A parametrization of sheets of conjugacy classes in bad characteristic

Filippo Ambrosio, Giovanna Carnovale, Francesco Esposito
Dipartimento di Matematica "Tullio Levi-Civita"
Torre Archimede - via Trieste 63 - 35121 Padova - Italy
email: ambrosio@math.unipd.it,
carnoval@math.unipd.it, esposito@math.unipd.it

Abstract

Let G be a simple algebraic group of adjoint type over an algebraically closed field k of bad characteristic. We show that its sheets of conjugacy classes are parametrized by G-conjugacy classes of pairs $(M, \mathcal{O})$ where M is the identity component of the centralizer of a semisimple element in G and $\mathcal{O}$ is a rigid unipotent conjugacy class in M, in analogy with the good characteristic case. We explicitly describe the possible choices for M.

1 Introduction

Sheets in a reductive algebraic group G are the irreducible components of the locally closed subsets of G consisting of conjugacy classes of the same dimension. They occur also as irreducible components of the strata the partition of G, defined in [9] in terms of Springer representations with trivial local system, [2, 3]. One of the most fascinating features of strata is that they are parametrized by a family of irreducible representations of the Weyl group which depends on the root system of G and not on the characteristic. It is therefore of interest to figure out the behaviour of the irreducible components of strata when the characteristic of the base field varies.

A description of sheets in good characteristic, and a parametrization of sheets in terms of G-conjugacy classes of triples $(M, Z(M)^\circ s, \mathcal{O})$ where M is the identity component of the centralizer of a semisimple element in G, $Z(M)^\circ s$ is a suitable coset in the component group $Z(M)/Z(M)^\circ$, and $\mathcal{O}$ is a rigid unipotent conjugacy class in M was given in [4] in good characteristic, and extended to the case of bad characteristic in [13]. A refinement of this parametrization in terms of pairs $(M, \mathcal{O})$ where M and $\mathcal{O}$ are as above was given in [3] under the assumption that G is simple of adjoint type and the characteristic of the base field is good for G. The present paper answers a question by G. Lusztig on the extension to arbitrary characteristic of this parametrization of sheets.

Observe that, even if the formulation of the statement is the same, the collection of possible centralizers of a semisimple element in G varies with the characteristic of the base field, as well as the collection of unipotent conjugacy classes. Centralizers of semisimple elements are fewer in bad characteristic than in good characteristic, [5, 6, 7] whilst the number of unipotent conjugacy classes may increase when passing from good to bad characteristic.

We therefore elaborate upon results in [5, 6, 7] in the spirit of [12] in order to provide a combinatorial description of the root systems of centralizers of semisimple elements, which will allow us to retrieve most ingredients that were necessary for the proof of [3, Theorem 4.1]. This will allow us to show that the number of sheets in type G_2 is the same in good characteristic and in characteristic 3, but it is smaller in characteristic 2. In [9] G. Lusztig defined sets of representations of the Weyl group $\mathcal{S}_2(W)$ and $\mathcal{S}_2^p(W)$ for every prime p. We wonder whether for those primes p for which $\mathcal{S}_2(W) = \mathcal{S}_2^p(W)$, the number of sheets for G simple of adjoint type in good characteristic and in characteristic p are equal.

2 Notation

Let G be a connected reductive algebraic group defined over an algebraically closed field k of characteristic exponent p. Let Φ be the root system of G and $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a fixed base of Φ. If Φ is irreducible, the numbering of simple roots will be as in [1] and we will denote by α_0 the opposite of the highest root in Φ and by d_i the coefficient of α_i in the expression of $-\alpha_0$. We set $d_0 := 1$ and $\widetilde{\Delta} = \Delta \cup \{\alpha_0\}$. For a subset $S \subset \widetilde{\Delta}$ we define $d_S := \gcd(d_i \mid \alpha_i \notin S)$. In particular, $d_S = 1$ if $S \subset \Delta$.

