Research Article

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Rectifiability of entropy defect measures in a micromagnetics model

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Abstract: We study the fine properties of a class of weak solutions u of the eikonal equation arising as asymptotic domain of a family of energy functionals introduced in [T. Rivière and S. Serfaty, Limiting domain wall energy for a problem related to micromagnetics, *Comm. Pure Appl. Math.* **54** (2001), no. 3, 294–338]. In particular, we prove that the entropy defect measure associated to u is concentrated on a 1-rectifiable set, which detects the jump-type discontinuities of u.

Keywords: Eikonal equation, rectifiability, Lagrangian representation, entropy production, kinetic formulation, micromagnetics

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1 Introduction

We consider a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ and we investigate the fine properties of the following class of divergence free unit vector fields:

Definition 1.1. We denote by $\mathcal{M}_{\mathrm{div}}(\Omega)$ the set of vector fields $u : \Omega \to \mathbb{C}$ for which the following conditions hold:

(i) $\operatorname{div} u = 0$ in the sense of distributions.

(ii) There exists $\phi \in L^\infty(\Omega)$ such that $u = e^{i\phi}$ and

$$\langle U_\phi, \psi(x,a)\rangle := \int_{\Omega\times\mathbb{R}} e^{i(\phi(x)\wedge a)} \cdot \nabla_x \psi(x,a)\,dx\,da \in \mathcal{M}(\Omega\times\mathbb{R}),$$

where $\mathcal{M}(\Omega\times\mathbb{R})$ denotes the set of finite Radon measures on $\Omega\times\mathbb{R}$.

The space $\mathcal{M}_{\mathrm{div}}(\Omega)$ is the conjectured asymptotic domain as $\varepsilon \to 0$ of the following family of energy functionals introduced in [20] in the context of micromagnetics:

$$E_\varepsilon(u) := \varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} |H_u|^2,$$

where $u \in W^{1,2}(\Omega, \mathbb{S}^1)$ and the so-called demagnetizing field $H_u \in L^2(\mathbb{R}^2;\mathbb{R}^2)$ is such that $\operatorname{curl} H_u = 0$ and $\operatorname{div}(\tilde u + H_u) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$, where

$$\tilde u(x) = \begin{cases} u(x) & \text{if } x \in \Omega,\\ 0 & \text{otherwise.}\end{cases}$$

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The following compactness result was proven in [21]: let ϕ_{ε_n} be a bounded sequence in $L^\infty(\Omega)$ such that $E_{\varepsilon_n}(u_{\varepsilon_n})$ is uniformly bounded, where $u_{\varepsilon_n} = e^{i\phi_{\varepsilon_n}}$ and $\varepsilon_n \to 0$; then ϕ_{ε_n} is relatively compact in $L^p(\Omega)$ for every $p \in [1, \infty)$ and for every limit point $\bar{\phi}$ it holds

$$e^{i\bar{\phi}} \in \mathcal{M}_{\mathrm{div}}(\Omega) \quad \text{and} \quad |U_{\bar{\phi}}|(\Omega \times \mathbb{R}) \le \liminf_{n\to\infty} E_{\varepsilon_n}(u_{\varepsilon_n}). \tag{1.1}$$

Although the Γ-lim inf inequality (1.1) was proved in full generality, the corresponding Γ-lim sup inequality was obtained only in special cases. In particular, the energy-minimizing configurations were characterized by the results in [5, 21]. It is expected that the energy E_ε is concentrated on lines at a scale $\varepsilon > 0$ around the lines, allowing for sharper and sharper jumps as $\varepsilon \to 0$; the latter correspond in three dimensions to jumps across surfaces, called *domain walls* in the theory of micromagnetism (see [20]). These lines are detected by the measure U_ϕ: in particular, if we denote by $p_x : \Omega \times \mathbb{R} \to \Omega$ the standard projection on the first component and if $\phi \in BV(\Omega)$, then the measure

$$\nu := (p_x)_\sharp |U_\phi|$$

is concentrated on the 1-rectifiable jump set of ϕ.

However, vector fields in $\mathcal{M}_{\mathrm{div}}(\Omega)$ do not have necessarily bounded variation and a study of their fine properties must therefore be independent of the theory of BV functions. This program was announced in [5] and carried on in [4] leading to the following result.

Theorem 1.2. *Let ϕ be a lifting of $u \in \mathcal{M}_{\mathrm{div}}(\Omega)$ as in Definition 1.1. Then the following assertions hold:*
(1) *The jump set J of ϕ is countably $\mathcal{H}^1$-rectifiable and coincides, up to $\mathcal{H}^1$-negligible sets, with*

$$\Sigma := \Big\{x \in \Omega : \limsup_{r\to 0} \frac{\nu(B_r(x))}{r} > 0\Big\}. \tag{1.2}$$

Moreover, for every $a \in \mathbb{R}$ it holds

$$\mathrm{div}(e^{i\phi\wedge a}) \llcorner J = \mathbf{1}_{\phi^- < a < \phi^+}(e^{ia} - e^{i\phi^-}) \cdot \mathbf{n}_J \mathcal{H}^1 \llcorner J,$$

where $\mathbf{n}_J$ denotes the normal to J.
(2) *Every $x \in \Omega \setminus \Sigma$ is a vanishing mean oscillation point of ϕ, namely*

$$\lim_{r\to 0} \frac{1}{r^2} \int_{B_r(x)} |\phi - \phi_r(x)| = 0,$$

where $\phi_r(x)$ is the average of ϕ on $B_r(x)$.
(3) *The measure $\nu \llcorner (\Omega \setminus J)$ is orthogonal to $\mathcal{H}^1$, namely*

$$B \subset (\Omega \setminus J) \text{ Borel with } \mathcal{H}^1(B) < \infty \quad \text{implies} \quad \nu(B) = 0.$$

We observe that for functions $\phi \in BV_{\mathrm{loc}}(\Omega)$ the above properties (2) and (3) can be improved to the following:
(2’) $\mathcal{H}^1$-a.e. point in $\Omega \setminus J$ is a Lebesgue point of ϕ.
(3’) The measure $\nu \llcorner (\Omega \setminus J)$ is identically 0.
In [4], it was conjectured that both (2’) and (3’) hold for every $u \in \mathcal{M}_{\mathrm{div}}(\Omega)$. The following weaker version of (2’) was recently obtained in [14] in the close setting of weak solutions u with finite entropy production of the Burgers equation:
(2*) The set of non-Lebesgue points of u has Hausdorff dimension at most 1.
This result was extended for general conservation laws in [18], implying in particular that property (2*) holds in the setting of this paper, namely for functions $\phi \in L^\infty$ corresponding to vector fields $u \in \mathcal{M}_{\mathrm{div}}(\Omega)$.

The main result of this paper is the proof of property (3’) for general vector fields $u \in \mathcal{M}_{\mathrm{div}}(\Omega)$.

Theorem 1.3. *Let ϕ be a lifting of $u \in \mathcal{M}_{\mathrm{div}}(\Omega)$ as in Definition 1.1. Then the measure ν is concentrated on the countably $\mathcal{H}^1$-rectifiable set Σ defined in* (1.2)*. In particular, for every $a \in \mathbb{R}$ it holds*

$$\mathrm{div}(e^{i\phi\wedge a}) = \mathbf{1}_{\phi^- < a < \phi^+}(e^{ia} - e^{i\phi^-}) \cdot \mathbf{n}_J \mathcal{H}^1 \llcorner J.$$

Theorem 1.3 establishes that the concentration property expected for the Γ-limit functional of E_ε as $\varepsilon \to 0$ holds for the candidate Γ-limit; this property is also considered as a fundamental step to complete the Γ-lim sup analysis (see [16]).

1.1 Main tool and strategy of the proof

The strategy of the proof of Theorem 1.3 was introduced in [19] to prove the analogous result for weak solutions with finite entropy production of the Burgers equation (or, more generally, 1D scalar conservation laws with uniformly convex flux). Indeed, there is a strong analogy between weak solutions to conservation laws with finite entropy production and the solutions to the eikonal equation arising in this model or the related model introduced by Aviles and Giga in [6]. In particular, Theorem 1.2 has an analogous version for scalar conservation laws (see [11, 15]) and for the model by Aviles and Giga [10]. In order to compare the setting of this paper and the one of conservation laws, we observe that for $u = e^{i\phi} \in \mathcal{M}_{\mathrm{div}}(\Omega)$ it holds

$$\partial_{x_1} \cos\phi + \partial_{x_2} \sin\phi = 0.$$

Let us assume that ϕ takes values in $(0, \pi)$ so that the cosine is invertible in the range of ϕ and $v = \cos\phi$ satisfies the equation

$$\partial_{x_1} v + \partial_{x_2}(\sin(\cos^{-1}(v))) = 0.$$

Since the map $\sin\circ\cos^{-1}$ is convex on $(-1, 1)$, it is possible to transfer the results obtained for conservation laws with convex fluxes to solutions of the eikonal equation taking values in $(0, \pi)$. When instead the oscillation of ϕ is larger than π, the approach above fails and more refined arguments are needed.

The main tool used to prove Theorem 1.3 is the so called Lagrangian representation, which was introduced in [7] for entropy solutions to general conservation laws and then extended in [18] to weak solutions with finite entropy production. This Lagrangian representation (see Definition 3.1) is an extension of the classical method of characteristics to this non-smooth setting and it is strongly inspired by Ambrosio's superposition principle in the context of positive measure valued solutions to the linear continuity equation (see, for example, [1]). Roughly speaking, the evolution of the solution is obtained as superposition of single trajectories traveling with characteristic speed. This tool is well suited for our purposes since also the kinetic measure U_ϕ can be decomposed along the characteristic trajectories detected by the Lagrangian representation. In Section 3, we prove the existence of a Lagrangian representation for vector fields in $\mathcal{M}_{\mathrm{div}}(\Omega)$ building on the following kinetic formulation obtained in [21] (see also [13] in the study of the model by Aviles and Giga, and the fundamental paper [17] in the setting of entropy solutions to scalar conservation laws): by setting $\chi(x, a) := \mathbf{1}_{\phi(x)\geq a}$, it holds

$$ie^{ia} \cdot \nabla_x \chi = -\partial_a U_\phi \quad \text{in } \mathcal{D}'(\Omega \times \mathbb{R}). \tag{1.3}$$

The proof of the existence of a Lagrangian representation follows the strategy of [18], but additional work is required since we consider here solutions on bounded domains instead of the whole $\mathbb{R}^2$.

Once a Lagrangian representation is available for vector fields in $\mathcal{M}_{\mathrm{div}}(\Omega)$, we implement the strategy introduced in [19] to prove Theorem 1.3. Since the oscillation of ϕ is bigger than π, the argument does not apply straightforwardly. Still a partial result is obtained in Section 4.2 by covering the image of ϕ with finitely many intervals $(I_l)_{l=1}^L$ of length less than π and appropriately localizing the argument of [19]. A new regularity estimate is proven in Section 4.3 and this allows to conclude the proof of Theorem 1.3, relying on Theorem 1.2.

2 Preliminaries

2.1 Duality for L^1-optimal transport

In this section, we recall a few facts about L^1-optimal transport. We state the results in the form that we will need in Section 3. Given a metric space X, we denote by $\mathcal{M}_+(X)$ the set of finite non-negative Borel measures on X.

Definition 2.1. Let (X, d) be a complete and separable metric space and let $\mu_1, \mu_2 \in \mathcal{M}_+(X)$ be such that $\mu_1(X) = \mu_2(X)$. The Wasserstein distance of order 1 between μ_1 and μ_2 is defined by

$$W_1(\mu_1, \mu_2) := \inf_{\pi \in \Pi(\mu_1,\mu_2)} \int_X d(x, y)\, d\pi(x, y), \tag{2.1}$$

where $\Pi(\mu_1, \mu_2)$ is the set of transport plans from μ_1 to μ_2, i.e.

$$\Pi(\mu_1, \mu_2) := \{\omega \in \mathcal{M}_+(X^2) : \pi_{1\sharp}\omega = \mu_1,\ \pi_{2\sharp}\omega = \mu_2\},$$

denoting by $\pi_1, \pi_2 : X^2 \to X$ the two natural projections.

Notice that W_1 can take the value $+\infty$.

In order to prove the existence of a Lagrangian representation for vector fields in $\mathcal{M}_{\mathrm{div}}(\Omega)$, we will take advantage of the dual formulation of the L^1-optimal transport. The following duality formula can be found, for example, in [22].

Proposition 2.2. *For any $\mu_1, \mu_2 \in \mathcal{M}_+(X)$ with $\mu_1(X) = \mu_2(X)$, it holds*

$$W_1(\mu_1, \mu_2) = \sup_{\psi \in L^1(\mu_1), \|\psi\|_{\mathrm{Lip}} \le 1} \Big(\int_X \psi\, d\mu_1 - \int_X \psi\, d\mu_2 \Big).$$

Since it will be convenient to allow that the two measures μ_1, μ_2 have different masses, we deduce from Proposition 2.2 the following result.

Corollary 2.3. *Let (X, d) be bounded and let $\mu_1, \mu_2 \in \mathcal{M}_+(X)$. Assume that there exist $C_1, C_2 > 0$ such that for every $\psi \in \mathrm{Lip}(X)$ it holds*

$$\Big| \int_X \psi\, d\mu_1 - \int_X \psi\, d\mu_2 \Big| \le C_1 |\psi|_{\mathrm{Lip}} + C_2 \|\psi\|_{L^\infty}. \tag{2.2}$$

Then there exist $\tilde\mu_1 \le \mu_1, \tilde\mu_2 \le \mu_2$ such that $\|\mu_1 - \tilde\mu_1\| \le C_2$, $\|\mu_2 - \tilde\mu_2\| \le C_2$ and

$$W_1(\tilde\mu_1, \tilde\mu_2) \le C_1 + C_2 \operatorname{diam}(X). \tag{2.3}$$

Proof. We assume without loss of generality that $\alpha := \|\mu_1\| - \|\mu_2\| \ge 0$. Let $\bar\mu_2 = \mu_2 + \alpha\delta_{\bar x}$ for some $\bar x \in X$. Then we have

$$\begin{aligned}
\Big| \int_X \psi\, d\mu_1 - \int_X \psi\, d\bar\mu_2 \Big| &= \Big| \int_X (\psi - \psi(\bar x))\, d\mu_1 - \int_X (\psi - \psi(\bar x))\, d\bar\mu_2 \Big| \\
&= \Big| \int_X (\psi - \psi(\bar x))\, d\mu_1 - \int_X (\psi - \psi(\bar x))\, d\bar\mu_2 \Big| \\
&\le C_1 |\psi|_{\mathrm{Lip}} + C_2 |\psi|_{\mathrm{Lip}} \operatorname{diam}(X).
\end{aligned}$$

By Proposition 2.2, it follows that $W_1(\mu_1, \bar\mu_2) \le C_1 + C_2 \operatorname{diam}(X)$. Let $\pi \in \mathcal{M}(X^2)$ be an optimal plan with marginals μ_1 and $\bar\mu_2$ and let $\tilde\pi \le \pi$ be such that $(p_2)_\sharp\tilde\pi = \mu_2$. Then we check that the statement is true for $\tilde\mu_1 = (p_1)_\sharp\tilde\pi$ and $\tilde\mu_2 = \mu_2$: indeed, $\tilde\pi$ is an admissible plan between $\tilde\mu_1$ and $\tilde\mu_2$ by construction. Since $\tilde\pi \le \pi$ and $W_1(\mu_1, \bar\mu_2) \le C_1 + C_2 \operatorname{diam}(X)$, inequality (2.3) holds true. Moreover,

$$\|\mu_2 - \tilde\mu_2\| = 0 \quad \text{and} \quad \|\mu_1 - \tilde\mu_1\| = \mu_1(X) - \tilde\mu_1(X) = \alpha$$

since $\tilde\mu_1 \le \mu_1$. Finally, we observe that, by choosing $\psi \equiv 1$ in (2.2), we obtain $\alpha \le C_2$. This shows that $\|\mu_1 - \tilde\mu_1\| \le C_2$ and completes the proof. □

The next theorem from [8] provides the existence of an L^1-optimal map with respect to quite general distances on $\mathbb{R}^N$.

