Arch. Rational Mech. Anal. 242 (2021) 1289–1316
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-021-01704-w

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Characterization of Minimizers of Aviles–Giga Functionals in Special Domains

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Communicated by M. Ortiz

Abstract

We consider the singularly perturbed problem $F_\varepsilon(u,\Omega) := \int_\Omega \varepsilon|\nabla^2 u|^2 + \varepsilon^{-1}|1-|\nabla u|^2|^2$ on bounded domains $\Omega \subset \mathbb{R}^2$. Under appropriate boundary conditions, we prove that if Ω is an ellipse, then the minimizers of $F_\varepsilon(\cdot,\Omega)$ converge to the viscosity solution of the eikonal equation $|\nabla u| = 1$ as $\varepsilon \to 0$.

1. Introduction

1.1. The Main Result

We consider the family of functionals

$$F_\varepsilon(u,\Omega) := \int_\Omega \left(\varepsilon|\nabla^2 u|^2 + \frac{1}{\varepsilon}\left|1-|\nabla u|^2\right|^2 \right) dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a C^2 bounded open set, $\varepsilon > 0$ and $u \in W_0^{2,2}(\Omega)$. These functionals were introduced in [4] and proposed as a model for blistering in [27]. In these cases we are interested in the minimizers u_ε of F_ε in the space

$$\Lambda(\Omega) := \left\{ u \in W_0^{2,2}(\Omega) : \frac{\partial u}{\partial n} = -1 \text{ on } \partial\Omega \right\},$$

where n denotes the outer normal to Ω. The final goal is the understanding of the behavior of u_ε as $\varepsilon \to 0$. In [27] (and more explicitly in [5]) it is conjectured that

$$u_\varepsilon \to \bar{u} := \text{dist}(\cdot, \partial\Omega), \tag{1.2}$$

The author has been supported by the SNF Grant 182565

at least for convex domains Ω. A first partial result in this direction was obtained in [16, Theorem 5.1], where the authors proved that if Ω is an ellipse, then

$$\lim_{\varepsilon\to 0} \min F_\varepsilon(\cdot,\Omega) = F_0(\bar{u},\Omega), \tag{1.3}$$

where F_0 is the candidate asymptotic functional that we are going to introduce in (1.4).

The main result of this paper is the proof of (1.2) in the same setting as in [16], namely

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an ellipse and, for every $\varepsilon > 0$, let u_ε be a minimizer of $F_\varepsilon(\cdot,\Omega)$. Then*

$$\lim_{\varepsilon\to 0} u_\varepsilon = \mathrm{dist}(\cdot,\partial\Omega) \qquad \text{in } W^{1,1}(\Omega).$$

This result is obtained as a corollary after showing that $\bar{u}$ is the unique minimizer of a suitable asymptotic problem for $F_\varepsilon(\cdot,\Omega)$ as $\varepsilon \to 0$. In order to rigorously introduce it, we recall some previous results (see also the introduction of [10] for a presentation of the history of the problem).

1.2. Previous Results

In what follows, Ω denotes a C^2 bounded open subset of $\mathbb{R}^2$. Independently from the validity of 1.2, it is conjectured already in [4] that

(1) if u_ε is such that $\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\Omega) < \infty$, then u_ε converges up to subsequences to a Lipschitz solution u of the eikonal equation $|\nabla u| = 1$;
(2) if u_ε is a sequence of minimizers of $F_\varepsilon(\cdot,\Omega)$, then any limit u of u_ε minimizes the functional

$$F_0(v,\Omega) := \frac{1}{3}\int_{J_{\nabla v}} |\nabla v^+ - \nabla v^-|^3 d\mathcal{H}^1, \tag{1.4}$$

among the solutions of the eikonal equation. Here, $J_{\nabla v}$ denotes the jump set of ∇v and $\nabla v^\pm$ the corresponding traces.

A positive answer to the first point was obtained independently in [12] and [2]. A fundamental notion in this analysis and in particular in [12] is the one of entropy, borrowed from the field of conservation laws.

Definition 1.2. We say that $\Phi \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ is an *entropy* if for every open set $\Omega \subset \mathbb{R}^2$ and every smooth $m : \Omega \to \mathbb{R}^2$ it holds that

$$\Big(\mathrm{div}\, m = 0 \text{ and } |m|^2 = 1\Big) \quad \Rightarrow \quad \mathrm{div}(\Phi(m)) = 0. \tag{1.5}$$

We will denote by $\mathcal{E}$ the set of entropies.

We will consider the following family of entropies introduced first in [6,16]:

$$\Sigma_{\alpha_1,\alpha_2}(z) := \frac{4}{3}\left((z\cdot\alpha_2)^3\alpha_1 + (z\cdot\alpha_1)^3\alpha_2\right).$$

there (α_1,α_2) is an orthonormal system in $\mathbb{R}^2$.

Collecting the results of [12] and [2] we get the following statement:

Theorem 1.3. *Let $\varepsilon_k \to 0$ and $u_k \in W_0^{2,2}(\Omega)$ be such that $\limsup_{k\to\infty} F_{\varepsilon_k}(u_k,\Omega) < \infty$. Then $m_k := \nabla^\perp u_k$ is pre-compact in $L^1(\Omega)$. Moreover if m_k converges to m in $L^1(\Omega)$, then $|m| = 1$ a.e. in Ω, for every entropy $\Phi \in \mathcal{E}$ it holds that*

$$\mu_\Phi := \operatorname{div}\Phi(m) \in \mathcal{M}(\Omega),$$

where $\mathcal{M}(\Omega)$ denotes the set of finite Radon measures on Ω, and

$$\left(\bigvee_{(\alpha_1,\alpha_2)} |\operatorname{div}\Sigma_{\alpha_1,\alpha_2}(m)|\right)(\Omega) \le \liminf_{k\to\infty} F_{\varepsilon_k}(u_k,\Omega),$$

where $\bigvee$ denotes the supremum operator on non-negative measures (see for example [3, Def. 1.68]).

Theorem 1.3 motivates the introduction of the following space of vector fields, which contains all the limits of sequences $\nabla^\perp u_{\varepsilon_k}$, where u_{ε_k} have equi-bounded energy.

Definition 1.4. We denote by $A(\Omega)$ the set of all $m \in L^\infty(\Omega;\mathbb{R}^2)$ such that

$$\operatorname{div} m = 0 \quad \text{in } \mathcal{D}'(\Omega), \qquad |m|^2 = 1 \quad \mathscr{L}^2\text{-a.e. in } \Omega$$

and such that for every entropy $\Phi \in \mathcal{E}$ it holds that

$$\mu_\Phi := \operatorname{div}(\Phi(m)) \in \mathcal{M}_{\mathrm{loc}}(\Omega),$$

namely μ_Φ is a locally finite Radon measure on Ω. We moreover set

$$\tilde{F}_0(u,\Omega) := \left(\bigvee_{(\alpha_1,\alpha_2)} \left|\operatorname{div}\Sigma_{\alpha_1,\alpha_2}\left(\nabla^\perp u\right)\right|\right)(\Omega).$$

Finally we denote by

$$\Lambda^0(\Omega) := \left\{u \in W_0^{1,\infty}(\Omega) : \nabla^\perp u \in A(\Omega)\right\}.$$

The functional $\tilde{F}_0(\cdot,\Omega)$ coincides with $F_0(\cdot,\Omega)$ in the subspace of $\Lambda^0(\Omega)$ whose elements have gradient in $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ (see [2]) and it is the natural candidate to be the Γ-limit of the functionals $F_\varepsilon(\cdot,\Omega)$ as $\varepsilon \to 0^+$.

Although $A(\Omega) \not\subset \mathrm{BV}_{\mathrm{loc}}(\Omega)$, elements of $A(\Omega)$ share with BV functions most of their fine properties.

Theorem 1.5. [11] *For every $m \in A(\Omega)$ there exists a $\mathscr{H}^1$-rectifiable set $J \subset \Omega$ such that*

(1) for $\mathscr{H}^1$-a.e. $x \notin J$ it holds that

$$\lim_{r\to 0} \frac{1}{r^2} \int_{B_r(x)} |m(y) - \bar{m}_{x,r}| dy = 0,$$

where $\bar{m}_{x,r}$ denotes the average of m on $B_r(x)$, namely x is a vanishing mean oscillation point of m;

(2) for $\mathscr{H}^1$-a.e. $x \in J$ there exist $m^+(x), m^-(x) \in \mathbb{S}^1$ such that

$$\lim_{r\to 0} \frac{1}{r^2} \left(\int_{B_r^+(x)} |m(y) - m^+(x)| dy + \int_{B_r^-(x)} |m(y) - m^-(y)| dy \right) = 0,$$

where $B^\pm(x) := \{y \in B_r(x) : \pm y \cdot \mathbf{n}(x) > 0\}$ and $\mathbf{n}(x)$ is a unit vector normal to J in x;

(3) for every $\Phi \in \mathcal{E}$ it holds that

$$\mu_\Phi \llcorner J = [\mathbf{n} \cdot (\Phi(m^+) - \Phi(m^-))] \mathscr{H}^1 \llcorner J,$$
$$\mu_\Phi \llcorner K = 0 \quad \forall K \subset \Omega \setminus J \text{ with } \mathscr{H}^1(K) < \infty.$$

The analogy with the structure of elements in $A(\Omega) \cap \mathrm{BV}_{\mathrm{loc}}(\Omega)$ is not complete: for these functions properties (1) and (3) can be improved to

(1') $\mathscr{H}^1$-a.e. $x \notin J$ is a Lebesgue point of m;
(3') for every $\Phi \in \mathcal{E}$

$$\mu_\Phi = [\mathbf{n} \cdot (\Phi(m^+) - \Phi(m^-))] \mathscr{H}^1 \llcorner J. \tag{1.6}$$

In order to prove (3') from (3) one should show that μ_Φ is concentrated on J. This is considered as a fundamental step towards the solution of the Γ-limit conjecture and it remains open. Notice moreover that by means of Theorem 1.5 we can give a meaning to the definition of the functional $F_0(\cdot, \Omega)$ even for solutions u to the eikonal equation with $\nabla^\perp u \in A(\Omega) \setminus \mathrm{BV}_{\mathrm{loc}}(\Omega)$; Property (3') would imply that F_0 coincides with $\tilde{F}_0$ on the whole $\Lambda^0(\Omega)$.

A fundamental tool in the study of fine properties of elements of $A(\Omega)$ is the kinetic formulation [18] (see also [23] in the framework of scalar conservation laws). Here we use a more recent version obtained in [13].

Theorem 1.6. *Let $m \in A(\Omega)$. Then there exists $\sigma \in \mathcal{M}_{loc}(\Omega \times \mathbb{R}/2\pi\mathbb{Z})$ such that*

$$e^{is} \cdot \nabla_x \chi = \partial_s \sigma \qquad \text{in } \mathcal{D}'(\Omega \times \mathbb{R}/2\pi\mathbb{Z}), \tag{1.7}$$

where $\chi : \Omega \times \mathbb{R}/2\pi\mathbb{Z}$ is defined by

$$\chi(x, s) = \begin{cases} 1 & \text{if } e^{is} \cdot m(x) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

We observe that if σ solves (1.7), then

$$\sigma + \mu \otimes \mathcal{L}^1$$

also solves (1.7) for every $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega)$. This ambiguity is resolved in [13] by considering the unique σ_0 solving (1.7) such that

$$\int_{\Omega\times\mathbb{S}^1} \varphi(x)d\sigma_0(x,s) = 0, \qquad \forall\varphi \in C_c^\infty(\Omega).$$

The above kinetic formulation encodes the entropy production of the family of entropies

$$\mathcal{E}_\pi := \left\{\Phi \in \mathcal{E} : \frac{d}{ds}\Phi(e^{is})|_{s=\bar{s}} = -\frac{d}{ds}\Phi(e^{is})|_{s=\bar{s}+\pi}\right\}.$$

Condition (1.5) is equivalent to $\frac{d}{ds}\Phi(e^{is})\cdot e^{is} = 0$ for every $s \in \mathbb{R}/2\pi\mathbb{Z}$, therefore for every $\Phi \in \mathcal{E}$ we can define $\psi_\Phi : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ such that

$$\frac{d}{ds}\Phi(e^{is}) = 2\psi_\Phi\left(s+\frac{\pi}{2}\right)e^{i(s+\frac{\pi}{2})} \qquad \forall s \in \mathbb{R}/2\pi\mathbb{Z}.$$

Notice that $\Phi \in \mathcal{E}_\pi$ if and only if ψ_Φ is π-periodic. Rephrasing the construction in [13], we have the following identity: for every $\Phi \in \mathcal{E}_\pi$ and every $\zeta \in C_c^1(\Omega)$ it holds that

$$\langle \mathrm{div}\Phi(m), \zeta\rangle = \langle \partial_s\sigma, \zeta \otimes \psi_\Phi\rangle, \tag{1.9}$$

namely

$$\int_\Omega \Phi(m)\cdot\nabla\zeta\,dx = \int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \zeta(x)\psi'_\Phi(s)d\sigma.$$

A possibly weaker version of (3') is the following:

(3") Eq. (1.6) holds for every $\Phi \in \mathcal{E}_\pi$.

