A FINITENESS CONDITION ON THE COEFFICIENTS OF THE PROBABILISTIC ZETA FUNCTION

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ABSTRACT. We discuss whether finiteness properties of a profinite group G can be deduced from the coefficients of the probabilistic zeta function $P_G(s)$. In particular we prove that if $P_G(s)$ is rational and all but finitely many non abelian composition factors of G are isomorphic to $PSL(2,p)$ for some prime p, then G contains only finitely many maximal subgroups.

1. Introduction

Let G be a finitely generated profinite group. As G has only finitely many open subgroups of a given index, for any $n \in \mathbb{N}$ we may define the integer $a_n(G)$ as $a_n(G) = \sum_H \mu_G(H)$, where the sum is over all open subgroups H of G with $|G : H| = n$. Here $\mu_G(H)$ denotes the Möbius function of the poset of open subgroups of G, which is defined by recursion as follows: $\mu_G(G) = 1$ and $\mu_G(H) = -\sum_{H<K} \mu_G(K)$ if $H < G$. Then we associate to G a formal Dirichlet series $P_G(s)$, defined as

$$P_G(s) = \sum_{n \in \mathbb{N}} \frac{a_n(G)}{n^s}.$$

Notice that if H is an open subgroup of G and $\mu_G(H) \neq 0$, then H is an intersection of maximal subgroups of G. Therefore the formal Dirichlet series $P_G(s)$ encodes information about the lattice generated by the maximal subgroups of G, just as the Riemann zeta function encodes information about the primes, and combinatorial properties of the probabilistic sequence $\{a_n(G)\}$ reflect on the structure of G.

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If G contains only finitely many maximal subgroups (i.e. if the Frattini subgroup $\operatorname{Frat} G$ of G has finite index in G), then there are only finitely many open subgroups H of G with $\mu_G(H) \neq 0$ and consequently $a_n(G) = 0$ for all but finitely many $n \in \mathbb{N}$ (i.e. $P_G(s)$ is a finite Dirichlet series). A natural question is whether the converse is true.

Let $\{G_n\}_{n\in\mathbb{N}}$ be a countable descending series of open normal subgroups with the properties that $G_1 = G$, $\bigcap_{n\in\mathbb{N}} G_n = 1$ and G_n/G_{n+1} is a chief factor of G for each $n \in \mathbb{N}$. The factor group G/G_n is finite, so the Dirichlet series $P_{G/G_n}(s)$ is also finite and belongs to the ring $\mathcal{D}$ of Dirichlet polynomials with integer coefficients. Actually, $P_{G/G_n}(s)$ is a divisor of $P_{G/G_{n+1}}(s)$ in the ring $\mathcal{D}$, i.e. there exists a Dirichlet polynomial $P_n(s)$ such that $P_{G/G_{n+1}}(s) = P_{G/G_n}(s)P_n(s)$. As explained in [3], the Dirichlet series $P_G(s)$ can be written as an infinite formal product

$$P_G(s) = \prod_{n\in\mathbb{N}} P_n(s),$$

and if we change the series $\{G_n\}_{n\in\mathbb{N}}$, the factorization remains the same up to reordering the factors. Moreover it turns out that $P_n(s) = 1$ if G_n/G_{n+1} is a Frattini chief factor (i.e. $G_n/G_{n+1} \leq \operatorname{Frat}(G/G_{n+1})$). Notice that G has finitely many open maximal subgroups if and only if the chief series $\{G_n\}_{n\in\mathbb{N}}$ contains only finitely many non-Frattini factors. This could suggest a wrong argument: if the product $P_G(s) = \prod_{n\in\mathbb{N}} P_n(s)$ is a Dirichlet polynomial, then $P_n(s) = 1$ for all but finitely many $n \in \mathbb{N}$ and consequently the series $\{G_n\}_{n\in\mathbb{N}}$ contains only finitely many non-Frattini factors. The problem is that it is possible that a Dirichlet polynomial can be written as a formal product of infinitely many non trivial elements of $\mathcal{D}$. To give an idea of what can occur, let us recall a related question, with an unexpected solution: if G is prosolvable, then we can consider the p-local factor

