Chapter V

Irregularity: formal theory

Introduction

With the example of Kummer hypergeometric equation at hand, we first recall some typical issues due to an irregular singularity: presence of divergent series, tension between the topological viewpoint (monodromy) and the formal algebraic viewpoint and its resolution in terms of asymptotic expansions in sectors and the Stokes phenomenon.

We then turn and settle to the algebraic viewpoint which is prevalent in the book. The central theme is the structure of formal differential modules in one variable: the slope filtration (and its associated Newton polygon) and the more refined Turrittin-Levelt decomposition.

We start by defining the maximal slope (the Poincaré rank) in terms of spectral norms of derivations, and study in detail the case of a cyclic differential module. We establish the Turrittin-Levelt decomposition and study a number of variants: notion of turning point in the presence of parameters, existence of a similar decomposition at a crossing point of the polar divisor.

14 Confluent hypergeometric equations and phenomena related to irregularity

14.1 Solutions of the confluent hypergeometric equation

Let us consider the confluent hypergeometric differential equation (1.2.1), for $c \notin \mathbb{Z}$, namely

$$L_{a,c}y = x\,\partial_x^2\,y + (c - x)\,\partial_x\,y - a\,y = 0. \tag{14.1.1}$$

By an easy computation, the Fuchs number at ∞ (see 6.3.3) is $i_\infty(L_{a,c}) = 1$, so that the differential equation is irregular at ∞.

It is easily seen that a basis of solutions at any point of the complex plane is represented by the two converging expressions

$$ {}_1F_1(a,c;x) = \sum_0^\infty \frac{(a)_n}{(c)_n n!} x^n \quad \text{and} \quad x^{1-c}{}_1F_1(a+1-c, 2-c; x). $$

It will be useful in the sequel to observe that *Kummer's transformation* $y(x) \mapsto e^x y(-x)$ transforms $L_{a,c}$ in $L_{c-a,c}$, so that it provides alternative expressions of the same solutions, namely

$$ \begin{aligned} {}_1F_1(a,c;x) &= e^x{}_1F_1(c-a,c;-x), \\ x^{1-c}{}_1F_1(a+1-c,2-c;x) &= x^{1-c}e^x{}_1F_1(1-a,2-c;-x). \end{aligned} $$

At ∞, a formal solution is given by

$$ x^{-a}{}_2F_0(a, a-c+1; x^{-1}), \tag{14.1.2} $$

where ${}_2F_0(a,a-c+1;x^{-1})$ is the divergent series

$$ \sum_{n=0}^\infty \frac{(a)_n(a-c+1)_n}{n!} \frac{(-1)^n}{x^n}. $$

To determine a second (formal) solution, we twist $xL_{a,c} = \vartheta_x^2 + (c-1-x)\vartheta_x - ax$, first by x^{-a} and then by e^x. We use the fact that $x^a \circ \vartheta_x \circ x^{-a} = \vartheta_x - a$ and $e^{-x} \circ \vartheta_x \circ e^x = \vartheta_x - x$ to find $x^a e^{-x} \circ xL_{a,c} \circ x^{-a}e^x = \vartheta_x^2 + (c-1-2a+x)\vartheta_x + (c-2a)x + a(a-c+1)$. We then determine the unique solution of this equation in $\mathbb{C}[[\frac{1}{x}]]$ and find a second solution at infinity of (14.1.1):

$$ x^{-a}e^x E(a, a-c+1; x^{-1}), \tag{14.1.3} $$

where $E(a,a-c+1;x^{-1})$ is the divergent series

$$ {}_3F_1(a, a-c+1, 1; 2a-c+1; x^{-1}) = \sum_{n=0}^\infty \frac{(a)_n(a-c+1)_n}{(2a-c+1)_n} x^{-n}. $$

This provides a full set of *formal solutions* of (14.1.1) (in a differential field extension of $\mathbb{C}((x^{-1}))$ containing x^{-a} and e^x), namely

$$ \begin{aligned} \hat{u}(a,c;x^{-1}) &:= x^{-a}{}_2F_0(a,a-c+1;-x^{-1}), \\ \hat{v}(a,c;x^{-1}) &:= x^{-a}e^x E(a,a-c+1;x^{-1}). \end{aligned} \tag{14.1.4} $$

Since this is a purely formal decomposition, it is not a priori clear that this might be of any use in the understanding of the complex-analytic theory of the equation (14.1.1).

14.2 Meromorphic coefficients and Stokes multipliers

Given a differential module M over $\mathbb{C}(\{x\})$, we may view it as a differential module with analytic coefficients over a small disk punctured at 0; the classification is then given by the monodromy (cf. 6.2.9).

On the other hand, we may look at its formal completion $\widehat{M}$ over $\mathbb{C}((x))$. In this chapter, we will study the classification of irregular differential modules over $\mathbb{C}((x))$.

In the case where 0 is a regular singularity, we have seen the formal classification in Chapter III and how it fits with the analytic viewpoint. In general, as we saw in the confluent hypergeometric case, divergent series occur and the relation to the analytic theory is much more delicate; it involves asymptotic expansions in suitable sectors, Gevrey series, and the Stokes phenomenon. This goes beyond the scope of this book (cf. [88]).

15 Poincaré rank

15.1 Spectral norms

15.1.1. Let $(K, |\text{-}|)$ be a valued field of characteristic 0, complete with respect to a *non archimedean* absolute value $|\text{-}| : K \to \mathbb{R}_{\geqslant 0}$. This means that the ultrametric inequality

$$|x+y| \leqslant \max\{|x|, |y|\} \ \forall x, y \in K,$$

holds. If the image of $|\text{-}|$ is $\{0,1\}$, we say that $|\text{-}|$ is the *trivial absolute value.* We do not exclude this case. We define $v(\text{-}) := -\log|\text{-}| : K \to \mathbb{R} \cup \{\infty\}$ as the corresponding valuation. In this situation K is called a *non-archimedean field.*

We will set $R_v := \{f \in K \mid |f| \leqslant 1\}$ = the local ring of v-integers of K, $\mathfrak{m}_v := \{f \in K \mid |f| < 1\}$ = the maximal ideal of R_v, $\kappa(v) := R_v/\mathfrak{m}_v$ = the residue field of K at v.

Example 15.1.2 (**x-adic valuation**)**.** Let K be a field of characteristic 0 and $F = K((x))$, endowed with the x-adic valuation $v(\text{-}) = \operatorname{ord}_x(\text{-})$. So, $|x| = a^{-1} < 1$, and for any $f \in F$, we have $|f(x)| = a^{-\operatorname{ord}_x f}$, where $a > 1$ is chosen arbitrarily. Then F is a non-archimedean field and $R_v = K[[x]]$, $\kappa(v) = K$. The closed subfield K of F is trivially valued. The field $(K((x)), |\text{-}|)$ is the basic object of this book.

15.1.3. Let K be a non-archimedean field. A non-archimedean K-Banach space is a K-vector space M endowed with a *Banach norm* (or *K-Banach norm*, for precision), that is, a map $|\text{-}|_M : M \to \mathbb{R}_{\geqslant 0}$ satisfying

(1) $|m|_M = 0$ if and only if $m = 0$;

(2) $|am|_M = |a||m|_M$, for $a \in K$, $m \in M$;

(3) $|m+n|_M \leqslant \max\{|m|_M, |n|_M\}$;

(4) M equipped with the distance $d_M(m,n) = |m-n|_M$ is a complete metric space.

15.1.4. A subset S of a non-archimedean K-Banach space $(M, |\text{-}|_M)$ is *orthonormal* if for any finite subset $\{v_1, \dots, v_n\} \subset S$, and coefficients $c_1, \dots, c_n \in K$,

$$\Big|\sum\nolimits_i c_i v_i\Big|_M = \max_i |c_i|.$$

Two K-Banach norms $|\text{-}|_1$ and $|\text{-}|_2$ on a K-vector space M are *equivalent* if there exist constants $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$|\text{-}|_1 \leqslant c_2|\text{-}|_2 \leqslant c_1|\text{-}|_1.$$

Any two K-Banach norms on a finite-dimensional K-vector space M are equivalent. In particular, on any finite-dimensional K-Banach space $(M, |\text{-}|_M)$ there exists an equivalent norm which admits an orthonormal basis.

Remark 15.1.5. It will be sometimes more convenient to use the *Banach valuation*

$$v_M(\text{-}) := -\log|\text{-}|_M$$

on M.

15.1.6 (**Operator norm**)**.** Let K be a non-archimedean field. For two K-Banach spaces $(M, |\text{-}|_M)$, $(N, |\text{-}|_N)$ we define $\mathcal{L}_K(M,N)$ as the K-Banach space of bounded K-linear maps, equipped with the *operator norm* $|\text{-}|_{M,N}$: for $\varphi \in \mathcal{L}_K(M,N)$,

$$\begin{aligned} |\varphi|_{\mathcal{L}_K(M,N)} &= \inf\{C > 0 \mid |\varphi m|_N \leqslant C|m|_M \ \forall m \in M\} \\ &= \sup\left\{\frac{|\varphi m|_N}{|m|_M} \mid m \in M,\ m \neq 0\right\}. \end{aligned} \tag{15.1.7}$$

For a further K-Banach space $(P, |\text{-}|_P)$ and $\varphi \in \mathcal{L}_K(M,N)$ and $\psi \in \mathcal{L}_K(N,P)$, we have

$$|\psi \circ \varphi|_{\mathcal{L}_K(M,P)} \leqslant |\psi|_{\mathcal{L}_K(N,P)}|\varphi|_{\mathcal{L}_K(M,N)}. \tag{15.1.8}$$

For $M = N$, we simply write $(\mathcal{L}_K(M), |\text{-}|_M)$ for this K-Banach space. It is in fact a *K-Banach algebra*, since for $\varphi, \psi \in \mathcal{L}_K(M)$, $|\psi \circ \varphi|_M \leqslant |\psi|_M|\varphi|_M$.

15.1.9 (**Tensor product**)**.** For two K-Banach spaces $(M, |\text{-}|_M)$, $(N, |\text{-}|_N)$ we define the topological tensor product $(M\widehat{\otimes}_K N, |\text{-}|_{M\widehat{\otimes}_K N})$ of $(M, |\text{-}|_M)$ and $(N, |\text{-}|_N)$ as the completion of $(M \otimes_K N, |\text{-}|_{M\otimes_K N})$, where

$$|\ell|_{M\otimes_K N} = \inf\{\max_i |m_i|_M |n_i|_N\},$$

over all representations of $\ell \in M \otimes_K N$ as $\ell = \sum_i m_i \otimes n_i$.

15.1.10. Let F/K be an extension of valued fields with F non-trivially valued, and let $\mathcal{L}_K(F)$ be the K-Banach algebra of bounded K-linear endomorphisms of F. If M and N are finite-dimensional F-vector spaces endowed with F-Banach ultranorms $|\text{-}|_M$ and $|\text{-}|_N$, then $(M,|\text{-}|_M)$ and $(N,|\text{-}|_N)$ are at the same time K-Banach spaces. We have a canonical continuous injection

$$\mathrm{Hom}_F(M,N) = \mathcal{L}_F(M,N) \hookrightarrow \mathcal{L}_K(M,N)$$

and, for any $L \in \mathcal{L}_F(M,N)$, $|L|_{\mathcal{L}_F(M,N)} = |L|_{\mathcal{L}_K(M,N)}$. Let $m_1,\dots,m_r$ (resp. $n_1,\dots,n_s$) be an orthonormal F-basis of M (resp. N). Then any $\varphi \in \mathcal{L}_K(M,N)$ can be uniquely expressed as

$$\varphi = \sum_{i,j} \varphi_{i,j}(m_i^\vee \widehat{\otimes}_K n_j)$$

with $\varphi_{i,j} \in \mathcal{L}_K(F)$. We have $|\varphi|_{\mathcal{L}_K(M,N)} = \max_{i,j}|\varphi_{i,j}|_{\mathcal{L}_K(F)}$.

Proposition 15.1.11. *Let $(M,|\text{-}|_M)$, $(N,|\text{-}|_N)$ be finite-dimensional F-Banach spaces which both admit F-orthonormal bases. Then, for any $m \in M$ and $n \in N$,*

$$|m \otimes_F n|_{M\otimes_F N} = |m \widehat{\otimes}_K n|_{M\widehat{\otimes}_K N} = |m|_M |n|_N .$$

Proof. Notation as above. Then there exist unique coefficients $a_1,\dots,a_r$ and $b_1,\dots,b_s$ in F such that $m = \sum_i a_i m_i$, $n = \sum_j b_j n_j$. Then $\{m_i \otimes_F n_j\}_{i,j}$ is an F-orthonormal basis of $M \otimes_F N$, and $|m \otimes_F n|_{M\otimes_F N} = \max_{i,j}|a_i||b_j| = \max_i |a_i| \max_j |b_j| = |m|_M|n|_N$. □

Remark 15.1.12. The second equality in the previous statement holds for any K-Banach spaces if K is non-trivially valued, but its proof is somewhat subtle. We refer the interested reader to [94, Prop. 17.4].

If M and N are finite-dimensional F-vector spaces endowed with F-Banach norms $|\text{-}|_M$ and $|\text{-}|_N$, the F-vector space $M \otimes_F N$ is already complete for the F-norm $|-|_{M\otimes_F N}$, and we have a canonical bounded projection

$$p_{M,N} : M \widehat{\otimes}_K N \longrightarrow M \otimes_F N \ .$$

In the previous situation, we have

Proposition 15.1.13. *Assumptions as in Proposition 15.1.11. Then, for any $\ell \in M \otimes_F N$,*

$$|\ell|_{M\otimes_F N} = \inf_{\ell' \mapsto \ell} |\ell'|_{M\widehat{\otimes}_K N} = \min_{\ell' \mapsto \ell} |\ell'|_{M\widehat{\otimes}_K N}.$$

Proof. Notation as in the proof of Proposition 15.1.11. We have uniquely $\ell = \sum_{i,j} a_{i,j} m_i \otimes_F n_j$ with $a_{i,j} \in F$, and

$$|\ell|_{M\otimes_F N} = \max_{i,j} |a_{i,j}|.$$

Suppose now $\ell' = \sum_\alpha x_\alpha \widehat{\otimes}_K y_\alpha \mapsto \ell$ with $x_\alpha \in M$ and $y_\alpha \in N$. We find uniquely determined coefficients $a_{\alpha,i}, b_{\alpha,j} \in F$ such that

$$x_\alpha = \sum_{i=1}^{r} a_{\alpha,i} m_i, \quad y_\alpha = \sum_{j=1}^{s} b_{\alpha,j} n_j.$$

Then

$$\ell' = \sum_{i,j} \Big(\sum_\alpha a_{\alpha,i} \widehat{\otimes}_K b_{\alpha,j}\Big)(m_i \widehat{\otimes}_K n_j) \longmapsto \ell = \sum_{i,j} a_{i,j} m_i \otimes_F n_j,$$

so that for the product map $\mu_F : F\widehat{\otimes}_K F \to F$,

$$\mu_F\Big(\sum_\alpha a_{\alpha,i} \widehat{\otimes}_K b_{\alpha,j}\Big) = \sum_\alpha a_{\alpha,i} b_{\alpha,j} = a_{i,j}.$$

The statement is then reduced to the case of $M = N = F$. But then for any convergent expression in F

$$a = \sum_\alpha a_\alpha b_\alpha$$

we have

$$|a| = \Big|\sum_\alpha a_\alpha b_\alpha\Big| \leqslant \sup_\alpha |a_\alpha||b_\alpha|.$$

The result follows. ☐

Definition 15.1.14 (Spectral norms and valuations). *For any K-Banach algebra $(\mathcal{A}, ||\text{-}||)$ one defines the* spectral norm *of $f \in \mathcal{A}$ as the number*

$$|f|_{\text{sp}} = \lim_{n\to\infty} ||f^n||^{1/n} = \inf_{n\geqslant 1} ||f^n||^{1/n}. \tag{15.1.15}$$

We define also the spectral valuation *of $f \in \mathcal{A}$ as*

$$v_{\text{sp}}(f) = -\log|f|_{\text{sp}} = \lim_{n\to\infty} \frac{1}{n} v(f^n) = \sup_{n\geqslant 1} \frac{1}{n} v(f^n),$$

for $v(\text{-}) = -\log||\text{-}||$.

