_\theta, +\infty[, \end{cases}$$

where $r_\theta \equiv e^{-1/\theta}$ for all $\theta \in]0, 1]$. Obviously, $\omega_\theta(\cdot)$ is concave and satisfies condition (2.2). We also note that if $\mathbb{D} \subseteq \mathbb{R}^n$, then the continuous embedding

$$C_b^{0,\theta}(\mathbb{D}) \subseteq C_b^{0,\omega_\theta(\cdot)}(\mathbb{D}) \subseteq C_b^{0,\theta'}(\mathbb{D})$$

holds for all $\theta' \in]0, \theta[$. Here the subscript b denotes that we are considering the intersection of a (generalized) Hölder space with the space $B(\mathbb{D})$ of the bounded functions in $\mathbb{D}$. Then we introduce the following result of [21, Thm. 6.3].

Theorem 3.12 *Let Y be a compact manifold of class C^1 that is imbedded in $\mathbb{R}^n$. Let $s_1 \in [0, (n-1)[$. Let $\beta \in]0, 1]$, $t_1 \in [\beta, (n-1)+\beta[$, $t_2 \in [\beta, +\infty[$, $t_3 \in]0, 1]$. Let the kernel $K \in \mathcal{K}_{s_1,s_1+1,1}(Y \times Y)$ satisfy the following assumption*

$$K(\cdot, y) \in C^1(Y \setminus \{y\}) \quad \forall y \in Y .$$

Then the following statements hold.

(i) *If* $t_1 < (n-1)$ *and* $\mathrm{grad}_{Y,x} K(\cdot,\cdot) \in (\mathcal{K}_{t_1,t_2,t_3}(Y \times Y))^n$, *then the following statements hold.*

(a) *If* $t_2 - \beta > (n-1)$, $t_2 < (n-1) + \beta + t_3$ *and*

$$\int_Y K(\cdot, y)\, d\sigma_y \in C^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ *to* $C^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)$ *that takes* μ *to the function* $\int_Y K(\cdot,y)\mu(y)\, d\sigma_y$ *is linear and continuous.*

(aa) *If* $t_2 - \beta = (n-1)$ *and*

$$\int_Y K(\cdot, y)\, d\sigma_y \in C^{1,\max\{r^\beta,\omega_{t_3}(\cdot)\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ *to* $C^{1,\max\{r^\beta,\omega_{t_3}(\cdot)\}}(Y)$ *that takes* μ *to the function* $\int_Y K(\cdot,y)\mu(y)\, d\sigma_y$ *is linear and continuous.*

(ii) *If* $t_1 = (n-1)$ *and* $\mathrm{grad}_{Y,x} K(\cdot,\cdot) \in \left(\mathcal{K}^\sharp_{t_1,t_2,t_3}(Y \times Y)\right)^n$, *then the following statements hold.*

(b) *If* $t_2 - \beta > (n-1)$, $t_2 < (n-1) + \beta + t_3$ *and*

$$\int_Y K(\cdot, y)\, d\sigma_y \in C^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ *to* $C_b^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(Y)$ *that takes* μ *to the function* $\int_Y K(\cdot,y)\mu(y)\, d\sigma_y$ *is linear and continuous.*

(bb) *If* $t_2 - \beta = (n-1)$ *and*

$$\int_Y K(\cdot, y)\, d\sigma_y \in C^{1,\max\{r^\beta,\omega_{t_3}(\cdot)\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ *to* $C^{1,\max\{r^\beta,\omega_{t_3}(\cdot)\}}(Y)$ *that takes* μ *to the function* $\int_Y K(\cdot,y)\mu(y)\, d\sigma_y$ *is linear and continuous.*

(iii) *If* $t_1 > (n-1)$ *and* $\mathrm{grad}_{Y,x} K(\cdot,\cdot) \in (\mathcal{K}_{t_1,t_2,t_3}(Y \times Y))^n$, *then the following statements hold.*

(c) *If* $t_2-\beta>(n-1)$, $t_2<(n-1)+\beta+t_3$ *and*

$$\int_Y K(\cdot,y)\,d\sigma_y\in C^{1,\min\{\beta,(n-1)+\beta-t_1,(n-1)+t_3+\beta-t_2\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ *to* $C^{1,\min\{\beta,(n-1)+\beta-t_1,(n-1)+t_3+\beta-t_2\}}(Y)$ *that takes* μ *to the function* $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ *is linear and continuous.*

(cc) *If* $t_2-\beta=(n-1)$ *and*

$$\int_Y K(\cdot,y)\,d\sigma_y\in C^{1,\max\{r^\beta,r^{(n-1)+\beta-t_1},\omega_{t_3}(\cdot)\}}(Y)\,,$$

then the map from $C^{0,\beta}(Y)$ *to* $C^{1,\max\{r^\beta,r^{(n-1)+\beta-t_1},\omega_{t_3}(\cdot)\}}(Y)$ *that takes* μ *to the function* $\int_Y K(\cdot,y)\mu(y)\,d\sigma_y$ *is linear and continuous.*

We also need to consider convolution kernels, thus we introduce the following notation. If $n\in\mathbb{N}\setminus\{0\}$, $m\in\mathbb{N}$, $h\in\mathbb{R}$, $\alpha\in]0,1]$, then we set

$$\mathcal{K}_h^{m,\alpha}\equiv\left\{k\in C^{m,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\}):\,k\text{ is positively homogeneous of degree }h\right\},\tag{3.13}$$

where $C^{m,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$ denotes the set of functions of $C^m(\mathbb{R}^n\setminus\{0\})$ whose restriction to $\overline{\Omega}$ is of class $C^{m,\alpha}(\overline{\Omega})$ for all bounded open subsets Ω of $\mathbb{R}^n$ such that $\overline{\Omega}\subseteq\mathbb{R}^n\setminus\{0\}$ and we set

$$\|k\|_{\mathcal{K}_h^{m,\alpha}}\equiv\|k\|_{C^{m,\alpha}(\partial\mathbb{B}_n(0,1))}\qquad\forall k\in\mathcal{K}_h^{m,\alpha}\,.$$

We can easily verify that $\left(\mathcal{K}_h^{m,\alpha},\|\cdot\|_{\mathcal{K}_h^{m,\alpha}}\right)$ is a Banach space. We also mention the following variant of a well known statement.

Lemma 3.14 *Let* $n\in\mathbb{N}\setminus\{0\}$, $h\in[0,+\infty[$. *If* $k\in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$ *is positively homogeneous of degree* $-h$, *then* $k(x-y)\in\mathcal{K}_{h,h+1,1}(\mathbb{R}^n\times\mathbb{R}^n)$. *Moreover, the map from* $\mathcal{K}^{0,1}_{-h}$ *to* $\mathcal{K}_{h,h+1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ *which takes* k *to* $k(x-y)$ *is linear and continuous (see (3.13) for the definition of* $\mathcal{K}^{0,1}_{-h}$*).*

Proof. Since k is positively homogeneous of degree $-h$, we have

$$|k(x-y)|\le\Big(\sup_{\partial\mathbb{B}_n(0,1)}|k|\Big)|x-y|^{-h}\qquad\forall(x,y)\in(\mathbb{R}^n\times\mathbb{R}^n)\setminus\mathbb{D}_{\mathbb{R}^n\times\mathbb{R}^n}\,.$$

Since k is positively homogeneous of degree $-(n-1)$, the inequality of Cialdea [3, VIII, p. 47] (see also [7, Lem. 4.14] with $\alpha = 1$) implies that if x', $x'' \in \mathbb{R}^n$, $x' \neq x''$, $y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x'-x''|)$, then

$$\begin{aligned}
&|k(x'-y) - k(x''-y)| \\
&\qquad \leq (2^1 + 2h) \max\{ \sup_{\partial\mathbb{B}_n(0,1)} |k|, |k : \partial\mathbb{B}_n(0,1)|_1 \} \\
&\qquad\quad \times |(x'-y) - (x''-y)| (\min\{|(x'-y)|, |(x''-y)|\})^{-h-1} .
\end{aligned}$$

Then Lemma 3.5 implies that $|x''-y| \geq \frac{1}{2}|x'-y|$, and thus we have

$$\begin{aligned}
&|k(x'-y) - k(x''-y)| \\
&\qquad \leq (2 + 2h) \max\{ \sup_{\partial\mathbb{B}_n(0,1)} |k|, |k : \partial\mathbb{B}_n(0,1)|_1 \} \frac{|x'-x''|}{|x'-y|^{h+1}} 2^{h+1}
\end{aligned}$$

and the proof is complete. □

If X and Y are subsets of $\mathbb{R}^n$, then the restriction operator

$$\text{from } \mathcal{K}_{h,h+1,1}(\mathbb{R}^n \times \mathbb{R}^n) \text{ to } \mathcal{K}_{h,h+1,1}(X \times Y)$$

is linear and continuous. Thus Lemma 3.14 implies that the map

$$\text{from the subspace } \mathcal{K}^{0,1}_{-h} \text{ of } C^{0,1}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) \text{ to } \mathcal{K}_{h,h+1,1}(X \times Y)\,,$$

which takes k to $k(x-y)$ is linear and continuous.