$$P_{G,p}(s) = \sum_{m\in\mathbb{N}} \frac{a_{p^m}}{p^{ms}}.$$

It turns out that $P_{G,p}(s) = \prod_{n\in\Omega_p} P_n(s)$ where Ω_p is the set of indices n such that G_n/G_{n+1} has p-power order. It is not difficult to prove that a finitely generated prosolvable group G contains only finitely many maximal subgroups whose index is a power of p if and only if a chief series of G contains only finitely many non-Frattini factors whose order is a p-power. Therefore the previous tempting wrong argument would suggest the following conjecture: if the p-factor $P_{G,p}(s)$ is a Dirichlet polynomial, then G has only finitely many maximal subgroups of p-power index. However this is false; in [4] a 2-generated prosolvable group G is constructed, such that, for any prime p, G contains infinitely many maximal subgroups of p-power index, while $P_{G,p}(s)$ is a finite Dirichlet series. Knowing that $P_{G,p}(s)$ can be a polynomial even when Ω_p is infinite, could lead to believe in the existence of a counterexample to the conjecture that $P_G(s) \in \mathcal{D}$ implies $G/\operatorname{Frat}(G)$ finite. However, using results from number theory, in [4] it was proved that if G is prosolvable and $P_{G,p}(s)$ is a polynomial, then either Ω_p is finite or, for every prime q, there exists $n \in \Omega_p$ such that the dimension of G_n/G_{n+1} as $\mathbb{F}_p G$-module is divisible by q; using standard arguments of modular representation theory one deduce that this is possible only if infinitely many primes appear among the divisors of the order of the finite images of G; but then $P_{G,r}(s) \neq 1$ for infinitely many primes r and $P_G(s)$ cannot be a polynomial. So $P_G(s)$ can be a polynomial only if Ω_p

is finite for every prime and empty for all but finitely many, and therefore only if $G/\operatorname{Frat}(G)$ is finite. Really a stronger result holds: if G is a finitely generated prosolvable group, then $P_G(s)$ is rational (i.e. $P_G(s) = A(s)/B(s)$ with $A(s)$ and $B(s)$ finite Dirichlet series) if and only if $G/\operatorname{Frat} G$ is a finite group. Partial generalization has been obtained in [5] and in [6]. All these results can be summarized in the following statement:

Theorem 1.1. *Let G be a finitely generated profinite group. Assume that there exist a prime p and a normal open subgroup N of G such that the set $\mathcal{S}$ of nonabelian composition factors of N satisfies one of the following properties:*

- *all the groups in $\mathcal{S}$ are alternating groups;*
- *all the groups in $\mathcal{S}$ are of Lie type over fields of characteristic p, where p is a fixed prime;*
- *all the groups in $\mathcal{S}$ are sporadic simple groups.*

Then $P_G(s)$ is rational if and only if $G/\operatorname{Frat}(G)$ is a finite group.

The main ingredient in the proof of the previous results is the following result, proved with the help of the Skolem-Mahler-Lech Theorem and where $\pi(G)$ is the set of the primes q with the property that G contains at least an open subgroup H whose index is divisible by q.

Proposition 1.2. *Let G be a finitely generated profinite group, assume that $\pi(G)$ is finite and let $\{r_i\}_{i \in \mathbb{N}}$ be the sequence of the composition lengths of the non-Frattini factors in a chief series of G. Assume that there exists a positive integer q and a sequence $\{c_i\}_{i \in \mathbb{N}}$ of nonnegative integers such that the formal product*

$$\prod_i \left(1 - \frac{c_i}{(q^{r_i})^s} \right)$$

is rational. Then $c_i = 0$ for all but finitely many indices i.

In our applications, $c_i/(q^{r_i})^s$ is one of the summands of the Dirichlet polynomial $P_i(s)$ associated to the chief factor G_i/G_{i+1} and must be chosen so that the polynomials $P_i^*(s) = 1 - c_i/(q^{r_i})^s$ satisfy two conditions:

    (1) if $P_i(s) \neq 1$ for infinitely many $i \in \mathbb{N}$, then also $P_i^*(s) \neq 1$ for infinitely many $i \in \mathbb{N}$;
    (2) if the infinite product $\prod_i P_i(s)$ is rational, then also $\prod_i P_i^*(s)$ is rational.