The existence of the limit and its coincidence with the inf are standard consequences of Fekete's lemma: for any integers $n \geqslant 0$ and $m > 0$ we can take the euclidean division $n = q(n)m + r(n)$ with $0 \leqslant r(n) < m$ so that $c||f^n||^{1/n} \leqslant ||f^m||^{q(n)/n}||f^{r(n)}||^{1/n} = (||f^m||^{q(n)m/n})^{1/m}||f^{r(n)}||^{1/n}$. Letting n go to ∞, we have $\limsup_n ||f^n||^{1/n} \leqslant ||f^m||^{1/m}$. Therefore,

$$\limsup_{n\to\infty} ||f^n||^{1/n} \leqslant \inf_m ||f^m||^{1/m} \leqslant \liminf_{n\to\infty} ||f^n||^{1/n},$$

from which the existence of the limit and the equality with the inf follow at once.

The choice of an equivalent norm on $\mathcal{A}$ does not affect $||\text{-}||_{\text{sp}}$.

Remark 15.1.16. For any $f \in \mathcal{A}$ we have $|f|_{\mathrm{sp}} \leqslant ||f||$ or, equivalently, $v_{\mathrm{sp}}(f) \geqslant v(f)$.

15.1.17. Notice that in general the "spectral norm" is not an algebra norm, nor even a semi-norm on the Banach space $\mathcal{A}$. However, it defines an algebra semi-norm on any commutative sub-algebra of $\mathcal{A}$, in the following sense.

15.1.18. For any non-archimedean K-Banach algebra $\mathcal{A}$, and any commuting elements $f, g \in \mathcal{A}$, we have the following properties:

(1) $|f+g|_{\mathrm{sp}} \leqslant \max\{|f|_{\mathrm{sp}}, |g|_{\mathrm{sp}}\}$, with $=$ holding if $|f|_{\mathrm{sp}} \neq |g|_{\mathrm{sp}}$,

(2) $|fg|_{\mathrm{sp}} \leqslant |f|_{\mathrm{sp}} |g|_{\mathrm{sp}}$,

(3) $|f^n|_{\mathrm{sp}} = |f|_{\mathrm{sp}}^n$.

See for instance [90, pp. 222–223].

Lemma 15.1.19. *Let $\mathcal{J}$ be a closed square-zero bilateral ideal of the non-archimedean K-Banach algebra $\mathcal{A}$. Let $(\overline{\mathcal{A}} = \mathcal{A}/\mathcal{J}, |\text{-}|_{\overline{\mathcal{A}}})$ be the quotient K-Banach algebra, and let $f \mapsto \overline{f}$ be the canonical projection. For any $f \in \mathcal{A}$ we have*

$$|f|_{A,\mathrm{sp}} = |\overline{f}|_{\overline{\mathcal{A}},\mathrm{sp}}.$$

Proof. Let us first show that $|f|_{A,\mathrm{sp}}$ depends only on $\overline{f}$. By symmetry, it suffices to show that for any $j \in \mathcal{J}$, $|f+j|_{A,\mathrm{sp}} \leqslant |f|_{A,\mathrm{sp}}$. Because $\mathcal{J}^2 = 0$, one has $(f+j)^n = f^n + \sum_{k=0}^{n-1} f^k j f^{n-1-k}$. In particular

$$|(f+j)^n|_A \leqslant \max\bigl\{|f^n|_A, \max_{k<n} |f^k|_A |f^{n-1-k}|_A |j|_A\bigr\},$$

which gives

$$\begin{aligned}|f+j|_{A,\mathrm{sp}} = \lim |(f+j)^n|_A^{1/n} &\leqslant \max\Bigl\{\lim_n |f^n|_A^{1/n},\ \lim_n \max_{k<n} (|f^k|_A |f^{n-1-k}|_A)^{1/n}\Bigr\} \\ &= \max\Bigl\{|f|_{A,\mathrm{sp}},\ \lim_m \max_{k\leqslant m/2} |f^k|_A^{\frac{1}{m}} \cdot |f^{m-k}|_A^{\frac{1}{m}}\Bigr\}.\end{aligned}$$

But

$$\begin{aligned}&\lim_m \max_{k\leqslant m/2} |f^k|_A^{\frac{1}{m}} \cdot |f^{m-k}|_A^{\frac{1}{m}} = \sup_\ell \lim_{m\geqslant \ell} \max_{k\leqslant m/2} |f^k|_A^{\frac{1}{m}} \cdot |f^{m-k}|_A^{\frac{1}{m}} \\ &= \max\Bigl\{\sup_\ell \lim_{m\geqslant\ell} \max_{\ell\leqslant k\leqslant m/2} \bigl(|f^k|_A^{\frac{1}{k}}\bigr)^{\frac{k}{m}} \cdot \bigl(|f^{m-k}|_A^{\frac{1}{m-k}}\bigr)^{1-\frac{k}{m}}, \\ &\qquad\qquad \sup_\ell \lim_{m\geqslant\ell} \max_{k<\ell} \bigl(|f^k|_A^{\frac{1}{k}}\bigr)^{\frac{k}{m}} \cdot \bigl(|f^{m-k}|_A^{\frac{1}{m-k}}\bigr)^{1-\frac{k}{m}}\Bigr\},\end{aligned}$$

and both terms are bounded by $|f|_{A,\mathrm{sp}}$, as wanted.

Now $|\overline{f}|_{\overline{\mathcal{A}},\mathrm{sp}} = \inf_n \inf_{j\in\mathcal{J}} |(f+j)^n|_A^{1/n} = \inf_{j\in\mathcal{J}} \inf_n |(f+j)^n|_A^{1/n} = \inf_{j\in\mathcal{J}} |f+j|_{A,\mathrm{sp}} = |f|_{A,\mathrm{sp}}$. □

Lemma 15.1.20. *Let M, N be finite-dimensional F-vector spaces and $|\text{-}|_M$, $|\text{-}|_N$ be F-Banach norms on them. For any $D \in \mathcal{L}_K(M)$ and $D' \in \mathcal{L}_K(N)$, we have*

$$|D \widehat{\otimes}_K D'|_{M \widehat{\otimes}_K N, \mathrm{sp}} = |D|_{M,\mathrm{sp}} |D'|_{N,\mathrm{sp}}$$

and

$$|D \widehat{\otimes}_K 1_N + 1_M \widehat{\otimes}_K D'|_{M \widehat{\otimes}_K N, \mathrm{sp}} \leqslant \max\{|D|_{M,\mathrm{sp}}, |D'|_{N,\mathrm{sp}}\}.$$

Moreover, let $\mathcal{L}_K(D, D') \in \mathcal{L}_K(\mathcal{L}_K(M, N))$ be $D' \circ - \circ D$; then one has

$$|\mathcal{L}_K(D, D')|_{\mathcal{L}_K(M,N),\mathrm{sp}} = |D|_{M,\mathrm{sp}} |D'|_{N,\mathrm{sp}}$$

and

$$|\mathcal{L}_K(1_M, D') - \mathcal{L}_K(D, 1_N)|_{\mathcal{L}_K(M,N),\mathrm{sp}} \leqslant \max\{|D|_{M,\mathrm{sp}}, |D'|_{N,\mathrm{sp}}\}.$$

If $|D|_{M,\mathrm{sp}} \neq |D'|_{N,\mathrm{sp}}$, equality holds in the previous formulas.

Proof. Under the assumptions of Proposition 15.1.11 the stronger formula

$$|D \widehat{\otimes}_K D'|_{M \widehat{\otimes}_K N} = |D|_M |D'|_N$$

holds. Indeed, by 15.1.10 and 15.1.11, the tensor product

$$(\mathcal{L}_K(M), |\text{-}|_M) \widehat{\otimes}_K (\mathcal{L}_K(N), |\text{-}|_N)$$

embeds isometrically as a closed subspace of $(\mathcal{L}_K(M \widehat{\otimes}_K N, |\text{-}|_{M \widehat{\otimes}_K N})$, so that the formula follows from Proposition 15.1.11 applied to the former tensor product. This proves the first part of the statement.

We next note that the operators $D \widehat{\otimes}_K 1_N$ and $1_M \widehat{\otimes}_K D'$ on $M \widehat{\otimes}_K N$ such that $D \widehat{\otimes}_K 1_N \circ 1_M \widehat{\otimes}_K D' = D \widehat{\otimes}_K D'$ (resp. $\mathcal{L}_K(1_M, D')$ and $\mathcal{L}_K(D, 1_N)$ on $\mathcal{L}_K(M, N)$, such that $\mathcal{L}_K(1_M, D') \circ \mathcal{L}_K(D, 1_N) = \mathcal{L}_K(D, D')$) commute. On the other hand, $|D \widehat{\otimes}_K 1_N|_{M \widehat{\otimes}_K N, \mathrm{sp}} = |D|_{M,\mathrm{sp}}$, and $|1_M \widehat{\otimes}_K D'|_{M \widehat{\otimes}_K N, \mathrm{sp}} = |D'|_{N,\mathrm{sp}}$ (resp. $|\mathcal{L}_K(D, 1_N)|_{\mathcal{L}_K(M,N),\mathrm{sp}} = |D|_{M,\mathrm{sp}}$, and $|\mathcal{L}_K(1_M, D')|_{\mathcal{L}_K(M,N),\mathrm{sp}} = |D'|_{N,\mathrm{sp}}$). So, the second and fourth assertions hold.

It remains to prove the third formula. Again, under the assumptions of Proposition 15.1.11 the stronger formula

$$|\mathcal{L}_K(D, D')|_{\mathcal{L}_K(M,N)} = |D|_M |D'|_N$$

holds. In fact, let us write

$$D = \sum_{i,j=1}^{r} D_{i,j}(m_i^\vee \widehat{\otimes}_K m_j) \quad \text{and} \quad D' = \sum_{h,k=1}^{s} D'_{h,k}(n_h^\vee \widehat{\otimes}_K n_k), \quad D_{i,j}, D'_{h,k} \in \mathcal{L}_K(F),$$

so that $|D|_M = \max_{i,j} |D_{i,j}|_{\mathcal{L}_K(F)}$ and $|D'|_N = \max_{h,k} |D'_{h,k}|_{\mathcal{L}_K(F)}$. On the other hand, a simple computation shows that

$$\mathcal{L}_K(D, D')(m_u^\vee \widehat{\otimes}_K n_v) = \sum_{u,v} D_{i,u} D'_{v,k} m_i^\vee \widehat{\otimes}_K n_k,$$

so that the formula follows.

The last assertion in the statement follows from the properties of the spectral norm recalled above. ☐

Lemma 15.1.21. *Let*

$$(P, |\text{-}|_P) = (M, |\text{-}|_M) \oplus (N, |\text{-}|_N) \tag{15.1.22}$$

be an orthogonal direct sum of K-Banach spaces and let $F \in \mathcal{L}_K(P)$ be an operator such that $F(M) \subset M$. Let $D = F_{|M} \in \mathcal{L}_K(M)$ and $D' \in \mathcal{L}_K(N)$ be the operators induced by F. Then

$$|F|_{P,\mathrm{sp}} = \max\{|D|_{M,\mathrm{sp}}, |D'|_{N,\mathrm{sp}}\}.$$

Proof. Consider the operator $L' = D \oplus D' \in \mathcal{L}_K(P)$. Obviously, $|L'|_{P,\mathrm{sp}} = \max\{|D'|_{N,\mathrm{sp}}, |D|_{M,\mathrm{sp}}\}$. On the other hand, $L = L' + H$, where H is an operator that kills M and sends P to M. Such operators form a bilateral ideal $\mathcal{J}$, with $\mathcal{J}^2 = 0$, in the sub-K-Banach algebra $\mathcal{A}$ of $\mathcal{L}_K(P)$ of operators preserving M. We may then apply Lemma 15.1.19 to $(\mathcal{A}, \mathcal{J})$ and to $L, L' \in \mathcal{A}$, which reduce modulo $\mathcal{J}$ to the same element $\overline{L} = \overline{L'} \in \overline{\mathcal{A}} = \mathcal{A}/\mathcal{J}$. We deduce that $|L|_{P,\mathrm{sp}} = |L'|_{P,\mathrm{sp}}$. ☐

15.1.23. Let E be a subfield of F: it inherits an absolute value induced by $|\text{-}|_F$. If the ramification index $e = (|F^*| : |E^*|)$ is finite (which in particular is the case if F is a finite extension of E), it will often be convenient to renormalize the absolute value on E by setting $|\text{-}|_E = |\text{-}|_F^{1/e}$. We write w for the associated valuation on E, and write $e = e(v/w)$.

Lemma 15.1.24. *Assume that $|\text{-}|$ is trivial on K. For any $L \in \mathcal{L}_K(M) = \mathcal{L}_K({}_EM)$, $|L|_{M,\mathrm{sp}} = |L|_{{}_EM,\mathrm{sp}}^{e(v/w)}$.*

Proof. Indeed, ${}_EM$ is the same K-space as M, endowed with the norm $|\text{-}|^{1/e}$. ☐

15.2 Christol-Dwork-Katz theorem

15.2.1. Let F/K be as in 15.1.10 with K a field of characteristic 0; in particular, the absolute value of F is non-trivial. Let ∂ be a bounded K-linear derivation of F and let (M, ∇_∂) be a differential module over (F, ∂) in the sense of 2.4. So, ∇_∂ is a K-linear endomorphism of M which satisfies the Leibniz rule w.r.t. the elements of F and *is bounded with respect to any F-Banach norm on M*. Since M is of finite dimension over F, all these norms are equivalent, so that the condition does not depend on the norm.

The operator norm of ∇_∂ depends on the norm $|\text{-}|_M$ we choose on M, but the spectral norm of ∇_∂ defined by (15.1.15) does not.

We now present the computation of the spectral norm of ∇_∂ in terms of its action on a cyclic vector of (M, ∇_∂).

