

Quasistatic evolution for a phase-transition model: a Young measure approach

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A quasistatic evolution problem for a phase transition model with nonconvex energy density is considered in terms of Young measures. We focus on the particular case of a finite number of phases. The new feature consists in the use of suitable regularity arguments in order to prove an existence result for a generalized notion of evolution.

1 Introduction

In the last years the energetic formulation of rate-independent processes has been widely used to describe mesoscopic models for the isothermal stress-induced transformation in crystalline materials (see [8] and references therein).

We consider a material which can assume only a finite number of different phases (or phase-variants). The *stored-energy* density is assumed to depend on the phase state and on the elastic deformation of the material, and has a multiple-well potential form (see [9], [7], [5]). More in general we deal with a density which does not satisfy any convexity assumption with respect to the parameter describing the phase state of the material. We assume that changes of the phase distribution of the material lead to an energy *dissipation*. Moreover, we require that the admissible deformations satisfy a prescribed time-dependent boundary condition, which we impose on the whole boundary of the reference configuration to avoid some technical difficulties. For the same reason, we neglect any contribution due to external forces.

As in [3] (where the case of a material with infinitely many phases is studied), the lack of convexity of the stored energy gives rise to many technical difficulties, making unsolvable in usual functional spaces the incremental minimum problems used in the construction of approximate solutions (see [8] and references therein).

To avoid any artificial regularization, in this paper we follow the same approach as in [3], and set the problem in a suitable space of Young measures, where the incremental minimum problems can be solved. To find a correct extension to the Young measure setting of the

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dissipation functional and of the corresponding notion of total dissipation on a time interval, we need to use the tool of compatible systems of Young measures (see [1] and [3]).

The aim of the paper is to prove an existence result for a notion of quasistatic evolution of Young measures, characterized by an admissibility condition and by suitably reformulated stability condition and energy equality. The stability condition is a global minimality condition, but the set of competitors is a proper subset of the admissible measures, represented by rearrangements of the phase distribution and translations of the Young measure corresponding to the deformation.

Thanks to the fact that the material can assume just a finite number of different phases, the notion of evolution introduced in this paper presents some improvements with respect to the one considered in [3]. Indeed, under weaker assumption on W , the stability condition considered here allows us to compare the evolution with a quite large set of competitors, including all possible rearrangements of the phase distribution. Moreover, it is possible to obtain not only an upper energy estimate as in [3], but a complete energy balance.

The proof of the existence theorem (Theorem 4.2) follows the classical scheme of time-discretization, resolution of incremental minimum problems, and passage to the limit of the approximate solutions.

The main new feature with respect to [3] concerns the use of regularity results for quasi-minima of integral functionals (see [6]) to select special solutions to the discretized minimum problems. The proof of the lower energy estimate for the limit of the approximate solutions presents some new aspects too.

For a more detailed discussion on these results we refer to [4].

2 The mechanical model

Assuming that the reference configuration of the crystalline material is a bounded region $D \subset \mathbb{R}^d$, the state of the system is determined by two functions: the deformation $v: D \rightarrow \mathbb{R}^N$ and the internal variable $z: D \rightarrow Z \subset \mathbb{R}$, which takes into account the phase transformations of the material.

In our framework, Z is the finite set $\{1, \dots, q\}$, representing the different phases (or phase variants) of the crystal, and $z: D \rightarrow Z$ represents the phase distribution of the material. Then the stored energy of the system is

$$\mathcal{W}(z, v) := \int_D W(z(x), \nabla v(x)) \, dx,$$

where $W: Z \times \mathbb{R}^{N \times d} \rightarrow [0, +\infty)$ is a C^1 -function satisfying

$$c|F|^2 - C \leq W(\alpha, F) \leq C(1 + |F|^2), \quad (1)$$

for suitable positive constants c and C , and for every $\alpha \in Z$ and $F \in \mathbb{R}^{N \times d}$. The energy dissipated when the phase distribution of the material changes is represented by

$$\int_D H(z_{new}(x), z_{old}(x)) \, dx,$$

where H is a metric distance on Z , z_{old} is the old phase distribution and z_{new} the new one.

The prescribed boundary datum on ∂D at time t is denoted by $\varphi(t)$; φ is assumed to be an absolutely continuous function on $[0, T]$ with values in $W^{1,p}(D; \mathbb{R}^N)$, with $2 < p < +\infty$.