Theorem 2.4. *Let $X = \mathbb{R}^N$ with $N \in \mathbb{N}$ be the Euclidean space equipped with the distance induced by a convex norm $|\cdot|_{D*}$. Let $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^N)$ be two probability measures such that $\mu_1 \ll \mathscr{L}^N$ and the infimum in (2.1) is finite. Then there exists an optimal plan π in (2.1) induced by a map, i.e. there exists a measurable map $T : \mathbb{R}^N \to \mathbb{R}^N$ such that $T_\sharp \mu_1 = \mu_2$ and*

$$W_1(\mu_1, \mu_2) = \int_X |T(x) - x|_{D*} \, d\mu_1(x).$$

2.2 Weak convergence of measures

We will say that a sequence of measures $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_+(X)$ is *narrowly convergent* to $\mu \in \mathcal{M}_+(X)$ if

$$\lim_{n\to\infty} \int_X f d\mu_n = \int_X f d\mu \quad \text{for all } f \in C_b(X),$$

where $C_b(X)$ denotes the set of continuous real-valued bounded functions on X. Moreover, we say that a bounded family $\mathscr{F} \subset \mathcal{M}_+(X)$ is *tight* if for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that for every $\mu \in \mathscr{F}$ it holds

$$\mu(X \setminus K) < \varepsilon.$$

The following classical theorem characterizes the relatively compact families in $\mathcal{M}_+(X)$ (see [9]).

Theorem 2.5 (Prokhorov). *Let X be a metric space. If a bounded family $\mathscr{F} \subset \mathcal{M}_+(X)$ is tight, then it is relatively compact with respect to the narrow convergence. If moreover X is complete and separable, then also the converse implication holds.*

3 Lagrangian representation for vector fields in $\mathcal{M}_{\mathrm{div}}$

In this section, we introduce the notions of Lagrangian representations of the hypograph and of the epigraph for the liftings ϕ of vector fields in $\mathcal{M}_{\mathrm{div}}(\Omega)$. Moreover, we provide a suitable decomposition along characteristics of the kinetic measure U_ϕ introduced in (1.3).

3.1 Notation and main definition

We will consider the standard decomposition of the measure $Df \in \mathcal{M}(\mathbb{R})$, where $f \in \mathrm{BV}(\mathbb{R}, \mathbb{R})$ (see, for example, [2]). We will adopt the following notation:

$$Df = D^{\mathrm{ac}} f + D^c f + D^j f = \tilde{D} f + D^j f,$$

where $D^{\mathrm{ac}} f$, $D^c f$ and $D^j f$ denote the absolutely continuous part, the Cantor part and the atomic part of Df, respectively; we refer to $\tilde{D} f$ as the diffuse part of Df.

For every function $\phi : \Omega \to [0, M]$, we denote its hypograph and its epigraph by

$$H_\phi := \{(x, a) \in \Omega \times [0, M] : a \leq \phi(x)\} \quad \text{and} \quad E_\phi := \{(x, a) \in \Omega \times [0, M] : a \geq \phi(x)\},$$

respectively.

We denote by B_R an open ball of radius R such that $\overline{B_R} \subset \Omega$ and we set

$$\Gamma := \{(\gamma, t_\gamma^-, t_\gamma^+) : 0 \leq t_\gamma^- \leq t_\gamma^+ \leq 1, \ \gamma \in \mathrm{BV}((t_\gamma^-, t_\gamma^+); B_R \times [0, M]), \ \gamma_x \text{ is Lipschitz}\}.$$

In order to fix a representative, we will always assume that γ is continuous from the right. For every $t \in (0, 1)$, we consider the section

$$\Gamma(t) := \{(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma : t \in (t_\gamma^-, t_\gamma^+)\}.$$

and we set

$$e_t : \Gamma(t) \to B_R \times [0, M],$$
$$(\gamma, t_\gamma^-, t_\gamma^+) \mapsto \gamma(t).$$

Sometimes we will identify the triple $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ with the curve γ itself to make the notation less heavy.

Definition 3.1. Suppose $u \in \mathcal{M}_{\mathrm{div}}(\Omega)$ and $\phi \in L^\infty(\Omega)$ as in Definition 1.1. We say that the Radon measure $\omega_h \in \mathcal{M}(\Gamma)$ is a *Lagrangian representation* of the hypograph of ϕ on B_R if the following conditions hold:
(1) For every $t \in (0, 1)$, it holds

$$(e_t)_\sharp[\omega_h \llcorner \Gamma(t)] = \mathscr{L}^3 \llcorner H_\phi.$$

(2) The measure ω_h is concentrated on the set of curves $\gamma \in \Gamma$ such that for $\mathscr{L}^1$-a.e. $t \in (t_\gamma^-, t_\gamma^+)$ the following characteristic equation holds:

$$\dot\gamma_x(t) = ie^{i\gamma_a(t)}. \tag{3.1}$$

(3) It holds the integral bound

$$\int_\Gamma \mathrm{TotVar}_{[0,1)}\, \gamma_a \, d\omega_h(\gamma) < \infty.$$

Similarly, we say that $\omega_e \in \mathcal{M}(\Gamma)$ is a *Lagrangian representation* of the epigraph of u on B_R if conditions (2) and (3) hold and (1) is replaced by

$$(e_t)_\sharp[\omega_e \llcorner \Gamma(t)] = \mathscr{L}^3 \llcorner E_\phi \quad \text{for every } t \in (0, 1).$$

In the following, we will adopt the slight abuse of notation

$$(e_t)_\sharp \omega_h := (e_t)_\sharp(\omega_h \llcorner \Gamma(t)).$$

A fundamental property of the Lagrangian representations ω_h, ω_e above is that it is possible to decompose the Radon measure U_ϕ along the characteristic curves.

Given $\gamma \in \Gamma$, we consider

$$\mu_\gamma = (\mathrm{I}, \gamma)_\sharp \tilde D_t \gamma_a + \mathscr{H}^1 \llcorner E_\gamma^+ - \mathscr{H}^1 \llcorner E_\gamma^- \in \mathcal{M}((0, 1) \times B_R \times [0, M]),$$

where

$$E_\gamma^+ := \{(t, x, a) : \gamma_x(t) = x,\ \gamma_a(t-) < \gamma_a(t+),\ a \in (\gamma_a(t-), \gamma_a(t+))\},$$
$$E_\gamma^- := \{(t, x, a) : \gamma_x(t) = x,\ \gamma_a(t+) < \gamma_a(t-),\ a \in (\gamma_a(t+), \gamma_a(t-))\},$$

$\mathrm{I} : [0, 1) \to [0, 1)$ denotes the identity and $\tilde D_t \gamma_a$ denotes the diffuse part of the measure $D_t \gamma_a$.

The main result of this section is the following theorem.

Theorem 3.2. *Let $u \in \mathcal{M}_{\mathrm{div}}(\Omega)$ and $\phi \in L^\infty(\Omega)$ as in Definition 1.1. Let B_R be an open ball of radius R such that $\overline{B_R} \subset \Omega$ and $\mathscr{H}^1$-a.e. $x \in \partial B_R$ is a Lebesgue point of ϕ. Then there exist Lagrangian representations ω_h, ω_e of the hypograph and of the epigraph of u, respectively, on B_R enjoying the additional properties:*

$$\int_\Gamma \mu_\gamma \, d\omega_h(\gamma) = \mathscr{L}^1 \times U_\phi = -\int_\Gamma \mu_\gamma \, d\omega_e(\gamma), \tag{3.2}$$

$$\int_\Gamma |\mu_\gamma| \, d\omega_h(\gamma) = \mathscr{L}^1 \times |U_\phi| = \int_\Gamma |\mu_\gamma| \, d\omega_e(\gamma). \tag{3.3}$$

Equations (3.2) and (3.3) are equalities in the space $\mathcal{M}((0, 1) \times B_R \times [0, M])$; equation (3.2) asserts that the measure $\mathscr{L}^1 \times U_\phi$ can be decomposed along characteristics and equation (3.3) says that it can be done minimizing

$$\int_\Gamma \mathrm{TotVar}_{(0,1)}\, \gamma_a \, d\omega_h(\gamma) \quad \text{and} \quad \int_\Gamma \mathrm{TotVar}_{(0,1)}\, \gamma_a \, d\omega_e(\gamma).$$

Moreover, it follows from (3.2) and (3.3) that we can separately represent the negative and the positive parts of $\mathscr{L}^1 \times U_\phi$ in terms of the negative and positive parts of the measures μ_γ:

$$\int_\Gamma \mu_\gamma^- \, d\omega_h(\gamma) = \mathscr{L}^1 \times U_\phi^- = \int_\Gamma \mu_\gamma^+ \, d\omega_e(\gamma) \quad \text{and} \quad \int_\Gamma \mu_\gamma^+ \, d\omega_h(\gamma) = \mathscr{L}^1 \times U_\phi^+ = \int_\Gamma \mu_\gamma^- \, d\omega_e(\gamma). \tag{3.4}$$

The proof of Theorem 3.2 follows the strategy used in [18] to deal with general conservation laws; some additional work is required to obtain representations of solutions defined on B_R and not on the whole Euclidean space.

3.2 An L^1-transport estimate

In this section, we prove an L^1-transport estimate that will be used as building block in the construction of approximate characteristics. First, we need the following lemma.

Lemma 3.3. *Let $\bar{x} \in \Omega$ and let B_R be an open ball of radius R centered at $\bar{x}$ such that $\overline{B_R} \subset \Omega$ and $\mathscr{H}^1$-a.e. $x \in \partial B_R$ is a Lebesgue point of ϕ. Let $\bar{t} > 0$ be such that $\bar{t} < \operatorname{dist}(B_R, \partial\Omega)$ and let χ, U_ϕ be as in* (1.3). *We define $\chi^1, \chi^2 : [0, \bar{t}] \times \Omega \times [0, M] \to \{0, 1\}$ by*

$$\chi^1(t, x, a) = \chi(x, a)\mathbf{1}_{B_R}(x) \quad \textit{and} \quad \chi^2(t, x, a) = \chi(x - ie^{ia}t, a)\mathbf{1}_{B_R}(x).$$

Then there exist two Radon measure $\mu_{\bar{t}}^1, \mu_{\bar{t}}^2 \in \mathcal{M}([0, \bar{t}] \times \Omega \times [0, M])$ absolutely continuous with respect to $\mathscr{H}^3 \llcorner([0, \bar{t}] \times \partial B_R \times [0, M])$ such that

$$\begin{cases} \partial_t \chi^1 + ie^{ia} \cdot \nabla_x \chi^1 = -\partial_a(\mathbf{1}_{[0,T] \times B_R \times [0,M]} U_\phi) + \mu_{\bar{t}}^1, \\ \partial_t \chi^2 + ie^{ia} \cdot \nabla_x \chi^2 = \mu_{\bar{t}}^2, \\ \varepsilon_{\bar{t}} := \dfrac{\|\mu_{\bar{t}}^1 - \mu_{\bar{t}}^2\|}{\bar{t}} \to 0 \quad \textit{as } \bar{t} \to 0. \end{cases} \tag{3.5}$$

Proof. Let $\delta \in (0, R)$ and let $\psi_\delta \in C^1([0, +\infty); \mathbb{R})$ be such that

$$\psi_\delta \equiv 1 \text{ in } (0, R_\delta), \quad \psi_\delta \equiv 0 \text{ in } (R, +\infty), \quad \|\psi_\delta'\|_{C^0} \le \frac{2}{\delta}.$$

Let us consider $\chi_\delta^1(t, x, a) := \chi(x, a)\psi_\delta(|x - \bar{x}|)$. Since $\mathscr{H}^1$-a.e. $x \in \partial B_R$ is a Lebesgue point of ϕ, we have that $\mathscr{H}^2$-a.e. $(x, a) \in \partial B_R \times [0, M]$ is a Lebesgue point of χ. Therefore, testing (1.3) with $\psi_\delta(|\cdot - \bar{x}|)$ and letting $\delta \to 0^+$, we get

$$ie^{ia} \cdot \nabla_x(\chi(x, a)\mathbf{1}_{B_R}(x)) = -\partial_a(\mathbf{1}_{B_R \times [0,M]} U_\phi) + g\mathscr{H}^2 \llcorner(\partial B_R \times [0, M]),$$

where $g(x, a) := ie^{ia} \cdot n(x)\chi(x, a)$ and n denotes the inner normal to B_R. Since $\chi^1(t, x, a) = \chi(x, a)\mathbf{1}_{B_R}(x)$ for every $t \in [0, T]$, system (3.5) holds for

$$\mu_{\bar{t}}^1 = \bar{g}\mathscr{H}^3 \llcorner([0, \bar{t}] \times \partial B_R \times [0, M]) \quad \text{with } \bar{g}(t, x, a) = ie^{ia} \cdot n(x)\chi(x, a).$$

From the definition of χ^2, the second equation in (3.5) holds with

$$\mu_{\bar{t}}^2 = ie^{ia} \cdot n(x)\chi(x - ie^{ia}t, a)\mathscr{H}^3 \llcorner([0, \bar{t}] \times \partial B_R \times [0, M]).$$

Since $\mathscr{H}^2$-a.e. $(x, a) \in \partial B_R \times [0, M]$ is a Lebesgue point of χ, for $\mathcal{L}^1$-a.e. $a \in [0, M]$ we have that $\mathscr{H}^1$-a.e. $x \in \partial B_R$ is a Lebesgue point of $\chi(\cdot, a)$. In particular, for $\mathcal{L}^1$-a.e. $a \in [0, M]$ it holds

$$\lim_{\bar{t} \to 0} \frac{1}{\bar{t}} \int_0^{\bar{t}} \int_{\partial B_R} |\chi(x, a) - \chi(x - ie^{ia}t, a)| \, d\mathscr{H}^1(x) \, dt = 0. \tag{3.6}$$