The group acts on itself by conjugation and we denote $g \cdot h = ghg^{-1}$ for $g, h \in G$ and $G \cdot g$ the G-conjugacy class of $g \in G$. For a closed subgroup $H \leq G$, the identity component will be denoted by H°, and for $g \in G$ the centralizer will be denoted by G_g. We will call G_g° the connected centralizer of g. The Jordan decomposition of an element $g \in G$ will be usually denoted by $g = su$.

For $m \in \mathbb{N}$ we set $G_{(m)} := \{g \in G \mid \dim(G \cdot g) = m\}$. These sets are locally-closed and their irreducible components are called the *sheets* of the G-action. For $Z \subset G$

we also define $m_Z := \max\{m \in \mathbb{N} \mid G_{(m)} \cap Z \neq \emptyset\}$ and $Z^{reg} := Z \cap G_{(m_Z)}$, and $C_G(Z)$ will indicate the centralizer of Z in G.

If we fix a maximal torus T of G and Φ is the root system of G with respect to T, then for $\alpha \in \Phi$ we indicate by X_α the corresponding root subgroup. For a closed subset $\Psi \subset \Phi$, (see [1, VI, n. 1.7, Définition 4]), and for $s \in T$ we set $G_\Psi := \langle T, X_\alpha \mid \alpha \in \Psi \rangle$.

2.1 Construction of sheets and a first parametrization

It was observed in [8, §3] that G has a partition into finitely many, locally closed, smooth, irreducible, G-stable sets, which we call Jordan classes, each contained in some $G_{(m)}$. As a set, the class containing $g = su$ is

$$J(su) = G \cdot ((Z(G_s^\circ)^\circ s)^{reg} u).$$

In other words, a G-conjugacy class lies in $J(su)$ if and only if it contains an element with Jordan decomposition $s'u$ with $G_s^\circ = G_{s'}^\circ$ and $s' \in Z(G_s^\circ)^\circ s$. The closure of a Jordan class is a union of Jordan classes, [8, §3], hence the same holds for the regular locus of the closure of a Jordan class. This gives a partial order on the set of Jordan classes given by $J_1 \leq J_2$ if and only if $J_1 \subset \overline{J_2}^{reg}$. The sheets in G are the locally closed sets of the form $\overline{J}^{reg}$ where J is maximal with respect to $\leq$. Hence the set of sheets in G is in bijection with the set $\mathcal{J}$ consisting of maximal Jordan classes [4, Proposition 5.1], [13, §3].

If $J = J(su)$ as above, then [4, Proposition 4.8],[3, Lemma 2.1] give:

$$\overline{J(su)}^{reg} = \bigcup_{z \in Z(G_s^\circ)^\circ s} G \cdot (z \mathrm{Ind}_{G_s^\circ}^{G_z^\circ}(G_s^\circ \cdot u))$$

where $\mathrm{Ind}_{G_s^\circ}^{G_z^\circ}(G_s^\circ \cdot u)$ is Lusztig-Spaltenstein's induced unipotent conjugacy class, [10]. The maximal Jordan classes are precisely those for which the class of u is *rigid* in G_s°, i. e., it is not induced from any unipotent class in a proper Levi subgroup of a parabolic subgroup of G_s°, [4, Proposition 5.3], [13, Lemma 2.4].

Sheets are then parametrized as follows.

Theorem 2.1. ([4, Theorem 5.6], [13, Theorem 3.1]) *The assignment $J = J(su) \mapsto (G_s^\circ, Z(G_s^\circ)^\circ s, G_s^\circ \cdot u)$ induces a bijection between $\mathcal{J}$ and the set of G-orbits of triples $(M, Z(M)^\circ r, \mathcal{O})$ where M is the connected centralizer of a semisimple element of G; $Z(M)^\circ r$ is a coset in $Z(M)/Z(M)^\circ$ satisfying $C_G(Z(M)^\circ r)^\circ = M$ and $\mathcal{O}$ is a rigid unipotent conjugacy class in M.*

□

We aim at a simpler parametrization for G simple and of adjoint type.