Theorem 15.2.2 (Christol-Dwork-Katz). *Let (M, ∇_∂) be a differential module over the differential field (F, ∂), as described above, and let $m \in M$ be a cyclic vector. Let us write*

$$\nabla_\partial^\mu(m) = \sum_{i=0}^{\mu-1} a_i \nabla_\partial^i(m), \quad a_i \in F,$$

so that, in terms of the basis $\mathbf{m} = (m, \nabla_\partial(m), \dots, \nabla_\partial^{\mu-1}(m))$,

$$\nabla_\partial\, \mathbf{m} = \mathbf{m} \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & a_{\mu-2} \\ 0 & 0 & \cdots & 1 & a_{\mu-1} \end{pmatrix}.$$

For $n = 0, 1, 2, \dots$, let us write $\nabla_\partial^n\, \mathbf{m} = \mathbf{m}\, H_n$, with $H_n \in M_\mu(F)$.

(1) *For any $\sigma \geqslant |\partial|$, the following conditions are equivalent:*

  (i) $|\nabla_\partial|_{\mathrm{sp}} \leqslant \sigma$;

  (ii) $|a_j| \leqslant \sigma^{\mu-j}$ *for any* $j = 0, 1, \dots, \mu - 1$,

  (iii) $|H_n| \leqslant \sigma^n \max\{\sigma, \sigma^{-1}\}^{\mu-1}$ *for any* $n \in \mathbb{N}$.

(2) *Suppose that there exists ξ in F with $|\xi| = \min\left\{1, |a_j|^{-\frac{1}{\mu-j}}\right\}$, and let L the matrix of $\xi\nabla_\partial$ in the basis $\mathbf{n} = \mathbf{m}\Xi$, where Ξ is the diagonal matrix with diagonal entries $1, \xi, \xi^2, \dots, \xi^{\mu-1}$. Then $|L| \leqslant 1$, and if $|\nabla_\partial|_{\mathrm{sp}} > 1$, the reduction $L(0)$ is not nilpotent.*

Proof. We follow the proof of [30, 1.5].
(1) (iii)⇒(i) is clear: $|\nabla_\partial^n| \leqslant \max_{i+j=n}(|\partial^i|\,|H_j|) \leqslant \sigma^n \max\{\sigma, \sigma^{-1}\}^{\mu-1}$.

(ii)⇒(iii) is seen by induction on n: define $a_{j,n} \in F$, $n \in \mathbb{N}$ by $\nabla_\partial^n(m) = \sum_{j=0}^{\mu-1} a_{j,n} \nabla_\partial^j(m)$. one has the recursion $a_{j,n} = \partial(a_{j,n-1}) + a_{j-1,n-1} + a_{\mu-1,n-1} a_j$. Since $|\partial(a)| \leqslant \sigma|a|$, using (ii) we get by induction on n that $|a_{j,n}| \leqslant \sigma^{n-j}$, so that $|H_n| \leqslant \max_{i,j \leqslant \mu-1} \sigma^{n+i-j} \leqslant \sigma^n \max\{\sigma, \sigma^{-1}\}^{\mu-1}$.

(i)⇒(ii) (by contraposition): suppose (ii) does not hold, so that there exists ξ in a finite separable extension of F with $|\xi|^{-1} = \max_j |a_j|^{1/\mu-j} > \sigma$. This implies

$$|a_j \xi^{\mu-j}| \leqslant 1, \quad |\xi^{-1}\partial\xi| < 1.$$

It suffices to show that $|\nabla_\partial|_{\mathrm{sp}} \geqslant |\xi|^{-1}$.

Consider the base change $\mathbf{n} = \mathbf{m}\Xi$ where Ξ is the diagonal matrix with entries $1, \dots, \xi^{\mu-1}$. Then the matrix H of ∇_∂ is changed to $H' = \Xi^{-1} H \Xi + \Xi^{-1}\partial(\Xi)$, explicitly $H' = \xi^{-1}(A+B)$, where

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0\xi^{\mu} \\ 1 & 0 & \cdots & 0 & a_1\xi^{\mu-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & a_{\mu-2}\xi^2 \\ 0 & 0 & \cdots & 1 & a_{\mu-1}\xi \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & & & & \\ & \partial(\xi) & & & \\ & & 2\partial(\xi) & & \\ & & & \ddots & \\ & & & & (\mu-1)\partial(\xi) \end{pmatrix}.$$

We define H'_n by $\nabla_\partial^n \mathbf{n} = \mathbf{n} H'_n$. Let us show that

$$|H'^n| = |H'|^n = |\xi|^{-n} = |H'_n|.$$

In fact, the first two equalities follow since the characteristic polynomial $t^\mu - \sum_i a_i t^i$ of ξA (equal to that of H) has the property that $|a_j| \leqslant |\xi|^{j-n}$ (with at least one equality), and the characteristic polynomial $t^\mu - \sum_i b_i t^i$ of H' has the same property. In particular, its Newton polygon has a side of slope $v(\xi)$, and H' has an eigenvalue (in some algebraic extension of F) of valuation $|\xi|^{-1}$. We conclude that $|\xi|^{-n} \leqslant |H'^n| \leqslant |H'|^n = |\xi|^{-n}$. The last equality follows by induction on n using the relations

$$H_n - H^n = \partial(H_{n-1}) + H(H_{n-1} - H^{n-1}).$$

From these equalities, using the inequalities $|H'_n| \leqslant |\nabla_\partial^n| \leqslant \max\{|\partial|, |H'|\}^n$, we deduce that $|\nabla_\partial^n| = |\xi|^{-n}$, and then $|\nabla_\partial|_{\mathrm{sp}} = |\xi|^{-1} > \sigma$, contradicting (i). This concludes the proof of the first part of the theorem.

(2) With the above notation, $L = \xi H' = A + B$, with $|A| = 1$ and $|\nabla_\partial|_{\mathrm{sp}} = |\xi|^{-1}$. Therefore $|\nabla_\partial|_{\mathrm{sp}} > 1$ implies $|\xi| < 1$ and $|B| < 1$. The reduction modulo x of the characteristic polynomial of L coincides with the reduction modulo x of the characteristic polynomial of A, which is different from t^μ because one of the coefficients has norm 1. □

Remark 15.2.3. Notice that this gives a new proof of some of results seen in Chapter III, in the regular case (cf. 7.5.1 and 8.3.3). The theorem (with the same proof) also holds for any valued field of characteristic 0 as in 15.1, see [6].

15.3 Poincaré rank

We now specialize the definitions and results presented in this section to the case of a trivially valued field K of characteristic 0, and of $F = K((x))$, as in example 15.1.2. We still allow some freedom in the choice of the F/K-derivation ∂.

Definition 15.3.1 (Poincaré rank). *Given a differential module (M, ∇_∂) over $(K((x)), \partial)$ as above (15.2.1), we define the* Poincaré rank *(a.k.a.* Poincaré-Katz rank*, a.k.a.* rank of irregularity*) of (M, ∇_∂) as*

$$\rho_v(M, \nabla_\partial) = \max\{0, \log|\nabla_\partial|_{\mathrm{sp}} - \log|\partial|_{\mathrm{sp}}\}.$$

Remark 15.3.2. Lemma 6.2.4 of [67] asserts that

$$|\nabla_\partial|_{\mathrm{sp}} \geqslant |\partial|_{\mathrm{sp}},$$

so that in fact

$$\rho_v(M, \nabla_\partial) = \log|\nabla_\partial|_{\mathrm{sp}} - \log|\partial|_{\mathrm{sp}} \geqslant 0.$$

Lemma 15.3.3. *Let M, M_1, M_2 be F/K-differential modules. Then*

(i) $\rho_v(M^\vee) = \rho_v(M)$;

(ii) $\rho_v(M_1 \otimes_F M_2) \leqslant \max\{\rho_v(M_1), \rho_v(M_2)\}$;

(iii) $\rho_v(\mathrm{Hom}_F(M_1, M_2)) \leqslant \max(\rho_v(M_1), \rho_v(M_2))$;

(iv) *if* $\rho_v(M_1) \neq \rho_v(M_2)$ *we have equality in* (ii) *and* (iii);

(v) *if* $0 \to M_1 \to M \to M_2 \to 0$ *is an exact sequence of differential modules, then* $\rho_v(M) = \max\{\rho_v(M_1), \rho_v(M_2)\}$.

Proof. Parts (i) to (iv) follow from 15.1.20; (v) follows from Lemma 15.1.21. □

Lemma 15.3.4. *Let F' be a finite field extension of F (endowed with the unique extension v' of the valuation of F) with ramification index e. Assume that $|\text{-}|$ is trivial on K. Then, using the normalization of 15.1.23, we have the following properties:*

(i) *if M is a F/K-differential module, then* $\rho_{v'}(M_{F'}) = e\rho_v(M)$;

(ii) *if M' is a F'/K-differential module, then* $\rho_v({}_F M') = e^{-1}\rho_{v'}(M')$.

Proof. (ii) follows from 15.1.24. For (i), note that $\rho_v(M) = \rho_v({}_F(M_{F'}))$ by 15.3.3 (iii) (with $M_2 = (F', d_{F'/K})$), so that (i) follows from (ii). □

Example 15.3.5 (Regularity). In case $\partial = \vartheta_x$ we obviously have $|\partial|_{\mathrm{sp}} = |\partial|_F = 1$. So, $\rho_v(M, \nabla_{\vartheta_x}) = \log|\nabla_{\vartheta_x}|_{\mathrm{sp}}$. Let us show that $(M, \nabla_{\vartheta_x})$ is regular if and only if $|\nabla_{\vartheta_x}|_{\mathrm{sp}} = 1$, i.e., if and only if $\rho_v(M, \nabla_{\vartheta_x}) = 0$.

In fact, from item (2) of 8.3.3 we see that the regularity condition for a differential module over $K((x))$ is equivalent to the condition that for any (or some) $K[[x]]$-lattice Λ of M the action of the operators $\nabla^n_{\vartheta_x}$ (for $n \in \mathbb{N}$) is bounded for the x-adic valuation of M defined by the lattice. More precisely, $(M, \nabla_{\vartheta_x})$ is regular if and only if for any (or some) $K[[x]]$-lattice Λ of M we have

$$\sup_n -v(\nabla^n_{\vartheta_x}(\Lambda)) \leqslant c$$

for a constant c, where v is the valuation on M defined by $v(m) = i$ if $m \in x^i\Lambda$ and $m \notin x^{i+1}\Lambda$ (two such valuations induced by different lattices differ by a constant). □

We list now some useful consequence of the Christol-Dwork-Katz theorem.

Corollary 15.3.6. *If* $|\partial|_{\mathrm{sp}} = |\partial|$*, then* $\rho_v(\nabla_\partial) = \max\left\{\log|\partial|_{\mathrm{sp}},\ \max_j \frac{-v(a_j)}{\mu-j}\right\}$*. If moreover* $|\partial| = 1$*, then* $\rho_v(\nabla_\partial) = \max\left\{0,\ \max_j \frac{-v(a_j)}{\mu-j}\right\}$ *is a rational number with denominator bounded by* μ*; in addition, the function*

$$n \longmapsto \log|\nabla_\partial^n| - n\rho_v(\nabla_\partial)$$

for $n \in \mathbb{N}$ *is non-negative and bounded.*

Proof. The first two assertions follow immediately from item (1)(i) of Theorem 15.2.2; the last one from item (1)(iii). ☐

Corollary 15.3.7. *We have* $\rho_v(\nabla_\partial) \leqslant \rho$ *if and only if there exists a basis of* M *such that the matrix of* ∇_∂ *in that basis has valuation* $\geqslant -\rho$*.*

Proof. It suffices to take the basis $\mathbf{n}$ as in Theorem 15.2.2. ☐

15.3.8. Let us choose $\partial = \frac{d}{dx}$ or $x\frac{d}{dx}$. In both cases $|\partial|_{\mathrm{sp}} = |\partial|$, and $|x\frac{d}{dx}| = 1$. The Poincaré rank was introduced by Poincaré himself [86, p. 305] for differential operators in $\mathbb{C}[z, \frac{d}{dz}]$, with respect to the singularity at ∞: his definition coincides with the formula in 15.3.7 for $x = 1/z$, the x-adic valuation of a polynomial in $\mathbb{C}[z]$ being identified with its degree.

16 Turrittin-Levelt decomposition and variants

In this section, we give the structure theorem for formal differential modules in one variable. Herein $F = K((x))$, with its x-adic valuation[1].

Any finite extension $K((x))$ is a complete valued field of ramification index e of the form $K'((x'))$, with $(x')^e = x$ and K' a finite extension field of K. The extension $K'((x'))/K((x))$ is Galois if and only if K'/K is a Galois extension containing the e-th roots of unity.

16.1 The Turrittin-Levelt decomposition

16.1.1. Consider now a finite extension $F' = K'((x^{1/e}))$ of F and the operator $\partial = \vartheta_x = x\partial_x$. We have that $\partial\log(F'^{\times}) = \frac{1}{e}\mathbb{Z} \oplus x^{1/e}K'[[x^{1/e}]]$. Notice that $\partial\log(F'^{\times}) \cap K'[x^{-1/e}] = \frac{1}{e}\mathbb{Z}$, and that $\partial\log(F'^{\times}) + K'[x^{-1/e}] = F'$, so that

$$F'/\partial\log(F'^{\times}) \;\cong\; K'[x^{-1/e}]/\tfrac{1}{e}\mathbb{Z} \;\cong\; (K'/\tfrac{1}{e}\mathbb{Z}) \oplus x^{-1/e}K'[x^{-1/e}].$$

As a consequence, if we choose a section τ of the canonical projection $K' \to K'/\frac{1}{e}\mathbb{Z}$, we may extend it canonically to a section τ of $F' \to F'/\partial\log(F'^{\times})$.

---

[1] For statements over more general valued differential fields, with the same proof, see [5, 2.3].

Theorem 16.1.2 (**Turrittin-Levelt-Jordan decompositions**). *Let* $(M, x\nabla_{\partial_x})$ *be a rank* μ *differential module over* F/K. *There is a finite extension* $F' = F(\phi_1, \dots, \phi_r)$ *of* F *over which* $(M_{F'}, x\nabla_{\partial_x})$ *admits a Jordan decomposition of* F'/K *differential modules*

$$M_{F'} = \bigoplus_{i=1}^{r} M_{\phi_i}^{(\mu)}$$

with characters $\overline{\phi}_i \in K'[x^{-1/e}]/\frac{1}{e}\mathbb{Z}$, *where* $M_{\phi_i}^{(\mu)} = F' \otimes_{K'} \mathrm{Ker}_{M_{F'}}(x\nabla_{\partial_x} - \phi_i)^{\mu}$.