This is equivalent to require that $\nu_0 := (p_x)_\sharp|\sigma_0| \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ is concentrated on J and moreover it would be sufficient to establish the equality $F_0 = \tilde{F}_0$. The following proposition is a partial result in this direction for general $m \in A(\Omega)$; we remark here that a key step of the proof of Theorem 1.1 is to establish (3") for a class of m including the limits of $\nabla^\perp u_\varepsilon$, where u_ε is a minimizer of $F_\varepsilon(\cdot,\Omega)$ and Ω is an ellipse.

Proposition 1.7. *Let $m \in A(\Omega)$ and $(\sigma_{0,x})_{x\in\Omega} \subset \mathcal{M}(\mathbb{R}/2\pi\mathbb{Z})$ be the disintegration of σ_0 with respect to ν_0 defined for ν_0-a.e. $x \in \Omega$ by the properties $|\sigma_{0,x}|(\mathbb{R}/2\pi\mathbb{Z}) = 1$ and*

$$\int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \varphi(x,s)d\sigma_0(x,s) = \int_\Omega\int_{\mathbb{R}/2\pi\mathbb{Z}} \varphi(x,s)d\sigma_{0,x}(s)d\nu_0(x)$$

for every $\varphi \in C_c^\infty(\Omega\times\mathbb{R}/2\pi\mathbb{Z})$. Then for ν_0-a.e. $x \in \Omega\setminus J$ there exists $\bar{s} = \bar{s}(x) \in \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\sigma_{0,x} = \pm\frac{1}{4}\left(\delta_{\bar{s}} + \delta_{\bar{s}+\pi} - \frac{1}{\pi}\mathcal{L}^1\right).$$

Among other results, the same expression for $\sigma_{0,x}$ has been obtained very recently in [22] under the additional assumption that div $\Phi(m) \in L^p(\Omega)$ for every $\Phi \in \mathcal{E}$. As the authors point out, it is still not known if this additional assumption is sufficient to establish that indeed σ_0 vanishes.

1.3. The Asymptotic Problem

Adapting the argument in [30] for scalar conservation laws to this context, it is possible to prove that the elements of $A(\Omega)$ with finite energy have strong traces in L^1 at the boundary of Ω. However, the conditions

$$u_\varepsilon \in \Lambda(\Omega), \qquad \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega) < \infty, \qquad \text{and} \qquad u = \lim_{\varepsilon\to 0} u_\varepsilon \quad \text{in } W^{1,1}$$

do not guarantee that $\frac{\partial u}{\partial n} = -1$ on $\partial\Omega$; in other words we can have boundary layers. In order to take them into account we slightly reformulate the minimum problem for $F_\varepsilon(\cdot, \Omega)$: given $\delta > 0$ we define

$$\Omega_\delta = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \Omega) < \delta\}, \qquad \text{and} \qquad S_\delta := \Omega_\delta \setminus \bar{\Omega}.$$

Being Ω of class C^2, we can take $\delta > 0$ sufficiently small so that the function $-\operatorname{dist}(x, \partial\Omega)$ belongs to $W^{2,2}(S_\delta)$. We therefore consider the minimum problems for the functionals $F_\varepsilon(\cdot, \Omega_\delta)$ on the space

$$\Lambda_\delta(\Omega) := \left\{ u \in W^{2,2}(\Omega_\delta) : u(x) = -\operatorname{dist}(x, \partial\Omega) \text{ for a.e. } x \in S_\delta \right\}.$$

Notice that for every $u \in \Lambda(\Omega)$ the function $u^\delta : \Omega_\delta \to \mathbb{R}$ defined by

$$u^\delta(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ -\operatorname{dist}(x, \partial\Omega) & \text{if } x \in \Omega_\delta \setminus \Omega \end{cases} \tag{1.10}$$

belongs to $\Lambda_\delta(\Omega)$ and

$$F_\varepsilon(u^\delta, \Omega_\delta) = F_\varepsilon(u, \Omega) + \varepsilon \int_{S_\delta} |\nabla^2 \operatorname{dist}(x, \partial\Omega)|^2 dx.$$

Similarly the restriction to Ω of any function in $\Lambda_\delta(\Omega)$ belongs to $\Lambda(\Omega)$, so that the two minimum problems are equivalent. We will also denote by

$$A_\delta(\Omega) := \left\{ m \in A(\Omega_\delta) : m = -\nabla^\perp \operatorname{dist}(\cdot, \partial\Omega) \text{ in } S_\delta \right\}.$$

We will prove the following result:

Theorem 1.8. *Let Ω be an ellipse. Then the function $\bar{u}^\delta$, defined by (1.10) with $\bar{u} = \operatorname{dist}(x, \partial\Omega)$, is the unique minimizer of $\tilde{F}_0(\cdot, \Omega_\delta)$ in the space*

$$\Lambda_\delta^0(\Omega) := \left\{ u \in W^{1,2}(\Omega_\delta) : \nabla^\perp u \in A(\Omega_\delta) \textit{ and } u = \bar{u}^\delta \textit{ in } S_\delta \right\}.$$

We show now that Theorem 1.1 is a corollary of Theorem 1.8 and the previous mentioned results: indeed let $\varepsilon_k \to 0$ as $k \to \infty$ and for any k let u_{ε_k} be a minimizer of $F_{\varepsilon_k}(\cdot, \Omega)$ on $\Lambda(\Omega)$. By Theorem 1.3 and (1.3) we have that every limit point u_0 of u_{ε_k} belongs to $\Lambda^0(\Omega)$ and moreover it holds

$$\begin{aligned}\tilde{F}_0(u_0^\delta, \Omega_\delta) &\leq \liminf_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k}^\delta, \Omega_\delta) = \liminf_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k}, \Omega)\\ &= \lim_{k\to\infty} \min_{\Lambda(\Omega)} F_{\varepsilon_k}(\cdot, \Omega) = \tilde{F}_0(\bar{u}, \Omega) = \tilde{F}_0(\bar{u}^\delta, \Omega_\delta).\end{aligned}$$

Since $\bar{u}^\delta$ is the only minimizer of $\tilde{F}_0(\cdot, \Omega_\delta)$ in $\Lambda_\delta^0(\Omega)$, then $u_0^\delta = \bar{u}^\delta$, namely $u_0 = \bar{u}$.

1.4. Related Results

1.4.1. Zero-Energy States The only case in which the behavior of minimizers of $F_\varepsilon(\cdot, \Omega)$ as $\varepsilon \to 0$ is completely understood is when $\lim_{\varepsilon\to 0} \min F_\varepsilon(\cdot, \Omega) = 0$. All the sets Ω admitting sequences with vanishing energy were characterized in [17] and with the appropriate boundary conditions the limit function is in these cases $\bar{u} = \mathrm{dist}(\cdot, \partial\Omega)$. A quantitative version of this result is proven in [20] (see also [19]). In a different direction, it was shown in [21] that the vanishing of the two entropy defect measures $\mathrm{div}\,\Sigma_{e_1,e_2}(m)$ and $\mathrm{div}\,\Sigma_{\varepsilon_1,\varepsilon_2}(m)$ is sufficient to establish $\mathrm{div}\,\Phi(m) = 0$ for every $\Phi \in \mathcal{E}$. Here we denoted by (e_1, e_2) the standard orthonormal system in $\mathbb{R}^2$ and by

$$(\varepsilon_1, \varepsilon_2) := \left(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right)$$

the orthonormal system obtained by performing a rotation of (e_1, e_2) by $\pi/4$.

1.4.2. States with a Vanishing Entropy Defect Measure The case when Ω is an ellipse is special since we know a priori that there exists an orthonormal system (α_1, α_2) in $\mathbb{R}^2$ for which the minimizers u^δ in $A(\Omega_\delta)$ of the asymptotic problem $\tilde{F}_0(\cdot, \Omega_\delta)$ satisfy

$$\mathrm{div}\,\Sigma_{\alpha_1,\alpha_2}\left(\nabla^\perp u^\delta\right) = 0 \qquad \text{in } \mathcal{D}'(\Omega_\delta). \tag{1.11}$$

This situation has been considered more extensively in [14,15], where in particular the authors proved the minimizing property of the viscosity solution (1.3) for more general domains and functionals. In this direction we only mention here that the same arguments of this paper allow to prove Theorem 1.1 also in the case where Ω is a stadium, namely a domain of the form

$$\Omega = \{x \in \mathbb{R}^2 : \mathrm{dist}(x, [0, L] \times \{0\}) < R\} \qquad \text{for some } L, R > 0.$$

We finally mention that under the additional assumption (1.11) we can prove Property (3”).

1.4.3. A Micromagnetics Model A family of functionals E_ε strictly related to (1.1) was introduced in [28,29] in the context of micro-magnetics. An analogous result to Theorem 1.1 was proved in [8] even for general smooth domains Ω, while the Γ-limit conjecture is still open also in this setting. Although Theorem 1.5 has a perfect analogue for the elements in the asymptotic domain of E_ε (see [7]), the main difficulty seems to be a still not complete understanding of the fine properties of these elements. In this direction we notice that the method used here to establish Proposition 1.7 gives the analogue in this setting of the concentration property (3') (see [25]).

2. Lagrangian Representation of Elements in $A(\Omega)$

The Lagrangian representation is an extension of the classical method of characteristics to the non-smooth setting: it was introduced in the framework of scalar conservation laws in [9,24] building on the kinetic formulation from [23]. This approach is strongly inspired by the decomposition in elementary solutions of non-negative measure valued solutions of the linear transport equation, called superposition principle (see [1]). Indeed by Theorem 1.6, the vector fields $m \in A(\Omega)$ are represented by the solution χ of the linear transport equation (1.7). The main difficulty in this case is due to the source term which is merely a derivative of a measure. This issue is reflected in the lack of regularity of the characteristics detected by our Lagrangian representation, which have bounded variation but they are in general not continuous. A fundamental feature for our analysis is that we can decompose the kinetic measure σ in (1.7) along the characteristics.