Since for every $t \in [0, \bar{t}]$ and every $a \in [0, M]$ it holds

$$\int_{\partial B_R} |\chi(x, a) - \chi(x - ie^{ia}t, a)| \, d\mathscr{H}^1(x) \le |\partial B_R|,$$

by integrating (3.6) with respect to a, it follows by the dominated convergence theorem that

$$\|\mu^1_{\bar t} - \mu^2_{\bar t}\| = \int_0^{\bar t}\int_0^M \int_{\partial B_R} |\chi(x,a) - \chi(x - ie^{ia}t, a)|\, d\mathscr{H}^1(x)\, da\, dt = o(\bar t) \quad \text{as } \bar t \to 0$$

since $\mathscr{H}^1$-a.e. $x \in \partial B_R$ is a Lebesgue point of ϕ, and therefore $\mathscr{H}^2$-a.e. $(x,a) \in \partial B_R \times [0,M]$ is a Lebesgue point of χ. □

Proposition 3.4. *In the setting of Lemma 3.3, let $\psi \in C^1_c(\Omega \times \mathbb{R})$. Then*

$$\int_{\Omega\times\mathbb{R}} \psi(x,a)(\chi^1(\bar t) - \chi^2(\bar t))\, dx\, da \le \Big(\bar t \|\partial_a \psi\|_{L^\infty} + \frac{\bar t^2}{2}\|\nabla_x \psi\|_{L^\infty}\Big)\nu(B_R) + \|\psi\|_{L^\infty}\varepsilon_{\bar t}\bar t.$$

Proof. We set $\tilde\chi := \chi^1 - \chi^2$ and $\tilde\psi(t,x,a) := \psi(x + ie^{ia}(\bar t - t), a)$. It is straightforward to check that

$$\partial_t(\tilde\chi\tilde\psi) + ie^{ia}\cdot\nabla_x(\tilde\chi\tilde\psi) = -\tilde\psi\partial_a(\mathscr{L}^1 \times U_\phi) + \tilde\psi(\mu^1_{\bar t} - \mu^2_{\bar t}) \quad \text{in } \mathcal{D}'((0,\bar t)\times\Omega\times\mathbb{R}). \tag{3.7}$$

Let $g : [0,\bar t] \to \mathbb{R}$ be defined by

$$g(t) = \int_{\Omega\times\mathbb{R}} \tilde\chi(t)\tilde\psi(t)\, dx\, da.$$

It follows from (3.7) that

$$g'(t) = -\int_{\Omega\times\mathbb{R}} \partial_a\tilde\psi(t)\, dU_\phi + \int_{\Omega\times\mathbb{R}} \tilde\psi(t)\, d(\mu^1_{\bar t} - \mu^2_{\bar t})_t$$

holds in the sense of distributions, where $(\mu^1_{\bar t} - \mu^2_{\bar t})_t$ denotes the disintegration of the measure $\mu^1_{\bar t} - \mu^2_{\bar t}$ in $t \in (0,\bar t)$ with respect to $\mathscr{L}^1 \llcorner (0,\bar t)$. Therefore, $g \in C^1([0,\bar t])$. Since $g(0) = 0$, it holds

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}} \psi(\chi^1(\bar t) - \chi^2(\bar t))\, dx\, da &= g(\bar t) - g(0) \\
&= \int_0^{\bar t} g'(t)\, dt \\
&= -\int_0^{\bar t}\int_{\Omega\times\mathbb{R}} \partial_a\tilde\psi(t)\, dU_\phi\, dt + \int_{(0,\bar t)\times\Omega\times\mathbb{R}} \tilde\psi\, d(\mu^1_{\bar t} - \mu^2_{\bar t}) \\
&= -\int_0^{\bar t}\int_{\Omega\times\mathbb{R}} (\partial_v\phi - (\bar t - t)e^{ia}\cdot\nabla_x\psi)\, dU_\phi\, dt + \int_{(0,\bar t)\times\Omega\times\mathbb{R}} \tilde\psi\, d(\mu^1_{\bar t} - \mu^2_{\bar t}) \\
&\le \Big(\bar t\|\partial_a\psi\|_{L^\infty} + \frac{\bar t^2}{2}\|\nabla_x\psi\|_{L^\infty}\Big)\nu(B_R) + \|\psi\|_{L^\infty}\|\mu^1_{\bar t} - \mu^2_{\bar t}\|,
\end{aligned}$$

and this concludes the proof. □

We set $L_{\bar t} = (\varepsilon_{\bar t} \vee \bar t)^{-\frac12}$ and we consider the anisotropic distance

$$\begin{aligned}
&d_{\bar t} : (B_R \times [0,M])^2 \to [0,+\infty), \\
&((x_1,a_1),(x_2,a_2)) \mapsto L_{\bar t}|x_1 - x_2| + |a_1 - a_2|.
\end{aligned}$$

A test function $\psi : B_R \times [0,M] \to \mathbb{R}$ is 1-Lipschitz with respect to $d_{\bar t}$ if and only if

$$\|\partial_a\psi\|_{L^\infty} \le 1 \quad \text{and} \quad \|\nabla_x\psi\|_{L^\infty} \le L_{\bar t}.$$

Applying Corollary 2.3 to $\mu^1 = \chi^1(\bar t)\mathscr{L}^3$ and $\mu^2 = \chi^2(\bar t)\mathscr{L}^3$ on the space $(B_R \times [0,M], d_{\bar t})$, we obtain the following result as a consequence of Proposition 3.4 and Theorem 2.4.

Corollary 3.5. *There exist $\rho^1_{\bar t} \le \chi^1(\bar t)$ and $\rho^2_{\bar t} \le \chi^2(\bar t)$ such that*

$$\int_{B_R\times[0,M]} (\chi^1(\bar t) - \rho^1_{\bar t})\,dx\,da \le \varepsilon_{\bar t}\bar t, \qquad \int_{B_R\times[0,M]} (\chi^2(\bar t) - \rho^2_{\bar t})\,dx\,da \le \varepsilon_{\bar t}\bar t$$

and

$$W_1(\rho^1_{\bar t}\mathscr{L}^3, \rho^2_{\bar t}\mathscr{L}^3) \le (\bar t + \bar t^{\frac{3}{2}})\nu(B_R) + \varepsilon_{\bar t}^{\frac{1}{2}}\bar t(2R + \varepsilon_{\bar t}^{\frac{1}{2}} M).$$

In particular, there exists

$$T = (T_x, T_a) : B_R \times [0, M] \to B_R \times [0, M]$$

such that $T_\sharp(\rho^2_{\bar t}\mathscr{L}^3) = \rho^1_{\bar t}\mathscr{L}^3$ and

$$\int_{B_R\times[0,M]} (L_{\bar t}|T_x(x, a) - x| + |T_a(x, a) - a|)\rho^2_{\bar t}(x, a)\,dx\,da \le (\bar t + \bar t^{\frac{3}{2}})\nu(B_R) + \varepsilon_{\bar t}^{\frac{1}{2}}\bar t(2R + \varepsilon_{\bar t}^{\frac{1}{2}} M). \tag{3.8}$$

Remark 3.6. Now, we observe that in order to build a Lagrangian representation as in Theorem 3.2, the use of Theorem 2.4 can be replaced by a more elementary argument: indeed, the infimum in (2.1) can be equivalently taken only on the plans induced by transport maps (see, for example, [3]). In particular, the second part of the statement in Corollary 3.5 can be replaced by the following slightly weaker version: for every $\varepsilon' > 0$, there exists a map

$$T = (T_x, T_a) : B_R \times [0, M] \to B_R \times [0, M]$$

such that $T_\sharp(\rho^2_{\bar t}\mathscr{L}^3) = \rho^1_{\bar t}\mathscr{L}^3$ and

$$\int_{B_R\times[0,M]} (L_{\bar t}|T_x(x, a) - x| + |T_a(x, a) - a|)\rho^2_{\bar t}(x, a)\,dx\,da \le (\bar t + \bar t^{\frac{3}{2}})\nu(B_R) + \varepsilon_{\bar t}^{\frac{1}{2}}\bar t(2R + \varepsilon_{\bar t}^{\frac{1}{2}} M) + \varepsilon'.$$

The only property that we will use of (3.8) is that the right-hand side is of the form $\bar t\nu(B_R) + o(\bar t)$ as $\bar t \to 0$. In particular, choosing $\varepsilon' = o(\bar t)$, we can avoid the use of Theorem 2.4.

3.3 Construction of approximate characteristics

3.3.1 Building block

For a fixed $\bar t > 0$, we consider the following sets:

$$\begin{aligned} E_1 &:= \{(x, a) \in B_R \times [0, M] : x + ie^{ia}\bar t \in B_R\},\\ E_2 &:= \{(x, a) \in B_R \times [0, M] : x + ie^{ia}\bar t \notin B_R\},\\ E_3 &:= \{(x, a) \in (\Omega \setminus B_R) \times [0, M] : x + ie^{ia}\bar t \in B_R\}. \end{aligned}$$

For every $(x, a) \in E_1$, we define $\gamma_{\bar t,x,a} : [0, \bar t] \to B_R \times [0, M]$ by

$$\gamma_{\bar t,x,a}(t) = \begin{cases} (x + ie^{ia}t, a) & \text{if } t \in [0, \bar t),\\ T(x + ie^{ia}\bar t, a) & \text{if } t = \bar t, \end{cases}$$

where the transport map T is defined in Corollary 3.5. For every $(x, a) \in E_2$, we set

$$t^+(x, a) := \sup\{t \in [0, \bar t] : x + ie^{ia}t \in B_R\}$$

and we define $\gamma_{\bar t,x,a} : [0, t^+(x, a)) \to B_R \times [0, M]$ by

$$\gamma_{\bar t,x,a}(t) = (x + ie^{ia}t, a).$$

For every $(x, a) \in E_3$, we set

$$t^-(x, a) := \inf\{t \in [0, \bar t] : x + ie^{ia}t \in B_R\}$$

and we define $\gamma_{\bar t,x,a} : (t^-(x, a), \bar t] \to B_R \times [0, M]$ by

$$\gamma_{\bar t,x,a}(t) = \begin{cases} (x + ie^{ia}t, a) & \text{if } t \in (t^-(x, a), \bar t)\\ T(x + ie^{ia}\bar t, a) & \text{if } t = \bar t. \end{cases}$$

3.3.2 Approximate characteristics

Fix $n \in N$ and set $\bar{t}_n = 2^{-n}$. For every $(x, a) \in E_2$, we consider the curve

$$\gamma^{0,n}_{x,a} : (t^-_{\gamma^{0,n}_{x,a}}, t^+_{\gamma^{0,n}_{x,a}}) \to B_R \times [0, M]$$

with

$$t^-_{\gamma^{0,n}_{x,a}} = 0, \quad t^+_{\gamma^{0,n}_{x,a}} = t^+(x, a), \quad \gamma^{0,n}_{x,a}(t) = \gamma_{2^{-n},x,a}(t) \qquad \text{for all } t \in (t^-_{\gamma^{0,n}_{x,a}}, t^+_{\gamma^{0,n}_{x,a}}).$$

For every $(x, a) \in E_1$, we define

$$\gamma^{0,n}_{x,a} : (t^-_{\gamma^{0,n}_{x,a}}, t^+_{\gamma^{0,n}_{x,a}}) \to B_R \times [0, M]$$

with

$$t^-_{\gamma^{0,n}_{x,a}} = 0, \quad t^+_{\gamma^{0,n}_{x,a}} \geq 2^{-n}$$

to be determined in the construction and

$$\gamma^{0,n}_{x,a}(t) = \gamma_{2^{-n},x,a}(t) \quad \text{for all } t \in (t^-_{\gamma^{0,n}_{x,a}}, 2^{-n}].$$

For every $k = 1, \dots, 2^n$ and for every $(x, a) \in E_3$, we introduce a curve

$$\gamma^{k,n}_{x,a} : (t^-_{\gamma^{k,n}_{x,a}}, t^+_{\gamma^{k,n}_{x,a}}) \to B_R \times [0, M]$$

with

$$t^-_{\gamma^{k,n}_{x,a}} = (k-1)2^{-n} + t^-(x, a), \quad t^+_{\gamma^{k,n}_{x,a}} \geq k2^{-n}$$

to be determined and

$$\gamma^{k,n}_{x,a}(t) = \gamma_{2^{-n},x,a}(t - (k-1)2^{-n}) \quad \text{for all } t \in (t^-_{\gamma^{k,n}_{x,a}}, k2^{-n}].$$

It remains to define the evolution of the curves $\gamma^{0,n}_{x,a}$ for $(x, a) \in E_1$ and $t \geq 2^{-n}$ and of the curves $\gamma^{k,n}_{x,a}$ for $(x, a) \in E_3$ and $t \geq k2^{-n}$. Let us fix $k = 1, \dots, 2^n$ and $(x, a) \in E_3$. We define the evolution of $\gamma^{k,n}_{x,a}$ by recursion: assume that $\gamma^{k,n}_{x,a}$ is defined on

$$(t^-_{\gamma^{k,n}_{x,a}}, l2^{-n}] \quad \text{for some } l \geq k.$$

If $l = 2^n$, then we set

$$t^+_{\gamma^{k,n}_{x,a}} = 1.$$

Otherwise, if $l < 2^n$, then we distinguish two cases.

If $\gamma^{k,n}_{x,a}(l2^{-n}) \in E_2$, then we set

$$t^+_{\gamma^{k,n}_{x,a}} = l2^{-n} + t^+(\gamma^{k,n}_{x,a}(l2^{-n}))$$

and

$$\gamma^{k,n}_{x,a}(t) = \gamma_{2^{-n},\gamma^{k,n}_{x,a}(l2^{-n})}(t - l2^{-n}) \quad \text{for all } t \in (l2^{-n}, t^+_{\gamma^{k,n}_{x,a}}).$$

If instead $\gamma^{k,n}_{x,a}(l2^{-n}) \in E_1$, then we extend $\gamma^{k,n}_{x,a}$ on the whole interval $(l2^{-n}, (l+1)2^{-n}]$ by setting

$$\gamma^{k,n}_{x,a}(t) = \gamma_{2^{-n},\gamma^{k,n}_{x,a}(l2^{-n})}(t - l2^{-n}) \quad \text{for all } t \in (l2^{-n}, (l+1)2^{-n}].$$

The extension of the curves $\gamma^{0,n}_{x,a}$ for $(x, a) \in E_1$ is defined by the same procedure described above for the curves $\gamma^{k,n}_{x,a}$ for $(x, a) \in E_3$ with $k = 1$.