Bringing together the summands $M_{\phi}^{(\mu)}$ *for which the characters* ϕ_i*'s differ only by the constant terms, we get the* Turrittin-Levelt decomposition *of* F'/K *differential modules*

$$M_{F'} = \bigoplus_{j} L_{\psi_j} \otimes_{F'} R_j,$$

where L_{ψ_j} *is of* F'*-dimension one,* $\psi_j \in x^{-1/e}K'[x^{-1/e}]$, *and* R_j *is regular.*

Remark 16.1.3. In the Turrittin-Levelt-Jordan decomposition the characters are parametrized by $\phi \in F'/\partial \log(F'^{\times})$, while the Turrittin-Levelt decomposition is indexed by $\psi \in F'/R_v$. There is a canonical projection $\pi : F'/\partial \log(F'^{\times}) \to F'/R_v$ induced by the inclusion of $\partial \log(F'^{\times})$ in R_v. Then the sum of the terms $M_{\phi}^{(\mu)}$ of the Turrittin-Levelt-Jordan decomposition of M with $\pi(\phi) = \psi$ gives the term $L_{\psi} \otimes R$ of the Turrittin-Levelt decomposition. Viceversa, a term $L_{\psi} \otimes R$ of the Turrittin-Levelt decomposition can be written according the Jordan decomposition of the regular part R (see 8.3.4), to obtain the Turrittin-Levelt-Jordan decomposition.

Proposition-Definition 16.1.4 (**Turrittin index**). *There is a unique minimal extension field* $K'((x'))$ *of* $K((x))$ *on which the Turrittin-Levelt-Jordan decomposition holds. It is a Galois extension and it is generated by* $\phi_1, \dots, \phi_r$ *over* $K((x))$. *The ramification index* e *of* $K'((x'))/K((x))$ (*a divisor of* l.c.m.$(2, \dots, \mu)$) *is called the* Turrittin index *of* M.

The extension K'/K *is generated by the primitive* e*-th roots of unity and the coefficients of the terms of the* ϕ_j*'s of degree* $\leqslant 0$ *in* x'.

Proof. The first part of the proposition follows from the general Jordan theory (see 8.1.19). For the ramification index, notice that in the proof of the decomposition the only ramification that needs to be introduced at each step arises from the denominator of the Poincaré rank of M. □

Definition 16.1.5 (**Turrittin exponents**). *The constant terms* (*in* $K'/(\frac{1}{e}\mathbb{Z})$) *of the* $\overline{\phi}_j$*'s* (*viewed as polynomials in* $x^{-1/e}$) *are called the* Turrittin exponents *of* M.

Proposition 16.1.6. *In the situation of Theorem* 16.1.2, *let* δ *be any* x*-adically continuous derivation of* F *and* ∇_{δ} *an action on* M *commuting with* $x\nabla_{\partial_x}$ *and making* (M, ∇_{δ}) *a differential module over* (F, δ). *Then the Turrittin-Level-Jordan decomposition is stable under* ∇_{δ}.

In particular, the submodules of the decomposition are defined over (F, δ)*, and* δ *kills the Turrittin exponents of* M*.*

Proof. From the general theory of Section 8.2 it follows that the Turrittin-Levelt-Jordan decomposition is stable under any ∇_δ for any derivation δ commuting with ϑ_x. □

16.2 Proof of the decomposition

Theorem 16.1.2 can be proved using the nilpotent orbit method of Babbitt and Varadarajan (see [10]), and also using Hensel's lemma for differential operators (see [89]). We don't follow these paths, but derive the theorem from the Dwork-Katz-Turrittin theorem using a decomposition lemma of van den Essen and Levelt (see [40]).

Proposition 16.2.1 (**Splitting lemma**)**.** *Let* R *be a complete noetherian local ring, with maximal ideal* $\mathfrak{m}$ *and residue field* k (*of any characteristic*)*. Let* δ *be a derivation of* R *such that* $\delta(R) \subseteq \mathfrak{m}$ *and* $\delta(\mathfrak{m}) \subseteq \mathfrak{m}^2$*. Let* E *be a free* R*-module of finite type, endowed with an additive action* ∇_δ *of* δ *satisfying the Leibniz rule* (*w.r.t.* δ)*, and let* $\overline{\nabla}_\delta$ *denote the induced* k*-linear action of* δ *on* $\overline{E} := E \otimes_R k$*. Let*

$$\overline{E} = \bigoplus \overline{E}_j$$

be a decomposition into k*-spaces such that the sets of eigenvalues of* $\overline{\nabla}_\delta$ (*in any extension of* k) *on the* $\overline{E}_j$*'s are pairwise disjoint. Then this decomposition lifts uniquely to a decomposition*

$$E = \bigoplus E_j$$

into ∇_δ*-stable* R*-submodules. Moreover, if* δ' *is another derivation of* R*, and* $\nabla_{\delta'}$ *is an additive action of* δ' *on* E *satisfying the Leibniz rule* (*w.r.t.* δ') *and commuting with* ∇_δ*, then the decomposition is stable under* $\nabla_{\delta'}$*.*

Proof. It suffices to treat the case of two factors.
Existence. Let $\mathbf{e} = (e_1, \ldots, e_\mu)$ be a basis of E such that the image of $(e_1, \ldots, e_\nu)$ (resp. $(e_{\nu+1}, \ldots, e_\mu)$) is a basis of $\overline{E}_1$ (resp. $\overline{E}_2$), and let us write the matrix of ∇_δ in this basis in block form $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$. Its reduction modulo $\mathfrak{m}$ is $\begin{pmatrix} \overline{P} & 0 \\ 0 & \overline{S} \end{pmatrix}$, where $\overline{P}$ and $\overline{S}$ have no common eigenvalues. This assumption will be used in the guise that the endomorphism

$$\overline{H} \longmapsto \overline{PH} - \overline{HS} \quad \text{of} \quad M_{\nu,\mu-\nu}(k)$$

is injective, hence surjective, which implies that, for any n, the endomorphism

$$H \longmapsto PH - HS \quad \text{of} \quad M_{\nu,\mu-\nu}(\mathfrak{m}^n)$$

is also surjective. We look for a matrix $T = \begin{pmatrix} I & X \\ Y & I \end{pmatrix}$, with $\overline{X} = 0$, $\overline{Y} = 0$, such that in the basis $\mathbf{e}T$, the matrix of ∇_δ takes the block form $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$. This amounts to solving the two equations

$$\begin{aligned} \delta(X) + PX - XS - XRX + Q &= 0, \\ \delta(Y) + SY - YP - YQY + R &= 0. \end{aligned}$$

Let us show that the first equation admits a solution $X \in M_{\nu,\mu-\nu}(\mathfrak{m})$ (the second equation is similar, after transposition). Since R is $\mathfrak{m}$-adically complete, one can proceed by successive approximation. Let $X \in M_{\nu,\mu-\nu}(\mathfrak{m})$ be a solution of the congruence

$$Z := \delta(X) + PX - XS - XRX + Q \equiv 0 \ \text{modulo } \mathfrak{m}^n,$$

and let us look for a solution $X + H$ of the congruence modulo $\mathfrak{m}^{n+1}$, with $H \in M_{\nu,\mu-\nu}(\mathfrak{m}^n)$. Since $\delta(\mathfrak{m}^n) \subseteq \mathfrak{m}^{n+1}$, the latter congruence amounts to

$$PH - HS + Z \equiv 0 \ \text{modulo } \mathfrak{m}^{n+1},$$

which is indeed solvable in H by the remark at the beginning of the proof.

Uniqueness. Let $E'_1 \oplus E'_2$ be another decomposition as in the proposition. The natural homomorphism $\phi : E_1 \to E'_2$ then commutes with ∇_δ. To show that $\phi = 0$, we proceed by induction, assuming that $\phi(E_1) \subseteq \mathfrak{m}^n E'_2$. Let

$$\overline{\phi} : \overline{E}_1 \longrightarrow \mathfrak{m}^n E'_2 \otimes_R k \cong \overline{E}_2 \otimes_R (\mathfrak{m}^n/\mathfrak{m}^{n+1})$$

be the induced homomorphism. Then $(\overline{\nabla}_{\delta|\overline{E}_2} \otimes 1) \circ \overline{\phi} = \overline{\phi} \circ \overline{\nabla}_{\delta|\overline{E}_1}$, and since $\overline{\nabla}_{\delta|\overline{E}_1}$ and $\overline{\nabla}_{\delta|\overline{E}_2}$ have no common eigenvalues, $\overline{\phi} = 0$, that is, $\phi(E_1) \subseteq \mathfrak{m}^{n+1}E'_2$. So we deduce that $\phi = 0$. Similarly, the canonical morphism $E_2 \to E'_1$ is zero, and the decompositions coincide.

Stability. Consider the morphism $\phi : E_1 \to E_2$ given by the restriction to E_1 of the composition of $\nabla_{\delta'}$ with the projection onto E_2. It is easy to see that ϕ is an R-linear morphism and commutes with ∇_δ. Then the previous argument show that $\phi = 0$, so that $\nabla_{\delta'}$ is stable on E_1, and similarly for E_2. ☐

Notation. We let L_ϕ denote the differential module over the differential field $(K((x)), \partial)$ generated by one element ℓ subject to the action $\nabla_\partial(\ell) = \phi\ell$. For example, if $\partial = \frac{d}{dx}$ (resp. $\partial = \vartheta_x = x\frac{d}{dx}$), then $\ell = \exp(\psi)$ where ψ is a primitive of ϕ (resp. ψ is a primitive of ϕ/x). The Poincaré rank of L_ϕ is $\max\{0, -\mathrm{ord}_x(\phi)\}$ (resp. $\max\{0, 1 - \mathrm{ord}_x(\phi)\}$).

Proof of Theorem 16.1.2. The proof proceeds by induction on pairs ($\mu \in \mathbb{N}$, $\rho \in \frac{1}{\mu!}\mathbb{N}$), using the lexicographic order: $(\mu, \rho) < (\mu', \rho')$ if either $\mu < \mu'$, or $\mu = \mu'$

and $\rho < \rho'$. In this discussion $\mu = \dim_{K((x))} M$, $\rho = \rho(M)$, applied to any M as before.

The induction starts at $(\mu, 0)$, for any μ (it is the regular case), and at $(1, \rho)$, for any $\rho \in \mathbb{N}$ (it is the rank-one case).

Write ρ as an irreducible fraction l/m, $m \leqslant \mu$. Choose a cyclic basis $\mathbf{m} = (m, \dots, \vartheta_x^{\mu-1} m)$, and write

$$\vartheta_x \mathbf{m} = \mathbf{m} \begin{pmatrix} 0 & \cdots & 0 & a_0 \\ 1 & \cdots & 0 & a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & a_{\mu-1} \end{pmatrix}.$$

According to Corollary 15.3.6, $\rho = \max\Big\{0, \max_{j=0,\dots,\mu-1}\big(-\frac{ord_x(a_j)}{\mu-j}\big)\Big\}$. As in the proof of Theorem 15.2.2 (using $\partial = \vartheta_x$ and $\xi = x^\rho$), we then modify $\mathbf{m}$, viewed as a basis of $M_{K((x'))}$, with $x' = x^{1/m}$, to $\mathbf{n} = \mathbf{m}\Xi$, where Ξ is the diagonal matrix with entries $1, x^\rho, \dots, x^{(\mu-1)\rho}$. The matrix of $\delta := x^{\rho+1}\frac{d}{dx} = x^\rho \vartheta_x = m^{-1}(x')^{m\rho+1}\frac{d}{dx'} = m^{-1}(x')^{m\rho}\vartheta_{x'}$ in this new basis is

$$B_{-\rho} = \begin{pmatrix} 0 & \cdots & 0 & x^{\mu\rho} a_0 \\ 1 & \cdots & 0 & x^{(\mu-1)\rho} a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & x^{rho} a_{\mu-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \rho x^\rho & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\mu-1)\rho x^\rho \end{pmatrix} \in M_\mu(K[[x^{1/m}]]).$$

If $\rho = 0$, we are in the regular case, which is the starting point of the induction.

If $\rho > 0$, we may apply the splitting lemma (with $F' = \overline{K}[[x^{1/m}]]$).

If $B_{-\rho}(0)$ has at least two distinct eigenvalues in $\overline{K}$, this reduces the situation to the case of rank $< \mu$, and the induction assumption applies.

If $B_{-\rho}(0)$ has only one eigenvalue $\zeta \in \overline{K}$, this eigenvalue cannot be 0 (since $\rho > 0$, cf. (2) of 15.2.2), and in fact one has

$$x^{(\mu-j)\rho} a_j + \binom{\mu}{j}(-\zeta)^{\mu-j} \quad \in x^{1/m} K[\zeta][[x^{1/m}]] \quad \text{for all } j = 0, \dots, \mu-1.$$

Therefore,

$$\rho = -\frac{\operatorname{ord}_x(a_j)}{\mu - j}, \quad \text{for all } j = 0, \dots, \mu-1,$$

whence $m = 1$, and $\rho = l$ is a positive integer in this case.

Tensoring M with the rank-one $K((x))/K$-differential module $L_{-\zeta x^{-\rho}}$ (with generator $\ell = \exp(\frac{\zeta}{\rho} x^{-\rho})$ and action $\vartheta_x \ell = -\zeta x^{-\rho}\ell$), one checks that $\mathbf{m}' = (m' := m \otimes \ell, \dots, (x\frac{d}{dx})^{\mu-1} m')$ is a cyclic basis, since it may be written using a triangular invertible matrix in terms of the basis $\mathbf{m}\otimes\ell$, and write

$$\vartheta_x\mathbf{m}' = \mathbf{m}'\begin{pmatrix} 0 & \cdots & 0 & a'_0 \\ 1 & \cdots & 0 & a'_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & a'_{\mu-1}\end{pmatrix}.$$

Using the basis $\mathbf{n}' = \mathbf{m}'\Xi$ with Ξ as before, we have that the matrix $B'_{-\rho}$ of the operator $\delta = x^{\rho+1}\frac{d}{dx} = x^\rho\vartheta_x$ has the property that $B'_{-\rho}(0)$ is nilponent, whence $\operatorname{ord}_x(x^{(\mu-j)\rho}a'_j) > 0$ for all $j = 0,\dots,\mu-1$. We deduce that

$$-\frac{\operatorname{ord}_x(a'_j)}{\mu-j} < \rho \quad \text{for all } j = 0,\dots,\mu-1\ ,$$

hence $\rho(M\otimes L) < \rho$ and the induction assumption applies. □

17 Slopes and Newton polygons

17.1 Slope decomposition

Definition 17.1.1 (**Slopes**). *For any element $\overline{\phi}$ of $F/\partial\log F^\times$, we define the* slope *of $\overline{\phi}$ to be 0 if $\overline{\phi}=0$, and to be $-v(\overline{\phi})$ if $\overline{\phi}\neq 0$.*

The definition is well-posed, since $v(\partial\log F^\times) = 0$.

17.1.2. In the case of $F = K((x))$, we have that the slope of ϕ is $-\operatorname{ord}_x(\phi)$. If $F' = K'((x'))$ with $x'^e = x$, then $\operatorname{ord}_{x'} = e\operatorname{ord}_x$, that is $v' = ev$.

