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# Direct sum decompositions of modules, almost trace ideals, and pullbacks of monoids

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**Abstract.** We show that a number of pullback diagrams appear naturally in the study of pre-ordered Grothendieck groups. The passage of projective modules from a ring $R$ to a factor ring $R/I$ turns out to be particularly good for a certain class of ideals, which we call almost trace ideals. We generalize to arbitrary rings a result by Goodearl concerning the lattice of the directed convex subgroups of $K_0(R)$. Finally, we show that a variant $K_0'(I)$ of the Grothendieck group of $I$, introduced by Quillen, has an easy description in terms of projective modules when $I$ is an almost trace ideal.

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## 1 Introduction

In this paper we show that pullbacks of monoids appear very frequently in the study of the pre-ordered stucture of the Grothendieck group $K_0(R)$. For instance, let $R$ be a ring with Jacobson radical $J(R)$ and $V(R)$ the monoid of finitely generated projective $R$-modules up to isomorphism, with the operation induced by direct sum. The Grothendieck group $K_0(R)$ is the universal enveloping group of $V(R)$, and the pre-order on $K_0(R)$ has the image of the universal mapping $\psi_R : V(R) \to K_0(R)$ as its positive cone. We prove that the canonical projection $p : R \to R/J(R)$ induces a pullback diagram

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$$\begin{array}{ccc} V(R) & \xrightarrow{V(p)} & V(R/J(R)) \\ {\scriptstyle \psi_R}\downarrow & & \downarrow{\scriptstyle \psi_{R/J(R)}} \\ K_0(R) & \xrightarrow{K_0(p)} & K_0(R/J(R)). \end{array}$$

We show that a lot of other pullback diagrams appear naturally in pre-ordered Grothendieck groups.

Trace induces a one-to-one correspondence between prime ideals $P$ of the commutative monoid $V(R)$ and trace ideals $I$ of the ring $R$, so that operations in the monoids $V(R)$, like localization $V(R)_P$ with respect to a prime ideal $P$ or reduced localization $(V(R)_P)_{\mathrm{red}}$, can be interpreted in terms of projective $R/I$-modules [8, Theorem 2.2]. Every ideal $I$ of $R$ contains a greatest trace ideal $\mathrm{Tr}(I)$, and we say that a two-sided ideal $I$ is an *almost trace ideal* if $I/\mathrm{Tr}(I)$ is contained in the Jacobson radical of $R/\mathrm{Tr}(I)$. For an almost trace ideal $I$ of $R$, the natural map $V(R) \to V(R/I)$ factors through $(V(R)_P)_{\mathrm{red}}$, where $P$ corresponds to $\mathrm{Tr}(I)$ in the above mentioned bijection, and the corresponding map $(V(R)_P)_{\mathrm{red}} \to V(R/I)$ is injective and induces a pull-back as above (Proposition 5.3; also see Proposition 5.7). Injective monoid homomorphisms that induce a pullback yield order-embeddings of Grothendieck groups (Lemma 6.1).

Using our results, we show that, for any almost trace ideal $I$, the natural map $K_0(R) \to K_0(R/I)$ factors through an order-embedding $G((V(R)_P)_{\mathrm{red}}) \to K_0(R/I)$. Moreover, we get an exact sequence of groups

$$G(V(I)) \to K_0(R) \to G((V(R)_P)_{\mathrm{red}}) \to 0$$

where $V(I)$ is the monoid defined by the finitely generated projective $R$-modules $A_R$ such that $A = AI$. In Section 7, the group $G(V(I))$ is compared with both the Grothendieck group $K_0(I)$ [19] and its variant $K_0'(I)$ defined by Quillen in [17]. We show that the groups $K_0'(I)$ and $G(V(I))$ are canonically isomorphic for every almost trace ideal $I$ and that $K_0(I)$ and $G(V(I))$ are isomorphic when $I$ is a trace ideal.

We also apply pullbacks of monoids and almost trace ideals to describe directed subgroups of $K_0(R)$. Generalizing a result of Goodearl [11, Theorem 15.20], we show that there is an order-preserving one-to-one correspondence between the set of directed convex subgroups of $K_0(R)$ and a suitable set of trace ideals of $R$.

## 2 Notations and preliminaries

All rings of this paper are assumed to be associative rings with identity, and all modules are right unital modules. All our monoids are commutative additive monoids, that is, commutative additive semigroups with a zero element. Let $M$ be a (commutative additive) monoid. We denote by $U(M)$ the subgroup of $M$ consisting of all elements $a \in M$ with an additive inverse $-a$ in $M$, and we call $M$ *reduced* if $U(M) = \{0\}$. For a monoid $M$, we denote by $M_{\mathrm{red}}$ the factor monoid $M/U(M)$, whose elements are all cosets $x + U(M)$ with $x \in M$. The monoid $M_{\mathrm{red}}$ is obviously a reduced monoid.

Let $R$ be a ring with identity. We shall denote by proj-$R$ both the class of all finitely generated projective right $R$-modules and the full subcategory of Mod-$R$ whose objects are all finitely generated projective right $R$-modules. For every ring $R$ under consideration, we fix a set $V(R)$ of representatives up to isomorphism of the finitely generated projective right $R$-modules. Thus it is possible to associate to every module $A_R \in$ proj-$R$ a unique module $\langle A_R \rangle \in V(R)$ isomorphic to $A_R$, and for all $A_R, B_R \in$ proj-$R$, $A_R \cong B_R$ if and only if $\langle A_R \rangle = \langle B_R \rangle$. The set $V(R)$ is a commutative monoid under the addition defined by $\langle A_R \rangle + \langle B_R \rangle = \langle A_R \oplus B_R \rangle$ for all $\langle A_R \rangle, \langle B_R \rangle \in V(R)$. The monoid $V(R)$ can also be viewed in an informal way as the monoid whose elements are the *isomorphism classes* $\langle A_R \rangle$ of the modules $A_R \in$ proj-$R$, but if we look at $V(R)$ in this way, the elements $\langle A_R \rangle$ of $V(R)$ are not sets, and thus $V(R)$ is not a set by Zermelo's Sum Axiom (Union Axiom) of General Set Theory ("for any set $S$ there exists the set whose elements are the elements of the elements of $S$"). To avoid this set theoretical difficulty, we have preferred not to introduce $V(R)$ as the class whose elements are the isomorphism classes of finitely generated projective modules, but to think of $V(R)$ as a fixed set of representatives of proj-$R$ up to isomorphism.

Any ring homomorphism $f : R \to S$ induces a monoid homomorphism $V(f) : V(R) \to V(S)$ defined by $V(f) : \langle A_R \rangle \mapsto \langle A \otimes_R S \rangle$ for all $\langle A_R \rangle \in V(R)$. Thus $V(-)$ is a functor from the category of rings with identity to the category of commutative monoids.

Clearly, the monoid $V(R)$ describes all direct sum decompositions of finitely generated projective right $R$-modules up to isomorphism, in the sense that to every decomposition of a projective module $A_R \in$ proj-$R$ as a direct sum of finitely many submodules there corresponds a decomposition of the element $\langle A_R \rangle$ of the monoid $V(R)$ as a sum of elements of $V(R)$, and two direct sum decompositions of $A_R$ are isomorphic in the sense of the Krull-Schmidt theorem if and only if they correspond to the same sum decomposition of $\langle A_R \rangle$ in the monoid $V(R)$, up to the order of the summands.

In particular, if we want to describe all direct sum decompositions of the module $R_R$, the convenient structure to consider over $V(R)$ is the structure of commutative monoid with order-unit, which is defined as follows. Recall that there is a natural pre-order (= reflexive and transitive relation) on any commutative additive monoid $M$, defined by $x \leq y$ if there exists $z \in M$ such that $x + z = y$. We shall call this pre-order $\leq$ on $M$ the *algebraic pre-order* on $M$. An element $u$ of $M$ is an *order-unit* if for every $x \in M$ there exists a positive integer $n$ such that $x \leq nu$. The category of commutative monoids with order-unit is defined as follows. Its objects are the pairs $(M, u)$, where $M$ is a commutative monoid and $u \in M$ is an order-unit. The morphisms $\varphi : (M, u) \to (M', u')$ are the monoid homomorphisms $\varphi : M \to M'$ such that $\varphi(u) = u'$. For instance, for every ring $R$ the element $\langle R_R \rangle$ is an order-unit in the monoid $V(R)$, and there is a functor from the category of rings with identity to the category of commutative monoids with order-unit that associates to each ring $R$ the commutative monoid with order-unit $(V(R), \langle R_R \rangle)$. The monoid with order-unit $(V(R), \langle R_R \rangle)$ describes all direct sum decompositions of the module $R_R$ up to isomorphism.

More generally, let $M_S$ be a right module over an arbitrary ring $S$, let add$(M_S)$ be the full subcategory of Mod-$S$ whose objects are all modules isomorphic to direct summands of direct sums $M^n$ of finitely many copies of $M$, and let $R = \mathrm{End}(M_S)$. The functors $\mathrm{Hom}_S(M_S, -) : \text{Mod-}S \to \text{Mod-}R$ and $- \otimes_R M_S : \text{Mod-}R \to \text{Mod-}S$ induce a category equivalence between the categories $\mathrm{add}(M_S)$ and proj-$R$ [7, Theorem 4.7]. Under this equivalence, $M_S$ corresponds to $R_R$, and direct sum decompositions of $M_S$ correspond to direct sum decompositions of $R_R$. As in the case of $V(R)$, we fix a set $V(\mathrm{add}(M_S))$ of representatives of the modules in $\mathrm{add}(M_S)$ up to isomorphism. Then $V(\mathrm{add}(M_S))$ becomes a commutative monoid with order-unit $\langle M_S \rangle$ isomorphic to the monoid with order-unit $(V(R), \langle R_R \rangle)$. Thus all direct sum decompositions of a module $M_S$ are described by the monoid with order-unit $(V(R), \langle R_R \rangle)$ for $R$ the endomorphism ring $\mathrm{End}(M_S)$ of $M_S$.

A submonoid $M'$ of a monoid $M$ is said to be *divisor-closed* if $x \in M$, $y \in M'$ and $x \le y$ in $M$ implies $x \in M'$. Divisor-closed submonoids of $M$ have also been called *order-ideals* (or *o-ideals*) in a number of places (e.g. [4]). A *prime ideal* of a monoid $M$ is a proper subset $P$ of $M$ such that $M \setminus P$ is a divisor-closed submonoid; that is, for any $x, y \in M$ one has $x + y \in P$ if and only if either $x \in P$ or $y \in P$. It is easy to see that the prime ideals of a commutative monoid $M$ are exactly the subsets $P$ of $M$ for which there exists a homomorphism $\varphi$ of $M$ into a reduced monoid $N$ with $P = \{x \in M \mid \varphi(x) \ne 0\}$. Equivalently, the prime ideals of a monoid $M$ are exactly the subsets $P$ of $M$ for which there exists a congruence $\sim$ on $M$ with $M/\sim$ reduced and $P = \{x \in M \mid x \not\sim 0\}$. Notice that the union of any family of prime ideals of a commutative monoid $M$ is a prime ideal, so that the set $\mathrm{Spec}(M)$ of all prime ideals of $M$, partially ordered by set inclusion, is a complete lattice whose greatest element is the prime ideal $M \setminus U(M)$ and whose least element is the empty ideal $\emptyset$. By passing to the complements, we can consider the complete lattice $Dc(M)$ of all divisor-closed submonoids of $M$. In this lattice $Dc(M)$, the greatest element is $M$ and the least element is $U(M)$. If $P$ is a prime ideal of $M$, then the *localization* $M_P$ of $M$ at $P$ is the monoid whose elements are all formal differences $x - s$ with $x \in M$ and $s \in M \setminus P$, and in which we define $x - s = x' - s'$, for all $x, x' \in M$ and $s, s' \in M \setminus P$, if and only if there exists $t \in M \setminus P$ such that $x + s' + t = x' + s + t$ [13, §4]. The monoid $(M_P)_{\mathrm{red}} = M_P/U(M_P)$ is called the *reduced localization* of $M$ at $P$. If $x, x' \in M$ and $s, s' \in M \setminus P$, then $x - s + U(M_P) = x' - s' + U(M_P)$ in $(M_P)_{\mathrm{red}}$ if and only if there exist elements $t, t' \in M \setminus P$ such that $x + t = x' + t'$. Notice that the canonical homomorphism $\varphi : M \to (M_P)_{\mathrm{red}}$, defined by $x \mapsto x - 0 + U(M_P)$, is surjective. The reduced localization $(M_P)_{\mathrm{red}}$ was denoted by $M/S$, where $S$ is the complement of $P$ in $M$, in the paper [4].