2.1. Lagrangian Representation

We introduce the following space of curves: given $T > 0$, we let

$$\Gamma := \Big\{(\gamma, t_\gamma^-, t_\gamma^+) : 0 \le t_\gamma^- \le t_\gamma^+ \le T, \gamma = (\gamma_x, \gamma_s) \in \mathrm{BV}((t_\gamma^-, t_\gamma^+); \\ \Omega \times \mathbb{R}/2\pi\mathbb{Z}), \gamma_x \text{ is Lipschitz}\Big\}.$$

We will always consider the right-continuous representative of the component γ_s. Moreover we will adopt the notation from [3] for the decomposition of the measure Dv where $v \in \mathrm{BV}(I; \mathbb{R})$ for some interval $I \subset \mathbb{R}$,

$$Dv = \tilde{D}v + D^j v,$$

where $\tilde{D}v$ denotes the sum of the absolutely continuous part and the Cantor part of Dv and $D^j v$ denotes the jump part of Dv. We will need to consider also $\tilde{D}v$ for functions $v \in \mathrm{BV}(I; \mathbb{R}/2\pi\mathbb{Z})$. In this case $\tilde{D}v = \tilde{D}w$ where w is any function in $\mathrm{BV}(I; \mathbb{R})$ such that for every $z \in I$ the value $w(z)$ belongs to the class $v(z)$ in $\mathbb{R}/2\pi\mathbb{Z}$. For every $t \in (0, T)$ we consider the section

$$\Gamma(t) := \left\{\left(\gamma, t_\gamma^-, t_\gamma^+\right) \in \Gamma : t \in \left(t_\gamma^-, t_\gamma^+\right)\right\},$$

and we denote

$$\begin{aligned} e_t : \Gamma(t) &\to \Omega \times \mathbb{R}/2\pi\mathbb{Z} \\ (\gamma, t_\gamma^-, t_\gamma^+) &\mapsto \gamma(t). \end{aligned}$$

Definition 2.1. Let $m \in A(\Omega)$ and Ω' be a $W^{2,\infty}$-open set compactly contained in Ω We say that a finite non-negative Radon measure $\omega \in \mathcal{M}(\Gamma)$ is a *Lagrangian representation* of m in Ω' if the following conditions hold:

(1) for every $t \in (0, T)$ it holds that

$$(e_t)_\sharp [\omega \llcorner \Gamma(t)] = \chi \mathscr{L}^2 \times \mathscr{L}^1, \tag{2.1}$$

where χ is defined in (1.8);

(2) the measure ω is concentrated on curves $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ such that for $\mathscr{L}^1$-a.e. $t \in (t_\gamma^-, t_\gamma^+)$ the following characteristic equation holds:

$$\dot{\gamma}_x(t) = e^{i\gamma_s(t)}; \tag{2.2}$$

(3) it holds the integral bound

$$\int_\Gamma \text{Tot.Var.}_{(0,T)} \gamma_s d\omega(\gamma) < \infty;$$

(4) for ω-a.e. $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ it holds that

$$t_\gamma^- > 0 \Rightarrow \gamma_x(t_\gamma^- +) \in \partial\Omega', \qquad \text{and} \qquad t_\gamma^+ < T \Rightarrow \gamma_x(t_\gamma^+ -) \in \partial\Omega'.$$

For every curve $\gamma \in \Gamma$ we define the measure $\sigma_\gamma \in \mathcal{M}((0, T) \times \Omega' \times \mathbb{R}/2\pi\mathbb{Z})$ by

$$\sigma_\gamma = (\text{id}, \gamma)_\sharp \tilde{D}_t \gamma_s + \mathscr{H}^1 \llcorner E_\gamma^+ - \mathscr{H}^1 \llcorner E_\gamma^-, \tag{2.3}$$

where

$$\begin{aligned} E_\gamma^+ &:= \{(t, x, s) \in (0, T) \times \Omega \times \mathbb{R}/2\pi\mathbb{Z} : \gamma_x(t) = x \text{ and } \gamma_s(t-) \\ &\quad \le s \le \gamma_s(t+) \le \gamma_s(t-) + \pi\}, \\ E_\gamma^- &:= \{(t, x, s) \in (0, T) \times \Omega \times \mathbb{R}/2\pi\mathbb{Z} : \gamma_x(t) = x \text{ and } \gamma_s(t+) \\ &\quad \le s \le \gamma_s(t-) < \gamma_s(t+) + \pi\}. \end{aligned} \tag{2.4}$$

Notice that since $\mathbb{R}/2\pi\mathbb{Z}$ is not ordered, given $s_1 \neq s_2 \in \mathbb{R}/2\pi\mathbb{Z}$ the condition $s_1 < s_2$ is not defined. Nevertheless we use the notation $s \in (s_1, s_2)$ or $s_1 < s < s_2$ to indicate the following condition (depending only on the orientation of $\mathbb{R}/2\pi\mathbb{Z}$): if $t_1, t_2 \in \mathbb{R}$ are such $t_1 < t_2 < t_1 + 2\pi$, $e^{it_1} = e^{is_1}$ and $e^{it_2} = e^{is_2}$ then there exists $t \in (t_1, t_2)$ such that $e^{it} = e^{is}$.

Lemma 2.2. *Let ω be a Lagrangian representation of $m \in A(\Omega)$ on Ω'. Let us denote by*

$$\sigma_\omega := -\int_\Gamma \sigma_\gamma d\omega$$

and by $\tilde{\chi} : (0,T) \times \Omega \times \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ the function defined by $\tilde{\chi}(t,x,s) = \chi(x,s)$ for every $t \in (0,T)$. Then it holds that

$$e^{is} \cdot \nabla_x \tilde{\chi} = \partial_s \sigma_\omega \quad \in \mathcal{D}'((0,T) \times \Omega' \times \mathbb{R}/2\pi\mathbb{Z}). \tag{2.5}$$

Proof. We show that (2.5) holds when tested with every function of the form $\phi(t,x,s) = \zeta(t)\varphi(x,s)$ with $\zeta \in C_c^\infty((0,T))$ and $\varphi \in C_c^\infty(\Omega' \times \mathbb{R}/2\pi\mathbb{Z})$. It follows from (2.1) and (2.2) that

$$\begin{aligned}\langle e^{is} \cdot \nabla_x \tilde{\chi}, \phi\rangle &= -\int e^{is} \cdot \nabla_x \varphi(x,s)\zeta(t)\tilde{\chi}(t,x,s)dtdxds \\ &= -\int_{(0,T)} \int_{\Gamma(t)} e^{i\gamma_s(t)} \cdot \nabla_x \varphi(\gamma(t))d\omega\zeta(t)dt \\ &= -\int_\Gamma \int_{t_\gamma^-}^{t_\gamma^+} \frac{d}{dt}\gamma_x(t) \cdot \nabla_x \varphi(\gamma(t))\zeta(t)dtd\omega.\end{aligned} \tag{2.6}$$

By the chain-rule for functions with bounded variation we have the following equality between measures:

$$\frac{d}{dt}\varphi \circ \gamma = \nabla_x \varphi(\gamma(t)) \cdot \frac{d}{dt}\gamma_x(t) + \partial_s \varphi(\gamma(t))\tilde{D}_t \gamma_s + \sum_{t_j \in J_\gamma} \big(\varphi(t_j, \gamma(t_j+)) \\ -\varphi(t_j, \gamma(t_j-))\big)\delta_{t_j},$$

where J_γ denotes the jump set of γ. Therefore, proceeding in the chain (2.6), we have

$$\langle e^{is} \cdot \nabla_x \tilde{\chi}, \phi\rangle = -\int_\Gamma \int_{t_\gamma^-}^{t_\gamma^+} \frac{d}{dt}\varphi(\gamma(t))\zeta(t)dtd\omega + \int_\Gamma \int_{t_\gamma^-}^{t_\gamma^+} \partial_s \varphi(\gamma(t))\zeta(t)d\tilde{D}_t\gamma_s(t)d\omega \\ + \int_\Gamma \sum_{t_j \in J_\gamma} \big(\varphi(t_j, \gamma(t_j+)) - \varphi(t_j, \gamma(t_j-))\big)\zeta(t_j)d\omega.$$

By definition of σ_γ in (2.3) it holds that

$$\int_{t_\gamma^-}^{t_\gamma^+} \partial_s \varphi(\gamma(t))\zeta(t)d\tilde{D}_t\gamma_s(t) + \sum_{t_j \in J_\gamma} \big(\varphi(t_j, \gamma(t_j+)) - \varphi(t_j, \gamma(t_j-))\big)\zeta(t_j)d\omega \\ = \int \partial_s \varphi(x,s)\zeta(t)d\sigma_\gamma,$$

therefore in order to establish $\langle e^{is} \cdot \nabla_x \tilde{\chi}, \phi\rangle = \langle \partial_s \sigma_\omega, \phi\rangle$ it suffices to prove that

$$\int_\Gamma \int_{t_\gamma^-}^{t_\gamma^+} \frac{d}{dt}\varphi(\gamma(t))\zeta(t)dtd\omega = 0.$$

By Point (4) in Definition 2.1 for ω-a.e. $\gamma \in \Gamma$ it holds that $\varphi(\gamma(t_\gamma^- +)) = \varphi(\gamma(t_\gamma^+ -)) = 0$, and, in particular,

$$\int_\Gamma \int_{t_\gamma^-}^{t_\gamma^+} \frac{d}{dt}\varphi(\gamma(t))\zeta(t)dtd\omega = -\int_\Gamma \int_{t_\gamma^-}^{t_\gamma^+} \varphi(\gamma(t))\zeta'(t)dtd\omega$$
$$= \int_{(0,T)\times\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \tilde{\chi}\varphi(x,s)\zeta'(t)dtdxds$$
$$= 0,$$

where we used (2.1) in the second equality and that $\tilde{\chi}$ does not depend on t in the last equality. This concludes the proof. □

Definition 2.3. We say that $\sigma \in \mathcal{M}_{\mathrm{loc}}(\Omega' \times \mathbb{R}/2\pi\mathbb{Z})$ is a *minimal kinetic measure* if it satisfies (1.7) and for every σ' solving (1.7) it holds that

$$\nu_\sigma := (p_x)_\sharp|\sigma| \le (p_x)_\sharp|\sigma'| =: \nu_{\sigma'}.$$

We moreover say that ω is a *minimal Lagrangian representation* of m if it is a Lagrangian representation of m according to Def. 2.1 and

$$\sigma_\omega = \mathscr{L}^1 \otimes \sigma_t$$

with σ_t minimal kinetic measure for $\mathscr{L}^1$-a.e. $t \in (0, T)$.

The existence of a minimal kinetic measure is proven in the following lemma:

Lemma 2.4. *For every $m \in A(\Omega)$ there exists a minimal kinetic measure σ. Moreover there exists $\nu_{\min} \in \mathcal{M}_{loc}(\Omega)$ such that for every minimal kinetic measure σ it holds that $\nu_{\min} = (p_x)_\sharp|\sigma|$.*

Proof. Since $\partial_s\sigma$ is uniquely determined by (1.7), we have that a kinetic measure σ is minimal if and only if for ν_σ-a.e. $x \in \Omega$ the disintegration σ_x satisfies the following inequality:

$$1 = \|\sigma_x\| \le \left\|\sigma_x + \alpha\mathscr{L}^1\right\| \qquad \forall \alpha \in \mathbb{R}. \tag{2.7}$$

Therefore all minimal kinetic measures are of the form

$$\nu_{\sigma_0} \otimes \left((\sigma_0)_x + \alpha(x)\mathcal{L}^1\right),$$

where $\alpha : \Omega \to \mathbb{R}$ is a measurable function such that for ν_{σ_0}-a.e. $x \in \Omega$ it holds that

$$\left\|(\sigma_0)_x + \alpha(x)\mathcal{L}^1\right\| \le \left\|(\sigma_0)_x + c\mathcal{L}^1\right\| \qquad \forall c \in \mathbb{R}. \tag{2.8}$$

The existence of such an α is trivial and in particular it holds that

$$\nu_{\min} = \left(\min_{\alpha\in\mathbb{R}} \left\|(\sigma_0)_x + \alpha\mathcal{L}^1\right\|\right)\nu_0.$$

In Sect. 3 we will show that for every $m \in A(\Omega)$ there exists a *unique* minimal kinetic measure $\sigma_{\min}$, namely that for $\nu_{\min}$-a.e. $x \in \Omega$ there exists a unique $\alpha(x)$ such that (2.8) holds.