Remark 3.15 *As Lemma 3.14 shows the convolution kernels associated to positively homogeneous functions of negative degree are standard kernels. We note however that there exist potential type kernels that belong to a class* $\mathcal{K}_{s_1,s_2,s_3}(X \times Y)$ *with* $s_2 \neq s_1 + s_3$.

4 Technical preliminaries on the differential operator

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^2. The kernel of the boundary integral operator corresponding to the double layer potential is the following

$$\overline{B^*_{\Omega,y}}\,(S_{\mathbf{a}}(x-y)) \equiv -\sum_{l,j=1}^{n} a_{jl}\nu_l(y) \frac{\partial S_{\mathbf{a}}}{\partial x_j}(x-y) \tag{4.1}$$

$$-\sum_{l=1}^{n}\nu_l(y)a_l S_{\mathbf{a}}(x-y) \qquad \forall (x,y)\in(\partial\Omega)^2\setminus\mathbb{D}_{\partial\Omega}$$

(cf. (1.4)). In order to analyze the kernel of the double layer potential, we need some more information on the fundamental solution $S_{\mathbf{a}}$. To do so, we introduce the fundamental solution S_n of the Laplace operator. Namely, we set

$$S_n(x)\equiv\begin{cases}\frac{1}{s_n}\ln|x| & \forall x\in\mathbb{R}^n\setminus\{0\}, \quad \text{if } n=2\,,\\ \frac{1}{(2-n)s_n}|x|^{2-n} & \forall x\in\mathbb{R}^n\setminus\{0\}, \quad \text{if } n>2\,,\end{cases}$$

where s_n denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n(0,1)$ and we follow a formulation of Dalla Riva [5, Thm. 5.2, 5.3] and Dalla Riva, Morais and Musolino [6, Thm. 5.5], that we state as in paper [8, Cor. 4.2] with Dondi (see also John [16], and Miranda [25] for homogeneous operators, and Mitrea and Mitrea [28, p. 203]).

Proposition 4.2 *Let* $\mathbf{a}$ *be as in (1.1), (1.2), (1.3). Let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a},D]$*. Then there exist an invertible matrix* $T\in M_n(\mathbb{R})$ *such that*

$$a^{(2)}=TT^t\,, \tag{4.3}$$

a real analytic function A_1 *from* $\partial\mathbb{B}_n(0,1)\times\mathbb{R}$ *to* $\mathbb{C}$ *such that* $A_1(\cdot,0)$ *is odd,* $b_0\in\mathbb{C}$*, a real analytic function* B_1 *from* $\mathbb{R}^n$ *to* $\mathbb{C}$ *such that* $B_1(0)=0$*, and a real analytic function* C *from* $\mathbb{R}^n$ *to* $\mathbb{C}$ *such that*

$$\begin{aligned} S_{\mathbf{a}}(x) = &\frac{1}{\sqrt{\det a^{(2)}}}S_n(T^{-1}x) \\ &+|x|^{3-n}A_1(\frac{x}{|x|},|x|)+(B_1(x)+b_0(1-\delta_{2,n}))\ln|x|+C(x)\,, \end{aligned} \tag{4.4}$$

for all $x\in\mathbb{R}^n\setminus\{0\}$*, and such that both* b_0 *and* B_1 *equal zero if* n *is odd. Moreover,*

$$\frac{1}{\sqrt{\det a^{(2)}}}S_n(T^{-1}x)$$

is a fundamental solution for the principal part of $P[\mathbf{a},D]$*.*

In particular for the statement that $A_1(\cdot,0)$ is odd, we refer to Dalla Riva, Morais and Musolino [6, Thm. 5.5, (32)], where $A_1(\cdot,0)$ coincides with $\mathbf{f}_1(\mathbf{a},\cdot)$ in that paper. Here we note that a function A from $(\partial\mathbb{B}_n(0,1))\times\mathbb{R}$ to $\mathbb{C}$ is said to be real analytic provided that it has a real analytic extension to an open neighbourhood of $(\partial\mathbb{B}_n(0,1))\times\mathbb{R}$ in $\mathbb{R}^{n+1}$. Then we have the following elementary lemma.

Lemma 4.5 *Let $n \in \mathbb{N} \setminus \{0,1\}$. A function A from $(\partial \mathbb{B}_n(0,1)) \times \mathbb{R}$ to $\mathbb{C}$ is real analytic if and only if the function $\tilde{A}$ from $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$ defined by*

$$\tilde{A}(x,r) \equiv A(\frac{x}{|x|}, r) \qquad \forall (x,r) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R} \tag{4.6}$$

is real analytic.

Proof. If A is real analytic then, it has a real analytic extension $A^\sharp$ to an open neighborhood U of $(\partial \mathbb{B}_n(0,1)) \times \mathbb{R}$ in $\mathbb{R}^{n+1}$. Since the function $\frac{x}{|x|}$ is real analytic in $x \in \mathbb{R}^n \setminus \{0\}$, then the composition $\tilde{A}$ of $A^\sharp$ and of $(\frac{x}{|x|}, r)$ is real analytic.

Conversely, if $\tilde{A}$ is real analytic, we note that $\tilde{A}$ is an extension of A to the open neighborhood $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$ of $(\partial \mathbb{B}_n(0,1)) \times \mathbb{R}$ in $\mathbb{R}^{n+1}$ and that accordingly A is real analytic. □

Then one can prove the following formula for the gradient of the fundamental solution (see reference [8, Lem. 4.3, (4.8) and the following 2 lines] with Dondi. Here one should remember that $A_1(\cdot, 0)$ is odd and that $b_0 = 0$ if n is odd).

Proposition 4.7 *Let $\mathbf{a}$ be as in (1.1), (1.2), (1.3). Let $T \in M_n(\mathbb{R})$ be as in (4.3). Let $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a}, D]$. Let B_1, C be as in Proposition 4.2. Then there exists a real analytic function A_2 from $\partial \mathbb{B}_n(0,1) \times \mathbb{R}$ to $\mathbb{C}^n$ such that*

$$\begin{aligned} DS_{\mathbf{a}}(x) = & \frac{1}{s_n \sqrt{\det a^{(2)}}} |T^{-1}x|^{-n} x^t (a^{(2)})^{-1} \\ & + |x|^{2-n} A_2(\frac{x}{|x|}, |x|) + DB_1(x) \ln|x| + DC(x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \tag{4.8}$$

Moreover, $A_2(\cdot, 0)$ is even.

Then one can prove the following formula for the kernel of the double layer potential

$$\begin{aligned} \overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y)) = & -DS_{\mathbf{a}}(x-y)a^{(2)}\nu(y) - \nu^t(y)a^{(1)}S_{\mathbf{a}}(x-y) \\ = & -\frac{1}{s_n \sqrt{\det a^{(2)}}} |T^{-1}(x-y)|^{-n}(x-y)^t \nu(y) \\ & -|x-y|^{2-n} A_2(\frac{x-y}{|x-y|}, |x-y|) a^{(2)}\nu(y) \\ & -DB_1(x-y)a^{(2)}\nu(y) \ln|x-y| - DC(x-y)a^{(2)}\nu(y) \end{aligned} \tag{4.9}$$

$$-\nu^t(y)a^{(1)}S_{\mathbf{a}}(x-y) \qquad \forall x,y\in\partial\Omega, x\neq y\,.$$

(see reference [8, (5.2) p. 86] with Dondi). Then the following statement holds (see reference [8, Lem. 5.1, inequality at line 13 of p. 86] with Dondi).

Lemma 4.10 *Let* $\mathbf{a}$ *be as in (1.1), (1.2), (1.3). Let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a},D]$*. Let* $\alpha\in]0,1]$*. Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$*. Then the following statements hold.*

(i) If $\alpha\in]0,1[$*, then*

$$b_{\Omega,\alpha}\equiv\sup\left\{|x-y|^{n-1-\alpha}|\overline{B^*_{\Omega,y}}\,(S_{\mathbf{a}}(x-y))\,|:\, x,y\in\partial\Omega, x\neq y\right\}<+\infty\,. \tag{4.11}$$

If $n>2$*, then (4.11) holds also for* $\alpha=1$*. If* $n=2$ *and* $DB_1(0)=0$*, then (4.11) holds also for* $\alpha=1$*.*

(ii) If $n=2$ *and* $\alpha=1$*, then*

$$b_{\Omega,\alpha}\equiv\sup\left\{\frac{|\overline{B^*_{\Omega,y}}\,(S_{\mathbf{a}}(x-y))\,|}{(1+|\ln|x-y||)}\,:\, x,y\in\partial\Omega, x\neq y\right\}<+\infty. \tag{4.12}$$

In particular, the kernel $\overline{B^*_{\Omega,y}}\,(S_{\mathbf{a}}(x-y))$ *belongs to* $\mathcal{K}_{\epsilon,(\partial\Omega)\times(\partial\Omega)}$ *for all* $\epsilon\in]0,+\infty[$*.*

(iii)

$$\tilde{b}_{\Omega,\alpha}\equiv\sup\left\{\frac{|x'-y|^{n-\alpha}}{|x'-x''|}|\overline{B^*_{\Omega,y}}\,\big(S_{\mathbf{a}}(x'-y)\big)-\overline{B^*_{\Omega,y}}\,\big(S_{\mathbf{a}}(x''-y)\big)\,|:\right.$$
$$\left. x',x''\in\partial\Omega, x'\neq x'', y\in\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)\right\}<+\infty\,.$$

By applying equality (4.9), we can compute a formula for the tangential gradient with respect to its first variable of the kernel of the double layer potential and establish some of its properties. To do so we introduce the following technical lemma (see reference [8, Lem. 3.2 (v), 3.3] with Dondi).