2.2 Connected centralizers of semisimple elements

In this subsection G is quasisimple. Identity components of centralizers of semisimple elements have been studied in [5, 6, 7, 12]. In the spirit of the latter, we give a combinatorial characterization of the root subsystem of such subgroups when the base field is an arbitrary algebraically closed field. We believe it to be well-known but we could not locate a proper reference.

Proposition 2.2. *Let G be quasisimple, T be a maximal torus in G and Ψ be a closed subset of Φ. Then G_Ψ is the connected centralizer of an element in T if and only if Ψ is conjugate to a root subsystem Ψ' admitting a base $\Delta_{\Psi'} \subset \widetilde{\Delta}$ and such that* $\gcd(p,\, d_i \mid i \in \widetilde{\Delta} \setminus \Delta_{\Psi'}) = 1$ *by an element in the normalizer $N(T)$ of T.*

Proof. If Ψ is $N(T)$-conjugate to a root subsystem Ψ' admitting a base $\Delta_{\Psi'} \subset \widetilde{\Delta}$ and such that $\gcd(p,\, d_i \mid i \in \widetilde{\Delta} \setminus \Delta_{\psi'}) = 1$, then there exists an $\alpha_{\bar{i}} \in \widetilde{\Delta} \setminus \Delta_{\Psi'}$ such that $p \not| d_{\bar{i}}$. Replacing a_1 with $d_{\bar{i}}$ in the proof of [12, Proposition 32], we obtain an element $s \in Z(G_\Psi)$ such that $G_s^\circ = G_\Psi$.

Assume now that $G_\Psi = G_s^\circ$ for some $s \in T$. By [12, Proposition 30] the subgroup G_Ψ is G-conjugate to some $G_{\Psi'}$ where Ψ' admits a base $\Delta_{\Psi'} \subset \widetilde{\Delta}$. Conjugacy of maximal tori in $G_{\Psi'}$ ensures that conjugation of the two subgroups, and of the corresponding root systems can be obtained using an element in $N(T)$. We show that $\gcd(p, d_{\Delta_{\Psi'}}) = 1$. If the characteristic exponent $p = 1$, then there is nothing to prove. Assume for a contradiction that $p > 1$ and divides d_i for every $\alpha_i \in \widetilde{\Delta} \setminus \Delta_{\Psi'}$. Since $d_0 = 1$, in this situation $\alpha_0 \in \Delta_{\Psi'}$ and $G_{\Psi'}$ is never the Levi subgroup of a parabolic subgroup of G.

Let $\{\omega_j^\vee,\, j = 1, \ldots, n\}$ be the basis for the cocharacters of T dual to Δ and let $t \in T$ be such that $G_{\Psi'} = G_t^\circ$. Since $\alpha_i(t) = 1$ for every $\alpha_i \in \Delta_{\Psi'}$, we have $t = \prod_{\substack{j=1,\ldots,n \\ \alpha_j \notin \Delta_\Psi}} \omega_j^\vee(\zeta_j)$, and $\alpha_0(t) = 1$ gives

$$\prod_{j\geq 1,\, \alpha_j \notin \Delta_{\Psi'}} \zeta_j^{d_j} = 1, \text{ so also } \prod_{j\geq 1,\, \alpha_j \notin \Delta_{\Psi'}} \zeta_j^{d_j/p^l} = 1 \text{ if } p^l | d_{\Delta_{\Psi'}}. \tag{2.1}$$

By [1, VI §1, n. 1.7, Prop 24] the system Ψ' is not $\mathbb{Q}$-closed, i.e., there exists an $\alpha \in (\mathbb{Q}\Delta_{\Psi'} \cap \Phi) \setminus \Psi'$. In other words, there exist $a,\, b,\, a_i,\, b_i \in \mathbb{Z}$, for $i = 1, \ldots, n$ such that $\alpha_i \notin \Delta_{\Psi'}$, with $b, b_i \neq 0$ and $z_j \in \mathbb{Z}$ for $j = 1, \ldots, n$ such that

$$\alpha = \frac{a}{b}\alpha_0 + \sum_{\alpha_i \in \Delta_{\Psi'} \cap \Delta} \frac{a_i}{b_i}\alpha_i = \sum_{j=1}^{n} z_j \alpha_j \in \Phi \setminus \Psi'.$$