The main result of this section is

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, $m \in A(\Omega)$ and Ω' be a $W^{2,\infty}$ open set compactly contained in Ω be such that $\mathscr{H}^1$-a.e. $x \in \partial\Omega'$ is a Lebesgue point of m. Then there exists a minimal Lagrangian representation ω of m on Ω'. In particular it holds that*

$$|\sigma_\omega| = \int_\Gamma |\sigma_\gamma| d\omega. \tag{2.9}$$

The existence of a Lagrangian representation for weak solutions with finite entropy production to general conservation laws on the whole $(0, T) \times \mathbb{R}^d$ has been proved in [24]. The case of bounded domains when Ω' is a ball was considered in [25] for the class of solutions to the eikonal equation arising in [29]. The extension to the case where Ω' is a $W^{2,\infty}$ open set does not cause any significant difficulty. In particular the argument proposed in [25] applies here with trivial modifications and leads to the following partial result:

Lemma 2.6. *In the setting of Proposition 2.5, let $\sigma \in \mathcal{M}_{loc}(\Omega \times \mathbb{R}/2\pi\mathbb{Z})$ be a locally finite measure satisfying (1.7). Then there exists a Lagrangian representation ω of m on Ω' such that*

$$\int_\Gamma \mathit{Tot.Var.}_{(t_\gamma^-, t_\gamma^+)} \gamma_s d\omega \le T|\sigma|(\Omega' \times \mathbb{R}/2\pi\mathbb{Z}).$$

We now prove Proposition 2.5 relying on Lemma 2.6.

Proof of Proposition 2.5. Let $m \in A(\Omega)$ and let $\bar{\sigma}$ be a minimal kinetic measure. By Lemma 2.6, there exists a Lagrangian representation ω of m such that

$$\int_\Gamma \text{Tot.Var.}_{(t_\gamma^-, t_\gamma^+)} \gamma_s d\omega \le T\|\bar{\sigma}\|. \tag{2.10}$$

By definition of σ_ω it holds that

$$\|\sigma_\omega\| \le \left(\int_\Gamma |\sigma_\gamma| d\omega\right)\left((0, T) \times \Omega \times \mathbb{R}/2\pi\mathbb{Z}\right) = \int_\Gamma \text{Tot.Var.}_{(t_\gamma^-, t_\gamma^+)} \gamma_s d\omega. \tag{2.11}$$

By Lemma 2.2, the measure σ_ω satisfies (2.5); being $\bar{\sigma}$ a minimal kinetic measure for m, it follows that $T\|\bar{\sigma}\| \le \|\sigma_\omega\|$. In particular the inequalities in (2.10) and (2.11) are equalities and (2.9) follows. □

The following lemma is a simple application of Tonelli theorem and (2.1); since it is already proven in [26], we refer to it for the details.

Lemma 2.7. *For ω-a.e. $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ it holds that for $\mathscr{L}^1$-a.e. $t \in (t_\gamma^-, t_\gamma^+)$*

(1) $\gamma_x(t)$ is a Lebesgue point of m;
(2) $e^{i\gamma_s(t)} \cdot m(\gamma_x(t)) > 0$.

We denote by Γ_g the set of curves $\gamma \in \Gamma$ such that the two properties above hold.

3. Structure of the Kinetic Measure

The main goal of this section is to prove Proposition 1.7. As a corollary we will obtain the concentration property (3”) presented in the introduction for solutions $m \in A(\Omega)$ with a vanishing entropy defect measure. The key step is the following regularity result (the strategy of the proof is borrowed from [25], where an analogous statement was proved for the solutions to the eikonal equation arising in the micromagnetics model mentioned in the introduction, and we finally observe that in that situation this result is sufficient to establish the concentration property (3’), while it is not the case here):

Lemma 3.1. *Let $\bar\gamma \in \Gamma_g$ and $\bar t \in (t^-_{\bar\gamma}, t^+_{\bar\gamma})$, and set $\bar x := \bar\gamma_x(\bar t)$ and $\bar s := \bar\gamma_s(\bar t+)$. Then there exists $c > 0$ such that for every $\delta \in (0, 1/2)$ we have at least one of the following:*

(1) the lower density estimate holds true:

$$\liminf_{r\to 0} \frac{\mathscr{L}^2\left(\left\{x \in B_r(\bar x) : e^{i\bar s} \cdot m(x) > -\delta\right\}\right)}{r^2} \geq c\delta;$$

(2) the following lower bound holds true:

$$\limsup_{r\to 0} \frac{\nu_{\min}(B_r(\bar x))}{r} \geq c\delta^3.$$

The same statement holds by setting $\bar s := \bar\gamma_s(\bar t-)$.

Proof. We prove the lemma only for $\bar s = \bar\gamma_s(\bar t+)$, being the case $\bar s = \bar\gamma_s(\bar t-)$ analogous. Let $\delta_1 > 0$ be sufficiently small so that for $\mathscr{L}^1$-a.e. $t \in (\bar t, \bar t + \delta_1)$ it holds that

$$e^{i\bar\gamma_s(t)} \cdot e^{i\bar s} \geq \cos\left(\frac{\delta}{5}\right). \tag{3.1}$$

Since $\bar\gamma_x$ satisfies (2.2), then for every $r \in \left(0, \frac{\delta_1}{2}\right)$ there exists $t_r \in (\bar t, \bar t + \delta_1)$ such that

$$\bar\gamma_x(t) \in B_r(\bar x) \quad \forall t \in (\bar t, t_r), \qquad \text{and} \qquad \bar\gamma_x(t_r) \in \partial B_r(\bar x).$$

Moreover since $\cos(\delta/5) \in (1/2, 1)$, then (3.1) implies

$$r \leq t_r - \bar t \leq 2r.$$

For every $r \in \left(0, \frac{\delta_1}{2}\right)$ we denote by

$$E_+(r) := \{t \in (\bar t, t_r) : m(\bar\gamma_x(t)) \cdot e^{i(\bar\gamma_s(t)+\delta)} > 0\},$$
$$E_-(r) := \{t \in (\bar t, t_r) : m(\bar\gamma_x(t)) \cdot e^{i(\bar\gamma_s(t)-\delta)} > 0\}.$$

Since $\gamma \in \Gamma_g$, for $\mathscr{L}^1$-a.e. $t \in (0, t_r)$ it holds that

$$m(\bar\gamma_x(t)) \cdot e^{i\bar\gamma_s(t)} > 0,$$

therefore, being $\delta \in \left(0, \frac{1}{2}\right)$, we have

$$(\bar{t}, t_r) \subset E_+(r) \cup E_-(r).$$

In particular

$$\mathscr{L}^1(E_+(r)) + \mathscr{L}^1(E_-(r)) \geq t_r - \bar{t} \geq r.$$

In the remaining part of the proof we assume that $\mathscr{L}^1(E_-(r)) > r/2$, being the case $\mathscr{L}^1(E_+(r)) > r/2$ analogous.

Given $\varepsilon > 0$, we consider the strip

$$S_{r,\varepsilon} := \left\{x \in \Omega_\delta : \exists t \in (\bar{t}, t_r) : |\bar{\gamma}_x(t) - x| < \varepsilon\right\}. \tag{3.2}$$

For every $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ let $(t_{\gamma,i}^-, t_{\gamma,i}^+)_{i=1}^{N_\gamma}$ be the nontrivial interiors of the connected components of $\gamma_s^{-1}\left((\bar{s} - \delta), \bar{s} - \frac{2}{5}\delta\right)$ which intersect $\gamma^{-1}\left(S_{r,\varepsilon} \times \left(\bar{s} - \frac{4}{5}\delta, \bar{s} - \frac{3}{5}\delta\right)\right)$. Notice that we have the estimate

$$N_\gamma \leq 1 + \frac{5}{\delta}\text{Tot.Var.}\gamma_s.$$

For every $i \in \mathbb{N}$ we consider

$$\Gamma_i := \{(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma : N_\gamma \geq i\}$$

and the measurable restriction map

$$\begin{aligned} R_i : \Gamma_i &\to \Gamma. \\ (\gamma, t_\gamma^-, t_\gamma^+) &\mapsto (\gamma, t_{\gamma,i}^-, t_{\gamma,i}^+) \end{aligned}$$

We finally consider the measure

$$\tilde{\omega} := \sum_{i=1}^{\infty} (R_i)_\sharp \left(\omega \llcorner \Gamma_i\right).$$

We observe that $\tilde{\omega} \in \mathcal{M}_+(\Gamma)$, since, for every $N > 0$,

$$\left\| \sum_{i=1}^{N} (R_i)_\sharp \left(\omega \llcorner \Gamma_i\right) \right\| \leq \int_\Gamma N_\gamma d\omega \leq \int_\Gamma \left(1 + \frac{5}{\delta}\text{Tot.Var.}\gamma_s\right) d\omega(\gamma) < \infty,$$

by Point (3) in Definition 2.1. The advantage of the measure $\tilde{\omega}$ is that it is concentrated on curves whose x-components are transversal to $\bar{\gamma}_x$ on the whole domain of definition. This property allows us to prove the following claim:

Claim 1. There exists an absolute constant $\tilde{c} > 0$ such that for $\tilde{\omega}$-a.e. $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ it holds

$$\mathscr{L}^1\left(\left\{t \in (t_\gamma^-, t_\gamma^+) : \gamma(t) \in S_{r,\varepsilon} \times \left(\bar{s} - \frac{4}{5}\delta, \bar{s} - \frac{3}{5}\delta\right)\right\}\right) \leq \tilde{c}\frac{\varepsilon}{\delta}.$$

Proof of Claim 1. It follows from (3.1) and the characteristic equation (2.2) that there exists a Lipschitz function $f_{\bar{\gamma}} : \mathbb{R} \to \mathbb{R}$ such that

$$\left\{\bar{\gamma}_x(t) : t \in (\bar{t}, \bar{t} + \delta_1)\right\} \subset \left\{ze^{i\bar{s}} + f_{\bar{\gamma}}(z)e^{i\left(\bar{s}+\frac{\pi}{2}\right)} : z \in \mathbb{R}\right\} \quad \text{and}$$
$$\mathrm{Lip}(f_{\bar{\gamma}}) \le \tan\left(\frac{\delta}{5}\right). \tag{3.3}$$

Similarly for $\tilde{\omega}$-a.e. $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ there exists a Lipschitz function f_γ such that

$$\left\{\gamma_x(t) : t \in (t_\gamma^-, t_\gamma^+)\right\} \subset \left\{ze^{i\bar{s}} + f_\gamma(z)e^{i\left(\bar{s}+\frac{\pi}{2}\right)} : z \in \mathbb{R}\right\} \quad \text{and}$$
$$\frac{d}{dz} f_\gamma(z) \in \left(-\tan\delta, -\tan\left(\frac{2}{5}\delta\right)\right)$$

for $\mathscr{L}^1$-a.e. $z \in \mathbb{R}$. By the definitions of $S_{r,\varepsilon}$ in (3.2) and of $f_{\bar{\gamma}}$ in (3.3), it easily follows that

$$S_{r,\varepsilon} \subset \left\{x \in \Omega_\delta : f_{\bar{\gamma}}\left(x \cdot e^{i\bar{s}}\right) - \varepsilon\left(\cos\left(\frac{\delta}{5}\right)\right)^{-1} \le x \cdot e^{i\left(\bar{s}+\frac{\pi}{2}\right)} \le f_{\bar{\gamma}}\left(x \cdot e^{i\bar{s}}\right) + \varepsilon\left(\cos\left(\frac{\delta}{5}\right)\right)^{-1}\right\}. \tag{3.4}$$

Given $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ let us consider the function $g_\gamma : (t_\gamma^-, t_\gamma^+) \to \mathbb{R}$ defined by

$$g_\gamma(t) = \gamma_x(t) \cdot e^{i\left(\bar{s}+\frac{\pi}{2}\right)}.$$

By construction of $\tilde{\omega}$, for $\tilde{\omega}$-a.e. $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ and $\mathscr{L}^1$-a.e. $t \in (t_\gamma^-, t_\gamma^+)$ it holds that

$$\frac{d}{dt} g_\gamma(t) \le -\sin\left(\frac{2}{5}\delta\right). \tag{3.5}$$

On the other hand

$$\frac{d}{dt} f_{\bar{\gamma}}(\gamma_x(t) \cdot e^{i\bar{s}}) \ge -\sin\left(\frac{\delta}{5}\right). \tag{3.6}$$