Lemma 4.13 *Let* Y *be a nonempty bounded subset of* $\mathbb{R}^n$*. Then the following statements hold.*

(i) Let $F\in\mathrm{Lip}(\partial\mathbb{B}_n(0,1)\times[0,\mathrm{diam}\,(Y)])$ *with*

$$\mathrm{Lip}(F)\equiv\left\{\frac{|F(\theta',r')-F(\theta'',r'')|}{|\theta'-\theta''|+|r'-r''|}\,:\right.$$

$$(\theta',r'),(\theta'',r'')\in\partial\mathbb{B}_n(0,1)\times[0,\mathrm{diam}\,(Y)],\ (\theta',r')\neq(\theta'',r'')\Big\}.$$

Then

$$\left|F\left(\frac{x'-y}{|x'-y|},|x'-y|\right)-F\left(\frac{x''-y}{|x''-y|},|x''-y|\right)\right| \tag{4.14}$$
$$\leq \mathrm{Lip}(F)(2+\mathrm{diam}\,(Y))\frac{|x'-x''|}{|x'-y|}\qquad \forall y\in Y\setminus\mathbb{B}_n(x',2|x'-x''|)\,,$$

for all $x',x''\in Y$, $x'\neq x''$. *In particular, if* $f\in C^1(\partial\mathbb{B}_n(0,1)\times\mathbb{R},\mathbb{C})$, *then*

$$M_{f,Y}\equiv\sup\Bigg\{\left|f\left(\frac{x'-y}{|x'-y|},|x'-y|\right)-f\left(\frac{x''-y}{|x''-y|},|x''-y|\right)\right|\frac{|x'-y|}{|x'-x''|}$$
$$:\ x',x''\in Y,x'\neq x'',y\in Y\setminus\mathbb{B}_n(x',2|x'-x''|)\Bigg\}$$

is finite and thus the kernel $f\left(\frac{x-y}{|x-y|},|x-y|\right)$ *belongs to* $\mathcal{K}_{0,1,1}(Y\times Y)$.

(ii) Let W *be an open neighbourhood of* $\overline{(Y-Y)}$. *Let* $f\in C^1(W,\mathbb{C})$. *Then*

$$\tilde{M}_{f,Y}\equiv\sup\Bigg\{|f(x'-y)-f(x''-y)|\,|x'-x''|^{-1}:$$
$$x',x''\in Y,x'\neq x'',y\in Y\Bigg\}<+\infty\,.$$

Here $Y-Y\equiv\{y_1-y_2:\ y_1,y_2\in Y\}$. *In particular, the kernel* $f(x-y)$ *belongs to the class* $\mathcal{K}_{0,0,1}(Y\times Y)$, *which is continuously imbedded into* $\mathcal{K}_{0,1,1}(Y\times Y)$.

(iii) The kernel $\ln|x-y|$ *belongs to* $\mathcal{K}_{\epsilon,1,1}(Y\times Y)$ *for all* $\epsilon\in]0,1[$.

Proof. For the proof of (i), the first part of (ii) and (iii), we refer to the above mentioned paper [8, Lem. 3.2 (v), 3.3]. The imbedding of the second part of (ii) follows by the imbedding Proposition 3.8 (ii). □

We are now ready to prove the following statement. For the definition of tangential gradient $\mathrm{grad}_{\partial\Omega}$ and tangential divergence $\mathrm{div}_{\partial\Omega}$, we refer to Kirsch and Hettlich [18, A.5], Chavel [1, Chap. 1].

Lemma 4.15 *Let* $\mathbf{a}$ *be as in (1.1), (1.2), (1.3). Let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a},D]$. *Let* $\alpha\in]0,1]$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$. *Then the following statements hold.*

(i) If $h \in \{1, \dots, n\}$, then

$$
\begin{aligned}
&(\mathrm{grad}_{\partial\Omega,x}\overline{B^*_{\Omega,y}}\,(S_{\mathbf{a}}(x-y)))_h \qquad (4.16)\\
&\qquad = \frac{\partial}{\partial x_h}\overline{B^*_{\Omega,y}}\,(S_{\mathbf{a}}(x-y))) - \nu_h(x)\sum_{l=1}^{n}\nu_l(x)\frac{\partial}{\partial x_l}\overline{B^*_{\Omega,y}}\,(S_{\mathbf{a}}(x-y)))\\
&= \frac{n}{s_n\sqrt{\det a^{(2)}}}\frac{(x-y)^t\cdot\nu(y)}{|T^{-1}(x-y)|^n}\\
&\times\sum_{l=1}^{n}\nu_l(x)\Big[\nu_l(x)\frac{\sum_{j,z=1}^{n}(T^{-1})_{jz}(x_z-y_z)(T^{-1})_{jh}}{|T^{-1}(x-y)|^2}\\
&-\nu_h(x)\frac{\sum_{j,z=1}^{n}(T^{-1})_{jz}(x_z-y_z)(T^{-1})_{jl}}{|T^{-1}(x-y)|^2}\Big]\\
&-\frac{\sum_{l=1}^{n}\nu_l(x)\big[\nu_l(x)\nu_h(y)-\nu_h(x)\nu_l(y)\big]}{s_n\sqrt{\det a^{(2)}}|T^{-1}(x-y)|^n}\\
&-(2-n)|x-y|^{1-n}A_2\left(\frac{x-y}{|x-y|},|x-y|\right)a^{(2)}\nu(y)\\
&\times\sum_{l=1}^{n}\nu_l(x)\big[\nu_l(x)\frac{x_h-y_h}{|x-y|}-\nu_h(x)\frac{x_l-y_l}{|x-y|}\big]\\
&-\sum_{j=1}^{n}\frac{\partial A_2}{\partial y_j}\left(\frac{x-y}{|x-y|},|x-y|\right)a^{(2)}\nu(y)|x-y|^{-n}\\
&\times\sum_{l=1}^{n}\nu_l(x)\Big[\nu_l(x)\Big(\delta_{jh}|x-y|-\frac{(x_j-y_j)(x_h-y_h)}{|x-y|}\Big)\\
&-\nu_h(x)\Big(\delta_{jl}|x-y|-\frac{(x_j-y_j)(x_l-y_l)}{|x-y|}\Big)\Big]\\
&-\frac{\partial A_2}{\partial r}\left(\frac{x-y}{|x-y|},|x-y|\right)a^{(2)}\nu(y)\\
&\times\sum_{l=1}^{n}\nu_l(x)\left[\nu_l(x)\frac{x_h-y_h}{|x-y|^{n-1}}-\nu_h(x)\frac{x_l-y_l}{|x-y|^{n-1}}\right]\\
&-\sum_{j,z=1}^{n}\sum_{l=1}^{n}\nu_l(x)\left[\nu_l(x)\frac{\partial^2 B_1}{\partial x_h\partial x_j}(x-y)-\nu_h(x)\frac{\partial^2 B_1}{\partial x_l\partial x_j}(x-y)\right]\\
&\times a_{jz}\nu_z(y)\ln|x-y|\\
&-DB_1(x-y)a^{(2)}\nu(y)\sum_{l=1}^{n}\nu_l(x)\left[\nu_l(x)\frac{x_h-y_h}{|x-y|^2}-\nu_h(x)\frac{x_l-y_l}{|x-y|^2}\right]
\end{aligned}
$$

$$
-\sum_{j,s=1}^{n}\sum_{l=1}^{n}\nu_l(x)\left[\nu_l(x)\frac{\partial^2 C}{\partial x_h\partial x_j}(x-y)-\nu_h(x)\frac{\partial^2 C}{\partial x_l\partial x_j}(x-y)\right]a_{js}\nu_s(y)
$$

$$
-\nu(y)^t\cdot a^{(1)}\sum_{l=1}^{n}\nu_l(x)\left[\nu_l(x)\frac{\partial S_{\mathbf{a}}}{\partial x_h}(x-y)-\nu_h(x)\frac{\partial S_{\mathbf{a}}}{\partial x_l}(x-y)\right]
$$

for all $(x,y)\in(\partial\Omega)^2\setminus\mathbb{D}_{\partial\Omega}$*, where we understand that the symbols*

$$
\frac{\partial A_2}{\partial y_j}\qquad \forall j\in\{1,\dots,n\}
$$

denote partial derivatives of any of the analytic extensions of A_2 *to an open neighborhood of* $(\partial\mathbb{B}_n(0,1))\times\mathbb{R}$ *in* $\mathbb{R}^{n+1}$*.*

(ii) The kernel $\mathrm{grad}_{\partial\Omega,x}\overline{B^*_{\Omega,y}}\,(S_{\mathbf{a}}(x-y))$ *belongs to* $(\mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega\times\partial\Omega))^n$*.*