Proposition-Definition 17.1.3. *Using the notation of 16.1.2, for $\lambda\in\mathbb{Q}_{\geqslant 0}$, let us set*

$$M'_{(\lambda)} = \begin{cases} \displaystyle\bigoplus_{\operatorname{ord}_{x'}(\phi_j)\geqslant 0} M^{(\mu)}_{\phi_j}, & \text{if } \lambda = 0,\\ \displaystyle\bigoplus_{\operatorname{ord}_{x'}(\phi_j) = -e\lambda} M^{(\mu)}_{\phi_j}, & \text{if } \lambda > 0.\end{cases}$$

Then the decomposition $M_{F'} = \bigoplus_\lambda M'_{(\lambda)}$ descends to a decomposition (*the* slope decomposition) *over $K((x))[x\frac{\partial}{\partial x}]$,*

$$M = \bigoplus_\lambda M_{(\lambda)},$$

where $M'_{(\lambda)} = (M_{(\lambda)})_{K'((x'))}$. The dimension μ_λ of $M_{(\lambda)}$ as a $K((x))$-vector space is called the multiplicity of the slope λ of M. *The Poincaré rank of $M_{(\lambda)}$ is λ.*

Proof. This follows from Galois descent, cf. 16.1.4. □

Remarks 17.1.4. (1) For any $\lambda \in \mathbb{Q}_{\geqslant 0}$, $F_\lambda(M) := \bigoplus_{\lambda' \leqslant \lambda} M_{(\lambda)}$ is the maximal differential $K((x))/K$-submodule of M, of Poincaré rank $\leqslant \lambda$; its elements are those $m \in M$ for which the monic operator of minimal order

$$\Gamma_m = \vartheta_x^n - \sum_{j=0}^{n-1} a_j \vartheta_x^j, \qquad a_j \in K((x)),$$

such that $\Gamma_m(m) = 0$, satisfies, for any j, $\frac{\operatorname{ord}_x(a_j)}{n-j} \geqslant -\lambda$.

Starting from the slope filtration, there is an abstract procedure, using duality, to recover the slope decomposition , see [7, §10].

(2) As a consequence of 16.1.4 and 16.1.5, the Turrittin index e_λ of $M_{(\lambda)}$ is a divisor of l.c.m.$(2, \dots, \mu_\lambda)$, and the Turrittin index of M is a divisor of l.c.m.$(2, \dots, \max_\lambda \mu_\lambda)$.

(3) Even if the slopes of M are all integral, one may need some ramification to obtain the Turrittin-Levelt decomposition. An example is obtained by starting with an M having Turrittin index > 1, and twisting it by any $L = L_{x^{-\lambda}}$, of positive integral slope λ strictly bigger than the slopes of M. The resulting $K((x))/K$-differential module $L \otimes M$ has the single slope λ, but its characters are given by $\phi_i + x^{-\lambda}$ (the ϕ_i being the characters of M), and in particular its Turrittin index e equals the Turrittin index of M.

Lemma 17.1.5. (1) *For every λ, $M^\vee_{(\lambda)}$ is dual to $M_{(\lambda)}$.*

(2) *Let M_1 and M_2 be $K((x))/K$-differential modules purely of slopes λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$; then $\operatorname{Hom}_{K((x))\langle\vartheta_x\rangle}(M_1, M_2) = 0$.*

(3) *If $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence of differential modules, then for every λ, there is an exact sequence $0 \to M_{1(\lambda)} \to M_{(\lambda)} \to M_{2(\lambda)} \to 0$.*

Proof. This follows from the Turrittin-Levelt-Jordan decomposition and 8.1.18. ☐

Definition 17.1.6. *The submodule $M_{(0)} = F_0(M)$ is called the* regular part *of M. The Turrittin exponents of the regular part of M are also called the* Fuchs exponents *of M.*

Proposition 17.1.7. (1) *Let M be the cyclic module attached to the differential operator $\Lambda \in K((x))\langle\frac{d}{dx}\rangle$ $(F = K((x)))$. The roots of $\operatorname{ind}_{\Lambda,0}(t)$ are the Fuchs exponents.*

(2) *Let us assume that for some set Δ of derivations of K, there is an action of $\delta \in \Delta$ on M commuting with $\frac{d}{dx}$ and which makes M a differential module over $(K((x)), \delta)$. Then the Fuchs exponents of M are in K^Δ.*

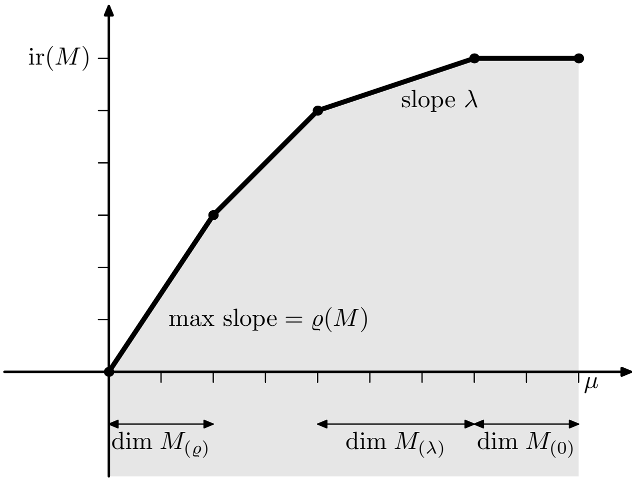
Proof. (1) One can write $\Lambda = \Lambda_{>0}\Lambda_0$, with $M_{>0} \cong F[\partial]/F[\partial]\Lambda_{>0}$, $M_{(0)} \cong F[\partial]/F[\partial]\Lambda_0$ (see 3.2.10). It is clear that $\operatorname{ind}_{\Lambda,0}(t) = \operatorname{ind}_{\Lambda_0,0}(t)$ up to a multiplicative constant.

(2) follows from (1) and Proposition 16.1.6 ☐

17.2 Newton polygons

Definition 17.2.1 (Newton polygon and irregularity). *Let* (M,∇) *be a* F/K*-differential module of rank* μ*, and let* $M=\bigoplus_\lambda M_{(\lambda)}$ *be its slope decomposition. The* Newton polygon *of* (M,∇) *is the convex hull* $\mathrm{NP}(M,\nabla)\subseteq[0,\mu]\times\mathbb{R}$ *of the vertical half-line* $\{\,x_1=0\,,\ x_2\leqslant 0\}$*, and the concave piecewise affine curve such that the edge of slope* λ *has horizontal length* μ_λ*.*

The irregularity *of* M*, denoted by* $\mathrm{ir}(M,\nabla)$*, is the maximum of the ordinates of the vertices of* $\mathrm{NP}(M,\nabla)$ (*alternatively, it is the ordinate of the extreme right vertex*)*.*



Lemma 17.2.2. *Let* M,M_1,M_2 *be* F/K*-differential modules. Then*

(i) $\mathrm{NP}(M^\vee)=\mathrm{NP}(M)$*;*

(ii) *if* $0\to M_1\to M\to M_2\to 0$ *is an exact sequence of differential modules, then* $\mathrm{NP}(M)=\mathrm{NP}(M_1)+\mathrm{NP}(M_2)$ (*algebraic sum of convex plane sets*)*. In particular,*

$$\mathrm{ir}(M)=\mathrm{ir}(M_1)+\mathrm{ir}(M_2).$$

Proof. Lemma 17.1.5 allows one to reduce to the case of a single slope, where this follows from the analogous properties for the Poincaré rank 15.3.3. ☐

Remark 17.2.3. This additivity property accounts in part for the fact that, as a measure of irregularity, $\mathrm{ir}(M)$ is more commonly used in the literature than the Poincaré rank $\rho(M)$, which is the maximal slope of $\mathrm{NP}(M)$. Another reason is its occurrence in index formulas, both in the local and the global cases [78], [35]. A third reason is the analogy between irregularity and Swan conductor in arithmetics.

For any $r, s \in \mathbb{Q}_{>0}$, let us denote by $\varphi_{r,s}$ the automorphism of $\mathbb{R}^2$ given by $\varphi_{r,s}(x, y) = (rx, sy)$.

Lemma 17.2.4. *Let F' be a finite field extension of F (endowed with the unique extension v' of the valuation of F) with ramification index e. Assume that $|-|$ is trivial on K. Then, using the notation of 15.1.23, we have the following properties:*

(i) *if M is a F/K-differential module, then $\mathrm{NP}(M_{F'}) = \varphi_{1,e}\mathrm{NP}(M)$;*

(ii) *if M' is a F'/K-differential module, then $\mathrm{NP}({}_F M') = \varphi_{d,d/e}\mathrm{NP}(M')$.*

Proof. Item (ii) of the preceding lemma allows one to reduce to the case of a single slope, where this follows from the analogous properties for the Poincaré rank 15.3.4. ☐

Remark 17.2.5 (Action of commuting derivations). Let $(M, \nabla_{x\partial_x})$ be a rank-μ differential module over F/K. Let δ be a derivation of $F = K((x))$ of norm 1 which commutes with $x\partial_x$, and let ∇_δ be an action on M commuting with $\nabla_{x\partial_x}$ such that (M, ∇_δ) is a differential module. Then we have

$$\rho_v(\nabla_\delta) \leqslant \rho_v(\nabla_{x\partial_x}) \quad \text{and} \quad \mathrm{NP}(M, \nabla_\delta) \subseteq \mathrm{NP}(M, \nabla_{x\partial_x}).$$

In particular, if $(M, \nabla_{x\partial_x})$ is regular, then is (M, ∇_δ).

Using the stability of the Turrittin-Levelt decomposition for ∇_δ, the proof is reduced to the case of an irreducible module, that is, of a Newton polygon with a single slope: then it is enough to verify the first assertion. Then by the μ-exterior product the proof is reduced to the rank-one case. In that case, using a generator m, we have $\nabla_{x\partial_x}(m) = \vartheta m$, $\nabla_\delta(m) = \eta m$ and $\delta(\vartheta) = d\partial_x(\eta)$. Then we have

$$\begin{aligned}\rho_v(\nabla_\delta) &= \max(0, -v(\eta)) = \max(0, -v(\partial_x \eta)) \\ &= \max(0, -v(\delta\vartheta)) \leqslant \max(0, -v(\vartheta)) = \rho_v(\nabla_{x\partial_x}).\end{aligned}$$

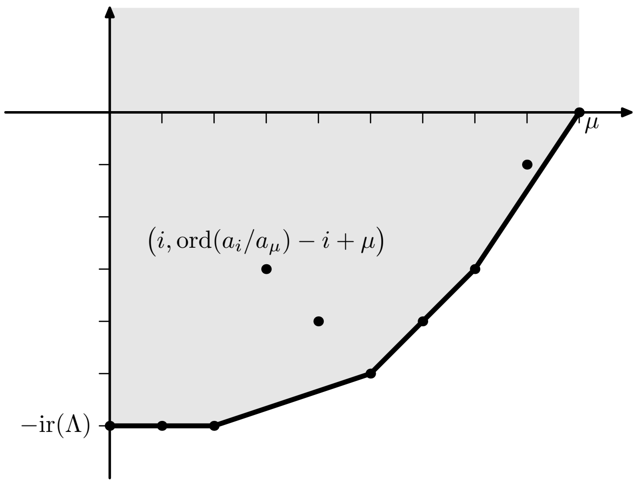
17.3 Newton polygons of cyclic modules

Definition 17.3.1 (Newton polygon and Fuchs number of differential operators). *Let $\Lambda = \sum_0^\mu a_i \frac{d^i}{dx^i}$, with $\alpha_\mu \neq 0$, be a differential operator with coefficients in $F = K((x))$. The* Newton polygon *of Λ is the convex hull $\mathrm{NP}(\Lambda)$ of the union of the vertical strips $S_0, \ldots, S_\mu$, where*

$$S_i = \{\, 0 \leqslant x_1 \leqslant i\,,\ x_2 \geqslant \mathrm{ord}_x(a_i/a_\mu) - i + \mu \,\} \subseteq \mathbb{R}^2, \quad \textit{for } i = 0, \ldots, \mu.$$

The height of $\mathrm{NP}(\Lambda)$ (i.e., the maximal distance between the ordinates of the vertices) is the Fuchs number *of Λ (cf. 6.3.3).*

Notice that the points $(0, -ir(\Lambda))$ and $(\mu, 0)$ are the extreme vertices of $\mathrm{NP}(\Lambda)$.



Remarks 17.3.2. (1) If we write

$$\frac{x^\mu}{a_\mu}\Lambda = \vartheta_x^\mu - \sum_{i=0}^{\mu-1} b_i \vartheta_x^i$$

and set $b_\mu = -1$, $\mathrm{NP}(\Lambda)$ can also be described as the convex hull of the union of the strips

$$S'_i = \{x_1 = i\,,\ x_2 \geqslant \mathrm{ord}_x(b_i)\,\} \subseteq \mathbb{R}^2, \quad \text{for } i = 0, \dots, \mu.$$

(2) The Newton polygon (resp. the irregularity) of a product of differential operators is the (algebraic) sum of the Newton polygons (resp. the irregularities) of the factors.

(3) Substituting the variable $x^{1/e}$ to x in Λ has the effect of applying $\varphi_{1,e}$ to $\mathrm{NP}(\Lambda)$ and multiplying the irregularity by e.

Theorem 17.3.3. *If $M = F\langle\partial\rangle/F\langle\partial\rangle\Lambda$ is the cyclic module attached to a differential operator Λ with respect to $\partial = x\frac{\partial}{\partial x}$, then $\mathrm{NP}(M)$ coincides with the reflection around the point $(\mu/2, 0)$ (sending (x, y) to $(\mu - x, -y)$) of the Newton polygon of the associated differential operator.*

In particular, $\mathrm{ir}(M)$ coincides with the Fuchs number of Λ and $\rho(M)$ with the maximal slope of $\mathrm{NP}(\Lambda)$.

With the above notation:

$$\begin{aligned}
\operatorname{ir}(M) &= \max\Big\{0, \max_{i=0,\dots,\mu-1}(-\operatorname{ord}_x(a_i/a_\mu)+i-\mu)\Big\}\\
&= \max\Big\{0, \max_{i=0,\dots,\mu-1}(-\operatorname{ord}_x(b_i))\Big\};\\
\rho(M) &:= \max\Big\{0, \max_{i=0,\dots,\mu-1}\big(-\frac{\operatorname{ord}_x(a_i/a_\mu)}{\mu-i}-1\big)\Big\}\\
&= \max\Big\{0, \max_{i=0,\dots,\mu-1}\big(-\frac{\operatorname{ord}_x(a_i)}{\mu-i}\big)\Big\}.
\end{aligned}$$

The reason for the reflection is a matter of conventions: for Λ, we have followed the traditional normalization of Newton polygon for (differential) polynomial, while for M we have followed the convention from the general theory of slope filtrations (see [7]). For more on the computation of these invariants, see [55].

Proof. We first recall that if $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence of F/K-differential modules, one can find Λ_1, Λ_2 and $\Lambda = \Lambda_1\Lambda_2$ in $F\langle\partial\rangle$, all monic polynomials in ∂, such that $M_i = F\langle\partial\rangle/F\langle\partial\rangle\Lambda_i$ and $M = F\langle\partial\rangle/F\langle\partial\rangle\Lambda$ (see 3.2.10). Using remarks 17.2.4 and 17.3.2, the Turrittin-Levelt-Jordan decomposition reduces the question to the rank-one case (after a ramification and taking successive extensions), which is immediate. ☐

Corollary 17.3.4. *The vertices of* $\mathrm{NP}(M)$ *have integral coordinates.*

Remark 17.3.5. Let M be a differential module over $K((X))$ with respect to $\partial = x\frac{\partial}{\partial x}$, and let L be the monic differential operator associated to a cyclic vector of M. Then, compatible with the slope decomposition

$$M = M_{(\lambda_1)} \oplus \cdots \oplus M_{(\lambda_r)}$$

with $\lambda_1 < \cdots < \lambda_r$, we have a factorization

$$L = L_{(\lambda_1)} \cdots L_{(\lambda_r)}$$

with $\mathrm{NP}(L_{(\lambda_i)}) = \mathrm{NP}(M_{(\lambda_i)})$ for $i = 1, \dots r$ (Newton polygons with only one slope). This follows from 3.2.10 and Theorem 17.3.3. For more on the formal factorization of differential operators, see [56].