3 Young measures and discrete sets of values

For the mathematical preliminaries about measures and Young measures we refer to [3, Section 2 and 4]. It is easy to see that any Young measure μ on D with values in Z can be written as $\mu = \sum_{\alpha \in Z} b_\alpha \delta_\alpha$ where b_α are functions in $L^\infty(D; [0, 1])$ satisfying the condition $\sum_{\alpha \in Z} b_\alpha(x) = 1$, for a.e. $x \in D$. Therefore the set $Y(D; Z)$ of Young measures on D with values in Z can be identified with the set of all families $b = (b_\alpha)_{\alpha \in Z}$ in $L^\infty(D; [0, 1])$ satisfying the condition above. Analogously, we can identify the space of compatible systems of Young measures on D with values in Z and time set $[0, T]$ with the set of all families $\mathbf{b} = (\mathbf{b}_{\alpha_1 \dots \alpha_n}^{t_1 \dots t_n})$ of functions in $L^\infty(D; [0, 1])$, with $t_1 < \dots < t_n$ varying in $[0, T]$ and $(\alpha_1, \dots, \alpha_n) \in Z^n$, satisfying the condition

$$\sum_{(\alpha_1, \dots, \alpha_n) \in Z^n} \mathbf{b}_{\alpha_1 \dots \alpha_n}^{t_1 \dots t_n}(x) = 1,$$

for every $t_1 < \dots < t_n$ in $[0, T]$ and a.e. $x \in D$, and a suitable projection property (for the precise definition, see [4, Section 3]).

The dissipation $\text{Diss}_H(\mathbf{b}; c, d)$ is defined by

$$\text{Diss}_H(\mathbf{b}; c, d) := \sup \sum_{i=1}^k \sum_{\alpha\beta} H(\beta, \alpha) \int_D \mathbf{b}_{\alpha\beta}^{s_{i-1}s_i}(x) dx, \quad (2)$$

where the supremum is taken over all partitions $c = s_0 < \dots < s_k = d$ of the interval $[c, d]$.

The above definition justifies the use of compatible systems of Young measures: the energetic effect of the phase transitions occurring from the instant s_{i-1} to the instant s_i can only be described by an object $\mathbf{b}_{\alpha\beta}^{s_{i-1}s_i}$, representing the volume fraction at x undergoing the phase transformation from α to β . The knowledge of $\mathbf{b}_\alpha^{s_{i-1}}$ and $\mathbf{b}_\beta^{s_i}$ separately does not keep the complete information about the energy spent in the transition. Indeed, if we consider the case of a homogeneous phase distribution $\mathbf{b}_\alpha^{s_{i-1}} = 1/q$ for every α , and we suppose that the material undergoes a transition from s_{i-1} to s_i just permuting the phases and leaving the volume fractions unchanged, we have $\mathbf{b}^{s_i} = \mathbf{b}^{s_{i-1}}$. Hence the dissipation computed using only $\mathbf{b}^{s_{i-1}}$ and \mathbf{b}^{s_i} is 0, while the dissipation energy computed using $\mathbf{b}^{s_{i-1}s_i}$ depends on the permutation and it is different from zero. Therefore, the previous description seems to give a more realistic picture of the dissipation phenomenon, if compared, for instance, with the one proposed in [7, Section 5], which only takes into account the contribution of single time instants.

4 Definition of quasistatic evolution and main result.

First of all we fix some notation and give the definition of quasistatic evolution.

We define the set of admissible pairs $Ad([0, T], \varphi)$ as the set of all pairs $(\mathbf{b}, \boldsymbol{\lambda})$ formed by a compatible system \mathbf{b} on D with values in Z and time set $[0, T]$ and a time-dependent family $\boldsymbol{\lambda}$ of Young measures on D with values in $\mathbb{R}^{N \times d}$, which can be suitably approximated by means of time-dependent functions satisfying the boundary condition (for the precise definition and the properties of the admissible set, see [4, Section 5]).

Given a measurable map M defined on D with values in the set of $q \times q$ -stochastic matrices, a function $\tilde{u} \in H_0^1(D; \mathbb{R}^{N \times d})$, an admissible pair $(\mathbf{b}, \boldsymbol{\lambda})$, and a time instant $t \in [0, T]$, we

define a competitor $(\tilde{b}^{M,\tilde{u}}, \tilde{\lambda}^{M,\tilde{u}})$ for the stability condition at time t by setting