Hence $b|d_i$ for all $\alpha_i \notin \Delta_{\Psi'}$, that is, $b|d_{\Delta_{\Psi'}}$. Observe that $d_{\Delta_{\Psi'}}$ is either p^l for some $l \geq 1$ or else $d_{\Delta_{\Psi'}} = 6$, $\Phi = E_8$, $p = 2$ or 3, and $\Delta_{\Psi'} = \widetilde{\Delta} \setminus \{\alpha_4\}$. In the first case (2.1) gives $\alpha(t) = 1$, a contradiction. In the second case, $t = \omega_4^\vee(\zeta_4)$ and (2.1) becomes $\zeta_4^{6/p} = 1$. Let $\beta \in \Phi \setminus \Psi'$ be such that its coefficient of α_4 in its expression as a sum of simple roots is $6/p$. Then again $\beta(t) = 1$, a contradiction.

□

Proposition 2.3. *Let G be simple of adjoint type, T be a maximal torus in G, and $\Psi \subset \Phi$ be a closed subsystem such that $G_\Psi = G_s^\circ$ for some $s \in T$. Assume in addition that Ψ admits a base $\Delta_\Psi \subset \widetilde{\Delta}$. Then*

(a) *The torsion subgroup of $\mathbb{Z}\Phi/\mathbb{Z}\Psi$ is $\mathbb{Z}/d_{\Delta_\Psi}\mathbb{Z}$.*

(b) *$Z(G_\Psi)/Z(G_\Psi)^\circ$ is cyclic of order d_{Δ_Ψ}.*

(c) *For $t \in Z(G_\Psi)$ we have $C_G(Z(G_\Psi)^\circ t) = G_\Psi$ if and only if $Z(G_\Psi)^\circ t$ is a generator of $Z(G_\Psi)/Z(G_\Psi)^\circ$.*

Proof. (a) This is observed in [14, §2].
(b) The order of the torsion subgroup of $\mathbb{Z}\Phi/\mathbb{Z}\Psi$ is coprime with p by Proposition 2.2. Hence we are in a position to use the argument in [14, §2.1], [12, Lemma 33], that we sketch for completeness. By construction and [11, Proposition 3.8] from which we borrow notation, $Z(G_\Psi) = (\mathbb{Z}\Psi)^\perp$, $Z(G_\Psi)^\perp = \mathbb{Z}\Psi$, and the character group $X(Z(G_\Psi))$ is $\mathbb{Z}\Phi/\mathbb{Z}\Psi$. Then

$$X(Z(G_\Psi)/Z(G_\Psi)^\circ) \simeq \{\chi \in X(Z(G_\Psi)) \mid \chi(z) = 1, \forall z \in Z(G_\Psi)^\circ\}$$

consists of those torsion elements in $\mathbb{Z}\Phi/\mathbb{Z}\Psi$ whose order is coprime with p, that is, $\mathbb{Z}/d_{\Delta_\Psi}\mathbb{Z}$.
(c) In good characteristic this is [12, Lemma 34 (1)], whose proof relies on the fact that every character of $Z(G_\Psi)/Z(G_\Psi)^\circ$ can be represented by an element in Φ. The proof of this statement depends on considerations on root systems which are characteristic-free and on a natural isomorphism between $X(Z(G_\Psi)/Z(G_\Psi)^\circ)$ and $\mathbb{Z}(\mathbb{Q}\Psi \cap \Phi)/\mathbb{Z}\Psi$, which remains valid because $X(Z(G_\Psi)/Z(G_\Psi)^\circ)$ is the torsion subgroup of $\mathbb{Z}\Phi/\mathbb{Z}\Psi$ also in our situation. □

2.3 The parametrization

In this Section G is simple of adjoint type. We are now in a position to prove the refinement of the parametrization of sheets of G. The general case can be readily deduced by standard arguments.