Proof. (i) By formula (4.9), we have

$$
\begin{aligned}
&\frac{\partial}{\partial x_h}\overline{B^*_{\Omega,y}}\,(S_{\mathbf{a}}(x-y)))\\
&\quad=-\frac{(-n)}{s_n\sqrt{\det a^{(2)}}}\sum_{j,z=1}^{n}\frac{(T^{-1})_{jz}(x_z-y_z)(T^{-1})_{jh}}{|T^{-1}(x-y)|^2}\frac{(x-y)^t\cdot\nu(y)}{|T^{-1}(x-y)|^n}\\
&\quad-\frac{1}{s_n\sqrt{\det a^{(2)}}}|T^{-1}(x-y)|^{-n}\nu_h(y)\\
&\quad-(2-n)|x-y|^{1-n}\frac{x_h-y_h}{|x-y|}A_2(\frac{x-y}{|x-y|},|x-y|)a^{(2)}\nu(y)\\
&\quad-\sum_{j=1}^{n}\frac{\partial A_2}{\partial y_j}(\frac{x-y}{|x-y|},|x-y|)a^{(2)}\nu(y)\frac{\delta_{jh}|x-y|-\frac{(x_j-y_j)(x_h-y_h)}{|x-y|}}{|x-y|^n}\\
&\quad-\frac{\partial A_2}{\partial r}(\frac{x-y}{|x-y|},|x-y|)a^{(2)}\nu(y)\frac{x_h-y_h}{|x-y|^{n-1}}\\
&\quad-\sum_{j,z=1}^{n}\frac{\partial^2 B_1}{\partial x_h\partial x_j}(x-y)a_{jz}\nu_z(y)\ln|x-y|\\
&\quad-DB_1(x-y)a^{(2)}\nu(y)\frac{x_h-y_h}{|x-y|^2}\\
&\quad-\sum_{j,s=1}^{n}\frac{\partial^2 C}{\partial x_h\partial x_j}(x-y)a_{js}\nu_s(y)-\nu(y)^t\cdot a^{(1)}\frac{\partial S_{\mathbf{a}}}{\partial x_h}(x-y)
\end{aligned}
$$

for all $(x,y)\in(\partial\Omega)^2\setminus\mathbb{D}_{\partial\Omega}$. Then the definition of tangential gradient implies the validity of formula (4.16).

We now turn to the proof of (ii). If suffices to show that if $h \in \{1, \ldots, n\}$, then each addendum in the right hand side of formula (4.16) belongs to the class $\mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega \times \partial\Omega)$.

By Lemma 3.14 the kernel $\frac{1}{|T^{-1}(x-y)|^n}$ belongs to $\mathcal{K}_{n,n+1,1}(\partial\Omega \times \partial\Omega)$. Since there exists $c_{\Omega,\alpha} \in]0, +\infty[$ such that

$$|\nu(y) \cdot (x-y)| \leq c_{\Omega,\alpha}|x-y|^{1+\alpha} \qquad \forall x, y \in \partial\Omega$$

the kernel $\nu(y) \cdot (x-y)$ belongs to $\mathcal{K}_{-1-\alpha,-\alpha,1}(\partial\Omega \times \partial\Omega)$ (cf. *e.g.*, reference [8, Lem. 3.4 and p. 87 line 8] with Dondi). Then the product Theorem 3.6 implies that the kernel $\frac{\nu(y)(x-y)}{|T^{-1}(x-y)|^n}$ belongs to $\mathcal{K}_{n-1-\alpha,n-\alpha,1}(\partial\Omega \times \partial\Omega)$. By Lemma 3.4, $\mathcal{K}_{n-1-\alpha,n-\alpha,1}(\partial\Omega \times \partial\Omega)$ is contained in $\mathcal{K}_{n-1-\alpha,n-1,\alpha}(\partial\Omega \times \partial\Omega)$.

By Lemma 3.14 the kernel $\frac{x_h - y_h}{|T^{-1}(x-y)|^2}$ belongs to $\mathcal{K}_{1,2,1}(\partial\Omega \times \partial\Omega)$. By Lemma 3.4, $\mathcal{K}_{1,2,1}(\partial\Omega \times \partial\Omega)$ is contained in $\mathcal{K}_{1,1+\alpha,\alpha}(\partial\Omega \times \partial\Omega)$. Then the α-Hölder continuity of ν and Propostion 3.7 imply that

$$\sum_{l=1}^{n} \nu_l(x) \Big[\nu_l(x) \frac{\sum_{j,z=1}^{n} (T^{-1})_{jz}(x_z - y_z)(T^{-1})_{jh}}{|T^{-1}(x-y)|^2} \\ -\nu_h(x) \frac{\sum_{j,z=1}^{n} (T^{-1})_{jz}(x_z - y_z)(T^{-1})_{jl}}{|T^{-1}(x-y)|^2} \Big]$$

belongs to $\mathcal{K}_{1,1+\alpha,\alpha}(\partial\Omega \times \partial\Omega)$. Then the product Theorem 3.6 (ii) implies that

$$\frac{(x-y)^t \cdot \nu(y)}{|T^{-1}(x-y)|^n} \sum_{l=1}^{n} \nu_l(x) \Big[\nu_l(x) \frac{\sum_{j,z=1}^{n} (T^{-1})_{jz}(x_z - y_z)(T^{-1})_{jh}}{|T^{-1}(x-y)|^2} \\ -\nu_h(x) \frac{\sum_{j,z=1}^{n} (T^{-1})_{jz}(x_t - y_t)(T^{-1})_{jl}}{|T^{-1}(x-y)|^2} \Big] \\ \in \mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega \times \partial\Omega)\,. \tag{4.17}$$

We now consider the second addendum in the right hand side of formula (4.16) and we observe that

$$\frac{\sum_{l=1}^{n} \nu_l(x) \big[\nu_l(x)\nu_h(y) - \nu_h(x)\nu_l(y) \big]}{s_n \sqrt{\det a^{(2)}} |T^{-1}(x-y)|^n} \\ = \frac{\sum_{l=1}^{n} \nu_l(x) \big[\nu_l(x)(\nu_h(y) - \nu_h(x)) - \nu_h(x)(\nu_l(y) - \nu_l(x)) \big]}{s_n \sqrt{\det a^{(2)}} |T^{-1}(x-y)|^n}$$

for all $(x, y) \in (\partial\Omega)^2 \setminus \mathbb{D}_{\partial\Omega}$. Since ν is α-Hölder continuous, Lemma 3.9 implies that $\nu_h(x) - \nu_h(y)$ belongs to $\mathcal{K}_{-\alpha,0,\alpha}(\partial\Omega \times \partial\Omega)$. By Lemma 3.14 the kernel $\frac{1}{|T^{-1}(x-y)|^n}$ belongs to $\mathcal{K}_{n,n+1,1}(\partial\Omega\times\partial\Omega) \subseteq \mathcal{K}_{n,n+1-(1-\alpha),1-(1-\alpha)}(\partial\Omega\times\partial\Omega)$. Then the product Theorem 3.6 (ii) implies that

$$\frac{\nu_h(x)-\nu_h(y)}{|T^{-1}(x-y)|^n} \in \mathcal{K}_{n-\alpha,n+\alpha-\alpha,\alpha}(\partial\Omega\times\partial\Omega)\,.$$

Then the α-Hölder continuity of ν and Propostion 3.7 implies that

$$\sum_{l=1}^{n} \frac{(\nu_l(x)-\nu_l(y))}{|T^{-1}(x-y)|^n}\nu_l(x)\nu_h(x) \in \mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega\times\partial\Omega)\,.$$

Hence,

$$\frac{\sum_{l=1}^{n}\nu_l(x)\big[\nu_l(x)\nu_h(y)-\nu_h(x)\nu_l(y)\big]}{|T^{-1}(x-y)|^n} \in \mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega\times\partial\Omega)\,. \tag{4.18}$$

We now consider the third addendum in the right hand side of formula (4.16). Since A_2 is real analytic in $\partial\mathbb{B}_n(0,1)\times\mathbb{R}$, Lemma 4.13 (i) implies that the kernel $A_2\left(\frac{x-y}{|x-y|},|x-y|\right)$ belongs to $\mathcal{K}_{0,1,1}(\partial\Omega\times\partial\Omega)$. Since the function $|\xi|^{1-n}\frac{\xi_h}{|\xi|}$ of the variable $\xi\in\mathbb{R}^n\setminus\{0\}$ is positively homogeneous of degree $-(n-1)$, Lemma 3.14 implies that the kernel $|x-y|^{1-n}\frac{x_h-y_h}{|x-y|}$ is of class $\mathcal{K}_{n-1,n,1}(\partial\Omega\times\partial\Omega)$. Then the product Theorem 3.6 (ii) and Proposition 3.7 (iii) imply that the kernel

$$-(2-n)|x-y|^{1-n}\frac{x_h-y_h}{|x-y|}A_2\left(\frac{x-y}{|x-y|},|x-y|\right)a^{(2)}\nu(y)$$

belongs to the class $\mathcal{K}_{n-1,n,1}(\partial\Omega\times\partial\Omega)$. By the imbedding Proposition 3.8 (ii) with

$$s_1=n-1\,,\quad s_2=n\,,\quad s_3=1\,,\qquad t_1=n-\alpha\,,\quad t_2=n\,,\quad t_3=\alpha\,,$$