## 3 The Grothendieck group

The localization $M_\emptyset$ of $M$ at its empty prime ideal $\emptyset$ is an abelian group, which is usually called the *Grothendieck group* of $M$, or the *group of differences* of $M$, and denoted by $G(M)$. There is a canonical monoid homomorphism $\psi_M : M \to G(M)$, and $G(-)$ turns out to be a functor of the category of commutative monoids into the category of abelian groups. The Grothendieck group $G(V(R))$ is usually denoted $K_0(R)$. For any projective $R$-module $A_R$, we shall denote by $[A_R]$ the image of $\langle A_R \rangle$

via the homomorphism $\psi_{V(R)} : V(R) \to K_0(R)$, and usually write $\psi_R$ instead of $\psi_{V(R)}$. For every $A_R, B_R \in \text{proj-}R$, we have $[A_R] = [B_R]$ if and only if $A_R$ and $B_R$ are *stably isomorphic*, that is, $A_R \oplus R_R^n \cong B_R \oplus R_R^n$ for some integer $n \geq 0$. Then $K_0(R) = \{[A_R] - [B_R] \mid A_R, B_R \in \text{proj-}R\}$. Any ring homomorphism $f : R \to S$ induces an abelian group homomorphism $K_0(f) : K_0(R) \to K_0(S)$ defined by $K_0(f) : [A_R] - [B_R] \mapsto [A \otimes_R S] - [B \otimes_R S]$. Thus $K_0(-)$ turns out to be a functor from the category of rings with identity to the category of abelian groups.

There is a one-to-one correspondence between the submonoids $M$ of an abelian group $G$ and the translation-invariant pre-orders $\leq$ on $G$ [10]. If $G$ is an abelian group with a translation-invariant pre-order $\leq$, an *order-unit* in $G$ is an element $u \in G$, $u \geq 0$, such that for every $x \in G$ there exists a positive integer $n$ with $x \leq nu$. For any ring $R$, there is a canonical translation-invariant pre-order on $K_0(R)$ whose positive cone is $K_0(R)^+ = \{[A_R] \mid A_R \in \text{proj-}R\}$. Relatively to this pre-order, $[R_R]$ turns out to be an order-unit in $K_0(R)$. If $F$ is the forgetful functor from the category of abelian groups to the category of commutative monoids, then both $V(-)$ and $F \circ K_0(-)$ are functors from rings to commutative monoids. For every ring homomorphism $f : R \to S$ the diagram

$$\begin{array}{ccc} V(R) & \xrightarrow{V(f)} & V(S) \\ \psi_R \downarrow & & \downarrow \psi_S \\ K_0(R) & \xrightarrow{K_0(f)} & K_0(S) \end{array}$$

is commutative. Thus the homomorphisms $\psi_R : V(R) \to K_0(R)$ define a natural transformation from the functor $V(-)$ to the functor $F \circ K_0(-)$.

We have already remarked that the monoid $V(R)$ is the algebraic object that describes the direct sum decompositions of the finitely generated projective $R$-modules, and that the monoid with order-unit $(V(R), \langle R_R \rangle)$ describes the direct sum decompositions of the $R$-module $R_R$ or, more generally, the direct sum decompositions of any right module $M_S$ with $R \cong \text{End}(M_S)$. The abelian group $K_0(R)$ does not have a similar property. For instance, when the monoid $V(R)$ is not cancellative, most information about direct sum decompositions is lost in the passage from $V(R)$ to $K_0(R)$. Even when $V(R)$ is cancellative, the monoid $V(R)$ contains information that is lost in $K_0(R)$. For instance, if $R$ is semilocal, then $V(R)$ can be any finitely generated Krull monoid, while $K_0(R)$ is a free abelian group, that is, $K_0(R)$ is isomorphic to $\mathbb{Z}^n$ for some $n$, and this is not sufficient to faithfully describe the wealth of behaviors that direct sum decompositions of finitely generated projective modules can have in this case [9]. To remedy this difficulty, it is necessary to consider not only the abelian group structure on $K_0(R)$, but also its structure of pre-ordered abelian group. This is because the category of commutative cancellative monoids is equivalent to the category of directed pre-ordered abelian groups, and the category of commutative cancellative monoids with order-unit is equivalent to the category of directed pre-ordered abelian groups with order-unit. (Recall that a pre-ordered abelian group $G$ is a *directed group* in case $G = G^+ - G^+$.)

Our first result describes the relation between the monoid $V(R)$, the abelian group

$K_0(R)$, and the reduction modulo the Jacobson radical $J(R)$. The relation is in terms of pullbacks. Notice that if

$$\begin{array}{ccc} & & M' \\ & & \downarrow{\scriptstyle \psi} \\ M & \xrightarrow{\varphi} & M'' \end{array} \tag{1}$$

are homomorphisms of commutative monoids (or of commutative monoids with order-unit), then the pullbacks of diagram (1) in the category of sets and in the category of monoids (or in the category of commutative monoids with order-unit) coincide. They are in one-to-one correspondence with the subset (isomorphic to the submonoid, submonoid with order-unit) of the product $M \times M'$ whose elements are all the pairs $(x, x')$ with $x \in M$, $x' \in M'$ and $\varphi(x) = \psi(x')$.

**Theorem 3.1.** *Let $R$ be a ring, $J(R)$ its Jacobson radical, and $p : R \to R/J(R)$ the canonical projection. Then the commutative diagram*

$$\begin{array}{ccc} V(R) & \xrightarrow{V(p)} & V(R/J(R)) \\ {\scriptstyle \psi_R}\downarrow & & \downarrow{\scriptstyle \psi_{R/J(R)}} \\ K_0(R) & \xrightarrow{K_0(p)} & K_0(R/J(R)) \end{array} \tag{2}$$

*is a pullback of monoids.*

*Proof.* We must prove that, for every pair $([A_R] - [B_R], \langle C_{R/J(R)} \rangle)$ of elements $[A_R] - [B_R] \in K_0(R)$ and $\langle C_{R/J(R)} \rangle \in V(R/J(R))$ with $[A/AJ(R)] - [B/BJ(R)] = [C_{R/J(R)}]$ in $K_0(R/J(R))$, there exists a unique $\langle X_R \rangle \in V(R)$ such that $[X_R] = [A_R] - [B_R]$ and $\langle X/XJ(R) \rangle = \langle C_{R/J(R)} \rangle$. From $[A/AJ(R)] - [B/BJ(R)] = [C_{R/J(R)}]$, we obtain that there exists $n \geq 0$ such that $A/AJ(R) \oplus (R/J(R))^n \cong B/BJ(R) \oplus C_{R/J(R)} \oplus (R/J(R))^n$. Replacing $A$ with $A \oplus R^n$ and $B$ with $B \oplus R^n$, we can assume that $A/AJ(R) \cong B/BJ(R) \oplus C_{R/J(R)}$.

Let $\alpha : A/AJ(R) \to B/BJ(R) \oplus C_{R/J(R)}$ be an isomorphism and $p_A : A_R \to A/AJ(R)$, $p_B : B_R \to B/BJ(R)$ and $\bar{p} : B/BJ(R) \oplus C_{R/J(R)} \to B/BJ(R)$ the canonical projections. As $A_R$ is projective, the mapping $\bar{p}\alpha p_A : A_R \to B/BJ(R)$ factors through $p_B$, that is, there exists $h : A_R \to B_R$ such that $p_B h = \bar{p}\alpha p_A$. In particular, $p_B h$ is epic. As the kernel of $p_B$ is superfluous, it follows that $h$ also is epic. Thus $h$ splits, that is, there is a direct sum decomposition $A_R = X_R \oplus E_R$ such that the restriction of $h$ to $X_R$ is zero and the restriction of $h$ to $E_R$ is an isomorphism of $E_R$ onto $B_R$. Notice that $A_R = X_R \oplus E_R \cong X_R \oplus B_R$, so that $[X_R] = [A_R] - [B_R]$. If $\bar{h} : A/AJ(R) \to B/BJ(R)$ denotes the mapping induced by $h$ modulo the Jacobson radical, the identity $p_B h = \bar{p}\alpha p_A$ implies that $\bar{h} = \bar{p}\alpha$. Moreover, there is a direct sum decomposition $A/AJ(R) = X/XJ(R) \oplus E/EJ(R)$ such that the restriction of $\bar{h}$ to $X/XJ(R)$ is zero and the restriction of $\bar{h}$ to $E/EJ(R)$ is an isomorphism of $E/EJ(R)$

onto $B/BJ(R)$. Thus $\ker \bar{h} = X/XJ(R)$. As $\ker(\bar{p}\alpha) \cong C_{R/J(R)}$, it follows that $X/XJ(R) \cong C_{R/J(R)}$. Hence $X_R$ is a projective $R$-module with the desired properties.

In order to prove that $X_R$ is the *unique* finitely generated projective $R$-module with the required properties up to isomorphism, notice that the condition $\langle X/XJ(R)\rangle = \langle C_{R/J(R)}\rangle$ implies that $X_R$ must be isomorphic to the projective cover of $C_{R/J(R)}$ viewed as an $R$-module. The uniqueness of $X_R$ now follows from the uniqueness of projective covers. □

Thus, by Theorem 3.1, the monoid homomorphism $\psi_{R/J(R)} : V(R/J(R)) \to K_0(R/J(R))$ and the abelian group homomorphism $K_0(p) : K_0(R) \to K_0(R/J(R))$ completely determine the monoid $V(R)$. Notice that all the morphisms in the commutative square (2) are morphisms of monoids with order-unit, so that (2) is a pullback in the category of commutative monoids with order-unit as well.

## 4 Pullbacks of monoids

Motivated by Theorem 3.1, we give the following definition.

**Definition 4.1.** We shall say that a homomorphism of monoids $\varphi : M \to M'$ *induces a pullback* if the associated commutative diagram

$$(3)\qquad \begin{array}{ccc} M & \xrightarrow{\varphi} & M' \\ {\scriptstyle\psi_M}\downarrow & & \downarrow{\scriptstyle\psi_{M'}} \\ G(M) & \xrightarrow{G(\varphi)} & G(M') \end{array}$$

is a pullback of monoids.

In this terminology, Theorem 3.1 says that for every ring $R$ the monoid homomorphism $V(p) : V(R) \to V(R/J(R))$, obtained applying the functor $V(-)$ to the canonical projection $p : R \to R/J(R)$, induces a pullback.

It is obviously possible to give a characterization of the monoid homomorphisms $\varphi : M \to M'$ that induce a pullback. This is done in the following elementary Lemma.

**Lemma 4.2.** *A monoid homomorphism $\varphi : M \to M'$ induces a pullback if and only if for every $x, y \in M$ and every $x', y' \in M'$ such that $\varphi(x) + x' + y' = \varphi(y) + y'$ there exists a unique element $t \in M$ satisfying both the following conditions:*

(a) *$\varphi(t) = x'$;*

(b) *there exists $z \in M$ such that $x + t + z = y + z$.*

*Proof.* Assume that (3) is a pullback. Let $x, y \in M$ and $x', y' \in M'$ be elements such that $\varphi(x) + x' + y' = \varphi(y) + y'$. Then $\psi_{M'}(x') = G(\varphi)([y] - [x])$ and so, by the pullback property, there is a unique $t \in M$ such that $\varphi(t) = x'$ and $\psi_M(t) = \psi_M(y) - \psi_M(x)$. But $\psi_M(t) = \psi_M(y) - \psi_M(x)$ if and only if $\psi_M(x + t) = \psi_M(y)$, if and only if there exists $z \in M$ with $y + z = x + t + z$.