3.4 Approximate Lagrangian representation

The approximate characteristics built in the previous section belong to the space

$$\tilde{\Gamma} := \{(\gamma, t^-_\gamma, t^+_\gamma) : 0 \leq t^-_\gamma \leq t^+_\gamma \leq 1,\ \gamma \in \mathrm{BV}((t^-_\gamma, t^+_\gamma); B_R \times [0, M])\}.$$

For every $n \in \mathbb{N}$ sufficiently large, we define $\omega_n \in \mathcal{M}(\tilde{\Gamma})$ by

$$\omega_n = \int_{(B_R\times[0,M])\cap H_\phi} \delta_{\gamma^{0,n}_{a,x}, t^-_{\gamma^{0,n}_{a,x}}, t^+_{\gamma^{0,n}_{a,x}}}\, dx\, da + \sum_{k=1}^{2^n} \int_{E_3\cap H_\phi} \delta_{\gamma^{k,n}_{a,x}, t^-_{\gamma^{k,n}_{a,x}}, t^+_{\gamma^{k,n}_{a,x}}}\, dx\, da, \tag{3.9}$$

where the curves $\gamma^{k,n}_{x,a}$ are defined in Section 3.3.2.

Lemma 3.7. *Let ω_n be defined in* (3.9). *Then the following estimates hold:*

$$e_h(n) := \int_{\tilde{\Gamma}} \sup_{t\in(t^-_\gamma,t^+_\gamma)} \Big|\gamma_x(t) - \gamma_x(t^-_\gamma) - \int_{t^-_\gamma}^{t} ie^{i\gamma_a(s)}\, ds\Big|\, d\omega_n(\gamma) = o(1) \quad \text{as } n \to \infty, \tag{3.10}$$

$$e_v(n) := \int_{\tilde{\Gamma}} \mathrm{TotVar}_{(t^-_\gamma,t^+_\gamma)}\, \gamma_a\, d\omega_n(\gamma) \le \nu(B_R) + o(1) \qquad \text{as } n \to \infty. \tag{3.11}$$

Proof. Since for ω_n-a.e. $(\gamma, t^-_\gamma, t^+_\gamma) \in \tilde{\Gamma}$ it holds

$$\dot{\gamma}_x(t) = ie^{i\gamma_a(t)} \quad \text{for all } t \in (\gamma, t^-_\gamma, t^+_\gamma) \setminus 2^{-n}\mathbb{N},$$

we have

$$\begin{aligned} \sup_{t\in(t^-_\gamma,t^+_\gamma)} \Big|\gamma_x(t) - \gamma_x(t^-_\gamma) - \int_{t^-_\gamma}^{t} ie^{i\gamma_a(s)}\, ds\Big| &\le \sum_{l^-(\gamma)}^{l^+(\gamma)} |\gamma_x(l2^{-n}) - \gamma_x(l2^{-n}-)| \\ &= \sum_{l^-(\gamma)}^{l^+(\gamma)} |T_x(\gamma(l2^{-n}-) - \gamma_x(l2^{-n}-)|, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} l^-(\gamma) &= 2^n \inf(2^{-n}\mathbb{Z} \cap (t^-_\gamma, t^+_\gamma)), \\ l^+(\gamma) &= 2^n \sup(2^{-n}\mathbb{Z} \cap (t^-_\gamma, t^+_\gamma)). \end{aligned}$$

Integrating (3.12) with respect to ω_n, it follows by Corollary 3.5 with $\bar{t} = 2^{-n}$ that

$$\begin{aligned} e_h(n) &\le \sum_{l=1}^{2^n-1} \int_X |T_x(x,a) - x|\, d(e_{l2^{-n}-})_\sharp \omega_n \\ &\le \sum_{l=1}^{2^n-1} \Big(\int_X |T_x(x,a) - x| \rho^2_{\bar{t}}(x,a)\, dx\, da + 2R \|((e_{l2^{-n}-})_\sharp \omega_n - \rho^2_{\bar{t}} \mathscr{L}^3)^+\| \Big) \\ &\le \frac{2^n}{L_{2^{-n}}} (2^{-n} + 2^{\frac{-3n}{2}}) \nu(B_R) + \varepsilon^{\frac{1}{2}}_{2^{-n}} (2R + \varepsilon^{\frac{1}{2}}_{2^{-n}} M) + 2R \sum_{l=1}^{2^n-1} \|((e_{l2^{-n}-})_\sharp \omega_n - \rho^2_{\bar{t}} \mathscr{L}^3)^+\|, \end{aligned} \tag{3.13}$$

where $X = B_R \times [0, M]$ and $e_{t-} : \tilde{\Gamma}(t) \to X$ is defined by $e_{t-}(\gamma) = \lim_{t'\to t-} \gamma(t')$. Since, by construction,

$$(e_{l2^{-n}-})_\sharp \omega_n \le \chi^2(\bar{t}) \mathscr{L}^3 \quad \text{and} \quad \rho^2_{\bar{t}} \le \chi^2(\bar{t})$$

with

$$\|(\chi^2(\bar{t}) - \rho^2_{\bar{t}}) \mathscr{L}^3\| \le 2^{-n} \varepsilon_{2^{-n}},$$

for every $l = 1, \dots, 2^n - 1$ it holds

$$\|((e_{l2^{-n}-})_\sharp \omega_n - \rho^2_{\bar{t}} \mathscr{L}^3)^+\| \le 2^{-n} \varepsilon_{2^{-n}}. \tag{3.14}$$

Plugging (3.14) into (3.13), we immediately get (3.10).

Now, we prove (3.11). Since γ_a is constant in each connected component of $(t^-_\gamma, t^+_\gamma) \setminus 2^{-n}\mathbb{N}$ for ω_n-a.e. γ, it follows by Corollary 3.5 that

$$\begin{aligned}
\int_{\tilde\Gamma} \mathrm{TotVar}_{(t^-_\gamma, t^+_\gamma)}\, \gamma_a \, d\omega_n(\gamma) &= \sum_{l=1}^{2^n-1} \int_{\tilde\Gamma(l2^{-n})} |T_a(\gamma(l2^{-n}-)) - \gamma_a(l2^{-n}-)| \, d\omega_n(\gamma) \\
&= \sum_{l=1}^{2^n-1} \int_X |T_a(x,a) - a| \, d(e_{l2^{-n}-})_\sharp \omega_n \\
&\le \sum_{l=1}^{2^n-1} \int_X |T_a(x,a) - a| \rho^2_{\bar t}(x,a)\, dx\, da + M\|((e_{l2^{-n}-})_\sharp \omega_n - \rho^2_{\bar t}\mathscr{L}^3)^+\| \\
&\le 2^n\big[(2^{-n} + 2^{\frac{-3n}{2}})\nu(B_R) + \varepsilon^{\frac12}_{2^{-n}} \bar t(2R + \varepsilon^{\frac12}_{2^{-n}} M)\big] + M\varepsilon_{2^{-n}},
\end{aligned}$$

which implies (3.11). □

Now, we show that $(e_t)_\sharp\omega_n$ approximates $\chi\mathscr{L}^3$ in the strong topology of measures for every $t \in 2^{-n}\mathbb{N} \cap [0,1)$. This property and the weak continuity estimate provided in Proposition 3.4 will guarantee Definition 3.1 (1).

Lemma 3.8. *For every $l = 0, \dots, 2^n - 1$, it holds*

$$\|(e_{l2^{-n}})_\sharp \omega_n - \chi\mathscr{L}^3\| \le 2^{-n+1} l \varepsilon_{2^{-n}}. \tag{3.15}$$

Moreover, for every $t \in [l2^{-n}, (l+1)2^{-n})$ and every $\psi \in C^\infty_c(B_R \times [0,M])$, it holds

$$\Big|\int_X \psi \, d(e_t)_\sharp\omega_n - \int_X \psi\, d(e_{l2^{-n}})_\sharp \omega_n\Big| \le 2^{-n}(2M\mathscr{H}^1(\partial B_R)\|\psi\|_{L^\infty} + \|\nabla\psi\|_{L^\infty}\mathscr{L}^3(H_\phi)). \tag{3.16}$$

Proof. First, the case $l = 0$ follows by the definition of ω_n. In order to get (3.15), we prove that for every $l = 0, \dots, 2^n - 2$ it holds

$$\|(e_{(l+1)2^{-n}})_\sharp \omega_n - \chi\mathscr{L}^3\| \le \|(e_{l2^{-n}})_\sharp\omega_n - \chi\mathscr{L}^3\| + 2^{-n+1}\varepsilon_{2^{-n}}.$$

Indeed,

$$\begin{aligned}
\|(e_{(l+1)2^{-n}})_\sharp \omega_n - \chi\mathscr{L}^3\| &\le \|(e_{(l+1)2^{-n}})_\sharp \omega_n - \rho^1_{\bar t}\mathscr{L}^3\| + \|\rho^1_{\bar t}\mathscr{L}^3 - \chi\mathscr{L}^3\| \\
&= \|T_\sharp(e_{(l+1)2^{-n}-})_\sharp\omega_n - T_\sharp(\rho^2_{\bar t}\mathscr{L}^3)\| + \|\rho^1_{\bar t}\mathscr{L}^3 - \chi\mathscr{L}^3\| \\
&\le \|(e_{(l+1)2^{-n}-})_\sharp\omega_n - \rho^2_{\bar t}\mathscr{L}^3\| + 2^{-n}\varepsilon_{2^{-n}} \\
&\le \|(e_{(l+1)2^{-n}-})_\sharp\omega_n - \chi^2(\bar t)\mathscr{L}^3)\| + \|(\chi^2(\bar t) - \rho^2_{\bar t}\mathscr{L}^3)\| + 2^{-n}\varepsilon_{2^{-n}} \\
&\le \|(e_{l2^{-n}})_\sharp\omega_n - \chi\mathscr{L}^3\| + 2\cdot 2^{-n}\varepsilon_{2^{-n}}.
\end{aligned}$$

Inequality (3.16) follows by

$$\begin{aligned}
\Big|\int_X \psi\, d(e_t)_\sharp\omega_n - \int_X \psi\, d(e_{l2^{-n}})_\sharp\omega_n\Big| &\le \Big|\int_X \psi\, d(e_t)_\sharp \omega_n \llcorner \{t^-_\gamma > l2^{-n}\}\Big| + \Big|\int_X \psi\, d(e_{l2^{-n}})_\sharp\omega_n\llcorner\{t^+_\gamma < t\}\Big| \\
&\quad + \Big|\int_X \psi\, d(e_t)_\sharp\omega_n\llcorner\{t^-_\gamma \le l2^{-n}\} - \int_X \psi\, d(e_{l2^{-n}})_\sharp\omega_n\llcorner\{t^+_\gamma > t\}\Big| \\
&\le 2\cdot 2^{-n}\mathscr{H}^1(\partial B_R) M\|\psi\|_{L^\infty} + \|\nabla\psi\|_{L^\infty} 2^{-n}\omega_n(\tilde\Gamma(t)) \\
&\le 2^{-n+1}\mathscr{H}^1(\partial B_R)M\|\psi\|_{L^\infty} + \|\nabla\psi\|_{L^\infty}2^{-n}\mathscr{L}^3(H_\phi),
\end{aligned}$$

as desired. □

3.5 Compactness of ω_n and existence of a Lagrangian representation

We consider on $\tilde\Gamma$ the topology τ that induces the following convergence: $(\gamma_n, t^-_{\gamma_n}, t^+_{\gamma_n})$ converges to $(\gamma, t^-_\gamma, t^+_\gamma)$ if $t^\pm_{\gamma_n} \to t^\pm_\gamma$ with respect to the Euclidean topology in $\mathbb{R}$ and there exist extensions $\tilde\gamma, \tilde\gamma_n$ of γ, γ_n defined on $(0,1)$

such that the horizonal components $\tilde{\gamma}_{n,x}$ converge to $\tilde{\gamma}_x$ uniformly and the vertical components $\tilde{\gamma}_{n,a}$ converge to $\tilde{\gamma}_a$ in $L^1(0, 1)$.

Lemma 3.9. *The sequence of measures ω_n defined in* (3.9) *is bounded and tight in $\mathcal{M}(\tilde{\Gamma})$, namely for every $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset \tilde{\Gamma}$ such that for every $n \in \mathbb{N}$ it holds*

$$\omega_n(\tilde{\Gamma} \setminus K_\varepsilon) < \varepsilon.$$

Proof. We prove first that the sequence ω_n is bounded: for every n it holds

$$|E_{3,n} \cap H_\phi| \leq |E_{3,n}| \leq M\mathscr{H}^1(\partial B_R)2^{-n}.$$

In particular,

$$\limsup_{n\to\infty}|\omega_n|(\tilde{\Gamma}) = \limsup_{n\to\infty} \mathscr{L}^3(H_\phi) + 2^n|E_{3,n} \cap H_\phi| \leq \mathscr{L}^3(H_\phi) + M\mathscr{H}^1(\partial B_R).$$

In order to prove the tightness of the sequence ω_n, we consider for every $n \in \mathbb{N}$ and $C > 0$ the set of curves $(\gamma, t^-_\gamma, t^+_\gamma) \in \tilde{\Gamma}_{n,C} \subset \tilde{\Gamma}$ satisfying the following properties:

(i) $\operatorname{TotVar}_{(t^-_\gamma, t^+_\gamma)} \gamma_a \leq C$.

(ii) One has

$$\sum_{k=l^-(\gamma)}^{l^+(\gamma)} |\gamma_x(2^{-n}k) - \gamma_x(2^{-n}k-)| \leq Ce_h(n)^{1/2},$$

where $e_h(n)$ is defined in Lemma 3.7, and

$$l^-(\gamma) := 2^n \inf 2^{-n}\mathbb{Z} \cap (t^-_\gamma, t^+_\gamma) \quad \text{and} \quad l^+(\gamma) := 2^n \sup 2^{-n}\mathbb{Z} \cap (t^-_\gamma, t^+_\gamma).$$

(iii) $\operatorname{Lip} \gamma \llcorner ([(k-1)2^{-n}, k2^{-n})) \leq 1$ for every $k = l^-(\gamma), \dots, l^+(\gamma)$.

Since $e_h(n)$ tends to 0 as $n \to \infty$, for every $C > 0$ the space

$$\tilde{\Gamma}(C) := \bigcup_{n=1}^{\infty} \tilde{\Gamma}_{n,C}$$

is compact with respect to the topology τ introduced above. Moreover, it follows by Lemma 3.7 and the Chebychev inequality that for every $\varepsilon > 0$ there exists $C > 0$ sufficiently large such that for every $n \in \mathbb{N}$,

$$\omega_n(\tilde{\Gamma} \setminus \tilde{\Gamma}(C)) \leq \varepsilon. \qquad \square$$

By Theorem 2.5, it follows that the sequence ω_n is precompact with respect to the narrow convergence. We show in the next lemma that every limit point of ω_n is a Lagrangian representation of the hypograph of ϕ on B_R.

Lemma 3.10. *Every limit point ω of the sequence ω_n is a Lagrangian representation of the hypograph of ϕ on B_R.*

Proof. We need to check that the three conditions in Definition 3.1 are satisfied and that $\omega \in \mathcal{M}_+(\Gamma)$, namely that ω is concentrated on Γ.