$\mathcal{K}_{n-1,n,1}(\partial\Omega\times\partial\Omega)$ is contained in $\mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega\times\partial\Omega)$. Since the components of ν are of class $C^{0,\alpha}$, the product Proposition 3.7 (ii) implies that

$$\begin{aligned}&-(2-n)|x-y|^{1-n}A_2\left(\frac{x-y}{|x-y|},|x-y|\right)a^{(2)}\nu(y)\\ &\qquad\sum_{l=1}^{n}\nu_l(x)\big[\nu_l(x)\frac{x_h-y_h}{|x-y|}-\nu_h(x)\frac{x_l-y_l}{|x-y|}\big] \in \mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega\times\partial\Omega)\,.\end{aligned} \tag{4.19}$$

We now consider the fourth addendum in the right hand side of formula (4.16). Let $j\in\{1,\dots,n\}$. Since $\frac{\partial A_2}{\partial y_j}$ is real analytic in $\partial\mathbb{B}_n(0,1)\times\mathbb{R}$, Lemma

4.13 (i) implies that the kernel $\frac{\partial A_2}{\partial y_j}\left(\frac{x-y}{|x-y|},|x-y|\right)$ belongs to $\mathcal{K}_{0,1,1}(\partial\Omega\times\partial\Omega)$. By Lemma 3.4,

$$\mathcal{K}_{0,1,1}(\partial\Omega\times\partial\Omega)\subseteq\mathcal{K}_{0,1-(1-\alpha),1-(1-\alpha)}(\partial\Omega\times\partial\Omega)=\mathcal{K}_{0,\alpha,\alpha}(\partial\Omega\times\partial\Omega)\,.$$

Since the functions $|\xi|^{-(n-1)}$ and $|\xi|^{-n-1}\xi_j\xi_l$ of the variable $\xi\in\mathbb{R}^n\setminus\{0\}$ are positively homogeneous of degree $-(n-1)$, Lemma 3.14 implies that the kernels $|x-y|^{-(n-1)}$ and $|x-y|^{-n-1}(x_j-y_j)(x_l-y_l)$ are of class $\mathcal{K}_{n-1,n,1}(\partial\Omega\times\partial\Omega)$. By Lemma 3.4, $\mathcal{K}_{n-1,n,1}(\partial\Omega\times\partial\Omega)$ is contained in $\mathcal{K}_{n-1,n-1+\alpha,\alpha}(\partial\Omega\times\partial\Omega)$. Then the product Theorem 3.6 (ii) implies that the product is continuous from

$$\mathcal{K}_{n-1,n-1+\alpha,\alpha}(\partial\Omega\times\partial\Omega)\times\mathcal{K}_{0,\alpha,\alpha}(\partial\Omega\times\partial\Omega)\quad\text{to}\quad\mathcal{K}_{n-1,n-1+\alpha,\alpha}(\partial\Omega\times\partial\Omega)\,.$$

Then the α-Hölder continuity of the components of ν, Proposition 3.7 (ii), (iii) and the imbedding Proposition 3.8 (iii) imply that

$$\begin{aligned}&-\sum_{j=1}^{n}\frac{\partial A_2}{\partial y_j}\left(\frac{x-y}{|x-y|},|x-y|\right)a^{(2)}\nu(y)|x-y|^{-n}\\&\qquad\times\sum_{l=1}^{n}\nu_l(x)\biggl[\nu_l(x)\biggl(\delta_{jh}|x-y|-\frac{(x_j-y_j)(x_h-y_h)}{|x-y|}\biggr)\\&\qquad-\nu_h(x)\biggl(\delta_{jl}|x-y|-\frac{(x_j-y_j)(x_l-y_l)}{|x-y|}\biggr)\biggr]\\&\qquad\in\mathcal{K}_{n-1,n-1+\alpha,\alpha}(\partial\Omega\times\partial\Omega)\subseteq\mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega\times\partial\Omega)\,.\end{aligned}\tag{4.20}$$

We now consider the fifth addendum in the right hand side of formula (4.16). Since $\frac{\partial A_2}{\partial r}$ is real analytic in $\partial\mathbb{B}_n(0,1)\times\mathbb{R}$, Lemma 4.13 (i) implies that the kernel $\frac{\partial A_2}{\partial r}\left(\frac{x-y}{|x-y|},|x-y|\right)$ belongs to $\mathcal{K}_{0,1,1}(\partial\Omega\times\partial\Omega)$ that is contained in $\mathcal{K}_{0,\alpha,\alpha}(\partial\Omega\times\partial\Omega)$ (cf. Lemma 3.4). Since the function $|\xi|^{-(n-1)}\xi_l$ of the variable $\xi\in\mathbb{R}^n\setminus\{0\}$ is positively homogeneous of degree $n-2$, Lemma 3.14 implies that the kernels $|x-y|^{-(n-1)}(x_l-y_l)$ are of class $\mathcal{K}_{n-2,n-1,1}(\partial\Omega\times\partial\Omega)$, that is contained in $\mathcal{K}_{n-2,n-2+\alpha,\alpha}(\partial\Omega\times\partial\Omega)$ (cf. Lemma 3.4). Then the product Theorem 3.6 (ii) implies that the product is continuous from

$$\mathcal{K}_{n-2,n-2+\alpha,\alpha}(\partial\Omega\times\partial\Omega)\times\mathcal{K}_{0,\alpha,\alpha}(\partial\Omega\times\partial\Omega)\quad\text{to}\quad\mathcal{K}_{n-2,n-2+\alpha,\alpha}(\partial\Omega\times\partial\Omega)\,.$$

Then the α-Hölder continuity of the components of ν, Proposition 3.7 (ii), (iii) and the imbedding Proposition 3.8 (iii) imply that

$$\frac{\partial A_2}{\partial r}\left(\frac{x-y}{|x-y|},|x-y|\right)a^{(2)}\nu(y)\tag{4.21}$$

$$\times \sum_{l=1}^{n} \nu_l(x) \left[\nu_l(x) \frac{x_h - y_h}{|x-y|^{n-1}} - \nu_h(x) \frac{x_l - y_l}{|x-y|^{n-1}} \right]$$
$$\in \mathcal{K}_{n-2,n-2+\alpha,\alpha}(\partial\Omega \times \partial\Omega) \subseteq \mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega \times \partial\Omega) .$$

We now consider the sixth addendum in the right hand side of formula (4.16). Since B_1 is analytic, Lemma 4.13 (ii) implies that the kernel $\frac{\partial^2 B_1}{\partial x_l \partial x_j}(x-y)$ belongs to $\mathcal{K}_{0,1,1}(\partial\Omega \times \partial\Omega)$ that is contained in $\mathcal{K}_{0,\alpha,\alpha}(\partial\Omega \times \partial\Omega)$ for each $j, l \in \{1, \dots, n\}$ (cf. Lemma 3.4). Then the α-Hölder continuity of the components of ν and the product Proposition 3.7 (ii), (iii) imply that

$$\sum_{j,t=1}^{n} \sum_{l=1}^{n} \nu_l(x) \left[\nu_l(x) \frac{\partial^2 B_1}{\partial x_h \partial x_j}(x-y) - \nu_h(x) \frac{\partial^2 B_1}{\partial x_l \partial x_j}(x-y) \right] a_{jt} \nu_t(y)$$
$$\in \mathcal{K}_{0,\alpha,\alpha}(\partial\Omega \times \partial\Omega) .$$

By Lemma 4.13 (iii) and by Lemma 3.4, we have

$$\ln|x-y| \in \mathcal{K}_{\epsilon,1,1}(\partial\Omega \times \partial\Omega) \subseteq \mathcal{K}_{\epsilon,\alpha,\alpha}(\partial\Omega \times \partial\Omega) \qquad \forall \epsilon \in]0,1[.$$

Theorem 3.6 (ii) implies that the product is continuous from

$$\mathcal{K}_{0,\alpha,\alpha}(\partial\Omega \times \partial\Omega) \times \mathcal{K}_{\epsilon,\alpha,\alpha}(\partial\Omega \times \partial\Omega) \quad \text{to} \quad \mathcal{K}_{\epsilon,\alpha+\epsilon,\alpha}(\partial\Omega \times \partial\Omega) .$$

Hence, inequalities $n - \alpha \geq \epsilon$, $\alpha \leq \alpha$ and the imbedding Proposition 3.8 (iii) imply that

$$\sum_{j,z=1}^{n} \sum_{l=1}^{n} \nu_l(x) \left[\nu_l(x) \frac{\partial^2 B_1}{\partial x_h \partial x_j}(x-y) - \nu_h(x) \frac{\partial^2 B_1}{\partial x_l \partial x_j}(x-y) \right] \tag{4.22}$$
$$\times a_{jz} \nu_z(y) \ln|x-y| \in \mathcal{K}_{\epsilon,\alpha+\epsilon,\alpha}(\partial\Omega \times \partial\Omega) \subseteq \mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega \times \partial\Omega) .$$