Conversely, assume that $\psi_M(y) - \psi_M(x) \in G(M)$ and let $x' \in M'$ be such that $G(\varphi)(\psi_M(y) - \psi_M(x)) = \psi_{M'}(x')$ in $G(M')$. Then $\psi_{M'}(\varphi(x) + x') = \psi_{M'}(\varphi(y))$, so that there is $y' \in M'$ with $\varphi(x) + x' + y' = \varphi(y) + y'$. By hypothesis, there is a unique $t \in M$ such that $\varphi(t) = x'$ and there exists $z \in M$ such that $x + t + z = y + z$, that is, $\psi_M(x + t) = \psi_M(y)$. Equivalently, there is a unique $t \in M$ such that $\varphi(t) = x'$ and $\psi_M(t) = \psi_M(y) - \psi_M(x)$. Thus diagram (3) is a pullback. □

The characterization of Lemma 4.2 can be improved in the case in which $\varphi : M \to M'$ is a homomorphism of monoids with $\varphi(M)$ cofinal in $M'$ (that is, for every $w' \in M'$ there exists $w \in M$ with $w' \leq \varphi(w)$ in $M'$). For instance, if $\varphi : M \to M'$ is a homomorphism of monoids with order-unit, then $\varphi(M)$ is cofinal in $M'$.

**Theorem 4.3.** *Let $\varphi : M \to M'$ be a homomorphism of monoids and suppose that $\varphi(M)$ is cofinal in $M'$. Then $\varphi : M \to M'$ induces a pullback if and only if for every $x, y \in M$ and every $x' \in M'$ such that $\varphi(x) + x' = \varphi(y)$, there exists a unique element $t \in M$ such that $\varphi(t) = x'$ and $x + t = y$.*

*Proof.* Assume that (3) is a pullback. Let $x, y \in M$ and $x' \in M'$ be elements such that $\varphi(x) + x' = \varphi(y)$. By Lemma 4.2 (applied with $y' = 0$), there exists at most one element $t \in M$ satisfying the two conditions $\varphi(t) = x'$ and $x + t = y$. This proves the uniqueness of the element $t$ with the properties required in the statement of the proposition, if such a $t$ exists. In order to prove the existence, notice that $\psi_{M'}(x') = G(\varphi)([y] - [x])$, and so, by the pullback property, there is a unique $t \in M$ such that $\varphi(t) = x'$ and $\psi_M(y) - \psi_M(x) = \psi_M(t)$. Thus $\varphi(x + t) = \varphi(y)$ and $\psi_M(x + t) = \psi_M(y)$. It follows that $x + t = y$. Notice that the proof of this implication does not need the cofinality of $\varphi(M)$ in $M'$.

For the converse, in order to prove that (3) is a pullback we shall apply Lemma 4.2. Let $x, y \in M$ and $x', y' \in M'$ be such that $\varphi(x) + x' + y' = \varphi(y) + y'$. As $\varphi(M)$ is cofinal in $M'$, there exists $z \in M$ with $y' \leq \varphi(z)$. Thus $\varphi(x) + x' + \varphi(z) = \varphi(y) + \varphi(z)$. By hypothesis, there exists a unique element $t \in M$ such that $\varphi(t) = x'$ and $x + z + t = y + z$. This proves the existence of the element $t$ with the property required in the statement of Lemma 4.2. To prove the uniqueness, suppose that $q \in M$ also has the properties required in the statement of Lemma 4.2, that is, $\varphi(q) = x'$ and there exists $w \in M$ such that $x + q + w = y + w$. Then $(x + z + w) + t = (y + z + w)$ and $(x + z + w) + q = (y + z + w)$. By hypothesis, $\varphi(x + z + w) + x' = \varphi(y + z + w)$ implies that there exists a unique element $t' \in M$ such that $\varphi(t') = x'$ and $(x + z + w) + t' = (y + z + w)$. Thus $q = t' = t$. This shows that diagram (3) is a pullback. □

If $M$ is a commutative monoid and $G(M)$ is its Grothendieck group, the kernel of the canonical monoid homomorphism $\psi_M : M \to G(M)$ is the congruence $\sim_M$ defined, for all $x, y \in M$, by $x \sim_M y$ if there exists $t \in M$ such that $x + t = y + t$. Let $\varphi : M \to M'$ be an arbitrary monoid homomorphism. Then, for each $x, y \in M$, $x \sim_M y$ implies $\varphi(x) \sim_{M'} \varphi(y)$, so that $\varphi : M \to M'$ induces by restriction a mapping $\varphi|_{[x]_{\sim_M}} : [x]_{\sim_M} \to [\varphi(x)]_{\sim_{M'}}$ of the congruence class $[x]_{\sim_M}$ of $x$ in $M$ into the congruence class $[\varphi(x)]_{\sim_{M'}}$ of $\varphi(x)$ in $M'$ for each element $x \in M$.

In the next proposition, we give a further characterization of monoid homomorphisms with cofinal image that induce a pullback. Since we shall not use it in this paper, we leave its direct proof to the reader. Recall that a monoid homomorphism $f : M \to M'$ is a *divisor homomorphism* if, for every $x, y \in M$, $f(x) \leq f(y)$ implies $x \leq y$.

**Proposition 4.4.** *Let $\varphi : M \to M'$ be a homomorphism of commutative monoids and suppose $\varphi(M)$ cofinal in $M'$. Then $\varphi$ induces a pullback if and only if both the following conditions hold:*

(a) *$\varphi$ is a divisor homomorphism;*

(b) *the restriction $\varphi|_{[x]_{\sim_M}} : [x]_{\sim_M} \to [\varphi(x)]_{\sim_{M'}}$ is a bijective mapping for every $x \in M$.*

**Example 1.** The two conditions of Proposition 4.4 are independent. For an example in which (a) does not hold, but (b) does, it is sufficient to take as $\varphi : M \to M'$ the embedding of $\mathbb{N}$ into $\mathbb{Z}$. An example in which (a) holds, and (b) does not, and in which $M$ and $M'$ are monoids of the type $V(R)$, will be given in Example 2.

If $I$ is a two-sided ideal of a ring $R$ and $V(p_I) : V(R) \to V(R/I)$ is the monoid homomorphism induced by the canonical projection $p_I : R \to R/I$, then $V(p_I)$ is a divisor homomorphism if and only if $A_R$ is a direct summand of $B_R$ whenever $A_R, B_R \in$ proj-$R$ and $A_R/A_RI$ is a direct summand of $B_R/B_RI$. Let $\mathscr{D}_R$ be the set of all the ideals $I \in \mathscr{I}_R$ for which the homomorphism $V(p_I)$ is a divisor homomorphism. Partially order $\mathscr{D}_R$ by set inclusion. If an ideal $I$ of $R$ belongs to $\mathscr{D}_R$, then every ideal of $R$ contained in $I$ belongs to $\mathscr{D}_R$ as well. The second author and Franz Halter-Koch proved in [8, Theorem 3.1] that $\mathscr{D}_R$ always has maximal elements. Moreover, $J(R) \subseteq I_0$ and $J(R/I_0) = 0$ for every maximal element $I_0$ of $\mathscr{D}_R$. It would be very natural to think that Theorem 3.1 holds not only for the Jacobson radical $J(R)$, which belongs to $\mathscr{D}_R$, but also for any other ideal $I \in \mathscr{D}_R$. This is false as the following example shows.

**Example 2.** *Example of a ring $R$ with an ideal $I \in \mathscr{D}_R$ for which the monoid homomorphism $V(p_I) : V(R) \to V(R/I)$ obtained applying the functor $V(-)$ to the canonical projection $p_I : R \to R/I$ does not induce a pullback.*

Let $F$ be a field, $x$ and $y$ two non-commuting indeterminates over $F$ and $R = F\langle x, y\rangle$ be the free associative $F$-algebra. Let $I$ be the principal two-sided ideal of $R$ generated by the element $xy - 1$ of $R$. In order to show that $I \in \mathscr{D}_R$, notice that any right or left ideal of $R$ is free [5, Corollary 2.4.3], so that $R$ is hereditary. But in a hereditary ring every projective module is a direct sum of finitely generated ideals [1], so that every right or left projective $R$-module is free. There is a unique surjective ring homomorphism $\varphi : R \to F$ that is the identity on $F$ and maps both $x$ and $y$ to 1. Clearly, $I$ is contained in the kernel of $\varphi$, so that there is a surjective homomorphism $R/I \to F$. Therefore if $A_R, B_R \in$ proj-$R$ and $A_R/A_RI$ is a direct summand of $B_R/B_RI$, then there exist non-negative integers $n$ and $m$ with $A_R \cong R_R^n$ and $B_R \cong R_R^m$. Then $A_R/A_RI \otimes_{R/I} F \cong F_F^n$ is a vector space over $F$ isomorphic to a direct summand of $B_R/B_RI \otimes_{R/I} F \cong F_F^m$. In particular, $n \leq m$, so that $A_R$ is isomorphic to a direct summand of $B_R$. This proves that $p_I$ is a divisor homomorphism, that is, $I \in \mathscr{D}_R$.

In order to prove that $V(p_I) : V(R) \to V(R/I)$ does not induce a pullback, we shall apply Theorem 4.3. Thus it is sufficient to show that there exist $A_R, B_R \in$ proj-$R$ and $C'_{R/I} \in$ proj-$R/I$ such that $A_R/A_RI \oplus C'_{R/I} \cong B_R/B_RI$, but there does not exist $C_R \in$ proj-$R$ with $C_R/C_RI \cong C'_{R/I}$. Set $A_R = B_R = R_R$. The ring $R/I$ is the prototype of a ring that is not directly finite. That is, if $\bar{x}, \bar{y}$ denote the images of $x, y$ in $R/I$, then left multiplication by $\bar{y}$ is an injective non-surjective endomorphism of the right $R/I$-module $R/I$, left multiplication by $\bar{x}$ is a surjective non-injective endomorphism of the module $R/I$, and the composition of these two endomorphisms is the identity endomorphism of $R/I$. Thus $R/I \cong R/I \oplus \mathrm{r.ann}_{R/I}(\bar{x})$. If we set $C'_{R/I} = \mathrm{r.ann}_{R/I}(\bar{x})$, we see that $C'_{R/I}$ is a non-zero cyclic projective $R/I$-module and $A_R/A_RI \oplus C'_{R/I} \cong B_R/B_RI$. Suppose that there exists $C_R \in$ proj-$R$ with $C_R/C_RI \cong C'_{R/I}$. Then $C_R \cong R_R^t$ for some $t$. Hence $C'_{R/I}$ is a free $R/I$-module of rank $t$. Tensoring by $F$ the isomorphism $A_R/A_RI \oplus C'_{R/I} \cong B_R/B_RI$ and comparing the dimensions, we see that $1 + t = 1$, so that $t = 0$, hence $C_R = 0$. Thus $0 = C_R/C_RI \cong C'_{R/I} = \mathrm{r.ann}_{R/I}(\bar{x})$, contradiction.

## 5 Trace ideals in *R*, prime ideals in *V*(*R*), and reduced localization

For any subset $\mathscr{U}$ of $V(R)$, we shall denote by $\mathrm{Tr}_R(\mathscr{U})$ the smallest two-sided ideal $I$ of $R$ such that $f(A_R) \subseteq I$ for every $A_R \in \mathscr{U}$ and every $f \in \mathrm{Hom}(A_R, R_R)$. The ideal $\mathrm{Tr}_R(\mathscr{U})$ is called the *trace* of $\mathscr{U}$. Obviously, $\mathrm{Tr}_R(\mathscr{U})$ is the sum of all the images $f(A_R)$ when $A_R$ ranges in the set $\mathscr{U}$ and $f$ ranges in the set of all homomorphisms of $A_R$ into $R_R$. Conversely, for any two-sided ideal $I$ of $R$, we shall denote by $T(I)$ the largest subset $\mathscr{U}$ of $V(R)$ such that $f(A_R) \subseteq I$ for every $A_R \in \mathscr{U}$ and every $f \in \mathrm{Hom}(A_R, R_R)$. Obviously, $T(I) = \{A_R \in V(R) \mid f(A_R) \subseteq I$ for every $f \in \mathrm{Hom}(A_R, R_R)\}$.

When $\mathscr{U}$ has a unique element $A_R$, we shall write $\mathrm{Tr}_R(A_R)$ instead of $\mathrm{Tr}_R(\mathscr{U})$. The trace $\mathrm{Tr}_R(\mathscr{U})$ is the smallest two-sided ideal $I$ of $R$ satisfying $A_RI = A_R$ (or, equivalently, $A \otimes_R R/I = 0$) for every $A_R \in \mathscr{U}$. Moreover, $\mathrm{Tr}_R(\mathscr{U})$ is an idempotent ideal. We call an ideal $I$ of $R$ a *trace ideal* if $I = \mathrm{Tr}_R(\mathscr{U})$ for some subset $\mathscr{U}$ of $V(R)$. For instance, let $A_R$ be a cyclic projective right $R$-module, so that $A_R \cong eR$ for some idempotent element $e \in R$. Then $\mathrm{Tr}_R(A_R) = \mathrm{Tr}_R(eR) = ReR$ is the two-sided ideal generated by $e$. The sum of trace ideals is a trace ideal, so that the set $\mathscr{T}(R)$ of all trace ideals of $R$, partially ordered by set inclusion, is a complete lattice whose greatest element is the trace ideal $R$ and whose least element is the zero ideal.