By (3.4), for every $t \in (t_\gamma^-, t_\gamma^+)$ such that $\gamma_x(t) \in S_{r,\varepsilon}$ it holds that

$$f_{\bar{\gamma}}(\gamma_x(t) \cdot e^{i\bar{s}}) - \varepsilon\left(\cos\left(\frac{\delta}{5}\right)\right)^{-1} \le g_\gamma(t) \le f_{\bar{\gamma}}(\gamma_x(t) \cdot e^{i\bar{s}}) + \varepsilon\left(\cos\left(\frac{\delta}{5}\right)\right)^{-1}.$$

Therefore, by (3.5) and (3.6), we have

$$\mathscr{L}^1\left(\{t : \gamma_x(t) \in S_{r,\varepsilon}\}\right) \le \frac{2\varepsilon\left(\cos\left(\frac{\delta}{5}\right)\right)^{-1}}{\left|\sin\left(\frac{2}{5}\delta\right) - \sin\left(\frac{\delta}{5}\right)\right|} \le \tilde{c}\frac{\varepsilon}{\delta},$$

for some universal $\tilde{c} > 0$. This concludes the proof of the claim. □

By construction we have

$$(e_t)_\sharp\tilde{\omega} \geq \mathscr{L}^3 \llcorner \left\{(x,s) \in S_{r,\varepsilon} \times \left(\bar{s} - \frac{4}{5}\delta, \bar{s} - \frac{3}{5}\delta\right) : m(x) \cdot e^{is} > 0\right\}$$

for every $t \in (0, T)$. Therefore

$$\begin{aligned} T\mathscr{L}^3 & \left(\left\{(x,s) \in S_{r,\varepsilon} \times \left(\bar{s} - \frac{4}{5}\delta, \bar{s} - \frac{3}{5}\delta\right) : m(x) \cdot e^{is} > 0\right\}\right) \\ & \leq \int_\Gamma \mathscr{L}^1 \left(\left\{t : \gamma(t) \in S_{r,\varepsilon} \times \left(\bar{s} - \frac{4}{5}\delta, \bar{s} - \frac{3}{5}\delta\right)\right\}\right) d\tilde{\omega} \\ & \leq \tilde{c}\frac{\varepsilon}{\delta}\tilde{\omega}(\Gamma). \end{aligned} \tag{3.7}$$

On the other hand, since $\bar{\gamma} \in \Gamma_g$ and $\mathscr{L}^1(E_-(r)) > r/2$ there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$ it holds that

$$\begin{aligned} \mathscr{L}^3 & \left(\left\{(x,s) \in S_{r,\varepsilon} \times \left(\bar{s} - \frac{4}{5}\delta, \bar{s} - \frac{3}{5}\delta\right) : m(x) \cdot e^{is} > 0\right\}\right) \\ & \geq \frac{1}{2}\mathscr{L}^3 \left(S_{r,\varepsilon} \times \left(\bar{s} - \frac{4}{5}\delta, \bar{s} - \frac{3}{5}\delta\right)\right) \\ & \geq \frac{\varepsilon r \delta}{5}. \end{aligned} \tag{3.8}$$

By (3.7) and (3.8) it follows that

$$\tilde{\omega}(\Gamma) \geq \frac{\varepsilon r \delta}{5} \cdot \frac{\delta T}{\tilde{c}\varepsilon} = \frac{r\delta^2}{5\tilde{c}}T.$$

We consider the split $\Gamma = \Gamma_> \cup \Gamma_<$, where

$$\begin{aligned} \Gamma_> &:= \{(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma : t_\gamma^+ - t_\gamma^- \geq r\}, \quad \text{and} \\ \Gamma_< &:= \{(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma : t_\gamma^+ - t_\gamma^- < r\}. \end{aligned}$$

We will prove the following claim, from which the lemma follows immediately:

Claim 2. There exists an absolute constant $c_1 > 0$ such that the two following implications hold true:

(1) if $\tilde{\omega}(\Gamma_>) \geq \frac{r\delta^2 T}{10\tilde{c}}$, then

$$\mathscr{L}^2\left(\left\{x \in B_{2r}(\bar{x}) : e^{i\bar{\gamma}_s(\bar{t}+)} \cdot m(x) > -\delta\right\}\right) \geq c_1\delta r^2;$$

(2) if $\tilde{\omega}(\Gamma_<) \geq \frac{r\delta^2 T}{10\tilde{c}}$, then

$$\nu(B_{2r}(\bar{x})) \geq c_1\delta^3 r.$$

Proof of (1). By definition of $\Gamma_>$ and the assumption in (1) we have

$$T\frac{r^2\delta^2}{10\tilde{c}} \le \int_\Gamma \mathscr{L}^1\left(\left\{t \in (t_\gamma^-, t_\gamma^+) : \gamma(t) \in B_{2r}(\bar{x}) \times \left(\bar{s}-\delta, \bar{s}-\frac{2}{5}\delta\right)\right\}\right) d\tilde{\omega}$$
$$\le T\mathscr{L}^3\left(\left\{(x,s) \in B_{2r}(\bar{x}) \times \left(\bar{s}-\delta, \bar{s}-\frac{2}{5}\delta\right) : m(x)\cdot e^{is} > 0\right\}\right)$$
$$\le T\delta\mathscr{L}^2\left(\left\{x \in B_{2r}(\bar{x}) : m(x)\cdot e^{i\bar{s}} > -\delta\right\}\right).$$

Proof of (2). For $\tilde{\omega}$-a.e. $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_<$, the image of γ_x is contained in $B_{2r}(\bar{x})$ and $\text{Tot.Var.}(\gamma_s) \ge \frac{\delta}{5}$. Since ω is a minimal Lagrangian representation, this implies that

$$T\nu_{\min}(B_{2r}(\bar{x})) = |\sigma_\omega|((0,T)\times B_{2r}(\bar{x})) \ge \int_{\Gamma_<} \text{Tot.Var.}\gamma_s d\tilde{\omega} \ge \frac{\delta}{5}\tilde{\omega}(\Gamma_<) \ge \frac{Tr\delta^3}{50\tilde{c}}.$$

Proposition 3.2. *Let $m \in A(\Omega)$ and $\sigma \in \mathcal{M}(\Omega \times \mathbb{R}/2\pi\mathbb{Z})$ be a minimal kinetic measure. Then for $\nu_{\min}$-a.e. $x \in \Omega \setminus J$ it holds that*

$$\operatorname{supp} \partial_s\sigma_x = \{s, s+\pi\} \quad \textit{for some } s \in \mathbb{R}/2\pi\mathbb{Z}. \tag{3.9}$$

Proof. Let ω be a minimal Lagrangian representation and let $s, s' \in \mathbb{R}/2\pi\mathbb{Z}$; from the explicit expression of σ_ω we have that for $\mathcal{L}^1 \times \nu_{\min}$-a.e. $(t,x) \in (0,T)\times\Omega$ such that $\operatorname{supp}(\partial_s(\sigma_\omega)_{t,x}) \cap (s,s') \ne 0$ there exists $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_g$ such that

$$t \in (t_\gamma^-, t_\gamma^+), \qquad \gamma_x(t) = x, \qquad \text{and} \qquad \left[\gamma_s(t-) \in (s,s') \quad \text{or} \quad \gamma_s(t+) \in (s,s')\right].$$

Given $s_1, s_2 \in \pi\mathbb{Q}/2\pi\mathbb{Z}$ with $s_1 \ne s_2$ and $s_1 \ne s_2 + \pi$, we set

$$\delta_{s_1,s_2} = \frac{1}{3}\min\{|s_1 - s_2|, |s_1 + \pi - s_2|\}$$

so that the intervals $I_1 := (s_1 - \delta_{s_1,s_2}, s_1 + \delta_{s_1,s_2})$, $I_2 := (s_2 - \delta_{s_1,s_2}, s_2 + \delta_{s_1,s_2})$, $I_3 := (s_1 + \pi - \delta_{s_1,s_2}, s_1 + \pi + \delta_{s_1,s_2})$ and $I_4 := (s_2 + \pi - \delta_{s_1,s_2}, s_2 + \pi + \delta_{s_1,s_2})$ are pairwise disjoint and the distance between any two of these intervals is at least δ_{s_1,s_2}. We denote by

$$E(s_1,s_2) := \left\{(t,x) \in (0,T)\times\Omega : \operatorname{supp}(\partial_s(\sigma_\omega)_{t,x}) \cap I_j \ne \emptyset \text{ for } j = 1,2,3,4\right\}.$$

It was shown in [13] that the constraint forces σ to be π-periodic in s, in particular for $\mathcal{L}^1 \times \nu_{\min}$-a.e. $(t,x) \in (0,T)\times\Omega$ the support of $\partial_s\sigma_\omega$ is π-periodic. Therefore if $(t,x) \in (0,T)\times\Omega$ is such that (3.9) does not hold, then there exist four distinct points $\bar{s}_1, \bar{s}_2, \bar{s}_1+\pi, \bar{s}_2+\pi \in \mathbb{R}/2\pi\mathbb{Z}$ belonging to $\operatorname{supp}(\partial_s(\sigma_\omega)_{t,x})$. In particular $\mathcal{L}^1 \times \nu_{\min}$-a.e. $(t,x) \in (0,T)\times\Omega$ for which (3.9) does not hold belongs to

$$\bigcup_{s_1,s_2 \in \pi\mathbb{Q}/2\pi\mathbb{Z}} E(s_1,s_2).$$

By the discussion at the beginning of the proof, we have that for $\mathcal{L}^1 \times \nu_{\min}$-a.e. $(t,x) \in E(s_1,s_2)$ and every $j = 1,2,3,4$ there exists $(\gamma_j, t^-_{\gamma_j}, t^+_{\gamma_j}) \in \Gamma_g$ such that

$$t \in (t^-_{\gamma_j}, t^+_{\gamma_j}), \qquad (\gamma_j)_x(t) = x, \qquad \text{and} \qquad \big[(\gamma_j)_s(t-) \in I_j \quad \text{or} \quad (\gamma_j)_s(t+) \in I_j \big].$$

We show that if $(t,x) \in E(s_1,s_2)$, then x is not a vanishing mean oscillation point of m. Let us assume by contradiction that x is a VMO point of m and there exists $t \in (0,T)$ such that $(t,x) \in E(s_1,s_2)$; by applying Lemma 3.1 for every $j = 1,2,3,4$ there exists $\bar{s}_j \in I_j$ such that

$$\liminf_{r \to 0} \frac{\mathscr{L}^2(\{x' \in B_r(x) : e^{i\bar{s}_j} \cdot m(x') > -\delta_{s_1,s_2}\})}{r^2} \geq c\delta_{s_1,s_2}.$$

Since it does not exist any value $\bar{m} \in \mathbb{R}^2$ with $|\bar{m}| = 1$ such that $\bar{m} \cdot e^{i\bar{s}_j} > -\delta_{s_1,s_2}$ for every $j = 1,2,3,4$, this proves that x is not a vanishing mean oscillation point of m. Thm 1.5 implies that $\mathcal{H}^1$-a.e. $x \in \Omega \setminus J$ is a VMO point of m, therefore since $\nu_{\min} \ll \mathcal{H}^1$, then the set of points $x \in \Omega \setminus J$ for which there exists $t \in (0,T)$ such that $(t,x) \in E(s_1,s_2)$ is $\nu_{\min}$-negligible.

Letting s_1, s_2 vary in $\pi\mathbb{Q}/2\pi\mathbb{Z}$, this proves the claim. □

Remark 3.3. Proposition 1.7 states for the measure σ_0 the same property we obtained here for a minimal kinetic measure σ. Although σ_0 is not always a minimal kinetic measure, the two statements are equivalent since $\nu_{\min} \leq \nu_0 \ll \nu_{\min}$ and $\partial_s \sigma_0 = \partial_s \sigma$ (see the discussion in Lemma 2.4).