We now consider the seventh addendum in the right hand side of formula (4.16). Since B_1 is analytic, Lemma 4.13 (ii) and the product Proposition 3.7 (iii) imply that $DB_1(x-y)a^{(2)}\nu(y)$ belongs to $\mathcal{K}_{0,1,1}(\partial\Omega \times \partial\Omega)$ that is contained in $\mathcal{K}_{0,\alpha,\alpha}(\partial\Omega \times \partial\Omega)$ (cf. Lemma 3.4). Since the functions $|\xi|^{-2}\xi_l$ of the variable $\xi \in \mathbb{R}^n \setminus \{0\}$ are positively homogeneous of degree -1, Lemma 3.14 implies that the kernels $|x-y|^{-2}(x_l - y_l)$ are of class $\mathcal{K}_{1,2,1}(\partial\Omega \times \partial\Omega)$ that is contained in $\mathcal{K}_{1,1+\alpha,\alpha}(\partial\Omega \times \partial\Omega)$ (cf. Lemma 3.4). Hence the α-Hölder continuity of the components of ν and the product Proposition 3.7 (ii) imply that

$$\sum_{l=1}^{n} \nu_l(x) \left[\nu_l(x) \frac{x_h - y_h}{|x-y|^2} - \nu_h(x) \frac{x_l - y_l}{|x-y|^2} \right] \in \mathcal{K}_{1,1+\alpha,\alpha}(\partial\Omega \times \partial\Omega).$$

Theorem 3.6 (ii) implies that the product is continuous from

$$\mathcal{K}_{0,\alpha,\alpha}(\partial\Omega \times \partial\Omega) \times \mathcal{K}_{1,1+\alpha,\alpha}(\partial\Omega \times \partial\Omega) \quad \text{to} \quad \mathcal{K}_{1,1+\alpha,\alpha}(\partial\Omega \times \partial\Omega)$$

and thus the imbedding Proposition 3.8 (iii) implies that

$$DB_1(x-y)a^{(2)}\nu(y)\sum_{l=1}^{n}\nu_l(x)\left[\nu_l(x)\frac{x_h-y_h}{|x-y|^2}-\nu_h(x)\frac{x_l-y_l}{|x-y|^2}\right] \tag{4.23}$$
$$\in \mathcal{K}_{1,1+\alpha,\alpha}(\partial\Omega \times \partial\Omega) \subseteq \mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega \times \partial\Omega)\,.$$

We now consider the eighth addendum in the right hand side of formula (4.16). Since C is analytic, Lemma 4.13 (ii) implies that the kernel $\frac{\partial^2 C}{\partial x_l \partial x_j}(x-y)$ belongs to $\mathcal{K}_{0,1,1}(\partial\Omega \times \partial\Omega)$ that is contained in $\mathcal{K}_{0,\alpha,\alpha}(\partial\Omega \times \partial\Omega)$ for each $j,l \in \{1,\ldots,n\}$ (cf. Lemma 3.4). Then the α-Hölder continuity of the components of ν, the product Proposition 3.7 (ii), (iii) and the imbedding Proposition 3.8 (iii) imply that

$$\sum_{j,s=1}^{n}\sum_{l=1}^{n}\nu_l(x)\left[\nu_l(x)\frac{\partial^2 C}{\partial x_h \partial x_j}(x-y)-\nu_h(x)\frac{\partial^2 C}{\partial x_l \partial x_j}(x-y)\right]a_{js}\nu_s(y) \tag{4.24}$$
$$\in \mathcal{K}_{0,\alpha,\alpha}(\partial\Omega \times \partial\Omega) \subseteq \mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega \times \partial\Omega)\,.$$

We now consider the nineth addendum in the right hand side of formula (4.16). By reference [8, Rmk. 6.1] with Dondi the kernels $\frac{\partial S_{\mathbf{a}}}{\partial x_l}(x-y)$ belong to the class $\mathcal{K}_{n-1,n,1}(\partial\Omega \times \partial\Omega)$ that is contained in $\mathcal{K}_{n-1,n-1+\alpha,\alpha}(\partial\Omega \times \partial\Omega)$ for each $l \in \{1,\ldots,n\}$ (cf. Lemma 3.4). Hence the α-Hölder continuity of the components of ν, the product Proposition 3.7 (ii), (iii) and the imbedding Proposition 3.8 (iii) imply that

$$-\nu(y)^t \cdot a^{(1)}\sum_{l=1}^{n}\nu_l(x)\left[\nu_l(x)\frac{\partial S_{\mathbf{a}}}{\partial x_h}(x-y)-\nu_h(x)\frac{\partial S_{\mathbf{a}}}{\partial x_l}(x-y)\right] \tag{4.25}$$
$$\in \mathcal{K}_{n-1,n-1+\alpha,\alpha}(\partial\Omega \times \partial\Omega) \subseteq \mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega \times \partial\Omega)\,.$$

By the memberships of (4.17)–(4.25), we conclude that each addendum in the right hand side of formula (4.16) belongs to the class $\mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega \times \partial\Omega)$ and thus the proof is complete. □

5 Continuity properties of the double layer potential

As a consequence of Lemmas 4.10 and 4.15, we can apply Theorem 3.12 and prove the following classical result on the continuity of the double layer potential on the boundary (see Miranda [26, 15.VI], where the author mentions a result of Giraud [11]. For the Laplace operator in case $n = 2$ see Fichera and De Vito [9, LXXXIII]).

Theorem 5.1 *Let $n \in \mathbb{N} \setminus \{0,1\}$. Let $\mathbf{a}$ be as in (1.1), (1.2), (1.3). Let $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a}, D]$. Let $\alpha \in]0,1[$, $\beta \in]0,1]$, $\alpha + \beta > 1$.*

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold.

(i) If $\beta < 1$, then the operator $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, \cdot]$ from $C^{0,\beta}(\partial\Omega)$ to $C^{1,\alpha+\beta-1}(\partial\Omega)$ defined by (1.4) for all $\mu \in C^{0,\beta}(\partial\Omega)$ is linear and continuous.

(ii) If $\beta = 1$, then the operator $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, \cdot]$ from $C^{0,\beta}(\partial\Omega) = C^{0,1}(\partial\Omega)$ to $C^{1,\omega_{\alpha+\beta-1}}(\partial\Omega) = C^{1,\omega_\alpha}(\partial\Omega)$ defined by (1.4) for all $\mu \in C^{0,1}(\partial\Omega)$ is linear and continuous.

Proof. By formula (4.9), we have $\overline{B^*_{\Omega,y}}\,(S_{\mathbf{a}}(\cdot - y))) \in C^1((\partial\Omega) \setminus \{y\})$ for all $y \in \partial\Omega$. By Lemmas 4.10 and 4.15, we know that the kernel of the double layer potential belongs to $\mathcal{K}_{n-1-\alpha,n-\alpha,1}(\partial\Omega \times \partial\Omega)$ and that its tangential gradient with respect to the variable x belongs to $(\mathcal{K}_{n-\alpha,n,\alpha}(\partial\Omega \times \partial\Omega))^n$. We now plan to apply Theorem 3.12 (iii). We first note that reference [8, Thm 9.2] with Dondi implies that $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, 1] \in C^{1,\alpha}(\partial\Omega)$. Moreover,

$$\beta \le 1 \le n-1 < n-\alpha \equiv t_1 = (n-1) + (1-\alpha) < (n-1) + \beta\,,$$
$$t_2 \equiv n \ge (n-1) + \beta\,, \quad 0 \le s_1 \equiv (n-1) - \alpha < n-1\,.$$

(i) If $\beta < 1$, then $t_2 - \beta = n - \beta = (n-1) + 1 - \beta > n-1$,

$$\beta \le 2 \le t_2 = n < n + \alpha + \beta - 1 = (n-1) + \beta + t_3\,, \quad \text{where } t_3 \equiv \alpha\,,$$

and

$$\min\{\beta, (n-1) + \beta - t_1, (n-1) + t_3 + \beta - t_2\}$$
$$= \min\{\beta, (n-1) + \beta - (n-\alpha), (n-1) + \alpha + \beta - n\} = \alpha + \beta - 1 \le \alpha\,.$$

Then

$$W_\Omega[\mathbf{a}, S_{\mathbf{a}}, 1] \in C^{1,\alpha}(\partial\Omega)$$
$$\subseteq C^{1,\alpha+\beta-1}(\partial\Omega) = C^{1,\min\{\beta,(n-1)+\beta-t_1,(n-1)+t_3+\beta-t_2\}}(\partial\Omega)$$

and Theorem 3.12 (iii) (c) implies that $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, \cdot]$ is linear and continuous from $C^{0,\beta}(\partial\Omega)$ to

$$C^{1,\min\{\beta,(n-1)+\beta-t_1,(n-1)+t_3+\beta-t_2\}}(\partial\Omega) = C^{1,\alpha+\beta-1}(\partial\Omega)\,.$$

(ii) If $\beta = 1$, then $t_2 - \beta = n - \beta = n - 1$ and

$$C^{1,\max\{r^\beta, r^{(n-1)+\beta-t_1}, \omega_{t_3}(\cdot)\}}(\partial\Omega) = C^{1,\max\{r, r^\alpha, \omega_\alpha(\cdot)\}}(\partial\Omega) = C^{1,\omega_\alpha(\cdot)}(\partial\Omega)\,.$$

Then

$$W_\Omega[\mathbf{a}, S_{\mathbf{a}}, 1] \in C^{1,\alpha}(\partial\Omega) \subseteq C^{1,\omega_\alpha(\cdot)}(\partial\Omega) = C^{1,\max\{r^\beta, r^{(n-1)+\beta-t_1}, \omega_{t_3}(\cdot)\}}(\partial\Omega)$$

and Theorem 3.12 (iii) (cc) implies that $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, \cdot]$ is linear and continuous from $C^{0,\beta}(\partial\Omega) = C^{0,1}(\partial\Omega)$ to

$$C^{1,\max\{r^\beta, r^{(n-1)+\beta-t_1}, \omega_{t_3}(\cdot)\}}(\partial\Omega) = C^{1,\omega_\alpha(\cdot)}(\partial\Omega)$$

and thus the proof is complete. □

Next we introduce the following two technical statements in case $n = 2$.