We recall some results of [8, Section 2]. Let $\mathscr{P}(V(R))$ denote the set of all subsets of $V(R)$ and $\mathscr{I}(R)$ the set of all two-sided ideals of $R$. Define two mappings

$$\Phi : \mathscr{I}(R) \to \mathscr{P}(V(R)) \text{ by } \Phi(I) := \{\langle A_R \rangle \in V(R) \mid A_RI \neq A_R\}$$

$$\text{for all } I \in \mathscr{I}(R)$$

and

$$\Psi : \mathscr{P}(V(R)) \to \mathscr{I}(R) \text{ by } \Psi(\mathscr{U}) := \mathrm{Tr}_R(V(R) \backslash \mathscr{U}) \quad \text{for all } \mathscr{U} \in \mathscr{P}(V(R)).$$

Then $\Phi$ and $\Psi$ are a Galois connection from $\mathscr{I}(R)$ to $\mathscr{P}(V(R))$ with the inverse order [14, Theorem IV.5.1], that is, for all $\mathscr{U} \in \mathscr{P}(V(R))$ and $I \in \mathscr{I}(R)$,

$$\Phi(I) \subseteq \mathscr{U} \quad \text{if and only if} \quad I \supseteq \Psi(\mathscr{U}),$$

as is immediately verified. The image of $\Phi$ is the lattice $\operatorname{Spec}(V(R))$ of all prime ideals of the monoid $V(R)$, and the image of $\Psi$ is $\mathscr{T}(R)$. Thus the lattices $\mathscr{T}(R)$ and $\operatorname{Spec}(V(R))$ are antiisomorphic via the restrictions of $\Phi$ and $\Psi$ [8, Theorem 2.2(c)].

**Lemma 5.1.** *The following conditions are equivalent for every $A_R \in$ proj-$R$ and every $I \in \mathscr{I}(R)$:*
(a) *$f(A_R) \subseteq I$ for every $f \in \operatorname{Hom}(A_R, R_R)$;*
(b) *$A_R I = A_R$.*

*Proof.* Statement (a) says that $\operatorname{Tr}_R(A_R)$ is contained in $I$. As the trace $\operatorname{Tr}_R(A_R)$ is the smallest ideal $J$ with $A_R J = A_R$, $\operatorname{Tr}_R(A_R)$ is contained in $I$ if and only if statement (b) holds. □

From Lemma 5.1 we immediately obtain that:

**Corollary 5.2.** *For every $I \in \mathscr{I}(R)$, the subset $\Phi(I)$ of $V(R)$ is the complement of $T(I)$ in $V(R)$.*

As $\Psi\Phi(I) \subseteq I$ for every ideal $I$ of $R$, the composition $\Psi\Phi : \mathscr{I}(R) \to \mathscr{I}(R)$ is a closure operation on the partially ordered set $\mathscr{I}(R)$ with the inverse inclusion (recall that a *closure operation* $t$ in a partially ordered set $P$ is a monotonic mapping $t : P \to P$ such that $x \leq t(x)$ and $t(t(x)) = t(x)$ for all $x \in P$ [14, p. 135]). We shall denote by $\operatorname{Tr}(I)$ the ideal $\Psi\Phi(I)$ for every $I \in \mathscr{I}(R)$, and call $\operatorname{Tr}(I)$ the *trace ideal* of $I$. Thus $\operatorname{Tr}(I)$ is the greatest trace ideal of $R$ contained in $I$ for every ideal $I$ of $R$. Equivalently, the trace ideal $\operatorname{Tr}(I)$ of an ideal $I$ is the trace of the set of all the finitely generated projective modules $P$ such that $P = PI$. Also, $I$ is a trace ideal if and only if $I = \operatorname{Tr}(I)$. (Notice that there is some ambiguity of terminology when $I$ is at the same time a two-sided ideal of $R$ and a projective right $R$-module. For instance, if $I$ is any proper non-zero ideal of the ring $\mathbf{Z}$, then $\operatorname{Tr}(I) = 0$, but the trace of $I$ viewed as a projective right $R$-module is $\mathbf{Z}$.) We shall say that an ideal $I$ of $R$ is an *almost trace ideal* if $I/\operatorname{Tr}(I)$ is contained in the Jacobson radical $J(R/\operatorname{Tr}(I))$ of $R/\operatorname{Tr}(I)$.

For any ring homomorphism $f : R \to S$, we shall denote by $P_f$ the set of all $\langle A_R \rangle \in V(R)$ with $V(f)(\langle A_R \rangle) \neq 0$. In the following proposition we shall see that $P_f$ is a prime ideal of $V(R)$.

**Proposition 5.3.** *Let $f : R \to S$ be a ring homomorphism, $I = \ker f$, and $V(f) : V(R) \to V(S)$ the monoid homomorphism induced by $f$. Then:*

(a) *The set $P_f$ is the prime ideal of $V(R)$ corresponding to the trace ideal $\operatorname{Tr}(I)$ of $I$ in the antiisomorphism between the lattices $\operatorname{Spec}(V(R))$ and $\mathscr{T}(R)$.*

(b) *The homomorphism $V(f) : V(R) \to V(S)$ factors uniquely through the canonical homomorphism $\varphi$ of $V(R)$ into its reduced localization $(V(R)_{P_f})_{\text{red}}$, that is, there exists* *a unique monoid homomorphism* $\omega : (V(R)_{P_f})_{\mathrm{red}} \to V(S)$ *with* $V(f) = \omega\varphi$*. The image of* $V(f)$ *is equal to the image of* $\omega$.

(c) *If* $f$ *is surjective and* $I$ *is an almost trace ideal, then* $\omega$ *is injective and induces a pullback.*

*Proof.* (a) The prime ideal of $V(R)$ corresponding to $\mathrm{Tr}(I) = \Psi\Phi(I)$ is $\Phi\Psi\Phi(I) = \Phi(I) = \{\langle A_R \rangle \in V(R) \mid A \otimes_R R/I \neq 0\} = \{\langle A_R \rangle \in V(R) \mid A \otimes_R S \neq 0\} = P_f$.

(b) It is sufficient to remark that $\omega$ is defined by $\omega(\langle A_R \rangle - \langle B_R \rangle + U(V(R)_{P_f})) = V(f)(\langle A_R \rangle)$ for every $\langle A_R \rangle, \langle B_R \rangle \in V(R)$, $\langle B_R \rangle \notin P_f$.

For the proof of (c), which is modelled on the proof of [20, Theorem 1.5], we need the following lemma.

**Lemma 5.4.** *Let* $I$ *be an almost trace ideal of a ring* $R$, $A_R, B_R \in \text{proj-}R$ *and let* $\bar{h} : A_R/A_R I \to B_R/B_R I$ *be an epimorphism. Then there exist a module* $C_R \in \text{proj-}R$ *and an epimorphism* $h : A_R \oplus C_R \to B_R$ *such that* $C_R \, \mathrm{Tr}(I) = C_R$, $\ker(\bar{h}) \cong \ker(h)/\ker(h)I$*, and the diagram*

$$\begin{array}{ccc} A_R \oplus C_R & \xrightarrow{h} & B_R \\ {\scriptstyle p_A \oplus 0}\downarrow & & \downarrow{\scriptstyle p_B} \\ A_R/A_R I & \xrightarrow{\bar{h}} & B_R/B_R I, \end{array}$$

*where* $p_A : A_R \to A_R/A_R I$ *and* $p_B : B_R \to B_R/B_R I$ *denote the canonical projections, is commutative.*

*Proof.* As $A_R$ is projective and $\bar{h} : A_R/A_R I \to B_R/B_R I$ is an epimorphism, there exists a homomorphism $h' : A_R \to B_R$ such that $p_B h' = \bar{h} p_A$. In particular, $B_R = h'(A_R) + B_R I$. Then $B_R/B_R \, \mathrm{Tr}(I) = (h'(A_R) + B_R \, \mathrm{Tr}(I)/B_R \, \mathrm{Tr}(I)) + (BI + B_R \, \mathrm{Tr}(I)/B_R \, \mathrm{Tr}(I)) = (h'(A_R) + B_R \, \mathrm{Tr}(I)/B_R \, \mathrm{Tr}(I)) + (B_R/B_R \, \mathrm{Tr}(I))(I/\mathrm{Tr}(I))$. As $I$ is an almost trace ideal, that is, $I/\mathrm{Tr}(I) \subseteq J(R/\mathrm{Tr}(I))$, we can apply Nakayama's Lemma, from which we get that $B_R/B_R \, \mathrm{Tr}(I) = h'(A_R) + B_R \, \mathrm{Tr}(I)/B_R \, \mathrm{Tr}(I)$, and thus $B_R = h'(A_R) + B_R \, \mathrm{Tr}(I)$.

Let $b_1, \dots, b_t$ be a set of generators of $B_R$, and write each $b_i$ as $b_i = h'(a_i) + b_1 r_{i1} + \cdots + b_t r_{it}$ for suitable $a_i \in A_R$ and $r_{ij} \in \mathrm{Tr}(I)$. As $\mathrm{Tr}(I)$ is a trace ideal, if $P = \Phi(I)$ is the prime ideal of $V(R)$ corresponding to $\mathrm{Tr}(I)$, then for each $r_{ij} \in \mathrm{Tr}(I)$ there is a homomorphism from a projective module $C_{ij} \in V(R) \backslash P$ to $R_R$ whose image contains $r_{ij}$. Thus there is a homomorphism from $C_{ij}$ to $B_R$ whose image contains $b_j r_{ij}$. Taking the direct sum, we find a homomorphism $g$ from a projective module $C_R = \bigoplus_{i,j} C_{ij} \in V(R) \backslash P$ to $B_R$ whose image contains all the elements $b_j r_{ij}$. Thus there is a surjective homomorphism $h := (h', g) : A_R \oplus C_R \to B_R$. From $C_R \notin P$, it follows that $C_R \, \mathrm{Tr}(I) = C_R$. Thus $C_R I = C_R$, from which $p_B g = 0$. The commutativity of the diagram follows. Finally, as $B_R$ is projective, the exact sequence $0 \to \ker h \to A_R \oplus C_R \xrightarrow{h} B_R \to 0$ splits. Therefore it remains exact when tensored with $R/I$. Thus $\ker(h)/\ker(h)I$ is isomorphic to the kernel of $h \otimes R/I : A_R/A_R I \oplus C_R/C_R I \to B_R/B_R I$, which is isomorphic to $\ker(\bar{h})$ because $C_R/C_R I = 0$. □

We are ready for the proof of Proposition 5.3(c). Assume $f$ surjective and $I$ an almost trace ideal. The prime ideal $P_f = \Phi(I)$ corresponds to the trace ideal $\mathrm{Tr}(I)$, that is, $\mathrm{Tr}(I)$ is the trace of the set $V(R)\backslash P_f$. Let $\langle A_R\rangle, \langle A'_R\rangle, \langle B_R\rangle, \langle B'_R\rangle$ be elements of $V(R)$ with $\langle A'_R\rangle, \langle B'_R\rangle \notin P_f$ and $\omega(\langle A_R\rangle - \langle A'_R\rangle + U(V(R)_{P_f})) = \omega(\langle B_R\rangle - \langle B'_R\rangle + U(V(R)_{P_f}))$. Then $A \otimes_R S \cong A' \otimes_R S$, i.e., there is an isomorphism $\bar{h} : A_R/A_R I \to B_R/B_R I$. By Lemma 5.4, there are $C_R \in$ proj-$R$ and an epimorphism $h : A_R \oplus C_R \to B_R$ with $C_R \,\mathrm{Tr}(I) = C_R$, $\ker(\bar{h}) \cong \ker(h)/\ker(h)I$ and $\bar{h}(p_A \oplus 0) = p_B h$. In particular, $C_R \notin P_f$. As $\bar{h}$ is an isomorphism, it follows that $\ker(h)/\ker(h)I = 0$, so that $\ker(h) \notin P_f$ either. Thus $A_R \oplus C_R \cong B_R \oplus \ker(h)$ implies that $\langle A_R\rangle + U(V(R)_{P_f}) = \langle B_R\rangle + U(V(R)_{P_f})$ in $(V(R)_{P_f})_{\mathrm{red}}$, from which $\langle A_R\rangle - \langle A'_R\rangle + U(V(R)_{P_f}) = \langle B_R\rangle - \langle B'_R\rangle + U(V(R)_{P_f})$. This shows that $\omega$ is injective.