Condition (1). We prove that for every $t \in (0, 1)$ the following two limits hold in the sense of distributions:

$$\lim_{n\to\infty} (e_t)_\sharp \omega_n = \mathscr{L}^3 \llcorner H_\phi \quad \text{and} \quad \lim_{n\to\infty} (e_t)_\sharp \omega_n = (e_t)_\sharp \omega. \tag{3.17}$$

For every $t = 2^{-k}\mathbb{N} \cap (0, 1)$ for some $k \in \mathbb{N}$, the first limit holds true thanks to Lemma 3.8 since $\chi\mathscr{L}^3 = \mathscr{L}^3 \llcorner H_\phi$ by the definition of χ. The continuity in time stated in (3.16) implies that the limit holds true therefore for every $t \in (0, 1)$ in the sense of distributions. We observe that the second limit in (3.17) is not trivial since e_t is not continuous on $\tilde{\Gamma}$ with respect to the topology introduced above. In order to establish it, we need to check that for every $\psi \in C^\infty_c(B_R \times [0, M])$ it holds

$$\lim_{n\to\infty} \int_{\tilde{\Gamma}(t)} \psi(\gamma(t))\, d\omega_n = \int_{\tilde{\Gamma}(t)} \psi(\gamma(t))\, d\omega.$$

Let $I \subset (0,1)$ be a non-empty open interval. Then consider the continuous and bounded function $T_{\psi,I} : \tilde{\Gamma} \to \mathbb{R}$ defined by

$$T_{\psi,I}(\gamma, t_\gamma^-, t_\gamma^+) := \int_{I \cap (t_\gamma^-, t_\gamma^+)} \psi(\gamma(t))\, dt.$$

By the definition of narrow convergence and by the Fubini theorem, it follows that

$$\lim_{n\to\infty} \int_I \int_{\tilde{\Gamma}(t)} \psi(\gamma(t))\, d\omega_n\, dt = \lim_{n\to\infty} \int_{\tilde{\Gamma}} T_{\psi,I}\, d\omega_n = \int_{\tilde{\Gamma}} T_{\psi,I}\, d\omega = \int_I \int_{\tilde{\Gamma}(t)} \psi(\gamma(t))\, d\omega\, dt.$$

This proves that the second limit in (3.17) holds for $\mathscr{L}^1$-a.e. $t \in (0,1)$. In order to prove that the limit is valid for every $t \in (0,1)$, we observe that ω is concentrated on curves with endpoints in ∂B_R and for every $t \in (0,1)$ it holds

$$\omega(\{(\gamma, t_\gamma^-, t_\gamma^+) \in \tilde{\Gamma} : t \in (t_\gamma^-, t_\gamma^+) \text{ and } \gamma_a(t-) \neq \gamma_a(t+)\}) = 0.$$

Indeed, assume by contradiction that there exist $\bar{t} \in (0,1)$ and $\varepsilon > 0$ such that

$$\omega(\{(\gamma, t_\gamma^-, t_\gamma^+) \in \tilde{\Gamma} : \bar{t} \in (t_\gamma^-, t_\gamma^+) \text{ and } |\gamma_a(\bar{t}-) - \gamma_a(\bar{t}+)| > \varepsilon\}) > \varepsilon. \tag{3.18}$$

Inequality (3.18) implies that for every $t_1, t_2 \in (0,1)$ such that $t_1 < \bar{t} < t_2$ it holds

$$e_\nu(t_1, t_2) := \int_{\tilde{\Gamma}} \operatorname{TotVar}_{(t_\gamma^-, t_\gamma^+) \cap (t_1, t_2)} \gamma_a\, d\omega(\gamma) \geq \varepsilon^2. \tag{3.19}$$

On the other hand, by localizing in (t_1, t_2) the same argument as in the proof of Lemma 3.7 to obtain (3.11), we have that

$$e_\nu(n, t_1, t_2) := \int_{\tilde{\Gamma}} \operatorname{TotVar}_{(t_\gamma^-, t_\gamma^+) \cap (t_1, t_2)} \gamma_a\, d\omega_n(\gamma) \leq (t_2 - t_1)\nu(B_R) + o(1) \quad \text{as } n \to \infty. \tag{3.20}$$

By choosing

$$t_2 - t_1 < \frac{\varepsilon^2}{1 + \nu(B_R)},$$

the two conditions in (3.19) and (3.20) contradict each other.

In particular, $t \mapsto (e_t)_\sharp \omega$ is continuous in the sense of distributions on $B_R \times [0, M]$, and therefore the second limit in (3.17) holds for every $t \in (0,1)$.

Condition (2). The function $g : \tilde{\Gamma} \to \mathbb{R}$ defined by

$$g(\gamma, t_\gamma^-, t_\gamma^+) := \sup_{t \in (t_\gamma^-, t_\gamma^+)} \left| \gamma_x(t) - \gamma_x(t_\gamma^-) - \int_{t_\gamma^-}^{t} i e^{i\gamma_a(s)}\, ds \right|$$

is lower semicontinuous. Therefore,

$$\int_{\tilde{\Gamma}} g(\gamma) d\omega \leq \lim_{n\to\infty} \int_{\tilde{\Gamma}} g(\gamma) d\omega_n,$$

which is equal to 0 by (3.10).

Condition (3). This follows similarly from (3.11). In particular, ω is concentrated on Γ and this concludes the proof. □

3.6 Representation of the defect measure and good curves selection

In the following proposition, we show that the kinetic measure U_ϕ can be decomposed along the characteristic trajectories detected by the Lagrangian representation ω_h.

Proposition 3.11. *Let ω_h be a Lagrangian representation of the hypograph of ϕ on B_R obtained as limit point of ω_n as in the previous section. Then*

$$\mathscr{L}^1 \times U_\phi = \int_\Gamma \mu_\gamma \, d\omega_h(\gamma) \quad \text{and} \quad \mathscr{L}^1 \times |U_\phi| = \int_\Gamma |\mu_\gamma| \, d\omega_h(\gamma).$$

Proof. Let

$$\bar{\psi}(t,x,a) = \varphi(t)\psi(x,a) \in C_c^\infty((0,1) \times B_R \times [0,M]).$$

Then

$$\begin{aligned} -\int_{(0,1)\times X} \varphi \partial_a \psi \, dU_\phi \, dt &= \int_{(0,1)\times H_\phi} ie^{ia} \cdot \nabla_x \psi \varphi \, dx \, da \, dt \\ &= \int_0^1 \int_{\Gamma(t)} ie^{i\gamma_a(t)} \cdot \nabla_x \psi(\gamma(t)) \varphi(t) \, d\omega_h(\gamma) \, dt \\ &= \int_0^1 \int_{\Gamma(t)} \dot{\gamma}_x(t) \cdot \nabla_x \psi(\gamma(t)) \, d\omega_h(\gamma) \varphi(t) \, dt \\ &= \int_\Gamma \int_{t_\gamma^-}^{t_\gamma^+} \dot{\gamma}_x(t) \cdot \nabla_x \psi(\gamma(t)) \varphi(t) \, dt \, d\omega_h(\gamma). \end{aligned} \tag{3.21}$$

For every $\gamma \in \Gamma$, we consider the map $\psi_\gamma := \psi \circ \gamma : (t_\gamma^-, t_\gamma^+) \to B_R \times [0,M]$. Since ω_h-a.e. $\gamma \in \Gamma$ has bounded variation on its domain, also $\psi_\gamma \in \mathrm{BV}((t_\gamma^-, t_\gamma^+); \mathbb{R})$, and we have the following chain rule:

$$\begin{aligned} D_t \psi_\gamma &= \nabla \psi(\gamma(t)) \cdot \tilde{D}_t \gamma + \sum_{t_j \in J_\gamma} (\psi(\gamma(t_j+)) - \psi(\gamma(t_j-)))\delta_{t_j} \\ &= \nabla_x \psi(\gamma(t)) \cdot \tilde{D}_t \gamma_x + \partial_a \psi(\gamma(t)) \tilde{D}_t \gamma_a + \sum_{t_j \in J_\gamma} (\psi(\gamma(t_j+)) - \psi(\gamma(t_j-)))\delta_{t_j}. \end{aligned} \tag{3.22}$$

Since for ω-a.e. γ it holds $D_t \gamma_x = \dot{\gamma}_x(t)\mathscr{L}^1$, plugging (3.22) into (3.21), we obtain

$$\begin{aligned} -\int_{(0,1)\times X} \varphi \partial_a \psi \, dU_\phi \, dt &= \int_\Gamma \Big(\int_{(t_\gamma^-, t_\gamma^+)} \varphi \Big(D_t \psi_\gamma - \partial_a \psi(\gamma(t)) \tilde{D}_t \gamma_a - \sum_{t_j \in J_\gamma} (\psi_\gamma(t_j+) - \psi_\gamma(t_j-))\delta_{t_j} \Big) \Big) d\omega_h \\ &= \int_\Gamma \Big(\int_{(t_\gamma^-, t_\gamma^+)} \varphi (D_t \psi_\gamma - \partial_a \psi(\gamma(t)) \tilde{D}_t \gamma_a) - \sum_{t_j \in J_\gamma} \varphi(t_j)(\psi_\gamma(t_j+) - \psi_\gamma(t_j-)) \Big) d\omega_h. \end{aligned}$$

We observe that, by construction, if $t_\gamma^- > 0$, then $\gamma(t_\gamma^-) \in \partial B_R \times [0,M]$, and therefore $\psi(\gamma(t_\gamma^-)) = 0$. Similarly, if $t_\gamma^+ < 1$, then $\psi(\gamma(t_\gamma^+)) = 0$. Therefore,

$$\begin{aligned} \int_\Gamma \int_{(t_\gamma^-, t_\gamma^+)} \varphi(t) D_t \psi_\gamma \, d\omega_h(\gamma) &= -\int_\Gamma \int_{(t_\gamma^-, t_\gamma^+)} \varphi'(t) \psi(\gamma(t)) \, dt \, d\omega_h(\gamma) \\ &= \int_0^1 \int_{H_\phi} \varphi'(t) \psi(x,a) \, dx \, da \, dt = 0. \end{aligned}$$

Since for ω_h-a.e. γ and every $t_j \in J_\gamma$ it holds $\gamma_x(t_j+) = \gamma_x(t_j-)$, it follows from the definition of μ_γ that

$$\begin{aligned} -\int_{(0,1)\times X} \varphi \partial_a \psi \, dU_\phi \, dt &= \int_\Gamma \Big(\int_{(t_\gamma^-, t_\gamma^+)} (-\varphi(t) \partial_a \psi(\gamma(t)) \tilde{D}_t \gamma_a) - \sum_{t_j \in J_\gamma} \varphi(t_j)(\psi(\gamma(t_j+)) - \psi(\gamma(t_j-))) \Big) d\omega_h \\ &= -\int_\Gamma \int_{(0,1)\times X} \varphi \partial_a \psi d\mu_\gamma d\omega_h(\gamma). \end{aligned}$$

This proves the first equality in the statement when tested with functions of the form $\varphi \partial_a \psi$ for two test functions φ, ψ. Since both U_ϕ and $\int \mu_\gamma \, d\omega_h$ are supported on $[0,1] \times B_R \times [0,M]$, the equality holds true for every test function.

The inequality

$$\mathscr{L}^1 \times |U_\phi| \leq \int_\Gamma |\mu_\gamma| \, d\omega_h$$

follows immediately from the already proved first equality in the statement. In order to prove the opposite inequality, it is enough to prove the global inequality

$$(\mathscr{L}^1 \times |U_\phi|)((0,1) \times B_R \times [0,M]) \geq \int_\Gamma |\mu_\gamma|((0,1) \times B_R \times [0,M]) \, d\omega_h.$$

We observe that $|\mu_\gamma|((0,1) \times B_R \times [0,M]) = \mathrm{TotVar}_{(t_\gamma^-, t_\gamma^+)} \gamma_a$ and that the map

$$(\gamma, t_\gamma^-, t_\gamma^+) \mapsto \mathrm{TotVar}_{(t_\gamma^-, t_\gamma^+)} \gamma_a$$

is lower semicontinuous on $\tilde{\Gamma}$. Therefore, it follows from (3.11) that

$$\begin{aligned}\int_\Gamma |\mu_\gamma|(B_R \times [0,M]) \, d\omega_h &= \int_\Gamma \mathrm{TotVar}_{(t_\gamma^-, t_\gamma^+)} \gamma_a \, d\omega_h \\ &\leq \liminf_{n\to\infty} \int_{\tilde{\Gamma}} \mathrm{TotVar}_{(t_\gamma^-, t_\gamma^+)} \gamma_a \, d\omega_n \\ &\leq (\mathscr{L}^1 \times |U_\phi|)((0,1) \times B_R \times [0,M]).\end{aligned}$$ □

With the result above, the proof of the part of Theorem 3.2 concerning the hypograph of ϕ is complete; the statement for the epigraph of ϕ can be proven in the same way.

The following lemma is an application of the Tonelli theorem and it is already proven in [19], to which we refer for the details.

Lemma 3.12. *For ω_h-a.e. $\gamma \in \Gamma$ and for $\mathscr{L}^1$-a.e. $t \in (t_\gamma^-, t_\gamma^+)$, the following assertions hold:*
(i) *$\gamma_x(t)$ is a Lebesgue point of ϕ.*
(ii) *$\gamma_a(t) < \phi(\gamma_x(t))$.*
We denote by Γ_h the set of curves $\gamma \in \Gamma$ such that the two properties above hold. Similarly, for ω_e-a.e. $\gamma \in \Gamma$ and for $\mathscr{L}^1$-a.e. $t \in (t_\gamma^-, t_\gamma^+)$, the following assertions hold:
(i) *$\gamma_x(t)$ is a Lebesgue point of ϕ.*
(ii) *$\gamma_a(t) > \phi(\gamma_x(t))$*
We denote the set of these curves by Γ_e.

4 Rectifiability of the measure ν

In this section, we prove that the measure $\nu := (p_x)_\sharp |U_\phi|$ is concentrated on a 1-rectifiable set. The rectifiability of ν is equivalent to the rectifiability of both the measures $(p_x)_\sharp U_\phi^-$ and $(p_x)_\sharp U_\phi^+$. Since these two cases are analogous, we only provide the proof of the rectifiability of $(p_x)_\sharp U_\phi^-$.

4.1 Pairing between ω_h and ω_e and its decomposition

In the following lemma, we introduce a pairing between the two representations $\omega_h \otimes \mu_\gamma^-$ and $\omega_e \otimes \mu_\gamma^+$ of the negative part of the defect measure $\mathscr{L}^1 \times U_\phi^-$. We will denote by X the set $B_R \times [0,M]$.