The choice of the "approximation" $P_i^*(s)$ of $P_i(s)$ is not easy and we are not able to use this strategy in the general case; roughly speaking, it requires an order on the set of the primes numbers with the property that "small" simple groups in $\mathcal{S}$ have order not divisible by "large" primes. When $\mathcal{S}$ is the set of the alternating groups, we just consider the natural order, but when $\mathcal{S}$ consists of simple groups of Lie type in characteristic p, we say that a prime q_1 is smaller than a prime q_2 if the multiplicative order of $p \mod q_1$ is smaller than its order modulo q_2. This order depends on the choice of p and our arguments do not work if $\mathcal{S}$ contains groups of different kinds, for example groups of Lie type in different characteristics. However we expect that our statement remains true in the general case. In the present paper we prove a new result in this direction:

Theorem 1.3. *Let G be a finitely generated profinite group. Assume that there exists a normal open subgroup N of G such that any nonabelian composition factor of N is isomorphic to $PSL(2, p)$ for some prime p. Then $P_G(s)$ is rational if and only if $G/\mathrm{Frat}(G)$ is a finite group.*

2. Preliminaries and notations

Let $\mathcal{R}$ be the ring of formal Dirichlet series with integer coefficients. We say that $F(s) = \sum_{n \in \mathbb{N}} a_n/n^s \in \mathcal{R}$ is a Dirichlet polynomial if $a_n = 0$ for all but finitely many $n \in \mathbb{N}$. The set $\mathcal{D}$ of the Dirichlet polynomials is a subring of $\mathcal{R}$. We will say that $F(s) \in \mathcal{R}$ is rational if there exist $A(s), B(s) \in \mathcal{D}$ with $F(s) = A(s)/B(s)$.

For every set π of prime number, we consider the ring endomorphism of $\mathcal{R}$ defined by:

$$F(s) = \sum_{n \in \mathbb{N}} \frac{a_n}{n^s} \mapsto F^\pi(s) = \sum_{n \in \mathbb{N}} \frac{a_n^*}{n^s}$$

where $a_n^* = 0$ if n is divisible by some prime $p \in \pi$, $a_n^* = a_n$ otherwise. We will use the following remark:

Remark 2.1. *For every set π of prime numbers, if $F(s)$ is rational then $F^\pi(s)$ is rational.*

The following result is a consequence of the Skolem-Mahler-Lech Theorem (see [4] for more details):

Proposition 2.2. *Let $I \subseteq \mathbb{N}$ and let q, r_i, c_i be positive integers for each $i \in I$. Assume that*

    (i) *for every $n \in \mathbb{N}$, the set $\{i \in I \mid r_i \text{ divides } n\}$ is finite;*

    (ii) *there exists a prime t such that t does not divide r_i for any $i \in I$.*

If the product

$$F(s) = \prod_{i \in I} \left(1 - \frac{c_i}{(q^{r_i})^s} \right)$$

is rational, then I is finite.

Proposition 2.3. [5, Corollary 5.2] *Let G be a finitely generated profinite group and assume that $\pi(G)$ is finite. For each n, there are only finitely many non-Frattini factors in a chief series whose composition length is at most n. Moreover there exists a prime t such that no non-Frattini chief factor of G has composition length divisible by t.*

Proof of Proposition 1.2. It follows immediately from Propositions 2.2 and 2.3.   □

Finally let us recall the following result.

Proposition 2.4. [5, Proposition 4.3] *Let $F(s)$ be a product of finite Dirichlet series:*

$$F(s) = \prod_{i \in I} F_i(s), \ \ where \ F_i(s) = \sum_{n \in \mathbb{N}} \frac{b_{i,n}}{n^s}.$$

Let q be a prime and Λ the set of positive integers divisible by q. Assume that there exists a set $\{r_i\}_{i \in I}$ of positive integers such that if $n \in \Lambda$ and $b_{i,n} \neq 0$ then n is an r_i-th power of some integer and $v_q(n) = r_i$ (where $v_q(n)$ is the q-adic valuation of n). Define

$$w = \min\{x \in \mathbb{N} \mid v_q(x) = 1 \text{ and } b_{i,x^{r_i}} \neq 0 \text{ for some } i \in I\}.$$

If $F(s)$ is rational, then the product

(2.1)
$$F^*(s) = \prod_{i \in I} \left(1 + \frac{b_{i,w^{r_i}}}{(w^{r_i})^s}\right)$$

is also rational.