In order to prove that $\omega$ induces a pullback, we shall apply Theorem 4.3. It is enough to show that, if $A_R, B_R$ are finitely generated projective $R$-modules and $Q_S$ is a finitely generated projective $S$-module such that $A/AI \cong B/BI \oplus Q$, then there exists a unique $\langle X\rangle + U(V(R)_{P_f}) \in (V(R)_{P_f})_{\mathrm{red}}$ such that $X/XI \cong Q$ as $S$-modules and $A \oplus T \cong B \oplus X \oplus T'$ for some $T, T' \in$ proj-$R$ with $\langle T\rangle, \langle T'\rangle \notin P_f$.

Let $\bar{h} : A/AI \to B/BI$ be an epimorphism with kernel isomorphic to $Q$, and apply Lemma 5.4. There are $C_R \in$ proj-$R$ and an epimorphism $h : A_R \oplus C_R \to B_R$ with $C_R \,\mathrm{Tr}(I) = C_R$ and $\ker(\bar{h}) \cong \ker(h)/\ker(h)I$. The module $X := \ker(h)$ has the required properties. The uniqueness of $\langle X\rangle + U(V(R)_{P_f})$ follows from the injectivity of $\omega$. □

Let $R$ be an exchange ring [7, p. 67]. Every projective right $R$-module is a direct sum of cyclic projective modules [7, Theorem 2.56]. Thus the (finitely generated) trace ideals are exactly the two-sided ideals generated by (finitely many) idempotents of $R$. In [4, Proposition 1.4], it is proved that if $R$ is an exchange ring and $I$ is a two-sided ideal of $R$, then $V(R/I)$ is isomorphic to the reduced localization of $V(R)$ at the prime ideal of all $\langle A_R\rangle \in V(R)$ with $A_R I \neq A_R$. The isomorphism is the isomorphism $\omega$ induced by the canonical projection $p_I : R \to R/I$. This can be generalized as follows. Let $I$ be an ideal of a ring $R$. We say that $I$ is an *exchange ideal* of $R$ in case for each $a \in I$ there are elements $r, s \in I$ and an idempotent $e \in I$ such that $ar = a + s - as = e$. This notion is left-right symmetric and depends only on the ring structure of $I$ and not on the particular embedding of $I$ as an ideal of a unital ring $R$; see [2]. Moreover, a ring $R$ is an exchange ring if and only if $R$ is an exchange ideal of $R$. (This follows from the well-known characterization of exchange rings obtained independently by Goodearl [12] and Nicholson [16].) By [2, Proposition 1.5(a)], if $I$ is an exchange ideal of a ring $R$, then each $A \in$ proj-$R$ with $A = AI$ is a direct sum of cyclic projective modules, so that the ideal $\mathrm{Tr}(I)$ is exactly the ideal generated by all the idempotents of $I$. Also, by [2, Theorem 2.2], the factor $I/\mathrm{Tr}(I)$ is an exchange ideal of $R/\mathrm{Tr}(I)$ with no nonzero idempotents, so $I/\mathrm{Tr}(I) \subseteq J(R/\mathrm{Tr}(I))$, i.e., $I$ is an almost trace ideal of $R$. It follows that we can apply Proposition 5.3(c). We have thus obtained the following:

**Corollary 5.5.** *Let $I$ be an exchange ideal of a ring $R$ and let $P$ be the prime ideal $\Phi(I)$ of $V(R)$. Then the map $\omega : (V(R)_P)_{\mathrm{red}} \to V(R/I)$ induces a pullback.*

Proposition 5.3(a) gives a characterization of prime ideals in the monoids $V(R)$, because:

**Proposition 5.6.** *Let R be a ring. A subset P of $V(R)$ is a prime ideal of $V(R)$ if and only if there exists a homomorphism f of R into some ring S such that $P = P_f$.*

*Proof.* In view of Proposition 5.3(a), it is sufficient to show that every prime ideal $P$ of $V(R)$ is of the type $P_f$ for some homomorphism $f : R \to S$. Let $P$ be a prime ideal and $I$ the trace ideal of $R$ corresponding to $P$ in the antiisomorphism $\Phi : \mathscr{T}(R) \to \mathrm{Spec}(V(R))$, so that $P = \Phi(I)$. Let $f : R \to R/I$ be the canonical projection. Then $P = \{\langle A_R \rangle \in V(R) \mid A_R I \neq A_R\} = \{\langle A_R \rangle \in V(R) \mid V(f)(\langle A_R \rangle) \neq 0\} = P_f$. □

**Proposition 5.7.** *For every ideal K of a ring R, let $p_K$ denote the canonical projection $R \to R/K$. Then:*

(a) *The trace ideal* $\mathrm{Tr}(I)$ *of an ideal I of R is the smallest among the ideals K of R with $P_{p_K} = P_{p_I}$.*

(b) *If I is an almost trace ideal of R and $\pi : R/\mathrm{Tr}(I) \to R/I$ is the canonical projection, then $V(\pi) : V(R/\mathrm{Tr}(I)) \to V(R/I)$ is injective and induces a pullback.*

*Proof.* (a) For every ideal $K$ of a ring $R$, one has $P_{p_K} = \{\langle A_R \rangle \in V(R) \mid A_R K \neq A_R\} = \Phi(K)$. As $\Phi(I) = \Phi\Psi\Phi(I) = \Phi(\mathrm{Tr}(I))$, it follows that $P_{p_I} = P_{p_{\mathrm{Tr}(I)}}$. Also, if $K$ is any ideal of $R$ such that $P_{p_K} = P_{p_I}$, then $\Phi(K) = \Phi(I)$, so that $\mathrm{Tr}(I) = \Psi\Phi(I) = \Psi\Phi(K) \subseteq K$.

(b) It is enough to show the result for an ideal $I$ contained in $J(R)$ (so that $\mathrm{Tr}(I) = 0$). It is well known that, for every ring $S$, the mapping $V(p_{J(S)}) : V(S) \to V(S/J(S))$ is injective (this follows from the uniqueness of projective covers). Applying this fact to the two rings $S = R$ and $S = R/I$, one sees that the two mappings

$$V(p_{J(R)}) : V(R) \to V(R/J(R))$$

and

$$V(p_{J(R/I)}) : V(R/I) \to V(R/J(R))$$

are both injective. From the equality $V(p_{J(R)}) = V(p_{J(R/I)}) \circ V(\pi)$ and the injectivity of $V(p_{J(R)})$, it follows that $V(\pi)$ is injective.

Theorem 3.1 says that $V(p_{J(R)}) : V(R) \to V(R/J(R))$ induces a pullback. From this fact, the injectivity of $V(p_{J(R)})$ and the equality $V(p_{J(R)}) = V(p_{J(R/I)}) \circ V(\pi)$, it easily follows that $V(\pi)$ also induces a pullback. □

## 6 Directed convex subgroups of $K_0(R)$

Let $I$ be an ideal of a ring $R$. The canonical projection $p_I : R \to R/I$ induces a group homomorphism $K_0(p_I) : K_0(R) \to K_0(R/I)$. We will prove that this homomorphism has good properties when $I$ is an almost trace ideal. We have seen in Proposition 5.3

that for such an ideal $I$, the map $(V(R)_P)_{\text{red}} \to V(R/I)$ induces a pullback, and this immediately gives us useful information on $K_0(p_I)$.

For any commutative monoid $M$, we endow its Grothendieck group $G(M)$ with the structure of pre-ordered group given by $G(M)^+ = \{[m] \mid m \in M\}$, where $[m]$ is the image of $m \in M$ under the canonical map $\psi_M : M \to G(M)$. Note that every monoid homomorphism $\varphi : M \to N$ induces a homomorphism of pre-ordered groups $G(\varphi) : G(M) \to G(N)$.

Let $G_1$ and $G_2$ be pre-ordered groups. We say that a map $f : G_1 \to G_2$ is an *order-embedding* if $f$ is an injective group homomorphism and $x \le y$ in $G_1$ if and only if $f(x) \le f(y)$ in $G_2$, for all $x, y \in G_1$. The latter property is equivalent to the statement $f(G_1^+) = f(G_1) \cap G_2^+$.

**Lemma 6.1.** *Let $\varphi : M \to N$ be an injective homomorphism of commutative monoids that induces a pullback. If $\varphi(M)$ is cofinal in N, then the induced map $G(\varphi) : G(M) \to G(N)$ is an order-embedding.*

*Proof.* Assume that $G(\varphi)([m_1] - [m_2]) = 0$. Then $\varphi(m_1) + v = \varphi(m_2) + v$ for some $v \in N$. Since $\varphi(M)$ is cofinal in $N$, there exists $w \in M$ with $v \le \varphi(w)$ in $N$. From this and the injectivity of $\varphi$, we deduce that $m_1 + w = m_2 + w$, and thus $[m_1] - [m_2] = 0$. This shows that $G(\varphi)$ is injective.

Now assume that $G(\varphi)([m_1] - [m_2]) = [n]$ in $G(N)$ for some $n \in N$ and $m_1, m_2 \in M$. By the pullback property, there is $m \in M$ such that $[m] = [m_1] - [m_2]$ and $n = \varphi(m)$. It follows that $[m_1] - [m_2] \in G(M)^+$ and so $G(\varphi)$ is an order-embedding. □

The converse of Lemma 6.1 does not hold, that is, there exist injective homomorphisms $\varphi : M \to N$ with $\varphi(M)$ cofinal in $N$ and $G(\varphi) : G(M) \to G(N)$ an order-embedding, but such that $\varphi$ does not induce a pullback. To see this, it is sufficient to take $N = \mathbb{N}/\sim$, where $\sim$ is the smallest congruence on $\mathbb{N}$ with $4 \sim 5$, so that $N$ is a monoid with five elements $0, 1, 2, 3, 4$, and $M$ the submonoid of $N$ whose elements are $0, 2, 4$.

A subgroup $H$ of a pre-ordered group $G$ is a pre-ordered group by taking $H^+ := H \cap G^+$. A *convex subset* of a partially ordered set $G$ is any subset $H$ with the property that whenever $x, z \in H$ and $y \in G$ with $x \le y \le z$, then $y \in H$. A *convex subgroup* of a pre-ordered group $G$ is any subgroup $H$ of $G$ which is also a convex subset of $G$. Clearly a subgroup $H$ of $G$ is convex if and only if whenever $0 \le a \le b$ with $b \in H$ and $a \in G$, then $a \in H$; see [10, p. 8]. We shall denote by $\mathscr{L}(G)$ the set of all directed convex subgroups of the pre-ordered group $G$. We shall say that a commutative monoid $M$ is *directly finite* if for every $x, y \in M$, $x + y = y$ implies $x = 0$ (cf. [18, p. 136]). Now let $M$ be a commutative monoid, $\operatorname{Spec}(M)$ the set of its prime ideals, and $G(M)$ the Grothendieck group of $M$. Let $\operatorname{Spec}'(M)$ denote the set of all $P \in \operatorname{Spec}(M)$ with $(M_P)_{\text{red}}$ directly finite. In the next proposition we shall describe the partially ordered set $\mathscr{L}(G(M))$.

**Proposition 6.2.** *Let $M$ be a commutative monoid. Then there is an order reversing bijection $f : \operatorname{Spec}'(M) \to \mathscr{L}(G(M))$ defined by $f(P) := \psi_M(M \setminus P) - \psi_M(M \setminus P)$ for* *every* $P \in \mathrm{Spec}'(M)$. *The inverse of* $f$ *is the order reversing mapping* $g : \mathscr{L}(G(M)) \to \mathrm{Spec}'(M)$ *defined by* $g(H) := \psi_M^{-1}(G(M)\backslash H)$ *for every* $H \in \mathscr{L}(G(M))$.

*Proof.* Let us prove that $f(P) := \psi_M(M\backslash P) - \psi_M(M\backslash P)$ belongs to $\mathscr{L}(G(M))$ for every $P \in \mathrm{Spec}'(M)$. It is easily seen that $f(P)$ is a directed subgroup of $G(M)$.