17.4 Index of operators and Malgrange's definition of irregularity

Definition 17.4.1. *For any field k and any morphism of k-vector spaces $\phi : E \to F$, we say that ϕ* has an index *if both* $\operatorname{Ker}\phi$ *and* $\operatorname{Coker}\phi$ *are finite-dimensional, and we define the* index of ϕ *as the integer*

$$\chi(\phi) = \dim_k \operatorname{Ker}\phi - \dim_k \operatorname{Coker}\phi.$$

The emphasis on index theorems for differential operators is due to Malgrange, cf. [78].

Remark 17.4.2. Let (M, ∇) be a $k((x))/k$-differential module. Let $M_1 \subset M_2$ be two $k[[x]]$-lattices in M such that $\nabla(M_1) \subseteq \frac{dx}{x} \otimes M_2$. Malgrange in [78, Prop. 5.1] shows that $\nabla : M_1 \to \frac{dx}{x} \otimes M_2$ has an index and that the integer

$$i(M,\nabla) := \chi\left(\nabla : M_1 \to \frac{dx}{x} \otimes M_2\right) + \dim_k M_2/M_1 \tag{17.4.3}$$

is independent of the choice of the pair (M_1, M_2) (cf. [78, (5.3)]). He also shows that $i(M, \nabla)$ is non-negative, by reduction to the cyclic case, where an explicit formula holds [78, (5.4)]. This formula coincides with the formula for irregularity deduced by Theorem 17.3.3, and shows that the Malgrange definition of irregularity $i(M, \nabla)$ coincides with the irregularity $\mathrm{ir}(M, \nabla)$ defined in (17.2.1).

Lemma 17.4.4. *Let (M, ∇) be a $k((x))/k$-differential module. Then, there exists a free $k[x^{-1}]$-submodule $M_0 \subset M$ such that $M = M_0 \otimes_{k[x^{-1}]} k((x))$ and such that*

$$\nabla(M_0) \subseteq \frac{d\,x}{x} \otimes M_0 .$$

Moreover, $\nabla : M \to \Omega^1_{k((x))/k} \otimes M$ has an index and that index is 0.

Proof. The first part follows from the Turrittin-Levelt formal decomposition by Galois descent to $k((x))$. The second part then follows from [78, Thm. 2.1(*b*)]. One can also proceed directly as follows. Clearly, for $F' = k((x'))$ and $(M' = M_{F'}, \nabla')$ as in the Turrittin-Levelt decomposition 16.1.2, we have:

$$\chi\big(\nabla : M \longrightarrow \Omega^1_{k((x))/k} \otimes M\big) = 0 \text{ if and only if } \chi\big(\nabla' : M' \longrightarrow \Omega^1_{k((x'))/k} \otimes M'\big) = 0.$$

So, it remains to prove the statement for a $k((x))/k$-differential module of the form $(k((x)), \nabla_\phi)$, with $\phi \in k[1/x]$ and $\nabla_\phi(1) = \phi \frac{dx}{x}$. Then, ∇_ϕ is bijective unless ϕ is an integer $n \in \mathbb{Z}$, in which case $\nabla_n x^i = (n + i)\, x^i \frac{d\,x}{x}$, so that $\ker \nabla_n = k\, x^{-n}$ and $\mathrm{coker} \nabla_n$ is generated by the class of x^{-n}. ☐

17.5 Variant with parameters. Turning points

17.5.1. We consider the situation where K is the fraction field of a noetherian, integrally closed k-algebra R, and $(M, \nabla_{\vartheta_x})$ a differential module over $(R((x)) = R[[x]][\frac{1}{x}], \vartheta_x)$. Let P be a point of $\mathrm{Spec}\, R$, and let $M_{(P)}$ be the specialization of M over the differential field $(\kappa(P)((x)), x\partial_x)$.

We explore here the following question: to which extent does the Turrittin-Levelt-Jordan decomposition 16.1.2 descend to $R'((x^{1/e}))$, where R' denotes the integral closure of R in K'?

Clearly, in the rank-one case the decomposition descends without further hypothesis. However, in the general case, suitable hypothesis are needed, as the following example shows. Let consider the rank-two differential module M over $R((x))$ with basis m_1, m_2 and

$$\nabla_{\vartheta_x} m_1 = \frac{y}{x} m_1, \quad \nabla_{\vartheta_x} m_2 = -m_1.$$

Then m_2 is a cyclic vector of $(M, \nabla_{\vartheta_x})$ over $(R((x)), \vartheta_x)$ with associated differential operator $\Lambda = x\vartheta_x^2 - y\vartheta_x$. Theorem 17.3.3 and and the calculation of Remark 17.3.2 (1) show that the slopes of M are 1 and 0, so that M decomposes over $(F = K((x)), \vartheta_x)$ into a direct sum of a summand of slope 1 and one of slope 0.

For $y = 0$ we obtain an indecomposable regular differential module, in particular the decomposition does not descend from F to $k[[y]]((x))$.

17.5.2. Let $(M, \nabla_{\vartheta_x})$ be as before. We assume that, when extended to $K((x))$, $(M, \nabla_{\vartheta_x})$ has Poincaré rank ρ and Turrittin index e. It follows from Theorem 16.1.2 that there is a finite extension K'/K such that $(M, \nabla_{\vartheta_x})$, extended over $F' = K'((x^{1/e}))$, is a Jordan module over (F', ϑ_x) in the sense of Definition 8.1.12. We let $\phi_1, \dots, \phi_r$ be representatives in $K'[x^{-1/e}]$ of the distinct characters $\overline{\phi}_1, \dots, \overline{\phi}_r$ of $(M, \nabla_{\vartheta_x})$ in $F'/\vartheta_x \log(F')^\times$. We recall that the constant terms (in K', well-defined modulo $\frac{1}{e}\mathbb{Z}$) of the polynomials $\phi_1, \dots, \phi_r$ are called the Turrittin exponents of M (cf. Definition 16.1.5)

Theorem 17.5.3. *In the previous situation*

1. *The coefficient of $x^{-\rho}$ in each of $\phi_1, \dots, \phi_r$ belongs to R'.*
2. *Assume*
    1. *the differences between Turrittin exponents are constant, i.e., belong to the algebraic closure k' of k in R' (modulo $\frac{1}{e}\mathbb{Z}$),*
    2. *condition*

       (17.5.4) *the polynomials $\phi_i \in K'[x^{-1/e}]$ and their differences are invertible elements of $R'((x^{1/e}))$ unless they are 0,*

       is satisfied,

   then the Turrittin-Levelt-Jordan decomposition 16.1.2 descends to a decomposition of $M_{R'((x^{1/e}))}$:

$$M_{R'((x^{1/e}))} = \bigoplus_{i=1}^{r} R'((x^{1/e})) \otimes_{R'} \operatorname{Ker}_{M_{R'((x^{1/e}))}} (x\nabla_{\partial_x} - \phi_i)^\mu. \tag{17.5.5}$$

Proof. The proof is similar to that of 8.4.2. After completion w.r.t. a divisorial valuation we can suppose $\widehat{R} \cong \kappa[[y]]$. As in the proof of the Turrittin-Levelt decomposition theorem, we choose an element $m \in \widehat{M}$ which generates a cyclic basis $\mathbf{m}$ of $\widehat{M} \otimes_{\kappa[[y]]((x))} \kappa((y))((x))$, and modify it into a basis $\mathbf{n} = \mathbf{m}\Xi$ with

$\Xi = \begin{pmatrix} 1 & 0 & \cdot & 0 \\ 0 & x^{\rho} & \vdots & 0 \\ \cdot & \cdot & \vdots & \cdot \\ 0 & 0 & \cdot & x^{(\mu-1)\rho} \end{pmatrix}$, in which the matrix $B_{-\rho}$ of $\delta := x^{\rho+1}\frac{\partial}{\partial x}$ has entries in $\kappa((y))[[x]]$. The eigenvalues of $B_{-\rho\,|x=0}$ are the coefficients of $x^{-\rho}$ in the ϕ_j's.

Actually, we choose m as in the proof of 8.4.2 to be a lifting of a cyclic vector $\overline{m}$ of $\widehat{M} \otimes_{\kappa[[y]]((x))} \kappa((x))$, and we eliminate the apparent singularities as in loc.cit. (via a matrix $Q'' \in \mathrm{GL}_{\mu}(\kappa((y))[x]_{(x)})$). We get in this way a basis $\mathbf{n}'$ of $\widehat{M}$ which generates a δ-stable $\kappa[[x,y]]$-lattice in $\widehat{M}$. The matrix of δ in this new basis is $B'_{-\rho} = (Q'')^{-1}B_{-\rho}Q'' + (Q'')^{-1}\delta Q''$. The eigenvalues of $B'_{-\rho\,|x=0} = (Q'')^{-1}_{|x=0}B_{-\rho\,|x=0}Q''_{|x=0}$ are again the coefficients $\phi_{j,-\rho}$ of $x^{-\rho}$ in the ϕ_j's (with some multiplicity). In particular $\phi_{j,-\rho} \in \kappa[[y]]$. This proves (1).

For item (2), if there are at least two slopes (i.e., if not all ϕ_j's are of degree $\rho > 0$ in $1/x$), condition (17.5.4) implies that $\phi_{j,-\rho}$ is a unit in $\kappa[[y]]$ if and only if it is non-zero. The Splitting Lemma 16.2.1 thus applies (with the two-dimensional local ring $\widehat{R} = \kappa[[x,y]]$) and separates out the slope-ρ part of $\widehat{M}$. We may thus assume that there is a unique slope $\rho > 0$.

If there are at least two ϕ_j's (of the same slope ρ), then (17.5.4) implies that the $\phi_{j,-\rho}$'s are not all equal modulo $\mathfrak{m}$ and the Splitting Lemma applies. So, we reduce to the case of a single ϕ_i which is obvious. ☐

Remark 17.5.6. By induction on ρ after twisting by $\kappa[[y]]((x)) \cdot \exp\big(-\frac{\phi_{i,-\rho}}{\rho}x^{-\rho}\big)$, we can also prove that the characters ϕ_j's belongs to $R'[\frac{1}{x}]$.

Condition (1) could actually be omitted, but it is traditionally part of the following definition.

Definition 17.5.7 (**Turning points**)**.** *A point P of* $\operatorname{Spec} R$ *for which the stability condition* (17.5.4) *does not hold is called a turning point.*

From Krull's Hauptidealsatz, one gets:

Lemma 17.5.8. *The set of turning points form a closed subset of* $\operatorname{Spec} R$ *which is either empty, or of codimension one.*

17.6 Variation of the Newton polygon

We consider again the situation where K is the fraction field of a noetherian, integrally closed k-algebra R. We consider a differential module $(M, x\nabla_{\partial_x})$ over $(R((x)) = R[[x]][\frac{1}{x}], x\partial_x)$. For any point P of $\operatorname{Spec} R$ we let $M_{(P)}$ denote the specialization of M over the differential field $(\kappa(P)((x)), x\partial_x)$.

Proposition 17.6.1 (**Turning points in terms of** NP)**.** *In the above situation, the following conditions are equivalent:*

(1) *P is not a turning point for $(M, x\nabla_{\partial_x})$ (see 17.5.7: the characters and their differences are invertible in $R'((x^{1/e}))$ in a Zariski neighborhood of P);*

(1′) *the coefficients of the terms of minimal order of the characters ϕ_i and their differences $\phi_i - \phi_j$ are invertible in R'_P (if not zero);*

(2) *we have equalities of Newton polygons*

$$\mathrm{NP}(M_{(P)}) = \mathrm{NP}(M) \quad \textit{and} \quad \mathrm{NP}(\mathrm{End}(M_{(P)})) = \mathrm{NP}(\mathrm{End}(M));$$

(3) *we have equalities of Newton polygons*

$$\mathrm{NP}(M_C) = \mathrm{NP}(M) \quad \textit{and} \quad \mathrm{NP}(\mathrm{End}(M_C)) = \mathrm{NP}(\mathrm{End}(M))$$

for any formal curve C in $\mathrm{Spf} R[[x]]$ cutting $\mathrm{Spec}(R)$ transversally in P.

Proof. The equivalence of (1) and (1′) is standard. The condition (1′) clearly implies (2) and (3) since the slopes of the Newton polygons $\mathrm{NP}(M)$ (resp. $\mathrm{NP}(M_{(P)})$, $\mathrm{NP}(\mathrm{End}(M))$, $\mathrm{NP}(\mathrm{End}(M_{(P)}))$) are given by the order in $1/x$ of the terms ϕ_i (resp. $\phi_i(P)$, $\phi_i - \phi_j$, $(\phi_i - \phi_j)(P)$). Condition (3) clearly implies (2), so it remains to prove that (2) implies (1).

Replacing R with the completion of R_P, we may suppose that R is a complete local ring, and P the closed point of $\mathrm{Spec}(R)$.