Corollary 3.4. *For every $m \in A(\Omega)$ there exists a unique minimal kinetic measure $\sigma_{\min}$ of m. In particular for every minimal Lagrangian representation ω of m on $\Omega' \subset \Omega$ it holds that*

$$\sigma_\omega = \mathscr{L}^1 \llcorner (0,T) \otimes \sigma_{\min} \llcorner \Omega'.$$

Moreover the disintegration of $\sigma_{\min}$ with respect to $\nu_{\min}$ has the following structure:

(1) for $\nu_{\min}$-a.e. $x \in \Omega \setminus J$ it holds that

$$(\sigma_{\min})_x = \frac{1}{2}(\delta_{\bar{s}-\frac{\pi}{2}} + \delta_{\bar{s}+\frac{\pi}{2}}), \qquad \text{or} \qquad (\sigma_{\min})_x = -\frac{1}{2}(\delta_{\bar{s}-\frac{\pi}{2}} + \delta_{\bar{s}+\frac{\pi}{2}})$$

for some $\bar{s} \in \mathbb{R}/2\pi\mathbb{Z}$.

(2) for $\nu_{\min}$-a.e. $x \in J$ let m^+, m^- and $\mathbf{n}$ denote the traces and the normal to J at x as in Theorem 1.5 and let $\beta \in (0,\pi)$ and $\bar{s} \in \mathbb{R}/2\pi\mathbb{Z}$ be uniquely determined by

$$m^+ = e^{i(\bar{s}+\beta)}, \qquad \text{and} \qquad m^- = e^{i(\bar{s}-\beta)}.$$

Then

$$(\sigma_{\min})_x = \mathbf{n} \cdot e^{i\bar{s}} \bar{g}_\beta(s - \bar{s}) \mathscr{L}^1,$$

where $\bar{g}_\beta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ *is* π*-periodic and for every* $s \in [0, \pi]$ *is defined by*

$$\bar{g}_\beta(s) := \begin{cases} c(\beta)\left[(\sin s - \cos\beta)\mathbb{1}_{[\pi/2-\beta,\pi/2+\beta]}(s)\right] & \text{if } \beta \in (0, \pi/4] \\ c(\beta)\left[(\sin s - \cos\beta)\mathbb{1}_{[\pi/2-\beta,\pi/2+\beta]}(s) + \cos\beta - \frac{\sqrt{2}}{2}\right] & \text{if } \beta \in (\pi/4, \pi/2] \\ \bar{g}_{\pi-\beta}(s) & \text{if } \beta \in (\pi/2, \pi), \end{cases} \tag{3.10}$$

and where $c(\beta) > 0$ *is such that*

$$\int_0^{2\pi} \left|\bar{g}_\beta(s)\right| ds = 1.$$

Proof. In particular let σ be a minimal kinetic measure; since σ is π-periodic in the variable s, it follows from Proposition 3.2 that for $\nu_{\min}$-a.e. $x \in \Omega \setminus J$ it holds that

$$\sigma_x = \frac{1}{2 + 2\pi c}(\delta_{\bar{s}-\frac{\pi}{2}} + \delta_{\bar{s}+\frac{\pi}{2}} + c\mathscr{L}^1), \qquad \text{or}$$
$$\sigma_x = -\frac{1}{2 + 2\pi c}(\delta_{\bar{s}-\frac{\pi}{2}} + \delta_{\bar{s}+\frac{\pi}{2}} + c\mathscr{L}^1)$$

for some $\bar{s} \in \mathbb{R}/2\pi\mathbb{Z}$ and some $c \in \mathbb{R}$ depending on x. The necessary and sufficient condition (2.7) for minimality trivially implies $c = 0$. By Theorem 1.5 and (1.9) it holds that

$$\mathbf{n} \cdot \big(\Phi(m^+) - \Phi(m^-)\big)\, \mathscr{H}^1 \llcorner J = (p_x)_\sharp \left(-\partial_s \psi_\Phi \sigma \llcorner J \times \mathbb{R}/2\pi\mathbb{Z}\right).$$

The following identity was obtained in Sect. 4.2 of [13]: for every $\beta \in [0, \pi/2]$ it holds that

$$e_1 \cdot (\Phi(e^{i\beta}) - \Phi(e^{-i\beta})) = -\int_0^{2\pi} g_\beta(s)\partial_s\psi_\Phi(s)ds, \tag{3.11}$$

where $g_\beta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ is a π-periodic defined by

$$g_\beta(s) = (\sin s - \cos\beta)\mathbb{1}_{[\pi/2-\beta,\pi/2+\beta]}(s) - \frac{2}{\pi}(\sin\beta - \beta\cos\beta) \qquad \forall s \in [0, \pi].$$

Observe that the constraint $\operatorname{div} m = 0$ implies that for $\mathscr{H}^1$-a.e. $x \in J$ it holds $m^+ \cdot \mathbf{n} = m^- \cdot \mathbf{n}$. Therefore, with the notation introduced in the statement, we have $\mathbf{n} = \pm e^{i\bar{s}}$. We prove (3.10) first in the case $\beta \in [0, \pi/2]$.

Choosing $\tilde{\Phi}$ such that $\psi_{\tilde{\Phi}}(s) = \psi_\Phi(s + \bar{s})$, we deduce from (3.11) that

$$\begin{aligned} \mathbf{n} \cdot \big(\Phi(m^+) - \Phi(m^-)\big) &= \left(\mathbf{n} \cdot e^{i\bar{s}}\right) e^{i\bar{s}} \cdot \left(\Phi\left(e^{i(\bar{s}+\beta)}\right) - \Phi\left(e^{i(\bar{s}-\beta)}\right)\right) \\ &= \left(\mathbf{n} \cdot e^{i\bar{s}}\right) e^{i\bar{s}} \cdot \int_{-\beta}^{\beta} \psi_\Phi\left(s + \bar{s} + \frac{\pi}{2}\right) e^{i(s+\bar{s}+\frac{\pi}{2})} ds \\ &= \left(\mathbf{n} \cdot e^{i\bar{s}}\right) e_1 \cdot \int_{-\beta}^{\beta} \psi_\Phi\left(s + \bar{s} + \frac{\pi}{2}\right) e^{i(s+\frac{\pi}{2})} ds \end{aligned}$$

$$= \left(\mathbf{n} \cdot e^{i\bar{s}}\right) e_1 \cdot \left(\tilde{\Phi}\left(e^{i\beta}\right) - \tilde{\Phi}\left(e^{-i\beta}\right)\right)$$
$$= -\left(\mathbf{n} \cdot e^{i\bar{s}}\right) \int_0^{2\pi} g_\beta(s) \psi'_{\tilde{\Phi}}(s) ds$$
$$= -\left(\mathbf{n} \cdot e^{i\bar{s}}\right) \int_0^{2\pi} g_\beta(s - \bar{s}) \psi'_{\Phi}(s) ds.$$

This shows that for $\nu_{\min}$-a.e. $x \in J$ with $\beta \in (0, \pi/2)$ there exist two constants $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that $\sigma_x = c_1(g_\beta(\cdot - \bar{s}) + c_2)\mathscr{L}^1$. It is a straightforward computation to check that the choice in (3.10) is the unique that satisfies the constraint in (2.7). In particular σ_x is uniquely determined for $\nu_{\min}$-a.e. $x \in J$ such that $\beta \in (0, \pi/2)$.

The case $\beta \in (\pi/2, \pi)$, can be reduced to the previous case exchanging m^+ with m^-, and therefore changing the sign of $\mathbf{n}$ and replacing $\bar{s}$ with $\bar{s} + \pi$. Since $\partial_s \psi_\Phi$ and g_β for $\beta \in (0, \pi/2]$ are π-periodic, then the same computations as above leads to

$$\mathbf{n} \cdot \left(\Phi(m^+) - \Phi(m^-)\right) = -\left(\mathbf{n} \cdot e^{i\bar{s}}\right) \int_0^{2\pi} g_{\pi-\beta}(s - \bar{s}) \partial_s \psi_\Phi(s) ds.$$

Similarly the choice in (3.10) is the unique that satisfies the constraint (2.7). σ_x being uniquely determined for $\nu_{\min}$-a.e. $x \in \Omega$, the measure $\sigma_{\min}$ is unique. □

The following lemma links the jump set of the characteristic curves with the jump set of $m \in A(\Omega)$:

Lemma 3.5. *Let $m \in A(\Omega)$ and Ω' be a $W^{2,\infty}$ open set compactly contained in Ω. Let moreover ω be a minimal Lagrangian representation of m on Ω'. Then for ω-a.e. $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ the following property holds: for every $t \in (t_\gamma^-, t_\gamma^+)$ such that $\gamma_s(t+) \neq \gamma_s(t-)$ it holds that $\gamma_x(t) \in J$.*

Proof. Since ω is a minimal Lagrangian representation, by Proposition 2.5 and Corollary 3.4 it holds that

$$\int_\Gamma |\sigma_\gamma| d\omega = |\sigma_\omega| = \mathcal{L}^1 \times |\sigma_{\min}| = \mathcal{L}^1 \times (\nu_{\min} \otimes |(\sigma_{\min})_x|)$$

as measures in $(0, T) \times \Omega' \times \mathbb{R}/2\pi\mathbb{Z}$. By Corollary 3.4 it follows that for $\mathcal{L}^1 \times \nu_{\min}$-a.e. $(t, x) \in (0, T) \times (\Omega \setminus J)$, it holds that

$$\operatorname{supp}\left(\mathcal{L}^1 \times |\sigma_{\min}|\right)_{t,x} \subset \{\bar{s}, \bar{s} + \pi\} \tag{3.12}$$

for some $\bar{s} \in \mathbb{R}/2\pi\mathbb{Z}$. Suppose by contradiction that there exists $G \subset \Gamma$ with $\omega(G) > 0$ and a measurable function $\tilde{t} : G \to (0, T)$ such that for every $(\gamma, t_\gamma^-, t_\gamma^+)$ in G it holds that

$$\tilde{t}(\gamma) \in (t_\gamma^-, t_\gamma^+), \qquad \gamma_s\left(\tilde{t}(\gamma)+\right) \neq \gamma_s\left(\tilde{t}(\gamma)+\right), \qquad \text{and} \qquad \gamma_x\left(\tilde{t}(\gamma)\right) \in \Omega' \setminus J.$$

For every $(\gamma, t_\gamma^-, t_\gamma^+) \in G$ we set

$$\tilde{\sigma}_\gamma = \mathcal{H}^1 \llcorner E_\gamma^+\left(\tilde{t}(\gamma)\right) - \mathcal{H}^1 \llcorner E_\gamma^-\left(\tilde{t}(\gamma)\right), \qquad \text{where}$$

$$E_\gamma^\pm\left(\tilde t(\gamma)\right) := \{(t,x,s)\in E_\gamma^\pm : t = \tilde t(\gamma)\},$$

and $E_\gamma^\pm$ are defined in (2.4). Let $\tilde\sigma_\omega := \int_\Gamma |\tilde\sigma_\gamma| d\omega \in \mathcal{M}^+((0,T)\times\Omega'\times\mathbb{R}/2\pi\mathbb{Z})$; by definition we have $\tilde\sigma_\omega \le |\sigma_\omega|$. Let us denote by $\tilde\nu := (p_{t,x})_\sharp\tilde\sigma_\gamma$. Then by definition of $\tilde\sigma_\omega$ we have that $\tilde\nu$ is concentrated on $(0,T)\times\Omega'\setminus J$ and for $\tilde\nu$-a.e. $(t,x)\in(0,T)\times\Omega'\setminus J$ there exist no $\bar s\in\mathbb{R}/2\pi\mathbb{Z}$ such that $\mathrm{supp}(\tilde\sigma_\omega)_{t,x}\subset\{\bar s,\bar s+\pi\}$. Since $\tilde\nu(\Omega')>0$, this is in contradiction with (3.12). □