Lemma 4.1. *Denote by $p_1, p_2 : (\Gamma \times [0,1] \times X)^2 \to \Gamma \times [0,1] \times X$ the standard projections. Then there exists a plan $\pi^- \in \mathcal{M}((\Gamma \times [0,1] \times X)^2)$ with marginals*

$$\begin{cases} (p_1)_\sharp \pi^- = \omega_h \otimes \mu_\gamma^-, \\ (p_2)_\sharp \pi^- = \omega_e \otimes \mu_\gamma^+, \end{cases} \tag{4.1}$$

concentrated on the set

$$\mathcal{G} := \{((\gamma, t_\gamma^-, t_\gamma^+, t, x, a), (\gamma', t_\gamma^{-\prime}, t_\gamma^{+\prime}, t', x', a')) \in (\Gamma \times X)^2 : t \in (t_\gamma^-, t_\gamma^+),\ t' \in (t_\gamma^{-\prime}, t_\gamma^{+\prime}),\ t = t',$$
$$\gamma_x(t) = x = x' = \gamma'_x(t'),\ a = a',\ a \in [\gamma_a(t+), \gamma_a(t-)] \cap [\gamma'_a(t'-), \gamma'_a(t'+)]\}.$$

Proof. First, we observe that, by definition, $\omega_h \otimes \mu_\gamma^-$ is concentrated on the set

$$\mathcal{G}_h^- := \{(\gamma, t_\gamma^-, t_\gamma^+, t, x, a) \in \Gamma \times [0,1] \times X : t \in (t_\gamma^-, t_\gamma^+),\ \gamma_x(t) = x,\ a \in [\gamma_a(t+), \gamma_a(t-)]\}$$

and $\omega_e \otimes \mu_\gamma^+$ is concentrated on the set

$$\mathcal{G}_e^+ := \{(\gamma, t_\gamma^-, t_\gamma^+, t, x, a) \in \Gamma \times [0,1] \times X : t \in (t_\gamma^-, t_\gamma^+),\ \gamma_x(t) = x,\ a \in [\gamma_a(t-), \gamma_a(t+)]\}.$$

By denoting by $p_{2,3} : \Gamma \times [0,1] \times X \to [0,1] \times X$ the standard projection, it follows from (3.4) that

$$(p_{2,3})_\sharp(\omega_h \otimes \mu_\gamma^-) = \mathscr{L}^1 \times U_\phi^- = (p_{2,3})_\sharp(\omega_e \otimes \mu_\gamma^+).$$

By the disintegration theorem (see, for example, [2]), there exist two measurable families of probability measures

$$(\mu_{t,x,a}^{-,h})_{(t,x,a)\in X}, (\mu_{t,x,a}^{+,e})_{(t,x,a)\in X} \in \mathcal{P}(\Gamma \times [0,1] \times X)$$

such that

$$\omega_h \otimes \mu_\gamma^- = \int_{[0,1]\times X} \mu_{t,x,a}^{-,h} d\mathscr{L}^1 \times U_\phi^- \quad \text{and} \quad \omega_e \otimes \mu_\gamma^+ = \int_{[0,1]\times X} \mu_{t,x,a}^{+,e} d\mathscr{L}^1 \times U_\phi^- \tag{4.2}$$

and for $\mathscr{L}^1 \times U_\phi^-$-a.e. (t, x, a) the measures $\mu_{t,x,a}^{-,h}$ and $\mu_{t,x,a}^{+,e}$ are concentrated on the set

$$p_{2,3}^{-1}(\{t, x, a\}) = \{(\gamma, t_\gamma^-, t_\gamma^+, t', x', a') \in \Gamma \times [0,1] \times X : t' = t,\ x' = x,\ a' = a\}.$$

Moreover, since $\omega_h \otimes \mu_\gamma^-$ is concentrated on the set $\mathcal{G}_h^-$ and $\omega_e \otimes \mu_\gamma^+$ is concentrated on the set $\mathcal{G}_e^+$, we have that for $\mathscr{L}^1 \times U_\phi^-$-a.e. (t, x, a) the measure $\mu_{t,x,a}^{-,h}$ is concentrated on $p_{2,3}^{-1}(\{t, x, a\}) \cap \mathcal{G}_h^-$ and $\mu_{t,x,a}^{+,e}$ is concentrated on $p_{2,3}^{-1}(\{t, x, a\}) \cap \mathcal{G}_e^+$. Eventually, we set

$$\pi^- := \int_{[0,1]\times X} (\mu_{t,x,a}^{-,h} \otimes \mu_{t,x,a}^{+,e})\, d(\mathscr{L}^1 \times U_\phi^-).$$

From (4.2), directly (4.1) follows. By the above discussion, for $\mathscr{L}^1 \times U_\phi^-$-a.e. $(t, x, a) \in [0,1] \times X$ the measure $\mu_{t,x,v}^{-,h} \otimes \mu_{t,x,v}^{+,e}$ is concentrated on $(p_{2,3}^{-1}(\{t, x, a\}) \cap \mathcal{G}_h^-) \times (p_{2,3}^{-1}(\{t, x, a\}) \cap \mathcal{G}_e^+)$. Therefore, π^- is concentrated on

$$\bigcup_{(t,x,a)\in[0,1]\times X} (p_{2,3}^{-1}(\{t, x, a\}) \cap \mathcal{G}_h^-) \times (p_{2,3}^{-1}(\{t, x, a\}) \cap \mathcal{G}_e^+) = \mathcal{G},$$

and this concludes the proof. □

Now, we split the set $\mathcal{G}$ introduced in Lemma 4.1 in finitely many components. First, we set

$$\mathcal{G}_{h,\text{jump}}^- := \{(\gamma, t_\gamma^-, t_\gamma^+, t, x, a) \in \mathcal{G}_h^- : \gamma_a(t+) < \gamma_a(t-)\},$$
$$\mathcal{G}_{e,\text{jump}}^+ := \{(\gamma, t_\gamma^-, t_\gamma^+, t, x, a) \in \mathcal{G}_e^+ : \gamma_a(t-) < \gamma_a(t+)\}.$$

Moreover, we consider the following covering with overlaps of $[0, M]$. Let $L = \lfloor \frac{2M}{\pi} \rfloor$ and for every $l = 0, \dots, L$ set

$$I_l = \Big(l\frac{\pi}{2} - \frac{\pi}{8}, (l+1)\frac{\pi}{2} + \frac{\pi}{8}\Big)$$

and

$$\mathcal{G}_{h,l}^- := \{(\gamma, t_\gamma^-, t_\gamma^+, t, x, a) \in \mathcal{G}_h^- : \gamma_a(t+),\ \gamma_a(t-) \in I_l\},$$
$$\mathcal{G}_{e,l}^+ := \{(\gamma, t_\gamma^-, t_\gamma^+, t, x, a) \in \mathcal{G}_e^+ : \gamma_a(t-),\ \gamma_a(t+) \in I_l\}.$$

Then we define

$$\pi_l^- := \pi^- \llcorner (\mathcal{G}_{h,l}^- \times \mathcal{G}_{e,l}^+), \quad \pi_{\text{jump}}^- = \pi^- \llcorner ((\mathcal{G}_{h,\text{jump}}^- \times \mathcal{G}_e^+) \cup (\mathcal{G}_h^- \times \mathcal{G}_{e,\text{jump}}^+)).$$

We prove separately that $\nu_{\text{jump}}^- := (p_x^1)_\sharp \pi_{\text{jump}}^-$ is 1-rectifiable and that $\nu_l^- := (p_x^1)_\sharp \pi_l^-$ is rectifiable for every $l = 0, \dots, L$.

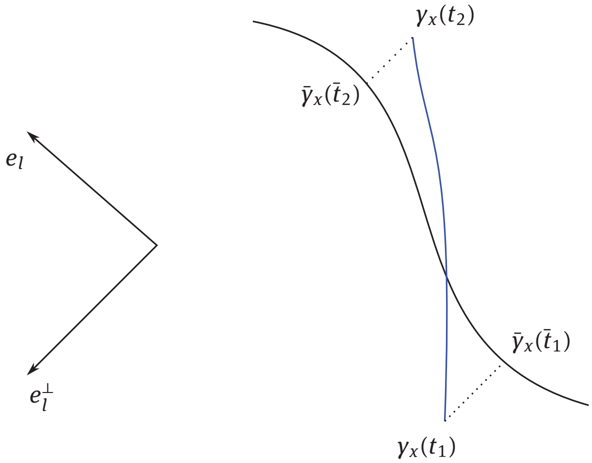

Figure 1: The blue curve γ represents an element of $G_{\mathrm{cr}}(\bar\gamma, \tilde t^-_{\bar\gamma}, \tilde t^+_{\bar\gamma})$: in this particular case, $l = 4k$ for some $k \in \mathbb{N}$.

4.2 Rectifiability of ν_l^-

The proof of the rectifiability of ν_l^- follows the strategy used in [19]. In particular, the first step is to identify a countable family of Lipschitz curves where we will prove that ν_l^- is concentrated.

4.2.1 Shock curves

For shortness, we set

$$e_l := ie^{i(l\frac{\pi}{2}+\frac{\pi}{4})} \quad \text{and} \quad e_l^\perp := ie_l.$$

The following proposition establishes the intuitive fact that a curve of the epigraph cannot cross from below a curve of the hypograph. Since the same proposition and the following corollary were proven in [19] in the case of the Burgers equation, we only sketch the arguments here.

Proposition 4.2. *Let $(\bar\gamma, t^-_{\bar\gamma}, t^+_{\bar\gamma}) \in \Gamma_h$ and let $(\tilde t^-_{\bar\gamma}, \tilde t^+_{\bar\gamma}) \subset (t^-_{\bar\gamma}, t^+_{\bar\gamma})$ be such that*

$$\bar\gamma_a((\tilde t^-_{\bar\gamma}, \tilde t^+_{\bar\gamma})) \subset I_l.$$

We denote by $G_{\mathrm{cr}}(\bar\gamma, \tilde t^-_{\bar\gamma}, \tilde t^+_{\bar\gamma})$ the set of curves

$$(\gamma, t^-_\gamma, t^+_\gamma) \in \Gamma_e$$

for which there exist $\bar t_1, \bar t_2 \in (\tilde t^-_{\bar\gamma}, \tilde t^+_{\bar\gamma})$ and $t_1, t_2 \in (t^-_\gamma, t^+_\gamma)$ such that the following conditions are satisfied (see Figure 1):

(i) $t_1 < t_2$ *and* $\bar t_1 < \bar t_2$.
(ii) $\gamma_a((t_1, t_2)) \subset I_l$.
(iii) $\gamma_x(t_1) \cdot e_l = \bar\gamma_x(\bar t_1) \cdot e_l$ *and* $\gamma_x(t_1) \cdot e_l^\perp > \bar\gamma_x(\bar t_1) \cdot e_l^\perp$.
(iv) $\gamma_x(t_2) \cdot e_l = \bar\gamma_x(\bar t_2) \cdot e_l$ *and* $\gamma_x(t_2) \cdot e_l^\perp < \bar\gamma_x(\bar t_2) \cdot e_l^\perp$.

Then

$$\omega_e(G_{\mathrm{cr}}(\bar\gamma, \tilde t^-_{\bar\gamma}, \tilde t^+_{\bar\gamma})) = 0.$$

Proof. Let

$$s^- := \bar\gamma_x(\tilde t^-_{\bar\gamma}) \cdot e_l \quad \text{and} \quad s^+ := \bar\gamma_x(\tilde t^+_{\bar\gamma}) \cdot e_l.$$

Since $\bar\gamma_a((\tilde t^-_{\bar\gamma}, \tilde t^+_{\bar\gamma})) \subset I_l$ and $\dot{\bar\gamma}_x(t) = ie^{i\bar\gamma_a(t)}$ for $\mathscr{L}^1$-a.e. $t \in (\tilde t^-_{\bar\gamma}, \tilde t^+_{\bar\gamma})$, the map

$$\begin{aligned} h_{\bar\gamma} : (\tilde t^-_{\bar\gamma}, \tilde t^+_{\bar\gamma}) &\to (s^-, s^+), \\ t &\mapsto \bar\gamma_x(t) \cdot e_l \end{aligned}$$

is bi-Lipschitz. For every $s \in (s^-, s^+)$, we set $g_{\bar\gamma}(s) = \gamma_x(h_{\bar\gamma}^{-1}(t)) \cdot e_l^\perp$. Let $\delta > 0$ and let $\psi_\delta : \mathbb{R} \to \mathbb{R}$ be the Lipschitz approximation of the Heaviside function defined by $\psi_\delta(v) = 0 \vee (v/\delta \wedge 1)$. Let us consider a measurable selection of t_1, t_2 in $G_{\mathrm{cr}}(\bar\gamma, \tilde t_{\bar\gamma}^-, \tilde t_{\bar\gamma}^+)$ and let us set

$$G_{\mathrm{cr}}(\bar\gamma, \tilde t_{\bar\gamma}^-, \tilde t_{\bar\gamma}^+, \delta) := \{(\gamma, t_\gamma^-, t_\gamma^+) \in G_{\mathrm{cr}}(\bar\gamma, \tilde t_{\bar\gamma}^-, \tilde t_{\bar\gamma}^+) : \gamma_x(t_{1,\gamma}) \cdot e_l^\perp - g_{\bar\gamma}(h_{\bar\gamma}(\gamma_x(t) \cdot e_l)) > \delta\}.$$

For every $t \in (0, 1)$ and $\gamma \in G_{\mathrm{cr}}(\bar\gamma, \tilde t_{\bar\gamma}^-, \tilde t_{\bar\gamma}^+, \delta)$, set

$$f(\gamma, t) := \begin{cases} 0 & \text{if } t < t_{1,\gamma}, \\ 1 - \psi_\delta(\gamma_x(t) \cdot e_l^\perp - g_{\bar\gamma}(\gamma_x(t) \cdot e_l)) & \text{if } t \in (t_{1,\gamma}, t_{2,\gamma}), \\ 1 & \text{if } t > t_{2,\gamma}. \end{cases}$$

Finally, we consider the functional

$$\Psi_\delta(t) := \int\limits_{\Gamma_{\mathrm{cr}}(\bar\gamma, \tilde t_{\bar\gamma}^-, \tilde t_{\bar\gamma}^+, \delta)} f(\gamma, t)\, d\omega_e(\gamma).$$

A straightforward computation shows that

$$\Psi_\delta'(t) \le \frac{C}{\delta} \int\limits_{G(\delta, t)} [\bar\gamma_a(h_{\bar\gamma}^{-1}(\gamma_x(t) \cdot e_l)) - \gamma_a(t)]^+\, d\omega_e(\gamma), \tag{4.3}$$

where

$$\begin{aligned} G(\delta, t) = \{(\gamma, t_\gamma^-, t_\gamma^+) \in G_{\mathrm{cr}}(\bar\gamma, \tilde t_{\bar\gamma}^-, \tilde t_{\bar\gamma}^+, \delta) : t \in (t_{1,\gamma}, t_{2,\gamma}) \text{ and} \\ \gamma_x(t) \cdot e_l^\perp \in (g_{\bar\gamma}(\gamma_x(t) \cdot e_l), g_{\bar\gamma}(\gamma_x(t) \cdot e_l) + \delta)\}. \end{aligned}$$

Let us set

$$S_\delta := \{x \in B_R : x \cdot e_l \in (g_{\bar\gamma}(x \cdot e_l), g_{\bar\gamma}(x \cdot e_l) + \delta)\}.$$