Theorem 2.4. *Let G be simple and of adjoint type. The sheets in G are in bijection with the G-conjugacy classes of pairs $(M, \mathcal{O})$ where M is the connected centralizer of a semisimple element in G and $\mathcal{O}$ is a rigid unipotent conjugacy class in M.*

Proof. In good characteristic this is [2, Theorem 4.1], so we assume that p is bad for G. Sheets are parametrized by triples $(M, Z(M)^{\circ}s, \mathcal{O})$ as in Theorem 2.1. The assignment $(M, Z(M)^{\circ}s, \mathcal{O}) \mapsto (M, \mathcal{O})$ induces a well-defined and surjective map between the set of G-conjugacy classes of triples and the set of G-conjugacy classes of pairs as above. We show injectivity of this map. If G is classical, then M is a Levi subgroup of a parabolic subgroup of G, for any pair $(M, \mathcal{O})$, hence $Z(M) = Z(M)^{\circ}$ and there is nothing to prove, so we assume that G is of exceptional type.

Let $(M, Z(M)^{\circ}s, \mathcal{O})$ and $(M, Z(M)^{\circ}r, \mathcal{O})$ be two triples inducing the same image. Without loss of generality $s \in T$, $M = G_{\Psi}$ where Ψ has base $\Delta_{\Psi} \subset \widetilde{\Delta}$, and $Z(M)^{\circ}s$, $Z(M)^{\circ}r \subset T$. If $d_{\Delta_{\Psi}} \leq 2$, then necessarily $Z(M)^{\circ}r = Z(M)^{\circ}s$, so we assume that $d_{\Delta_{\Psi}} \geq 3$.

By [14, Proposition 7] there is a w in the stabilizer $N_W(\Delta_{\Psi})$ of Δ_{Ψ} in W whose action on $\mathbb{Z}\Phi/\mathbb{Z}\Psi$ generates the automorphism group of the torsion subgroup of $\mathbb{Z}\Phi/\mathbb{Z}\Psi$, which is isomorphic to $Z(M)/Z(M)^{\circ}$ by Propoisition 2.3 (a). We claim that any representative of w in $N(T)$ preserves $\mathcal{O}$. Since rigid unipotent classes in type A are trivial, it is enough to consider only the case in which Δ_{Ψ} contains a (necessarily unique) component of type different from A, and $\mathcal{O}$ is non-trivial in the corresponding subgroup. Such a component is always preserved by the action of w. The list of Δ_{Ψ} with $d_{\Delta_{\Psi}} \geq 3$ in the proof of [14, Proposition 7] shows that we only need to consider two cases for G of type E_8, namely $\Delta_{\Psi} = A_3 + D_5$ which may occur only when $p = 3, 5$, and $\Delta_{\Psi} = A_2 + E_6$. Unipotent conjugacy classes in type D for $p = 3$ and 5 are characteristic unless their Jordan form corresponds to a very even partition, i.e., a partition with only even terms, each occurring an even number of times. Such partitions never occur in D_n for n odd. Rigid unipotent conjugacy classes in E_6 in arbitrary characteristic can be deduced from [15, Chapitre II. Appendice] and they are characteristic for dimensional reasons. □

Fixing a maximal torus T in G, by standard arguments we retrieve a parametrization of sheets by orbits of the Weyl group. We set $\mathcal{T}$ to be the set of pairs $(M, \mathcal{O})$ where M is the connected centralizer of an element in T and $\mathcal{O}$ is a rigid unipotent class in M, so $N(T)$ and $W = N(T)/T$ naturally act on $\mathcal{T}$.

Corollary 2.5. *Let G be simple and of adjoint type. The sheets in G are parametrized by elements in $\mathcal{T}/W$.*

□

2.4 On the number of sheets in G

It was observed in [13, Remark 3.3] that for G of type B_2, the number of sheets is independent of the characteristic and it was suggested this to hold in general for G connected and simply connected. This fails in general because there exist sheets that are obtained from one another by multiplication by a central element, and such central element might no longer exist in bad characteristic: for example in $G = SL_2(k)$, the sheets are: $\{\mathrm{id}\}$, $\{-\mathrm{id}\}$ and G^{reg} for $p \neq 2$ and $\{\mathrm{id}\}$ and G^{reg} for $p = 2$. The following remark shows that the number of sheets depends on p also for G simple of adjoint type.