3.1. Solutions with a Single Vanishing Entropy

The goal of this section is to prove the following result about solutions with vanishing entropy production:

Proposition 3.6. *Let $\Omega\subset\mathbb{R}^2$ be an open set and $m\in A(\Omega)$ be such that $\mathrm{div}\,\Sigma_{\varepsilon_1,\varepsilon_2}(m)=0$. Then J is contained in the union of countably many horizontal and vertical segments. Moreover $\nu_{\min}$ is concentrated on J.*

The result follows from Proposition 3.2 and the following general result about BV functions for which we refer to [3, Proposition 3.92]:

Lemma 3.7. *Let $f\in \mathrm{BV}((0,T);\mathbb{R})$ be continuous from the right. Then for every $E\subset\mathbb{R}$ at most countable it holds*

$$\left|\tilde D f\right|\left(f^{-1}(E)\right)=0.$$

Proof of Proposition 3.6. We recall from [13] that

$$\mathrm{div}\,\Sigma_{\varepsilon_1,\varepsilon_2}(m) = -2(p_x)_\sharp\left[\sin(2s)\sigma\right].$$

For $\nu_{\min}$-a.e. $x\in J$ it holds $n=\pm e^{i\bar s}$, therefore in order to show that J is contained in a countable union of horizontal and vertical segments, it is sufficient to observe that for every $\beta\in(0,\pi)$ it holds that

$$\int_{\mathbb{R}/2\pi\mathbb{Z}} g_\beta(s-\bar s)\sin(2s)ds = 0 \qquad\Longrightarrow\qquad \bar s\in\frac{\pi}{2}\mathbb{Z}. \tag{3.13}$$

This can be proven directly by using the explicit expression of g_β in (3.10). Alternatively, we refer to [6, Lemma 2.4], where the authors show that for $m\in A(\Omega)\cap BV(\Omega)$ it holds that

$$|\mathrm{div}\,\Sigma_{\varepsilon_1,\varepsilon_2}(m)|\llcorner J = \frac{1}{3}\cos(2\alpha)|m^+-m^-|^3\mathcal{H}^1\llcorner J,$$

where $\alpha\in\mathbb{R}/2\pi\mathbb{Z}$ is such that $n=\pm e^{i(\alpha+\frac{\pi}{4})}$. Theorem 1.5 implies that the same computation is valid for every $m\in A(\Omega)$. Since $\cos(2\alpha)=0\Rightarrow\alpha\in\frac{\pi}{4}+\frac{\pi}{2}\mathbb{Z}$, then $\mathrm{div}\,\Sigma_{\varepsilon_1,\varepsilon_2}(m)=0$ implies that $n=e^{i\bar s}$ with $\bar s\in\frac{\pi}{2}\mathbb{Z}$ a.e. with respect to the measure $\nu_{\min}\llcorner J$.

Now we prove that $\nu_{\min}$ is concentrated on J: by Corollary 3.4, for $\nu_{\min}$-a.e. $x \in \Omega \setminus J$ it holds that

$$\int_{\mathbb{R}/2\pi\mathbb{Z}} \sin(2s) d(\delta_{\bar{s}} + \delta_{\bar{s}+\pi}) = 0,$$

which trivially implies $\bar{s} \in \frac{\pi}{2}\mathbb{Z}$. By Lemma 3.5, we have

$$\mathscr{L}^1 \times \nu_{\min} \llcorner (\Omega \setminus J) \leq \int_\Gamma (\gamma_x)_\sharp |\tilde{D}_t \gamma_s| \left(\gamma_s^{-1} \left(\frac{\pi}{2}\mathbb{Z} \right) \right) d\omega(\gamma) = 0,$$

where in the last equality we used Lemma 3.7. □

Remark 3.8. The same argument shows that, in order to prove that $\nu_{\min}$ is concentrated on J, the assumption $\operatorname{div} \Sigma_{\varepsilon_1,\varepsilon_2}(m) = 0$ can be replaced with $\operatorname{div} \Phi(m) = 0$ for any $\Phi \in \mathcal{E}_\pi$ such that $\{s : \partial_s \psi_\Phi(s) = 0\}$ is at most countable.

4. Uniqueness of Minimizers on Ellipses

The goal of this section is to prove Theorem 1.8. Since the functional $\tilde{F}_0$ is invariant by rotations, then we will assume without loss of generality that the major axis of the ellipse is parallel to x-axis in the plane.

The next result is essentially contained in [16] (see also [15]); for completeness, we give the proof here.

Proposition 4.1. *Let $\bar{u}^\delta$ be defined as in Theorem 1.8. Then $\bar{u}^\delta$ is a minimizer of $\tilde{F}_0(\cdot, \Omega_\delta)$ in $\Lambda^0_\delta(\Omega)$. Moreover, for every minimizer u^δ of $\tilde{F}_0(\cdot, \Omega_\delta)$ in $\Lambda^0_\delta(\Omega)$ the function $m = \nabla^\perp u^\delta$ satisfies*

$$\operatorname{div} \Sigma_{\varepsilon_1,\varepsilon_2}(m) = 0 \quad \textit{and} \quad \operatorname{div} \Sigma_{e_1,e_2}(m) \geq 0 \quad \textit{in } \mathcal{D}'(\Omega_\delta).$$

Proof. In [2], the authors noticed that for every $u \in A(\Omega_\delta)$ it holds that

$$\begin{aligned} \tilde{F}_0(u, \Omega_\delta) &= \left\| \begin{pmatrix} \operatorname{div} \Sigma_{e_1,e_2}(\nabla^\perp u) \\ \operatorname{div} \Sigma_{\varepsilon_1,\varepsilon_2}(\nabla^\perp u) \end{pmatrix} \right\| (\Omega_\delta) \\ &\geq \left(\left(\left| \operatorname{div} \Sigma_{e_1,e_2}(\nabla^\perp u) \right| (\Omega_\delta) \right)^2 + \left(\left| \operatorname{div} \Sigma_{\varepsilon_1,\varepsilon_2}(\nabla^\perp u) \right| (\Omega_\delta) \right)^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{4.1}$$

Let us denote by $\bar{m} := \nabla^\perp \bar{u}^\delta$. Since for every $u \in \Lambda_\delta(\Omega)$ it holds $\nabla^\perp u = \bar{m}$ in S_δ, then it follows from (4.1) that

$$\begin{aligned} \tilde{F}_0(u, \Omega_\delta) &\geq \left(\left(\left| \operatorname{div} \Sigma_{\varepsilon_1,\varepsilon_2}(\nabla^\perp u) \right| (\Omega_\delta) \right)^2 + \left(\left| \operatorname{div} \Sigma_{e_1,e_2}(\nabla^\perp u) \right| (\Omega_\delta) \right)^2 \right)^{\frac{1}{2}} \\ &\geq \operatorname{div} \Sigma_{e_1,e_2}(\nabla^\perp u)(\Omega_\delta) \\ &= \int_{\partial\Omega_\delta} \Sigma_{e_1,e_2}(\nabla^\perp u) \cdot n d\mathcal{H}^1 \\ &= \operatorname{div} \Sigma_{e_1,e_2}(\bar{m})(\Omega_\delta) \\ &= \tilde{F}_0(\bar{u}^\delta, \Omega_\delta), \end{aligned} \tag{4.2}$$

where in the last equality we used $\operatorname{div}\Sigma_{\varepsilon_1,\varepsilon_2}(\bar{m}) = 0$ and $\operatorname{div}\Sigma_{e_1,e_2}(\bar{m}) \geq 0$. This shows, in particular, that $\bar{u}^\delta$ is a minimizer of $\tilde{F}_0(\cdot, \Omega_\delta)$ in $\Lambda^0_\delta(\Omega)$. Moreover for every minimizer u of $\tilde{F}_0(\cdot, \Omega_\delta)$ in $\Lambda^0_\delta(\Omega)$, the inequality in (4.2) is an equality and this completes the proof. □

Theorem 4.2. *Let Ω be an ellipse, and $m \in A_\delta(\Omega)$ be such that*

$$\operatorname{div}\Sigma_{\varepsilon_1,\varepsilon_2}(m) = 0, \qquad \textit{and} \qquad \operatorname{div}\Sigma_{e_1,e_2}(m) \geq 0. \tag{4.3}$$

Then

$$m \llcorner \Omega = \nabla^\perp \operatorname{dist}(\cdot, \partial\Omega). \tag{4.4}$$

Proof. The proof is divided into three steps: in Step 1 we link the assumptions in (4.3) with the sign of $\partial_s \sigma_{\min}$ relying on Corollary 3.4 and Proposition 3.6. Then we will prove in Step 2 that the entropy defect measures of every m as in the statement are concentrated on the axis of the ellipse. We finally prove in Step 3 that this last condition forces m to satisfy (4.4).

Step 1. Let $m \in A_\delta(\Omega)$ be as in the statement and $\sigma_{\min}$ be its minimal kinetic measure. Then, for every $\phi \in C^1_c(\Omega_\delta \times \mathbb{R}/2\pi\mathbb{Z})$ such that $\phi \geq 0$ and

$$\operatorname{supp}\phi \subset \Omega_\delta \times \left(\left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3}{2}\pi\right)\right),$$

it holds that

$$\langle \partial_s \sigma_{\min}, \phi \rangle = -\int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \partial_s \phi \, d\sigma_{\min} \geq 0.$$

Proof of Step 1. Since $\operatorname{div}\Sigma_{\varepsilon_1,\varepsilon_2}(m) = 0$, it follows from Proposition 3.6 that for $\nu_{\min}$-a.e. $x \in J$ the normal to J at x is $\mathbf{n}(x) = e^{is(x)}$ for some $s(x) \in \frac{\pi}{2}\mathbb{Z}$. Up to exchange m^+ and m^-, we can therefore assume without loss of generality that $\mathbf{n}(x) = (1, 0)$ or $\mathbf{n}(x) = (0, 1)$ for $\nu_{\min}$-a.e. $x \in J$. We denote by $J_h \subset J$ the points for which $\mathbf{n} = (0, 1)$ and $J_v \subset J$ the points with $\mathbf{n}(x) = (1, 0)$. We consider these two cases separately.

If $\mathbf{n}(x) = (0, 1)$, then

$$\left(\operatorname{div}\Sigma_{e_1,e_2}(m)\right) \llcorner J_h = \frac{1}{3}\left((m_1^+)^3(x) - (m_1^-)^3(x)\right) \mathscr{H}^1 \llcorner J_h,$$

therefore $m_1^+(x) = -m_1^-(x) > 0$ for ν-a.e. $x \in J_h$. In particular, using the same notation as in Corollary 3.4, we have $\bar{s} = \frac{3}{2}\pi$. We observe that by the definition of $\bar{g}_\beta$ in (3.10), for every $\beta \in (0, \pi)$ it holds $\partial_s \bar{g}_\beta(s) \geq 0$ for $\mathcal{L}^1$- a.e. $s \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3}{2}\pi\right)$ and $\partial_s \bar{g}_\beta(s) \leq 0$ for $\mathcal{L}^1$- a.e. $s \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3}{2}\pi, 2\pi\right)$. In particular for every $\beta \in (0, \pi)$ and $\mathcal{L}^1$- a.e. $s \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3}{2}\pi\right)$ it holds that

$$\left(\mathbf{n} \cdot e^{i\bar{s}}\right) \partial_s \bar{g}_\beta(s - \bar{s}) = -\partial_s \bar{g}_\beta(s - \bar{s}) \geq 0.$$

Similarly, if $\mathbf{n} = (1, 0)$, then

$$\left(\operatorname{div}\Sigma_{e_1,e_2}(m)\right) \llcorner J_v = \frac{1}{3}\left((m_2^+)^3(x) - (m_2^-)^3(x)\right)\mathscr{H}^1 \llcorner J_v,$$

therefore $m_2^+(x) = -m_2^-(x) > 0$ for $\nu_{\min}$-a.e. $x \in J_v$. In particular $\bar{s} = 0$ so that for every $\beta \in (0, \pi)$ and $\mathcal{L}^1$- a.e. $s \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3}{2}\pi\right)$ it holds that

$$\left(\mathbf{n} \cdot e^{i\bar{s}}\right)\partial_s \bar{g}_\beta(s - \bar{s}) = \partial_s \bar{g}_\beta(s) \geq 0.$$

Therefore by Corollary 3.4, it follows that

$$\langle \partial_s \sigma_{\min}, \phi\rangle = \int_\Omega \int_0^{2\pi} \left(\mathbf{n} \cdot e^{i\bar{s}}\right) \bar{g}'_\beta(s - \bar{s})\phi \, ds \, d\nu_{\min} \geq 0.$$

Step 2. We prove that $\nu_{\min}$ is concentrated on the axis of the ellipse.