$$\begin{aligned}\tilde{b}_\beta^{M,\tilde{u}} &:= \sum_\alpha M_{\beta\alpha} \mathbf{b}_\alpha^t \quad \text{a.e. in } D \\ (\tilde{\lambda}_\beta^{M,\tilde{u}})^x &:= \frac{\sum_\alpha M_{\beta\alpha}(x) \mathbf{b}_\alpha^t(x) T_{\nabla\tilde{u}(x)}(\boldsymbol{\lambda}_\alpha^t)^x}{\sum_\alpha M_{\beta\alpha}(x) \mathbf{b}_\alpha^t(x)} \quad \text{if } \sum_\alpha M_{\beta\alpha}(x) \mathbf{b}_\alpha^t(x) > 0,\end{aligned}$$

for a.e. $x \in D$ and every $\beta = 1, \dots, q$, where $T_{\nabla\tilde{u}(x)}$ represents a translation operator on the set of probability measures on $\mathbb{R}^{N \times d}$.

We also set

$$\begin{aligned}\langle W, (\mathbf{b}^t, \boldsymbol{\lambda}^t) \rangle &:= \sum_\alpha \int_{D \times \mathbb{R}^{N \times d}} \mathbf{b}_\alpha^t(x) W(\alpha, F) \, d\boldsymbol{\lambda}_\alpha^t(x, F), \\ \mathbb{H}(\tilde{b}^{M,\tilde{u}}, \mathbf{b}^t) &:= \sum_{\alpha,\beta} H(\beta, \alpha) \int_D M_{\beta\alpha}(x) \mathbf{b}_\alpha^t(x) \, dx.\end{aligned}$$

Definition 4.1 Given $\varphi \in AC([0, T]; W^{1,p}(D; \mathbb{R}^N))$, for $2 < p < +\infty$, $T > 0$, $z_0 \in L^\infty(D; Z)$, and $v_0 \in \varphi(0) + H_0^1(D; \mathbb{R}^N)$, a *quasistatic evolution of Young measures* with boundary datum φ and initial condition (z_0, v_0) , in the time interval $[0, T]$, is an admissible pair $(\mathbf{b}, \boldsymbol{\lambda}) \in Ad(D, \varphi)$, satisfying the following conditions:

(ev0) *initial condition*: with $D_\alpha^0 := \{x \in D : z_0(x) = \alpha\}$, we have $\mathbf{b}_\alpha^0 = 1_{D_\alpha^0}$ and $(\boldsymbol{\lambda}_\alpha^0)^x = \delta_{\nabla v_0(x)}$ if $x \in D_\alpha^0$, for every α ;

(ev1) *partial-global stability*: for every $t \in [0, T]$, we have

$$\langle W, (\mathbf{b}^t, \boldsymbol{\lambda}^t) \rangle \leq \langle W, (\tilde{b}^{M,\tilde{u}}, \tilde{\lambda}^{M,\tilde{u}}) \rangle + \mathbb{H}(\tilde{b}^{M,\tilde{u}}, \mathbf{b}^t),$$

for every $\tilde{u} \in H_0^1(D; \mathbb{R}^N)$, and every measurable map M on D with values in the set of $q \times q$ -stochastic matrices.

(ev2) *energy equality*: for every $t \in [0, T]$, we have

$$\langle W, (\mathbf{b}^t, \boldsymbol{\lambda}^t) \rangle + \text{Diss}_H(\mathbf{b}; 0, t) = \mathcal{W}(z_0, v_0) + \int_0^t \langle \boldsymbol{\sigma}(s), \nabla \dot{\varphi}(s) \rangle_2 \, ds,$$

where $\boldsymbol{\sigma}$ represents the stress of the system and $\text{Diss}_H(\mathbf{b}; t_1, t_2)$ is defined by (2).

Theorem 4.2 Let $\varphi \in AC([0, T]; H^1(D; \mathbb{R}^N))$ and $T > 0$. Assume that the partial-global stability condition is satisfied by $(z_0, v_0) \in L^\infty(D; Z) \times (\varphi(0) + H_0^1(D; \mathbb{R}^N))$. Then there exists a quasistatic evolution of Young measures with boundary datum φ and initial condition (z_0, v_0) in the time interval $[0, T]$.

5 Proof of the main theorem

The proof is obtained via time-discretization, resolution of incremental minimum problems, and passing to the limit.

5.1 The incremental minimum problem and the regularity argument

The first step of the proof consists in the resolution of a chain of discretized minimum problems.