We proceed by induction on the rank μ of M (notice that the condition (2) is stable by taking direct summand). For $\mu = 1$ or $\rho = \rho(M) = 0$ the result is obvious, so we may suppose $\rho > 0$. If there is only one character ϕ, by 17.5.3 (part (1) and the remark following the theorem) the character has coefficients in $R' \cap K = R$ and $\phi_{j,-\rho}$ is a unit in R_P. If there are at least two different characters, we choose ϕ_j one of the characters and put

$$\rho' = \max_{i \neq j} -v_x(\phi_i - \phi_j) \in (0, \rho].$$

Let $\phi_j^{\leqslant \rho'}$ denote the component of ϕ_j of degree $\leqslant \rho'$; again by 17.5.3, $\phi_j^{\leqslant \rho'}$ has coefficients in R'. Now, the differential module (over $R'((x))$)

$$M' = M \otimes L_{-\phi_j^{\leqslant \rho'}}$$

has Poincaré rank equal to ρ', equal to the Poincaré rank of $\mathrm{End}(M')$ (since M' has at least two slopes). Using the hypothesis (2), we have that $\mathrm{NP}(\mathrm{End}(M'_{(P)})) = \mathrm{NP}(\mathrm{End}(M'))$, so that the coefficients $\phi_{i,-\rho'} - \phi_{j,-\rho'}$ are zero or invertible in R'. Then we can apply the Splitting Lemma 16.2.1 using the derivation $\delta = x^{\rho'+1}\partial_x$ of $R'[[x^{1/e}]]$ to the lattice of M' generated over $R'[[x^{1/e}]]$ by a basis of M' in which the matrix of $\nabla_{x\partial_x}$ is of the form $x^{-\rho'}G(x)$ with $G(x) \in M_\mu(R'[[x^{1/e}]])$. In this way we obtain a decomposition of $M'_{R'[[x^{1/e}]]}$, hence of $M_{R'[[x^{1/e}]]}$, while respecting the hypothesis on Newton polygons, which permits to reduce μ. ☐

It turns out that at turning points, the Newton polygon can only drop. We start with an easier observation:

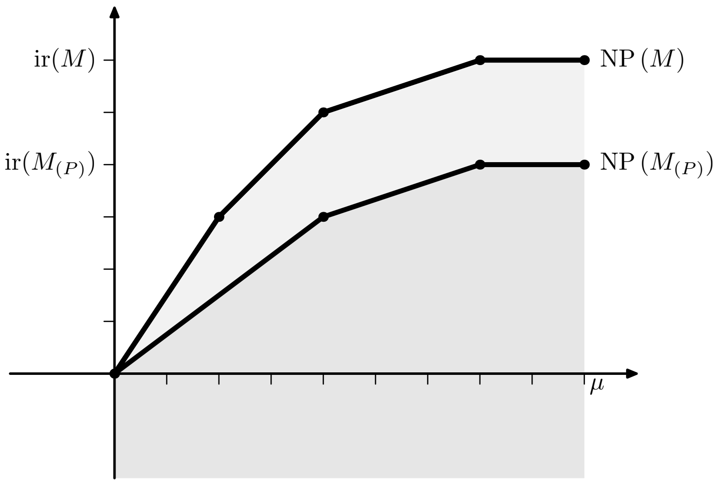
Lemma 17.6.2. $\rho(M_{(P)}) \leqslant \rho(M)$ *(in particular, if* M *is regular, so is* $M_{(P)}$*).*

Proof. Let us consider an x-adic norm for M, and the quotient norm for $M_{(P)}$. Then for every n we have the inequality

$$|\nabla^n_{(P),\vartheta_x}|_{M_{(P)}} \leqslant |\nabla^n_{\vartheta_x}|_M$$

and passing to the limit, we have the result. $\square$

Proposition 17.6.3. *In the above situation, we have* $\mathrm{NP}(M_{(P)}) \leqslant \mathrm{NP}(M)$*, that is,* $\mathrm{NP}(M_{(P)})$ *lies below* $\mathrm{NP}(M)$*, that is,* $\mathrm{NP}(M_{(P)}) \subseteq \mathrm{NP}(M)$*.*



Proof. [2] By successive specializations, we may assume that R has dimension one, in fact a discrete valuation ring with fraction field K and residue field k, and that P is the closed point of $\operatorname{Spec} R$. Then M is free of rank μ over $R((x))$, and we denote by M_K and $M_k = M_{(P)}$ the corresponding differential modules over $K((x))$ and $k((x))$, respectively. After ramification of x, we may assume that their slopes are integers, and we set

$$\begin{aligned} \mu_{(\lambda)} &= \text{ rank of the } \lambda\text{-slope part of } M \\ \overline{\mu}_{(\lambda)}, &= \text{ rank of the } \lambda\text{-slope part of } M_k\ . \end{aligned}$$

We may also suppose, eventually after ramification, that the slopes λ are integers. Following the strategy of [43], we choose an $R[[x]]$-lattice Λ of M, and define by

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[2] Taken from [6]. The "proof" proposed in the first edition of this book (II.4.2.3) was not correct, because of the use of an incorrect cyclic vector lemma, see note at the proof of 3.3.2.

induction a double sequence of lattices:

$$\begin{aligned}\Lambda_{m,0} &= \Lambda \text{ for all } m \in \mathbb{N}\ ,\\ \Lambda_{m,n+1} &= \Lambda_{m,n} + \nabla_{x^{m+1}\partial_x}(\Lambda_{m,n}).\end{aligned}$$

We notice that the quotients $\Lambda_{m,n}/\Lambda$ are R-modules of finite type, and define

$$\begin{aligned}\lambda_m &= \lim_n \frac{1}{n} \dim_K (\Lambda_{m,n})_K/\Lambda_K,\\ \overline{\lambda}_m &= \lim_n \frac{1}{n} \dim_k \operatorname{Im}((\Lambda_{m,n})_k \to M_k/\Lambda_k).\end{aligned}$$

Using the Turrittin-Levelt decomposition 16.1.2 and 15.3.6 we get the equalities

$$\lambda_m = \sum_{\lambda > m} (\lambda - m)\mu_{(\lambda)}, \quad \overline{\lambda}_m = \sum_{\lambda > m} (\lambda - m)\overline{\mu}_{(\lambda)}.$$

Moreover,

$$\lambda_m \geqslant \overline{\lambda}_m \geqslant 0.$$

Indeed[3], the $R[[x]]$-bidual $\Lambda_{m,n}^{**}$ of $\Lambda_{m,n}$ is free as an $R[[x]]$-module (since $R[[x]]$ is regular of dimension 2), and the quotient $\Lambda_{m,n}^{**}/\Lambda$ is flat as an R-module (since it is a submodule of M/Λ, which has no torsion over the discrete valuation ring R). From the exact sequence $\Lambda_{m,n}/\Lambda \to \Lambda_{m,n}^{**}/\Lambda \to \Lambda_{m,n}^{**}/\Lambda_{m,n} \to 0$ tensored with k, we get $\operatorname{Ker}((\Lambda_{m,n}^{**})_k/\Lambda_k \to (\Lambda_{m,n}^{**}/\Lambda_{m,n})_k) = \operatorname{Im}((\Lambda_{m,n})_k \to M_k/\Lambda_k)$. Therefore,

$$\begin{aligned}\dim_K((\Lambda_{m,n})_K/\Lambda_K) &= \dim_k((\Lambda_{m,n}^{**})_k/\Lambda_k)\\ &= \dim_k \operatorname{Im}((\Lambda_{m,n})_k \to M_k/\Lambda_k) + \dim_k(\Lambda_{m,n}^{**}/\Lambda_{m,n})_k,\end{aligned}$$

from which the inequality follows.

Consider now the functions $f(x)$ (resp. $\overline{f}(x)$) defined by the convex Newton polygon associated to $\mathrm{NP}(M)$ (resp. $\mathrm{NP}(M_k)$), that is, the Newton polygon associated to the corresponding differential operator. Their Legendre transforms

$$f^*(\xi) = \sup_t (t\xi - f(t)), \quad \overline{f}^*(\xi) = \sup_t (t\xi - \overline{f}(t))$$

(for $\xi \in [0, \infty[$), satisfy the Young inequalities

$$t\xi \leqslant f(t) + f^*(\xi), \quad t\xi \leqslant \overline{f}(t) + \overline{f}^*(\xi).$$

Using the previous results we have that

$$f^*(m) = \mu m + \lambda_m, \quad \overline{f}^*(m) = \mu m + \overline{\lambda}_m$$

and the Young inequality gives

$$\lambda_m \geqslant -f(t) - (\mu - t)m, \quad \overline{\lambda}_m \geqslant -\overline{f}(t) - (\mu - t)m,$$

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[3] This argument is the Deligne lemma in [36], lettre à N. Katz (1/12/1976)

for all $t \in [0, \mu]$. For $t = t_m < \mu$ a vertex of $f(t)$, and $\xi = m$ the right-slope at that vertex, the left inequality is an equality and since $\lambda_m \geqslant \overline{\lambda}_m$ we deduce that

$$-f(t_m) - (\mu - t_m)m \geqslant -\overline{f}(t_m) - (\mu - t_m)m.$$

In particular, we obtain that $f(t_m) \leqslant \overline{f}(t_m)$, from which the result follows. □

18 Varia

18.1 Cyclic vectors in the neighborhood of a non-turning singular point

Theorem 18.1.1. *Let M be a $R((x))/R$-differential module. Assume that the Turrittin index is 1, and that there is no turning point. Then M has a cyclic vector.*

Remark 18.1.2. Remark 3.3.3 shows that the assumption that there is no turning point cannot be dropped.

Proof. Let μ be the rank of M. We apply Theorem 17.5.3 to obtain a decomposition (taking into account that the Turrittin index is 1)

$$M = \bigoplus_{i=1}^{r} R((x)) \otimes_R \operatorname{Ker}_M(\nabla_{\vartheta_x} - \phi_i)^{\mu},$$

where $\nabla_{\vartheta_x} = x\nabla_{\partial_x}$, the characters ϕ_i belong to $R\big[\frac{1}{x}\big]$ and the differences $\phi_i - \phi_j$ for $i \neq j$ are invertible in $R((x))$ and not integers.

For any $\nu = 1, \ldots, \mu$, there exist elements $e_{\nu,j,1}, \ldots, e_{\nu,j,s_{\nu,j}}$ in $\operatorname{Ker}_M(\nabla_{\vartheta_x} - \phi_j)^{\nu}$ whose classes form an R-basis in the quotient $\operatorname{Ker}_M(\nabla_{\vartheta_x} - \phi_j)^{\nu}/\operatorname{Ker}_M(\nabla_{\vartheta_x} - \phi_j)^{\nu-1}$. Altogether, the elements $e_{\nu,j,k}$ form an $R((x))$-basis of M. We reorder them in lexicographical order:

$$(\nu, j, k) < (\nu', j', k') \Leftrightarrow ((\nu > \nu') \text{ or } (\nu = \nu' \,,\, j > j') \text{ or } (\nu = \nu' \,,\, j = j', \, k > k')),$$

and rename them $(m_0, \ldots, m_{\mu-1})$, so that $\nabla_{\vartheta_x} m_j = \psi_j m_j + \sum_{k>j} a_{kj} m_k$, where ψ_j is one of the ϕ_l, and $a_{kj} \in R$.

We now choose an integer $n > \mu \cdot \rho(M)$, and set

$$m := m_0 + x^{-n} m_1 + \cdots + x^{-nj} m_j + \cdots + x^{-n(\mu-1)} m_{\mu-1} \in M.$$

We shall prove that m is a cyclic vector, with the help of an auxiliary sequence $m^{(0)} := m, m^{(1)}, \ldots$. Let us set $m_j^{(0)} = x^{-nj} m_j$, and rewrite $m^{(0)} = m_0 + \sum_{j=1}^{\mu-1} m_j^{(0)}$, $\psi_j^{(0)} := \psi_j - nj$, so that

$$\nabla_{\vartheta_x} m_j^{(0)} = \psi_j^{(0)} m_j^{(0)} + \sum_{k>j} a_{kj}^{(0)} m_k^{(0)},$$

with $\mathrm{ord}_x\, a_{kj}^{(0)} > 0$ and, for $j \neq k$, $\psi_j^{(0)} - \psi_k^{(0)}$ is a unit in $R((x))$, with $\mathrm{ord}_x\, (\psi_j^{(0)} - \psi_k^{(0)}) \leqslant 0$.

Next, we define

$$m^{(1)} := \left(\psi_1^{(0)} - \psi_0^{(0)} + a_{1,0}^{(0)}\right)^{-1}\left(\nabla_{\vartheta_x} - \psi_0^{(0)}\right)m = m_1^{(0)} + \sum_{k>1} u_{k,1} m_k^{(0)},$$

where $u_{k,1} = \left(\psi_1^{(0)} - \psi_0^{(0)} + a_{1,0}^{(0)}\right)^{-1}\left(\psi_k^{(0)} - \psi_0^{(0)} + \sum_{k>j} a_{kj}^{(0)}\right)$ is a unit in $R((x))$, of x-order between $-\rho(M)$ and $+\rho(M)$. It is clear that $(m^{(0)}, m^{(1)})$, and hence also $(m, \nabla_{\vartheta_x} m)$, generate $M/\sum_{k>1} R((x))m_k$.

If we set $m_j^{(1)} = u_{j,1} m_j^{(0)}$, so that $m^{(1)} = m_1^{(0)} + \sum_{j>1} m_j^{(1)}$, $\psi_j^{(1)} = \psi_j^{(0)} + \frac{\nabla_{\vartheta_x}(u_{j,1})}{u_{j,1}}$ for $j > 1$, we get

$$\nabla_{\vartheta_x} m_j^{(1)} = \psi_j^{(1)} m_j^{(1)} + \sum_{k>j} a_{kj}^{(1)} m_k^{(1)},$$

with $a_{kj}^{(1)} = u_{k,1} a_{kj}^{(0)}$. Note that $\mathrm{ord}_x\, a_{kj}^{(1)} > 0$ (using the fact that $n > \rho(M)$) and, for $j \neq k$, $\psi_j^{(1)} - \psi_k^{(1)}$ is a unit in $R((x))$, with $\mathrm{ord}_x\left(\psi_j^{(1)} - \psi_k^{(1)}\right) \leqslant 0$.

We then construct

$$m^{(2)} := \left(\psi_2^{(1)} - \psi_1^{(1)} + a_{2,1}^{(1)}\right)^{-1}\left(\nabla_{\vartheta_x} - \psi_1^{(1)}\right)m^{(1)} = m_2^{(1)} + \sum_{k>2} u_{k,2} m_k^{(1)}.$$

It is clear that $(m^{(0)}, m^{(1)}, m^{(2)})$, hence also $(m, \nabla_{\vartheta_x} m, \nabla_{\vartheta_x}^2 m)$, generate $M/\sum_{k>2} R((x))m_k$.

Then we construct $m_j^{(2)}$, $\psi_j^{(2)}$, $a_{jk}^{(2)}$ (of positive x-order since $n > 2\rho(M)$), as before, and iterate $\mu - 1$ times. Iteration step ν shows that $(m^{(0)}, \dots, m^{(\nu)})$, hence also $(m, \dots, \nabla_{\vartheta_x}^{\nu} m)$, generate $M/\sum_{k>\nu} R((x))m_k$. For $\nu = \mu$, we conclude that m is cyclic. ☐

18.2 Turrittin decomposition around crossing points of the polar divisor

18.2.1. We finish this chapter with an exploration of the Turrittin decomposition around a crossing point of the polar divisor of an integrable connection in two variables. We will need these considerations in the next chapter.

For deeper work in this direction (notably Sabbah's conjecture and its proof), we refer to [91], [83] and [66]).

We set $Y = \mathrm{Spf}\, k[[y_1, y_2]]$, $D =$ the divisor $xy = 0$, $P = Y_{\mathrm{red}}$ the crossing point $(y_1 = y_2 = 0)$, and a finitely generated $k[[y_1, y_2]][\frac{1}{y_1 y_2}]$-module M with integrable connection ∇ (w.r.t. continuous derivations of $k[[y_1,y_2]]$). Any such module is projective, and even free since $k[[y_1,y_2]][\frac{1}{y_1y_2}]$ is a principal domain.

In the rank-one case, such a module is determined by $\omega := \nabla(1) \in \Omega^1_Y(*D)$, a closed form. If we set $\vartheta_i = y_i\partial_{y_i}$, $\psi_i = \nabla_{\vartheta_i}(\omega) = \langle\vartheta_i,\omega\rangle \in \mathcal{O}_Y(*D)$ for $i=1,2$, the integrability condition translates into the equation $\vartheta_1(\psi_2) = \vartheta_2(\psi_1)$. We denote this module by L_ω.