Now let G be a finitely generated profinite group and let $\{G_i\}_{i \in \mathbb{N}}$ be a fixed countable descending series of open normal subgroups with the property that $G_0 = G$, $\bigcap_{i \in \mathbb{N}} G_i = 1$ and G_i/G_{i+1} is a chief factor of G/G_{i+1} for each $i \in \mathbb{N}$. In particular, for each $i \in \mathbb{N}$, there exist a simple group S_i and a positive integer r_i such that $G_i/G_{i+1} \cong S_i^{r_i}$. Moreover, as described in [3], for each $i \in \mathbb{N}$ a finite Dirichlet series

(2.2)
$$P_i(s) = \sum_{n \in \mathbb{N}} \frac{b_{i,n}}{n^s}$$

is associated with the chief factor G_i/G_{i+1} and $P_G(s)$ can be written as an infinite formal product of the finite Dirichlet series $P_i(s)$:

(2.3)
$$P_G(s) = \prod_{i \in \mathbb{N}} P_i(s).$$

Moreover, this factorization is independent on the choice of chief series (see [2, 3]) and $P_i(s) = 1$ unless G_i/G_{i+1} is a non-Frattini chief factor of G.

We recall some properties of the series $P_i(s)$. If S_i is cyclic of order p_i, then $P_i(s) = 1 - c_i/(p_i^{r_i})^s$, where c_i is the number of complements of G_i/G_{i+1} in G/G_{i+1}. It is more difficult to compute the series $P_i(s)$ when S_i is a non-abelian simple group. In that case a relevant role is played by the group $L_i = G/C_G(G_i/G_{i+1})$. This is a monolithic primitive group and its unique minimal normal subgroup is isomorphic to $G_i/G_{i+1} \cong S_i^{r_i}$. If $n \neq |S_i|^{r_i}$, then the coefficient $b_{i,n}$ in (2.2) depends only on the knowledge of L_i; more precisely we have

$$b_{i,n} = \sum_{\substack{|L_i:H|=n \\ L_i = H\operatorname{soc}(L_i)}} \mu_{L_i}(H).$$

Some help in computing the coefficients $b_{i,n}$ comes from the knowledge of the subgroup X_i of $\operatorname{Aut}S_i$ induced by the conjugation action of the normalizer in L_i of a composition factor of the socle S_i^r (note that X_i is an almost simple group with socle isomorphic to S_i). More precisely, given an almost simple group X with socle S, we can consider the following Dirichlet polynomial:

(2.4)
$$P_{X,S}(s) = \sum_n \frac{c_n(X)}{n^s}, \quad \text{where } c_n(X) = \sum_{\substack{|X:H|=n \\ X=SN}} \mu_X(H).$$

The following can be deduced from [7]:

Lemma 2.5. *If S_i is nonabelian and π is a set of primes containing at least one divisor of $|S_i|$ then*

$$P_i^\pi(s) = P_{X_i,S_i}^\pi(r_i s - r_i + 1).$$

Moreover, if n is not divisible by any prime in π, then either $b_{i,n} = 0$ or there exists $m \in \mathbb{N}$ with $n = m^{r_i}$ and $b_{i,n} = c_m(X_i) \cdot m^{r_i - 1}$.

For an almost simple group X, let $\Omega(X)$ be the set of the odd integers $m \in \mathbb{N}$ such that

- X contains at least one subgroup Y such that $X = Y\mathrm{soc}X$ and $|X : Y| = m$;
- if $X = Y\mathrm{soc}X$ and $|X : Y| = m$, then Y is a maximal subgroup if X.