In order to show that $f(P)$ is convex, suppose that $0 \le x \le y$ with $x \in G(M)$ and $y \in f(P)$. Then $x = \psi_M(m)$ and $y = \psi_M(m') - \psi_M(m'')$ for suitable $m, m', m'' \in M$, $m', m'' \notin P$. Moreover, $x + z = y$ for some $z \in G(M)^+$, that is, $z = \psi_M(n)$ for some $n \in M$. As $x + z = y$, it follows that $\psi_M(m) + \psi_M(n) + \psi_M(m'') = \psi_M(m')$, i.e., $m + n + m'' + n' = m' + n'$ for a suitable $n' \in M$. Thus $\overline{m+n} + \overline{n'} = \overline{n'}$ in $(M_P)_{\mathrm{red}}$. As $(M_P)_{\mathrm{red}}$ is directly finite, it follows that $\overline{m+n} = \bar{0}$ in $(M_P)_{\mathrm{red}}$, i.e., there exists $n'' \in M\backslash P$ with $m + n + n'' \notin P$. Since $P$ is a prime ideal, we get that $m \notin P$. Thus $x \in f(P)$. This shows that $f$ is a well defined mapping of $\mathrm{Spec}'(M)$ into $\mathscr{L}(G(M))$. Clearly, $f$ is order reversing.

Let us prove that $g(H) := \psi_M^{-1}(G(M)\backslash H)$ belongs to $\mathrm{Spec}'(M)$ for every $H \in \mathscr{L}(G(M))$. The subset $g(H)$ of $M$ is proper, because $0 \notin g(H)$.

Suppose that $m, m' \in M\backslash g(H)$. Then $\psi_M(m), \psi_M(m') \in H$, so that $\psi_M(m + m') \in H$, and thus $m + m' \notin g(H)$. Conversely, if $m, m' \in M$ and $m + m' \notin g(H)$, then $\psi_M(m) + \psi_M(m') \in H$. Thus $\psi_M(m)$ and $\psi_M(m')$ are elements of $G(M)$ that are $\ge 0$ and $\le$ than the element $\psi_M(m) + \psi_M(m')$ of $H$. As $H$ is convex, it follows that both $\psi_M(m)$ and $\psi_M(m')$ belong to $H$. Thus $m, m' \notin g(H)$. This proves that $g(H) \in \mathrm{Spec}(M)$. Let us show that the monoid $(M_{g(H)})_{\mathrm{red}}$ is directly finite. Suppose that $x, y \in (M_{g(H)})_{\mathrm{red}}$ and $x + y = y$. Then $x = \bar{m}$ and $y = \bar{n}$ for some $m, n \in M$, and there exist $d, d' \in M\backslash g(H)$ with $m + n + d = n + d'$. Then $\psi_M(m) + \psi_M(d) = \psi_M(d')$, and $\psi_M(d), \psi_M(d') \in H$. Hence $\psi_M(m) = \psi_M(d') - \psi_M(d) \in H$, so that $m \notin g(H)$. In particular, $x = \bar{m} = \bar{0}$. Thus $g(H) \in \mathrm{Spec}'(M)$, and $g$ is a well defined mapping of $\mathscr{L}(G(M))$ into $\mathrm{Spec}'(M)$, clearly order reversing.

We shall now show that $gf(P) = P$ for every $P \in \mathrm{Spec}'(M)$. Suppose $m \in gf(P)$. Then $\psi_M(m) \notin f(P)$, so that $\psi_M(m) \ne \psi_M(m') - \psi_M(m'')$ for every $m', m'' \in M\backslash P$. In particular, $\psi_M(m) \ne \psi_M(m')$ for every $m' \in M\backslash P$, and so $m \in P$. Conversely, suppose $m \in M\backslash gf(P)$. Then $\psi_M(m) \in f(P)$, so that $\psi_M(m) = \psi_M(m') - \psi_M(m'')$ for suitable $m', m'' \in M\backslash P$. Thus $m + m'' + m_0 = m' + m_0$ for some $m_0 \in M$. In the directly finite monoid $(M_P)_{\mathrm{red}}$ we have that $\bar{m} + \overline{m_0} = \overline{m_0}$, hence $\bar{m} = \bar{0}$. Therefore $m + m_0' = m_0''$ for suitable $m_0', m_0'' \in M\backslash P$. From $m + m_0' \notin P$ it follows that $m \notin P$, as desired.

We shall now show that $fg(H) = H$ for every $H \in \mathscr{L}(G(M))$. As $fg(H)$ and $H$ are directed subgroups of $G(M)$, it suffices to show that $fg(H)^+ = H^+$. Suppose $x \in H^+$. Then $x = \psi_M(m)$ for some $m \in M$. From $\psi_M(m) \in H$, it follows that $m \notin g(H)$. Thus $m \in M\backslash g(H)$, so $x = \psi_M(m) \in \psi_M(M\backslash g(H)) \subseteq fg(H)$. Conversely, suppose $x \in fg(H)^+$. Then $x = \psi_M(m)$ for some $m \in M$, and $x = \psi_M(m) \in \psi_M(M\backslash g(H)) - \psi_M(M\backslash g(H))$. Thus $\psi_M(m) = \psi_M(m') - \psi_M(m'')$ for suitable $m', m'' \in M\backslash g(H)$. Thus $\psi_M(m'), \psi_M(m'') \in H$, so that $x = \psi_M(m') - \psi_M(m'')$ also belongs to $H$. □

**Example 3.** The greatest element of $\mathscr{L}(G(M))$ is $G(M)$, and the least element is the convex subgroup $H_0 = \{x \in G(M) \mid 0 \le x \le 0\}$ of $G(M)$. Correspondingly, the least element of $\mathrm{Spec}'(M)$ is the empty ideal $\emptyset = g(G(M))$. Notice that $\emptyset \in \mathrm{Spec}'(M)$ because $(M_\emptyset)_{\mathrm{red}} = G(M)_{\mathrm{red}} = 0$ is directly finite. The greatest element of $\mathrm{Spec}'(M)$ is the prime ideal $g(H_0) = \{m \in M \mid m + n + n' \ne n' \text{ for every } n, n' \in M\}$. To see this, notice that $m \in M$ belongs to $g(H_0)$ if and only if $\psi_M(m) \notin H_0$, that is, if and only if $\psi_M(m) + \psi_M(n) \ne 0$ for every $n \in M$, i.e., if and only if $m + n + n' \ne n'$ for every $n, n' \in M$.

**Example 4.** If $M$ is cancellative and $P$ is any prime ideal of $M$, then $(M_P)_{\mathrm{red}}$ is always cancellative. In particular, it is always directly finite. Thus $\mathrm{Spec}'(M) = \mathrm{Spec}(M)$ for $M$ cancellative. In this case, the least element of $\mathscr{L}(G(M))$ is the convex subgroup $U(M)$ of $G(M)$.

Proposition 6.2 allows us to describe the directed convex subgroups of $K_0(R)$ for a ring $R$. This is done in the next theorem, which is a generalization of [11, Theorem 15.20 and Corollary 15.21] to arbitrary rings. For a ring $R$, we shall denote by $\mathscr{T}'(R)$ the set of all trace ideals $I$ of $R$ such that for every $A_R, B_R \in \text{proj-}R$, $A_R/A_RI \oplus B_R/B_RI \cong B_R/B_RI$ implies $A_R = A_RI$.

**Theorem 6.3.** *Let $R$ be a ring. Then there is an order preserving one-to-one correspondence $h : \mathscr{T}'(R) \to \mathscr{L}(K_0(R))$ defined by $h(I) := \{[A_R] - [B_R] \mid A_R, B_R \in \text{proj-}R$ and $A_R = A_RI, B_R = B_RI\}$ for every $I \in \mathscr{T}'(R)$. The inverse of $h$ is the order preserving mapping $\ell : \mathscr{L}(K_0(R)) \to \mathscr{T}'(R)$ defined by $\ell(H) := \mathrm{Tr}_R(\{\langle A_R\rangle \mid A_R \in \text{proj-}R, [A_R] \in H\})$ for every $H \in \mathscr{L}(G(M))$.*

*Proof.* The mapping $h$ is the composite mapping of the mapping $f$ of Proposition 6.2 and the bijection $\Phi : \mathscr{T}(R) \to \mathrm{Spec}(V(R))$ restricted to $\mathscr{T}'(R)$. The mapping $\ell$ is the composite mapping of the bijection $\Psi : \mathrm{Spec}(V(R)) \to \mathscr{T}(R)$, restricted to $\mathrm{Spec}'(V(R))$, and the mapping $g$ of Proposition 6.2. By [8, Theorem 2.1(d)], for every $I \in \mathscr{T}(R)$, the image of the canonical homomorphism $V(R) \to V(R/I)$, $A_R \in \text{proj-}R \mapsto A_R/A_RI$, is canonically isomorphic to $((V(R))_{\Phi(I)})_{\mathrm{red}}$. Thus $((V(R))_{\Phi(I)})_{\mathrm{red}}$ is directly finite if and only if for every $A_R, B_R \in \text{proj-}R$, $A_R/A_RI \oplus B_R/B_RI \cong B_R/B_RI$ implies $A_R = A_RI$. Therefore an ideal $I \in \mathscr{T}(R)$ belongs to $\mathscr{T}'(R)$ if and only if $\Phi(I) \in \mathrm{Spec}'(V(R))$. □

There is another description of the directed convex subgroup $h(I)$ of $K_0(R)$ corresponding to an ideal $I \in \mathscr{T}'(R)$. It says that $h(I)$ is the kernel of the group homomorphism $K_0(p_I) : K_0(R) \to K_0(R/I)$. To show this, we first prove a proposition that holds not only for the trace ideals $I \in \mathscr{T}'(R)$, but for all almost trace ideals of $R$.

**Proposition 6.4.** *Let $I$ be an almost trace ideal of a ring $R$, and let $p_I : R \to R/I$ be the canonical projection. Then the following properties hold:*

(a) $\ker K_0(p_I) = \{[A_R] - [B_R] \mid A_R, B_R \in \text{proj-}R$ *and* $A_R = A_RI, B_R = B_RI\}$.

(b) $\ker K_0(p_I)$ *is a directed subgroup of* $K_0(R)$.

(c) $K_0(p_I)(K_0(R)^+) = K_0(p_I)(K_0(R)) \cap K_0(R/I)^+$.

*Proof.* (a) Set $P := \Phi(I) \in \operatorname{Spec}(V(R))$, $M := (V(R)_P)_{\mathrm{red}}$ and $N := V(R/I)$. By Proposition 5.3(c), the map $\omega : M \to N$ is injective and induces a pullback. So by Lemma 6.1, the map $G(\omega) : G(M) \to G(N) = K_0(R/I)$ is an order-embedding. Now the map $K_0(p_I)$ factors as $K_0(p_I) = G(\omega) \circ \tau$, where $\tau : K_0(R) \to G(M)$ is a surjection with $\ker(\tau) = \{[A_R] - [B_R] \mid A_R, B_R \in$ proj-$R$ and there exist $C_R, D_R, D'_R \in$ proj-$R$ with $D_R = D_R I, D'_R = D'_R I$ and $A_R \oplus C_R \oplus D_R \cong B_R \oplus C_R \oplus D'_R\} = \{[D'_R] - [D_R] \mid D_R, D'_R \in$ proj-$R$ and $D_R = D_R I, D'_R = D'_R I\}$. But $\ker(K_0(p_I)) = \ker(\tau)$.

(b) follows from (a).