Let us fix a partition of $[0, T]$, $0 = t^0 < t^1 < \dots < t^k = T$. We set $\mathbf{b}_\alpha^0 \boldsymbol{\lambda}_\alpha^0 := 1_{D_\alpha^0} \boldsymbol{\delta}_{\nabla v_0}$, where $D_\alpha^0 := \{x \in D : z_0(x) = \alpha\}$, and define inductively $(\mathbf{b}^{t^i}, \boldsymbol{\lambda}^{t^i})$ as a minimizer of the functional

$$\begin{aligned} & \langle W, (\mathbf{b}^{t^i}, \boldsymbol{\lambda}^{t^i}) \rangle + \langle H, \mathbf{b}^{t^{i-1}t^i} \rangle \\ := & \sum_\alpha \int_{D \times \mathbb{R}^{N \times d}} \mathbf{b}_\alpha^{t^i}(x) W(\alpha, F) d\boldsymbol{\lambda}_\alpha^{t^i}(x, F) + \sum_{\alpha, \beta} H(\beta, \alpha) \int_D \mathbf{b}_{\alpha\beta}^{t^{i-1}t^i}(x) dx, \end{aligned} \quad (3)$$

among the admissible pairs $(\mathbf{b}, \boldsymbol{\lambda})$, satisfying suitable “memory properties” with respect to $(\mathbf{b}^{t^{i-1}}, \boldsymbol{\lambda}^{t^{i-1}})$ (see [4, Section 7.1]).

The inductive procedure produces a sequence of minimizers. Ekeland Variational Principle allows us to manipulate it in order to obtain minimizers which can be approximated by quasi-minima of the integral functional $\mathcal{F}(v) = \int_D (1 + |\nabla v(x)|^2) dx$.

The regularity results proven by Giaquinta and Giusti for quasi-minima of this functional (see [6], and [4, Appendix] for further details concerning our specific functional) ensures a uniform bound on the r -moments of the selected minimizers, for some $r > 2$, more precisely we have

$$\begin{aligned} & \sum_\alpha \int_{D \times \mathbb{R}^{N \times d}} \mathbf{b}_\alpha^{t^i}(x) |F|^r d\boldsymbol{\lambda}_\alpha^{t^i}(x, F) \\ \leq & \gamma \left[1 + \left(\int_{D \times \mathbb{R}^{N \times d}} \mathbf{b}_\alpha^{t^i}(x) |F|^2 d\boldsymbol{\lambda}_\alpha^{t^i}(x, F) \right)^{r/2} \right], \end{aligned} \quad (4)$$

for a positive constant γ independent of i and of the partition t^0, \dots, t^k .

5.2 The passage to the limit and the continuity of the energy functional

A suitable interpolation of the sequence of minimizers constructed in the previous section gives an approximate solution for the given time discretization.

We consider now a sequence of finer and finer subdivisions of the time interval and the related sequence of approximate solutions. The minimality property of these approximations provides an a priori estimate on their second moments and on the total variation of the compatible systems. These uniform bounds ensure the convergence of a suitably chosen subsequence. Thanks to (4), the a priori estimate on the second moments gives a uniform bound on the r -moments, which improves the convergence of the approximate solutions. In particular, since the stored energy density has growth $2 < r$ (see assumption (1)), the functional $(\mathbf{b}^t, \boldsymbol{\lambda}^t) \mapsto \langle W, (\mathbf{b}^t, \boldsymbol{\lambda}^t) \rangle$ is continuous along the sequence of approximations (see [3, Remark 4.3]).

5.3 Stability condition and upper energy estimate

To prove that the limit of the approximate solutions satisfies the stability conditions, we fix \tilde{u} and M as in (ev1), and we construct a “recovery sequence” converging to $(\tilde{b}^{M, \tilde{u}}, \tilde{\lambda}^{M, \tilde{u}})$

and made by good tests for the minimality property satisfied by the approximations (see [4, Section 5]).

The continuity of the energy functional along the sequence of approximate solutions implies the continuity of the same functional along the recovery sequence, and this allows us to obtain the required stability in the passage to the limit.

A careful choice of the converging subsequence of approximate solutions is necessary in order to apply the argument in [2, Section 7]. This allows us to treat the term involving the stress in the energy balance and to obtain the upper energy estimate for the limit of the approximations.

5.4 Lower energy estimate

The proof of the lower energy estimate requires a more delicate argument than in the standard case (see e.g. [5, Step 5, p. 7]). Usually the proof of this estimate is based on a suitable minimality property guaranteed by the stability condition. In our case, due to the restriction of the set of competitors in the partial-global stability, we can only prove a weaker version of this minimality property, using the continuity provided by the regularity argument. This fact makes more delicate the last step of the proof, where we need to approximate a Lebesgue integral with Riemann sums.

The interested Reader will find more details in [4, Section 7.6].

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