Remark 2.6. Let $\Phi = G_2$. We use the parametrization in Theorem 2.4. The semisimple parts of the connected centralizers for p good are of type: G_2, A_2, $A_1 + \widetilde{A}_1$, A_1, $\widetilde{A}_1$ or conjugate to T. In type A all rigid unipotent classes are trivial, and there are 3 rigid unipotent classes in G, [15, Chapitre II. Appendice]. Hence, there are 8 sheets for p good.

According to Proposition 2.7 the semisimple parts of the connected centralizers for $p = 2$ are of type: G_2, A_2, A_1, $\widetilde{A}_1$ or conjugate to T, and there are 3 rigid unipotent classes in G, [15, Chapitre II. Appendice]. Hence, there are 7 sheets for $p = 2$.

According to Proposition 2.7 the semisimple parts of the connected centralizers for $p = 3$ are of type: G_2, $A_1 + \widetilde{A}_1$, A_1, $\widetilde{A}_1$ or conjugate to T, and there is an extra unipotent conjugacy class in G_2 which is rigid, as it can be deduced from the list of induced classes in [15, Chapitre II. Appendice]. Hence, the number of sheets for $p = 3$ equals the number of sheets for p good.

Appendix

For the reader's convenience and for further reference we list the possible connected centralizers in bad characteristic, obtained making use of the analysis of W-conjugacy classes of subsets of $\widetilde{\Delta}$ in [14, §2.2]. In most cases these classes are determined by their isomorphism type and the root lengths. For $\Phi = E_7$ and E_8 we remove ambiguities adopting, as in *loc. cit.*, Dynkin's convention. Namely, for $n = 7, 8$ we decorate with one prime the root subsystems which can be embedded in the subsystem of type A_n within E_n while we decorate with two primes the root subsystems with the same label which cannot be embedded in $A_n \subset E_n$.

Proposition 2.7. *Let G be quasisimple with p bad for G. Let T be a maximal torus in G and $\Psi \subset \Phi$ be a closed subsystem with base $\Delta_\Psi \subset \widetilde{\Delta}$.*

If Φ is of classical type, then G_Ψ is the connected centralizer of an element in T if and only if it is the Levi subgroup of a parabolic subgroup of G, [6].

If Φ is of exceptional type, then G_Ψ is the connected centralizer of an element in T unless G, p and Ψ occur in Table 1.

G	p	Ψ
E_6	2	$4A_1$, A_3+2A_1, A_5+A_1
E_6	3	$3A_2$
E_7	2	$(4A_1)'$, $(A_3+2A_1)'$, $(A_5+A_1)'$, A_7, $5A_1$, A_3+3A_1, $2A_3$, D_4+2A_1, D_6+A_1, $2A_3+A_1$
E_7	3	$3A_2$, $3A_2+A_5$
E_8	2	$(4A_1)''$, $(A_3+2A_1)''$, $(2A_3)''$, $(A_1+A_5)''$, A_7'', $5A_1$, $3A_1+A_3$, $2A_1+D_4$, $4A_1+A_2$, $2A_1+A_5$, A_3+D_4, A_1+D_6, A_1+2A_3, $2A_1+A_2+A_3$, $2A_1+D_5$, D_8, A_1+A_7, $A_1+A_2+A_5$, A_3+D_5, A_1+E_7
E_8	3	$3A_2$, A_2+A_5, $3A_2+A_1$, A_8, $A_1+A_2+A_5$, A_2+E_6
E_8	5	$2A_4$
F_4	2	$2A_1$, $2A_1+\tilde{A}_1$, A_1+B_2, A_3, A_1+C_3, $A_3+\tilde{A}_1$, B_4
F_4	3	$A_2+\tilde{A}_2$
G_2	2	$A_1+\tilde{A}_1$
G_2	3	A_2

Table 1: Root subsystems to be discarded

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