Note that if $m \in \Omega(X)$, $X = Y\mathrm{soc}X$ and $|X : Y| = m$, then $\mu_X(Y) = -1$: in particular $c_m(X) < 0$. Combined with Lemma 2.5, this implies:

Remark 2.6. *If $m \in \Omega(X_i)$, then $b_{i,m^{r_i}} < 0$.*

Lemma 2.7. *Let X be an almost simple group with $\mathrm{soc}(X) = PSL(2,q)$, where $q \geq 5$ is an odd prime. Then q divides the indices of all the subgroups of X with odd index.*

Proof. Assume that $S \cong PSL(2,q)$ where $q \geq 5$ is an odd prime: $\mathrm{Aut}(S) = PGL(2,q)$ and $X = PSL(2,q)$ or $X = PGL(2,q)$. In both the cases the conclusion follows easily form the list of maximal subgroups of X given in [1]. $\square$

3. **Proof of Theorem**1.3

We start now the proof of our main result. We assume that G is a finitely generated profinite group G with the properties that $P_G(s) = \sum_n a_n/n^s$ is rational. As described in Section 2, $P_G(s)$ can be written as a formal infinite product of Dirichlet polynomials $P_i(s) = \sum_{n \in \mathbb{N}} b_{i,n}/n^s$ corresponding to the factors G_i/G_{i+1} of a chief series of G. Let J be the set of indices i such that G_i/G_{i+1} is a non-Frattini chief factor of G. Since $P_i(s) = 1$ if $i \notin J$, we have

$$P_G(s) = \prod_{j \in J} P_j(s).$$

For $C(s) = \sum_{n \in \mathbb{N}} c_n/n^s \in \mathcal{R}$, we define $\pi(C(s))$ to be the set of the primes q for which there exists at least one multiple n of q with $c_n \neq 0$. Notice that if $C(s) = A(s)/B(s)$ is rational then $\pi(C(s)) \subseteq \pi(A(s)) \bigcup \pi(B(s))$ is finite. Let $\mathcal{S}$ be the set of the finite simple groups that are isomorphic to a composition factor of some non-Frattini chief factor of G. The first step in the proof of Theorem 1.3 is to show that $\mathcal{S}$ is finite. The proof of this claim requires the following result.

Lemma 3.1 ([5, Lemma 3.1]). *Let G be a finitely generated profinite group and let q be a prime with $q \notin \pi(P_G(s))$. If q divides the order of a non-Frattini chief factor of G, then this factor is not a q-group.*

Lemma 3.2. *If G satisfies the hypothesis of Theorem 1.3, then the sets $\mathcal{S}$ and $\pi(G)$ are finite.*

Proof. Since $P_G(s)$ is rational, we have that $\pi(P_G(s))$ is finite. Therefore, it follows from Lemma 3.1 that $\mathcal{S}$ contains only finitely many abelian groups. Assume by contradiction that $\mathcal{S}$ is infinite. This is possible only if the subset $\mathcal{S}^*$ of the simple groups in $\mathcal{S}$ that are isomorphic to $PSL(2,p)$ for some prime p is infinite. Let

$$I := \{j \in J \mid S_j \in \mathcal{S}^*\}, \ A(s) := \prod_{i \in I} P_i(s) \ \text{ and } \ B(s) := \prod_{i \notin I} P_i(s).$$

Notice that $\pi(B(s)) \subseteq \bigcup_{S \in \mathcal{S} \setminus \mathcal{S}^*} \pi(S)$ is a finite set. Since $P_G(s) = A(s)B(s)$ and $\pi(P_G(s))$ is finite, if follows that the set $\pi(A(s))$ is finite. In particular, there exists a prime number $q \geq 5$ such that $q \notin \pi(A(s))$ but $PSL(2,q) \in \mathcal{S}^*$. Let Λ be the set of the odd integers n divisible by q but not divisible by any prime strictly greater than q and set

$$r := \min\{r_i \mid S_i = PSL(2,q)\},$$
$$I^* := \{i \in I \mid S_i = PSL(2,q) \text{ and } r_i = r\},$$
$$w := \min\{w(X_i) \mid S_i = PSL(2,q) \text{ and } r_i = r\},$$
$$\beta := \min\{n > 1 \mid n \in \Lambda, v_q(n) = r \text{ and } b_{i,n} \neq 0 \text{ for some } i \in I\}.$$