Since

$$(e_t)_\sharp \omega_e \llcorner G(\delta, t) \le \mathcal{L}^3 \llcorner (E_\phi \cap (S_\delta \times [0, M]))$$

and for $\mathscr{L}^1$-a.e. $t \in (\tilde t_{\bar\gamma}^-, \tilde t_{\bar\gamma}^+)$ the point $\bar\gamma_x(t)$ is a Lebesgue point of ϕ with value larger than $\bar\gamma_a(t)$, we obtain from (4.3) that $\Psi_\delta'(t) \le o(1)$ as $\delta \to 0$. By the definition of the functional Ψ_δ it holds

$$\omega_e(G_{\mathrm{cr}}(\bar\gamma, \tilde t_{\bar\gamma}^-, \tilde t_{\bar\gamma}^+)) \le \liminf_{\delta \to 0} \Psi_\delta(1) = 0. \qquad \square$$

Corollary 4.3. *Let $\bar x \in B_R$ and denote by $\Gamma_l^-(\bar x)$ the set of curves $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_h$ for which there exists $t_1 \in (t_\gamma^-, t_\gamma^+)$ such that*

$$\gamma_x(t_1) \cdot e_l = \bar x \cdot e_l \quad \text{and} \quad \gamma_x(t_1) \cdot e_l^\perp < \bar x \cdot e_l^\perp.$$

Similarly, let $\Gamma_l^+(\bar x)$ be the set of curves $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_e$ for which there exists $t_1' \in (t_\gamma^-, t_\gamma^+)$ such that

$$\gamma_x(t_1') \cdot e_l = \bar x \cdot e_l \quad \text{and} \quad \gamma_x(t_1') \cdot e_l^\perp > \bar x \cdot e_l^\perp.$$

Then there exists a Lipschitz function $f_{\bar x, l} : [\bar x \cdot e_l, +\infty) \to \mathbb{R}$ such that

$$\begin{cases} \omega_h(\{(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_l^-(\bar x) : \exists t_2 \in (t_1, t_\gamma^+) \text{ s.t. } \gamma_a((t_1, t_2)) \subset I_l \text{ and } \gamma_x(t_2) \cdot e_l^\perp > f_{\bar x, l}(\gamma_x(t_2) \cdot e_l)\}) = 0, \\ \omega_e(\{(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_l^+(\bar x) : \exists t_2' \in (t_1', t_\gamma^+) \text{ s.t. } \gamma_a((t_1', t_2')) \subset I_l \text{ and } \gamma_x(t_2') \cdot e_l^\perp < f_{\bar x, l}(\gamma_x(t_2') \cdot e_l)\}) = 0, \end{cases} \tag{4.4}$$

where t_1, t_1' are as above.

Proof. Let $I \subset [\bar x \cdot e_l, +\infty)$ be the set of values y for which there exist $\gamma \in \Gamma_l^-(\bar x)$ and $t \in (t_1, t_2)$ such that $\gamma_x(t) \cdot e_l = y$, where $t_1 < t_2$ are such that $\gamma_a((t_1, t_2)) \subset I_l$. Let $\tilde f_{\bar x, l}$ be defined on I by

$$\tilde f_{\bar x, l}(s) := \sup\{\gamma_x(t) \cdot e_l^\perp : \gamma \in \Gamma_l^-(\bar x),\ t \in (t_1, t_2),\ \gamma_x(t) \cdot e_l = s\} = \sup g_\gamma(s), \tag{4.5}$$

where we used the same notation as in the proof of Proposition 4.2: we set $g_\gamma(s) := \gamma_x(t) \cdot e_l^\perp$, where t is the unique value in (t_1, t_2) for which $\gamma_x(t) \cdot e_l = s$. Since $\gamma_a((t_1, t_2)) \subset I_l$, it is straightforward to check that g_γ is Lipschitz with Lipschitz constant bounded by $\tan(\frac{3\pi}{8})$. The function $f_{\bar{x},l}$ is then defined as the smallest biggest C-Lipschitz function such that $f_{\bar{x},l} \geq \tilde{f}_{\bar{x},l}$ on I and $f_{\bar{x},l}(\bar{x} \cdot e_l) = \bar{x} \cdot e_l^\perp$, where $C > \tan(3\pi/8)$. The first equation in (4.4) follows from the fact that $f_{\bar{x},l} \geq \tilde{f}_{\bar{x},l}$ on I. Now, we prove the second equation in (4.4): given $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_l^-(\bar{x})$ and t, t_1, t_2 as above, let us consider the set $\Gamma'(\gamma, \bar{x})$ of curves $(\gamma', t_\gamma'^-, t_\gamma'^+) \in \Gamma$ for which there exist $t_1' < t_2'$ in $(t_\gamma'^-, t_\gamma'^+)$ such that the following conditions are satisfied:
(i) $\gamma_a'(t) \in I_l$ for every $t \in (t_1', t_2')$.
(ii) $\gamma_x'(t_1') \cdot e_l = \bar{x} \cdot e_l$ and $\gamma_x'(t_1') \cdot e_l^\perp > \bar{x} \cdot e_l^\perp$.
(iii) $\gamma_x'(t_2') \cdot e_l < \gamma_x(t_2) \cdot e_l$.
By Proposition 4.2, it follows that

$$\omega_e(\Gamma'(\gamma, \bar{x})) = 0. \tag{4.6}$$

Since the functions g_γ in (4.5) are equi-Lipschitz, the supremum in (4.5) can be realized by taking only countably many curves in $\Gamma_l^+(\bar{x})$. Therefore, it follows from (4.6) that

$$\omega_e(\{(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_l^+(\bar{x}) : \exists t_2' \in (t_1', t_\gamma^+) \text{ such that } \gamma_a((t_1', t_2')) \subset I_l \text{ and } \gamma_x(t_2') \cdot e_l^\perp < \bar{f}_{\bar{x},l}(\gamma_x(t_2') \cdot e_l)\}) = 0. \tag{4.7}$$

Finally, since for every $(\gamma', t_\gamma'^-, t_\gamma'^+) \in \Gamma'(\gamma, \bar{x})$ the associated map $g_{\gamma'} : (\gamma_x'(t_1') \cdot e_l, \gamma_x'(t_2') \cdot e_l) \to \mathbb{R}$ is Lipschitz with Lipschitz constant bounded by $\tan(\frac{3\pi}{8})$, we can replace $\bar{f}_{\bar{x},l}$ with $f_{\bar{x},l}$ in (4.7). This gives the second equation in (4.4) and it concludes the proof. □

The following elementary lemma is about functions of bounded variation of one variable: we refer to [2] for the theory of BV functions.

Lemma 4.4. *Let $v : (a, b) \to \mathbb{R}$ be a* BV *function and denote by D^-v the negative part of the measure Dv. Then for $\tilde{D}^-v$-a.e. $\bar{x} \in (a, b)$ there exists $\delta > 0$ such that*

$$\bar{v}(x) > \bar{v}(\bar{x}) \quad \text{for all } x \in (\bar{x} - \delta, \bar{x}), \qquad \text{and} \qquad \bar{v}(x) < \bar{v}(\bar{x}) \quad \text{for all } x \in (\bar{x}, \bar{x} + \delta).$$

We are now in the position to prove the rectifiability of ν_l^-.

Proposition 4.5. *The measure ν_l^- is concentrated on the set*

$$\bigcup_{\bar{x} \in \mathbb{Q}^2 \cap B_R} C_{f_{\bar{x},l}}, \quad \text{where} \quad C_{f_{\bar{x},l}} := B_R \cap \bigcup_{s > \bar{x} \cdot e_l^\perp} \{s e_l^\perp + f_{\bar{x},l}(s) e_l\}.$$

Proof. We prove this proposition in four steps.

Step 1. For every $\bar{x} \in B_R \cap \mathbb{Q}^2$ and every $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_h$, we consider the open set $I_{\bar{x},l,\gamma}^+ \subset (t_\gamma^-, t_\gamma^+)$ defined by the following property: we say that $t \in I_{\bar{x},l,\gamma}^+$ if there exists $t' \in (t_\gamma^-, t)$ such that

$$\gamma_a((t', t)) \subset I_l, \quad \gamma_x(t') \cdot e_l^\perp = \bar{x} \cdot e_l^\perp, \quad \gamma_x(t') \cdot e_l > \bar{x} \cdot e_l.$$

Moreover, we set

$$\mathcal{G}_{\bar{x},l}^> := \{(\gamma, t_\gamma^-, t_\gamma^+, t, x, a) \in \Gamma_h \times (0, 1) \times B_R \times [0, M] : t \in I_{\bar{x},l,\gamma}^+\}.$$

Similarly, for every $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_e$, we let $I_{\bar{x},l,\gamma}^- \subset (t_\gamma^-, t_\gamma^+)$ be the set of t for which there exists $t' \in (t_\gamma^-, t)$ such that

$$\gamma_a((t', t)) \subset I_l, \quad \gamma_x(t') \cdot e_l^\perp = \bar{x} \cdot e_l^\perp, \quad \gamma_x(t') \cdot e_l < \bar{x} \cdot e_l$$

and we set

$$\mathcal{G}_{\bar{x},l}^< := \{(\gamma, t_\gamma^-, t_\gamma^+, t, x, a) \in \Gamma_e \times (0, 1) \times B_R \times [0, M] : t \in I_{\bar{x},l,\gamma}^-\}.$$

We consider

$$\pi_{\bar{x},l}^- := \pi^- \llcorner (\mathcal{G}_{\bar{x},l}^> \times \mathcal{G}_{\bar{x},l}^<)$$

and we prove that $(p_x^1)_\sharp \pi_{\bar{x},l}^-$ is concentrated on $C_{f_{\bar{x},l}}$, where

$$p_x^1 : (\Gamma \times (0, 1) \times B_R \times [0, M])^2 \to B_R,$$
$$(\gamma, t_\gamma^-, t_\gamma^+, t, x, a, \gamma', t_\gamma^{-\prime}, t_\gamma^{+\prime}, t', x', a') \mapsto x.$$

Trivially, it holds

$$(p_x^1)_\sharp \pi^-_{\bar{x},l} \le (p_x^1)_\sharp [\pi^- \llcorner (\mathcal{G}^>_{\bar{x},l} \times (\Gamma \times (0,1) \times B_R \times [0,M]))].$$

From Corollary 4.3 it follows that for ω_h-a.e. $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_h$ it holds

$$\gamma_x(t) \cdot e_l^\perp > \bar{x} \cdot e_l^\perp \quad \text{and} \quad \gamma_x(t) \cdot e_l \ge f_{\bar{x},l}(\gamma_x(t) \cdot e_l^\perp) \qquad \text{for all } t \in I_{\bar{x},l,\gamma}.$$

Therefore,

$$(p_x^1)_\sharp \pi^-_{\bar{x},l}(\{x \in B_R : x \cdot e_l^\perp \le \bar{x} \cdot e_l^\perp\} \cup \{x \in B_R : x \cdot e_l^\perp > \bar{x} \cdot e_l^\perp \text{ and } x \cdot e_l < f_{\bar{x},l}(x \cdot e_l^\perp)\}) = 0. \tag{4.8}$$

In the same way, we get

$$(p_x^2)_\sharp \pi^-_{\bar{x},l}(\{x \in B_R : x \cdot e_l^\perp \le \bar{x} \cdot e_l^\perp\} \cup \{x \in B_R : x \cdot e_l^\perp > \bar{x} \cdot e_l^\perp \text{ and } x \cdot e_l > f_{\bar{x},l}(x \cdot e_l^\perp)\}) = 0, \tag{4.9}$$

where

$$p_x^2 : (\Gamma \times (0,1) \times B_R \times [0,M])^2 \to B_R,$$
$$(\gamma, t_\gamma^-, t_\gamma^+, t, x, a, \gamma', t_\gamma^{-\prime}, t_\gamma^{+\prime}, t', x', a') \mapsto x'.$$

Finally, since π^- is concentrated on $\mathcal{G}$,

$$(p_x^1 \otimes p_x^2)_\sharp \pi^- \in \mathcal{M}(([0,T] \times \mathbb{R})^2)$$

is concentrated on the graph of the identity on B_R and in particular $(p_x^1)_\sharp \pi^-_{\bar{x},l} = (p_x^2)_\sharp \pi^-_{\bar{x},l}$. Therefore, it follows from (4.8) and (4.9) that $(p_x^1)_\sharp \pi^-_{\bar{x},l}$ is concentrated on

$$\{x \in B_R : x \cdot e_l > \bar{x} \cdot e_l \text{ and } x \cdot e_l^\perp = f_{\bar{x},l}(x \cdot e_l)\} = C_{f_{\bar{x},l}}.$$

Step 2. We prove that for π_l^--a.e.

$$\mathcal{Z} = (\gamma, t_\gamma^-, t_\gamma^+, t, x, a, \gamma', t_\gamma^{-\prime}, t_\gamma^{+\prime}, t', x', a') \in (\Gamma \times (0,1) \times B_R \times [0,M])^2$$

there exists $\delta > 0$ such that for every $s \in (t-\delta, t)$ and $s' \in (t'-\delta, t')$ the following properties hold:
(i) $\gamma_a(s) \in I_l$ and $\gamma_a(s) > a$.
(ii) $\gamma_a'(s') \in I_l$ and $\gamma_a'(s') < a'$.
It is sufficient to prove the properties in (i) since the ones in (ii) can be shown analogously. The statement is trivial for elements $\mathcal{Z}$ for which $\gamma_a(t-) > a$, and it follows immediately by Lemma 4.4 applied to γ_a if γ_a is continuous at t. Since π^- is concentrated on points $\mathcal{Z}$ for which $\gamma_a(t+) \le \gamma_a(t-)$, it is sufficient to check that

$$\pi_l^-(\{\mathcal{Z} \in (\Gamma \times (0,1) \times B_R \times [0,M])^2 : \gamma_a(t-) = a > \gamma_a(t+)\}) = 0.$$

This follows immediately from the facts that for ω_h-a.e. γ the measure μ_γ^- has no atoms and the set

$$(t, x, a) \in (0,1) \times B_R \times [0,M]$$

for which $\gamma_x(t) = x$ and $\gamma_a(t-) = a > \gamma_a(t+)$ is at most countable.