(c) Since $\tau(K_0(R)^+) = G(M)^+$, we have, using Lemma 6.1,

$$K_0(p_I)(K_0(R)^+) = G(\omega)(G(M)^+)$$

$$= G(\omega)(G(M)) \cap K_0(R/I)^+ = K_0(p_I)(K_0(R)) \cap K_0(R/I)^+. \quad \square$$

**Theorem 6.5.** *For every ideal $I$ of a ring $R$, let $p_I : R \to R/I$ be the canonical projection. The following conditions are equivalent for a subset $H$ of $K_0(R)$:*

(a) *$H$ is a directed convex subgroup of $K_0(R)$.*

(b) *There exists an ideal $I \in \mathcal{T}'(R)$ such that $H = \{[A_R] - [B_R] \mid A_R, B_R \in$ proj-$R$ and $A_R = A_R I, B_R = B_R I\}$.*

(c) *There exists an ideal $I \in \mathcal{T}'(R)$ such that $H = \ker K_0(p_I)$.*

*Proof.* The equivalence of (a) and (b) follows from Theorem 6.3. The equivalence of (b) and (c) follows from Proposition 6.4(a). $\square$

## 7 The groups $K_0(I)$

For an ideal $I$ of a ring $R$, consider the submonoid $V(I)$ of $V(R)$ consisting of the elements $\langle A \rangle \in V(R)$ such that $AI = A$. Note that $V(I) = V(\operatorname{Tr}(I)) = V(R) \setminus P$, where $P$ is the prime ideal of $V(R)$ associated to $\operatorname{Tr}(I)$. The sequence

$$V(I) \to V(R) \to (V(R)_P)_{\mathrm{red}}$$

gives rise to an exact sequence of Grothendieck groups

$$G(V(I)) \to K_0(R) \to G((V(R)_P)_{\mathrm{red}}) \to 0$$

as has been shown in the proof of Proposition 6.4(a). The map $G(V(I)) \to K_0(R)$ is not injective in general, and its failure to be injective is related to the exact sequence in algebraic $K$-theory

$$K_1(R) \to K_1(R/I) \to K_0(I) \to K_0(R) \to K_0(R/I).$$

Recall that $K_0(I)$ is defined as the kernel of the natural map $K_0(I^1) \to K_0(\mathbb{Z})$, where $I^1 = I \oplus \mathbb{Z}$ is the unitization of $I$.

There is a natural map $\psi : G(V(I)) \to K_0(I)$ sending the class of a finitely generated projective module $A_R$ such that $A = AI$ to the corresponding class $[A \otimes_I I^1]$ in

$K_0(I)$. It is easy to check that if $I$ is an (almost) trace ideal of $R$, then $I$ is an (almost) trace ideal of any ring in which it can be embedded as an ideal. In particular, $I$ is an (almost) trace ideal of $I^1$ for any (almost) trace ideal $I$ of $R$.

The following proposition generalizes [3, Proposition 4.4], where the same result is obtained for exchange ideals.

**Proposition 7.1.** *Assume that $I$ is an almost trace ideal of $R$. Then the natural map $\psi : G(V(I)) \to K_0(I)$ is surjective.*

*Proof.* Set $S = I^1$, and note that $I$ is an almost trace ideal of $S$ by our previous observation. Observe that $K_0(I)$ consists of elements of the form $[A] - [S^r]$, where $A$ is a finitely generated projective $S$-module such that $A/AI \cong A \otimes_S \mathbb{Z} \cong \mathbb{Z}^r$. Let $\bar{h} : A/AI \to S^r/S^rI \cong \mathbb{Z}^r$ be an isomorphism. By Lemma 5.4, there exist a module $C_S \in \text{proj-}S$ and an epimorphism $h : A_S \oplus C_S \to S_S^r$ such that $C_S \operatorname{Tr}(I) = C_S$, $\ker(\bar{h}) \cong \ker(h)/\ker(h)I$, and the diagram

$$\begin{array}{ccc} A_S \oplus C_S & \xrightarrow{\ h\ } & S_S^r \\ \Big\downarrow{\scriptstyle p_A \oplus 0} & & \Big\downarrow{\scriptstyle p_{S^r}} \\ A_S/A_SI & \xrightarrow{\ \bar{h}\ } & S_S^r/S_S^rI, \end{array}$$

where $p_A : A_S \to A_S/A_SI$ and $p_{S^r} : S_S^r \to S_S^r/S_S^rI$ denote the canonical projections, is commutative.

Set $D = \ker(h)$. Then $A \oplus C \cong S^r \oplus D$ and, since $\bar{h}$ is an isomorphism, we get $D = DI$. Consequently $[A] - [S^r] = [D] - [C]$, with $C = CI$ and $D = DI$, which proves that $[A] - [S^r]$ is in the image of the map $\psi : G(V(I)) \to K_0(I)$. □

It was asked in [3] whether the map $\psi : G(V(I)) \to K_0(I)$ is injective for any exchange ideal $I$. We now provide a counterexample.

**Example 5.** We consider a modification of an example due to Chuang and Lee, cf. [6]. Let $F$ be a countable field and let $F(t)$ be the field of rational functions in the indeterminate $t$. Chuang and Lee construct in [6] a (von Neumann) regular ring $S$ which is an $F$-subalgebra of the algebra $\mathbb{B}(F)$ of row-and-column finite matrices over $F$ and contains the ideal $A = M(F)$ of matrices with only a finite number of nonzero entries. Moreover there is a surjective $F$-algebra homomorphism $\pi : S \to F(t)$ with kernel $A$, and there are elements $a, b \in S$ such that $\pi(a) = t$, $\pi(b) = t^{-1}$, $ba = 1$ and $1 - ab$ is a one-dimensional idempotent in $A$. Now consider $D = MF[t]_M$, where $M$ is the ideal of $F[t]$ generated by $1 - t$. As $F[t]_M$ is a local ring, $D$ is a radical ring. Set $I = \pi^{-1}(D)$. By [2, Corollary 2.5], $I$ is a non-unital exchange ring. Note that the elements $a - 1$ and $b - 1$ are in $I$.

All the idempotents of $M(I)$ are in $M(A)$ and so $G(V(A)) = G(V(I)) = \mathbb{Z}$. Since $A$ is a regular ring we have $G(V(A)) = K_0(A)$ (cf. [15, Proposition 1.2]). The generator of $K_0(A)$ is given by $[1 - ab]$. Set $B = I^1$. To compute $K_0(I)$ we use the exact sequence in $K$-theory associated to the ideal $A$ of $B$. Note that $B/A \cong D^1$, so the fact that $D$ is a radical ring gives us that $K_0(D^1) = K_0(\mathbb{Z})$, and we obtain an exact sequence

$$K_0(A) \to K_0(B) \to K_0(\mathbb{Z}) \to 0.$$

Thus the map $K_0(A) \to K_0(I)$ is surjective. But since $a-1, b-1 \in I$, the image of the generator $[1-ab]$ of $K_0(A)$ under that map is 0 and so $K_0(I) = 0$. Therefore the map $\psi : G(V(I)) \to K_0(I)$ is the map $\mathbb{Z} \to 0$, and it is not injective.

In [17], D. Quillen introduced a variant $K_0'(I)$ of $K_0(I)$ for a nonunital ring $I$ in the following way. Let $R$ be a unital ring containing $I$ as an ideal, for instance $R$ can be the unitization $I^1 = I \oplus \mathbb{Z}$ of $I$. The group $K_0'(I)$ is the abelian group generated by elements $[f : P \to Q]$, where $P$ and $Q$ are finitely generated projective right $R$-modules, $f : P \to Q$ is a homomorphism and $\bar{f} : P/PI \to Q/QI$ is an isomorphism, subject to the relations:

(1) $[f : P \to Q] + [f' : P' \to Q'] = [f \oplus f' : P \oplus P' \to Q \oplus Q']$;
(2) $[f : P' \to P] + [g : P \to Q] = [gf : P' \to Q]$;
(3) $[f : P \to Q] = 0$ when $f$ is an isomorphism.

It can be shown that $K_0'(I)$ does not depend on the unital ring $R$ in which $I$ is embedded [17, pp. 197–198]. There is a canonical surjection $K_0'(I) \to K_0(I)$ that maps $[f : P \to Q]$ to $[Q] - [P]$. This map is not injective in general [17, p. 208 and §7]. We next show that, for an almost trace ideal $I$, there is a natural isomorphism $K_0'(I) \to G(V(I))$. The canonical surjection $K_0'(I) \to K_0(I)$ factors as the composite mapping of the isomorphism $K_0'(I) \to G(V(I))$ and the natural map $\psi : G(V(I)) \to K_0(I)$. Thus Example 5 shows that $K_0'(I) \to K_0(I)$ is not necessarily injective for an almost trace ideal $I$.

**Theorem 7.2.** *For an almost trace ideal $I$ of a unital ring $R$, the groups $K_0'(I)$ and $G(V(I))$ are canonically isomorphic.*

*Proof.* Define $\varphi : K_0'(I) \to G(V(I))$ as follows. Given a generator $[f : P \to Q]$ of $K_0'(I)$, we can construct by using Lemma 5.4 an exact sequence

$$0 \to H \to P \oplus C \xrightarrow{(f,\tau)} Q \to 0$$

of finitely generated projective modules with $CI = C$. As $\bar{f}$ is an isomorphism, we get that $H = HI$. The mapping $\varphi$ sends $[f : P \to Q]$ to $[C] - [H] \in G(V(I))$. We have to show that this definition does not depend on the choice of the sequence $0 \to H \to P \oplus C \xrightarrow{(f,\tau)} Q \to 0$. If $0 \to H' \to P \oplus C' \xrightarrow{(f,\tau')} Q \to 0$ is another exact sequence with $C'I = C'$ and $H' = H'I$, there is a homomorphism $\binom{\alpha}{\beta} : C \to P \oplus C'$ such that $\tau = (f, \tau')\binom{\alpha}{\beta}$. Similarly, we get a map $\binom{\alpha'}{\beta'} : C' \to P \oplus C$ such that $\tau' = (f, \tau)\binom{\alpha'}{\beta'}$. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & P \oplus C & \xrightarrow{(f,\tau)} & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} & & \| & & \\ 0 & \longrightarrow & H' & \longrightarrow & P \oplus C' & \xrightarrow{(f,\tau')} & Q & \longrightarrow & 0. \end{array}$$

So we have an exact sequence

$$0 \to H \xrightarrow{\gamma} (P \oplus C) \oplus H' \xrightarrow{\begin{pmatrix} 1 & \alpha & * \\ 0 & \beta & * \end{pmatrix}} P \oplus C' \to 0,$$

which splits, and the splitting is determined by a map $P \oplus C' \to P \oplus C \oplus H'$ of the form $A = \begin{pmatrix} 1 & \alpha' \\ 0 & \beta' \\ \delta & \varepsilon \end{pmatrix}$. Let $\Delta$ be the automorphism $\begin{pmatrix} 1_P & 0 & 0 \\ 0 & 1_C & 0 \\ -\delta & 0 & 1_{H'} \end{pmatrix}$ of $P \oplus C \oplus H'$. Then we have

$$P \oplus C \oplus H' = \Delta(A(P \oplus C') \oplus \gamma(H))$$

$$= \Delta A(P) \oplus \Delta A(C') \oplus \Delta\gamma(H) = P \oplus C'' \oplus H''$$

with $C'' \cong C'$ and $H'' \cong H$. Thus $C \oplus H' \cong C'' \oplus H'' \cong C' \oplus H$. This shows that $[C] - [H] = [C'] - [H']$, as desired.

We must now show that $\varphi$ respects the defining relations (1), (2) and (3). This is obvious for (1) and (3). For (2), fix $f : P' \to P$ and $g : P \to Q$. Construct the corresponding exact sequences $0 \to H \xrightarrow{\binom{\alpha}{\beta}} P' \oplus C \xrightarrow{(f,\tau)} P \to 0$ and $0 \to H' \xrightarrow{\binom{\alpha'}{\beta'}} P \oplus C' \xrightarrow{(g,\tau')} Q \to 0$ via Lemma 5.4. As $H'$ is projective and $(f,\tau) \oplus 1_{C'} : P' \oplus C \oplus C' \to P \oplus C'$ is onto, there exists a map $\begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} : H' \to P' \oplus C \oplus C'$ that composed with $(f,\tau) \oplus 1_{C'}$ gives $\binom{\alpha'}{\beta'}$. The exact sequence corresponding to $gf : P' \to Q$ is

$$0 \to H \oplus H' \xrightarrow{\begin{pmatrix} \alpha & \delta_1 \\ \beta & \delta_2 \\ 0 & \delta_3 \end{pmatrix}} P' \oplus C \oplus C' \xrightarrow{(gf, g\tau, \tau')} Q \to 0.$$

This completes the proof that $\varphi$ is a well defined homomorphism.