Assume $i \in I$, $n \in \Lambda$ and $b_{i,n} \neq 0$. We have that $S_i \cong PSL(2,q_i)$ for a suitable prime q_i and, by Lemma 2.5, $n = x_i^{r_i}$ and X_i contains a subgroup whose index divides x_i. By Lemma 2.7 q_i divides x_i and consequently n. Since q_i is the largest prime divisor of $|S_i|$ and q is the largest prime divisor of n, we deduce that $q = q_i$. It follows that $\beta = w^r$ and $b_{i,\beta} \neq 0$ if and only if $i \in I^*$ and $w(X_i) = w$; moreover in this last case $b_{i,\beta} < 0$. Hence the coefficient c_β of $1/\beta^s$ in $A(s)$ is

$$c_\beta = \sum_{i \in I^*, w(X_i) = w} b_{i,\beta} < 0.$$

This implies that $q \in \pi(A(s))$ which is a contradiction. So we have proved that $\mathcal{S}$ is finite. By [5, Lemma 3.2], if follows that $\pi(G)$ is also finite. $\qquad \square$

Proof of Theorem 1.3. Let $\mathcal{T}$ be the set of the almost simple groups X such that there exists infinitely many $i \in J$ with $X_i \cong X$ and let $I = \{i \in J \mid X_i \in \mathcal{T}\}$. By Lemma 3.2, $J \setminus I$ is finite. We have to prove that J is finite; this is equivalent to showing that $I = \emptyset$. Let $i \in I$: by the hypothesis of Theorem 1.3, there exists a prime q_i such that $S_i \in \{C_{q_i}, PSL(2,q_i)\}$. Set $q = \max\{q_i \mid i \in I\}$ and let Λ be the set of odd integers n divisible by q. Assume $n \in \Lambda$ and $b_{i,n} \neq 0$ for some $i \in I$. If S_i is cyclic, then $P_i(s) = 1 - c_n/n^s$ where $n = |G_i/G_{i+1}| = q_i^{r_i}$ and c_n is the number of complements of G_i/G_{i+1} in G/G_{i+1} : this implies $q = q_i$. Otherwise $S_i = PSL(2,q_i)$ with $q_i \leq q$ and, by Lemma 2.5, $n = x_i^{r_i}$ and X_i contains a subgroup whose index divides x_i; since x_i divides $|SL(2,q_i)|$, q divides x_i and $q_i \leq q$, we must have $q = q_i$. In both the cases, we have $n_i = x_i^{r_i}$ where x_i a positive integer with $v_q(x_i) = 1$. Let

$$w = \min\{x \in \Lambda \mid v_q(x) = 1 \text{ and } b_{i,x^{r_i}} \neq 0 \text{ for some } i \in I\}.$$

Since $J \setminus I$ is finite and $P_G(s) = \prod_{i \in J} P_i(s)$ is rational, also $\prod_{i \in I} P_i(s)$ is rational. In particular, by Remark 2.1, taking $\pi = \{2\}$ we have that the following series is rational:

$$Q(s) = \prod_{i \in I} P_i^{\{2\}}(s).$$

Let $I^* = \{i \in I \mid b_{i,w^{r_i}} \neq 0\}$. By the above considerations and Remark 2.6, $i \in I^*$ if and only if either $S_i \cong C_q$ and $w = q$ or $\mathrm{soc}(X_i) = PSL(2,q)$ and $w(X_i) = w$. In particular if $i \in I^*$ then there exist infinitely many $j \in I$ with $X_i \cong X_j$ and all of them are in I^*, hence I^* is an infinite set. Moreover, by Remark 2.6, we have that $b_{i,w^{r_i}} < 0$ for every $i \in I^*$, and therefore applying Proposition 2.4 to the Dirichlet series $Q(s)$, we deduce that the product

$$H(s) = \prod_{i \in I} \left(1 + \frac{b_{i,w^{r_i}}}{w^{r_i s}} \right) = \prod_{i \in I^*} \left(1 + \frac{b_{i,w^{r_i}}}{w^{r_i s}} \right)$$

is rational. By Proposition 1.2 the set I^* must be finite, a contradiction.   □

Acknowledgments

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