18.2.2 (**Nice formal structure).** Any M as above gives rise to a differential module M_{F_1} over $F_1 = k((y_2))((y_1))$ (resp. M_{F_2} over $F_2 = k((y_1))((y_2))$). After ramification $(y_1,y_2) \mapsto (y_1^{\frac{1}{e}}, y_2^{\frac{1}{e}})$ (with an index e dividing $\mu!$), one has the Turrittin decomposition $M_{F_1} = \bigoplus_j M^{(\mu)}_{\phi_{1,j}}$, with $\phi_{1,j} \in \frac{1}{y_1}k((y_2))[\frac{1}{y_1}]$ of order at least $-\rho_1$ w.r.t. y_1 and equal to $-\rho_1$ for at least one index; similarly for M_{F_2}.

We say that M has *a nice formal structure at* P[4] if there exists a decomposition

$$M = \bigoplus_h M_{\overline{\omega}_h} \quad \text{with} \quad M_{\overline{\omega}_h} \cong L_{\omega_h} \otimes R_h,$$

where $\omega_h \in \Omega^1_Y(*D)$ are closed and have distinct classes $\overline{\omega}_h$ modulo $\Omega^1_Y(\log D)$, L_{ω_h} is the rank-one integrable connection associated to ω_h, and R_h is a free module of finite rank over $\mathcal{O}_Y(*D)$ with a regular integrable connection (w.r.t. ϑ_1, ϑ_2).

In this situation, the decomposition is unique and induces, after extension to F_1 and F_2, a refinement of the Turrittin-Levelt-Jordan decompositions:

$$M^{(\mu)}_{\phi_{1,j}} = \bigoplus_{h:\pi_1(\psi_{1,h})=\phi_{1,j}} M_{\overline{\omega}_h} \otimes F_1 \quad \text{and} \quad M^{(\mu)}_{\phi_{2,j}} = \bigoplus_{h:\pi_2(\psi_{2,h})=\phi_{2,j}} M_{\overline{\omega}_h} \otimes F_2. \tag{18.2.3}$$

Here π_1 and π_2 refer to the canonical projections

$$\begin{aligned} \pi_1 &: \frac{k[[y_1,y_2]]\big[\frac{1}{y_1y_2}\big]}{k[[y_1,y_2]]} \longrightarrow \tfrac{1}{y_1}k((y_2))\big[\tfrac{1}{y_1}\big] \quad \text{and} \\ \pi_2 &: \frac{k[[y_1,y_2]]\big[\frac{1}{y_1y_2}\big]}{k[[y_1,y_2]]} \longrightarrow \tfrac{1}{y_2}k((y_1))\big[\tfrac{1}{y_2}\big]. \end{aligned}$$

18.2.4. Let us consider a sequence of formal blow-ups

$$\pi : Y' \longrightarrow Y$$

of P, then of crossing points of successive exceptional divisors. One knows that such a sequence of *toric* blow-ups corresponds to a regular fan of the first quadrant of $\mathbb{R}^2$. Toric charts are isomorphic to $\mathbb{A}^2$ (completed). The trace of the pull-back of D in such a chart is the union of the coordinate axes. In the one associated

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[4] cf. [5]; this is equivalent to saying that P is semi-stable for M in the sense of [6], but weaker than saying that M has a good formal structure in the sense of [91].

to the cone bounded by edges passing through $(a,b)\in\mathbb{N}^2$ and $(c,d)\in\mathbb{N}^2$, with $ad-bc=1$, adapted coordinates (y_1',y_2') are given by

$$y_1=(y_1')^a(y_2')^b,\quad y_2=(y_1')^c(y_2')^d. \tag{18.2.5}$$

If $\omega=\phi_1\frac{dy_1}{y_1}+\phi_2\frac{dy_2}{y_2}$, then $\pi^*\omega=\phi_1'\frac{dy_1'}{y_1'}+\phi_2'\frac{dy_2'}{y_2'}$, where

$$\phi_1'=a.\phi_1+c.\phi_2\ ,\qquad \phi_2'=b.\phi_1+d.\phi_2. \tag{18.2.6}$$

Dually,

$$y_1\frac{\partial}{\partial y_1}=\pi_*\Big(d\,y_1'\frac{\partial}{\partial y_1'}-c\,y_2'\frac{\partial}{\partial y_2'}\Big),\qquad y_2\frac{\partial}{\partial y_2}=\pi_*\Big(-b\,y_1'\frac{\partial}{\partial y_1'}+a\,y_2'\frac{\partial}{\partial y_2'}\Big). \tag{18.2.7}$$

Note also that the ramification $(y_1,y_2)\mapsto(y_1^{\frac1e},y_2^{\frac1e})$ commutes with toric blow-ups.

Proposition 18.2.8 (Nice formal structure by toric blow-ups). *There exists a finite sequence of formal toric blow-ups $\pi:Y'\to Y$ (starting with a blow-up of P) such that π^*M has a nice formal structure at any crossing point P' of $\pi^{-1}(P)$ (after ramification in y_1 and y_2).*

This result was first proved by Sabbah (in [91, III,4.3.1]) using a generalization in dimension 2 of the nilpotent orbit method of Babbitt and Varadarajan. We sketch here the proof given in [6, 5.4], which only uses elementary calculations on toric blow-ups.

Proof (Sketch). We proceed by induction on the rank μ of M (there is nothing to prove for $\mu=1$). For any $\Lambda\subseteq k$, we denote by $k[\lambda]_\Lambda$ the localization $k[\lambda,\frac{1}{\lambda-\lambda_0}]_{\lambda_0\in\Lambda}$.

18.2.9. *Claim. After ramification, there exist a sequence π' of toric blow-ups, a finite set $\Lambda\subseteq k$ and, in each toric chart, a basis $\mathbf{n}(\lambda)$ of the pull-back of*

$$M^\Lambda:=M\otimes_{k[[y_1,y_2]][\frac{1}{y_1y_2}]}k[\lambda]_\Lambda[[y_1,y_2]]\big[\frac{1}{y_1y_2}\big]$$

in which the matrix of

$$\vartheta(\lambda):=\vartheta_1+\lambda\vartheta_2$$

has no pole or can be written, in adapted coordinates (y_1',y_2'), in the form

$$(y_1')^{-r_1}(y_2')^{-r_2}G(\lambda,y_1',y_2'),$$

where $r_1,r_2\geqslant 0$ (not both 0), $G(\lambda,y_1',y_2')\in M_\mu(k[\lambda]_\Lambda[[y_1',y_2']])$, and where for any $\lambda_0\notin\Lambda$, $G(\lambda_0,0,0)\in M_\mu(k)$ is not nilpotent.

Proof of the claim. Let $m \in M$ be a cyclic vector of $M \otimes \mathrm{Frac}(k[[y_1, y_2]])$ w.r.t. $\vartheta(0) = \vartheta_1$. In some basis of $\bigwedge^\mu M \cong k[\lambda][[y_1, y_2]]\big[\frac{1}{y_1 y_2}\big]$, one can write

$$m \wedge \vartheta(0)m \wedge \cdots \wedge \vartheta(0)^{\mu-1}m = g(y_1, y_2) \in k[[y_1, y_2]],$$

where g is not divisible by y_1 or y_2. On the other hand, we may write $m \wedge \vartheta(\lambda)m \wedge \cdots \wedge \vartheta(\lambda)^{\mu-1}m = y_1^{-s_1} y_2^{-s_2}.g(\lambda, y_1, y_2) \in k[\lambda][[y_1, y_2]]$, where $g(\lambda, y_1, y_2)$ is not divisible by y_1 or y_2 and $g(0, y_1, y_2) = y_1^{s_1} y_2^{s_2} g(y_1, y_2) \neq 0$.

Then for every λ_0 outside a finite set $\Lambda' \subseteq k$,

$$\mathrm{ord}_{y_1} g(\lambda_0, y_1, 0) = \mathrm{ord}_{y_1} g(\lambda, y_1, 0), \quad \mathrm{ord}_{y_2} g(\lambda_0, 0, y_2) = \mathrm{ord}_{y_2} g(\lambda, 0, y_2),$$

and there exists a sequence π'' of toric blow-ups such that in each chart, and for every $\lambda_0 \notin \Lambda'$, the strict transform of $g(\lambda_0, y_1, y_2) = 0$ does not meet any crossing of $(\pi'')^{-1}(P)$. It follows that

$$\mathbf{m}(\lambda) = (m, \vartheta(\lambda)(m), \ldots, \vartheta(\lambda)^{\mu-1}(m))$$

is a cyclic basis of the pull-back of $M^{\Lambda'}$ w.r.t. $\vartheta(\lambda)$, and induces a cyclic basis after specialization at every $\lambda_0 \notin \Lambda'$. The same remains true if one performs a ramification $(y_1, y_2) \mapsto (y_1^{\frac{1}{e}}, y_2^{\frac{1}{e}})$ at the beginning ($\vartheta(\lambda)$ becomes $\vartheta(\lambda)/e$), which allows us to assume that the Poincaré ranks ρ_1, ρ_2 along $y'_1 = 0, y'_2 = 0$ (in each toric chart) are integers.

Let us then modify $\mathbf{m}(\lambda)$ into a basis

$$\mathbf{m}(\lambda) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & (y'_1)^{\rho_1}(y'_2)^{\rho_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (y'_1)^{\rho_1(\mu-1)}(y'_2)^{\rho_2(\mu-1)} \end{pmatrix}.$$

Let us write the matrix of $\vartheta(\lambda)$ in this new basis in the form

$$y'^{-\rho_1}_1 y'^{-\rho_2}_2 H(\lambda, y'_1, y'_2)$$

with $H(\lambda, y'_1, y'_2) \in M_\mu(k[\lambda]_{\Lambda'}[[y'_1, y'_2]][\frac{1}{y'_1 y'_2}])$. The theorem of Dwork, Katz and Turrittin 15.2.2 applies (over the complete field $k(\lambda)((y'_2))((y'_1))$ endowed with the y'_1-adic valuation), and shows that the entries of $H(\lambda, y'_1, y'_2)$ lie in $k(\lambda)((y'_2))[[y'_1]]$ and in case $\rho_1 > 0$, $H(\lambda, 0, y'_2)$ is not nilpotent (item 2.). Idem by exchanging y'_2 and y'_1.

Moreover, if $\rho_1 = 0$ (and similarly if $\rho_2 = 0$), one can still reduce to the case where $H(\lambda, 0, y'_2)$ is not nilpotent on multiplying the basis by y'_1 (which adds to H the matrix $(y'_2)^{\rho_2} \frac{\vartheta(\lambda)(y'_1)}{y'_1} I_\mu$).

One thus finds a basis $\mathbf{n}(\lambda)$ in which the matrix of $\vartheta(\lambda)$ has no pole if $\rho_1 = \rho_2 = 0$, and otherwise has the form $y'^{-\rho_1}_1 y'^{-\rho_2}_2 H(\lambda, y'_1, y'_2)$, with $H(\lambda, y'_1, y'_2) \in M_\mu(k[\lambda]_{\Lambda'}[[y'_1, y'_2]])$, $H(\lambda, 0, y'_2)$ being $H(\lambda, y'_1, 0)$ non-nilpotent. In particular,

$$h(\lambda_0, y'_1, y'_2, t) := \det(t.I_\mu - H(\lambda_0, y'_1, y'_2)) - t^\mu \in k[[y'_1, y'_2]][t]$$

is not divisible by ${y'}_1$ or ${y'}_2$, and the same holds if one specializes t at some $t_0 \in k$. Therefore, for every λ_0 outside a finite set $\Lambda \supseteq \Lambda'$, one has

$$\begin{aligned}\operatorname{ord}_{y'_1} h(\lambda_0, y'_1, 0, t_0) &= \operatorname{ord}_{y'_1} h(\lambda, y'_1, 0, t_0),\\ \operatorname{ord}_{y'_2} h(\lambda_0, 0, y'_2, t_0) &= \operatorname{ord}_{y'_2} h(\lambda, 0, y'_2, t_0).\end{aligned}$$

Thanks to these uniform bounds, there exists a sequence of toric blow-ups such that in each chart and for every $\lambda_0 \notin \Lambda$, the strict transform of $h(\lambda_0, {y'}_1, {y'}_2, t_0) = 0$ does not meet any crossing of the divisor above P. Each of these charts admits adapted coordinates y''_1, y''_2 with $y'_1 = (y''_1)^a (y''_2)^b$, $y'_2 = (y''_1)^c (y''_2)^d$, $a, b, c, d \geqslant 0$, $ad - bc = 1$. It follows that in the basis $\mathbf{n}(\lambda)$, the matrix of $\vartheta(\lambda)$ is of the form $(y''_1)^{-r_1}(y''_2)^{-r_2} G(\lambda, y''_1, y''_2)$ with $G(\lambda, y''_1, y''_2) = H(\lambda, {y'}_1, {y'}_2) \in M_\mu(k[\lambda]_\Lambda[[y''_1, y''_2]])$, and

$$\det(t.I_\mu - G(\lambda_0, y''_1, y''_2)) - t^\mu = h(\lambda_0, {y'}_1, {y'}_2, t) \in k[[y''_1, y''_2]][t]$$

does not vanish at $y''_1 = y''_2 = 0$. Therefore, for every $\lambda_0 \notin \Lambda$, $G(\lambda_0, 0, 0) \in M_\mu(k)$ is not nilpotent, which proves the claim. $\square$

From here, the proof of the proposition follows [91, III.4.3.1]: twisting M by a suitable L_ω, we may assume that $\bigwedge^\mu M$ is regular. One completes the sequence of toric blow-ups by refining the corresponding fan in such a way that each cone has an edge with primitive vector (a, c), the image of a in k being non-zero and the image of c/a being outside Λ, and such that the associated divisor is a component of the polar divisor of $\vartheta(c/a)$ (if the latter is non-empty).

Let us fix a toric chart, with adapted coordinates y'_1, y'_2. Assume that y'_1 corresponds to an edge whose slope $\lambda_0 = c/a$ lies outside Λ. In the basis $\mathbf{n}(c/a)$ of the inverse of M, the matrix of $y'_1 \frac{\partial}{\partial y'_1} = a.\vartheta(c/a)$ is of the form $(y'_1)^{-r_1}(y'_2)^{-r_2} G(y'_1, y'_2)$ with $G(y'_1, y'_2) \in M_\mu(k[[y'_1, y'_2]])$, and $G(0,0)$ is not nilpotent if r_1 and r_2 are not both 0. Since $\bigwedge^\mu M$ is regular, $G(0,0)$ has trace 0, hence has two distinct eigenvalues. If r_1 and r_2 are not both 0, the Splitting Lemma 16.2.1 applies to the derivations $\delta = (y'_1)^{r_1+1}(y'_2)^{r_2} \frac{\partial}{\partial y'_1}$ and $\delta' = (y'_1)^{s_1}(y'_2)^{s_2+1} \frac{\partial}{\partial y'_2}$ of $k[[y'_1, y'_2]]$ (for suitable $s_1, s_2 \geqslant 0$) and to the $k[[y'_1, y'_2]]$-lattice spanned by $\mathbf{n}(c/a)$. This yields a decomposition of M, which allows us to reduce to smaller rank μ. If, on the contrary, $r_1 = r_2 = 0$, the pull-back of M is still regular along $y'_1 = 0$ and $y'_2 = 0$. $\square$