Now define a homomorphism $V(I) \to K_0'(I)$, $\langle C \rangle \mapsto [0 \to C]$. This monoid homomorphism induces a group homomorphism $\psi : G(V(I)) \to K_0'(I)$ such that $\psi([C] - [H]) = [0 : H \to C]$ for every $[C] - [H] \in G(V(I))$. It is obvious that $\varphi \circ \psi = \mathrm{Id}_{G(V(I))}$. To check that $\psi \circ \varphi = \mathrm{Id}_{K_0'(I)}$, we first consider the case of a split monomorphism $f : P \to Q$ with cokernel $C$ such that $C = CI$. We then have

$$[f : P \to Q] = [f : P \to f(P)] + [0 \to C] = [0 \to C].$$

For the general case, consider a generator $[f : P \to Q]$ of $K_0'(I)$. Consider the corresponding exact sequence

$$0 \to H \to P \oplus C \xrightarrow{(f,\tau)} Q \to 0.$$

Then there is a right inverse $g : Q \to P \oplus C$ of $(f,\tau)$, which is a split monomorphism with cokernel $H$. By the previous case, $[g : Q \to P \oplus C] = [0 \to H]$. Thus $[(f,\tau) : P \oplus C \to Q] = -[g : Q \to P \oplus C] = -[0 \to H] = [H \to 0]$. But $[(f,\tau) : P \oplus C \to Q] = [f \oplus 1_C : P \oplus C \to Q \oplus C] + [(1,\tau) : Q \oplus C \to Q] = [f : P \to Q] -$

$\left[\binom{1}{0} : Q \to Q \oplus C\right] = [f : P \to Q] - [0 \to C]$. We conclude that $[f : P \to Q] = [H \to 0] + [0 \to C] = [0 : H \to C]$. This completes the proof. □

We conclude our paper showing that, if $I$ is a trace ideal of a unital ring $R$, then the natural map $\psi : G(V(I)) \to K_0(I)$ is an isomorphism. It follows then from Theorem 7.2 that the canonical map $K_0'(I) \to K_0(I)$ also is an isomorphism for a trace ideal $I$.

We will use the description of $K_0(R)$ in terms of idempotent matrices over $R$; see for example [19, Section 2]. If $X$ is any additive subgroup of a ring $R$, we denote by $M(X)$ the set of all infinite matrices, indexed by $\mathbb{N} \times \mathbb{N}$, having their entries in $X$ almost all zero. Note that $M(X)$ can be identified with the direct limit $\varinjlim M_n(X)$, via the embeddings $M_n(X) \to M_{n+1}(X)$ given by $a \mapsto \mathrm{diag}(a, 0)$. If $A$ and $B$ are matrices of finite size, then we will denote by $A \oplus B$ the (block) diagonal matrix $\mathrm{diag}(A, B)$. If $e$ is an idempotent in $R$, we will use the notation $e_r$ for the diagonal matrix $\mathrm{diag}(e, \dots, e)$ ($r$ times). If $E, F$ are idempotent matrices in $M(R)$, we write $E \sim F$ if the corresponding finitely generated projective $R$-modules are isomorphic.

**Lemma 7.3.** *Let e be a non-zero idempotent in a ring R. Then $K_0(ReR) \cong K_0(eRe)$. More precisely, the map $\varphi : K_0(eRe) \to K_0((ReR))$, defined by $\varphi([g]_{K_0(eRe)}) = [g]_{K_0((ReR)^1)}$ for all idempotents $g \in M(eRe)$, is an isomorphism.*

*Proof.* Clearly, we have a monoid homomorphism $\gamma : V(eRe) \to V(ReR)$ sending the class in $V(eRe)$ of an idempotent $g$ in $M(eRe)$ to the class of $g$ in $V(ReR)$. It is readily seen that $\gamma$ is injective. To see that it is also surjective, let $E \in M(ReR)$ be an idempotent matrix. Then there exist $r \geq 0$ and $A, B \in M(R)$ such that $E = Ae_rB$. As $E$ is idempotent, $E = EAe_rBE$, so that $e_rBEAe_r \in M(eRe)$ is an idempotent matrix equivalent to $E$, which proves that $\gamma$ is onto. Thus $\gamma$ is an isomorphism, and therefore $G(\gamma) : K_0(eRe) = G(V(eRe)) \to G(V(ReR))$ is a group isomorphism.

Let $\varphi : K_0(eRe) \to K_0(ReR)$ be the composition of the isomorphism $\gamma : K_0(eRe) \to G(V(ReR))$ and the map $\psi : G(V(ReR)) \to K_0(ReR)$. Clearly $\varphi([g]_{K_0(eRe)}) = [g]_{K_0((ReR)^1)}$ for every idempotent $g$ in $M(eRe)$. By Proposition 7.1, the map $\psi$ is surjective, hence so is the map $\varphi$. It only remains to show that $\varphi$ is injective. Let $E, F \in M_n(eRe)$ be two idempotents such that $\varphi([E] - [F]) = 0$. Then there exists $m \geq 1$ with $E \oplus 1_m \sim F \oplus 1_m$ in $M_{n+m}((ReR)^1)$. Let $A, B \in M_{n+m}((ReR)^1)$ be such that

$$(4) \qquad (E \oplus 1_m)A = A, \quad A(F \oplus 1_m) = A, \quad (F \oplus 1_m)B = B, \quad B(E \oplus 1_m) = B$$

and

$$(5) \qquad AB = E \oplus 1_m, \quad BA = F \oplus 1_m.$$

Let $\pi : (ReR)^1 \to \mathbb{Z}$ be the canonical map. Observe that $\pi(A)\pi(B) = 0_n \oplus 1_m$ and $\pi(B)\pi(A) = 0_n \oplus 1_m$. Moreover, by conditions (4), we have $\pi(A) = 0_n \oplus Z_1$ and $\pi(B) = 0_n \oplus Z_2$ for some matrices $Z_1, Z_2 \in M_m(\mathbb{Z})$, so that $Z_2 = Z_1^{-1}$. Replacing $A$ with $(1_n \oplus Z_1^{-1})A$ and $B$ with $B(1_n \oplus Z_1)$, we see that, in addition to the other properties, we can assume that $(e_n \oplus 1_m) - A \in M_{n+m}(ReR)$ and $(e_n \oplus 1_m) - B \in$

$M_{n+m}(ReR)$. Write $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 1_m + a_{22} \end{pmatrix}$, where $a_{11} \in EM_n(eRe)F$, $a_{12} \in EM_{n\times m}(ReR)$, $a_{21} \in M_{m\times n}(ReR)F$ and $a_{22} \in M_m(ReR)$. Note that

$$(6) \qquad (e_n \oplus 1_m) - A = \begin{pmatrix} e_n - a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}.$$

One can find a positive integer $r$ and matrices $a'_{21} \in M_{m\times r}(Re)$ and $a''_{21} \in M_{r\times n}(eR)$ and $a''_{22} \in M_{r\times m}(eR)$ such that $a_{21} = a'_{21}a''_{21}$ and $a_{22} = a'_{21}a''_{22}$. Moreover, since $a_{21} = a_{21}F$, we can assume that $a''_{21} = a''_{21}F$. Now set

$$(7) \qquad Y = \begin{pmatrix} e_n & 0 \\ 0 & a'_{21} \end{pmatrix} \in M_{(n+m)\times(n+r)}(Re),$$

$$(8) \qquad X = \begin{pmatrix} e_n - a_{11} & -a_{12} \\ -a''_{21} & -a''_{22} \end{pmatrix} \in M_{(n+r)\times(n+m)}(eR).$$

Note that, by (6), (7) and (8), we have $YX = (e_n \oplus 1_m) - A$. Also observe that

$$(9) \qquad (E \oplus 1_m)Y = Y(E \oplus e_r), \quad (F \oplus 1_m)Y = Y(F \oplus e_r).$$

Consider the following matrices in $M_{n+r}(eRe)$:

$$A' = e_{n+r} - XY, \quad B' = (E \oplus e_r) + XBY.$$

Then

$$\begin{aligned} A'B' &= (E \oplus e_r) - XY(E \oplus e_r) + XBY - X(YX)BY \\ &= (E \oplus e_r) - XY(E \oplus e_r) + XBY - X[(e_n \oplus 1_m) - A]BY \\ &= (E \oplus e_r) - XY(E \oplus e_r) + XBY - XBY + X(E \oplus 1_m)Y \\ &= (E \oplus e_r) - XY(E \oplus e_r) + X(E \oplus 1_m)Y. \end{aligned}$$

From the first equation in (9), we get $A'B' = (E \oplus e_r)$.

On the other hand, we have

$$\begin{aligned} B'A' &= (E \oplus e_r) + XBY - (E \oplus e_r)XY - XB(YX)Y \\ &= (E \oplus e_r) + XBY - (E \oplus e_r)XY - XB[(e_n \oplus 1_m) - A]Y \\ &= (E \oplus e_r) + XBY - (E \oplus e_r)XY - XBY + X(F \oplus 1_m)Y \\ &= (E \oplus e_r) - (E \oplus e_r)XY + X(F \oplus 1_m)Y. \end{aligned}$$

From the second equation in (9), we get

$$B'A' = (E \oplus e_r) - (E \oplus e_r)XY + XY(F \oplus e_r)$$
$$= (F \oplus e_r) + (E \oplus e_r)(e_{n+r} - XY) - (e_{n+r} - XY)(F \oplus e_r).$$

Now

$$e_{n+r} - XY = \begin{pmatrix} a_{11} & a_{12}a'_{21} \\ a''_{21} & e_r + a''_{22}a'_{21} \end{pmatrix}.$$

As $Ea_{11} = a_{11}$, $Ea_{12} = a_{12}$, $e_r a''_{21} = a''_{21}$ and $e_r a''_{22} = a''_{22}$, we get that $(E \oplus e_r) \cdot (e_{n+r} - XY) = e_{n+r} - XY$. Similarly, from $a_{11} = a_{11}F$, $a'_{21} = a'_{21}e_r$ and $a''_{21}F = a''_{21}$, we obtain that $(e_{n+r} - XY)(F \oplus e_r) = e_{n+r} - XY$. Therefore $B'A' = F \oplus e_r$.

This shows that $[E] - [F] = 0$ in $K_0(eRe)$, and $\varphi$ is injective. □

**Theorem 7.4.** *Let $I$ be a trace ideal of a ring $R$. Then the natural map $\psi : G(V(I)) \to K_0(I)$ is a group isomorphism.*

*Proof.* By Proposition 7.1, the map $\psi$ is surjective. It remains to show that it is injective. Let $E$ and $F$ be idempotents in $M_n(I)$, for some $n \geq 1$, such that $\psi([E] - [F]) = 0$. Then there exists $m \geq 1$ such that $E \oplus 1_m \sim F \oplus 1_m$ in $M_{n+m}(I^1)$. As in the proof of Lemma 7.3, we can assume that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 1_m + a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & 1_m + b_{22} \end{pmatrix}$$

for some $a_{ij} \in M_*(I)$ and $b_{ij} \in M_*(I)$. (Here the notation $x \in M_*(X)$ means that $x$ is a matrix with entries in $X$ of suitable size.)

Since $I$ is a trace ideal, the non-unital ring $M(I)$ is generated as a two-sided ideal by its idempotents. It follows that there are $k \geq \max\{n, m\}$ and an idempotent $e \in M_k(I)$ such that $a_{ij} \oplus 0, b_{ij} \oplus 0 \in M_k(R)eM_k(R)$ for all $i, j \in 1, 2$, for suitably sized zero matrices. In particular, identifying $E \oplus 0_{k-n}$ with $E$ and $F \oplus 0_{k-n}$ with $F$, we see that $E, F \in M_k(R)eM_k(R)$. Put $S = M_k(R)$ and observe that $E \oplus 1 \sim F \oplus 1$ in $M_2((SeS)^1)$.

There are $x_i, y_i, z_i, t_i \in S$, for $i = 1, \ldots, r$, such that $E = \sum_{i=1}^r x_i e y_i$ and $F = \sum_{i=1}^r z_i e t_i$. Set $E' = e_r(y_1, \ldots, y_r)^T E(x_1, \ldots, x_r)e_r \in M_r(eSe)$, and $F' = e_r(t_1, \ldots, t_r)^T F(z_1, \ldots, z_r)e_r \in M_r(eSe)$. Then $E'$ and $F'$ are idempotents with $E' \sim E$ and $F' \sim F$ in $M_r(SeS)$. Let $\varphi : K_0(eSe) \to K_0(SeS)$ be the map defined in Lemma 7.3. Then $\varphi([E'] - [F']) = [E'] - [F'] = [E] - [F] = 0$ in $K_0(SeS)$. By Lemma 7.3, we get $[E'] - [F'] = 0$ in $K_0(eSe)$ and it follows that $[E] - [F] = [E'] - [F'] = 0$ in $K_0(I)$. This completes the proof. □

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