Let us denote by

$$\Omega = \left\{x \in \mathbb{R}^2 : x_1^2 + a x_2^2 < r^2\right\}$$

with $r > 0$ and $a \geq 1$. Let us assume by contradiction that $\nu_{\min}(J_h \cap \{x \in \mathbb{R}^2 : x_2 > 0\}) > 0$. Then there exists $b > 0$ such that $\nu_{\min}(\{x \in J_h : x_2 = b\}) > 0$. By the analysis in the proof of Step 1 there exists $A \subset \mathbb{R}$ such that $\mathscr{L}^1(A) > 0$ and for $\mathscr{H}^1$-a.e. $x \in A \times \{b\}$ it holds $m_1^-(x) < 0$. In particular we can choose $\alpha \in (\pi, 3\pi/2)$ such that

$$|\tan\alpha| \leq \frac{b}{2(r+\delta)} \quad \text{and}$$
$$\eta := \mathscr{H}^1\left(\left\{x \in \Omega \cap J_h : x_2 = b \text{ and } e^{i\alpha} \cdot m^-(x) > 0\right\}\right) > 0. \tag{4.5}$$

Let $\bar{x}_1 > 0$ be such that $(\bar{x}_1, b) \in \partial\Omega_\delta$ and denote by

$$E := \{x \in \Omega_\delta : x_2 \in (g(x_1), b)\}, \tag{4.6}$$

where $g(x_1) = \tan(\alpha)(x_1 - \bar{x}_1) + b$. The first constraint in (4.5) implies that $E \subset \{x_2 > 0\}$ (see Fig. 1).

We consider the following Lipschitz approximation of the characteristic function of E:

$$\psi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \notin E \\ \min\left\{1, \frac{1}{\varepsilon}\operatorname{dist}(x, \partial E)\right\} & \text{if } x \in E. \end{cases}$$

We moreover consider $\rho \in C_c^\infty(\pi + \frac{\alpha - \pi}{2}, \alpha)$ such that $\rho \geq 0$ and $\int_{\mathbb{R}} \rho(s)ds = 1$ and we test (1.7) with $\varphi_\varepsilon(s, x) = \psi_\varepsilon(x)\rho(s)$. If $\varepsilon < \delta$, then the choice of α in (4.5) and of ρ implies that

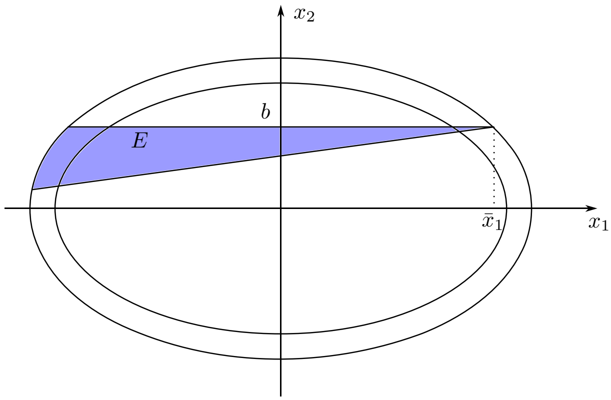

Fig. 1. The figure illustrates the definition of E in (4.6)

$$\{(x,s) \in \Omega_\delta \times \mathrm{supp}(\rho) : e^{is} \cdot \nabla_x \psi_\varepsilon < 0\} \subset \{(x,s) \in (\Omega_\delta \setminus \Omega) \times \mathrm{supp}(\rho) : x_2 > 0 \text{ and } x_1 < 0\}.$$

Since $m = \bar{m}$ on $\Omega_\delta \setminus \Omega$, then for $\mathscr{L}^2 \times \mathscr{L}^1$-a.e. $(x,s) \in (\Omega_\delta \setminus \Omega) \times \mathrm{supp}(\rho)$ it holds $\chi(x,s) = \mathbb{1}_{e^{is}\cdot m(x)>0} = 0$. In particular, by the second condition in (4.5), we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} e^{is} \cdot \nabla_x \psi_\varepsilon(x)\rho(s)\chi(x,s)dsdx \\
&\geq \int_{\{x\in\Omega: x_2=b\}\times\mathbb{R}/2\pi\mathbb{Z}} (-\sin s)\rho(s)\mathbb{1}_{e^{is}\cdot m^-(x)>0}(x)dsd\mathcal{H}^1(x) \\
&\geq \eta \sin\left(\frac{\alpha-\pi}{2}\right) \\
&> 0.
\end{aligned}$$

This contradicts Step 1, which implies that

$$\int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} e^{is} \cdot \nabla_x \psi_\varepsilon(x)\rho(s)\chi(x,s)dsdx = -\langle \partial_s \sigma_{\min}, \rho \otimes \psi_\varepsilon \rangle \leq 0.$$

A similar argument excludes that $\nu_{\min}(\{x \in J_h : x_2 = b\}) > 0$ if $b < 0$ and that $\nu_{\min}(\{x \in J_v : x_1 = a\}) > 0$ if $a \neq 0$; see Fig. 2 which illustrates the sets E that need to be considered in these cases.

Step 3. We prove that the unique $m \in A_\delta(\Omega)$ for which $\nu_{\min}$ is concentrated on the axis of the ellipse satisfies (4.4). In particular we show that $m = \bar{m}$ on

$$\tilde{\Omega}_\delta = \{x \in \Omega_\delta : x_1 < 0, x_2 > 0\},$$

this being the argument for the other analogous quadrants.

Let $\bar{x} \in \Omega$ be a Lebesgue point of m and let $\bar{s}(\bar{x}) \in (\pi/2, \pi)$ be such that

$$e^{i\bar{s}(\bar{x})} = -\nabla \operatorname{dist}(\bar{x}, \partial\Omega).$$

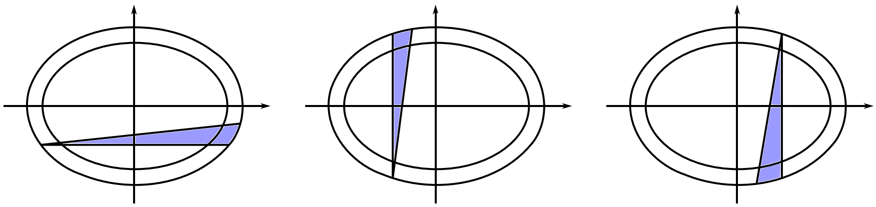

Fig. 2. The regions in blue indicate the sets E to be considered in order to repeat the presented argument in the three cases not addressed in details

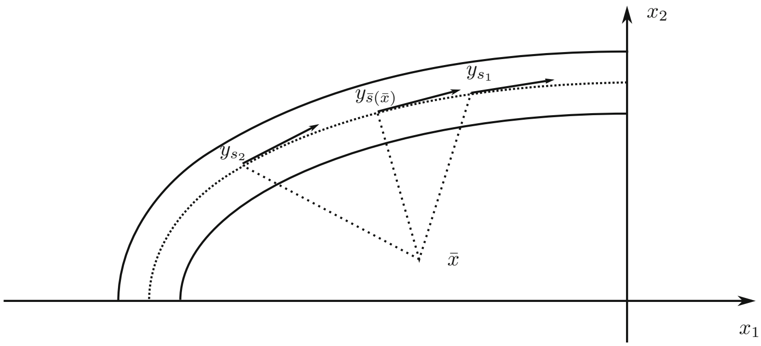

Fig. 3. The picture represents the points y_{s_1}, $y_{\bar{s}(\bar{x})}$, y_{s_2}, while the arrows represent the values of $\bar{m}$ at these points

For every $s \in (\pi/2, \pi)$ let $t_s > 0$ be the unique value such that

$$y_s := \bar{x} + t_s e^{is} \in \partial\Omega_{\delta/2} \cap \tilde{\Omega}_\delta.$$

By elementary geometric considerations (see Fig. 3) the following properties hold:

(1) $\bar{m}(y_s) \cdot e^{is} > 0$ for every $s \in (\pi/2, \bar{s}(\bar{x}))$;
(2) $\bar{m}(y_s) \cdot e^{is} < 0$ for every $s \in (\bar{s}(\bar{x}), \pi)$.

In particular for every $\varepsilon \in \left(0, \frac{1}{2}\min\{\bar{s}(\bar{x}) - \pi/2, \pi - \bar{s}(\bar{x})\}\right)$ there exists $r \in (0, \frac{\delta}{2})$ such that

(1) for every $s \in (\bar{s}(\bar{x}) - 2\varepsilon, \bar{s}(\bar{x}) - \varepsilon)$ and every $y \in B_r(y_s)$ it holds $\bar{m}(y) \cdot e^{is} > 0$;
(2) for every $s \in (\bar{s}(\bar{x}) + \varepsilon, \bar{s}(\bar{x}) + 2\varepsilon)$ and every $y \in B_r(y_s)$ it holds $\bar{m}(y) \cdot e^{is} < 0$.

By Step 2 we have that

$$e^{is} \cdot \nabla_x \chi = 0 \qquad \text{in } \mathcal{D}'(\tilde{\Omega}_\delta)$$

therefore for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}/2\pi\mathbb{Z}$ the sets $\left\{x \in \tilde{\Omega}_\delta : e^{is} \cdot m(x) > 0\right\}$ and $\left\{x \in \tilde{\Omega}_\delta : e^{is} \cdot m(x) < 0\right\}$ are invariant by translations in the direction e^{is} up to negligible sets. Since $m = \bar{m}$ in $\tilde{\Omega}_\delta \setminus \Omega$, then it follows by the previous analysis that for every $\varepsilon > 0$ there exists $r > 0$ such that for $\mathcal{L}^2$-a.e. $x \in B_r(\bar{x})$ the following two inequalities hold:

$$\begin{aligned} m(x) \cdot e^{is} > 0 \quad & \text{for } \mathcal{L}^1\text{-.a.e. } s \in (\bar{s}(\bar{x}) - 2\varepsilon, \bar{s}(\bar{x}) - \varepsilon), \\ m(x) \cdot e^{is} < 0 \quad & \text{for } \mathcal{L}^1\text{-.a.e. } s \in (\bar{s}(\bar{x}) + \varepsilon, \bar{s}(\bar{x}) + 2\varepsilon). \end{aligned} \tag{4.7}$$

The two conditions in (4.7) implies that for $\mathcal{L}^2$-a.e. $x \in B_r(\bar{x})$ it holds $m(x) = e^{is(x)}$ for some $s(x) \in [\bar{s}(\bar{x}) - \pi/2 - \varepsilon, \bar{s}(\bar{x}) - \pi/2 + \varepsilon]$. Since $\bar{x}$ is a Lebesgue point of m, letting $\varepsilon \to 0$, we obtain

$$m(\bar{x}) = \bar{s}(\bar{x}) - \frac{\pi}{2} = \bar{m}(\bar{x}).$$

This concludes the proof. □

Funding Open Access funding provided by EPFL Lausanne.

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(*Received April 21, 2021 / Accepted August 6, 2021*)
Published online August 23, 2021