Step 3. We prove that for π_l^--a.e. $\mathcal{Z} \in (\Gamma \times (0,1) \times B_R \times [0,M])^2$ there exists $\bar{x} \in \mathbb{Q}^2 \cap B_R$ such that

$$\mathcal{Z} \in \mathcal{G}^>_{\bar{x},l} \times \mathcal{G}^<_{\bar{x},l}.$$

Let us consider $\delta > 0$ from step 2. From property (i) and (3.1), it follows that for every $s \in (t-\delta, t)$ it holds

$$\gamma_x(s) \cdot e_l > \gamma_x(t) \cdot e_l - ie^{ia} \cdot e_l(\gamma_x(t) \cdot e_l^\perp - \gamma_x(s) \cdot e_l^\perp), \tag{4.10}$$

and similarly for every $s' \in (t'-\delta, t')$,

$$\gamma_x'(s') \cdot e_l < \gamma_x'(t') \cdot e_l - ie^{ia'} \cdot e_l(\gamma_x'(t') \cdot e_l^\perp - \gamma_x'(s') \cdot e_l^\perp). \tag{4.11}$$

Since π^- is concentrated on $\mathcal{G}$, for π_l^--a.e.

$$\mathcal{Z} \in (\Gamma \times (0,1) \times B_R \times [0,M])^2$$

it also holds $a = a'$ and $\gamma_x(t) = x = \gamma_x'(t')$. Let us consider

$$y \in \Big(\gamma_x(t)\cdot e_l^\perp - \frac{\delta}{100}, \gamma_x(t)\cdot e_l^\perp\Big) \cap \sqrt{2}\mathbb{Q}.$$

Then there exist $s \in (t-\delta, t)$ and $s' \in (t'-\delta, t')$ such that $\gamma_x(s)\cdot e_l^\perp = y = \gamma_x'(s')\cdot e_l^\perp$. It follows from (4.10) and (4.11) that

$$\begin{aligned}
\gamma_x'(s')\cdot e_l &< \gamma_x'(t')\cdot e_l - ie^{ia'}\cdot e_l(\gamma_x'(t')\cdot e_l^\perp - \gamma_x'(s')\cdot e_l^\perp)\\
&= x\cdot e_l - ie^{ia'}\cdot e_l(x\cdot e_l^\perp - y)\\
&= \gamma_x(t)\cdot e_l - ie^{ia}\cdot e_l(\gamma_x(t)\cdot e_l^\perp - \gamma_x(s)\cdot e_l^\perp)\\
&< \gamma_x(s)\cdot e_l.
\end{aligned}$$

Let $z \in (\gamma_x'(s')\cdot e_l, \gamma_x(s)\cdot e_l) \cap \sqrt{2}\mathbb{Q}$ and set $\bar{x} = ze_l + ye_l^\perp$. By construction, it holds

$$\mathcal{Z} \in \mathcal{G}^>_{\bar{x},l} \times \mathcal{G}^<_{\bar{x},l}.$$

Since $e_l, e_l^\perp \in (\sqrt{2}\mathbb{Q})^2$, we obtain $\bar{x} \in \mathbb{Q}^2$.

Step 4. It follows by step 3 that

$$\pi_l^- \le \pi^-\llcorner\Big(\bigcup_{\bar{x}\in\mathbb{Q}^2\cap B_R} \mathcal{G}^>_{\bar{x},l} \times \mathcal{G}^<_{\bar{x},l}\Big). \tag{4.12}$$

Since by step 1 we have that $(p_x^1)_\sharp \pi^-_{\bar{x},l}$ is concentrated on $C_{f_{\bar{x},l}}$, the statement of the proposition follows from (4.12). □

4.3 Rectifiability of v^-_{jump}

In the next lemma, we prove a regularity density estimate at a point $\bar{x}$ provided that the entropy dissipation measure decays faster than in a shock point.

Lemma 4.6. *Let $(\bar{\gamma}, t^-_{\bar{\gamma}}, t^+_{\bar{\gamma}}) \in \Gamma_h$, $\bar{t} \in (t^-_{\bar{\gamma}}, t^+_{\bar{\gamma}})$, and set $\bar{x} = \bar{\gamma}_x(\bar{t})$ and $\bar{a} = \bar{\gamma}_a(\bar{t}-) \vee \bar{\gamma}_a(\bar{t}+)$. Then there exists an absolute constant $c > 0$ such that for every $\delta \in (0, \pi/2)$ at least one of the following holds true:*

$$\liminf_{r\to 0} \frac{\mathcal{L}^2(\{x \in B_r(\bar{x}) : \phi(x) \ge \bar{a} - \delta\})}{r^2} \ge c\delta,$$

$$\limsup_{r\to 0} \frac{\nu(B_r(\bar{x}))}{r} \ge c\delta^3. \tag{4.13}$$

Proof. We assume without loss of generality that $\bar{a} = \bar{\gamma}_a(\bar{t}-)$, and we let $\delta_1 > 0$ be such that for every $t \in (\bar{t} - \delta_1, \bar{t})$ it holds

$$\bar{\gamma}_a(t) \in \Big(\bar{a} - \frac{\delta}{5}, \bar{a} + \frac{\delta}{5}\Big).$$

Moreover, we set $\bar{r} = \delta_1/2$ so that for every $r \in (0, \bar{r})$ there exists $t_r \in (\bar{t} - \delta_1, \bar{t})$ such that $\bar{\gamma}_x(t_r) \in \partial B_r(\bar{x})$ and $\gamma_x(t) \in B_r(\bar{x})$ for every $t \in (t_r, \bar{t})$. Since $\bar{\gamma} \in \Gamma_h$ and $\bar{\gamma}_a(t) \ge \bar{a} - \delta/5$ for every $t \in (t_r, \bar{t})$, there exists $\varepsilon > 0$ (possibly depending on r) such that

$$\mathscr{L}^2\Big(\Big\{x \in S_{\varepsilon,r} : \phi(x) \ge \bar{a} - \frac{\delta}{5}\Big\}\Big) \ge \varepsilon r, \quad \text{where} \quad S_{\varepsilon,r} := \bar{\gamma}_x((t_r, \bar{t})) + B_\varepsilon(0).$$

For every $(\gamma, t^-_\gamma, t^+_\gamma) \in \Gamma$, we consider the nontrivial interiors $(t^-_{\gamma,i}, t^+_{\gamma,i})_{i=1}^{N_\gamma}$ of the connected components of $\gamma_a^{-1}((\bar{a} - \delta), \bar{a} - \frac{2}{5}\delta)$, which intersect

$$\gamma^{-1}\Big(S_{\varepsilon,r} \times \Big(\bar{a} - \frac{4}{5}\delta, \bar{a} - \frac{3}{5}\delta\Big)\Big).$$

Notice that we have the estimate

$$N_\gamma \le 1 + \frac{5}{\delta}\,\mathrm{TotVar}\,\gamma_a.$$

For every $i \in \mathbb{N}$, we consider

$$\Gamma_i := \{(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma : N_\gamma \geq i\}$$

and the measurable restriction map

$$R_i : \Gamma_i \to \Gamma,$$
$$(\gamma, t_\gamma^-, t_\gamma^+) \mapsto (\gamma, t_{\gamma,i}^-, t_{\gamma,i}^+).$$

We finally consider the measure

$$\tilde{\omega}_h := \sum_{i=1}^{\infty} (R_i)_\sharp (\omega_h \llcorner \Gamma_i).$$

We observe that $\tilde{\omega}_h \in \mathcal{M}_+(\Gamma)$ since for every $N > 0$,

$$\Big\| \sum_{i=1}^{N} (R_i)_\sharp (\omega_h \llcorner \Gamma_i) \Big\| \leq \int_\Gamma N_\gamma \, d\omega_h \leq \int_\Gamma \Big(1 + \frac{5}{\delta} \operatorname{TotVar} \gamma_a\Big) \, d\omega_h(\gamma) < \infty.$$

The advantage of using the restrictions introduced above is in the following estimate: by an elementary transversality argument, there exists an absolute constant $\tilde{c} > 0$ such that for $\tilde{\omega}_h$-a.e. $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ it holds

$$\mathscr{L}^1\Big(\Big\{t \in (t_\gamma^-, t_\gamma^+) : \gamma(t) \in S_{\varepsilon,r} \times \Big(\bar{a} - \frac{4}{5}\delta, \bar{a} - \frac{3}{5}\delta\Big)\Big\}\Big) \leq \tilde{c}\frac{\varepsilon}{\delta}. \tag{4.14}$$

By construction, we have that for every $t \in (0, 1)$ it holds

$$(e_t)_\sharp \tilde{\omega}_h \geq \mathscr{L}^3 \llcorner \Big\{(x, a) \in S_{\varepsilon,r} \times \Big(\bar{a} - \frac{4}{5}\delta, \bar{a} - \frac{3}{5}\delta\Big) : \phi(x) \geq a\Big\}. \tag{4.15}$$

Since the measure of this set is at least $\varepsilon r\delta/5$, it follows by (4.14) and (4.15) that

$$\tilde{\omega}_h(\Gamma) \geq \varepsilon r \frac{\delta}{5} \cdot \frac{\delta}{\tilde{c}\varepsilon} = \frac{\delta^2}{5\tilde{c}} r. \tag{4.16}$$

We consider $\Gamma = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 := \{(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma : t_\gamma^+ - t_\gamma^- \geq r\} \quad \text{and} \quad \Gamma_2 := \{(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma : t_\gamma^+ - t_\gamma^- < r\}.$$

For $\tilde{\omega}_h$-a.e. $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_1$ it holds

$$\mathscr{L}^1\Big(\Big\{t \in (t_\gamma^-, t_\gamma^+) : \gamma(t) \in B_{2r}(\bar{x}) \times \Big(\bar{a} - \delta, \bar{a} - \frac{2}{5}\delta\Big)\Big\}\Big) \geq r,$$

while for $\tilde{\omega}_h$-a.e. $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_2$ we have

$$\gamma_x(t_\gamma^-, t_\gamma^+) \subset B_{2r}(\bar{x}) \quad \text{and} \quad \operatorname{TotVar} \gamma_a \geq \frac{2}{5}\delta.$$

It follows from (4.16) that at least one of the following holds:

$$\tilde{\omega}_h(\Gamma_1) \geq \frac{\delta^2}{10\tilde{c}} r \quad \text{or} \quad \tilde{\omega}_h(\Gamma_2) \geq \frac{\delta^2}{10\tilde{c}} r. \tag{4.17}$$

If the second condition holds, then we have that

$$\nu(B_{2r}(\bar{x})) \geq |U_\phi|(B_{2r}(\bar{x}) \times (\bar{a} - \delta, \bar{a})) \geq \frac{\delta^3}{25\tilde{c}} r,$$

so that the second condition in the statement is satisfied. Otherwise, we assume that the first condition in (4.17) holds: since for every $t \in (0, 1)$,

$$(e_t)_\sharp \tilde{\omega}_h \leq \chi \mathscr{L}^3$$

it follows from (4.17) and the Fubini theorem that

$$\mathscr{L}^2(\{x \in B_{2r}(\bar{x}) : \phi(x) \geq \bar{a} - \delta\}) \geq \frac{\delta^2}{10\tilde{c}} r \cdot \frac{5r}{3\delta} = \frac{\delta}{6\tilde{c}} r^2,$$

so that the first condition in the statement holds true. □

Remark 4.7. We observe that the third power in (4.13) is optimal; this is related to the fact that the optimal regularity of ϕ is $B^{1/3,3}_{\infty,\mathrm{loc}}(\Omega)$; see [12].

We also state the same result for curves in Γ_e, whose proof is analogous to the one of Lemma 4.6.

Lemma 4.8. *Let $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma_e$, $t \in (t_\gamma^-, t_\gamma^+)$, and set $\bar{x} = \gamma_x(t)$ and $\bar{a} = \gamma_a(t-) \wedge \gamma_a(t+)$. Then there exists an absolute constant $c > 0$ such that for every $\delta \in (0, \pi/2)$ at least one of the following holds true:*

$$\liminf_{r\to 0} \frac{\mathcal{L}^2(\{x \in B_r(\bar{x}) : \phi(x) \leq \bar{a} + \delta\})}{r^2} \geq c\delta,$$
$$\limsup_{r\to 0} \frac{\nu(B_r(\bar{x}))}{r} \geq c\delta^3.$$

The main result of this section is the following proposition.

Proposition 4.9. *For ν_{jump}^--a.e. $x \in B_R$,*

$$\limsup_{r\to 0} \frac{\nu(B_r(x))}{r} > 0. \tag{4.18}$$

Proof. For ν_{jump}^--a.e. $\bar{x} \in B_R$, one of the following assertions holds:
(i) There exist $(\gamma, t_\gamma^-, t_\gamma^+, t, x, a) \in \mathcal{G}_{h,\text{jump}}^-$ and $(\gamma', t_\gamma^{-\prime}, t_\gamma^{+\prime}, t', x', a') \in \Gamma_e$ such that

$$x = x' = \bar{x} \quad \text{and} \quad \gamma_a'(t'+) \leq a' = a < \gamma_a(t-).$$

(ii) There exist $(\gamma', t_\gamma^{-\prime}, t_\gamma^{+\prime}, t', x', a') \in \mathcal{G}_{e,\text{jump}}^+$ and $(\gamma, t_\gamma^-, t_\gamma^+, t, x, a) \in \Gamma_e$ such that

$$x = x' = \bar{x} \quad \text{and} \quad \gamma_a'(t'+) < a' = a \leq \gamma_a(t-).$$

Since the two cases are equivalent, we consider only the first one. We apply Lemma 4.6 to the curve γ and Lemma 4.8 to the curve γ' with $\delta = (\gamma_a(t-) - a)/3$. If condition (ii) holds in at least one of the two cases, then the statement follows; otherwise both of the following inequalities are satisfied:

$$\liminf_{r\to 0} \frac{\{x \in B_r(\bar{x}) : \phi(x) \geq \gamma_a(t-) - \delta\}}{r^2} \geq c\delta^2, \quad \liminf_{r\to 0} \frac{\{x \in B_r(\bar{x}) : \phi(x) \leq \gamma_a(t-) - 2\delta\}}{r^2} \geq c\delta^2.$$

This condition excludes that $\bar{x}$ is a point of vanishing mean oscillation of ϕ. Therefore, $\bar{x} \in \Sigma$ by Theorem 1.2, i.e. (4.18) holds true. □

4.4 Conclusion

Collecting the results in Sections 4.2 and 4.3, we obtain the rectifiability of the measure $(p_x)_\sharp U_\phi^-$.

Proposition 4.10. *The measure $(p_x)_\sharp U_\phi^-$ is 1-rectifiable.*

Proof. First, we observe that since π^- is concentrated on $\mathcal{G}$ and

$$\mathcal{G} \subset (\mathcal{G}_{h,\text{jump}}^- \times \mathcal{G}_e^+) \cup (\mathcal{G}_h^- \times \mathcal{G}_{e,\text{jump}}^+) \cup \Big(\bigcup_{l=0}^{L} (\mathcal{G}_{h,l}^- \times \mathcal{G}_{e,l}^+)\Big),$$

it follows from the definitions of π_l^- and π_{jump}^- that

$$\pi^- \leq \pi_{\text{jump}}^- + \sum_{l=0}^{L} \pi_l^-.$$

In particular,

$$(p_x)_\sharp U_\phi^- = (p_x^1)_\sharp \pi^- \leq (p_x^1)_\sharp \pi_{\text{jump}}^- + \sum_{l=0}^{L} (p_x^1)_\sharp \pi_l^-.$$

Since $(p_x^1)_\sharp \pi_l^-$ is 1-rectifiable for every $l = 0, \dots, L$ by Proposition 4.5 and $(p_x^1)_\sharp \pi_{\text{jump}}^-$ is 1-rectifiable by Proposition 4.9 and Theorem 1.2, also $(p_x)_\sharp U_\phi^-$ is 1-rectifiable. □

As mentioned at the beginning of this section, the rectifiability of the positive part $(p_x)_\sharp U_\phi^+$ can be proven following the same procedure. Therefore, this concludes the proof of Theorem 1.3.

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