Lemma 5.2 *Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^2$. Then*

$$c^{(v)}_\Omega \equiv \sup_{x\in\partial\Omega, s\in]0,1/e[} |s\log s|^{-1} \int_{(\partial\Omega)\cap\mathbb{B}_2(0,s)} |\log|x-y||\,d\sigma_y < +\infty\,.$$

Proof. By the Lemma of the uniform cylinders, there exist r, $\delta \in]0, 1/e[$ such that if $x \in \partial\Omega$, then there exist a 2×2 orthogonal matrix R_x such that

$$C(x, R_x, r, \delta) \equiv x + R_x^t(\mathbb{B}_{2-1}(0,r)\times]-\delta,\delta[)$$

is a coordinate cylinder for Ω around x, *i.e.*, there exists $\gamma_x \in C^{0,1}(\overline{\mathbb{B}_1(0,r)})$ such that

$$\begin{aligned} &R_x(\Omega - x) \cap (\mathbb{B}_{2-1}(0,r)\times]-\delta,\delta[) \qquad (5.3)\\ &\qquad = \{(\eta, y) \in \mathbb{B}_{2-1}(0,r)\times]-\delta,\delta[:\ y < \gamma_x(\eta)\} \equiv \text{hypograph}_s(\gamma_x)\,,\\ &\quad |\gamma_x(\eta)| < \delta/2 \qquad \forall \eta \in \mathbb{B}_{2-1}(0,r)\,, \qquad \gamma_x(0) = 0\,, \end{aligned}$$

and the corresponding function γ_x satisfies the inequality

$$A \equiv \sup_{x\in\partial\Omega} \|\gamma_x\|_{C^{0,1}(\overline{\mathbb{B}_1(0,r)})} < +\infty$$

(cf. [20, Defn. 10.1, Lem. 10.1]). By the continuity of the logarithm, it suffices to show that the supremum of the statement is finite with $s \in]0, r[$ and we note that $(\partial\Omega) \cap \mathbb{B}_2(x, s) \subseteq (\partial\Omega) \cap C(x, R_x, r, \delta)$ for all $s \in]0, r[$. Then we have

$$\int_{(\partial\Omega)\cap\mathbb{B}_2(x,s)} |\log|x-y||\,d\sigma_y$$

$$
\begin{aligned}
&\leq \int_{\{\eta\in]-r,r[:|\eta|^2+\gamma_x(\eta)^2<s^2\}} |\log|(\eta,\gamma_x(\eta))||\, d\eta \sqrt{1+\operatorname{ess\,sup}|\gamma_x'|^2} \\
&\leq \int_{\{\eta\in]-r,r[:|\eta|<s\}} |\log|\eta||\, d\eta \sqrt{1+A^2} \leq 2\left[\eta-\eta\log\eta\right]_{\eta=0^+}^{\eta=s} \sqrt{1+A^2} \\
&\leq 4|s\log s|\sqrt{1+A^2} \qquad \forall x\in\partial\Omega,\, s\in]0,1/e[\,.
\end{aligned}
$$

□

Proposition 5.4 *Let $n=2$. Let $\mathbf{a}$ be as in (1.1), (1.2). Let $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a},D]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^2$. Let $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a},D]$. Then $v_\Omega[S_{\mathbf{a}},\cdot]$ is continuous from $L^\infty(\partial\Omega)$ to $C^{0,\omega_1(\cdot)}(\partial\Omega)$.*

Proof. By Theorem 7.2 of [8] we already know that $v_\Omega[S_{\mathbf{a}},\cdot]$ is continuous from $L^\infty(\partial\Omega)$ to $C^0(\partial\Omega)$. We now take $\mu\in L^\infty(\partial\Omega)$ and we turn to estimate the Hölder constant of $v_\Omega[S_{\mathbf{a}},\mu]$. By formula (4.4) above, by the inequality $|T^{-1}x|\geq|T|^{-1}|x|$ for $x\in\mathbb{R}^2\setminus\{0\}$ and by Lemma 4.2 (ii) of [8], there exists a constant $c\in]0,+\infty[$ such that

$$
\begin{aligned}
&|\log|\xi||^{-1}|S_{\mathbf{a}}(\xi)|\leq c \quad \forall\xi\in\mathbb{B}_2(0,1/e)\setminus\{0\}\,, \\
&\frac{|x'-y|}{|x'-x''|}|S_{\mathbf{a}}(x'-y)-S_{\mathbf{a}}(x''-y)|\leq c \\
&\qquad\qquad \forall x',x''\in\partial\Omega, x'\neq x'', y\in(\partial\Omega)\setminus\mathbb{B}_n(x',2|x'-x''|)\,.
\end{aligned}
$$

Let $x',x''\in\partial\Omega$, $x'\neq x''$. By Remark 2.3, there is no loss of generality in assuming that $0<3|x'-x''|\leq 1/e$. Then the inclusion $\mathbb{B}_2(x',2|x'-x''|)\subseteq\mathbb{B}_2(x'',3|x'-x''|)$ and the triangular inequality imply that

$$
\begin{aligned}
&|v_\Omega[S_{\mathbf{a}},\mu](x')-v_\Omega[S_{\mathbf{a}},\mu](x'')| \\
&\qquad \leq \|\mu\|_{L^\infty(\partial\Omega)}\Bigg\{\int_{\mathbb{B}_2(x',2|x'-x''|)\cap\partial\Omega}|S_{\mathbf{a}}(x'-y)|\,d\sigma_y \\
&\qquad\quad +\int_{\mathbb{B}_2(x'',3|x'-x''|)\cap\partial\Omega}|S_{\mathbf{a}}(x''-y)|\,d\sigma_y \\
&\qquad\quad +\int_{\partial\Omega\setminus\mathbb{B}_2(x',2|x'-x''|)}|\,S_{\mathbf{a}}(x'-y)-S_{\mathbf{a}}(x''-y)\,|\,d\sigma_y\Bigg\}\,.
\end{aligned} \tag{5.5}
$$

Then Lemma 5.2 implies that

$$
\int_{\mathbb{B}_2(x',2|x'-x''|)\cap\partial\Omega}|S_{\mathbf{a}}(x'-y)|\,d\sigma_y \tag{5.6}
$$

$$
\begin{aligned}
&\qquad + \int_{\mathbb{B}_2(x'',3|x'-x''|)\cap\partial\Omega} |S_{\mathbf{a}}(x''-y)|\, d\sigma_y \\
&\leq c\Big\{ \int_{\mathbb{B}_2(x',2|x'-x''|)\cap\partial\Omega} |\log|x'-y||d\sigma_y \\
&\qquad + \int_{\mathbb{B}_2(x'',3|x'-x''|)\cap\partial\Omega} |\log|x''-y||d\sigma_y \Big\} \\
&\leq c2c_{\Omega}^{(v)}3|x'-x''||\log(3|x'-x''|)| \\
&\leq 6cc_{\Omega}^{(v)}|x'-x''|(|\log 3|+|\log|x'-x''||) \\
&\leq 6cc_{\Omega}^{(v)}|\log 3|2|x'-x''||\log|x'-x''||\,.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\int_{\partial\Omega\setminus\mathbb{B}_2(x',2|x'-x''|)} &|\, S_{\mathbf{a}}(x'-y) - S_{\mathbf{a}}(x''-y)\,|\, d\sigma_y \\
&\leq c \int_{\partial\Omega\setminus\mathbb{B}_2(x',2|x'-x''|)} \frac{|x'-x''|}{|x'-y|}\, d\sigma_y
\end{aligned}
\tag{5.7}
$$

Then Lemma 3.5 (iv) of [8] implies that there exists $c_{\Omega}^{iv} \in]0,+\infty[$ such that

$$
\int_{\partial\Omega\setminus\mathbb{B}_2(x',2|x'-x''|)} \frac{d\sigma_y}{|x'-y|} \leq c_{\Omega}^{iv}|\log|x'-x''||
$$

for all $x', x'' \in \partial\Omega$, $0 < |x'-x''| \leq 1/e$. Hence, the statement holds true. □

Next we prove a regularity statement for the double layer potential of a constant function. To do so, we need to exploit the tangential derivatives of a function defined on the boundary of an open set of class C^1. If $l, r \in \{1, \ldots, n\}$, then M_{lr} denotes the tangential derivative operator from $C^1(\partial\Omega)$ to $C^0(\partial\Omega)$ that takes f to

$$
M_{lr}[f] \equiv \nu_l \frac{\partial \tilde{f}}{\partial x_r} - \nu_r \frac{\partial \tilde{f}}{\partial x_l} \qquad \text{on } \partial\Omega\,, \tag{5.8}
$$

where $\tilde{f}$ is any continuously differentiable extension of f to an open neighborhood of $\partial\Omega$. We note that $M_{lr}[f]$ is independent of the specific choice of $\tilde{f}$ (cf. *e.g.*, reference [7, §2.21] with Dalla Riva and Musolino). Then we can state the following.

Lemma 5.9 *Let $n \in \mathbb{N}\setminus\{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,1}$. Let $\mathbf{a}$ be as in (1.1), (1.2), (1.3). Let $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a}, D]$. Then $W_{\Omega}[\mathbf{a}, S_{\mathbf{a}}, 1] \in C^{1,\omega_1(\cdot)}(\partial\Omega)$.*

Proof. By reference [8, Thm. 9.1] with Dondi, we know that $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, 1] \in C^1(\partial\Omega)$ and that the tangential derivatives of $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, 1]$ are delivered by the following formula.

$$
\begin{aligned}
&M_{lj}[W_\Omega[\mathbf{a}, S_{\mathbf{a}}, 1]] \qquad (5.10)\\
&\quad = \nu_l Q_j\left[\nu \cdot a^{(1)}, 1\right] - \nu_j Q_l\left[\nu \cdot a^{(1)}, 1\right]\\
&\qquad + \nu \cdot a^{(1)}\left\{Q_l[\nu_j, 1] - Q_j[\nu_l, 1]\right\} + R[\nu_l, \nu_j, 1] \qquad \text{on } \partial\Omega\,,
\end{aligned}
$$

where

$$
Q_j[g, \mu](x) = \int_{\partial\Omega} (g(x) - g(y)) \frac{\partial S_{\mathbf{a}}}{\partial x_j}(x - y)\mu(y)\, d\sigma_y \quad \forall x \in \partial\Omega\,,
$$

for all $(g, \mu) \in C^{0,1}(\partial\Omega) \times L^\infty(\partial\Omega)$ and

$$
\begin{aligned}
R[\nu_l, \nu_j, 1] &\equiv \sum_{r=1} a_r \left\{Q_r[\nu_l\nu_j, 1] - \nu_l Q_r[\nu_j, 1] - Q_r[\nu_j, \nu_l]\right\}\\
&\qquad + a\left\{\nu_l v_\Omega[S_{\mathbf{a}}, \nu_j] - \nu_j v_\Omega[S_{\mathbf{a}}, \nu_l]\right\} \qquad \text{on } \partial\Omega\,,\\
v_\Omega[S_{\mathbf{a}}, \nu_j](x) &\equiv \int_{\partial\Omega} S_{\mathbf{a}}(x - y)\nu_j(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n
\end{aligned}
$$

for all $l, j \in \{1, \ldots, n\}$. By the Lipschitz continuity of the components of ν, Proposition 5.4 above and Theorem 7.2 of [8] imply that $v_\Omega[S_{\mathbf{a}}, \nu_j]$ belongs to $C^{0,\omega_1(\cdot)}(\partial\Omega)$. By the Lipschitz continuity of the components of ν, Theorem 8.2 (i) of [8] implies that $Q_r[\nu_l\nu_j, 1]$, $Q_r[\nu_j, 1]$, $Q_j\left[\nu \cdot a^{(1)}, 1\right]$, $Q_r[\nu_j, \nu_l]$, belong to $C^{0,\omega_1(\cdot)}(\partial\Omega)$ for all j, l, $r \in \{1, \ldots, n\}$. Hence, the tangential derivatives $M_{lj}[W_\Omega[\mathbf{a}, S_{\mathbf{a}}, 1]]$ belong to $C^{0,\omega_1(\cdot)}(\partial\Omega)$ for all j, $l \in \{1, \ldots, n\}$, and accordingly $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, 1]$ belongs to $C^{1,\omega_1(\cdot)}(\partial\Omega)$ (cf. *e.g.*, [8, Lem. 2.2]). □

As a consequence of Lemmas 4.10, 4.15, 5.9, we can apply Theorem 3.12 and prove the following theorem on the continuity of the double layer potential on the boundary.

Theorem 5.11 *Let $\beta \in]0, 1]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,1}$. Assume that the following condition holds*

$$
\sup_{x \in \partial\Omega} \sup_{r \in]0,+\infty[} \left| \int_{(\partial\Omega)\setminus\mathbb{B}_n(x,r)} \mathrm{grad}_{\partial\Omega,x} \overline{B^*_{\Omega,y}}\, (S_{\mathbf{a}}(x - y))\, d\sigma_y \right| < +\infty\,, \qquad (5.12)
$$

i.e., the maximal function of the tangential gradient of the kernel of the double layer potential with respect to its first variable is bounded.

Let $\mathbf{a}$ *be as in (1.1), (1.2), (1.3). Let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$*. Then the following statements hold.*

(i) *If* $\beta < 1$*, then the operator* $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, \cdot]$ *from* $C^{0,\beta}(\partial\Omega)$ *to* $C^{1,\beta}(\partial\Omega)$ *defined by (1.4) for all* $\mu \in C^{0,\beta}(\partial\Omega)$ *is linear and continuous.*

(ii) *If* $\beta = 1$*, then the operator* $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, \cdot]$ *from* $C^{0,1}(\partial\Omega)$ *to* $C^{1,\omega_1(\cdot)}(\partial\Omega)$ *defined by (1.4) for all* $\mu \in C^{0,1}(\partial\Omega)$ *is linear and continuous.*

Proof. By formula (4.9), we have $\overline{B^*_{\Omega,y}}\,(S_{\mathbf{a}}(\cdot - y))) \in C^1((\partial\Omega) \setminus \{y\})$ for all $y \in \partial\Omega$. If $n = 2$, we choose $\epsilon \in]0, 1[$ and Lemma 4.10 (ii), (iii) implies that the kernel of the double layer potential belongs to $\mathcal{K}_{\epsilon,1,1}(\partial\Omega \times \partial\Omega)$. Then the imbedding Proposition 3.8 (ii) implies that $\mathcal{K}_{\epsilon,1,1}(\partial\Omega \times \partial\Omega)$ is contained in $\mathcal{K}_{\epsilon,1+\epsilon,1}(\partial\Omega \times \partial\Omega)$.

If $n \geq 3$ Lemma 4.10 (i), (iii) implies that the kernel of the double layer potential belongs to the class $\mathcal{K}_{n-2,n-1,1}(\partial\Omega \times \partial\Omega)$.

Then if $n \geq 2$ Lemma 4.15 and condition (5.12) imply that the tangential gradient with respect to the variable x of the kernel of the double layer potential belongs to the class $(\mathcal{K}^\sharp_{n-1,n,1}(\partial\Omega \times \partial\Omega))^n$. We now plan to apply Theorem 3.12 (ii). By Lemma 5.9, we have

$$W_\Omega[\mathbf{a}, S_{\mathbf{a}}, 1] \in C^{1,\omega_1(\cdot)}(\partial\Omega) \subseteq C^{1,\alpha}(\partial\Omega) \qquad \forall \alpha \in]0, 1[\,.$$

Moreover,

$$\beta \leq 1 \leq n - 1 \equiv t_1 < (n - 1) + \beta\,,$$
$$t_2 \equiv n \geq 2 > \beta\,, \quad s_1 \equiv \begin{cases} \epsilon < 2 - 1 = n - 1 & \text{if } n = 2\,, \\ (n - 1) - 1 < n - 1 & \text{if } n \geq 3\,. \end{cases}$$

(i) If $\beta < 1$, then

$$t_2-\beta = n-\beta > n-1\,, \quad t_2 = n < (n-1)+\beta+1 = (n-1)+\beta+t_3 \text{ where } t_3 \equiv 1\,.$$

and $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, 1] \in C^{1,\omega_1(\cdot)}(\partial\Omega) \subseteq C^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(\partial\Omega)$. Thus Theorem 3.12 (ii) (b) implies that $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, \cdot]$ is linear and continuous from $C^{0,\beta}(\partial\Omega)$ to

$$C^{1,\min\{\beta,(n-1)+t_3+\beta-t_2\}}(\partial\Omega) = C^{1,\min\{\beta,(n-1)+1+\beta-n)\}}(\partial\Omega) = C^{1,\beta}(\partial\Omega)\,.$$

(ii) If $\beta = 1$, then $t_2 - \beta = n - \beta = n - 1$ and $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, 1] \in C^{1,\omega_1(\cdot)}(\partial\Omega) \subseteq C^{1,\max\{r^\beta,\omega_1(r)\}}(\partial\Omega)$. Thus Theorem 3.12 (ii) (bb) implies that $W_\Omega[\mathbf{a}, S_{\mathbf{a}}, \cdot]$ is linear and continuous from $C^{0,\beta}(\partial\Omega) = C^{0,1}(\partial\Omega)$ to

$$C^{1,\max\{r^\beta,\omega_1(r)\}}(\partial\Omega) = C^{1,\max\{r^1,\omega_1(r)\}}(\partial\Omega) = C^{1,\omega_1(\cdot)}(\partial\Omega)\,.$$

and thus the proof is complete. □

For the validity of condition (5.12), we refer to [22].

6 Backmatter

Funding and/or Conflicts of interests/Competing interests

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